Maximum Flow

The Maximum Flow Problem

Given

- Directed graph $G = (V, E)$
- A capacity function $c \colon E \longrightarrow \mathbb{R}^+$
- Two vertices s and t.

s, t-Flow

- Function $f \colon E \longrightarrow \mathbb{R}^+$ such that
- $f(e) \leq c(e)$,
- for all $v \in V \setminus \{s, t\}$, $\sum_{uv \in E} f(uv) = \sum_{vw \in E} f(vw)$,
- $\sum_{vs \in E} f(vs) = 0$, and $\sum_{tv \in E} f(tv) = 0$.

The value $|f|$ of an s, t-flow f is $\sum_{sv \in E} f(s, v)$

The Maximum Flow Problem

Maximum Flow Problem

For a given directed graph and two given vertices s and t, find an s, t-flow f with maximum value.

Multi-Source and Multi-Sink Flow

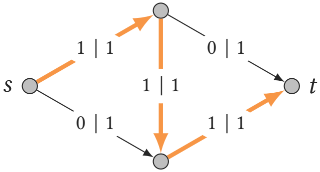
- Sources: $s_1, s_2, \dots, s_k$
- Sinks: $t_1, t_2, \dots, t_l$
- Add vertex s' and edges $s's_i$ with $c(s's_i) = \infty$ ($1 \leq i \leq k$).
- Add vertex t' and edges (t_it') with $c(t_it') = \infty$ ($1 \leq i \leq l$).
- Find maximum s', t'-flow.

Max Flow Algorithm

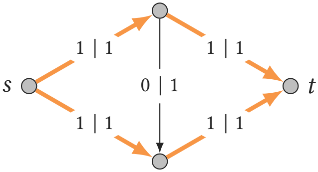
"Naive" Approach

- Find some path P from s to t where each edge e has $f(e) < c(e)$.
- Add increase $f(e)$ as much as possible for each edge of P.
- Repeat.

This approach will not always create an optimal solution



algorithm output



optimal solution

Residual Graph

Observation

- If $f(uv) = k$, we can assume there is an augmented edge vu with $c(vu) \geq k$, even if $vu \notin E$.

Residual Graph $G_f = (V, E_f)$

- $E_f = \{\, uv \mid uv \in E, f(uv) < c(uv) \,\} \cup \{\, vu \mid uv \in E, f(uv) > 0 \,\}$
- $c_f(uv) = \begin{cases} c(uv) - f(uv) & \text{if } uv \in E \\ f(vu) & \text{if } vu \in E \end{cases}$

f' is an s,t-flow for G_f if and only if $f + f'$ is an s,t-flow for G.

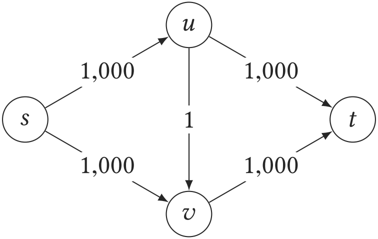
Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$.
- Find a simple s, t-path P in G_f.
- Update f with maximum flow for P.
- Update G_f.
- Repeat.

What is the worst case runtime for this algorithm?

Ford-Fulkerson Algorithm – Runtime

Consider the following case:



If the algorithm picks uv or vu in each iteration, it takes $2{,}000$ iterations to find a maximum flow.

Worst Case Runtime

- If $c\colon E \longrightarrow \mathbb{N}$, $\mathcal{O}(m|f_{\max}|)$

Edmonds-Karp Algorithm

Edmonds-Karp Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$.
- Find a *shortest* s,t-path P in G_f.
- Update f with maximum flow for P.
- Update G_f.
- Repeat.

Ford-Fulkerson

- Finds *some* s,t-path in G_f.
- Runtime: $\mathcal{O}(m|f_{\max}|)$ (integer capacities)

Edmonds-Karp Algorithm

- Finds *a shortest* s,t-path in G_f.
- Runtime: $\mathcal{O}(nm^2)$

Best known result today: $\mathcal{O}(nm)$ (complicated)

Max-Flow Min-Cut Theorem

Cut

s,t-Cut

An s,t-*cut* (S, T) in a graph $G = (V, E)$ is a partition of V into S and T such that $s \in S$ and $t \in T$.

The flow $f(S, T)$ is defined as

$$f(S, T) = \sum_{x \in S, y \in T, xy \in E} f(xy) - \sum_{x \in S, y \in T, yx \in E} f(yx).$$

The capacity $c(S, T)$ is defined as

$$c(S, T) = \sum_{x \in S, y \in T, xy \in E} c(xy).$$

Flows and Cuts

Lemma

Maximum s,t-flow $\leq$ minimum s,t-cut, i. e., $\max_f |f| \leq \min_{(S,T)} c(S,T)$.

Proof. Follows from definition. □

Lemma

For each s,t-cut (S,T), $f(S,T) = |f|$.

Proof.

- Let $V = \{s = v_0, v_1, v_2, \ldots, v_{n-1}, v_n = t\}$ and $V_i = \{v_i, v_{i+1}, \ldots, v_n\}$.
- $|f| = \sum_{sv \in E} f(s,v)$ and, for all $v \in V \setminus \{s,t\}$, $\sum_{uv \in E} f(uv) = \sum_{vw \in E} f(vw)$.
- Thus, $f(\{s\}, V_1) = f(\{s, v_1\}, V_2) = \ldots = f(S,T)$. □

Max-Flow Min-Cut Theorem

Theorem

For any s,t-flow f, the following three conditions are equivalent.

- (i) f is a maximum s,t-flow in G.
- (ii) There is no augmenting s,t-path in G_f.
- (iii) There is an s,t-cut (S, T) in G with $c(S, T) = |f|$.

$(i) \rightarrow (ii)$: If there would be an s,t-path P in G_f, $|f|$ could be increased by the capacity of P.

$(ii) \rightarrow (iii)$: There is no augmenting s,t-path in G_f. Let S be the set of vertices reachable from s in G_f and $T = V \setminus S$. If $c(S, T) > f(S, T)$, there is an edge xy in G_f with $x \in S$ and $y \in T$. Thus, y is reachable from s. Contradiction.

$(iii) \rightarrow (i)$: There is an s,t-cut (S, T) in G with $c(S, T) = |f|$. Because $\max_f |f| \leq \min_{(S,T)} c(S, T)$, f is a maximum s,t-flow in G. $\square$