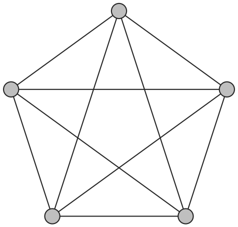
Clique, Vertex Cover, and Independent Set

Clique

Clique

A *clique* is a (sub)graph induced by a vertex set K in which all vertices are pairwise adjacent, i. e., for all distinct $u, v \in K$, $uv \in E$. A clique of size k is denoted as K_k.

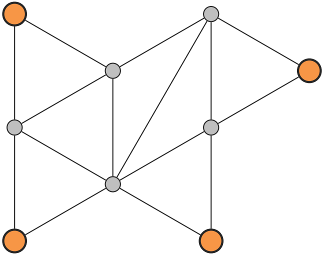


K_5

Independent Set

Independent Set

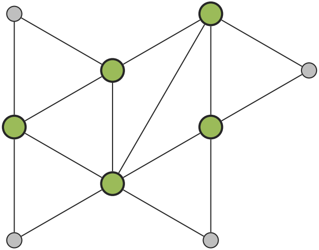
An *independent set* is vertex set S in which no two vertices are adjacent, i. e., for all distinct $u, v \in S$, $uv \notin E$.



Vertex Cover

Independent Set

A *vertex cover* is vertex set C such that each edge contains at least one vertex in C, i. e., for all distinct $uv \in E$, $u \in C \vee v \in C$.



Connection between Clique, VC, and IS

Theorem

For a graph $G = (V, E)$ and a set $S \subseteq V$, the following are equivalent

(i) S is an independent set in G.

(ii) S induces a clique in $\overline{G}$.

(iii) $C = V \setminus S$ is a vertex cover for G.

Proof.

$(i) \leftrightarrow (ii)$: By definition of complement, $uv \in E \leftrightarrow uv \notin \overline{E}$.

$(i) \longrightarrow (iii)$: Assume C is not a vertex cover. Then, there is an edge uv with $u, v \notin C$. Thus, $u, v \in S$. This is in contradiction with S being an independent set.

$(iii) \longrightarrow (i)$: Assume S is not an independent set. Then, there is an edge uv with $u, v \in S$. Thus, $u, v \notin C$. This is in contradiction with C being a vertex cover. $\square$

Connection between Clique, VC, and IS

Theorem

If there is an algorithm that solves one of the problems in $\mathcal{O}\left(f\left(|V|,|E|\right)\right)$ time on any given graph, then there is an algorithm which solves the other two problems in $\mathcal{O}\left(f\left(|V|,|V|^2\right)\right)$ time.

Theorem

There is (probably) no polynomial time algorithm to find a maximum clique, a maximum independent set or a minimum vertex cover in a given graph.

2-Approximation for Vertex Cover

Algorithm

- Pick an arbitrary edge uv.
- Add u and v to a set C.
- Remove u and v from G.
- Repeat until G has no edges left.

Can be implemented to run in linear time.

Theorem

If a graph has a minimum vertex cover C^*, then the algorithm creates a vertex cover C such that $|C^*| \leq |C| \leq 2|C^*|$.

There is (probably) no polynomial time algorithm to find a constant factor approximation for the maximum independent set problem.

Independent Set for Trees

Lemma

If u is a pendant vertex in a graph G, then there is a maximum independent set I with $u \in I$.

Proof.

- Let v be the neighbour of u, and I be a max. IS for G.
- Since I is a max. IS, $u \notin I$ if and only if $v \in I$.
  (Otherwise $I \cup \{u\}$ would be a larger IS.)
- If $u \notin I$, let $I' := (I \cup \{u\}) \setminus \{v\}$.
- $|I'| = |I|$ and there is no $w \in I$ with $uw \in E$.
- Thus, I' is a maximum independent set. □

Lemma

I is a maximum independent set for a graph G with $u \in I$ if and only if $I \setminus \{u\}$ is a maximum independent set for $G\big[V \setminus N[u]\big]$.

Independent Set for Trees

Algorithm

- Add all leaves to the set I.
- Remove all leaves and their neighbours from the tree.
- Repeat until tree has no vertices left.

Theorem

The algorithm finds a maximum independent set in a tree in linear time.