Dominating Set and p-Center Problem

Dominating Set

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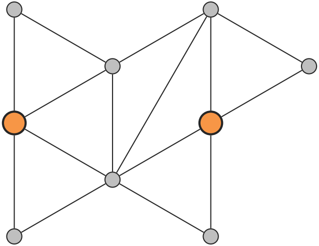
For a graph $G = (V, E)$, a vertex set D is called *dominating set* if

$$\bigcup_{v \in D} N[v] = V.$$

If $v \in D$ and $u \in N[v]$, we say that v *dominates* u.



Domonating Set Problem

Dominating Set Problem

Given a graph G find a dominating set D such that $|D|$ is minimal.

Theorem

There is (probably) no polynomial time algorithm to find a minimum dominating set.

We can get a reasonable approximation in linear time.

Approximation for Dominating Set

Algorithm

- Pick a vertex v such that the number of non-dominated vertices in $N[v]$ is maximal and add v to the set D.
- Repeat until each vertex is dominated.

Theorem

If D^* is a minimum dominating set, the algorithm computes a dominating set D such that $|D^*| \leq |D| \leq (1 + \ln |V|)|D^*|$.

Theorem

There is (probably) no polynomial time algorithm which guaranties to find a dominating set D such that $|D| \leq c \log |V||D^*|$ for some $c > 0$.

Dominating Set for Trees

Lemma

Let v be a leaf in T with the parent u. There is a minimum dominating set D with $u \in D$.

Proof.

- Assume that $u \notin D$. Then, $v \in D$.
- Because $N[v] \subseteq N[u]$, $D' := D + u - v$ is a minimum dominating set. $\square$

Note. This only works, because v has only one neighbour. To make it work for multiple neighbours, we would require that, for all $w \in N[v]$, $N[w] \subseteq N[u]$. In this case, u is called am *maximum neighbour* of v.

Dominating Set for Trees

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Pick an arbitrary vertex s.
Compute a BFS-order σ = ⟨s = v_1, v_2, ..., v_n⟩.
For i := n downto 1
    If v_i is not dominated Then
        Add the parent u of v to the set D and mark all neighbours of u
          as dominated. (We consider s as its own parent.)
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Runs in linear time.

Variants of Dominating Sets

Independent Domination

- D is an independent set.

Connected Domination]

- D induces a connected subgraph.

r-Domination

- $r : V \longrightarrow \mathbb{N}$. For all v, there is a $u \in D$ with $d(u, v) \leq r(v)$.

Total Domination

- For all v, $N(v) \cap D \neq \emptyset$, i. e., a vertex v does not dominate itself.

Perfect Domination

- For all $v \notin D$, $\left|N[v] \cap D\right| = 1$.

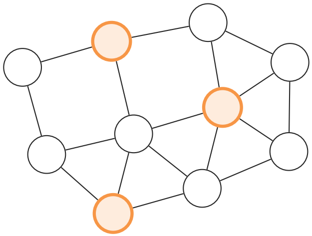
Efficient Domination

- For all v, $\left|N[v] \cap D\right| = 1$.

p-Center

p-Center Problem

For a given graph $G = (V, E)$ and given integer p, determine a vertex set D with $|D| \leq p$ such that $\mathrm{ecc}(D) = \max_{v \in V} d(v, D)$ is minimal.



An optimal 3-center.

r-Domination vs. p-Center

Two versions of the same problem.

r-Domination

- Given: Maximal distance.
- Find: Best cardinality.

p-Center

- Given: Maximal cardinality.
- Find: Lowest maximum distance.

An algorithm for one problem gives an algorithm for the other problem with low computational overhead.

2-Approximation

Input: A graph G and an integer p.

Pick an arbitrary vertex v and set $D := \{v\}$.

For $i := 2$ **To** p

  Find a vertex u with maximum distance to D.

  Add u to D, i. e., set $D := D \cup \{u\}$.

Output D.

Runs in $\mathcal{O}(pm)$ time.

2-Approximation

Theorem

For a given graph G, the algorithm computes a 2-approximation. That is, if C is an optimal p-cernter for G, then the algorithm computes a set D such that $|C| = |D|$ and $\mathrm{ecc}(C) \leq \mathrm{ecc}(D) \leq 2\,\mathrm{ecc}(C)$.

Proof.

- Let $C = \{c_1, c_2, \ldots, c_p\}$ and $\mathcal{V} = \{V_1, V_2, \ldots, V_p\}$ be a partition of V such that, for all i and all $v \in V_i$, $d(v, c_i) = d(v, C)$.
- By pidginhole principle, either (i) each set V_i contains a vertex $u_i \in D$, or (ii) there is a set V_i containing at least two vertices of D.
- *Case* (i). Since $d(v, c_i) \leq \mathrm{ecc}(C)$ for each $v \in V_i$, $d(v, u_i) \leq 2\,\mathrm{ecc}(C)$. Hence, $ecc(D) \leq 2\,\mathrm{ecc}(C)$.
- *Case* (ii). There is a set V_i containing two vertices $u, w \in D$. W. l. o. g., let u be added to D after w. Clearly (see case (i)), $d(u, w) \leq 2\,\mathrm{ecc}(C)$. Thus, by choice of u (vertex with max. distance), $\mathrm{ecc}(D) \leq 2\,\mathrm{ecc}(C)$.

Additive Approximation using r-Domination

$(r + \phi)$-Dominating Set

Let D be an optimal r-dominating set for a graph G. Then, D' is an $(r + \phi)$-*dominating set* for G if $|D'| \leq |D|$ and, for each vertex v, $d(v, D') \leq r(v) + \phi$.

Theorem

If an $\mathcal{O}\big(T(G)\big)$ time algorithm computing a $(r + \phi)$-dominating set is given, one can compute a $+\phi$-approximation for the p-Center problem in $\mathcal{O}\big(T(G) \log n\big)$ time.

Proof (outline).

- Make binary search on r to find smallest r such that, for the $(r + \phi)$-dominating set D', $|D'| \leq p$.
- Always successful if $r \geq \mathrm{ecc}(C)$ where C is optimal p-center.