\documentclass[12pt]{article}

\usepackage{graphicx}

\usepackage{amsmath, bm, amssymb, amsthm}

\usepackage{setspace}

\doublespacing

\usepackage[a4paper, total={6in, 8in}]{geometry}

\usepackage{comment}

\usepackage{setspace}

\AtBeginDocument{%

\addtolength\abovedisplayskip{-0.5\baselineskip}%

\addtolength\belowdisplayskip{-0.5\baselineskip}%

% \addtolength\abovedisplayshortskip{-0.5\baselineskip}%

% \addtolength\belowdisplayshortskip{-0.5\baselineskip}%

}

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Textclass specific LaTeX commands.

\newtheorem{theorem}{Theorem}[section]

\newtheorem{corollary}{Corollary}[section]

\newtheorem{definition}{Definition}[section]

\newtheorem{assumption}{Assumption}[section]

\newtheorem{lemma}{Lemma}[section]

\begin{document}

\author{Andre Hood}

\title{The Importance of Exponentially Weighted Moving Averages in Forecasting}

\maketitle

\section{Introduction}

Forecasting is one of the most critical and important parts of Statistics in all areas of the professional world. Forecasts are generally made

and analyzed by people in the business realm. But what are they, and why are they important?

Often times, people who operate in the business world, use them and have absolutely no idea what they really are.

In reality, this is an importance that needs to be brought to the forefront of businesses and corporations looking to improve productivity.

Most of the time, forecasts are done on hundreds, if not thousands, of different products at a time. Therefore we, as mathematicians,

must use the most efficient and cost effective methods possible. One very effective forecasting method uses what's known as

exponentially weighted moving averages. We will explore the various applications of this method, and what makes each desirable

in terms of forecasting. To be fair, we will also weigh-on some of the negative trade-offs of using this particular set of applications.

As always, we will start off with a hypothesis, analyze our findings in research, perform statistical testing, and come to a set of results based

on our findings.

\section{Simple Exponential System}

Let's start off with a definition of what Statisticians call an exponential system. An exponential system takes the weight of current observed values, as

well as previously observed values for a period of time. These observations are then used to make a forecast of future observations for a period of time.

A very simple application of an exponentially weighted moving average would

be making a forecast of the mean of a stochastic

variable. That mean would not change between successive

drawings. This would correspond to predicting the expected sales for a product

which had no definite seasonal pattern and no long-run trend.

\subsection{EWMA estimator}

Consider if we take a weighted average of all past observations and use

this as a prediction of the current mean of the distribution, as

\begin{align}

\tilde{S}\_t &= AS\_t + (1-A)\tilde{S}\_{t-1}

\end{align}

where

\begin{align\*}

S\_t &= \text{actual sales during the $t^{th}$ period} \\

\tilde{S}\_t &= \text{forecast of expected sales in the $t^{th}$ period} \\

0 &< A \leq 1, ~ t \in \{1,2, \ldots, M\}

\end{align\*}

then

\begin{align}

\tilde{S}\_{t-1} = AS\_{t-1} + (1-A)\tilde{S}\_{t-2}

\end{align}

so that

\begin{align}

\tilde{S}\_t = AS\_t + A(1-A)S\_{t-1} + (1-A)^2\tilde{S}\_{t-2}

\end{align}

we can continue this process until we reach the general form

\begin{align}

\tilde{S}\_t = A\sum\_{n=0}^{M}(1-A)^{n}S\_{t-n} + (1-A)^{M+1}\tilde{S}\_b.

\end{align}

\begin{align\*}

M &= \text{number of observations up to and including time t} \\

\tilde{S}\_b &= \text{the initial observed value of } \tilde{S}

\end{align\*}

If there isn't a clear seasonal pattern or trend in process which generates the sales data $S\_t$, we can assume that $S\_1, S\_2, \ldots, S\_M$ are independent and identically distributed random variables.

If $E(S)$ initial specified, then we set $\tilde{S}\_b = E(S)$ to be the initial observed value. Consequently, we will show in Theorem 3.1 that $\tilde{S}\_t $ is an unbiased estimator of $E(S)$. However, if $E(S)$ is unknown, then we can set $\tilde{S}\_b$ with an an initial guess. In this case, $\tilde{S}\_t \approx E(S)$ when $M$ is large. The statistical properties of $\tilde{S}\_t$ including its mean and variance will be derived in Section 3.

\subsection{Role of the exponential weight}

If the distribution mean is susceptible to a variety

of short-term and long-term changes, then model itself has some

intuitive advantages. If the distribution mean exhibits a slow change, then $A$ should be

a smaller value, keeping the effect of older observations. If the distribution mean

changes quickly, then $A$ should be a larger value minimizing the effect of

older observations, but not too large, or else $\tilde{S}\_t$ will be subject to more

random variation than we want. The problem of finding a satisfactory value of the weighting

parameter, $A$, will not be solved in detail for the simple exponential model.

\begin{comment}

\section{Ratio Seasonal Forecasting}

It is possible to develop a forecasting model with either a multiplicative or an

additive seasonal effect. If the amplitude of the seasonal pattern is independent

of the level of sales, then an additive model is appropriate. More often, however,

the amplitude of the seasonal pattern is proportional to the level of sales. This

would indicate using the multiplicative, or ratio, seasonal effect. This represents the sales for an individual product over a period of time. The actual sales in

period t is given by $S\_t$. The estimate of the smoothed and seasonally adjusted

sales rate in period t is given by $S\_t$ . The periodicity of the seasonal effect is L;

$S\_t$ is

a weighted sum of the current estimate obtained by de-seasonalizing the current

sales, $S\_t$, and last period's estimate, $S\_{t-n}$, of the smoothed and seasonally adjusted sales rate for the series. (Note that in de-seasonalizing current sales by

$S\_t/F\_{t-L}$, the most recent estimate of the seasonal effect for periods in this position in the cycle has been used; the seasonal factor computed for May last year

would be used to seasonally adjust this year's May data.) The value of $S\_t$ is then used in forming a new estimate of the seasonal factor. This new estimate, $F\_t$, is again a weighted sum of the current estimate, $S\_t/S\_t$, and the previous estimate, $F\_{t-L}$ .

In practice, the forecasting system would be used as follows to predict the

sales of an individual product:

\begin{enumerate}

\item At the end of the $t^{th}$ (or current) period the actual sales of the product

during the period, St, is recorded.

\item The simple exponential equation is applied to evaluate $S\_{t}$, using $S\_{t-n}$, and $A\_{t-n}$, from the last

period and the appropriate $F\_{t-L}$ computed during the previous cycle.

\item $S\_t$ is then used to evaluate $F\_t$, which can now replace $

F\_{t-L}$

\item $F\_t$ is used to determine $R\_t$, which can now replace $R\_{t-1}$ .

\item Forecasts of future sales can then be recorded and analyzed.

\item Finally, the value of $S\_{t-n}$ is replaced by $S\_t$ and the data is ready for use at the end

of the coming period.

\end{enumerate}

\end{comment}

\section{Statistical Properties of EWMA estimators}

If there is no seasonal pattern or trend in process which generates the sales data $S\_t$, we can assume that $S\_1, S\_2, \ldots, S\_M$ are independent and identically distributed random variables.

\begin{lemma}

Let $0< A \leq 1$,

\begin{equation}

\sum\_{n=0}^{M}A(1-A)^{n} + (1-A)^{M+1} = 1.

\end{equation}

\end{lemma}

\proof Let $r=1-A$, $0<r\leq 1$. We know from finite geometric sum that

\begin{equation}

\sum^M\_{n=0}r^n = (1-r^{M+1})/(1-r).

\end{equation}

It follows that

\begin{align\*}

\sum\_{n=0}^{M}A(1-A)^{n} + (1-A)^{M+1} &= A\sum\_{n=0}^{M}(1-A)^{n} + (1-A)^{M+1} \\

&= (1-r)\sum\_{n=0}^{M}r^{n} + r^{M+1} \\

&= 1 - r^{M+1} + r^{M+1} = 1.

\end{align\*}

\endproof

\begin{theorem}

Let $\tilde{S}\_t=AS\_t + (1-A)\tilde{S}\_{t-1}$, $0 < A \leq 1$, be the EWMA statistic. Then

\begin{itemize}

\item Mean of $\tilde{S}\_t$:

\begin{itemize}

\item $E(\tilde{S}\_t) = E(S)$ if ~ $\tilde{S}\_b = E(S)$

\item $E(\tilde{S}\_t) \approx E(S)$ for large $M$

\end{itemize}

\item Variance of $\tilde{S}\_t$:

$$

Var(\tilde{S}\_t) = \sigma^2 \frac{A}{2-A}[1-(1-A)^{2(M+1)}]

$$

\end{itemize}

\end{theorem}

\proof

Mean of $S\_t$ \\

Suppose $S\_t$ is an independently and identically distributed

random variable where $t$ is time index.

\begin{align\*}

E(\tilde{S\_t}) &= E\Big[\sum\_{n=0}^{M}A(1-A)^{n}S\_{t-n} + (1-A)^{M+1}\tilde{S\_b}\Big] \\

&= \sum\_{n=0}^{M}A(1-A)^{n}E(S) + (1-A)^{M+1}E(S) \\

&= E(S)[\sum\_{n=0}^{M}A(1-A)^{n} + (1-A)^{M+1}] = E(S)

\end{align\*}

Now, suppose $\tilde{S\_b} = E(S)$ and $M$ is large.

\begin{align\*}

E(\tilde{S\_t}) &= E[\sum\_{n=0}^{M}A(1-A)^{n}S\_{t-n} + (1-A)^{M+1}\tilde{S\_b}] \\

&= \sum\_{n=0}^{M}A(1-A)^{n}E(S) + (1-A)^{M+1}E(S) \\

&= E(S)[\sum\_{n=0}^{M}A(1-A)^{n}] + 0 \approx E(S)

\end{align\*}

Now, let's proceed with the proof for the variance of $\tilde{S}\_t$

\begin{align\*}

Var(\tilde{S\_t}) &= Var[\sum\_{n=0}^{M}A(1-A)^{n}S\_{t-n}+ (1-A)^{M+1}\tilde{S\_b}] \\ = Var[\sum\_{n=0}^{M}{A(1-A)^{n}S\_{t-n}}] + 0

\end{align\*}

as $m \longrightarrow \infty$,

$$(1-A)^{2(M+1)} \longrightarrow{0}.$$

Therefore, the variance of $\tilde{S\_t}$ goes to $\sigma^2 \frac{A}{2-A}.$

We will have then achieved our unbiased estimator.

\endproof

\section{Holt Winter's Method for Seasonality Is Now Possible}

There are two different versions of this method that differ in the nature of the seasonal component. The additive method is used when the seasonal variations are extremely close to constant through the series, while the multiplicative method is implemented when the variations in seasonality are changing proportional to the level of the time series.

Thanks to the work done with the simple exponential smoothing, we are now able to extend our forecasting abilities to different versions of the same smoothing equations, thus giving us more accurate and precise

estimates and estimators for our forecasting.

\section{Bibliography}

\begin{enumerate}

\item[1.] Forecasting Sales by Exponentially Weighted Moving Averages

Author(s): Peter R. Winters

Source: Management Science, Vol. 6, No. 3 (Apr., 1960), pp. 324-342

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Stable URL: https://www.jstor.org/stable/2627346

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\end{enumerate}

\end{document}