Donsker’s theorem

Donsker's theorem is a fundamental result in probability theory and mathematical statistics that describes the convergence of stochastic processes to Brownian motion.

Donsker's theorem provides the conditions under which certain types of stochastic processes, particularly empirical distribution functions, converge to a specific distribution known as the Wiener process or Brownian motion.

Brownian motion is a continuous-time stochastic process with independent and normally distributed increments, making it a fundamental model for various phenomena.

Donsker’s theorem definition

The theorem states that if we have a sequence of independent and identically distributed random variables, and we consider the empirical distribution function associated with these random variables, when appropriately normalized and scaled, they converge in the distribution to a Brownian motion.

Consider a sequence of independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$ with a common distribution function F. Suppose F_n represents the empirical distribution function associated with the first n observations, defined as:

$$F_n(x) = \sum_{i=1}^{n} X_i$$ known as random walk

If the random variables $X_1, X_2, \ldots$, are bounded and certain normalization and scaling conditions are applied to the empirical distribution function F_n, then when n tends to infinity, the sequence of processes $\sqrt{n}(F_n(x) - F(x))$ converges in the distribution to a Brownian motion or Wiener process.

In simpler terms, when you take a sequence of independent and identically distributed random variables, form their empirical distribution functions, and appropriately scale and normalize these functions by $\sqrt{n}$, as n grows, the resulting sequence of processes converges to a Brownian motion in its distribution.

Donsker's theorem is significant in probability theory and statistics because it extends the **Central Limit Theorem** by demonstrating convergence to a continuous-time stochastic process for a wider range of empirical processes beyond simple sums of random variables.

Relationship between Donsker’s theorem and Central Limit Theorem (CTL)

The relationship between Donsker's theorem and the Central Limit Theorem (CLT) lies in their connections regarding convergence to Gaussian processes.

The CLT is a fundamental theorem in probability theory that describes the behavior of the sum of independent and identically distributed random variables. It states that as the sample size increases, the distribution of the sample mean approaches a normal distribution.

Donsker's theorem can be seen as an extension of the CLT to more general types of random variables or stochastic processes. While the CLT specifically deals with the convergence of sums of random variables to a normal distribution, Donsker's theorem focuses on the convergence of empirical processes to a Brownian motion, which is a continuous-time Gaussian process.

Both theorems are crucial in statistical theory, describing the behavior of different types of random variables and processes as sample sizes grow, with the CLT focusing on sums of random variables and Donsker's theorem extending this concept to empirical processes.

References:

Donker’s Theorem - *Wikipedia*
https://en.wikipedia.org/wiki/Donsker%27s_theorem#History_and_related_results