Itô Calculus

Itô calculus, named after the Japanese mathematician Kiyoshi Itô, is a branch of mathematics that deals with stochastic processes and their integration. It is a mathematical theory used to handle computation for stochastic processes involving random changes over time, such as Brownian motion.

Stochastic processes are mathematical models used to describe the evolution of random phenomena over time.

Ito calculus extends classical calculus to handle stochastic processes, allowing mathematicians and researchers to perform calculus operations involving these processes. It introduces the concept of stochastic integrals and differential equations, which are essential tools for modeling and analyzing systems subject to random fluctuations.

The central concept in Ito calculus is the stochastic differential equation (SDE). A stochastic differential equation is an equation that involves both deterministic differentials and stochastic differentials (dW_t), where dW_t represents an infinitesimal increment of a Wiener process or Brownian motion.

The basic form of an Ito stochastic differential equation is:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Where:

• dX_t is the infinitesimal change in the stochastic process X_t.

• $\mu(t, X_t)$ is the drift term, representing the deterministic rate of change of the process

• $\sigma(t, X_t)$ is the diffusion term, indicating the volatility or randomness in the process

• dW_t is the stochastic differential associated with the Wiener process

Ito calculation consists of three phases

1. **Defining the SDE:**
   Formulate a stochastic differential equation representing the dynamics of the stochastic process.

2. **Solving the SDE:**
   Find solutions to the stochastic differential equation using various techniques like numerical methods, simulations, or analytical approaches when possible.

3. **Analysis:**
   Use the solutions to understand the behavior of the stochastic process, study its properties, and make predictions or inferences about the system it models.

Itô Integration

Ito integration is a fundamental concept in Ito calculus. It involves integrating stochastic processes with respect to a continuous-time stochastic process, often a Wiener process (Brownian motion).

The Ito integral generalizes the concept of integration from deterministic calculus to handle integrals involving stochastic terms or random variables. The process of Ito integration allows for the calculus of integrals with respect to stochastic processes, enabling the manipulation and analysis of random processes.

The formal definition of the Ito integral involves approximating the integral by summing up the products of random increments (associated with the stochastic process) and a function evaluated at different points. The process relies on constructing stochastic Riemann sums and taking limits to define the integral.

Ito integration plays a crucial role in stochastic differential equations (SDEs), as solutions to SDEs often involve Ito integrals.

The Ito integral is denoted by:

$$\int_0^t H_s \, dX_s$$

Where:

• dX_s represents the stochastic increment of the stochastic process at time s.

The Ito integral is defined using limits of stochastic Riemann sums, similar in principle to Riemann integration in classical calculus. The integrand is multiplied by the stochastic increment dX_s, and the sum is taken over infinitesimal increments of the stochastic process within the integration interval. The limit of these sums, as the partition becomes finer, defines the Ito integral.

The main properties of the Ito integral include linearity and Ito's lemma, which is a fundamental result in stochastic calculus. Ito's lemma relates the behavior of a function of a stochastic process to the behavior of the process itself, allowing for the analysis of transformations of stochastic processes.

References

Itô calculus - *Wikipedia*
https://en.wikipedia.org/wiki/It%C3%B4_calculus
https://it.wikipedia.org/wiki/Lemma_di_It%C5%8D