Analysis and application of Stochastic process

Introduction

In Statistics, a stochastic process is a concept that describes the evolution of a series of random variables over time or space.

It is a mathematical model that represents the random change of a system over time, where future conditions depend on both present conditions and random influences. This allows us to understand dynamic phenomena characterized by uncertainty and randomness.

Stochastic processes are often used to model and analyze dynamic and variable phenomena in which uncertainty or randomness play a significant role, such as in finance, weather forecasting, engineering, and many other disciplines. Their versatility makes them essential tools for understanding and modeling complex phenomena characterized by random fluctuations, making it possible to predict, optimize and manage risk in situations where uncertainty plays a significant role.

Definitions and key concepts

Stochastic processes are based on a number of fundamental concepts that form the basis for their understanding.

• Random Variables:
They are mathematical entities that represent the results of a random experiment. They can take on different values based on the outcome of the experiment, following a probability distribution.

• Conditional Probability:
It represents the probability that an event will occur, considering that another event has already occurred previously. It is essential for understanding how random variables can be affected by previous events.

• Stochastic Process:
It is a set of random variables that represent the evolution of a system over time. These variables depend on temporal or spatial parameters and can be used to model random changes over time.

• Conditional Probability Function:
It expresses the probability that a stochastic process takes on certain values at a certain instant of time, given the previous values. It is useful for understanding how the process evolves temporally.

• Probability Distribution:
Represents the probability associated with each value that a random variable can take on. In the case of stochastic processes, this distribution can vary in time or space.

• Random Sampling:
It is the process of extracting random data from a probability distribution. In the context of stochastic processes, random sampling is often used to simulate the evolution of the process over time.

Classification of stochastic processes

Stochastic processes can be classified into different categories based on their properties and characteristics that define them. The main categories of stochastic processes are:

Stationary Processes:
These processes have statistical properties that remain constant over time. They can be stationary in the sense:

- **strong**, in which all statistical properties are constant with respect to time.
- **weak**, in which only means and variances are constant over time.

Markov processes:
A Markov process is characterized by the fact that future conditions depend only on the present state of the system, ignoring the past as well as the present. This property is known as the "Markov property" or "bounded memory property".

Discrete and Continuous Time Processes:
This classification refers to the nature of the time in which the evolution of the process occurs. Discrete-time processes are defined over discrete intervals of time (for example, discrete time instants such as days or hours), while continuous-time processes are defined over continuous intervals of time.

Finite and Infinite Time Processes:
Indicates the duration of time over which the process is defined. Finite time processes are defined over a finite time interval, while infinite time processes are defined over a time interval that extends to infinity.

Ergodic Processes:
These processes exhibit a particular property called ergodicity, which implies that time averages converge to statistical averages over a long period of time. In other words, the average behavior of the prxocess over time is representative of its statistical behavior.

Fixed and Variable Time Stochastic Processes:
This classification is based on the nature of the temporal variation of process properties. Some processes maintain the same statistical properties over fixed time intervals, while others can vary their properties over time.

The classification of stochastic processes is fundamental to understand the specific properties of a process and to select the most appropriate model to describe a real or theoretical phenomenon accurately.

Stochastic Differential Equations (SDEs)

Stochastic differential equations constitute a fundamental tool for describing the evolution of stochastic processes over time. Unlike ordinary differential equations that involve deterministic functions, SDEs also involve stochastic components, that is, terms that include randomness.

These equations represent a complex tool used to describe the evolution of systems under the influence of random fluctuations, allowing the understanding and modeling of real phenomena involving uncertainty and randomness.

The general form of a stochastic differential equation is:

$$dX(t) = a(X(t), t)dt + b(X(t), t)dW(t)$$

Where:

• $(X(t)$ is the stochastic process.

• $a(X(t), t)$ is the deterministic term that can depend on the process state and time.

• $b(X(t), t)$ is the stochastic term involving the **Wiener process** $dW(t)$**.**

• dt e $dW(t)$ represent the time differential and **the Wiener process**.

SDEs are often solved using specific techniques, including the **Itō integral method**, which generalizes Riemann-Stieltjes integration to stochastic variables. This method allows you to correctly manipulate the stochastic terms and their differentials in the process of solving SDEs.

SDEs solutions can have different forms:

Strong Solutions: A strong solution of an SDE is a solution that exists and is unique for deterministically defined initial conditions.

Weak Solutions: Weak solutions consider the statistical aspect of the process, allowing a broader interpretation of the SDE solutions. They focus on the distribution of the random variables involved in the equation.

Simulation methods and numerical approximation

Numerical simulation and approximation methods are used to study and analyze stochastic processes when it is not possible to obtain exact analytical solutions or when it is desired to evaluate the behavior of such processes in a computationally efficient manner. However, it is important to note that the accuracy of such methods is affected by the need to generate a large number of simulations.

Monte Carlo Simulation:
This method involves the generation of random numbers to simulate the evolution of the process over time. It uses a large number of random samples to statistically approximate the properties of the stochastic process. Monte Carlo simulation is extremely flexible and can be applied to a wide range of stochastic processes.

Euler-Maruyama method:
The Euler-Maruyama method is a numerical approximation method used to solve stochastic differential equations. It is based on a time discretization and approximation of the stochastic process increment using the Euler method, adding the **Wiener term** to capture the stochastic aspect of the process.

SDEs usually have a deterministic component $a(X(t),\ t)dt$ and a stochastic component $b(X(t),\ t)dW(t)$.

The term $dW(t)$ represents the Wiener process increment, and dt is the time increment.
The Euler-Maruyama method approximates the evolution of the stochastic process from t_n to t_{n+1} incrementally, using the following iterative formula:

$$X_{n+1} = X_n + a(X_n, t_n)\Delta t + b(X_n, t_n)\Delta W_n$$

Where:

- X_n is the approximate value of the process at time t_n.

- X_{n+1} is the approximate value of the process at the next time t_{n+1}.

- $a(X_n, t_n)$ and $b(X_n, t_n)$ represent the deterministic and stochastic terms of the SDE evaluated at time t_n and X_n.

- Δt is the time increment, calculated as $t_{n+1} - t_n$.

- ΔW_n represents the increment of the Wiener process between t_n and t_{n+1}.

The quantity ΔW_n follows a normal distribution with zero mean and variance Δt, and is therefore generated as a random number drawn from a normal distribution.

The Euler-Maruyama method is simple to implement and computationally efficient. However, it is important to note that it can present numerical stability problems when used with very small time steps or with particularly complex stochastic equations.

Therefore, the choice of time step is crucial to ensure the accuracy and reliability of the approximations obtained with this method.

Wiener process (Brownian motion)

The Wiener process, also known as Brownian motion, is a fundamental continuous-time stochastic process used in various fields, including mathematics, physics, finance, and more.

Brownian motion takes its name from Robert Brown, a Scottish botanist, who, observing pollen particles suspended in water under a microscope, noticed a chaotic and irregular movement of the particles.

This observation became a fundamental object of study, stimulating questions and research in the scientific field.

Norbert Wiener was the first to develop a mathematical model used to describe the irregular movement of particles suspended in a fluid through what is called the Wiener process. The Wiener process is considered one of the fundamental pillars in the understanding of random phenomena and their mathematical representation.

The Wiener process is often denoted as W_t , where t represents time.

The Wiener process is characterized by the following conditions:

• $W_0 = 0$ (Starting point is zero)

• $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ for $0 <= s < t$

• the increments $W_t - W_s$ are independent for different time intervals.

Wiener process: Key Characteristics

Continuous Paths:
The Wiener process is characterized by continuous paths without any breaks or discontinuities. This means that the process evolves smoothly as time progresses

Independent and Stationary Increments:
The increments of the Wiener process over non-overlapping intervals are independent of each other and have a Gaussian (normally distributed) distribution. Additionally, these increments are stationary, meaning that their statistical properties (mean and variance) remain constant over time.

Gaussian Distribution:
At any given time, the value of the Wiener process follows a normal distribution. This property is crucial in many statistical and mathematical analyses involving random variables.

Randomness and Unpredictability:
The key defining feature of the Wiener process is its **"random walk"** behavior, where the process shows continuous but irregular movements, making its future path unpredictable. This unpredictability is a fundamental characteristic of the Wiener process and is integral to its applications in various fields.

Geometric Brownian Motion (GBM)

Geometric Brownian Motion is a continuous-time stochastic process that extends the Brownian motion (Wiener process) to model the dynamics of various quantities that exhibit random behavior over time, particularly in finance.

Continuous and Exponential Growth or Decay:
GBM models the exponential growth or decay of a quantity over time. This property makes it suitable for representing processes where the rate of change is proportional to the current value, leading to exponential growth or decay.

Drift and Volatility:
GBM incorporates two parameters:

- **Drift (μ):** Refers to the average rate of change of the stochastic process value over time. Represents the expected growth (or decay) of the modeled quantity.
- **Volatility (σ):** refers to the degree to which a value changes over time.

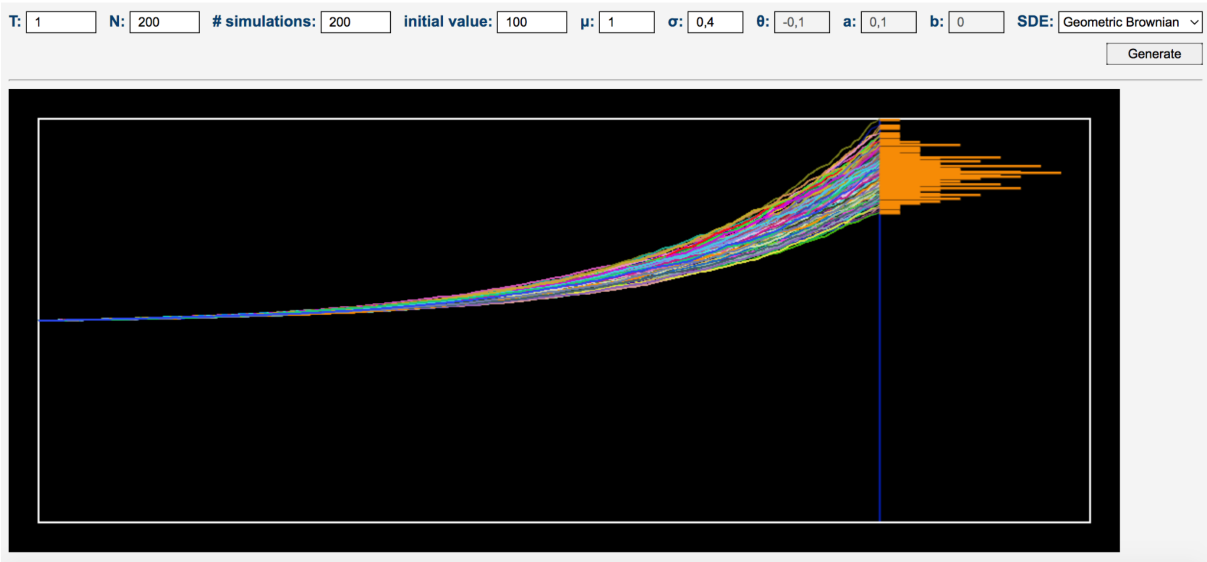
Continuous Paths:
Similar to the Wiener process, GBM exhibits continuous paths without breaks and discontinuities.

The equation describing Geometric Brownian Motion is:

$$dS = \mu S dt + \sigma S dW$$

where:

- dS represents the change in the value of the quantity;
- μ is the drift coefficient
- σ is the volatility coefficient
- S is the current value of the quantity.
- dt is an increment time
- dW represents the Wiener process.



this simulation, and subsequent ones, was calculated using the code I developed for the seventh homework

You can find and directly try the SDEs simulator at the following link https://andi2ews.github.io/Statistics/HW7/SDE%20simulation/

Ornstein–Uhlenbeck process

The Ornstein-Uhlenbeck process is a stochastic process used in financial mathematics and other fields to model the movement of variables that exhibit regression behavior toward a mean over time. It is named after scientists Leonard Ornstein and George Uhlenbeck who helped develop the model.

The Ornstein-Uhlenbeck process is defined by the following stochastic differential equation:

$$dX(t) = \; - \; \Theta X(t)dt \; + \; \sigma dW(t)$$

To which a further drift term is sometimes added, which makes it:

$$dX(t) = \; \Theta(\mu - X(t)dt \; + \; \sigma dW(t)$$

where:

• $X(t)$ represents the random variable at time t

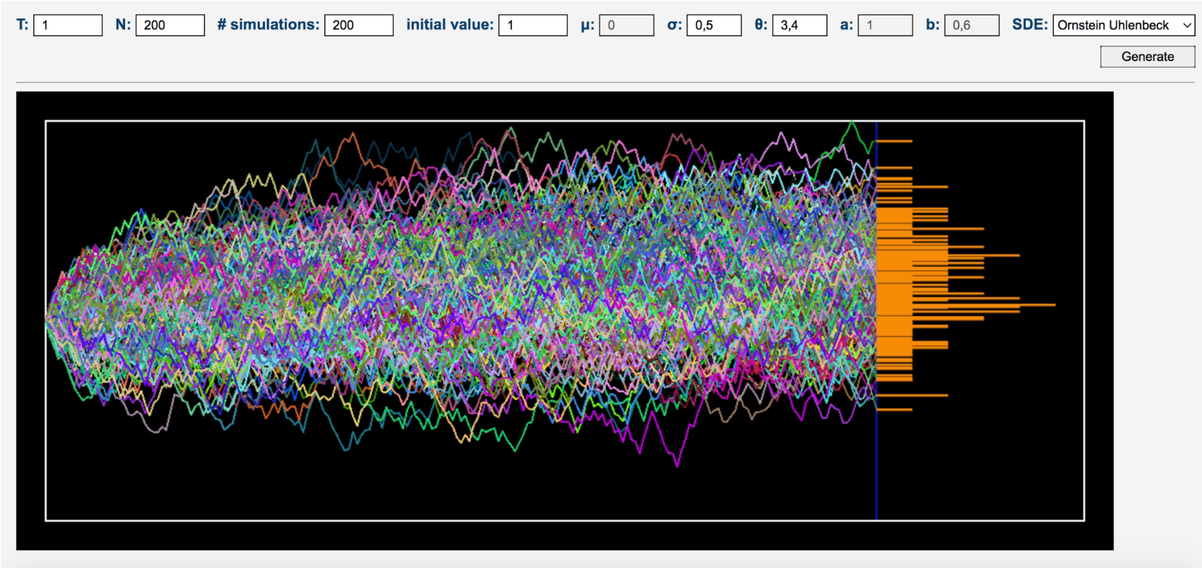
• Θ is the parameter of reversibility or speed of regression towards the mean μ

• μ is the mean value towards which the process tends over time

• σ is the volatility of the process

• $dW(t)$ represents a variation of the Wiener process

This process describes the behavior of a random variable that tends to return towards the mean value μ over time, with a speed determined by the parameter Θ. The term $\sigma dW(t)$ represents the stochastic component of the process, indicating the random variation over time.



Vasicek model

The Vasicek model is a stochastic model used to describe the evolution of interest rates over time. It was developed by Oldrich Vasicek, a financial economist, and is one of the pioneering models in the field of short-term interest rate modeling.

The stochastic differential equation of the Vasicek model for the evolution of the interest rate is expressed as:

$$dr(t) = a(b - r(t))dt + \sigma dW(t)$$

where:

• $r(t)$ is the interest rate at time t

• a it is the reversibility parameter or speed of regression towards the mean value b

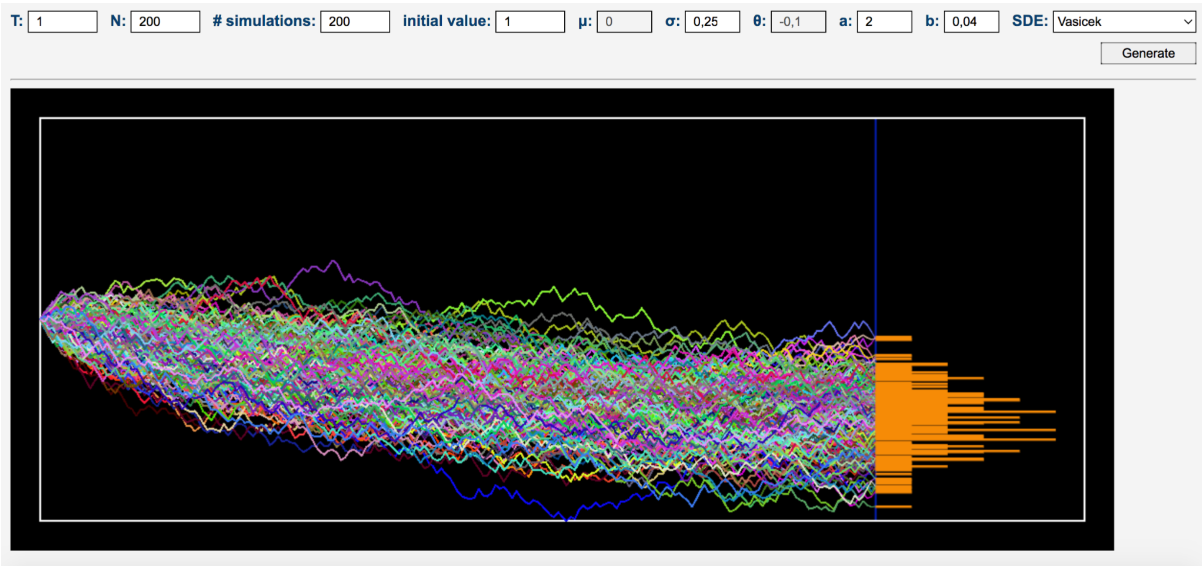
• b is the level to which the interest rate is expected to converge over time

• σ is the volatility of the interest rate

• $dW(t)$ represents a variation of the Wiener process

This model proposes that the interest rate has a tendency to regress towards the average level b, with a speed determined by the parameter a. Volatility σ adds an element of randomness or uncertainty to the process, representing the random variation of the interest rate over time.

Vasicek's model has been used to value bonds, interest rate options, and other financial instruments. However, it has some limitations, such as the assumption of constant volatility over time and the tendency for interest rates to potentially turn negative in extreme situations, which may not be realistic in current financial markets.



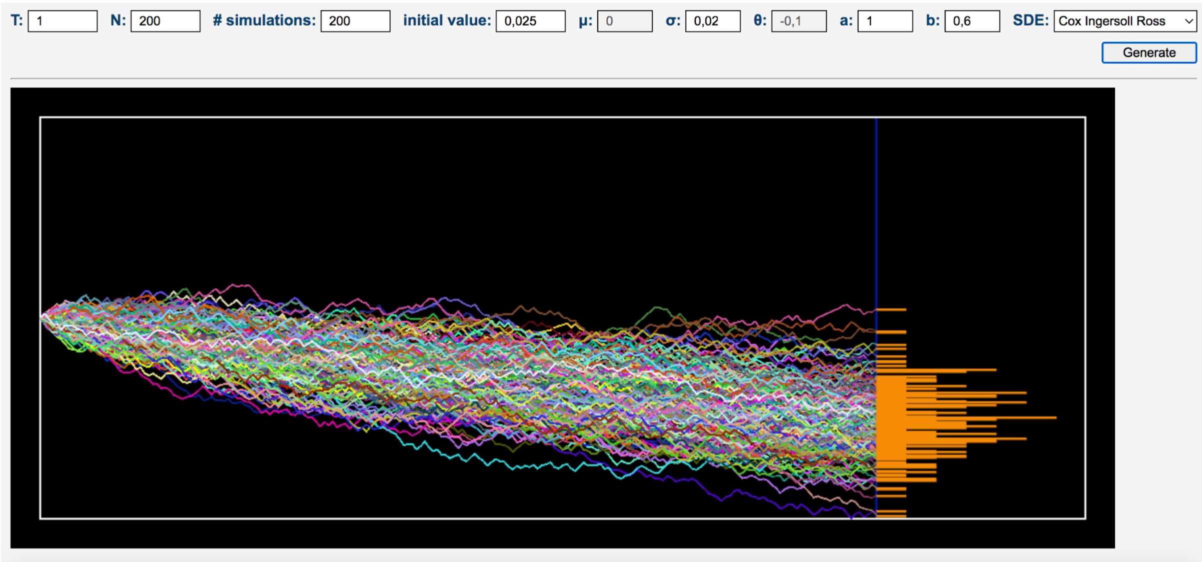
Cox-Ingersoll-Ross (CIR) model

The Cox-Ingersoll-Ross model is another widely used model for modeling short-term interest rates. This model was proposed by John C. Cox, Jonathan E. Ingersoll Jr. and Stephen A. Ross.

The stochastic differential equation of the Cox-Ingersoll-Ross model for the evolution of the interest rate is the following:

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

What distinguishes the CIR model from the Vasicek model is that volatility is a non-constant function of the interest rate itself, being proportional to the square root of the current interest rate. This behavior allows the model to better capture the tendency for interest rates to vary less volatilely when they are at lower levels.



Hull-White model

The Hull-White model is an interest rate model widely used in finance to estimate the evolution of interest rates over time. It is named after John Hull and Alan White, its creators.

This model is a variant of Vasicek's short-term interest rate model and can be used to predict the behavior of short-term and long-term interest rates. It uses a one-factor approach that considers uncertainty and change in interest rates by adding an average reversibility component.

The Hull-White model can be implemented to calculate the prices of bonds, interest rate options, and other interest rate-related financial instruments.

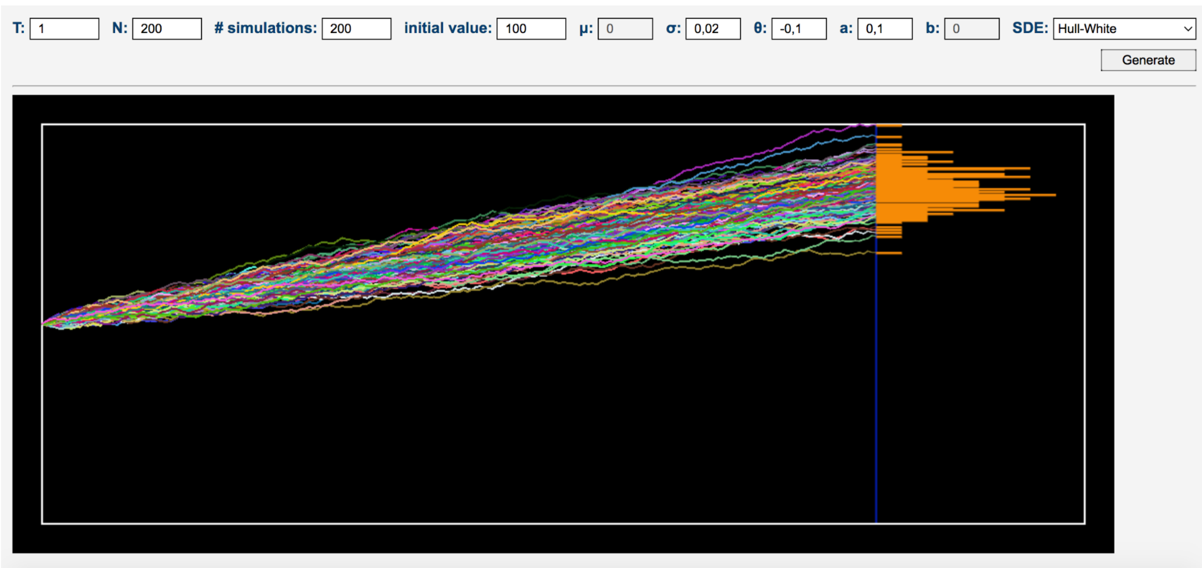
The main formula of the Hull-White model for the evolution of the interest rate is given by the following stochastic differential equation:

$$dr(t) = [\Theta(t) - a(t)r(t)]dt + \sigma(t)dW(t)$$

where:

- $r(t)$ is the interest rate at time t
- $\Theta(t)$ is the level to which the interest rate is expected to converge over time
- $a(t)$ is the reversibility parameter or speed of regression towards Θ
- $\sigma(t)$ is the volatility of the interest rate
- $dW(t)$ represents a variation of the Wiener process

This equation describes the evolution of the interest rate over time, where the first term represents the return towards the long-term Θ level, the second term is the stochastic deviation of the process, and the third term is a random term associated with the Brownian motion .



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