Wiener process (Brownian motion)

The Wiener process, also known as Brownian motion, is a fundamental continuous-time stochastic process used in various fields, including mathematics, physics, finance, and more.

Brownian motion takes its name from Robert Brown, a Scottish botanist, who, observing pollen particles suspended in water under a microscope, noticed a chaotic and irregular movement of the particles.

This observation became a fundamental object of study, stimulating questions and research in the scientific field.

Norbert Wiener was the first to develop a mathematical model used to describe the irregular movement of particles suspended in a fluid through what is called the Wiener process. The Wiener process is considered one of the fundamental pillars in the understanding of random phenomena and their mathematical representation.

The Wiener process is often denoted as W_t, where t represents time.

The Wiener process is characterized by the following conditions:

- $W_0 = 0$ (Starting point is zero)
- $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ for $0 <= s < t$
- the increments $W_t - W_s$ are independent for different time intervals.

Wiener process: Key Characteristics

Continuous Paths:
The Wiener process is characterized by continuous paths without any breaks or discontinuities. This means that the process evolves smoothly as time progresses

Independent and Stationary Increments:
The increments of the Wiener process over non-overlapping intervals are independent of each other and have a Gaussian (normally distributed) distribution. Additionally, these increments are stationary, meaning that their statistical properties (mean and variance) remain constant over time.

Gaussian Distribution:
At any given time, the value of the Wiener process follows a normal distribution. This property is crucial in many statistical and mathematical analyses involving random variables.

Randomness and Unpredictability:
The key defining feature of the Wiener process is its **"random walk"** behavior, where the process shows continuous but irregular movements, making its future path unpredictable. This unpredictability is a fundamental characteristic of the Wiener process and is integral to its applications in various fields.

Geometric Brownian Motion (GBM)

Geometric Brownian Motion is a continuous-time stochastic process that extends the Brownian motion (Wiener process) to model the dynamics of various quantities that exhibit random behavior over time, particularly in finance.

Continuous and Exponential Growth or Decay:
GBM models the exponential growth or decay of a quantity over time. This property makes it suitable for representing processes where the rate of change is proportional to the current value, leading to exponential growth or decay.

Drift and Volatility:
GBM incorporates two parameters:

- **Drift (μ):** Refers to the average rate of change of the stochastic process value over time. Represents the expected growth (or decay) of the modeled quantity.
- **Volatility (σ):** refers to the degree to which a value changes over time.

Continuous Paths:
Similar to the Wiener process, GBM exhibits continuous paths without breaks and discontinuities.

The equation describing Geometric Brownian Motion is:

$$dS = \mu S dt + \sigma S dW$$

where:

- dS represents the change in the value of the quantity;
- μ is the drift coefficient
- σ is the volatility coefficient
- S is the current value of the quantity.
- dt is an increment time
- dW represents the Wiener process.

Relationship between Wiener process and Geometric Brownian Motion

The link between the two lies in the representation of the stochastic component within the GBM equation. The term dW in the GBM equation represents the Wiener process, which introduces the stochastic or random behavior into the geometric growth or decay of the quantity being modeled.

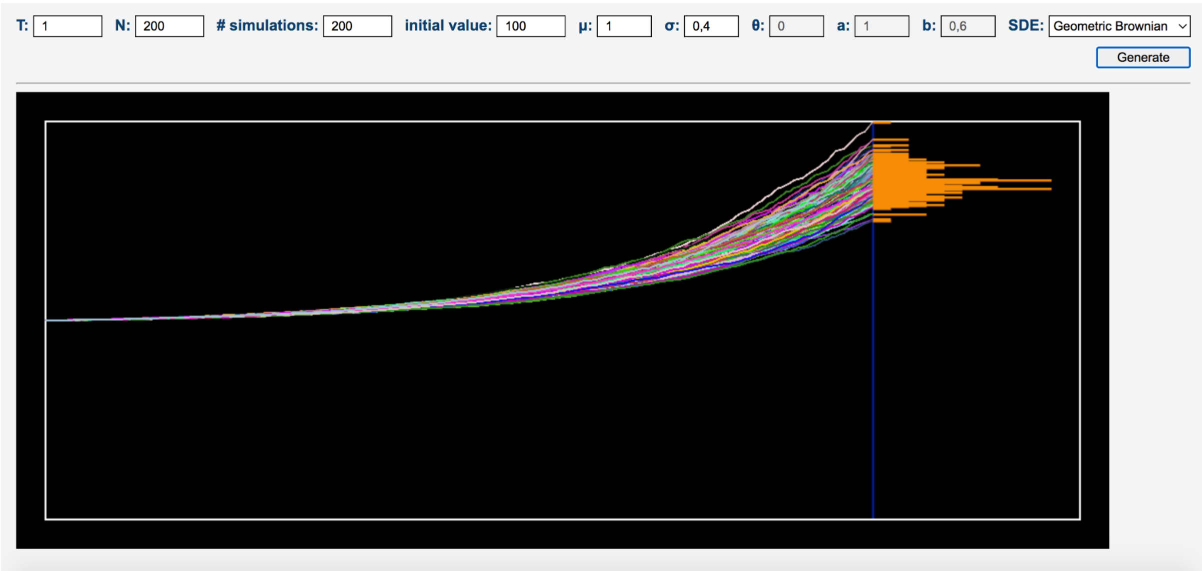
In fact, Geometric Brownian Motion extends the Wiener process by incorporating the stochastic component into a formula that describes the exponential growth or decline of a variable over time.

Therefore, the Wiener process is a crucial component of the Geometric Brownian Motion model, providing the randomness that drives the variability of the quantity being modeled.

Geometric Brownian Motion Simulation

This simulation was calculated with the code developed for the seventh homework.

You can find and directly try the SDEs simulator at the following link
https://andi2ews.github.io/Statistics/HW7/SDE%20simulation/



The growth of the initial value is simulated 200 times in 200 time intervals through the use of the Euler-Maruyama method.

References:

Brownian motion & Wiener process - *Wikipedia, Youtube*
https://it.wikipedia.org/wiki/Moto_browniano
https://en.wikipedia.org/wiki/Wiener_process
https://www.youtube.com/watch?v=YZL2xdhh7qE

Geometric Brownian motion - *Wikipedia*
https://en.wikipedia.org/wiki/Geometric_Brownian_motion

Euler-Maruyama simulation - *Youtube*
https://www.youtube.com/watch?v=4IM2BAG11go&t=596s