Law of Large Numbers (LLN)

The law of large numbers is one of the fundamental concepts of probability theory and statistics. It is a principle that describes the statistical behavior of a series of repeated random events and the tendency of the results to stabilize around an average value.

The Law of Large Numbers indicates that as the number of trials or random events increases, the average of the observed results tends to approach the expected value or theoretical average.

The law of large numbers is essential in statistical analysis as it provides a fundamental understanding of how the averages of a data set behave as observations increase. Its usefulness lies in the fact that it provides a guiding principle for understanding randomness and its relationship to expected long-term outcomes.

The Weak Law of Large Numbers

The weak law of large numbers states that, given a set of independent and identically distributed random variables, the mean of such variables tends, with a large number of trials, to the expected mean of the distribution from which they come.

Formally, if $X_1, X_2, \dots, X_n$ are independent random variables and identically distributed with mean μ then the sample mean M_n of the first n random variables tends to the mean μ as n increases according to the formula:

$$\lim_{n \to \infty} P(|M_n - \mu| >= \varepsilon) = 0$$

where P indicates the probability, M_n is the sample mean of the first n random variables, μ is the mean of the distribution and ε is a small positive number.

The Strong Law of Large Numbers

The strong law of large numbers states that the sample mean approaches the expected mean not only in probability, but almost certainly. In essence, it states that with probability 1, the average of the observed values converges to the expected average as the number of trials increases.

In formulas, the strong law of large numbers states that, given the same set of independent and identically distributed random variables with mean μ, then the sample mean M_n converges to the mean μ with probability 1 as n increases:

$$\lim_{n \to \infty} P(|M_n - \mu| >= \varepsilon) = 0$$

$$P(\lim_{n \to \infty} M_n = \mu) = 1$$

Proof of the Weak Law of Large Numbers

The weak law of large numbers is based on the idea that, with a sufficiently large number of trials, the mean of a set of random variables converges to the expected mean of those variables.

For the demonstration it is necessary to first introduce two key concepts:

- **Chebyshev's inequality:**
  - Chebyshev's inequality provides a relationship between the standard deviation of a random variable and the probability that the variable deviates from its mean.
  - It states that the probability that a random variable deviates from its mean by more than a certain number of standard deviations is limited.

- **Convergence Properties in Probability:**
  - Convergence in probability states that, as the number of observations increases, the probability that a certain quantity deviates from its expected value becomes smaller and smaller.

The proof of the weak law of large numbers uses **Chebyshev's inequality** to show that the deviation of the sample mean from the expected mean decreases as the number of trials increases. This implies that the probability that the sample mean deviates from the expected mean becomes smaller and smaller as the number of trials increases.

definitions

• M_n is the sample mean of a set of independent and identically distributed random variables.

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables with mean μ. The sample mean M_n of the first n random variables is defined as:

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The objective is to demonstrate that, as n increases, the sample mean M_n converges to the expected mean μ in probability.

Chebyshev's inequality states that for any random variable with finite variance, the probability that the variable will deviate from its mean by more than a certain value is limited.

for $a > 0$:

$$P(|X - \mu| >= a) <= \frac{\sigma^2}{a^2}$$

where X is the random variable, μ is the mean and σ^2 is the variance.

Now, consider the deviation of the sample mean M_n from the expected mean μ

$$|M_n - \mu| = |\frac{1}{n}\sum_{i=1}^{n} X_i - \mu|$$

$$|M_n - \mu| = |\frac{1}{n}\sum_{i=1}^{n} (X_i - \mu)|$$

Applying **Chebyshev's inequality:**

$$|M_n - \mu| <= \frac{1}{n}\sum_{i=1}^{n} |X_i - \mu|$$

Using **Chebyshev's inequality** on each term:

$$|M_n - \mu| <= \frac{1}{n}\sum_{i=1}^{n} \frac{\sigma^2}{a^2}$$

where $a >= 0$ is an arbitrary value.

$$|M_n - \mu| <= \frac{n\sigma^2}{na^2}$$

$$|M_n - \mu| <= \frac{\sigma^2}{a^2}$$

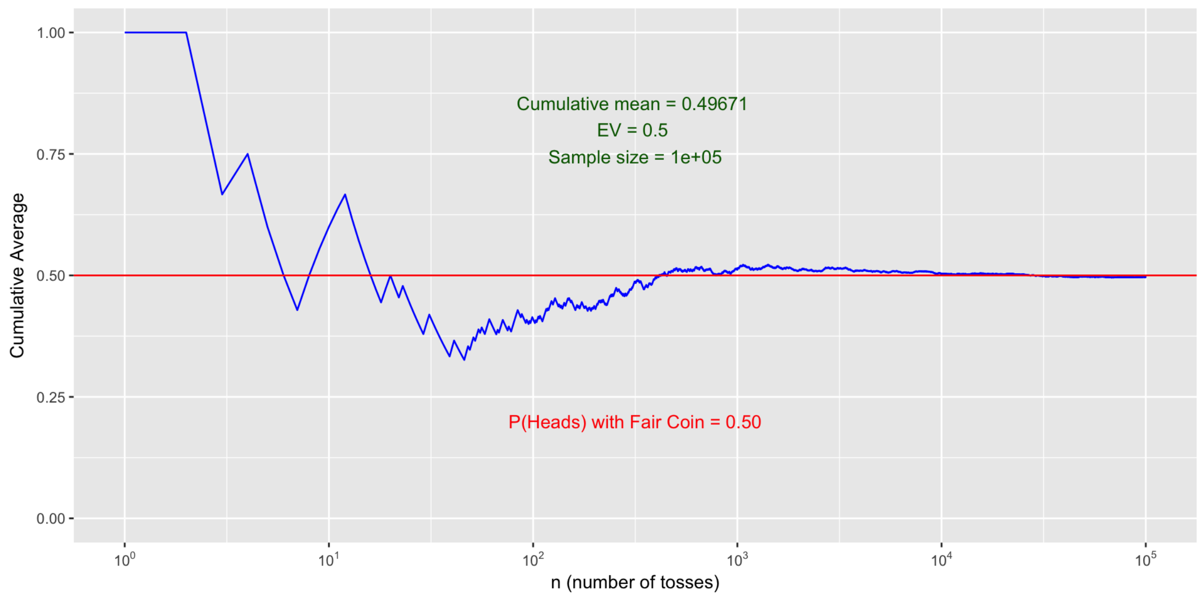
given $\varepsilon > 0$ and $a = \frac{\sigma}{\varepsilon}$,we obtain:

$$P(|M_n - \mu| >= \varepsilon) <= \frac{\sigma^2}{(a\sigma)^2} = \frac{\sigma^2}{(\frac{\sigma}{\varepsilon})^2} = \frac{\sigma^2\varepsilon^2}{\sigma^2} = \varepsilon^2$$

Therefore, for every $\varepsilon > 0$, the probability that $|M_n - \mu| > \varepsilon$ tends to zero as n increases.

This demonstrates the convergence in probability of the sample mean M_n to the expected mean μ according to the weak law of large numbers.

In conclusion the Law of Large Numbers is a fundamental concept in the theory of probability and statistics, providing a solid basis for understanding the behavior of averages in random processes and their convergence to expected values. The difference between the weak and strong versions lies in the certainty of convergence: the first asserts convergence in probability, while the second asserts convergence almost certainly.



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