Convex Relaxations for Cubic Polynomials Problems

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Introduction

We consider a problem in the form :

$$\begin{array}{lll} \min\limits_{x} & c'x & \\ \text{subject to} & g_s(x) & \leq 0, \quad s \in S \\ & 0 \leq x^l \leq x \leq x^u < \infty & \end{array}$$

where $g(x)$ is a polynomial function of degree up to 3:

$$g_s(x) = \sum_{i \in I} a_i x^i, \quad s \in S$$

with a_i a real coefficient and
$x^i = \prod_{k=1}^{n} x_k^{i_k}$, $i_k \in \mathbf{N_0}$ and $\sum_{k=1}^{n} i_k \leq 3$.

Introduction

- If all $g_s(x)$ are convex functions then problem is convex, so any local optimum solution is also global.
- In general functions $g_s(x)$ are not convex.

Approach:
For nonconvex $g_s(x)$ replace it by convex approximation.

talk about branch and bound

Convex under approximation

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Picture of polyhedral under approximation

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Figure of convex relaxation

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(the convex should in this case be better than polyhedral)

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Convex Relaxations

We consider for simplicity a single constraint

$$g(x) \leq 0$$

We want to build a convex relaxation for the set

$$S = \{x \mid g(x) \leq 0\}$$

that is a set S^C such that

- S^C is convex.
- $S \subseteq S^C$

Ideally we would like to have that if $S^{C'}$ is another convex set such that $S \subseteq S^{C'}$ then $S^C \subseteq S^{C'}$, that is S^C is the convex hull of S.

Convex Relaxations

Unfortunately this approach not attainable in general.

(Mention Sahinidis and Bao here?)
###

What can be done?
build relaxations for individual terms for which usually a tight convex relaxation is know.

Quadratic Case

We consider for simplicity a single constraint

$$g(x) \leq 0$$

If there are no terms of order 3, then we can write

$$g(x) = a_0 + \sum_{i \neq j} b_{ij} x_i x_j + \sum_{i=1}^{n} c_i x_i^2$$

Quadratic Case

In this case we have to deal with terms of the form $\alpha x \cdot y$ and βx^2.
Consider $x \cdot y$ with $x \in [x^l, x^u]$ and $y \in [y^l, y^u]$. To build a relaxation of this term we introduce a new variable z and the two linear constraints

$$
\begin{aligned}
z \geq &\quad y^l \cdot x + x^l \cdot y - x^l \cdot y^l \\
z \geq &\quad y^u \cdot x + x^u \cdot y - x^u \cdot y^u
\end{aligned}
$$

In the case of a $-x \cdot y$ term the relaxation is

$$
\begin{aligned}
z \leq &\quad y^l \cdot x + x^u \cdot y - x^u \cdot y^l \\
z \leq &\quad y^u \cdot x + x^l \cdot y - x^l \cdot y^u
\end{aligned}
$$

Quadratic Case

Previous relaxations are know as McCormic convex and concave envelopes of $x \cdot y$ in $[x^l, x^u] \times [y^l, y^u]$.

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picture of relaxations of x y

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For the terms x^2 we have that this term is convex, so we can leave it unchanged a possible relaxation.
However if we require the relaxation to be polyhedral we can use:

$$z \geq (x^i)^2 + 2x^i \cdot (x - x^i) = -(x^i)^2 + 2x^i \cdot x, \quad i = 1, \ldots N_i$$

as an polyhedral approximation.
We note that each of the previous inequalities is simply a subgradient inequality for x^2 at a point x^i.

$-x^2$

For $-x^2$ the best approximation we can have is the secant approximation

$$z \leq (x^l)^2 - \frac{(x^u)^2 - (x^l)^2}{x^u - x^l}\left(x - x^l\right) \tag{1}$$

or equivalently

$$z \leq \left(x^l + x^u\right)x - x^l x^u \tag{2}$$

Beyond the Quadratic Case

mention factorable programming ###

mention in some cases might be better direct approach ###

Convex Relaxations for Cubic Problems

From the Factorable Programming approach we can easily get the following relaxations for terms of order 3:
To relax the terms $x \cdot y \cdot z$ we introduce the additional variables $w_{xyz}, w_{xy}, w_{xz}, w_{yz}$ and the constraints :

$$
\begin{aligned}
w_{xyz} &\geq z^l \cdot w_{xy} + y^l \cdot w_{xz} + x^l \cdot w_{yz} \\
&\quad -y^l \cdot z^l \cdot x - x^l \cdot z^l \cdot y - y^l \cdot z^l z + x^l \cdot y^l \cdot z^l \\
w_{xyz} &\geq z^l \cdot w_{xy} + y^u \cdot w_{xz} + x^u \cdot w_{yz} \\
&\quad -y^u \cdot z^l \cdot x - x^u \cdot z^l \cdot y - y^u \cdot z^l z + x^u \cdot y^u \cdot z^l \\
w_{xyz} &\geq z^u \cdot w_{xy} + y^l \cdot w_{xz} + x^u \cdot w_{yz} \\
&\quad -y^l \cdot z^u \cdot x - x^u \cdot z^u \cdot y - y^l \cdot z^u z + x^u \cdot y^l \cdot z^u \\
w_{xyz} &\geq z^u \cdot w_{xy} + y^u \cdot w_{xz} + x^l \cdot w_{yz} \\
&\quad -y^u \cdot z^u \cdot x - x^l \cdot z^u \cdot y - y^u \cdot z^u z + x^l \cdot y^u \cdot z^u \\
z_{\gamma_1\gamma_2} &\geq \gamma_2^l \cdot \gamma_1 + \gamma_1^l \cdot \gamma_2 - \gamma_1^l \cdot \gamma_2^l \\
z_{\gamma_1\gamma_2} &\geq \gamma_2^u \cdot \gamma_1 + \gamma_1^u \cdot \gamma_2 - \gamma_1^u \cdot \gamma_2^u \\
&\quad (\gamma_1, \gamma_2) \in \{(x, y), (x, z), (y, z)\}
\end{aligned}
$$

Relaxations of Cubic Terms – Trilinear Terms

We can also write relaxations for trilinear terms without the use of the additional variables w_{xy}, w_{zx}, w_{yz}:

$$w_{xyz} \geq \dots$$

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note: it will be quite hard to write all these constraints. are they necessary to show here?

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Cubic Terms – $-x^3$

Here again we have the secant equation for the concave function $-x^3$
(Remember that $x >= 0$)

$$z \leq (x^l)^3 + \frac{(x^u)^3 - (x^l)^3}{x^u - x^l}\left(x - x^l\right) \tag{3}$$

or equivalently,

$$z \leq \left((x^l)^2 + (x^u)^2 + x^l x^u\right) x - x^l x^u \left(x^l + x^u\right) \tag{4}$$