Convex Relaxations for Cubic Polynomials Problems

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Introduction

We consider a problem in the form :

$$\begin{array}{lll} \min\limits_{x} & c'x & \\ \text{subject to} & g_s(x) \quad \leq 0, & s \in S \\ & 0 \leq x^l \leq x \leq x^u < \infty & \end{array}$$

where $g(x)$ is a polynomial function of degree up to 3:

$$g_s(x) = \sum_{i \in I} a_i x^i, \quad s \in S$$

with a_i a real coefficient and
$x^i = \prod_{k=1}^{n} x_k^{i_k}$, $i_k \in \mathbf{N_0}$ and $\sum_{k=1}^{n} i_k \leq 3$.

Introduction

- If all $g_s(x)$ are convex functions then problem is convex, so any local optimum solution is also global.
- In general functions $g_s(x)$ are not convex.

Approach:
For nonconvex $g_s(x)$ replace it by convex approximation.

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talk about branch and bound

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Convex under approximation

Picture of polyhedral under approximation

Figure of convex relaxation

(the convex should in this case be better than polyhedral)

Convex Relaxations

We consider for simplicity a single constraint

$$g(x) \leq 0$$

We want to build a convex relaxation for the set

$$S = \{x \mid g(x) \leq 0\}$$

that is a set S^C such that

- S^C is convex.
- $S \subseteq S^C$

Ideally we would like to have that if $S^{C'}$ is another convex set such that $S \subseteq S^{C'}$ then $S^C \subseteq S^{C'}$, that is S^C is the convex hull of S.

Convex Relaxations

Unfortunately this approach not attainable in general.

(Mention Sahinidis and Bao here?)
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What can be done?
build relaxations for individual terms for which usually a tight convex relaxation is know.

Quadratic Case

We consider for simplicity a single constraint

$$g(x) \leq 0$$

If there are no terms of order 3, then we can write

$$g(x) = a_0 + \sum_{i \neq j} b_{ij} x_i x_j + \sum_{i=1}^{n} c_i x_i^2$$

Quadratic Case

In this case we have to deal with terms of the form $\alpha x \cdot y$ and βx^2.
Consider $x \cdot y$ with $x \in [x^l, x^u]$ and $y \in [y^l, y^u]$. To build a relaxation of this term we introduce a new variable z and the two linear constraints

$$\begin{aligned} z \geq \quad & y^l \cdot x + x^l \cdot y - x^l \cdot y^l \\ z \geq \quad & y^u \cdot x + x^u \cdot y - x^u \cdot y^u \end{aligned}$$

In the case of a $-x \cdot y$ term the relaxation is

$$\begin{aligned} z \leq \quad & y^l \cdot x + x^u \cdot y - x^u \cdot y^l \\ z \leq \quad & y^u \cdot x + x^l \cdot y - x^l \cdot y^u \end{aligned}$$

Quadratic Case

Previous relaxations are know as McCormic convex and concave envelopes of $x \cdot y$ in $[x^l, x^u] \times [y^l, y^u]$.

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picture of relaxations of x y

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For the terms x^2 we have that this term is convex, so we can leave it unchanged a possible relaxation.

However if we require the relaxation to be polyhedral we can use:

$$z \geq (x^i)^2 + 2x^i \cdot (x - x^i) = -(x^i)^2 + 2x^i \cdot x, \quad i = 1, \ldots N_i$$

as an polyhedral approximation.

We note that each of the previous inequalities is simply a subgradient inequality for x^2 at a point x^i.

For $-x^2$ the best approximation we can have is the secant approximation

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formula for secant approximation

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Beyond the Quadratic Case

mention factorable programming

mention in some cases might be better direct approach

Convex Relaxations for Cubic Problems

From the Factorable Programming approach we can easily get the following relaxations for terms of order 3:
To relax the terms $x \cdot y \cdot z$ we introduce the additional variables $w_{xyz}, w_{xy}, w_{xz}, w_{yz}$ and the constraints :

$$\begin{aligned}
w_{xyz} &\geq z^l \cdot w_{xy} + y^l \cdot w_{xz} + x^l \cdot w_{yz} \\
&\quad -y^l \cdot z^l \cdot x - x^l \cdot z^l \cdot y - y^l \cdot z^l z + x^l \cdot y^l \cdot z^l \\
w_{xyz} &\geq z^l \cdot w_{xy} + y^u \cdot w_{xz} + x^u \cdot w_{yz} \\
&\quad -y^u \cdot z^l \cdot x - x^u \cdot z^l \cdot y - y^u \cdot z^l z + x^u \cdot y^u \cdot z^l \\
w_{xyz} &\geq z^u \cdot w_{xy} + y^l \cdot w_{xz} + x^u \cdot w_{yz} \\
&\quad -y^l \cdot z^u \cdot x - x^u \cdot z^u \cdot y - y^l \cdot z^u z + x^u \cdot y^l \cdot z^u \\
w_{xyz} &\geq z^u \cdot w_{xy} + y^u \cdot w_{xz} + x^l \cdot w_{yz} \\
&\quad -y^u \cdot z^u \cdot x - x^l \cdot z^u \cdot y - y^u \cdot z^u z + x^l \cdot y^u \cdot z^u \\
z_{\gamma_1\gamma_2} &\geq \gamma_2^l \cdot \gamma_1 + \gamma_1^l \cdot \gamma_2 - \gamma_1^l \cdot \gamma_2^l \\
z_{\gamma_1\gamma_2} &\geq \gamma_2^u \cdot \gamma_1 + \gamma_1^u \cdot \gamma_2 - \gamma_1^u \cdot \gamma_2^u
\end{aligned}$$

Relaxations of Cubic Terms – Trilinear Terms