L11 Homework 4

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1 When is $(S \times T, \rtimes_{\rhd})$ a semigroup?

Def 1.1 (Semi-direct product). *Assume* $(S, \otimes_S), (T, \otimes_T)$ *be semigroups,* $\rhd \in T \to (S \to S)$
Let $\rtimes_{\rhd} : (S \times T) \times (S \times T) \to S \times T$ *be a binary operator such that*

$$\forall (s_1, t_1), (s_2, t_2) \in S \times T, (s_1, t_1) \rtimes_{\rhd} (s_2, t_2) = (s_1 \otimes_S (t_1 \rhd s_2), t_1 \otimes_T t_2)$$

Def 1.2.

$$\begin{aligned}
\mathbb{D}(S, T, \otimes_S, \rhd) \equiv & \forall a, b \in S, t \in T, t \rhd (a \otimes_S b) = (t \rhd a) \otimes_S (t \rhd b) \\
\mathbb{P}(S, T, \otimes_T, \rhd) \equiv & \forall a, b \in T, s \in S, a \rhd (b \rhd s) = (a \otimes_T b) \rhd s
\end{aligned}$$

Prop 1.1.

$$\mathbb{D}(S, T, \otimes_S, \rhd) \wedge \mathbb{P}(S, T, \otimes_T, \rhd) \Rightarrow (S \times T, \rtimes_{\rhd}) \textit{ is a semigroup.}$$

Proof. Let $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T$, then

$$\begin{aligned}
(s_1, t_1) \rtimes_{\rhd} ((s_2, t_2) \rtimes_{\rhd} (s_3, t_3)) =& (s_1, t_1) \rtimes_{\rhd} (s_2 \otimes_S (t_2 \rhd s_3), t_2 \otimes_T t_3) \\
=& (s_1 \otimes_S (t_1 \rhd (s_2 \otimes_S (t_2 \rhd s_3))), t_1 \otimes_T (t_2 \otimes_T t_3)) \\
\overset{\mathbb{D}(S,T,\otimes_S,\rhd)}{=\!=\!=\!=\!=}& (s_1 \otimes_S ((t_1 \rhd s_2) \otimes_S (t_1 \rhd (t_2 \rhd s_3))), t_1 \otimes_T (t_2 \otimes_T t_3)) \\
\overset{\mathbb{P}(S,T,\otimes_T,\rhd)}{=\!=\!=\!=\!=}& (s_1 \otimes_S ((t_1 \rhd s_2) \otimes_S ((t_1 \otimes_T t_2) \rhd s_3)), t_1 \otimes_T (t_2 \otimes_T t_3)) \\
\underset{\mathbb{AS}(T,\otimes_T)}{\overset{\mathbb{AS}(S,\otimes_S)}{=\!=\!=\!=}}& ((s_1 \otimes_S (t_1 \rhd s_2)) \otimes_S ((t_1 \otimes_T t_2) \rhd s_3), (t_1 \otimes_T t_2) \otimes_T t_3) \\
=& (s_1 \otimes_S (t_1 \rhd s_2), t_1 \otimes_T t_2) \rtimes_{\rhd} (s_3, t_3) \\
=& ((s_1, t_1) \rtimes_{\rhd} (s_2, t_2)) \rtimes_{\rhd} (s_3, t_3)
\end{aligned}$$

Hence we have $\mathbb{AS}(S \times T, \rtimes_{\rhd})$. Then by Def 1.1, it is clear that $(S \times T, \rtimes_{\rhd})$ is closure.
So $(S \times T, \rtimes_{\rhd})$ is a semigroup. $\square$

2 When is $(S, \oplus_S, \otimes_S) \rtimes_{\rhd} (T, \oplus_T, \otimes_T)$ left distributive?

Def 2.1. *Assume $\oplus_S$ is commutative and selective, $(S, \oplus_S, \otimes_S), (T, \oplus_T, \otimes_T)$ are bi-semigroups,*

$$(S, \oplus_S, \otimes_S) \rtimes_{\rhd} (T, \oplus_T, \otimes_T) \equiv (S \times T, \oplus, \rtimes_{\rhd})$$

where $\oplus = \oplus_S \vec{\times} \oplus_T$

Def 2.2.

$$\begin{aligned}
\mathbb{D}(S, T, \oplus_S, \rhd) \equiv & \forall a, b \in S, t \in T, t \rhd (a \oplus_S b) = (t \rhd a) \oplus_S (t \rhd b) \\
\mathbb{LD}(S, \oplus_S, \otimes_S) \equiv & \forall a, b, c \in S, a \otimes_S (b \oplus_S c) = (a \otimes_S b) \oplus_S (a \otimes_S c) \\
\mathbb{LD}(T, \oplus_T, \otimes_T) \equiv & \forall a, b, c \in T, a \otimes_T (b \oplus_T c) = (a \otimes_T b) \oplus_T (a \otimes_T c) \\
\mathbb{LC}(S, \otimes_S) \equiv & \forall a, b, c \in S, c \otimes_S a = c \otimes_S b \Rightarrow a = b
\end{aligned}$$

Prop 2.1.

$$\left.\begin{array}{l}
\mathbb{D}(S, T, \oplus_S, \rhd) \\
\mathbb{LD}(S, \oplus_S, \otimes_S) \\
\mathbb{LD}(T, \oplus_T, \otimes_T) \\
\mathbb{LC}(S, \otimes_S) \\
\forall t \in T, \rhd(t) \textit{ is injection}
\end{array}\right\} \Rightarrow \mathbb{LD}(S \times T, \oplus, \rtimes_{\rhd})$$

Proof. Let $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T$.
Let

$$\begin{aligned}
LHS = & (s_1, t_1) \rtimes_{\rhd} ((s_2, t_2) \oplus (s_3, t_3))) \\
RHS = & ((s_1, t_1) \rtimes_{\rhd} (s_2, t_2)) \oplus ((s_1, t_1) \rtimes_{\rhd} (s_3, t_3))
\end{aligned}$$

Then,

$$RHS = (s_1 \otimes_S (t_1 \rhd s_2), t_1 \otimes_T t_2) \oplus (s_1 \otimes_S (t_1 \rhd s_3), t_1 \otimes_T t_3)$$

`Case 1.` $s_1 \otimes_S (t_1 \rhd s_2) = (s_1 \otimes_S (t_1 \rhd s_2)) \oplus_S (s_1 \otimes_S (t_1 \rhd s_3)) = s_1 \otimes_S (t_1 \rhd s_3)$
Then,

$$\begin{aligned}
s_1 \otimes_S (t_1 \rhd s_2) = & s_1 \otimes_S (t_1 \rhd s_3) & \\
t_1 \rhd s_2 = & t_1 \rhd s_3 & \text{By } \mathbb{LC}(S, \otimes_S) \\
s_2 = & s_3 & \text{As } \rhd (t) \text{ is injection } \forall t \in T \\
s_2 = s_3 = & s_2 \oplus_S s_3 & \text{By } \mathbb{SL}(S, \oplus_S)
\end{aligned}$$

So,

$$\begin{aligned}
RHS &= ((s_1 \otimes_S (t_1 \rhd s_2)) \oplus_S (s_1 \otimes_S (t_1 \rhd s_3)), (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \\
& \overset{\mathbb{LD}(S, \oplus_S, \otimes_S)}{\underset{\mathbb{LD}(T, \oplus_T, \otimes_T)}{=\!=\!=}} (s_1 \otimes_S ((t_1 \rhd s_2) \oplus_S (t_1 \rhd s_3)), t_1 \otimes_T (t_2 \oplus_T t_3)) \\
& \overset{\mathbb{D}(S, T, \oplus_S, \rhd)}{=\!=\!=} (s_1 \otimes_S (t_1 \rhd (s_2 \oplus_S s_3)), t_1 \otimes_T (t_2 \oplus_T t_3)) \\
& = (s_1, t_1) \rtimes_{\rhd} (s_2 \oplus_S s_3, t_2 \oplus_T t_3) \\
& \overset{s_2 = s_3 = s_2 \oplus_S s_3}{\underset{\oplus = \oplus_S \vec{\times} \oplus_T}{=\!=\!=}} (s_1, t_1) \rtimes_{\rhd} ((s_2, t_2) \oplus (s_3, t_3)) = LHS
\end{aligned}$$

`Case 2.` $s_1 \otimes_S (t_1 \rhd s_2) = (s_1 \otimes_S (t_1 \rhd s_2)) \oplus_S (s_1 \otimes_S (t_1 \rhd s_3)) \neq s_1 \otimes_S (t_1 \rhd s_3)$
Then,

$$\begin{array}{ll} s_1 \otimes_S (t_1 \rhd s_2) = s_1 \otimes_S ((t_1 \rhd s_2) \oplus_S (t_1 \rhd s_3)) \neq s_1 \otimes_S (t_1 \rhd s_3) & \text{By } \mathbb{LD}(S, \oplus_S, \otimes_S) \\ s_1 \otimes_S (t_1 \rhd s_2) = s_1 \otimes_S (t_1 \rhd (s_2 \oplus_S s_3)) \neq s_1 \otimes_S (t_1 \rhd s_3) & \text{By } \mathbb{D}(S, T, \oplus_S, \rhd) \end{array}$$

$$\begin{array}{ll} t_1 \rhd s_2 = t_1 \rhd (s_2 \oplus_S s_3) \neq t_1 \rhd s_3 & \text{By } \mathbb{LC}(S, \otimes_S) \text{ and Def of } \otimes_S \\ s_2 = s_2 \oplus_S s_3 \neq s_3 & \text{By injection } \rhd(t) \text{ and Def of } \rhd(t) \end{array}$$

So,

$$\begin{aligned} RHS &= ((s_1 \otimes_S (t_1 \rhd s_2)) \oplus_S (s_1 \otimes_S (t_1 \rhd s_3)), t_1 \otimes_T t_2) \\ &\xlongequal{\mathbb{LD}(S,\oplus_S,\otimes_S)} (s_1 \otimes_S ((t_1 \rhd s_2) \oplus_S (t_1 \rhd s_3)), t_1 \otimes_T t_2) \\ &\xlongequal{\mathbb{D}(S,T,\oplus_S,\rhd)} (s_1 \otimes_S (t_1 \rhd (s_2 \oplus_S s_3)), t_1 \otimes_T t_2) \\ &= (s_1, t_1) \rtimes_{\rhd} (s_2 \oplus_S s_3, t_2) \\ &\xlongequal[\oplus = \oplus_S \vec{\times} \oplus_T]{s_2 = s_2 \oplus_S s_3 \neq s_3} (s_1, t_1) \rtimes_{\rhd} ((s_2, t_2) \oplus (s_3, t_3)) = LHS \end{aligned}$$

`Case 3.` $s_1 \otimes_S (t_1 \rhd s_2) \neq (s_1 \otimes_S (t_1 \rhd s_2)) \oplus_S (s_1 \otimes_S (t_1 \rhd s_3)) = s_1 \otimes_S (t_1 \rhd s_3)$
Similar to Case 2.

Hence in all of the cases we have $RHS = LHS$, we have $\mathbb{LD}(S \times T, \oplus, \rtimes_{\rhd})$.

□