L11 Homework 3

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1 An Interesting Property of Lift

Def 1.1 (Lifted product semigroup)**.** *Assume* $(S, \bullet)$ *is a semigroup. Let* $lift(S, \bullet) \equiv (fin(2^S), \hat{\bullet})$*, where*

$$X \hat{\bullet} Y = \{x \bullet y | x \in X, y \in Y\}$$

Prop 1.1. *Assume* $(S, \bullet)$ *is a semigroup and* $|S| \geq 2$*, we have*

$$\mathbb{SL}(lift(S, \bullet)) \Rightarrow \mathbb{IP}(S, \bullet)$$

Proof. $\mathbb{SL}(lift(S, \bullet)) \Rightarrow \forall X, Y \in fin(2^S), X \hat{\bullet} Y = \{x \bullet y | x \in X, y \in Y\} \in \{X, Y\}$, so $X \hat{\bullet} Y = X$ or Y
Let $a \in S$, then $a \in \{a\}$. $\{a \bullet a\} = \{a\} \hat{\bullet} \{a\} = \{a\}$ or $\{a\} = \{a\}$, so $a \bullet a = a$, we have $\mathbb{IP}(S, \bullet)$ □

Prop 1.2. *Assume* $(S, \bullet)$ *is a semigroup and* $|S| \geq 2$*, we have*

$$\mathbb{SL}(lift(S, \bullet)) \Rightarrow \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \wedge |S| = 2)$$

Proof. Suppose (for contradiction that)

$$\begin{aligned}
\neg(\mathbb{SL}(lift(S, \bullet)) &\Rightarrow \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \wedge |S| = 2)) \\
\neg(\neg\mathbb{SL}(lift(S, \bullet)) &\vee (\mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \wedge |S| = 2))) \\
\mathbb{SL}(lift(S, \bullet)) &\wedge \neg(\mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \wedge |S| = 2)) \\
\mathbb{SL}(lift(S, \bullet)) &\wedge \neg\mathbb{IL}(S, \bullet) \wedge \neg\mathbb{IR}(S, \bullet) \wedge \neg(\mathbb{IP}(S, \bullet) \wedge |S| = 2) \\
\mathbb{SL}(lift(S, \bullet)) &\wedge \neg\mathbb{IL}(S, \bullet) \wedge \neg\mathbb{IR}(S, \bullet) \wedge (\neg\mathbb{IP}(S, \bullet) \vee |S| > 2)
\end{aligned}$$

`Case 1.`$\mathbb{SL}(lift(S, \bullet)) \wedge \neg\mathbb{IL}(S, \bullet) \wedge \neg\mathbb{IR}(S, \bullet) \wedge \neg\mathbb{IP}(S, \bullet)$
By contraposition of Prop 1.1, we have $\neg\mathbb{IP}(S, \bullet) \Rightarrow \neg\mathbb{SL}(lift(S, \bullet))$, giving a contradiction!
`Case 2.`$\mathbb{SL}(lift(S, \bullet)) \wedge \neg\mathbb{IL}(S, \bullet) \wedge \neg\mathbb{IR}(S, \bullet) \wedge |S| > 2$
$\neg\mathbb{IL}(S, \bullet) \Rightarrow \exists a, b \in S, a \bullet b \neq a$
Claim. $a \neq b$

Proof of Claim. Suppose (for contradiction) that $a = b$, then

$$\{a \bullet b\} = \{a\} \hat{\bullet} \{b\} = \{a\} \hat{\bullet} \{a\} \xlongequal{\mathbb{SL}(lift(S, \bullet))} \{a\}$$

So $a \bullet b = a$, contradict to the fact that $a \bullet b \neq a$ □

Then we have $\left.\begin{array}{l} a\bullet b\neq a \\ \{a\bullet b\}=\{a\}\hat{\bullet}\{b\} \xlongequal{\mathbb{SL}(lift(S,\bullet))} \{a\} \text{ or } \{b\}\end{array}\right\} \Rightarrow a\bullet b = b$

Similarly, we have $\neg\mathbb{IR}(S,\bullet) \Rightarrow \exists c,d\in S, c\bullet d\neq d$, then $c\neq d$ and $c\bullet d = c$.
Now we have

$$\begin{cases} \exists a,b\in S, a\neq b, a\bullet b = b \\ \exists c,d\in S, c\neq d, c\bullet d = c \end{cases}$$

`Case 2.1.` $b\neq c$
We have

$$\{a,c\}\hat{\bullet}\{b,d\} = \{a\bullet b, a\bullet d, c\bullet b, c\bullet d\} = \{b,c,a\bullet d,c\bullet b\}$$

Then

$$\left.\begin{array}{l} \left.\begin{array}{l} b\neq a \\ b\neq c\end{array}\right\} \Rightarrow b\notin\{a,c\} \Rightarrow \{b,c,a\bullet d,c\bullet b\}\neq\{a,c\} \\ \left.\begin{array}{l} c\neq b \\ c\neq d\end{array}\right\} \Rightarrow c\notin\{b,d\} \Rightarrow \{b,c,a\bullet d,c\bullet b\}\neq\{b,d\} \end{array}\right\} \Rightarrow \text{ contradict to } \mathbb{SL}(lift(S,\bullet))$$

`Case 2.2.` $b = c$
Claim. $a\neq d$.

Proof of Claim. Suppose (for contradiction) that $a = d$, then $\left.\begin{array}{l} a=d \\ b=c\end{array}\right\} \Rightarrow \begin{cases} a\bullet b = b \\ b\bullet a = b\end{cases}$
$|S|>2 \Rightarrow \exists e\in S, e\neq a, e\neq b$.
Then

$$\{a\}\hat{\bullet}\{b,e\} = \{a\bullet b, a\bullet e\} \xlongequal{a\bullet b=b} \{b, a\bullet e\} \xRightarrow[b\neq a]{\mathbb{SL}(lift(S,\bullet))} a\bullet e = e$$

$$\{a,b\}\hat{\bullet}\{e\} = \{a\bullet e, b\bullet e\} \xlongequal{a\bullet e=e} \{e, b\bullet e\} \xRightarrow[e\neq a, e\neq b]{\mathbb{SL}(lift(S,\bullet))} b\bullet e = e$$

$$\{b\}\hat{\bullet}\{a,e\} = \{b\bullet a, b\bullet e\} \xlongequal{b\bullet e=e} \{b\bullet a, e\} \xRightarrow[e\neq b]{\mathbb{SL}(lift(S,\bullet))} b\bullet a = a$$

Hence $b = b\bullet a = a$, contradict to the fact that $a\neq b$! ☐

We have

$$\{a\}\hat{\bullet}\{b,d\} = \{a\bullet b, a\bullet d\} \xlongequal{a\bullet b=b} \{b, a\bullet d\} \xRightarrow[b\notin\{a\}]{\mathbb{SL}(lift(S,\bullet))} a\bullet d = d$$

Then,

$$\{a,b\}\hat{\bullet}\{d\} = \{a\bullet d, b\bullet d\} \xlongequal[c\bullet d=c, b=c]{a\bullet d=d} \{d,b\} = \{b,d\}$$

We have

$$\left.\begin{array}{l} \left.\begin{array}{l} d\neq c=b \\ d\neq a\end{array}\right\} \Rightarrow d\notin\{a,b\} \Rightarrow \{b,d\}\neq\{a,b\} \\ b=c\neq d \Rightarrow b\notin\{d\} \Rightarrow \{b,d\}\neq\{d\} \end{array}\right\} \Rightarrow \text{ contradict to } \mathbb{SL}(lift(S,\bullet))$$

Hence we have got contradictions in all cases! ☐

Prop 1.3. *Assume that* $(S,\bullet)$ *is a semigroup and* $|S|\geq 2$*, we have*

$$\mathbb{IL}(S,\bullet) \Rightarrow \mathbb{SL}(lift(S,\bullet))$$

Proof. Let $X, Y \in lift(S, \bullet)$
`Case 1.` $X \neq \{\}$ and $Y \neq \{\}$
then $X = \{x_1, x_2, ..., x_m\}$, $Y = \{y_1, y_2, ..., y_n\}$, we have

$$\begin{aligned}
X \hat{\bullet} Y = \{&x_1 \bullet y_1, x_1 \bullet y_2, ..., x_1 \bullet y_n,\\
&x_2 \bullet y_1, x_2 \bullet y_2, ..., x_2 \bullet y_n,\\
&\vdots\\
&x_m \bullet y_1, x_m \bullet y_2, ..., x_m \bullet y_n\}\\
=&\{x_1, ..., x_m\} && \text{(By } \mathbb{IL}(S, \bullet))\\
=&X
\end{aligned}$$

`Case 2.` $X = \{\}$ and $Y \neq \{\}$, then $X \hat{\bullet} Y = \{\} = X$
`Case 3.` $X \neq \{\}$ and $Y = \{\}$, then $X \hat{\bullet} Y = \{\} = Y$
`Case 4.` $X = \{\}$ and $Y = \{\}$, then $X \hat{\bullet} Y = \{\} = X$ or Y
In all cases, $X \hat{\bullet} Y \in \{X, Y\}$, we have $\mathbb{SL}(lift(S, \bullet))$ □

Prop 1.4. *Assume that* $(S, \bullet)$ *is a semigroup and* $|S| \geq 2$*, we have*

$$\mathbb{IR}(S, \bullet) \Rightarrow \mathbb{SL}(lift(S, \bullet))$$

Proof. Similar to Prop 1.3. □

Prop 1.5. *Assume that* $(S, \bullet)$ *is a semigroup and* $|S| \geq 2$*, we have*

$$(\mathbb{IP}(S, \bullet) \wedge |S| = 2) \Rightarrow \mathbb{SL}(lift(S, \bullet))$$

Proof. $|S| = 2 \Rightarrow S = \{a, b\}, a \bullet b \in \{a, b\}$ by closure, Let $X, Y \in lift(S, \bullet)$
`Case 1.` $X = \{a\}$ and $Y = \{a\}$, then $X \hat{\bullet} Y = \{a \bullet a\} \xlongequal{\mathbb{IP}(S,\bullet)} \{a\} = X$ or Y
`Case 2.` $X = \{a\}$ and $Y = \{b\}$, then $X \hat{\bullet} Y = \{a \bullet b\} \xlongequal{a \bullet b \in \{a,b\}} X$ or Y
`Case 3.` $X = \{a\}$ and $Y = \{a, b\}$, then
$X \hat{\bullet} Y = \{a \bullet a, a \bullet b\} \xlongequal{\mathbb{IP}(S,\bullet)} \{a, a \bullet b\} \xlongequal{a \bullet b \in \{a,b\}} X$ or Y
`Case 4.` $X = \{b\}$ and $Y = \{a\}$, then $X \hat{\bullet} Y = \{b \bullet a\} \xlongequal{b \bullet a \in \{a,b\}} X$ or Y
`Case 5.` $X = \{b\}$ and $Y = \{b\}$, then $X \hat{\bullet} Y = \{b \bullet b\} \xlongequal{\mathbb{IP}(S,\bullet)} \{b\} = X$ or Y
`Case 6.` $X = \{b\}$ and $Y = \{a, b\}$, then
$X \hat{\bullet} Y = \{b \bullet a, b \bullet b\} \xlongequal{\mathbb{IP}(S,\bullet)} \{b \bullet a, b\} \xlongequal{b \bullet a \in \{a,b\}} X$ or Y
`Case 7.` $X = \{a, b\}$ and $Y = \{a\}$, then
$X \hat{\bullet} Y = \{a \bullet a, b \bullet a\} \xlongequal{\mathbb{IP}(S,\bullet)} \{a, b \bullet a\} \xlongequal{b \bullet a \in \{a,b\}} X$ or Y
`Case 8.` $X = \{a, b\}$ and $Y = \{b\}$, then
$X \hat{\bullet} Y = \{a \bullet b, b \bullet b\} \xlongequal{\mathbb{IP}(S,\bullet)} \{a \bullet b, b\} \xlongequal{a \bullet b \in \{a,b\}} X$ or Y
`Case 9.` $X = \{a, b\}$ and $Y = \{a, b\}$, then
$X \hat{\bullet} Y = \{a \bullet a, a \bullet b, b \bullet a, b \bullet b\} \xlongequal{\mathbb{IP}(S,\bullet)} \{a, b, a \bullet b, b \bullet a\} \xlongequal[b \bullet a \in \{a,b\}]{a \bullet b \in \{a,b\}} X$ or Y
`Case 10.` $X = \{\}$ and $Y \neq \{\}$, then $X \hat{\bullet} Y = \{\} = X$
`Case 11.` $X \neq \{\}$ and $Y = \{\}$, then $X \hat{\bullet} Y = \{\} = Y$
`Case 12.` $X = \{\}$ and $Y = \{\}$, then $X \hat{\bullet} Y = \{\} = X$ or Y
In all cases, $X \hat{\bullet} Y \in \{X, Y\}$, we have $\mathbb{SL}(lift(S, \bullet))$ □

Prop 1.6. *Assume that* $(S, \bullet)$ *is a semigroup and* $|S| \geq 2$*, we have*

$$\mathbb{SL}(lift(S, \bullet)) \Leftrightarrow \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \wedge |S| = 2)$$

Proof. By Prop 1.2, Prop 1.3, Prop 1.4 and Prop 1.5, we have

$$\left.\begin{array}{l}\left.\begin{array}{r}\text{Prop 1.3}\\ \text{Prop 1.4}\\ \text{Prop 1.5}\end{array}\right\} \Rightarrow (\mathbb{SL}(lift(S,\bullet)) \Leftarrow \mathbb{IL}(S,\bullet) \vee \mathbb{IR}(S,\bullet) \vee (\mathbb{IP}(S,\bullet) \wedge |S| = 2))\\ \text{Prop 1.2} \Rightarrow (\mathbb{SL}(lift(S,\bullet)) \Rightarrow \mathbb{IL}(S,\bullet) \vee \mathbb{IR}(S,\bullet) \vee (\mathbb{IP}(S,\bullet) \wedge |S| = 2))\end{array}\right\}$$

$$\Rightarrow (\mathbb{SL}(lift(S,\bullet)) \Leftrightarrow \mathbb{IL}(S,\bullet) \vee \mathbb{IR}(S,\bullet) \vee (\mathbb{IP}(S,\bullet) \wedge |S| = 2))$$

□

2 When $union\_lift(S,\bullet)$ is a semiring

Def 2.1. $union\_lift(S,\bullet) \equiv (\mathcal{P}_{fin}(S), \cup, \hat{\bullet})$

Prop 2.1. *The identity of monoid $(S,\bullet)$ is unique.*

Proof. Let α, β be two identities of $(S,\bullet)$. Then $\forall x \in S, x \bullet \alpha = x = \beta \bullet x$, so

$$\left.\begin{array}{l}\alpha \bullet \alpha = \alpha = \beta \bullet \alpha\\ \beta \bullet \alpha = \beta = \beta \bullet \beta\end{array}\right\} \Rightarrow \alpha = \beta \bullet \alpha = \beta$$

□

Prop 2.2. *Assume $(S,\bullet)$ is a semigroup and $|S| \geq 2$,*

$$union\_lift(S,\bullet) \textit{ is a semiring} \Rightarrow (S,\bullet) \textit{ has identity}$$

Proof.

$$union\_lift(S,\bullet) \text{ is a semiring} \Rightarrow (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is a monoid}$$

$$\Rightarrow \exists \bar{1} = \{\alpha_1, \alpha_2, ..., \alpha_n\} \in \mathcal{P}_{fin}(S) \text{ s.t. } \forall X \in \mathcal{P}_{fin}(S), \bar{1}\hat{\bullet}X = X\hat{\bullet}\bar{1} = X$$

$\bar{1}$ is unique by Prop 2.1.

Claim. $\bar{1} \neq \{\}$

Proof of Claim. Suppose $\bar{1} = \{\}$, then $\forall X \in \mathcal{P}_{fin}(S), \bar{1}\hat{\bullet}X = \{\}$, giving contradiction. □

Let $x \in S$, then $\{x\} \in \mathcal{P}_{fin}(S)$, we have

$$\bar{1}\hat{\bullet}\{x\} = \{x\}\hat{\bullet}\bar{1} = \{x\}$$

$$\{\alpha_1, \alpha_2, ..., \alpha_n\}\hat{\bullet}\{x\} = \{x\}\hat{\bullet}\{\alpha_1, \alpha_2, ..., \alpha_n\} = \{x\}$$

$$\{\alpha_1 \bullet x, \alpha_2 \bullet x, ..., \alpha_n \bullet x\} = \{x \bullet \alpha_1, x \bullet \alpha_2, ..., x \bullet \alpha_n\} = \{x\}$$

Hence,

$$\alpha_1 \bullet x = ... = \alpha_n \bullet x = x = x \bullet \alpha_1 = ... = x \bullet \alpha_n \Rightarrow \forall i \in [1,n], \alpha_i \bullet x = x = x \bullet \alpha_i$$

$$\left.\begin{array}{l}\Rightarrow \forall i \in [1,n], \alpha_i \text{ is the identity.}\\ \text{Prop. 2.1, identity is unique}\end{array}\right\} \Rightarrow \alpha_1 = ... = \alpha_n$$

Hence we have $\bar{1} = \{\alpha_1\}$, where α_1 is the identity of $(S,\bullet)$ □

Prop 2.3. *Assume $(S, \bullet)$ is a semigroup and $|S| \geq 2$,*

$$(\mathcal{P}_{fin}(S), \cup, \{\}) \text{ is a commutative monoid.}$$

Proof.

$$\left.\begin{array}{l} \left.\begin{array}{l} \forall A, B, C \in \mathcal{P}_{fin}(S), (A \cup B) \cup C = A \cup (B \cup C) \\ \forall A, B \in \mathcal{P}_{fin}(S), A \cup B \in \mathcal{P}_{fin}(S) \end{array}\right\} \Rightarrow (\mathcal{P}_{fin}(S), \cup) \text{ is semigroup} \\ \exists \{\} \in \mathcal{P}_{fin}(S), \forall A \in \mathcal{P}_{fin}(S), \{\} \cup A = A \cup \{\} = A \Rightarrow \{\} \text{ is the identity} \end{array}\right\}$$

$$\left.\begin{array}{l} \Rightarrow (\mathcal{P}_{fin}(S), \cup, \{\}) \text{ is monoid} \\ \forall A, B \in \mathcal{P}_{fin}(S), A \cup B = B \cup A \end{array}\right\} \Rightarrow (\mathcal{P}_{fin}(S), \cup, \{\}) \text{ is a commutative monoid.}$$

□

Prop 2.4. *Assume $(S, \bullet)$ is a semigroup and $|S| \geq 2$,*

$$(\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is a semigroup.}$$

Proof.

$$\begin{aligned} \text{Let } A, B, C \in \mathcal{P}_{fin}(S), \text{ then } (A \hat{\bullet} B) \hat{\bullet} C =& \{a \bullet b\}_{a \in A, b \in B} \hat{\bullet} \{c\}_{c \in C} = \{a \bullet b \bullet c\}_{a \in A, b \in B, c \in C} \\ =& \{a\}_{a \in A} \hat{\bullet} \{b \bullet c\}_{b \in B, c \in C} = A \hat{\bullet} (B \hat{\bullet} C) \\ \Rightarrow& (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is associative} \end{aligned}$$

$$\text{Let } A, B \in \mathcal{P}_{fin}(S), \text{ then } A \hat{\bullet} B = \{a \bullet b\}_{a \in A, b \in B} \in \mathcal{P}_{fin}(S) \Rightarrow (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is closure}$$

$$\left.\begin{array}{l} (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is associative} \\ (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is closure} \end{array}\right\} \Rightarrow (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is a semigroup.}$$

□

Prop 2.5. *Assume $(S, \bullet)$ is a semigroup and $|S| \geq 2$,*

$$(\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ is a pre-semiring.}$$

Proof.

$$\left.\begin{array}{l} \text{By Prop 2.3, } (\mathcal{P}_{fin}(S), \cup) \text{ is a commutative monoid.} \\ \text{By Prop 2.4, } (\mathcal{P}_{fin}(S), \hat{\bullet}) \text{ is a semigroup.} \end{array}\right\} \Rightarrow (\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ is a bi-semigroup}$$

Claim. $\mathbb{LD}(\mathcal{P}_{fin}(S), \cup, \hat{\bullet})$

Proof of Claim. Let $A, B, C \in \mathcal{P}_{fin}(S)$, then $\exists m, n, k \in \mathbb{N}^0$, $A = \{a_1, ..., a_m\}$, $B = \{b_1, ..., b_n\}$, $C = \{c_1, ..., c_k\}$

$$\begin{aligned} A \hat{\bullet} (B \cup C) =& \{a_1, ..., a_m\} \hat{\bullet} (\{b_1, ..., b_n\} \cup \{c_1, ..., c_k\}) \\ =& \{a_1, ..., a_m\} \hat{\bullet} \{b_1, ..., b_n, c_1, ..., c_k\} \\ =& \{a_1 \bullet b_1, ..., a_1 \bullet b_n, a_m \bullet b_1, ..., a_m \bullet b_n, a_1 \bullet c_1, ..., a_1 \bullet c_k, a_m \bullet c_1, ..., a_m \bullet c_k\} \\ =& \{a_1 \bullet b_1, ..., a_1 \bullet b_n, a_m \bullet b_1, ..., a_m \bullet b_n\} \cup \{a_1 \bullet c_1, ..., a_1 \bullet c_k, a_m \bullet c_1, ..., a_m \bullet c_k\} \\ =& (A \hat{\bullet} B) \cup (A \hat{\bullet} C) \end{aligned}$$

Hence we have $\mathbb{LD}(\mathcal{P}_{fin}(S), \cup, \hat{\bullet})$. □

$\mathbb{RD}(\mathcal{P}_{fin}(S), \cup, \hat{\bullet})$ is similar.
Hence,

$$\left.\begin{array}{l}(\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ is a bi-semigroup} \\ \cup \text{ is clearly commutative} \\ \mathbb{LD}(\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ and } \mathbb{RD}(\mathcal{P}_{fin}(S), \cup, \hat{\bullet})\end{array}\right\} \Rightarrow (\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ is a pre-semiring.}$$

□

Prop 2.6. *Assume $(S, \bullet)$ is a semigroup and $|S| \geq 2$,*

$$(S, \bullet) \textit{ has identity} \Rightarrow \textit{union\_lift}(S, \bullet) \textit{ is a semiring}$$

Proof. By Prop 2.3, we know that $\{\}$ is the identity of $(\mathcal{P}_{fin}(S), \hat{\bullet})$. Let $A \in \mathcal{P}_{fin}(S)$, then $\{\}\hat{\bullet}A = A\hat{\bullet}\{\} = \{\}$, so $\{\}$ is an annihilator for $\hat{\bullet}$.
As we assumed, $(S, \bullet)$ has identity, then by Prop 2.1, it has a unique identity α.
Claim. $(\mathcal{P}_{fin}(S), \hat{\bullet}, \{\alpha\})$ is a monoid.

Proof of Claim. By Prop 2.4, $(\mathcal{P}_{fin}(S), \hat{\bullet})$ is a semi-group.
Let $A \in \mathcal{P}_{fin}(S)$, then $\exists n \in \mathbb{N}^0, A = \{a_0, ..., a_n\}$, then

$$\begin{aligned}\{\alpha\}\hat{\bullet}A =&\{\alpha\}\hat{\bullet}\{a_1, ..., a_n\} \\ =&\{\alpha \bullet a_1, ..., \alpha \bullet a_n\} \\ =&\{a_1, ..., a_n\} \\ =&A \\ =&\{a_1, ..., a_n\} \\ =&\{a_1 \bullet \alpha, ..., a_n \bullet \alpha\} \\ =&A\hat{\bullet}\{\alpha\}\end{aligned}$$

Hence $\{\alpha\}$ is the unique identity of $(\mathcal{P}_{fin}(S), \hat{\bullet})$ by Prop 2.1.
So $(\mathcal{P}_{fin}(S), \hat{\bullet}, \{\alpha\})$ is a monoid. □

Hence we have

$$\left.\begin{array}{l}\{\} \text{ is an annihilator for } \hat{\bullet} \\ (\mathcal{P}_{fin}(S), \hat{\bullet}, \{\alpha\}) \text{ is a monoid} \\ \text{By Prop 2.5, } (\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ is a pre-semiring.} \\ \text{By Prop 2.3, } (\mathcal{P}_{fin}(S), \cup, \{\}) \text{ is a commutative monoid.}\end{array}\right\} \Rightarrow (\mathcal{P}_{fin}(S), \cup, \hat{\bullet}) \text{ is a semiring.}$$

So $\textit{union\_lift}(S, \bullet) \equiv (\mathcal{P}_{fin}(S), \cup, \hat{\bullet})$ is a semiring. □

Prop 2.7. *Assume $(S, \bullet)$ is a semigroup and $|S| \geq 2$,*

$$\textit{union\_lift}(S, \bullet) \textit{ is a semiring} \Leftrightarrow (S, \bullet) \textit{ has identity}$$

Proof. By Prop 2.2 and Prop 2.6, we have

$$\left.\begin{array}{r}\textit{union\_lift}(S, \bullet) \text{ is a semiring} \Rightarrow (S, \bullet) \text{ has identity} \\ (S, \bullet) \text{ has identity} \Rightarrow \textit{union\_lift}(S, \bullet) \text{ is a semiring}\end{array}\right\}$$

$$\Rightarrow (\textit{union\_lift}(S, \bullet) \text{ is a semiring} \Leftrightarrow (S, \bullet) \text{ has identity})$$

□