L11 Homework 2

Andi Zhang

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1 Minimum Spanning Tree

Def 1.1 (Minimum Spanning Tree)**.** *A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.*

Def 1.2. *Let $G = (V, E)$ be a connected minimax graph and each arc weight is unique. Let $S = \{(i, j) \in E | A(i, j) = A^*(i, j)\}$, where $A^*(i, j) = \min_{p \in \pi(i,j)} \max_{(u,v) \in p} A(u, v)$*

Prop 1.1. *S is a tree.*

Proof. Suppose (for contradiction) that S is not a tree, then there is a cycle $a_1 ... a_n a_1$ in S. Let q be the path $a_1 ... a_n$. As each arc weight is unique, $\exists i \in [1, n-1], w(a_i, a_{i+1}) = \max_{(u,v) \in q} w(u, v)$.
We have $(a_1, a_n) \in S$, then

$$\begin{aligned} w(a_1, a_n) =& A(a_1, a_n) = A^*(a_1, a_n) = \min_{p \in \pi(a_1, a_n)} \max_{(u,v) \in p} A(u, v) \\ &\le \max_{(u,v) \in q} A(u, v) = \max_{(u,v) \in q} w(u, v) = w(a_i, a_i + 1) \end{aligned}$$

So $w(a_1, a_n) \le w(a_i, a_{i+1})$.

Let k be path $a_i a_{i-1} ... a_1 a_n a_{n-1} ... a_{i+1}$.

$$\begin{aligned} w(a_i, a_{i+1}) =& A(a_i, a_{i+1}) = A^*(a_i, a_{i+1}) = \min_{p \in \pi(a_i, a_{i+1})} \max_{(u,v) \in p} A(u, v) \\ &\le \max_{(u,v) \in k} A(u, v) = \max_{(u,v) \in k} w(u, v) \end{aligned}$$

As we already known, $\forall (u, v) \in k \setminus \{(a_1, a_n)\}, w(u, v) < w(a_i, a_{i+1})$, so $w(a_1, a_n)$ is the only choice for $\max_{(u,v) \in k} w(u, v)$
So $w(a_1, a_n) = \max_{(u,v) \in k} w(u, v)$, $w(a_i, a_{i+1}) \le w(a_1, a_n)$

$$\left. \begin{aligned} w(a_1, a_n) \le w(a_i, a_{i+1}) \\ w(a_i, a_{i+1}) \le w(a_1, a_n) \end{aligned} \right\} \Rightarrow w(a_1, a_n) = w(a_i, a_{i+1})$$

Contradict to the fact that each arc weight is unique! □

Prop 1.2. *$\forall a \in V, \exists a' \in V$ adjacent to a, such that $(a, a') \in S$.*

Proof. Let $a \in V$, as G is connected, we have a_1, a_2, $...a_n$ which are adjacent to a. Then $\exists i \in [1, n]$ such that $w(a, a_i) = \min_{j \in [1,n]} w(a, a_j)$ since each arc weight is unique.
Claim. $(a, a_i) \in S$

Proof of Claim. Suppose (for contradiction) that $(a, a_i) \notin S$
Then $\exists q \in \pi(a, a_i), \max_{(u,v)\in q} w(u, v) = \min_{p\in\pi(a,a_i)} \max_{(u,v)\in p} w(u, v) < w(a, a_i)$ ($<$ here since the weights are unique)
So $\exists k \in [1, n], k \neq i, (a, a_k) \in q$. Then $\max_{(u,v)\in q} w(u, v) \geq w(a, a_k) > w(a, a_i)$

$$\left.\begin{array}{l}\max_{(u,v)\in q} w(u, v) < w(a, a_i) \\ \max_{(u,v)\in q} w(u, v) > w(a, a_i)\end{array}\right\} \Rightarrow \text{Contradiction!}$$

□

□

Prop 1.3. *S is a spanning tree.*

Proof.

$$\left.\begin{array}{l}\text{By Prop 1.1, } S \text{ is a tree.} \\ \text{By Prop 1.2, } \forall a \in V, \exists a' \in V \text{ adjacent to a, such that } (a, a') \in S\end{array}\right\} \Rightarrow S \text{ is a spanning tree.}$$

□

Prop 1.4 (Cycle Property of MST)**.** *For any cycle C in the graph, if the weight of an edge e of C is larger than the individual weights of all other edges of C, then this edge cannot belong to a MST.*

Proof. Refer to Wikipedia. □

Prop 1.5. *For any cycle C in the graph, if the edge $e \in MST$, then $w(e)$ is not the largest in C.*

Proof. This is just the contraposition of Prop 1.4. □

Prop 1.6. $MST(G) \subseteq S$

Proof. Let $e \in MST(G)$, suppose (for contradiction) that $e \notin S$. Then $S \cup \{e\}$ must has a cycle $C = \{e, d_1, d_2, ..., d_n\}$, where $d_i \in S, \forall i \in [1, n]$.

$$d_1 \in S \Rightarrow w(d_1) = A^*(d_1) < \max\{w(d_2)...w(d_n), w(e)\}$$
$$d_2 \in S \Rightarrow w(d_2) = A^*(d_2) < \max\{w(d_1)...w(d_n), w(e)\}$$
$$\vdots$$
$$d_n \in S \Rightarrow w(d_n) = A^*(d_n) < \max\{w(d_1)...w(d_{n-1}), w(e)\}$$

(We use $<$ but not $\leq$ because the weight of arcs are unique)

Since weight of each arc is unique, $\exists d_i \in C$ such that $w(d_i) > w(d_j), \forall j \neq i$

$$\left.\begin{array}{l}w(d_i) < \max\{w(d_1), ..., w(d_{i-1}), w(d_{i+1}), ...w(d_n), w(e)\} \\ w(d_i) > w(d_j), \forall j \neq i\end{array}\right\} \Rightarrow w(e) \text{ is the largest in } C$$

However, By Prop 1.5, we know $w(e)$ is not the largest in C, giving a contradiction!
Hence $MST(G) \subseteq S$ □

Prop 1.7. $S = MST(G)$ *(S is the minimum spanning tree)*

Proof. $|S| = |V - 1|$, $|MST(G)| = |V - 1|$ as S and $MST(G)$ are spanning trees, then

$$\left.\begin{array}{l}MST(G) \subseteq S \\ |S| = |MST(G)|\end{array}\right\} \Rightarrow S = MST(G)$$

□

2 Distributivity for Martelli's semiring

Def 2.1 (Martelli's Semiring)**.** *Let $G = (V, E)$ be a directed graph.*

$$\begin{aligned} M &\equiv (S, \oplus, \otimes, \bar{0}, \bar{1}) \\ S &\equiv \{X \in 2^{2^E} | \forall U, V \in X, U \subseteq V \Rightarrow U = V\} \\ X \oplus Y &\equiv \textit{remove all supersets from } \{U \cup V | U \in X, V \in Y\} \\ X \otimes Y &\equiv \textit{remove all supersets from } X \cup Y \\ \bar{0} &\equiv \{\{\}\} \\ \bar{1} &\equiv \{\} \end{aligned}$$

Def 2.2. *Let $+ : 2^{2^E} \times 2^{2^E} \to 2^{2^E}$ be a binary operator such that*

$$\forall A, B \in 2^{2^E}, A + B = \{U \cup V | U \in A, V \in B\}$$

Def 2.3. *Let $\times : 2^{2^E} \times 2^{2^E} \to 2^{2^E}$ be a binary operator such that*

$$\forall A, B \in 2^{2^E}, A \times B = A \cup B$$

Def 2.4. *Let $R : 2^{2^E} \to S$ be a function such that*

$$\forall A \in 2^{2^E}, R(A) = \textit{remove all supersets from } A$$

Def 2.5. *Let $Su : 2^{2^E} \to 2^{2^E}$ be a function such that*

$$\forall A \in 2^{2^E}, Su(A) = \textit{all the supersets from } A$$

Then $\forall A \in 2^{2^E}, A = R(A) \times Su(A)$ and $\forall X \in Su(A), \exists Y \in R(A)$ such that $Y \subseteq X$

It is clear that $+$, $\times$, R and Su are well-defined. (I define the function as Su because the sign S is already used in Def 2.1)
It is easy to check the commutativity and association of $+$ and $\times$.

Prop 2.1. *$\forall X, Y \in S$, we have:*

$$X \oplus Y = R(X + Y)$$

$$X \otimes Y = R(X \times Y)$$

Proof. This is clear by the definitions. □

Prop 2.2. *Let $A \in 2^{2^E}$, then*

$$P \in R(A) \Leftrightarrow \forall X \in A, X \not\subset P$$

Proof. This is clear by the definition of R (Def 2.4). □

Prop 2.3. *$\forall A \in S, R(A) = A$*

Proof. Let $A \in S$
$R(A) \subseteq A$ by Def 2.5.
By Def 2.1 we have $\forall U, V \in A, U \subseteq V \Rightarrow U = V$. Let $P \in A$, then $\forall Q \in A$, $Q \subseteq P \Rightarrow Q = P$, that is to say $Q \not\subset P$, by Prop 2.2, we have $P \in R(A)$, so $A \subseteq R(A)$.
Hence $A = R(A)$ □

Prop 2.4. *Let $T, P, Q \in 2^E$, then $(T \not\subset P$ and $T \subseteq Q) \Rightarrow Q \not\subset P$*

Proof. We have $(T \subseteq P \Rightarrow T = P)$ as $T \not\subset P$.
Let $Q \subseteq P$, then $T \subseteq Q \subseteq P$, so $T = P$, we have $P = T \subseteq Q \subseteq P$, then $Q = P$.
In all, $Q \subseteq P \Rightarrow Q = P$, that is to say, $Q \not\subset P$. □

Prop 2.5. *Let $C, C' \in 2^{2^E}$, if $\forall A \in C', \exists B \in C$ s.t. $B \subseteq A$, then $R(C \times C') = R(C)$*

Proof. Let $P \in R(C \times C')$, then $\forall X \in C \times C', X \not\subset P$ by Prop 2.2
Let $Y \in C$, then $Y \in C \times C'$ as $C \subseteq C \times C'$. So $Y \not\subset P$. By Prop 2.2 we have $P \in R(C)$.
Hence $R(C \times C') \subseteq R(C)$.

Let $P \in R(C)$, then $\forall X \in C, X \not\subset P$ by Prop 2.2
Let $Q \in C$, we have $Q \not\subset P$.
Let $Q \in C'$, then $\exists T \in C$ such that $T \subseteq Q$ by assumption. As $T \not\subset P$ and $T \subseteq Q$, we know $Q \not\subset P$ by Prop 2.4.
Hence $\forall Q \in C \times C', Q \not\subset P$, by Prop 2.2 we have $P \in R(C \times C')$. So $R(C) \subseteq R(C \times C')$

In all we have $R(C) = R(C \times C')$ □

Prop 2.6. $\forall A, B \in 2^{2^E}, R(R(A) \times R(B)) = R(A \times B)$

Proof.

$$\begin{aligned} R(A \times B) &= R((R(A) \times Su(A)) \times (R(B) \times Su(B))) && \text{(By Def 2.5)} \\ &= R(R(A) \times R(B) \times Su(A) \times Su(B)) && \text{(Commutativity)} \end{aligned}$$

By Def 2.5, $\forall X \in Su(B), \exists Y \in R(B) \subseteq R(A) \times R(B) \times Su(A), Y \subseteq X$, then by Prop 2.5, we have:

$$R(R(A) \times R(B) \times Su(A) \times Su(B)) = R(R(A) \times R(B) \times Su(A))$$

Similarly,

$$R(R(A) \times R(B) \times Su(A)) = R(R(A) \times R(B))$$

Hence,

$$R(A \times B) = R(R(A) \times R(B))$$

□

Prop 2.7. $\forall A, B \in 2^{2^E}, R(R(A) + R(B)) = R(A + B)$

Proof.

$$\begin{aligned} R(A + B) =& R((R(A) \times Su(A)) + (R(B) \times Su(B))) && \text{(By Def 2.5)} \\ =& R(\{r_a \cup r_b\}_{r_a \in R(A), r_b \in R(B)} \cup \{r_a \cup s_b\}_{r_a \in R(A), s_b \in Su(B)} \\ & \cup \{s_a \cup r_b\}_{s_a \in Su(A), r_b \in R(B)} \cup \{s_a \cup s_b\}_{s_a \in Su(A), s_b \in Su(B)}) && \text{(By Def of } +, \times) \\ =& R(\{r_a \cup r_b\}_{r_a \in R(A), r_b \in R(B)} \times \{r_a \cup s_b\}_{r_a \in R(A), s_b \in Su(B)} \\ & \times \{s_a \cup r_b\}_{s_a \in Su(A), r_b \in R(B)} \times \{s_a \cup s_b\}_{s_a \in Su(A), s_b \in Su(B)}) && \text{(By Def of } \times) \end{aligned}$$

Let $X \in \{r_a \cup s_b\}_{r_a \in R(A), s_b \in Su(B)}$, then $X = r'_a \cup s'_b$ for some $r'_a \in R(A), s'_b \in Su(B)$.
It is clear that $\exists r''_a \in R(A), r''_a = r'_a$.
By Def 2.5, $\exists s''_b \in R(B), s''_b \subseteq s'_b$.
Hence there is a $Y = r''_a \cup s''_b \in \{r_a \cup r_b\}_{r_a \in R(A), r_b \in R(B)}$, $Y \subseteq X$ as $r''_a \cup s''_b \subseteq r'_a \cup s'_b$

By Prop 2.5 and commutativity, the term $\{r_a \cup s_b\}_{r_a \in R(A), s_b \in Su(B)}$ is eliminated.

Similarly, the term $\{s_a \cup r_b\}_{s_a \in Su(A), r_b \in R(B)}$ is eliminated.

Let $X \in \{s_a \cup s_b\}_{s_a \in Su(A), s_b \in Su(B)}$, then $X = s'_a \cup s'_b$ for some $s'_a \in Su(A), s'_b \in Su(B)$.
By Def 2.5, $\exists s''_a \in R(A), s''_a \subseteq s'_a$.
By Def 2.5, $\exists s''_b \in R(B), s''_b \subseteq s'_b$.
Hence there is a $Y = s''_a \cup s''_b \in \{r_a \cup r_b\}_{r_a \in R(A), r_b \in R(B)}$, $Y \subseteq X$ as $s''_a \cup s''_b \subseteq s'_a \cup s'_b$
By Prop 2.5 and commutativity, the term $\{s_a \cup s_b\}_{s_a \in Su(A), s_b \in Su(B)}$ is eliminated.

Hence,

$$\begin{aligned}
R(A+B) =& R(\{r_a \cup r_b\}_{r_a \in R(A), r_b \in R(B)}) \\
=& R(R(A) + R(B)) && \text{(By Def of +)}
\end{aligned}$$

□

Prop 2.8. $\forall A, B, C \in 2^{2^E}, R(A \times (B + C)) = R((A \times B) + (A \times C))$

Proof.

$$\begin{aligned}
R((A \times B) + (A \times C)) =& R(\{a \cup a'\}_{a \in A, a' \in A} \cup \{a \cup c\}_{a \in A, c \in C} \\
& \cup \{b \cup a\}_{b \in B, a \in A} \cup \{b \cup c\}_{b \in B, c \in C}) && \text{(By Def of +, } \times\text{)} \\
=& R(\{a \cup a\}_{a \in A} \cup \{a \cup a'\}_{a, a' \in A, a \neq a'} \cup \{a \cup c\}_{a \in A, c \in C} \\
& \cup \{b \cup a\}_{b \in B, a \in A} \cup \{b \cup c\}_{b \in B, c \in C}) \\
=& R(\{a\}_{a \in A} \cup \{a \cup a'\}_{a, a' \in A, a \neq a'} \cup \{a \cup c\}_{a \in A, c \in C} \\
& \cup \{b \cup a\}_{b \in B, a \in A} \cup \{b \cup c\}_{b \in B, c \in C}) \\
=& R(\{a\}_{a \in A} \times \{a \cup a'\}_{a, a' \in A, a \neq a'} \times \{a \cup c\}_{a \in A, c \in C} \\
& \times \{b \cup a\}_{b \in B, a \in A} \times \{b \cup c\}_{b \in B, c \in C}) && \text{(By Def of } \times\text{)}
\end{aligned}$$

Let $X \in \{a \cup c\}_{a \in A, c \in C}$, then $X = a' \cup c'$ for some $a' \in A, c' \in C$.
It is clear that $\exists Y \in \{a\}_{a \in A}$, $Y = a' \subseteq a' \cup c' = X$
By Prop 2.5 and commutativity, the term $\{a \cup c\}_{a \in A, c \in C}$ is eliminated.
Similarly, the term $\{a \cup a'\}_{a, a' \in A, a \neq a'}$, $\{b \cup a\}_{b \in B, a \in A}$ are eliminated.
Hence,

$$\begin{aligned}
R((A \times B) + (A \times C)) =& R(\{a\}_{a \in A} \times \{b \cup c\}_{b \in B, c \in C}) \\
=& R(\{a\}_{a \in A} \cup \{b \cup c\}_{b \in B, c \in C}) && \text{(By Def of } \times\text{)} \\
=& R(A \times (B + C)) && \text{(By Def of +, } \times\text{)}
\end{aligned}$$

□

Prop 2.9. $\forall A, B, C \in S, A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$

Proof. Let $A, B, C \in S$,

$$
\begin{aligned}
A \otimes (B \oplus C) &= A \otimes R(B + C) && \text{(by Prop 2.1)} \\
&= R(A \times R(B + C)) && \text{(by Prop 2.1)} \\
&= R(R(A) \times R(B + C)) && \text{(by Prop 2.3)} \\
&= R(A \times (B + C)) && \text{(by Prop 2.6)} \\
&= R((A \times B) + (A \times C)) && \text{(by Prop 2.8)} \\
&= R(R(A \times B) + R(A \times C)) && \text{(by Prop 2.7)} \\
&= R((A \otimes B) + (A \otimes C)) && \text{(by Prop 2.1)} \\
&= (A \otimes B) \oplus (A \otimes C) && \text{(by Prop 2.1)}
\end{aligned}
$$

□

Hence the left distributivity hold for Martelli's semiring. Right distributivity is similar!