Rotation

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1 Introduction

In rotating frames of reference, there exist new quantities that we find more useful than linear mechanical quantities (i.e. linear momentum, velocity).

Table 1: Mechanical Quantities and their Rotational Analogues[1]

Linear Quantity	Rotational Analogue	Relationship
Position s	Angle θ	$s = \theta \times r$
Velocity v	Angular velocity ω	$v = \omega \times r$
Acceleration a	Angular acceleration α	$a = \alpha \times r$
Mass m	Moment of inertia I	$I = mr^2$
Force F	Torque τ	$\tau = r \times F_\perp$
Momentum p	Angular Momentum L	$L = r \times p_\perp$

We hence try to derive a few of these quantities.

2 Center of Mass

Every body has a point where its mass can be taken to be 'concentrated' at - in other words, under many circumstances, it will act as if all its mass is located at a single point. We can calculate the location of this center of mass R by taking the weighted average of all points on the body by mass:

$$R = \sum \left(r_i \frac{m_i}{M} \right)$$

For two bodies of mass M_A and mass M_B, the center of mass of the two is found by simply taking the weighted average of their two own centers of mass. If one's center of mass is R_A and the other is R_B, the overall center of mass is $R = \frac{R_A M_A + R_B M_B}{M_A + M_B}$. We show this as follows:

$$\begin{aligned} R &= \sum_i \left(r_i \frac{m_i}{M_A + M_B} \right) \\ &= \frac{\sum_{j \epsilon A} (r_j m_j) + \sum_{k \epsilon B} (r_k m_k)}{M_A + M_B} \\ &= \frac{R_A M_A + R_B M_B}{M_A + M_B} \end{aligned}$$

2.1 Pappus' Theorem

Pappus' theorem states that the geometric centroid of any plane figure (a 2D shape) can be found by taking the plane figure, rotating it about any external axis, and the following relationship will result. The volume

[1] $\perp$ means the quantity at 90 degrees to r.

of the solid of revolution V is related to the area of the shape A and the distance d travelled by the centroid such that $V = Ad$.

Alternately, for a plane curve (any line or curve), the center of mass is such that the area swept out by the plane curve when rotated about any external axis is equal to the product of the length of the curve and the distance travelled by the centroid.

Of course, we can apply this to finding the center of mass only when the object in question is of uniform density, since a geometric centroid only averages the positions of every point in a shape (it does not take into account different weightings).

3 Torque

We wish to show that torque $\tau = F \times r$, or that $\tau = F_x r_y - F_y r_x$. We proceed by using the following definition of torque: $W = \tau\Delta \cdot \theta$. So:

$$\begin{aligned} W &= F \cdot \Delta s \\ &= F \cdot (\Delta\theta \times r) \\ &= (F \times r) \cdot \Delta\theta \\ &= \tau \cdot \Delta\theta \end{aligned}$$

3.1 Equilibrium

We wish to formalize the notion of equilibrium in a rotation system - we propose that equilibrium is achieved only if $\sum F_i = 0$ and $\sum \tau_i = 0$. We note that the τ can be with respect to **any** axis. In other words, if total torque is 0 about some axis, it will be 0 about every axis. We show this as follows. Since torque is 0 about some axis, we can say $\sum r_i \times F_i = 0$. About some other axis, we write our new torque τ_1 as $\sum(r_i + R) \times F_i$, where R is the vector from the original axis to the new axis. Thus:

$$\begin{aligned} \tau_1 &= \sum(r_i + R) \times F_i \\ &= \sum r_i \times F_i + \sum R \times F_i \\ &= \tau + R \times F \\ &= 0 + R \times 0 \\ &= 0 \end{aligned}$$

Thus the torque τ_1 about every axis is 0 if it is 0 about one given axis.

4 Angular Momentum

We arrive at an equation for angular momentum by investigating the formula for torque $r \times F$. Since $p = \int F dt$, it is not unreasonable to assume $L = \int \tau dt = \int r \times F dt$. We can evaluate this through integration by parts (since the product rule holds for cross products, integration by parts will work as well):

$$\begin{aligned} L &= \int r \times F dt \\ &= r \times \int F dt - \int \frac{dr}{dt} \times (\textstyle\int F dt) dt \\ &= r \times p - \int v \times p \; dt \\ &= r \times p - m \int v \times v \; dt \\ &= r \times p \end{aligned}$$

We can further investigate this by writing L as $r \times mv = r \times m\omega r$. Since r and mv are at 90 degrees to each other, we can say their cross product is in the z direction (assuming the plane being rotated is the xy plane). Thus we can write $L = mr^2\omega\hat{z}$. If we remember ω is the rotational analogue of v, it seems clear that mr^2 is the rotational analogue of mass, which we call moment of inertia I.

4.1 Conservation of Angular Momentum

By Newton's laws, the torques caused by interactions between individual particles of a system should not manifest themselves in the 'sum' of torques - this is because if one particle exerts a torque on another, the other particle exerts an equal and opposite torque on the original particle. In this manner, internal torques cancel out and the change in total torque of a system can be caused only by external torques. Thus, if external torque is 0, the angular momentum of a system will remain constant.

5 Coriolis Force

In every system rotating at constant angular velocity ω there exists two 'pseudoforces': centrifugal force and the Coriolis force. The centrifugal force, of course, is $m\omega^2 r$. However, for the Coriolis force, we must have a **moving** particle (that is, moving relative to the spinning plane).

5.1 Radial Movement

We imagine a scenario where we have something we wish to make go in a straight line along a the radius, going from the center to the edge of the spinning plane. If we launch the particle outwards with a radial velocity of v_r, there is a tangential force we must exert to make the particle continue in the straight radial line. Intuitively this makes sense because the plate spinning beneath the particle has a tangential velocity, while the particle itself has no tangential velocity. Thus our force must be tangential. We can get an intuitive understanding for this force as such. The angular momentum L of the particle is given by $r \times p$. We know the derivative $\frac{dL}{dt}$ is equal to torque, or $r \times F_c$. The F_c here is the Coriolis force. We proceed[2]:

$$
\begin{aligned}
r \times F_c &= \frac{dL}{dt} \\
r \times F_c &= \frac{dr}{dt} \times p_\perp + r \times \frac{dp_\perp}{dt} \\
r \times F_c &= v_r \times p_\perp + r \times \frac{d(m\omega r)}{dt} \\
r \times F_c &= v_r \times m\omega r + r \times m\omega v_r \\
|r||F_c| &= m|v_r||\omega r| + m|r||\omega v_r| \\
|F_c| &= 2mv_r\omega
\end{aligned}
\tag{1}
$$

This explanation is slightly unintuitive. We can give a better one by breaking down what's really happening at in the second step of the above working, equation (1). The Corilis force is thus broken down into two components: $\frac{dr}{dt} \times p_\perp$ and $r \times \frac{dp_\perp}{dt}$. We label these components A and B respectively.

Component B makes intuitive sense. If we want our particle to have 0 velocity relative to the plate beneath it, it must have velocity ωr, and thus as it moves out, it must increase its velocity by ωdr for each small radial movement dr. Thus: $dv = \omega dr$, and so acceleration due to our need for an 'increasing velocity' is $m\frac{dv}{dt} = m\omega\frac{dr}{dt} = m\omega v_r$. The original formula for the magnitude of component B was $r\frac{dp}{dt} = r\frac{d(m\omega r)}{dt} = m\omega v_r$. So component B is explained by the need to increase our velocity as we move further out.

Component A arises from a more subtle cause. Since the plate is spinning, as our particle moves out, the outer place will move upwards by $\omega(r + dr)dt$, while the current section of the plate moves upwards only by $\omega r dt$. So we have to somehow move our particle up by $\omega dr dt = \omega v_r dt^2$. Looking back at component

[2]We note that $\frac{dr}{dt}$ is not purely v_r, of course the particle's radius vector is changing other than simply moving outwards. However, if we expand the vector $\frac{dr}{dt}$ we will see that since p is taken purely in the tangential direction (and thus the particle's momentum outwards radially is not counted), the tangential component of $\frac{dr}{dt}$ never shows up.

B, we are accelerating our particle by ωv_r, and thus distance travelled upwards (relative to the plate) will be $\frac{1}{2}\omega v_r dt^2$. So Component B only pushes our particle up **halfway** to where it needs to be. If we want to push it fully to where it needs to be, we need to deliver another acceleration of ωv_r. Thus the magnitude of Component A is equal to $m\omega v_r$ as well! This is where the factor of 2 arises in the formula for Coriolis force: out of a need to simultaneously increase the velocity of the particle, and to move the particle tangentially as it moves outwards (to keep up with the more rapidly spinning outwards portions of the plate).

5.1.1 Rigorous Vector Analysis

To see why the fact that p is taken only tangentially, we may write our proof out long form vector notation starting from (1). For the purposes of this proof, we take the i direction to be outwards radially, the j direction to be the tangent, and k to be through the axis of rotation.

$$\begin{aligned} r \times F_c &= \frac{dr}{dt} \times p_\perp + r \times \frac{dp_\perp}{dt} \\ \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} &= \begin{pmatrix} v_r \\ \omega r \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ m\omega r \\ 0 \end{pmatrix} + \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ m\omega \frac{dr}{dt} \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ rF_y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ v_r m\omega r - 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ rm\omega v_r - 0 \end{pmatrix} \\ rF_y &= 2m\omega r v_r \\ F_y &= 2m\omega v_r \end{aligned}$$

Now we see the Coriolis force is in the y-direction (tangential), with a magnitude of $2m\omega v_r$!

5.2 Tangential Movement

We imagine a new scenario. We have someone standing at radius R on a plate. He throws particle such that he sees it going around the plate with constant radius R and a velocity (relative to him) of v_t. To someone standing on the outside, he sees the particle going at velocity $\omega R + v_t$. Thus, there must be a force directed radially inwards of $m\frac{(\omega R+v_t)^2}{R}$. We expand this:

$$\begin{aligned} F &= m\frac{(\omega R + v_t)^2}{R} \\ &= m\frac{\omega^2 R^2 + 2\omega R v_t + v_t^2}{R} \\ &= m\omega^2 R + 2m\omega v_t + m\frac{v_t^2}{R} \end{aligned}$$

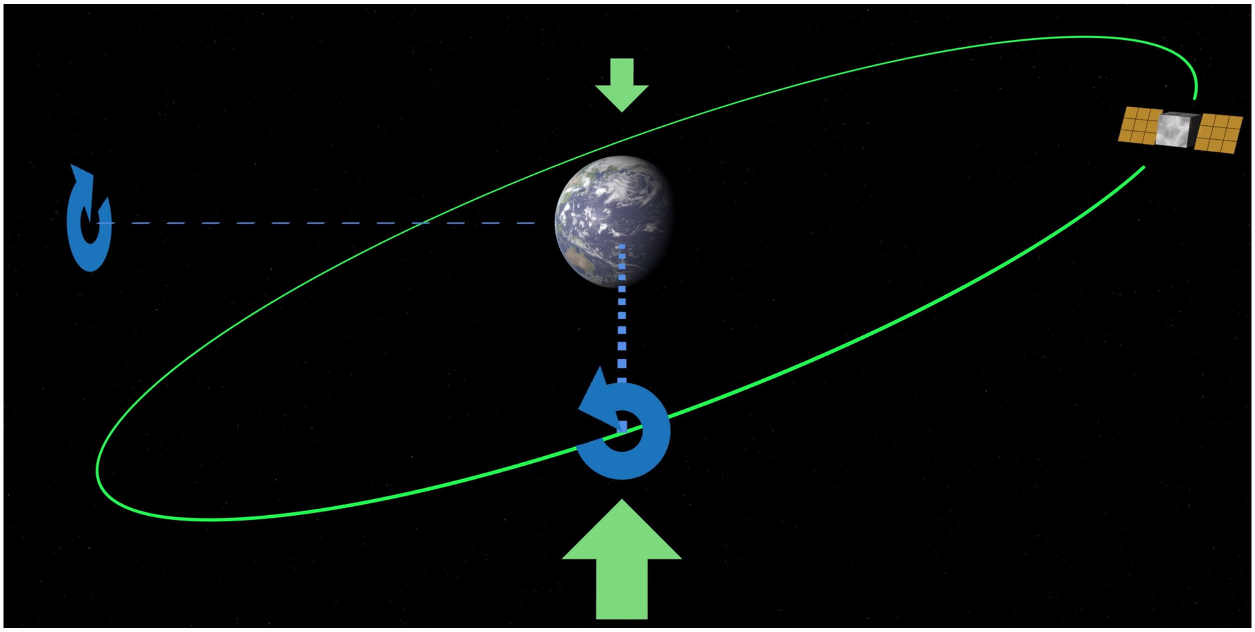
The first and third components of this force are to be expected. $m\omega^2 R$ is just the force required to keep anything on the turning plane at radius R - even if the particle was standing still relative to our observer, it would require this force radially. $m\frac{v_t^2}{R}$ is the force required for anything moving at speed v_t to go in a circle of radius R. However, there is an additional force required $2m\omega v_t$ - this is the Coriolis force! Thus it seems clear that the Coriolis force is always given by $2m\omega v$, where v is the relative velocity of the particle to a point on the plate, and the direction of the force is at 90 degrees to v.

6 Gyroscopes

We see that if we up and down at two points radially opposite on another in a rotating object's path, the path does not change as we expect initially. The path rotates about the line connecting the two points where we push! We see intuitively now why the bicycle wheel does not rotate as we expect. If we push one handle

[3] For a detailed explanation refer back to this.

Figure 1: Intuition Behind Torques and Changing Rotations[3]



of the bicycle wheel away from us, and pull one towards us, this exerts a lateral force on the spokes closest and farthest away from us. We get a scenario like the one depicted above – the path does not change as expected.