Special Relativity

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1 Introduction

Einstein proposed two postulates:

1. The laws of physics remain the same in every inertial reference frame (a coordinate system moving with constant velocity u).
2. The speed of light in a vacuum c is constant for all observers, regardless of the frame of reference.

Both of these have been experimentally verified. This presents a problem - the classical Newtonian notions of coordinate transforms do not hold anymore:

$$x' = x - ut$$

$$t' = t$$

However, we see that if this transform holds, one of Einstein's postulates is broken - we observe that

$$\frac{dx'}{dt} = \frac{dx}{dt} - u$$

Thus,

$$v' = v - u$$

In the specific case of light, this tells us that the speed of a beam of light, as seen by a moving observer, will be u less than the speed measured by a stationary observer. Thus, we must redefine our coordinate transforms.

2 Lorentz Transforms

To find the new coordinate transform, we take the following scenario. We have an observer standing at $x = 0$ in a stationary frame of reference S. We have another observer in the frame of reference S' moving at uniform velocity u in the $+x$ direction. At time $t = 0$, two things happen:

- The moving observer in S' arrives at $x = 0$. That is, $x' = x = 0$. At this point, the moving observer sets their clock to read $t' = 0$.
- A photon is shot out from $x = 0$, in the $+x$ direction.

By Einstein's second postulate, we observe that for the beam of light

$$c = \frac{x'}{t'} = \frac{x}{t} \tag{1}$$

Where x and x' denote the position of the photon). Since we already know that simply using a Newtonian transform will not work here, we hypothesize that there is a certain 'fudge factor' γ involved in the transform for length:

$$x' = \gamma(x - ut) \tag{2}$$

Also, we observe that the way in which we have set up our reference frames is symmetrical - we can reverse it such that S' becomes the stationary frame of reference, and S is now moving in the $-x'$ direction at speed u. Thus, by this symmetry,

$$x = \gamma(x' + ut') \tag{3}$$

Our goal is now to solve for γ, and we proceed by multiply equations (2) and (3):

$$x'x = \gamma^2(xx' + xut' - utx' + u^2tt') \tag{4}$$

$$1 = \gamma^2(1 + \frac{ut'}{x'} - \frac{ut}{x} + \frac{u^2tt'}{xx'}) \tag{5}$$

Substituting equation (1):

$$1 = \gamma^2(1 + \frac{u}{c} - \frac{u}{c} + \frac{u^2}{c^2}) \tag{6}$$

$$\gamma = \sqrt{1 - \frac{u^2}{c^2}} \tag{7}$$

With these, we can find our transform for not only x', but t' as well:

$$\begin{aligned} t' &= \frac{x'}{c} \\ &= \gamma\frac{(x-ut)}{c} \\ &= \gamma\frac{(ct-ut)}{c} \\ &= \gamma\Big(t - \frac{ut}{c}\Big) \\ &= \gamma\Big(t - \frac{ux}{c^2}\Big) \end{aligned}$$

And thus we have our new coordinate transform:

$$\boxed{x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}} \tag{8}$$

$$\boxed{t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}} \tag{9}$$

3 Consequences

It is often useful to take these transforms in a new format:

$$\Delta x' = \gamma(\Delta x - u\Delta t) \tag{10}$$

$$\Delta t' = \gamma\Big(\Delta t - \frac{u\Delta x}{c^2}\Big) \tag{11}$$

These can be derived simply by considering two events occurring at (x_1, t_1), (x'_1, t'_1), and (x_2, t_2), (x'_2, t'_2), and then taking the difference between x'_2 and x'_2 (and doing the same for time).

Using these transforms alone, it is possible to predict a wide variety of phenomena.

3.1 Length Dilation

Imagine there is a rod, stationary in the S frame of reference. The stationary observer will measure the length of the rod by finding the x coordinates of both ends of the rod. These measurements will be taken simultaneously - that is, $\Delta x = L$, where L is the length of the rod as measured by the stationary observer, and $\Delta t = 0$. We now use equation (10) to find the length of the rod $\Delta x' = L'$ as measured by an observer in the moving frame of reference S':

$$\begin{aligned} L' &= \gamma(\Delta x - u\Delta t) \\ &= \gamma L \end{aligned}$$

This tells us that the moving observer sees the rod to be longer than the stationary observer sees it to be. In other words, the moving observer's unit of length (e.g. meter) has **shortened** relative to the stationary observer's unit of length.

So, if someone carried a meter stick onto a spaceship moving at uniform velocity u, the meter stick would appear, from an outsider's perspective, to shorten by a factor $\frac{1}{\gamma}$. This phenomena is known as *length dilation*.

3.2 Time Dilation

Imagine there is a clock, stationary in the S frame of reference. It is a simple clock - it remains at $x = 0$, and the time interval between consecutive ticks is $\Delta t = \tau$. Given that $\Delta x = 0$ between consecutive ticks, we can find the time interval $\Delta t' = \tau'$ between ticks as measured by a moving observer in the frame of reference S'.

$$\begin{aligned} \tau' &= \gamma(\tau - \frac{u\Delta x}{c^2}) \\ &= \gamma\tau \end{aligned}$$

We see that the moving observer sees the interval between ticks to be **longer** than the stationary observer sees them to be. So the moving observer accuses the stationary observer of having clocks that run slow!

However, often times the clock is not stationary in the S frame of reference - rather, it is stationary in the S' frame of reference. For example, take a muon. When it is stationary, its lifespan is extremely short, say a. However, it flies towards the earth at extraordinary speeds. We wish to find how long it takes before the observer **on earth** sees the muon decompose.

We think of our 'clock' as being located on the muon. It emits one event at $(x, t) = (x_0, t_0)$ when the muon is formed. It emits one event at $(x, t) = (x_1, t_1)$ when the muon decomposes. We note that these two events are also at $(x', t') = (0, t_0')$, and $(x', t') = (0, t_0' + a)$. Our goal is to find $\Delta t = t_1 - t_0$, thus we may do the following:

$$\begin{aligned} \Delta t &= \gamma(a - \frac{u\Delta x'}{c^2}) \\ &= \gamma a \end{aligned}$$

For a perhaps more roundabout way of doing it, we may also do:

$$\begin{aligned} a &= \gamma(\Delta t + \frac{u\Delta x}{c^2}) \\ a &= \gamma\Big(\Delta t + \frac{u\Delta t(-u)}{c^2}\Big) \end{aligned}$$

**We substitute $\Delta x = -u\Delta t$ because the reference frame S is moving at $-u$ relative to S', while S' moves at $+u$ relative to S.*

$$\begin{aligned} a &= \gamma\Delta t\Big(1 - \frac{u^2}{c^2}\Big) \\ a &= \frac{1}{\gamma}\Delta t \\ \Delta t &= \gamma a \end{aligned}$$

And we find the same result, regardless of which way we solve for Δt.

3.3 Relativistic Mass

The conservation of momentum allows us to analyze what happens to mass when objects begin to move.

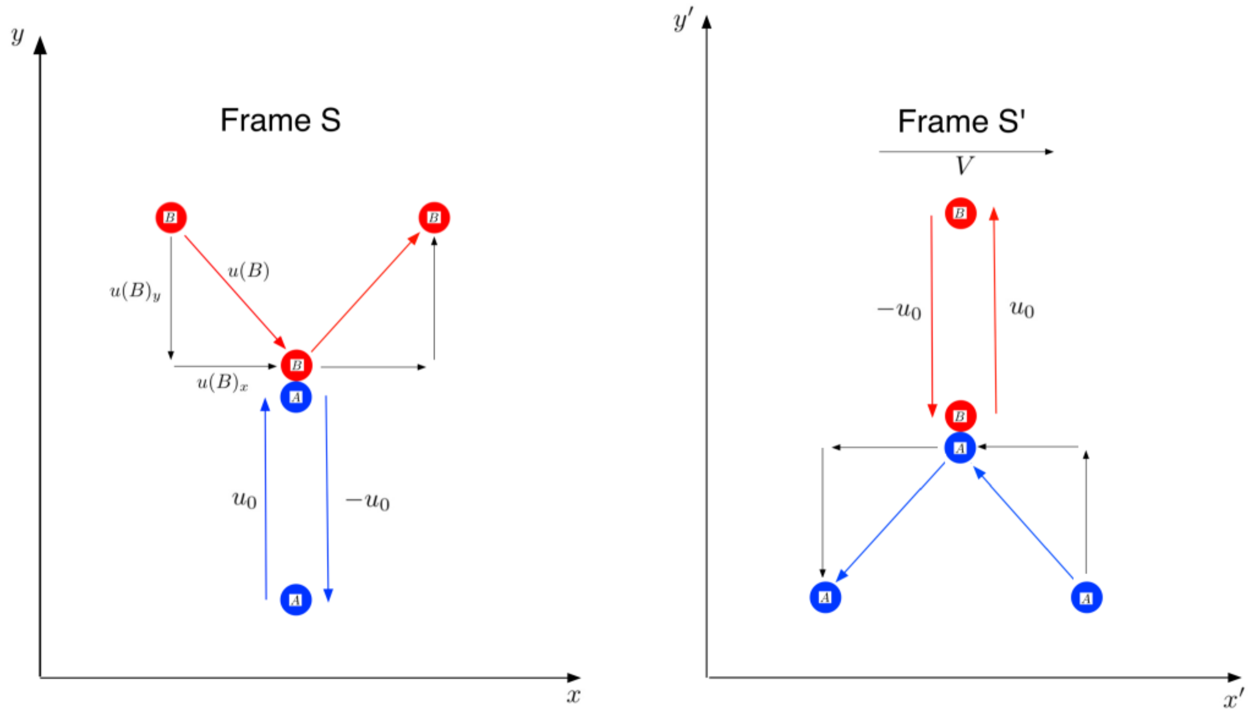


Figure 1: Two balls colliding

We observe the results of 2 balls colliding, ball A and ball B. Their 'rest masses' (mass measured when they are stationary) are identical. In the stationary frame of reference S, ball B moves with speed u, separated into components u_y and u_x. Meanwhile, ball A moves with speed u_0. However, in the S' frame of reference, the observer moves at uniform velocity V, such that ball B appears to stop moving in the horizontal direction, while ball A appears to maintain a constant horizontal velocity $-u_x$. We note, by symmetry, ball B must move with speed u_0 in the S' frame of reference.

To calculate relativistic mass, we first find u_x and u_y. First, we note that $u_x = V$, since ball B appears to stop moving in the horizontal direction in S'. Next, we note that u_y is not affected by length dilation, since $\vec{y}$ is perpendicular to $\vec{V}$. In other words, $\Delta y = \Delta y'$. We relate u_y and u_0 as follows:

$$\begin{aligned} u_y &= \frac{\Delta y}{\Delta t} \\ &= \frac{u_0 \Delta t'}{\Delta t} \\ &= \frac{u_0}{\gamma} \end{aligned}$$

Thus,

$$\frac{u_0}{u_y} = \gamma \tag{12}$$

**We can see why* $\frac{\Delta t'}{\Delta t} = \gamma$ *by using the Lorentz transform. We note that* $\Delta x' = 0$, *but* $\Delta x \neq 0$, *so we use the equation* $\Delta t = \gamma(\Delta t' + \frac{u \Delta x'}{c^2}) = \gamma \Delta t'$

Denoting the rest mass of each ball as m_0 and the relativistic mass of ball B in frame S as m_v:

$$
\begin{aligned}
m_0u_0 - u_ym_v &= -m_0u_0 + u_ym_v \\
2m_0u_0 &= 2u_ym_v \\
m_v &= m_0\frac{u_0}{u_y}
\end{aligned}
$$

**We substitute equation* (12).

$$m_v = \gamma m_0$$

3.4 Addition of Velocities

To see what happens when we need to 'add' velocities, we look again to the Lorentz transform. Supposed we have an object moving with velocity v' in the S' frame of reference (which is itself moving at u relative to the S frame of reference). We want to find the object's velocity in the S frame of reference, so we simply take $v = \frac{x}{t}$:

$$
\begin{aligned}
v &= \frac{\gamma(x' + ut')}{\gamma(t' + \frac{ux'}{c^2})} \\
&= \frac{v't' + ut'}{t' + \frac{uv't'}{c^2}} \\
&= \frac{v' + u}{1 + \frac{uv'}{c^2}}
\end{aligned}
$$

And thus we observe that velocities to not add in the conventional sense of addition in special relativity.

3.5 $E = mc^2$

It is possible to derive the famous equation $E = mc^2$ simply by using the equation for relativistic mass. We denote an object's rest mass as m_0 and its mass in motion (as a function of its velocity v) as simply m. We begin by using the equations $F = \frac{d(mv)}{dt}$ and $E_k = \int Fds$:

$$
\begin{aligned}
F &= \frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= m_0\left[\frac{dv}{dt}\sqrt{1-\frac{v^2}{c^2}} - \frac{v}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\frac{(-2v)}{c^2}\frac{dv}{dt}\right] \\
&= m_0\frac{dv}{dt}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left[\left(1-\frac{v^2}{c^2}\right)+\frac{v^2}{c^2}\right] \\
&= m_0\frac{dv}{dt}\gamma^3
\end{aligned} \tag{13}
$$

We now sub in equation (13) to $E_k = \int Fds$:

$$
\begin{aligned}
E_k &= \int Fds \\
&= \int m_0\frac{dv}{dt}\gamma^3 ds \\
&= \int m_0\gamma^3 v dv \\
&= m_0c^2\gamma + C \\
&= mc^2 + C
\end{aligned}
$$

We now see that $\Delta E_k = (m_1 - m_0)c^2$, where the change in energy is the change in mass times the speed of light squared. If we imagine what would happen if this object was broken down entirely into energy, we see that $\Delta m = m_0$, so its 'rest energy' is $E = m_0 c^2$.

4 Minkowski Diagrams

We wish to 'visualize' the effects of special relativity by putting it onto a diagram. We imagine there is only one 'space' dimension x and of course, time dimension t.

First we must find the two axes x' and t'. To do this, we make two observations:

1. The origins of both coordinate axes coincide. That is, $(x,t) = (0,0)$ and $(x',t') = (0,0)$ are at the same point.
2. The 'equation' for the x' axis is given by solving for $t' = 0$, and the equation for the t' is given by solving for $x' = 0$.

We begin by solving for the x' axis by plugging in $t' = 0$:

$$\begin{aligned} ct' &= \gamma(ct - \frac{ux}{c}) \\ 0 &= \gamma(ct - \frac{ux}{c}) \\ ct &= x\frac{u}{c} \end{aligned}$$

And then we solve for the t' axis by plugging in $x' = 0$:

$$\begin{aligned} x' &= \gamma(x - \frac{uc}{c}(ct)) \\ 0 &= \gamma(x - \frac{u}{c}(ct)) \\ \frac{u}{c}(ct) &= x \\ ct &= x\frac{c}{u} \end{aligned}$$

Now that we have the shape of the two axes, we must find the scale of the two.

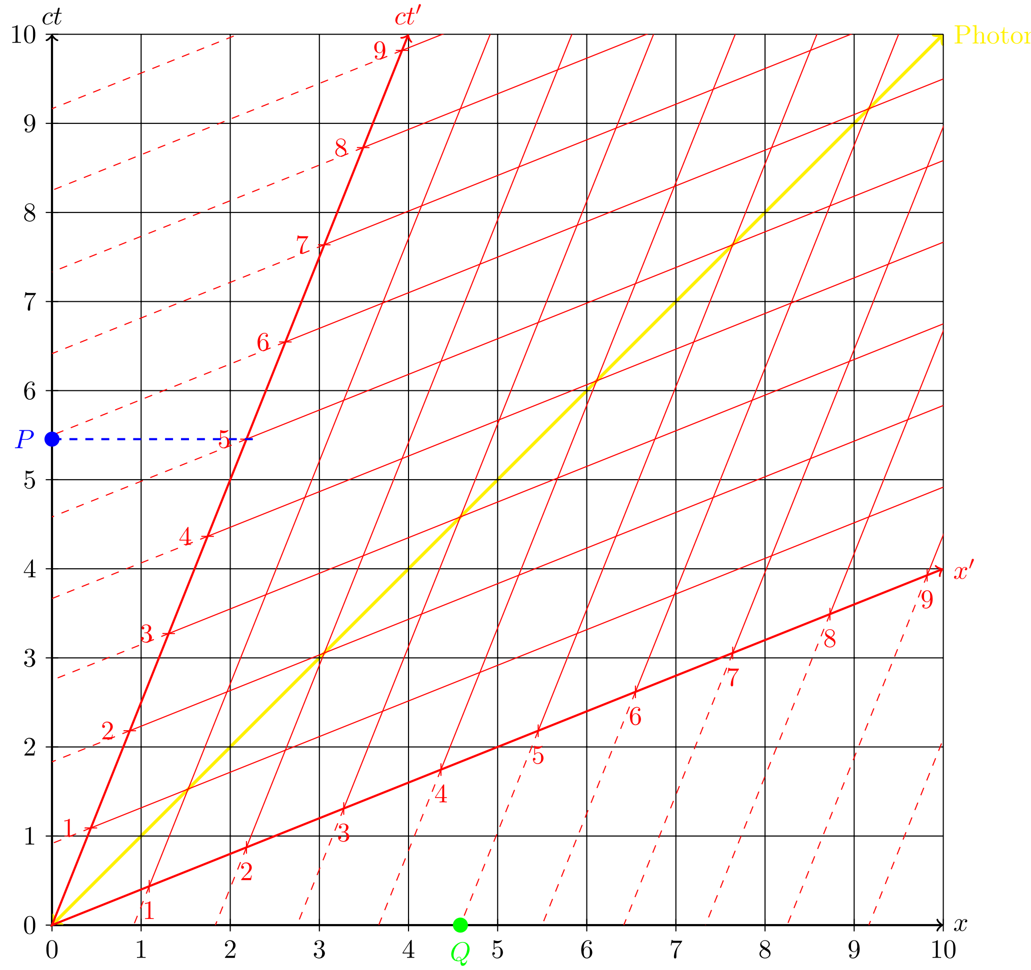
We find the 'unit' length of x' in terms of the unit length on x by plugging in $x = 1$ and $t' = 0$.

$$\begin{aligned} x &= \gamma(x' - ut') \\ 1 &= \gamma(x') \\ x' &= \frac{1}{\gamma} \end{aligned}$$

We can also do something similar for t', by plugging in $x' = 0$ and $t = 1$

$$\begin{aligned} t &= \gamma(t' - \frac{ux'}{c^2}) \\ 1 &= \gamma(t') \\ t' &= \frac{1}{\gamma} \end{aligned}$$

So we see that when $x = 1$, $x' = \frac{1}{\gamma}$. Conversely, we have that when $x' = 1$, $x = \gamma$ - so the unitary length of x' γ times larger than that of x! We also see the unitary length of the t' axis is γ times larger than that of t. Now we can begin to plot this on a Minkowski diagram. For simplicity's sake, we plot it with an S' frame of reference moving at $u = 0.4c$.



4.1 Visualizing Relativistic Effects on Time and Length

Using the Minkowski diagram, we can see where length contraction comes from. If we imagine a rod in the S' frame of reference of length 5, we see that its two ends will be located at $(x', t') = (0, a_1)$ and $(x', t') = (5, a_2)$, where $a_1 = a_2$ in the S' frame of reference. However, we see that in the S frame of reference, the rods two ends must be measured **at the same time -** and since $a_1 \neq a_2$ in S, we must make an adjustment. Instead, we measure the position of the rod at $t = 0$ (this will give us $x = 0$). We also measure the position of the other end of the rot at $t = 0$, which will give us $x = Q$. We see that $Q < 5$, and thus a rod of length 5 in a moving frame of reference appears to be shorter from the stationary frame of reference! We can verify this is true with a Lorentz transform.

We say that we measure both ends of the rod at $t = 0$. We know that our measurement for one end of the rod is $x = 0$, so we calculate for the other end:

$$x' = \gamma(x - ut) \tag{14}$$

$$x' = \gamma x \tag{15}$$

$$x = \frac{x'}{\gamma} \tag{16}$$

Thus, the length of the rod in S is $\frac{L}{\gamma}$.

We can also see where time dilation comes from. We place a clock at $x' = 0$ and let say it ticks once at $t' = t = 0$ and once at $t' = 5$. Since we place no such restrictions on measuring the clock's ticks at the same location, we can draw a **straight** line from $t' = 5$ to the t axis, instead of drawing a slanted line like before from $x' = 5$ to Q. Thus, we can now see that the observer in S sees the clock tick at $t = 0$ and $t = P$, where clearly $P > 5$. Thus the moving observer's clock seems to run slow. Mathematically:

$$t = \gamma(t' - \frac{ux'}{c^2}) \tag{17}$$

$$t = \gamma t' \tag{18}$$

We note that $x' = 0$, and thus $t = \gamma t'$. t is greater than t'!

5 My Derivation of the Time Transformation

The time transformation is $t' = \gamma(t - \frac{vx}{c^2})$. If it is not clear why time is related to position, we provide the following (informal) derivation that takes as assumptions time dilation (moving clocks run slow) and length contraction.

Observers agree on a scheme to keep track of time by placing a series of clocks at coordinates and synchronizing them appropriately. The synchronization process is done as follows. If a clock is placed at x, it is frozen to read $\frac{x}{c}$. At $t = 0$, a light pulse is emitted from the origin, and once the clock sees the light pulse, it begins ticking. This is the case for moving frames of reference as well.

Now, let us look at the moving clock placed at x'. It is initialized to read $t'_0 = \frac{x'}{c}$, but from the rest frame, the light pulse sent out at $t = 0$ will hit the moving clock at $ct_0 = vt_0 + \gamma x' \implies t_0 = \frac{\gamma}{1-\beta}\frac{x'}{c}$. Then, the moving clock will read its frozen time plus the time elapsed since the pulse hit it (dilated by an appropriate factor): $t' = \frac{x'}{c} + \frac{t-t_0}{\gamma}$. This reduces to the familiar formula:

$$\begin{aligned} t' &= \frac{x'}{c} + \frac{t - t_0}{\gamma} \\ &= \left(1 + \frac{1}{1-\beta}\right)\frac{x'}{c} + \frac{t}{\gamma} \\ &= \frac{\beta x'}{c} + \frac{t}{\gamma} \end{aligned}$$

We now make use of the intuitive $x' = \gamma(x - vt)$.

$$\begin{aligned} &= \frac{\beta\gamma(x - vt)}{c} + \frac{t}{\gamma} \\ &= t\left(\frac{1}{\gamma} + \frac{\beta v \gamma}{c}\right) - \frac{v\gamma x}{c^2} \\ &= t\left(\frac{1 + \beta^2\gamma^2}{\gamma}\right) - \frac{v\gamma x}{c^2} \\ &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$

The reason t' has x dependence is that the clocks in the moving frame are running away from the synchronizing source.