Electromagnetism and Magnetism

Andi Gu

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Introduction

This note is largely focused on Maxwell's Equations:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \vec{\mathbf{E}} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \cdot \vec{\mathbf{B}} &= 0 \\
\boldsymbol{\nabla} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\
\boldsymbol{\nabla} \times \vec{\mathbf{B}} &= \mu_0 \vec{\mathbf{J}} + \frac{1}{c^2} \frac{\partial \vec{\mathbf{E}}}{\partial t}
\end{aligned}
$$

It is hard to say that this is a note 'deriving' these laws, since they are based entirely on experimental observation. However, it is possible to motivate them semi-rigorously starting with a few basic axioms. These axioms are the axioms of special relativity, coupled with charge invariance and some physical intuition. There is also some discussion of the consequences of these laws and some brief examples.

1 Divergence of Electric Field

An important note in proving each of these laws is that $\nabla\cdot$ is a linear operator, so we can use superposition in proving each of these laws. If we have a point charge at rest, we draw a hollow sphere around the point charge, and the flux through this sphere is $\frac{q}{4\pi\epsilon_0 r^2}\cdot 4\pi r^2 = \frac{q}{\epsilon_0}$. What is the flux through another arbitrary surface? Say this surface is parameterized in spherical coordinates by $r(\theta,\phi)$. We find the surface area element $\mathrm{d}\vec{\mathbf{S}}$:

$$\begin{aligned}
\hat{\mathbf{r}}(\theta,\phi) &= \begin{bmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{bmatrix}\\
\frac{\partial\hat{\mathbf{r}}}{\partial\theta} &= \begin{bmatrix}-\sin\theta\sin\phi\\ \cos\theta\sin\phi\\ 0\end{bmatrix}\\
\frac{\partial\hat{\mathbf{r}}}{\partial\phi} &= \begin{bmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ -\sin\phi\end{bmatrix}\\
\vec{\mathbf{r}}(\theta,\phi) &= r(\theta,\phi)\hat{\mathbf{r}}(\theta,\phi)\\
\frac{\partial\vec{\mathbf{r}}}{\partial\theta} &= \frac{\partial r}{\partial\theta}\hat{\mathbf{r}} + \frac{\partial\hat{\mathbf{r}}}{\partial\theta}r\\
\frac{\partial\vec{\mathbf{r}}}{\partial\phi} &= \frac{\partial r}{\partial\phi}\hat{\mathbf{r}} + \frac{\partial\hat{\mathbf{r}}}{\partial\phi}r
\end{aligned}$$

We must be careful to note that by defining ϕ as the azimuthal angle (between the $\hat{\mathbf{z}}$ axis and the radius vector, the cross product below is oriented inwards.

$$\begin{aligned}
-\mathrm{d}\vec{\mathbf{S}} &= \frac{\partial\vec{\mathbf{r}}}{\partial\theta}\,\mathrm{d}\theta \times \frac{\partial\vec{\mathbf{r}}}{\partial\phi}\,\mathrm{d}\phi\\
&= \left(\frac{\partial r}{\partial\theta}\hat{\mathbf{r}} + \frac{\partial\hat{\mathbf{r}}}{\partial\theta}r\right)\times\left(\frac{\partial r}{\partial\phi}\hat{\mathbf{r}} + \frac{\partial\hat{\mathbf{r}}}{\partial\phi}r\right)\mathrm{d}\theta\,\mathrm{d}\phi\\
&= r\frac{\partial r}{\partial\theta}\left(\hat{\mathbf{r}}\times\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\right) + r\frac{\partial r}{\partial\phi}\left(\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\times\hat{\mathbf{r}}\right) + r^2\left(\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\times\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\right)\\
&= \left[r\frac{\partial r}{\partial\theta}\begin{bmatrix}-\sin\theta\sin^2\phi-\sin\theta\cos^2\phi\\ \cos\theta\sin^2\phi+\cos\theta\cos^2\phi\\ 0\end{bmatrix} + r\frac{\partial r}{\partial\phi}\begin{bmatrix}\cos\theta\sin\phi\cos\phi\\ \sin\theta\sin\phi\cos\phi\\ -\sin^2\theta\sin^2\phi-\cos^2\theta\sin^2\phi\end{bmatrix} + r^2\begin{bmatrix}-\cos\theta\sin^2\phi\\ -\sin\theta\sin^2\phi\\ -\sin\phi\cos\phi\end{bmatrix}\right]\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \left[r\frac{\partial r}{\partial\theta}\begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix} + r\sin\phi\frac{\partial r}{\partial\phi}\begin{bmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ -\sin\phi\end{bmatrix} - r^2\sin\phi\begin{bmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{bmatrix}\right]\mathrm{d}\theta\,\mathrm{d}\phi
\end{aligned}$$

We take the dot product with $\hat{\mathbf{r}}$, which proves to be useful later.

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\mathrm{d}\vec{\mathbf{S}} &= -\big(0+0-r^2\sin\phi\big(\cos^2\theta\sin^2\phi+\sin^2\theta\sin^2\phi+\cos^2\phi\big)\big)\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= r^2\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi
\end{aligned}$$

This is remarkably similar to the volume element $\mathrm{d}V$ in spherical coordinates. Now, we integrate over the entire surface:

$$\begin{aligned}\Phi &= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\cdot \mathrm{d}\vec{\mathbf{S}} \\ &= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{q\sin\phi}{4\pi\epsilon_0}\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \frac{q}{4\pi\epsilon_0}\int_0^{2\pi} 2 \\ &= \frac{q}{\epsilon_0} \quad \text{where } q \text{ is charge enclosed in } V\end{aligned}$$

Since this is true for a superposition of charges, we can do it with a charge distribution is ρ:

$$\oiint_S \vec{\mathbf{E}}\cdot \mathrm{d}\vec{\mathbf{S}} = \iiint_V \frac{\rho}{\epsilon_0}\,\mathrm{d}V$$

Now we can use divergence theorem.

$$\begin{aligned}\iiint_V \boldsymbol{\nabla}\cdot\vec{\mathbf{E}} &= \iiint_V \frac{\rho}{\epsilon_0}\,\mathrm{d}V \\ \boldsymbol{\nabla}\cdot\vec{\mathbf{E}} &= \frac{\rho}{\epsilon_0}\end{aligned}$$

2 Applying Special Relativity

Magnetic fields may seem mysterious at first, but their derivation is actually reasonable if one takes into account the laws of special relativity.

Electric Fields and Moving Charge

We have found how forces transform in a frame moving with speed u and where the object in the rest frame has a velocity $v = 0$.

$$\begin{aligned}F'_x &= F_x \\ F'_y &= \frac{F_y}{\gamma} \\ F'_z &= \frac{F_z}{\gamma}\end{aligned}$$

We must be careful that this set of transformations for force only applies because the object is at rest in the unprimed frame. The full transformation is:

$$\begin{aligned}F'_\parallel &= F_\parallel - \frac{\frac{u}{c^2}\vec{\mathbf{F}}_\perp\cdot\vec{\mathbf{v}}_\perp}{1-\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{c^2}} \\ \vec{\mathbf{F}}_\perp &= \frac{\vec{\mathbf{F}}_\perp}{\gamma\left(1-\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{c^2}\right)}\end{aligned}$$

We have a charge q moving at a speed v in the $\hat{x}$ direction. In the frame of the charge, the distance between the charge and a point is $\begin{bmatrix} r'_x & r'_y & r'_z \end{bmatrix}^T$ and the distance in a frame at rest is $\begin{bmatrix} r_x & r_y & r_z \end{bmatrix} = \begin{bmatrix} \gamma r_x & r_y & r_z \end{bmatrix}$.

$$\begin{aligned}
E_x(\vec{\mathbf{r}}) = E'_x &= \frac{qr'_x}{4\pi\|\vec{\mathbf{r}}\|^3} = \frac{q\gamma r_x}{4\pi\epsilon_0(\gamma^2 r_x^2 + r_y^2 + r_z^2)^{\frac{3}{2}}} \\
E_y(\vec{\mathbf{r}}) = \gamma E'_y &= \frac{q\gamma r_y}{4\pi\epsilon_0(\gamma^2 r_x^2 + r_y^2 + r_z^2)^{\frac{3}{2}}} \\
E_z(\vec{\mathbf{r}}) = \gamma E'_z &= \frac{q\gamma r_y}{4\pi\epsilon_0(\gamma^2 r_x^2 + r_y^2 + r_z^2)^{\frac{3}{2}}}
\end{aligned}$$

The electric field points away from the charge as before. We try to find the magnitude of the electric field as a function of (r_x, r_y, r_z):

$$\begin{aligned}
\left\|\vec{\mathbf{E}}\right\| &= \frac{q\gamma}{4\pi\epsilon_0}\sqrt{\frac{r_x^2 + r_y^2 + r_z^2}{(\gamma^2 r_x^2 + r_y^2 + r_z^2)^3}} \\
&= \frac{\gamma q\|\vec{\mathbf{r}}\|}{4\pi\epsilon_0\left(\gamma^2 r_x^2 + r_y^2 + r_z^2\right)^{\frac{3}{2}}} \\
&= \frac{\gamma q\|\vec{\mathbf{r}}\|}{4\pi\epsilon_0\left(\|\vec{\mathbf{r}}\|^2\left(1 - \frac{(1-\gamma^2)r_x^2}{\|\vec{\mathbf{r}}\|^2}\right)\right)^{\frac{3}{2}}}
\end{aligned}$$

Let θ be the angle between the radius vector and the $\hat{\mathbf{x}}$ axis.

$$\begin{aligned}
&= \frac{\gamma q}{4\pi\epsilon_0\|\vec{\mathbf{r}}\|^2\left(1 + \frac{\beta^2}{1-\beta^2}\cos^2\theta\right)^{\frac{3}{2}}} \\
&= \frac{q}{4\pi\epsilon_0\|\vec{\mathbf{r}}\|^2}\frac{(1-\beta^2)^{\frac{3}{2}}}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}\sqrt{1-\beta^2}} \\
&= \frac{q}{4\pi\epsilon_0\|\vec{\mathbf{r}}\|^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}
\end{aligned}$$

We have:

$$\begin{aligned}
F'_x &= F_x = qE_x = qE'_x \\
F'_y &= \frac{F_y}{\gamma} = \frac{qE_y}{\gamma} = qE'_y \\
F'_z &= qE'_z
\end{aligned}$$

Remarkably, $\vec{\mathbf{F}}' = q\vec{\mathbf{E}}'$.

3 Magnetic Field

For now, we forgo a rigorous derivation of the magnetic field. However, it is helpful to interpret as simply another aspect of the electromagnetic field, that manifests itself differently in different reference frames (like the momentum-energy four vector). At rest, there is no magnetic field – but when one begins moving, due to special relativity, a magnetic field is produced. The force on a particle in an electronic field $\vec{\mathbf{E}}$ and magnetic field $\vec{\mathbf{B}}$ is $\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$. Although I do not give a full derivation of magnetic field, it is possible to find the magnetic field due to a wire of infinite length.

Magnetic Field of a Wire

Say we have a wire with current I in the $+\hat{\mathbf{x}}$ direction, and a test charge q at $\vec{\mathbf{r}}$ relative to the wire, moving with velocity v in the $+\vec{\mathbf{x}}$ direction. In the rest frame, there is a force $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. The goal is to find what $\vec{\mathbf{B}}$ is.

In the rest frame, the charge density on the electrons is $-\lambda_0$ and the charge density on the protons is λ_0, with electrons moving at $-v_0$ along the wire. In the rest frame, *of the electrons* they have charge density $\frac{\lambda_0}{\gamma_{v_0}}$. When we start moving at v in the $+\hat{\mathbf{x}}$ direction, the electrons have charge density:

$$\begin{aligned}
\lambda' &= \frac{\gamma_{v'}\lambda_0}{\gamma_{v_0}} \quad \text{with} \quad v' = \frac{v + v_0}{1 + \frac{vv_0}{c^2}} \\
\lambda' &= \frac{\sqrt{1 - \frac{v_0^2}{c^2}}}{\sqrt{1 - \frac{c^4(v+v_0)^2}{c^2(c^2+vv_0)^2}}}\lambda_0 \\
&= \sqrt{\frac{c^2 - v_0^2}{c^2 - \frac{c^4(v+v_0)^2}{(c^2+vv_0)^2}}}\lambda_0 \\
&= \sqrt{\frac{(c^2 - v_0^2)(c^2 + vv_0)^2}{c^6 + 2c^4vv_0 + c^2v^2v_0^2 - c^4v^2 - 2c^4vv_0 - c^4v_0^2}}\lambda_0 \\
&= \sqrt{\frac{(c^2 - v_0^2)(c^2 + vv_0)^2}{c^6 - c^4(v^2 + v_0^2) + c^2v^2v_0^2}}\lambda_0 \\
&= \sqrt{\frac{(c^2 - v_0^2)(c^2 + vv_0)^2}{c^2(c^2 - v_0^2)(c^2 - v^2)}}\lambda_0 \\
&= \left(1 + \frac{vv_0}{c^2}\right)\gamma_v\lambda_0
\end{aligned}$$

The charge density on the protons is simply $\gamma\lambda_0$. Overall charge density is thus:

$$\lambda'_{eff} = -\frac{vv_0\gamma_v}{c^2}\lambda_0$$

With Gauss' law, the force on the test charge is thus:

$$F' = \frac{\lambda'_{eff}}{2\pi r\epsilon_0} \quad \text{pointing towards the wire}$$

In unprimed coordinates, $F_\perp = \frac{F'_\perp}{\gamma}$, and the parallel component stays the same.

$$\begin{aligned}\vec{\mathbf{F}} &= -\frac{vv_0\lambda_0}{2\pi r\epsilon_0 c^2}\hat{\mathbf{r}} \\ &= -\frac{Iv}{2\pi r\epsilon_0 c^2}\hat{\mathbf{r}}\end{aligned}$$

We define a quantity $\mu_0 = \frac{1}{\epsilon_0 c^2}$

$$= -\frac{Iv\mu_0}{2\pi r}\hat{\mathbf{r}}$$

Therefore, the magnetic field is counter-clockwise and has magnitude $\frac{I\mu_0}{2\pi r}$.

Properties of Magnetic Field

Observe that the line integral of electric field in a circle of radius r around a wire is always $I\mu_0$. Combined with the fact that the line integral around any path that does not contain the wire is zero, we hypothesize that the line integral of magnetic field around *any* current is the current contained multiplied by μ_0. In short:

$$\oint_C \vec{\mathbf{B}} \cdot \mathrm{d}\mathbf{s} = \mu_0 \oiint_S \vec{\mathbf{J}} \cdot \mathrm{d}\mathbf{A}$$

Equivalently, by Stokes' theorem,

$$\boldsymbol{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$

For the case of the wire, $\boldsymbol{\nabla} \cdot \vec{\mathbf{B}} = 0$ – we again hypothesize that this holds for any current distribution.

Motivated by similarities to the electric field, we define a quantity $\vec{\mathbf{A}}$ that is the 'vector potential' of $\vec{\mathbf{B}}$ such that $\vec{\mathbf{B}} = \boldsymbol{\nabla} \times \vec{\mathbf{A}}$ (similar to electric field, where $\vec{\mathbf{E}} = -\boldsymbol{\nabla}\phi$). We try to find what $\vec{\mathbf{A}}$ is.

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\mathbf{A}} = \mu_0 \vec{\mathbf{J}}$$

With the triple vector product identity,

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}\right) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\vec{\mathbf{A}} = \mu_0 \vec{\mathbf{J}}$$

Motivated by similarities in form to the electric potential, we require that $\boldsymbol{\nabla} \cdot A = 0$, reducing to:

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\vec{\mathbf{A}} + -\mu_0 \vec{\mathbf{J}}$$

In long form:

$$\begin{aligned}\nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z\end{aligned}$$

This is of course the same problem posed by the electric potential ($\nabla^2\phi = -\frac{\rho}{\epsilon_0}$), to which we have a solution.

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{\mathbf{J}}}{r_{12}}\,\mathrm{d}V \quad \text{with } r_{12} \text{ the relative position of the volume element}$$

We need to make sure that $\boldsymbol{\nabla}\cdot\vec{\mathbf{A}} = 0$, as we specified.

$$\boldsymbol{\nabla}\cdot\vec{\mathbf{A}}(x_1,y_1,z_1) = \frac{\mu_0}{4\pi}\iiint \boldsymbol{\nabla}\cdot\left(\frac{\vec{\mathbf{J}}}{r_{12}}\right)\mathrm{d}V$$

We note that the divergence is with respect to the frame of reference '1'. Since it is not with respect to the coordaintes of the integral, we can bring the divergence inside the integral.

$$= \frac{\mu_0}{4\pi}\iiint \vec{\mathbf{J}}\cdot\boldsymbol{\nabla}_1\frac{1}{r_{12}}\,\mathrm{d}V$$

We note $\boldsymbol{\nabla}_1\cdot\frac{1}{r_{12}} = -\boldsymbol{\nabla}_2\cdot\frac{1}{r_{12}}$.

$$= -\frac{\mu_0}{4\pi}\iiint \vec{\mathbf{J}}\boldsymbol{\nabla}_2\frac{1}{r_{12}}\,\mathrm{d}V$$

We note that $\boldsymbol{\nabla}\cdot\frac{\vec{\mathbf{J}}}{r_{12}} = \vec{\mathbf{J}}\boldsymbol{\nabla}\frac{1}{r_{12}} + \frac{\boldsymbol{\nabla}\cdot\vec{\mathbf{J}}}{r_{12}} = \vec{\mathbf{J}}\boldsymbol{\nabla}\frac{1}{r_{12}}$, since $\boldsymbol{\nabla}\cdot\vec{\mathbf{J}} = 0$.

$$= -\frac{\mu_0}{4\pi}\iiint \boldsymbol{\nabla}_2\cdot\frac{\vec{\mathbf{J}}}{r_{12}}\,\mathrm{d}V$$

We apply divergence theorem:

$$= -\frac{\mu_0}{4\pi}\iint_\infty \frac{\vec{\mathbf{J}}}{r_{12}}\cdot\mathrm{d}\vec{\mathbf{S}}$$

At the boundary of an infinite volume, the current is zero. Thus the integral is simply zero.

Finding $\vec{\mathbf{B}}$ from $\vec{\mathbf{A}}$

$$\begin{aligned}
\vec{\mathbf{B}} &= \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\left(\iiint\frac{\vec{\mathbf{J}}}{r_{12}}\,\mathrm{d}V_2\right)\\
&= \frac{\mu_0}{4\pi}\iiint\boldsymbol{\nabla}\times\left(\frac{\vec{\mathbf{J}}}{r_{12}}\right)\mathrm{d}V_2\\
&= \frac{\mu_0}{4\pi}\iiint\frac{\boldsymbol{\nabla}\times\vec{\mathbf{J}}}{r_{12}} + \left(\boldsymbol{\nabla}\frac{1}{r_{12}}\right)\times\vec{\mathbf{J}}\,\mathrm{d}V_2\\
&= \frac{\mu_0}{4\pi}\iiint\frac{\boldsymbol{\nabla}\times\vec{\mathbf{J}}}{r_{12}} + \frac{\hat{\mathbf{r_{12}}}\times\vec{\mathbf{J}}}{r_{12}^2}\,\mathrm{d}V_2
\end{aligned}$$

We note that $\boldsymbol{\nabla} \times \vec{J} = 0$ since $\vec{J}$ is in the '2' coordinate system and $\boldsymbol{\nabla}\times$ is taken with respect to the '1' coordinate system.

$$= -\frac{\mu_0}{4\pi} \iiint \frac{\hat{\mathbf{r}}_{12} \times \vec{\mathbf{J}}}{r_{12}^2} \, \mathrm{d}V_2$$

We can reduce this integral for a wire, since $\vec{\mathbf{J}} \, \mathrm{d}V_2 = I \, \mathrm{d}\mathbf{l}$.

$$= \frac{\mu_0}{4\pi} \int \frac{I \, \mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2}$$

Fields of Rings and Solenoids

The magnetic field of a ring with radius b and current I, at a height z above the ring is:

$$\begin{aligned}
\vec{\mathbf{B}}(z) &= \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{I(b \, \mathrm{d}\boldsymbol{\theta}) \times \vec{\mathbf{r}}}{b^2 + z^2} \\
&= \frac{\mu_0}{4\pi} \frac{2\pi I b \sin\theta}{r^2}
\end{aligned}$$

With $\sin\theta = \frac{b}{\sqrt{b^2+z^2}}$

$$= \frac{\mu_0 I b^2}{2(b^2+z^2)^{\frac{3}{2}}} \hat{\mathbf{z}}$$

We try to find the magnetic field of a solenoid. We approximate the field inside the solenoid as constant, and the solenoid as infinite in length. Let the current through the solenoid be I and the rings per unit length be n.

$$\begin{aligned}
\vec{\mathbf{B}} &= \int_{-\pi/2}^{\pi/2} \frac{\mu_0 I b^2}{2(b^2+z^2)^{\frac{3}{2}}} \cdot \frac{n b \, \mathrm{d}\theta}{\frac{b^2}{b^2+z^2}} \hat{\mathbf{z}} \\
&= \int_{-\pi/2}^{\pi/2} \frac{\mu_0 I n b \hat{\mathbf{z}}}{2\sqrt{b^2+z^2}} \, \mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{\mu_0 I n \cos\theta \hat{\mathbf{z}}}{2} \, \mathrm{d}\theta \\
&= \mu_0 I n \hat{\mathbf{z}}
\end{aligned}$$

4 How Fields Transform

Say we have a charged wire with charges flowing in the $+\hat{\mathbf{x}}$ direction so that charge velocity is v_e and charge density is λ (now $I = \lambda v_e$). Electric field is $\frac{\lambda}{2\pi\epsilon_0 r}$ radially and magnetic field is $\frac{\mu_0 \lambda v_e}{2\pi r}$. We see what happens if we transform to a frame of reference moving at velocity $-v$ in the $\hat{\mathbf{x}}$ direction.

The linear charge density is found as follows. The charge density in the rest frame of the electrons is $\gamma_{v_e}\lambda$, so the charge density in the new frame is:

$$\lambda' = \frac{\gamma_{v_e}\lambda}{\gamma_{v'_e}}$$
$$v'_e = \frac{v_e + v}{1 + \frac{vv_e}{c^2}}$$

Therefore,

$$\begin{aligned}
\lambda' &= \left(1 + \frac{vv_e}{c^2}\right)\gamma_v\lambda \\
E' &= \frac{\left(1 + \frac{vv_e}{c^2}\right)\gamma_v\lambda}{2\pi\epsilon_0 r} \\
&= \gamma_v\left(E + v\frac{v_e\lambda}{c^2(2\pi\epsilon_0 r)}\right) \\
&= \gamma_v(E + vB) \\
B' &= \frac{\mu_0\lambda'}{2\pi r}\left(\frac{v_e + v}{1 + \frac{vv_e}{c^2}}\right) \\
&= \frac{\mu_0\gamma_v\lambda(v_e + v)}{2\pi r} \\
&= \gamma_v\left(B + v\frac{\mu_0\lambda}{2\pi r}\right) \\
&= \gamma_v\left(B + \frac{vE}{c^2}\right)
\end{aligned}$$

Thus in general for *positive* velocity v:

$$E' = \gamma_v(E - vB)$$
$$B' = \gamma_v\left(B - \frac{vE}{c^2}\right)$$

These are only the magnitudes of the electric and magnetic field. However, by looking at the directions of each of the fields, we see the electric and magnetic field in the direction of motion is unchanged, and the components perpendicular are altered as follows:

$$\vec{\mathbf{E}}'_\perp = \gamma_v\left(\vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_\perp\right)$$
$$\vec{\mathbf{B}}'_\perp = \gamma_v\left(\vec{\mathbf{B}}_\perp - \frac{\vec{\mathbf{v}} \times \vec{\mathbf{E}}_\perp}{c^2}\right)$$

Connection with Lorentz Force

The relativistic transformation for $\mathbf{E}_\perp$ is reminiscent of the Lorentz force formula. The connection is not coincidental. Take a particle moving in an electric field $\vec{\mathbf{E}}$ and magnetic field $\vec{\mathbf{B}}$ at velocity

$\vec{\mathbf{v}}$. Then we transform to a frame of reference such that the particle is at rest – in this frame, the electric field parallel to $\vec{\mathbf{v}}$ is the same as before, and the electric field perpendicular to it is $\gamma(\vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_\perp)$. What is this force in this frame? Since it's not moving, the magnetic field has no effect. Then $\vec{\mathbf{F}}'_\parallel = q\vec{\mathbf{E}}_\parallel$ and $\vec{\mathbf{F}}'_\perp = q\gamma(\vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_\perp)$. Transforming back using $\vec{\mathbf{F}}_\parallel = \vec{\mathbf{F}}'_\parallel$ and $\gamma\vec{\mathbf{F}}_\perp = \vec{\mathbf{F}}'_\perp$, we see:

$$\begin{aligned}
\vec{\mathbf{F}}_\parallel &= q\vec{\mathbf{E}}_\parallel \\
\vec{\mathbf{F}}_\perp &= q(\vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_\perp) \\
\vec{\mathbf{F}} &= \vec{\mathbf{F}}_\parallel + \vec{\mathbf{F}}_\perp \\
&= q\Big(\vec{\mathbf{E}}_\parallel + \vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_\perp\Big)
\end{aligned}$$

We note $\vec{\mathbf{v}} \times \vec{\mathbf{B}}_\parallel = \vec{\mathbf{0}}$.

$$= q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$$

5 Induction

One can imagine a rectangular loop of wire moving through a space-varying (but time constant) magnetic field, and easily show that the voltage is equal to the rate at which magnetic flux through the loop changes. We will not go through this example here, it is simple enough. It is also true that for a loop of *any* shape, the relation $\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$ (where Φ is the magnetic flux through the loop) holds. With Stoke's theorem, we can put this equation into another form:

$$\begin{aligned}
\int_C \vec{\mathbf{E}} \cdot \mathrm{d}r &= -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \cdot \mathrm{d}S \\
\boldsymbol{\nabla} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t}
\end{aligned}$$

However, due to relativity, it is not possible to tell whether the changes in Φ are due to the loop's movement, or due to the magnetic field itself changing – for this reason, we propose that the rule $\boldsymbol{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$ holds no matter what the reason is for the changing magnetic field (i.e. in full generality).

6 Adjusting the Curl of the Magnetic Field

We run into a problem with our equation for curl when we have discontinuities in our circuit.

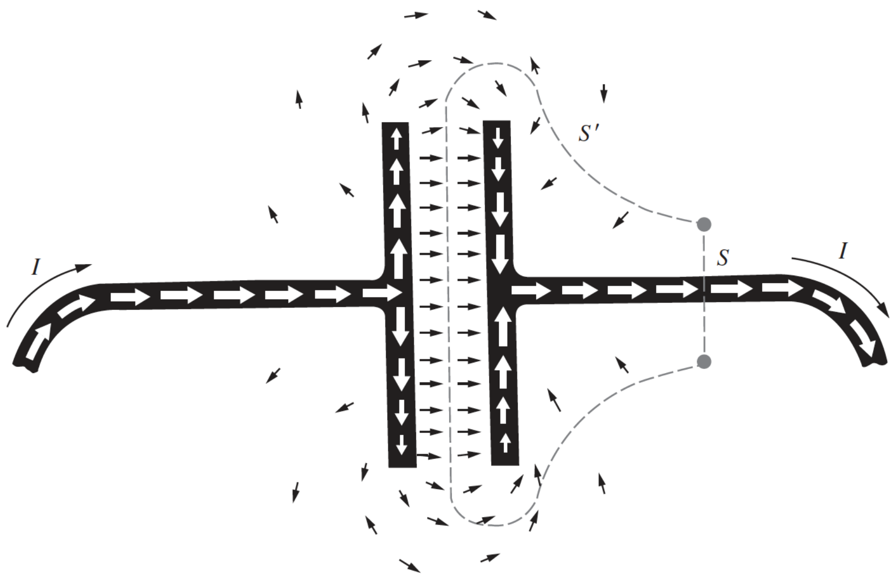

Figure 1: From Purcell E&M, p.434

We know the line integral around C (which is the boundary of the disk S) is $\mu_0 I$. We note that C is also the boundary of S' – but S' has no current flow through it, which would imply the line integral of magnetic field around C is zero. Clearly, we need an adjustment – we need to add another factor. We propose that this factor come from the changing electric field due to the capacitor, so that we have something similar in form to $\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$, but for the curl of magnetic field. So, if this additional factor is $\alpha \frac{\partial \vec{\mathbf{E}}}{\partial t}$, we see what α needs to be to make the line integral around C agree. Let the area of the capacitor plates by A.

$$\alpha A \frac{\partial E}{\partial t} = \mu_0 I$$
$$\frac{\alpha}{\epsilon_0} \frac{\partial A\sigma}{\partial t} = \mu_0 I$$

But we see $\frac{\partial A\sigma}{\partial t} = I$.

$$\alpha = \mu_0 \epsilon_0$$

Therefore:

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \frac{1}{c^2} \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

It is common to introduce another way of framing the time derivative term as a 'continuation current' of the actual conduction current (although there is no real current, it is helpful to think it as such).

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\mathbf{J}} + \vec{\mathbf{J}}_d \right)$$
$$\vec{\mathbf{J}}_d = \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

We have now arrived at all four of Maxwell's equations.

7 Light

Now, we are free to imagine an electric and magnetic field that varies in time and space as follows:

$$\vec{\mathbf{E}} = \hat{\mathbf{z}} E_0 \sin(y - vt)$$
$$\vec{\mathbf{B}} = \hat{\mathbf{x}} B_0 \sin(y - vt)$$

Knowing $\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$:

$$\hat{\mathbf{x}} E_0 \cos(y - vt) = v \hat{\mathbf{x}} B_0 \cos(y - vt)$$
$$E_0 = v B_0$$

Also, $\nabla \times \vec{\mathbf{B}} = \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ (since we are working in vaccuum, $\vec{\mathbf{J}} = 0$).

$$-\hat{\mathbf{z}} B_0 \cos(y - vt) = -\hat{\mathbf{z}} \mu_0 \epsilon_0 v E_0 \cos(y - vt)$$
$$B_0 = \mu_0 \epsilon_0 v^2 B_0$$
$$v = \pm \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

There are many things of note here. First and foremost, we have rediscovered the speed of light $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$. Also, we note the relation $E_0 = \pm c B_0$. Finally, the direction of propogation is perpendicular to $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ – more concretely, it is in the direction $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$.

We also use the relations for energy per volume $\mathrm{d}V$ in a field, $U = \frac{\epsilon_0 E^2}{2} \, \mathrm{d}V$ and $U = \frac{B^2}{2\mu_0} \, \mathrm{d}V$, so that we can find the energy contained in this electromagnetic wave.

$$U = \frac{\epsilon_0}{2} \left(E^2 + (cB)^2 \right)$$

We have found that the magnitude of B is $\frac{1}{c}$ that of E.

$$= \epsilon_0 E^2$$

For a typical sine wave, this is just

$$= \frac{\epsilon_0 E_0^2}{2}$$

We also seek to find the power density (rate of energy density change).

$$\frac{\partial U}{\partial t} = \epsilon_0 \left(\vec{\mathbf{E}} \cdot \frac{\partial \vec{\mathbf{E}}}{\partial t} + c\vec{\mathbf{B}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t} \right)$$

Applying Maxwell's equations:

$$= \frac{1}{\mu_0} \Big((\boldsymbol{\nabla} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{E}} - (\boldsymbol{\nabla} \times \vec{\mathbf{E}}) \cdot \vec{\mathbf{B}} \Big)$$

$$= -\frac{\boldsymbol{\nabla} \cdot \Big(\vec{\mathbf{E}} \times \vec{\mathbf{B}} \Big)}{\mu_0}$$

We define a vector $\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0}$, called the Poynting vector so that $\frac{\partial U}{\partial t} = -\boldsymbol{\nabla} \cdot S$. This vector also gives the direction of propogation of a light wave.