Geometric Algebra

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This is a collection of notes from my study of the *Geometric Algebra for Physicists* textbook (Doran and Lasenby 2007).

1 Foundations

The entire structure of geometric algebra can be boiled down to the geometric product, which is a binary operation between two vectors a and b. The usual rules of a vector space apply to these vectors, and the usefulness of geometric algebra is in the structure revealed by the product ab. This product satisfies three fairly natural axioms.

1. It is associative: $(ab)c = a(bc)$
2. It distributes over addition: $a(b + c) = ab + ac$
3. The square of any vector is real: $a^2 \in \mathbb{R}$. We do not make the assumption that $a^2 \geq 0$ to allow for the possibility of spaces with mixed signature (e.g., Minkowski space).

It is important not to interpret this product uv as another vector. In fact, uv is a sum of a scalar element $u \in \mathbb{R}$ and a 'bivector' V: $ab = u + V$ – this should be understood as something like a complex number $u + iv$. In fact, for a vector space of dimension 2, this analogy is exact. In general, the bivector should be pictured as a directed area, or an object like the angular momentum in classical mechanics. This analogy extends further: we can have a trivector, which is an oriented volume, and so on and so forth.

However, this geometric product is not so exotic. We can still understand it in terms of some familiar concepts. For one, we observe that $(a + b)^2$ is always real, so we must have $a^2 + ab + ba + b^2 \in \mathbb{R} \implies ab + ba \in \mathbb{R}$. We have found that the symmeterized geometric product is a binary relation between vectors that outputs a real number – that is, a dot product! So we define

$$a \cdot b = \frac{(ab + ba)}{2}.$$

We define the remaining part of the product as the exterior product (which might be thought of as replacing the role of the cross product):

$$a \wedge b = \frac{ab - ba}{2}.$$

This then gives $ab = a \cdot b + a \wedge b$: a sum of a scalar and a bivector, as promised.

Well, technically, not yet: we do not have a good reason to believe we should understand $a \wedge b$ as a bivector (or a directed area) yet. So we define it as such, and show this definition conforms to our intuition later.

More precisely, we define exterior product of r vectors as the full anti-symmeterized product over all of them:

$$a_1 \wedge a_2 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(r)},$$

where $\mathbb{S}_r$ is the set of all permutations on r elements and $(-1)^\sigma$ is the sign of a permutation. A consequence of this definition is that the product reverses sign under exchange of any two vectors. Then, we see that if any vector is repeated, the exterior product must be zero, which then implies if the vectors are linearly dependent, the exterior product is also zero (this follows simply by distributivity of the geometric product under addition). Therefore, the outer product can be understood to measure the dimensionality of a set of vectors. We say that the outer product of r vectors has grade r (if it does not vanish). A multivector that can be written purely as an outer product is called a blade.

Lemma 1. *Any blade $a_1 \wedge \ldots \wedge a_r$ can be written simply as a product of orthogonal vectors $e_1 \ldots e_r$. This justifies our idea that a blade with r vectors can be interpreted as a directed area for $r = 2$, volume for $r = 3$, and so on.*

Proof. Let $A_{i,j} = a_i \cdot a_j$. Since $A_{i,j}$ is a symmetric matrix, it can be diagonalized with $P^T D P$, where P is orthogonal. Letting $e_i = P_{i,k} a_k$, we see that $e_i \cdot e_j = P_{i,k} P_{j,\ell} a_k \cdot a_\ell = P_{i,k} A_{k,\ell} (P^T)_{\ell,j} = D_{i,j}$. Therefore, these vectors obey $e_i e_j = -e_j e_i$ for $i \neq j$. Finally, since $e_i = P_{i,k} a_k \implies a_k = P_{i,k} e_i$:

$$a_1 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma P_{i_1,\sigma(1)} P_{i_2,\sigma(2)} \ldots P_{i_r,\sigma(r)} e_{i_1} e_{i_2} \ldots e_{i_r}$$

Note that we have the restriction $i_1 \neq i_2 \neq \ldots \neq i_r$ due to the antisymmeterization of the sum. Therefore, we can rewrite as:

$$\begin{aligned}
a_1 \wedge \ldots \wedge a_r &= \frac{1}{r!} \sum_{\rho,\sigma \in \mathbb{S}_r} (-1)^\sigma P_{\rho(1),\sigma(1)} P_{\rho(2),\sigma(2)} \ldots P_{\rho(r),\sigma(r)} e_{\rho(1)} e_{\rho(2)} \ldots e_{\rho(r)} \\
&= \frac{1}{r!} \sum_{\rho,\sigma \in \mathbb{S}_r} (-1)^{\rho^{-1}} (-1)^\sigma P_{1,\sigma(\rho^{-1}(1))} P_{2,\sigma(\rho^{-1}(2))} \ldots P_{r,\sigma(\rho^{-1}(r))} e_1 e_2 \ldots e_r \\
&= \sum_{\alpha \in \mathbb{S}_r} (-1)^\alpha P_{1,\alpha(1)} P_{2,\alpha(2)} \ldots P_{r,\alpha(r)} e_1 e_2 \ldots e_r \\
&= \det(P) e_1 \ldots e_r
\end{aligned}$$

In the second line, the factor $(-1)^{\rho^{-1}}$ comes from unshuffling $e_{\rho(1)} \ldots e_{\rho(r)} \to e_1 \ldots e_r$, and in the third line, we use the fact that for any function f that depends only on $\sigma \circ \rho^{-1}$: $\sum_{\rho,\sigma \in \mathbb{S}_r} f(\sigma \circ \rho^{-1}) = r! \sum_{\alpha \in \mathbb{S}_r} f(\alpha)$, where α plays the role of $\sigma \circ \rho^{-1}$. Now, observe that $\det(P) = \pm 1$, and in the case where $\det(P) = -1$, we can simply reorder $e_1, \ldots, e_r$ to get rid of the negative sign.

□

In general, multivectors can be comprised of elements with different grades. We define $\langle \cdot \rangle_r$ to be the component of a multivector with grade r, so that in general a multivector A in a geometric algebra $\mathcal{G}_n$ (whose underlying vector space has dimension n) can be written $A = \sum_{r=0}^{n} \langle A \rangle_r$. A multivector that satisfies $\langle A \rangle_r = A$ (for some r) is called homogenous.

We denote the subspace of $\mathcal{G}_n$ of grade r as $\mathcal{G}_n^r$. The dimensionality of $\mathcal{G}_n^r$ is $\binom{n}{r}$, because the basis of $\mathcal{G}_n^r$ can be formed by choosing r items from the n basis vectors. The overall dimensionality is therefore $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

References

Doran, Chris, and A. N. Lasenby. 2007. *Geometric Algebra for Physicists*. 1st ed. Cambridge ; New York: Cambridge University Press.