# The Fastest Strategy for Emptying a Water Bottle

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## Contents

# Nomenclature

### Physical Constants

$g$ Acceleration due to gravity

$\rho$ Density of water

$\rho_{air}$ Density of air

$P_0$ Atmospheric pressure

### General Model Constants

$r_T$ Radius of the top of the bottle (when inverted)

$r_N$ Radius of the neck of the bottle

$\mu$ Ratio $\frac{r_T^2}{r_N^2}$

$\gamma$ Initial height of the water when the bottle is not spinning

$\Gamma$ Total height of the bottle

### Vortex Model Notation

$r_C(z)$ Radius of the core at a height $z$

$H_0$ Height of water after bottle begins spinning

$H$ Height of water at a certain time

$v_T$ Velocity of water on the free surface at the top of the vortex

$r_0$ Radius of vortex core at neck of bottle

$v_{air}$ Maximum velocity of air through the 'air hole' at the bottom of the vortex

### Oscillating Bubble Model Notation

$H(t_0)$ Water height at the start of a bubble cycle

$H(t_0 + \tau)$ Water height at $\tau$ seconds after the height of a bubble cycle

$T$ Period of a bubble cycle

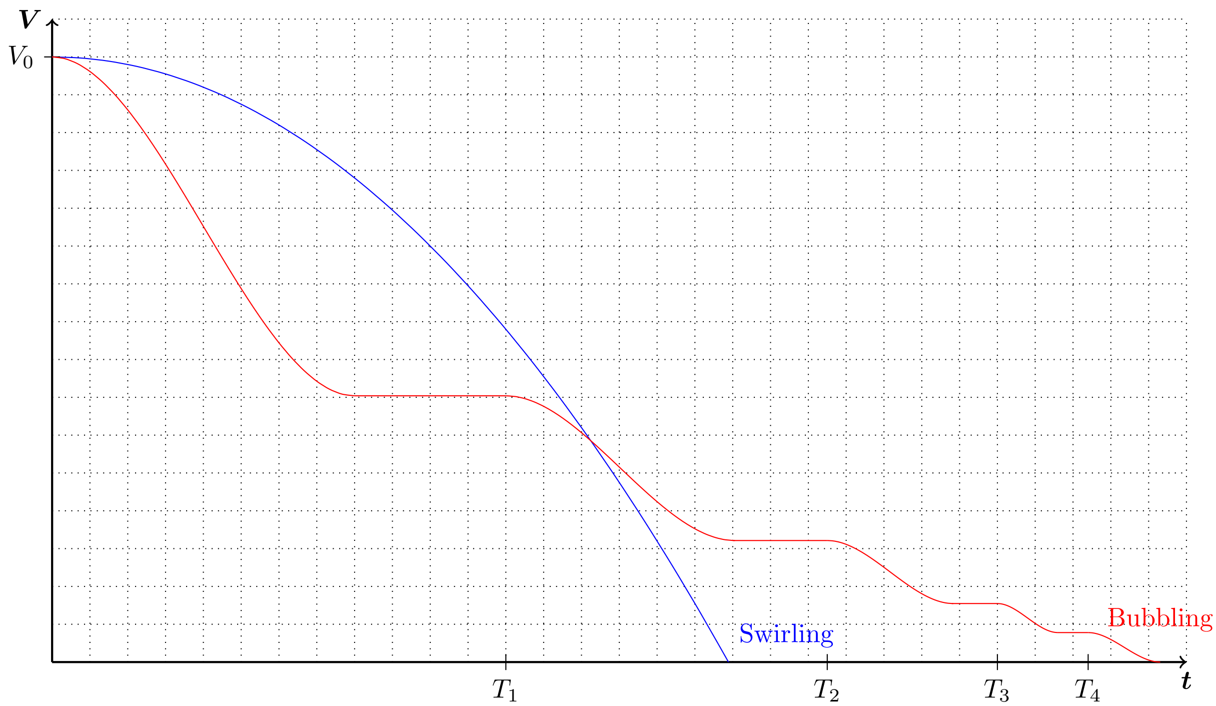
$\Delta V(\tau)$ The volume of water that left bottle $\tau$ seconds after the start of the cycle

# 1 Introduction

One day, as I was watching the water swirl down a sink, I began to wonder why it did so. It turns out that it is due to a little bit of initial angular momentum in the water, which manifests itself as a water vortex. However, this led me to a slightly more nuanced question. Was it advantageous for this water to drain in a vortex form? In other words, would the water drain faster if it were swirled or simply left alone to drain? Running a quick, informal experiment, I filled up a bottle with water, flipped it upside down, and swirled it as I began emptying it. I then dumped it upside down without swirling. The results were starkly different – the swirled bottle took around 5 seconds to empty while the one without swirling took nearly 15! Intrigued into why this was, I decided to analyze the theory behind this phenomenon.

Building a model for the water vortex, I was able to predict the theoretical shape of the water's free surface given certain parameters. This gave me the tools to find the flow rate of water at a certain time. After making a small restriction on the air inflow rate, an optimal set of parameters for the vortex was found. On the other hand, building a model for simply draining the water bottle proved to be much more difficult. As the water drains, there are periodically bubbles due to differences in air pressure in the top of the bottle and the outside air – this leads to a cyclic phenomena. Thus, the relation between volume of water and time looked something like this:

Figure 1: Volume Over Time




Despite the complex nature of the simple draining strategy, it was possible to calculate the period ($T_1$, $T_2$, etc.) of each of these 'bubble cycles', and writing a simulation enabled me to find the total time for the bottle to drain in this way as well. This simulation was run over several different parameters to investigate the effectiveness of each emptying strategy in different cases.

## 2 Vortex

In order to find the rate of flow $Q$ when there is a vortex, we must first find the radius of the vortex at its base $r_0$. We begin by making a few assumptions about the vortex:

- The top of the vortex has a radius $r_T$ equal to the radius of the water bottle: $r_T$. The velocity at the free surface at the top of the vertex is $v_T$.
- Radial water velocity is negligible.
- The free surface is isobaric (having a uniform pressure).

### 2.1 Pressure Gradient

We begin by discussing the pressure gradient with respect to the radius, $\frac{\partial P}{\partial r}$. Since we observe an infinitesimal parcel of water with cross sectional area $\mathrm{d}A$, width $\mathrm{d}r$, and at a radius $r$ away from the center of the vortex. In order to execute circular motion, the net force on this parcel must be $F_{net} = m\frac{v^2}{r}$. Since we are discussing an infinitesimal parcel, $F_{net}$ is really an infinitesimal $dF$. So:

$$\begin{aligned}
\mathrm{d}F &= \rho\,\mathrm{d}r\,\mathrm{d}A\,\frac{v^2}{r} \\
\mathrm{d}P &= \rho\,\mathrm{d}r\,\frac{v^2}{r} \quad \text{since } \mathrm{d}P = \frac{\mathrm{d}F}{\mathrm{d}A} \\
\frac{\partial P}{\partial r} &= \frac{\rho v^2}{r}
\end{aligned} \tag{1}$$

We wish now to find an expression for $v$ in terms of $r$. By using Bernoulli's principle, we can see that the pressure at two points $A$ and $B$ very close to each other, at $r$ and $r + \mathrm{d}r$ respectively are related as such that:

$$\begin{aligned}
P(r) + \rho\frac{v^2}{2} + \rho g z_A &= P(r + \mathrm{d}r) + \rho\frac{(v + \mathrm{d}v)^2}{2} + \rho g z_B \quad \text{where } P(r) \text{ is pressure at } r \\
P(r) + \rho\frac{v^2}{2} &= P(r + \mathrm{d}r) + \rho\frac{(v + \mathrm{d}v)^2}{2} \quad \text{since } z_A = z_B \\
\mathrm{d}P &= \rho\frac{v^2 - (v + \mathrm{d}v)^2}{2} \\
\mathrm{d}P &= -\rho\frac{2v\,\mathrm{d}v + (\mathrm{d}v)^2}{2} \\
\rho\frac{v^2\,\mathrm{d}r}{r} &= -\rho\left(v\,\mathrm{d}v + \frac{(\mathrm{d}v)^2}{2}\right) \quad \text{by substituting (1).}
\end{aligned}$$

Note that $(\mathrm{d}v)^2$ falls off rapidly as $\mathrm{d}v \to 0$.

$$\begin{aligned}
\frac{v^2}{r}\,\mathrm{d}r &= -v\,\mathrm{d}v \\
-\frac{\mathrm{d}v}{v} &= \frac{dr}{r}
\end{aligned} \tag{2}$$

We see that equation (2) is a linear differential equation with the solution $v = \frac{C}{r}$, where $C$ is some constant. By studying the boundary conditions at the free surface, we can find $C$ in terms of the velocity $v_C$ and radius $r_C$ at the free surface at a certain height $z$. By simple analysis, it becomes clear that $C = v_C r_C$. Thus:

$$v = \frac{v_C r_C}{r} \tag{3}$$

## 2.2 Vortex Shape

If we plug this back into (1), we may begin to find the equation for the radius of the free surface as a function of height $z$:

$$\begin{aligned}
\frac{\partial P}{\partial r} &= \rho \frac{v_C^2 r_C^2}{r^3} \\
\mathrm{d}P &= \rho \frac{v_C^2 r_C^2}{r^3}\,\mathrm{d}r \\
\int dP &= \rho \int \frac{v_C^2 r_C^2}{r^3}\,\mathrm{d}r \\
P(r,z) &= -\rho \cdot v_C^2(z) \cdot r_C^2(z) \frac{1}{r^2} + C(z)
\end{aligned}$$

The constant of integration $C$ is a function of $z$ since we were integrating over a partial derivative with respect to $\partial r$. We now wish to find $C(z)$, which is possible by analyzing the case where $r \to \infty$. This is because $\lim_{r\to\infty} v(r) = 0$, so as $r \to \infty$, we see that this is simply the case where the water is **standing still**. Now, $\lim_{r\to\infty} P(r,z) = C(z)$. So we see that the pressure as $r \to \infty$ is purely pressure due to the still water on top of a certain point. Of course, this is simply hydrostatic pressure! So, $C(z) = \rho g \cdot (H - z)$, where $H$ is the height of the entire body of water. Now, we have a full expression for pressure:

$$P(r,z) = -\rho \cdot v_C^2(z) \cdot r_C^2(z) \frac{1}{r^2} + \rho g \cdot (H - z)$$

Now, we recognize that $v_C$ is purely a function of height $z$, as well as $r_C$. Keeping this in mind, we simply write $v_C$ instead of $v_C(z)$, and likewise for $r_C$.

$$P(r,z) = -\frac{\rho \cdot v_C^2 \cdot r_C^2}{r^2} + \rho g \cdot (H - z) \tag{4}$$

Since we have assumed that along the entire free surface of the vortex, we have an identical pressure, we can break down equation (4) to find the relationship between the core radius and height $z$. We do this by moving along the free surface. Since the pressure along this surface is constant, we see that if we start on the free surface, then we move up a little bit, then move outwards radially a little bit, the pressure will be the same. We wish to quantify what this 'little bit' really is. Mathematically, $P(r_C + \mathrm{d}r_C\,, z + \mathrm{d}z) = P(r_C, z)$.

$$\begin{aligned}
-\frac{\rho \cdot v_C^2 \cdot r_C^2}{r_C^2} + \rho g \cdot (H - z) &= -\frac{\rho \cdot (v_C + \mathrm{d}v_C)^2 \cdot (r_C + \mathrm{d}r_C)^2}{(r_C + \mathrm{d}r_C)^2} + \rho g \cdot (H - z - \mathrm{d}z) \\
-v_C^2 + g \cdot (H - z) &= -(v_C + \mathrm{d}v_C)^2 + g \cdot (H - z - \mathrm{d}z) \\
2v_C\,\mathrm{d}v_C + (\mathrm{d}v_C)^2 &= -g\,\mathrm{d}z \\
2v_C\,\mathrm{d}v_C &= -g\,\mathrm{d}z \quad \text{since } (\mathrm{d}v_C)^2 \text{ falls off}
\end{aligned}$$

We note now that it is possible that $v_C$ be expressed in terms of the free surface velocity at the *top* of the vortex, called $v_T$, and the radius at the top of the vortex called $r_T$. We do this by saying $v_C = \frac{v_T r_T}{r_C}$ by using (3). Now $\mathrm{d}v_C = -\frac{v_T r_T}{r_C^3}\,\mathrm{d}r_C$.

$$\begin{aligned}
-g\,\mathrm{d}z &= 2\frac{v_T r_T}{r_C}\left(-\frac{v_T r_T}{r_C^2}\,\mathrm{d}r_C\right) \\
\mathrm{d}z &= \frac{2v_T^2 r_T^2}{g r_C^3}\,\mathrm{d}r_C
\end{aligned}$$

This is a linear differential equation with the solution $z(r_C) = -\frac{v_T^2 r_T^2}{g \cdot r_C^2} + C$. We have the boundary condition that $z(r_T) = H$, so we can solve for $C$:

$$\begin{aligned}
H &= -\frac{v_T^2 r_T^2}{g \cdot r_T^2} + C \\
C &= H + \frac{v_T^2}{g}
\end{aligned}$$

We now finally arrive at an equation for height $z$ given a core radius $R$:

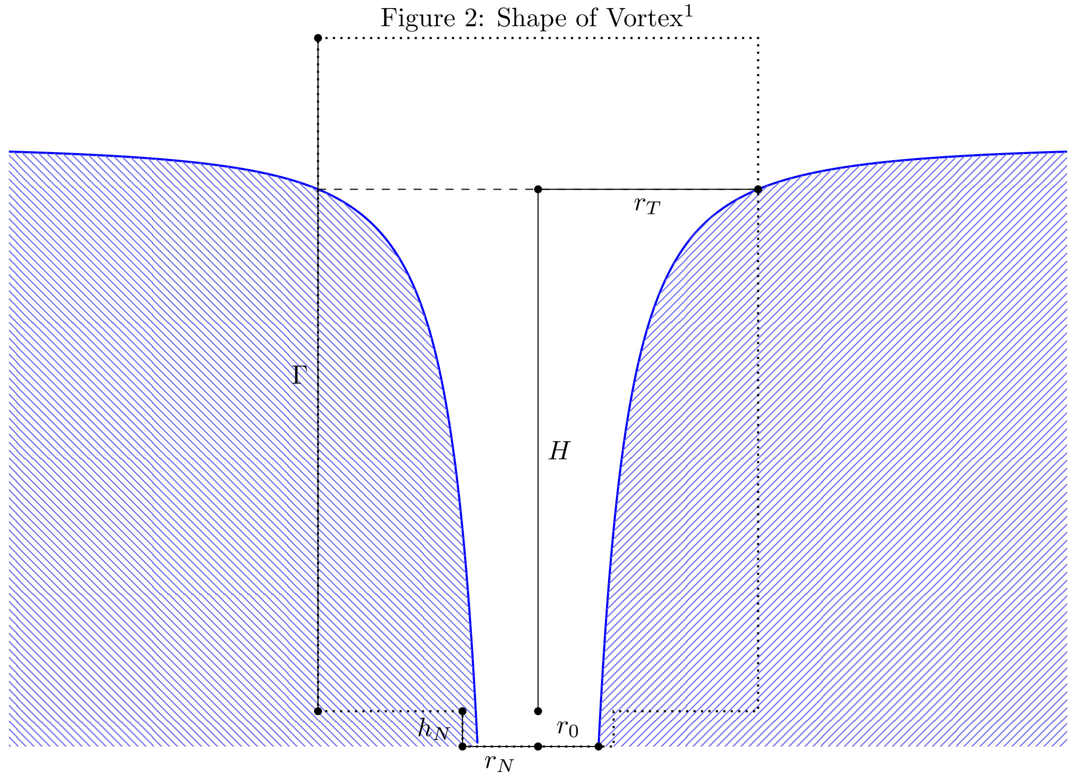
$$z(r_C) = H - \frac{v_T^2}{g}\left(\frac{r_T^2}{r_C^2} - 1\right) \tag{5}$$

We now have the necessary tools to find the core radius at the bottom of the bottle, called $r_0$, simply by setting $z = 0$:

$$\begin{aligned} 0 &= H - \frac{v_T^2}{g}\left(\frac{r_T^2}{r_0^2} - 1\right) \\ r_T^2 &= \left(\frac{gH}{v_T^2} + 1\right) r_0^2 \\ r_0^2 &= \frac{r_T^2}{\frac{gH}{v_T^2} + 1} \\ r_0 &= r_T v_T \sqrt{\frac{1}{gH + v_T^2}} \end{aligned} \tag{6}$$

If we graph equation (5) in a two dimensional plane with $z$ against $R$, the characteristics of the curve will be revealed.

Figure 2: Shape of Vortex[1]




## 2.3 Minimizing Emptying Time

We now observe that the solid of revolution of equation (5) is the vortex in three dimensions. Thus, the cross-sectional area of water able to flow out of the bottle is simply the $\pi r_0^2$ subtracted from the area of the

[1]It is not entirely accurate to say that $H$ does not include the height $h_N$ of the bottle. However, in realistic situations, $h_N \ll H$, so we may approximate $H \approx H - h_N$, and say that $z = 0$ occurs just above the neck of the water bottle.

neck of the bottle (given by $\pi r_N^2$). If the downwards velocity of the water is $v(t)$ at a given time, we now see that:

$$\begin{aligned} \mathrm{d}V &= (\pi r_N^2 - \pi r_0^2) v \,\mathrm{d}t \\ \mathrm{d}V &= -\pi g t\big(r_N^2 - r_0^2\big)\,\mathrm{d}t \quad \text{since } v = -gt \end{aligned}$$

We must now make an approximation: $r_0$ is constant for over the small range of values allowed for $H$. We see that with realistic values for $r_T$ and $v_T$, the maximum possible $r_0$ is not too different from its minimum possible value. Thus, we say that $r_0(H) \approx r_0(H_0), 0 \leq H \leq H_0$, where $H_0$ is the initial height of the water. So,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi g t\left(r_N^2 - \frac{r_T^2 v_T^2}{gH_0 + v_T^2}\right) \tag{7}$$

We see now that the flow rate depends on the initial height $H_0$ of water. We must find an expression for the volume of liquid in the bottle. We do this by first finding the inverse of $z(r_C)$, $r_C(z)$, and integrating from $z = 0$ to $z = H$. The amount of fluid in the neck of the bottle is negligible.

$$\begin{aligned} z &= H - \frac{v_T^2 r_T^2}{g r_C^2} + \frac{v_T^2}{g} \\ \frac{v_T^2}{g} - z + H &= \frac{v_T^2 r_T^2}{g r_C^2} \\ r_C(z) &= \sqrt{\frac{1}{\frac{1}{r_T^2} - \frac{g(z-H)}{v_T^2 r_T^2}}} \\ r_C(z) &= v_T r_T \sqrt{\frac{1}{v_T^2 - g(z-H)}} \end{aligned}$$

If we integrate the solid of revolution $\int_0^H \pi r_C^2 \,\mathrm{d}z$, we will find the volume of *air* in the bottle, not liquid. Since we may also equate this result to the initial volume of *air* in the bottle, we will be able to find $H_0$ just as well.

$$\begin{aligned} V_{air} &= \pi \int_0^H r_C^2 \,\mathrm{d}z \\ &= \pi v_T^2 r_T^2 \int_0^H \frac{1}{v_T^2 + gH - gz} \,\mathrm{d}z \\ &= \pi v_T^2 r_T^2 \left[ -\frac{\ln\left|v_T^2 + gH - gz\right|}{g} \right|_{z=0}^{z=H} \\ &= \pi v_T^2 r_T^2 \left( -\frac{\ln v_T^2}{g} + \frac{\ln\left(v_T^2 + gH\right)}{g} \right) \\ &= \frac{\pi v_T^2 r_T^2}{g} \ln\left(1 + \frac{gH}{v_T^2}\right) \end{aligned}$$

We now plug in 'initial air volume', which is equal to volume of water subtracted from the total volume of the bottle:

$$\pi r_T^2 \Gamma - \pi r_T^2 \gamma = \frac{\pi v_T^2 r_T^2}{g} \ln\left(1 + \frac{gH_0}{v_T^2}\right)$$

$\Gamma$ is the total height of the water bottle and $\gamma$ is the initial height of the water **before** rotating.

$$\begin{aligned} \frac{g(\Gamma - \gamma)}{v_T^2} &= \ln\left(1 + \frac{gH_0}{v_T^2}\right) \\ H_0 &= \frac{v_T^2}{g}\left(e^{\frac{g(\Gamma-\gamma)}{v_T^2}} - 1\right) \end{aligned}$$

We substitute this back into equation (7) to see that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi g t\left(r_N^2 - \frac{r_T^2 v_T^2}{g\frac{v_T^2}{g}\left(e^{\frac{g(\Gamma-\gamma)}{v_T^2}}-1\right)+v_T^2}\right)$$

$$Q = -\pi g t\left(r_N^2 - \frac{r_T^2}{e^{\frac{g(\Gamma-\gamma)}{v_T^2}}}\right)$$

$$Q = -\pi g\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right)\cdot t$$

We now see that the rate of flow is proportional to time, so that water leaves faster and faster over time. The constant of proportionality is $-\pi g\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right)$. We now want to find the time $t_0$ it takes for the bottle to empty by integrating flow over time:

$$\pi r_T^2\gamma = \int_0^{t_0}\pi g\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right)\cdot t\,\mathrm{d}t$$

$$\pi r_T^2\gamma = \frac{\pi g}{2}\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right)t_0^2$$

$$t_0 = \sqrt{\frac{2r_T^2\gamma}{g\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right)}} \tag{8}$$

We can see that if we wish to minimize $t_0$, we must set $v_T$ to 0. This makes sense, since a large $v_T$ will increase the size of the air 'hole' at the bottom of the vortex, leaving less room for water to flow out. However, setting $v_T$ to 0 will completely remove this air hole, leaving us with a scenario in which air must slowly bubble up through the bottle. But this is exactly what we were trying to avoid in the first place! So instead of saying $v_T$ is 0, we will say that we want $v_T$ to be the smallest possible value so that there is still an 'air hole' at the bottom of the vortex.

## 2.4 Minimizing $v_T$

Since the whole purpose behind swirling the water was to create an opening for air to enter, and thus equalize the air pressure in the bottle, having a velocity $v_T = 0$ is unacceptable – it leaves no opening for air to enter. So instead, we wish to find the lowest possible $v_T$ where this air hole is created and allows sufficient flow of air to equalize pressure. Since the outflow of water is the greatest towards the end of the emptying, we study this case. Also, since $H$ is so low towards the end, we see that $r_0$ approaches larger values. At some point, it can be approximated to be $r_N$ – thus the flow of air at the end of the emptying is approximated by $\pi r_N^2 \cdot v_{air}$, where $v_{air}$ is the velocity of the 'wind' blowing into the bottle. Since this inflow of air must be equal to the outflow of water at all times, we study the most extreme case. Water is flowing out the fastest at $t = t_0$, and so we say

$$\begin{aligned}
\pi g\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right) \cdot t_0 &= \pi r_N^2 \cdot v_{air} \\
\sqrt{2r_T^2\gamma g\left(r_N^2 - r_T^2 e^{\frac{g(\gamma-\Gamma)}{v_T^2}}\right)} &= r_N^2 v_{air} \quad \text{subbing in (8)} \\
e^{\frac{g(\gamma-\Gamma)}{v_T^2}} &= \frac{r_N^2 - \frac{r_N^4 v_{air}^2}{2r_T^2\gamma g}}{r_T^2} \\
\frac{g(\gamma-\Gamma)}{v_T^2} &= \ln \frac{2r_N^2 r_T^2 \gamma g - r_N^4 v_{air}^2}{2r_T^4\gamma g} \\
v_T &= \sqrt{\frac{g(\gamma-\Gamma)}{\ln \frac{2r_N^2 r_T^2 \gamma g - r_N^4 v_{air}^2}{2r_T^4\gamma g}}} \qquad (9)
\end{aligned}$$

Now, $v_{air}$ must be below a certain value. For starters, it clearly cannot exceed the speed of sound. It should also not exceed the speed of storm winds (around 30m/s), and it would even be a stretch to say it exceeds average wind speeds (approximately 15 m/s), since this would have a visible effect blowing the water away from the neck. It is not unreasonable to say that the air can flow into the bottle no faster than 5m/s. We see now that if we plug in $v_{air} = 5$, this expression restricts $v_T$ to at least a certain value. Since our goal is to minimize $v_T$, this will be the value that we use to swirl the water from now on. If we plug (9) back into (8), we will find our optimal time to empty the bottle.

$$\begin{aligned}
t_{best} &= \sqrt{\frac{2r_T^2\gamma}{g\left(r_N^2 - r_T^2 e^{\ln \frac{2r_N^2 r_T^2\gamma g - r_N^4 v_{air}^2}{2r_T^4\gamma g}}\right)}} \\
&= \sqrt{\frac{2r_T^2\gamma}{g\left(r_N^2 - \frac{2r_N^2 r_T^2\gamma g - r_N^4 v_{air}^2}{2r_T^2\gamma g}\right)}} \\
&= \sqrt{\frac{2r_T^2\gamma}{\frac{r_N^4 v_{air}^2}{2r_T^2\gamma}}} \\
&= \frac{2\gamma r_T^2}{v_{air} r_N^2}
\end{aligned}$$

This is our final expression for the best possible emptying time of a swirling bottle.

# 3 Bubbles

If instead of swirling the bottle, we simply hold it still, the water will experience a 'sputtering' effect. It will begin to flow freely - however, a pressure gradient begins to develop across the water bottle. The bottom experiences atmospheric pressure, while the top experiences a drop in air pressure due to the water that just exited – this causes the water to stop flowing. Once flow stops, a bubble forms and rises through the bottle, once again raising pressure at the top of the bottle to atmospheric pressure, and flow can begin again. In order to find the amount of time it takes for the bottle to completely empty, we must first find the magnitude of this pressure gradient when the water stops flowing.

## 3.1 Pressure Gradient

We study the workings of one 'cycle' in the bubbling process. If the system was just equalized (air pressure in the bottle is equal to atmospheric pressure) at a time $t_0$, we wish to study what happens in the system at a later time $t_0 + \tau$. If the height of the water at time $t_0$ is $H(t_0)$, the height of the air above it must be $\Gamma - H(t_0)$, where $\Gamma$ is the total height of the water bottle. The volume of air is thus $\pi r_T^2(\Gamma - H(t_0))$. Now, although it may tempting to describe the emptying process as adiabatic and use $PV^\gamma = constant$, the process need not be adiabatic. Instead, the process is isothermal. Any small changes in the temperature of the gas in the bottle will immediately absorbed by the water and plastic of the bottle. Thus, we can say that $PV$ (rather than $PV^\gamma$) is a constant. We see that the initial pressure of the gas (if it was *just* equalized to atmospheric pressure) is $P_0$, where $P_0$ is atmospheric pressure. Its volume is $\pi r_T^2(\Gamma - H(t_0))$. At a later time ($\tau$ later), its pressure will be some $P$ that we wish to find, and its volume will be $\pi r_T^2(\Gamma - H(t_0 + \tau))$.

$$P_0 \pi r_T^2(\Gamma - H(t_0)) = P\pi r_T^2(\Gamma - H(t_0 + \tau))$$
$$P = P_0 \frac{\Gamma - H(t_0)}{\Gamma - H(t_0 + \tau)}$$

The air pressure gradient $P_{grad}$ is $P_0 - P$.

$$P_{grad} = P_0 \left(1 - \frac{\Gamma - H(t_0)}{\Gamma - H(t_0 + \tau)}\right)$$
$$P_{grad} = P_0 \frac{H(t_0) - H(t_0 + \tau)}{\Gamma - H(t_0 + \tau)}$$

The flow rate will be regulated by the pressure gradient. The higher the gradient, the lower the flow rate. However, there is an additional downwards pressure on the water at the neck due to all the water on top of it. This pressure will be given by $\rho g \cdot H(t_0 + \tau)\frac{r_T^2}{r_N^2}$ (since all the hydrostatic pressure is concentrated on the neck of the bottle). The total pressure is thus

$$P_{net} = P_0 \frac{H(t_0) - H(t_0 + \tau)}{\Gamma - H(t_0 + \tau)} - \rho g \frac{r_T^2}{r_N^2} \cdot H(t_0 + \tau)$$

The water will stop flowing when this pressure is 0. So we solve for this case:

$$\rho g \frac{r_T^2}{r_N^2} \cdot H(t_0 + \tau) = P_0 \frac{H(t_0) - H(t_0 + \tau)}{\Gamma - H(t_0 + \tau)}$$

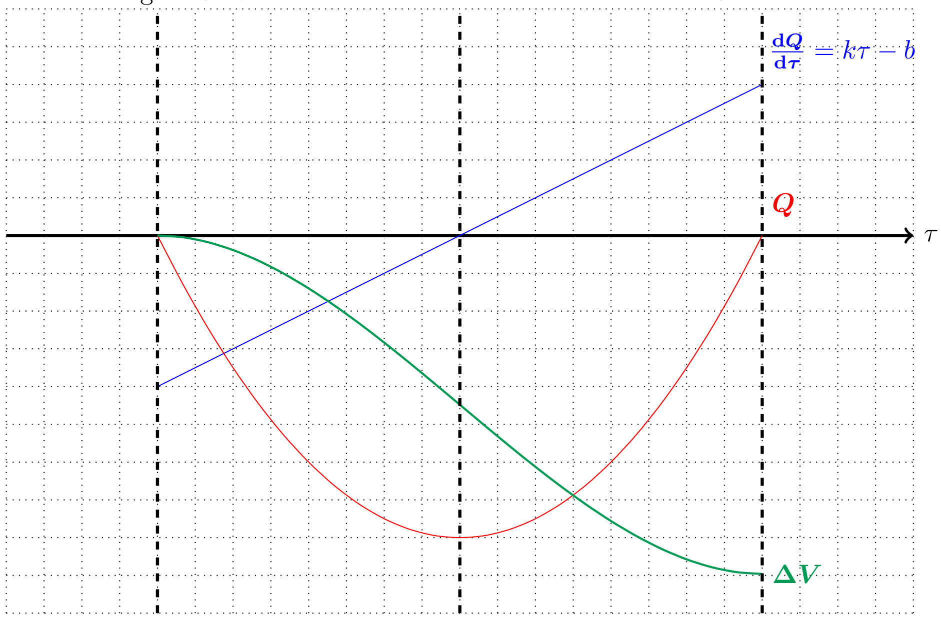
Since this is actually a quadratic equation, it can be solved. For brevity, the shorthand $A$ will be used to denote $\rho g \frac{r_T^2}{r_N^2}$, and $H(t_0 + \tau)$ will be set to $x$.

$$Ax = P_0 \frac{H(t_0) - x}{\Gamma - x}$$
$$\Gamma Ax - Ax^2 = P_0 \cdot H(t_0) - P_0 x$$
$$0 = Ax^2 - x(P_0 + \Gamma A) + P_0 \cdot H(t_0)$$
$$x = H(t_0 + \tau) = \frac{P_0 + \Gamma A - \sqrt{(P_0 + \Gamma A)^2 - 4AP_0 \cdot H(t_0)}}{2A}$$

## 3.2 Rate of Flow

We now study the rate of change in flow $\frac{\mathrm{d}Q}{\mathrm{d}\tau}$. The flow itself will start out as 0, but it seems likely that the *rate of change* will start out very negative (since the only pressure on the water will be downwards due to hydrostatic pressure). However, at some point, the flow will reach the **highest** possible magnitude (but lowest possible value, e.g. -0.5 liters per second). At this point, the rate of change in flow is 0. Finally, it will begin increasing again until the flow reaches 0 once more at a certain time $\tau = T$. These features can be modelled as a linear relation with $\frac{\mathrm{d}Q}{\mathrm{d}t} = k\tau - b$.

Figure 3: Variance of Flow and Derivative of Flow Over Time




$\tau = 0$. Flow begins, $Q = 0$. Since it is rapidly increasing in magnitude, $\frac{dQ}{d\tau}$ is very large in magnitude as well.

$\tau = T/2$. The magnitude of flow has reached a maximum. Thus $\frac{dQ}{d\tau}$ is at 0. From here, flow begins to slowly shrink in magnitude.

$\tau = T$. Flow has stopped ($Q = 0$) and one full 'bubble cycle' has completed.

Since $Q(\tau) = \int_0^\tau k\tau - b\,\mathrm{d}\tau = \frac{k}{2}\tau^2 - b\tau$, we see that $Q$ is a parabola. Furthermore, if the change in volume from $t$ to $t + \tau$ is $\Delta V$, we see that $\Delta V(\tau) = \int_0^\tau Q = \frac{k}{6}\tau^3 - \frac{b}{2}\tau^2$. We now see two things. $Q(T) = 0$, where $T$ is the length of the cycle, so:

$$\frac{k}{2}T^2 = bT$$
$$T = \frac{2b}{k} \tag{10}$$

Since $b$ is derivative of flow at $\tau = 0$, it is quite easy to find. We see that at $\tau = 0$, the pressure on the water at the neck of the bottle is simply $\rho g \frac{r_T^2}{r_N^2} \cdot H(t_0)$ since there is no pressure gradient due to atmospheric pressure. Thus the force on the water is $\pi \rho g r_T^2 \cdot H(t_0)$, and its acceleration is this force divided by the total mass of water in the bottle (given by $\pi \rho r_T^2 \cdot H(t_0)$). So acceleration is simply $g$! Thus, in the first infinitesimal $\mathrm{d}\tau$, the water changes its velocity by $\mathrm{d}v = g\,\mathrm{d}\tau$. We now that $\mathrm{d}Q = \pi r_N^2 \cdot \mathrm{d}v = \pi r_N^2 g\,\mathrm{d}\tau$, and thus $\frac{\mathrm{d}Q}{\mathrm{d}\tau} = \pi r_N^2 g$ at first. We have now found the initial rate of change in flow: $b = \pi r_N^2 g$.

Finding $k$ involves using the solution for $x$ that we found previously (the height of the water when it stops flowing). We use this $x$ to express the change in volume $\Delta V$ as:

$$\begin{aligned}
\pi r_T^2(x - H(t_0)) &= \Delta V(T) \\
\pi r_T^2(x - H(t_0)) &= \frac{k}{6}T^3 - \frac{b}{2}T^2 \\
\pi r_T^2(x - H(t_0)) &= \frac{8b^3 \cdot k}{6k^3} - \frac{4b^2 \cdot b}{2k^2} \quad \text{by plugging in (10)} \\
\pi r_T^2(x - H(t_0)) &= \frac{-4b^3}{6k^2} \\
k &= \frac{\pi r_N^3 g}{r_T}\sqrt{\frac{2g}{3(H(t_0)-x)}}
\end{aligned}$$

We finally substitute this back into (10) to find $T$.

$$\begin{aligned}
T &= 2\pi r_N^2 g \cdot \frac{r_T}{\pi r_N^3 g}\sqrt{\frac{3(H(t_0)-x)}{2g}} \\
T &= \frac{2r_T}{r_N}\sqrt{\frac{3(H(t_0)-x)}{2g}}
\end{aligned}$$

We add one small final adjustment to $T$. We have forgotten to mention the bubble that floats upwards after the flow stops. The bubble that rises will have volume $V_{bub} = \Delta V(T) = \pi r_T^2(x - H(t_0))$ since it needs to fill the gap for all the water that just left. It will experience an upwards force due to buoyancy of $\rho V_{bub}$, but its mass is $\rho_{air} V_{bub}$, where $\rho_{air}$ is the density of air. Thus its acceleration is simply $\frac{\rho}{\rho_{air}}$. By simple kinematics, the bubble takes $\sqrt{\frac{2\rho_{air}}{\rho}x}$ seconds to rise to the top of the water. Now we see that

$$T = \frac{2r_T}{r_N}\sqrt{\frac{3(H(t_0)-x)}{2g}} + \sqrt{\frac{2\rho_{air}}{\rho}x} \tag{11}$$

where

$$x = \frac{P_0 + \Gamma\rho g\frac{r_T^2}{r_N^2} - \sqrt{(P_0 + \Gamma\rho g\frac{r_T^2}{r_N^2})^2 - 4\rho g\frac{r_T^2}{r_N^2}P_0 H(t_0)}}{2\rho g\frac{r_T^2}{r_N^2}} \tag{12}$$

This is the time it takes for $\Delta V(T)$ of water to empty, including the time it takes for the bubble to reach the top of the bottle.

# 4 Simulation Results

It is clear that the emptying time for the bubbling cannot be solved analytically, since it requires repeated application of (11) Thus, a simulation was written, iterating over each successive 'bubble cycle' until the bottle was empty. Physical constants $P_0$, $\rho$, $\rho_{air}$, and $g$ were all set to SATP conditions. One last change was made to each of the formulae. This model is clearly scale free, since $r_T$ and $r_N$ always appear in ratios. Thus, a value $\mu$ is used to denote the ratio between the cross sectional area of the cylinder body to the cross sectional area of the neck of the cylinder. $\mu = \frac{\pi r_T^2}{\pi r_N^2} = \frac{r_T^2}{r_N^2}$. We now replace all ratios in our equations with this $\mu$.

For our swirling:

$$t_{best} = \frac{2\gamma\mu}{v_{air}} \tag{13}$$

And for the bubble cycles:

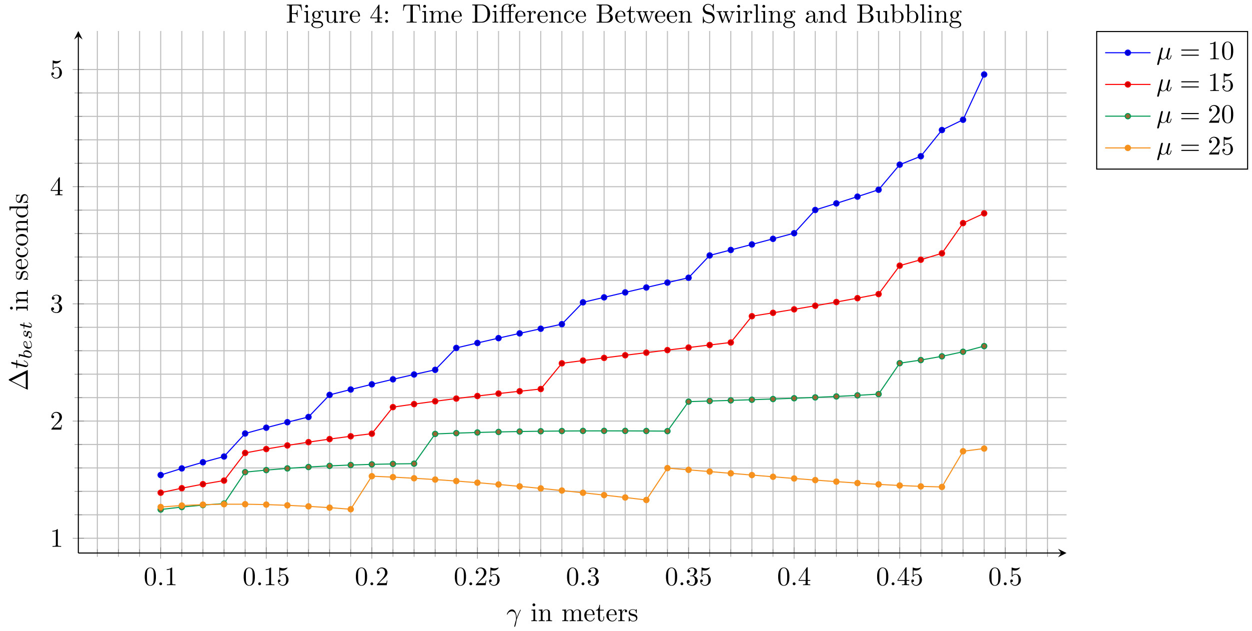
$$T = 2\sqrt{\frac{3\mu \cdot (H(t_0) - x)}{2g}} + \sqrt{\frac{2\rho_{air}}{\rho} x} \tag{14}$$

where

$$x = \frac{P_0 + \Gamma\rho g\mu - \sqrt{(P_0 + \Gamma\rho g\mu)^2 - 4\rho g\mu P_0 H(t_0)}}{2\rho g\mu} \tag{15}$$

This model was implemented into a simulation to investigate the difference between swirling the bottle and simply letting it drain with bubbling. Implementing the model for the swirling situation was trivial: the best time $t_{best,\ swirl}$ has been solved for with a direct formula. However, for bubbling action, iterations were run until the height of the water reached a negligible height (less than 0.001m). By repeatedly applying the bubbling action formulae (14) and (15), the total time $t_{best,\ no\ swirl}$ to empty the bottle with no draining was found. By comparing these two times, we can quantify which is the 'better' strategy to empty the bottle. The difference in emptying times $\Delta t_{best} = t_{best,\ no\ swirl} - t_{best,\ swirl}$ was found for different initial water heights, ranging from $\gamma = 0.1$m to $\gamma = 0.49$m. The total height of the water bottle $\Gamma$ was set to 0.5m. This was then ran for different values of $\mu$ (ratio $\frac{r_T^2}{r_N^2}$), from 10 to 25 in increments of 5. It was found that directly emptying the bottle took much longer, up to 5 seconds for specific cases! The implementation can be found at https://github.com/andigu/bottle-emptying-sim.

Figure 4: Time Difference Between Swirling and Bubbling




We see two things of interest here. First, the difference in time seems to increase in discrete segments. These may indicate an additional 'bubble cycle' as the bottle is emptying, which adds a significant difference in time for the directly draining bottle. It is this presence of cycles that causes these series of discrete steps. Secondly, as $\mu$ increases, the advantage of using a swirling technique gradually decreases. This may be due to the fact that the neck of the bottle gets gradually more restrictive as $\mu$ increases, and so the bottom of the vortex is restricted to a very small area. The volume of water that can flow out per second is drastically decreased. On the other hand, as we shrink the neck of the bottle for simple draining, we see that the upwards pressure (due to the air pressure gradient) acts over a much smaller area. This decreases resistance to flow and in fact allows the simple draining to finish faster relative to the swirling strategy.

So we can conclude two things. When the bottle has a relatively wide neck (such that its radius is at most 5 times smaller than its body's radius), it is best to swirl the water. On the other hand, when the neck shrinks, it is better to leave the bottle to drain on its own.

It should be noted that this model is not without limitations. Quite a few assumptions were made throughout the development of the model. First, to reduce several otherwise unsolvable differential equations, some variables were approximated to be constant. For example, the radius of the vortex core at the bottom of the bottle was assumed to be constant, and equal to its initial value, even though it could quite plausibly change. Secondly, the emptying process was assumed to be isothermal (and thus the ideal gas law was used to solve for air pressure). This may not have exactly reflected reality, since it is quite possible that any temperature changes in the water or air were not perfectly absorbed by the surrounding environment.