Geometric Algebra

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This is a collection of notes from my study of the *Geometric Algebra for Physicists* textbook (Doran and Lasenby 2007).

1   Foundations

The entire structure of geometric algebra can be boiled down to the geometric product, which is a binary operation between two vectors a and b. The usual rules of a vector space apply to these vectors, and the usefulness of geometric algebra is in the structure revealed by the product ab. This product satisfies three fairly natural axioms.

1. It is associative: $(ab)c = a(bc)$
2. It distributes over addition: $a(b + c) = ab + ac$
3. The square of any vector is real: $a^2 \in \mathbb{R}$. We do not make the assumption that $a^2 \geq 0$ to allow for the possibility of spaces with mixed signature (e.g., Minkowski space).

It is important not to interpret this product uv as another vector. In fact, uv is a sum of a scalar element $u \in \mathbb{R}$ and a 'bivector' V: $ab = u + V$ – this should be understood as something like a complex number $u + iv$. In fact, for a vector space of dimension 2, this analogy is exact. In general, the bivector should be pictured as a directed area, or an object like the angular momentum in classical mechanics. This analogy extends further: we can have a trivector, which is an oriented volume, and so on and so forth.

However, this geometric product is not so exotic. We can still understand it in terms of some familiar concepts. For one, we observe that $(a + b)^2$ is always real, so we must have $a^2 + ab + ba + b^2 \in \mathbb{R} \implies ab + ba \in \mathbb{R}$. We have found that the symmeterized geometric product is a binary relation between vectors that outputs a real number – that is, a dot product! So we define

$$a \cdot b = \frac{(ab + ba)}{2}.$$

We define the remaining part of the product as the exterior product (which might be thought of as replacing the role of the cross product):

$$a \wedge b = \frac{ab - ba}{2}.$$

This then gives $ab = a \cdot b + a \wedge b$: a sum of a scalar and a bivector, as promised.

Well, technically, not yet: we do not have a good reason to believe we should understand $a \wedge b$ as a bivector (or a directed area) yet. So we define it as such, and show this definition conforms to our intuition later.

More precisely, we define exterior product of r vectors as the full anti-symmeterized product over all of them:

$$a_1 \wedge a_2 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(r)},$$

where $\mathbb{S}_r$ is the set of all permutations on r elements and $(-1)^\sigma$ is the sign of a permutation. A consequence of this definition is that the product reverses sign under exchange of any two vectors. Then, we see that if any vector is repeated, the exterior product must be zero, which then implies if the vectors are linearly dependent, the exterior product is also zero (this follows simply by distributivity of the geometric product under addition). Therefore, the outer product can be understood to measure the dimensionality of a set of vectors. We say that the outer product of r vectors has grade r (if it does not vanish). A multivector that can be written purely as an outer product is called a blade.

Lemma 1. *Any blade $a_1 \wedge \ldots \wedge a_r$ can be written simply as a product of orthogonal vectors $e_1 \ldots e_r$. This justifies our idea that a blade with r vectors can be interpreted as a directed area for $r = 2$, volume for $r = 3$, and so on.*

Proof. Let $A_{i,j} = a_i \cdot a_j$. Since $A_{i,j}$ is a symmetric matrix, it can be diagonalized with $P^T D P$, where P is orthogonal. Letting $e_i = P_{i,k} a_k$, we see that $e_i \cdot e_j = P_{i,k} P_{j,\ell} a_k \cdot a_\ell = P_{i,k} A_{k,\ell} (P^T)_{\ell,j} = D_{i,j}$. Therefore, these vectors obey $e_i e_j = -e_j e_i$ for $i \neq j$. Finally, since $e_i = P_{i,k} a_k \implies a_k = P_{i,k} e_i$:

$$a_1 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma P_{i_1,\sigma(1)} P_{i_2,\sigma(2)} \ldots P_{i_r,\sigma(r)} e_{i_1} e_{i_2} \ldots e_{i_r}$$

Note that we have the restriction $i_1 \neq i_2 \neq \ldots \neq i_r$ due to the antisymmeterization of the sum. Therefore, we can rewrite as:

$$a_1 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{\rho,\sigma \in \mathbb{S}_r} (-1)^\sigma P_{\rho(1),\sigma(1)} P_{\rho(2),\sigma(2)} \ldots P_{\rho(r),\sigma(r)} e_{\rho(1)} e_{\rho(2)} \ldots e_{\rho(r)}$$

$$= \frac{1}{r!} \sum_{\rho,\sigma \in \mathbb{S}_r} (-1)^{\rho^{-1}} (-1)^\sigma P_{1,\sigma(\rho^{-1}(1))} P_{2,\sigma(\rho^{-1}(2))} \ldots P_{r,\sigma(\rho^{-1}(r))} e_1 e_2 \ldots e_r$$

$$= \sum_{\alpha \in \mathbb{S}_r} (-1)^\alpha P_{1,\alpha(1)} P_{2,\alpha(2)} \ldots P_{r,\alpha(r)} e_1 e_2 \ldots e_r$$

$$= \det(P) e_1 \ldots e_r$$

In the second line, the factor $(-1)^{\rho^{-1}}$ comes from unshuffling $e_{\rho(1)} \ldots e_{\rho(r)} \to e_1 \ldots e_r$, and in the third line, we use the fact that for any function f that depends only on $\sigma \circ \rho^{-1}$: $\sum_{\rho,\sigma \in \mathbb{S}_r} f(\sigma \circ \rho^{-1}) = r! \sum_{\alpha \in \mathbb{S}_r} f(\alpha)$, where α plays the role of $\sigma \circ \rho^{-1}$. Now, observe that $\det(P) = \pm 1$, and in the case where $\det(P) = -1$, we can simply reorder $e_1, \ldots, e_r$ to get rid of the negative sign.

$\square$

In general, multivectors can be comprised of elements with different grades. For a set of orthogonal vectors, we say that e_i has grade 1, $e_i e_j$ (for $i \neq j$) has grade 2, and so on. We define $\langle \cdot \rangle_r$ to be the component of a multivector with grade r, so that in general a multivector A in a geometric algebra $\mathcal{G}_n$ (whose underlying vector space has dimension n) can be written $A = \sum_{r=0}^{n} \langle A \rangle_r$. A multivector that satisfies $\langle A \rangle_r = A$ (for some r) is called homogenous.

We denote the subspace of $\mathcal{G}_n$ of grade r as $\mathcal{G}_n^r$. The dimensionality of $\mathcal{G}_n^r$ is $\binom{n}{r}$, because the basis of $\mathcal{G}_n^r$ can be formed by choosing r items from the n basis vectors.[1] The overall dimensionality is therefore $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

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[1]It is important to note that not every multivector in $\mathcal{G}_n^r$ is a blade. The simplest nontrivial example is $e_1 e_2 + e_3 e_4$ in $\mathcal{G}_n^4$ – there is simply no way to write this as a blade because e_1, e_2 and e_3, e_4 are orthogonal to each other.

1.1 Properties of the Geometric Product

We now study the behavior of aB_r, where a is some vector and B_r is a homogenous multivector of grade r. More specifically, we will show that we can define $a \cdot B_r$ and $a \wedge B_r$ in terms of aB_r and $B_r a$ in a way that is similar to the original definition for $\cdot$ and $\wedge$ between two ordinary vectors.

Theorem 2. *For any $a \in \mathcal{G}_n^1$ and $B_r \in \mathcal{G}_n^r$, $(aB_r - (-1)^r B_r a)$ is a homogenous multivector with grade $r-1$. This motivates the definition*

$$a \cdot B_r := \frac{1}{2}(aB_r - (-1)^r B_r), \tag{1}$$

since now $\cdot$ lowers grade by 1.

Proof. We assume B_r is a blade, since all of the above statements are linear in B_r. That is, it suffices to show that the above statement is true for $B_r = e_1 e_2 \ldots e_r$, for any choice of orthogonal vectors $\{e_1, \ldots, e_r\}$. We repeatedly apply $ab = 2a \cdot b - ba$. Observe that $aB_r = 2(a \cdot e_1)e_2 \ldots e_r - e_1 a e_2 \ldots e_r$. We can repeatedly do this, shifting a further back in the chain, to get:

$$aB_r = 2\sum_{k=1}^{r}(-1)^{k+1}(a \cdot e_k)(e_1 \ldots \check{e}_k \ldots e_r) + (-1)^r B_r a,$$

where $\check{e}_k$ denotes the fact that e_k is omitted from the product (i.e. $e_1 \check{e}_2 e_3 = e_1 e_3$). A simple rearrangement gives:

$$\frac{1}{2}(aB_r - (-1)^r B_r a) = \sum_{k=1}^{r}(-1)^{k+1}(a \cdot e_k)(e_1 \ldots \check{e}_k \ldots e_r)$$

Note that in the sum, $e_1 \ldots \check{e}_k \ldots e_r$ is grade $r-1$. So, $\frac{1}{2}(aB_r - (-1)^r B_r a)$ is a linear combination of blades, each of which are grade $r-1$ – so it is on the whole a homogenous multivector of grade $r-1$. $\square$

A similar result holds for the wedge product $\wedge$.

Theorem 3. *For any blade $B_r = b_1 \wedge \ldots \wedge b_r$ and any vector a:*

$$a \wedge b_1 \wedge \ldots \wedge b_r = \frac{1}{2}(aB_r + (-1)^r B_r a)$$

Proof. Let $e_1, \ldots e_r$ be an orthogonalization of $b_1, \ldots b_r$ using the same technique as Lemma 1, such that $b_1 \wedge \ldots \wedge b_r = e_1 \ldots e_r$. Then, let $a_\perp = a - \sum_{k=1}^{r}\beta_k e_k$, where:

$$\beta_k = \begin{cases} \frac{a \cdot e_k}{(e_k \cdot e_k)^2} & \text{if } e_k \cdot e_k \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Letting $a_\parallel = \sum_{k=1}^{r}\beta_k e_k$, we can write $a = a_\parallel + a_\perp$, and it is now simple to see that $a_\perp \cdot e_k = 0$ for all $k = 1, \ldots, r$. Furthermore, since each of the e_k are merely linear combinations of b_k, $a_\perp \cdot b_k = 0$ for all $k = 1, \ldots, r$ as well. Therefore, $a \wedge b_1 \wedge \ldots \wedge b_r = a_\perp \wedge b_1 \wedge \ldots \wedge b_r$.

Now, observe that:

$$a_\parallel B_r + (-1)^r B_r a_\parallel = \sum_{k=1}^{r}\beta_k \big((-1)^{k+1}(e_k)^2 e_1 \ldots \check{e}_k \ldots e_r + (-1)^r(-1)^{r-k}e_1 \ldots \check{e}_k \ldots e_r(e_k)^2\big)$$

$$= 0$$

Therefore, using the crucial fact that $a_\perp b_k = -b_k a_\perp$:

$$
\begin{aligned}
aB_r + (-1)^r B_r a &= a_\perp B_r + (-1)^r B_r a_\perp \\
&= \frac{1}{r!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma \big(a_\perp b_{\sigma(1)} \ldots b_{\sigma(r)} + (-1)^r b_{\sigma(1)} \ldots b_{\sigma(r)} a_\perp \big) \\
&= \frac{2}{r!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma a_\perp b_{\sigma(1)} \ldots b_{\sigma(r)} \\
&= \frac{2}{(r+1)!} \sum_{k=1}^{r+1} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma a_\perp b_{\sigma(1)} \ldots b_{\sigma(r)}
\end{aligned}
$$

Now, observe that we are free to scramble $a_\perp$ somewhere into the kth position of $b_{\sigma(1)} \ldots b_{\sigma(r)}$ – that is, $a_\perp b_{\sigma(1)} \ldots b_{\sigma(r)} = (-1)^k b_{\sigma(1)} \ldots b_{\sigma(k-1)} a_\perp b_{\sigma(k)} \ldots b_{\sigma(r)}$. But then, viewing $a_\perp$ as the first element of the set $S = [a_\perp, b_1, \ldots, b_r]$, so that we identify $b_0 = a_\perp$ and $S \cong [0, \ldots, r]$, we see that the permutation that sends S to $[b_{\sigma(1)}, \ldots, b_{\sigma(k-1)}, a_\perp, b_{\sigma(k)}, \ldots, b_{\sigma(r)}] \cong [\sigma(1), \ldots, \sigma(k-1), 0, \sigma(k), \ldots, \sigma(r)]$ has parity $(-1)^k (-1)^\sigma$! Therefore, the double sum over $\sum_{k=1}^{r+1} \sum_{\sigma \in \mathbb{S}_r}$ can be rewritten as a sum over permutations mapping $[0, \ldots, r]$ onto itself – that is, $\mathbb{S}_{r+1}$. We are then left with:

$$
\begin{aligned}
aB_r + (-1)^r B_r a &= \frac{2}{(r+1)!} \sum_{\sigma \in \mathbb{S}_{r+1}} b_{\sigma(0)} \ldots b_{\sigma(r)} \\
&= 2a \wedge b_1 \wedge \ldots \wedge b_r,
\end{aligned}
$$

as required.

$\square$

Corollary 1. *For any multivector $B_r \in \mathcal{G}_n^r$, let $a \wedge B_r$ be defined by:*

$$
a \wedge B_r := \frac{1}{2}(aB_r + (-1)^r B_r), \tag{2}
$$

Then, $a \wedge B_r$ is always a homogenous multivector of grade $r + 1$. That is, $\wedge$ raises grade by 1.

Proof. This follows simply by decomposing B_r into a linear combination of blades and applying Theorem 3 to each blade (observing that $a \wedge B_r$ is bilinear)

$\square$

It follows trivially from the definitions in Equation (1) and Equation (2) that $aB_r = a \cdot B_r + a \wedge B_r$, which says that the product aB_r is a sum of homogenous multivectors with grade $r \pm 1$. This can be generalized for a general multivector A_s. Since A_s can be written as a linear combination of blades, and each blade can be written as a product of anticommuting (orthogonal) vectors, $A_s B_r$ is composed of multivectors with grades $|r - s|, |r - s| + 2, \ldots, r + s$. To further generalize the dot and wedge product notation, when we write $A_s \cdot B_r$, we mean the lowest grade component of $A_s B_r$, and the highest for $A_s \wedge B_r$.

References

Doran, Chris, and A. N. Lasenby. 2007. *Geometric Algebra for Physicists.* 1st ed. Cambridge ; New York: Cambridge University Press.