Light

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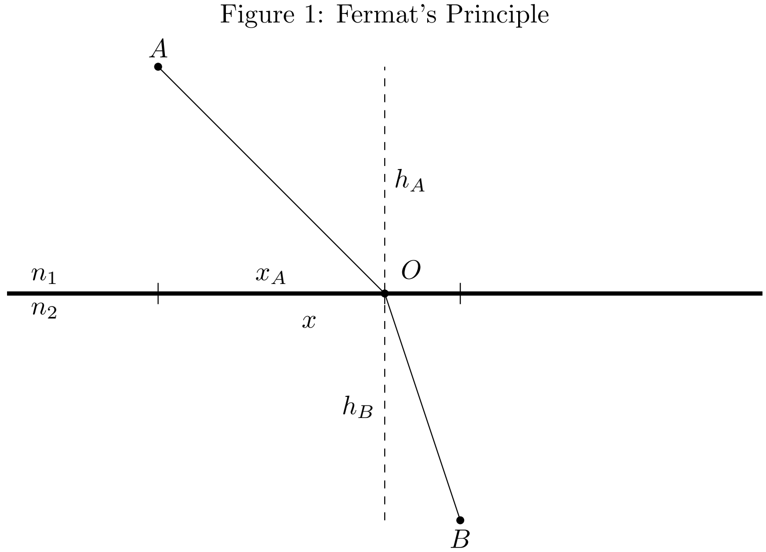
1 Fermat's Principle of Least Time

Initially, Fermat proposed that light will always take the path such that it gets from start to destination in the least possible time. In modern times, we restate it so that light will take a path such that there are other paths, arbitrarily nearby on either side, along which the ray would take almost exactly the same time to traverse.

Figure 1: Fermat's Principle



We can derive Snell's law by using Fermat's principle. We see that the time for light to travel from A to B is:

$$t(x_A) = \frac{\sqrt{x_A^2 + h_A^2}}{c/n_1} + \frac{\sqrt{(x - x_A)^2 + h_B^2}}{c/n_2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x_A} = \frac{n_1}{c}\frac{x_A}{\sqrt{x_A^2 + h_A^2}} + \frac{n_2}{c}\frac{x - x_A}{\sqrt{(x - x_A)^2 + h_B^2}}$$

We set the derivative to 0 to find the minimum.

$$\frac{n_1 x_A}{h_A\sqrt{x_A^2/h_A^2 + 1}} = \frac{n_2(x - x_A)}{h_B\sqrt{(x - x_A)^2 + 1}}$$

$$\frac{n_1 x_A}{h_A \sec\theta_I} = \frac{n_2(x - x_A)}{h_B \sec\theta_R}$$

$$n_1 \tan\theta_I \cos\theta_I = n_2 \tan\theta_R \cos\theta_R$$

$$n_1 \sin\theta_I = n_2 \sin\theta_R$$

2 Electromagnetism

Coulomb's law for static electric charges is correct in that the electric field E falls off as the inverse square of distance. However, this law is not true for moving charges. We know it isn't, since light is in part due to an electric field, and light certainly doesn't fall off as the inverse square of distance. Maxwell proposed a law for electric field:

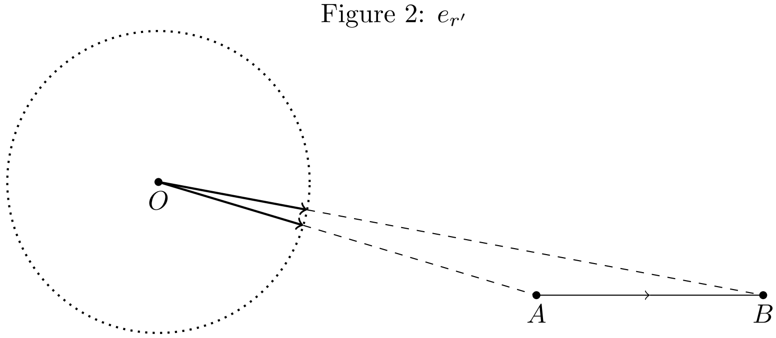
$$E = -\frac{q}{k}\left(\frac{e_{r'}}{r'^2} + \frac{r'}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(e_{r'})\right)$$

Here $e_{r'}$ is the unit direction vector from where E is being measured to where the charge was r'/c seconds ago. We have this odd delay r'/c because no influence can travel faster than the speed of light. We see that the first term is Coulomb's law. The second is some effort to make up for the delay – nature takes the rate of change in electric field r'/c seconds ago, and multiplies it by r'/c to estimate the difference between 'real' current electric field and the one that we are receiving. We see that the first two terms fall off as the inverse square of distance, so only the last should go off as the inverse of distance. So we study this:

$$E \propto \frac{\mathrm{d}^2}{\mathrm{d}t^2} e_{r'}$$

We imagine that $e_{r'}$ is restricted by a sphere, and observe this:

Figure 2: $e_{r'}$



We see that when a charge moves radially from O, it barely changes $e_{r'}$. So instead, we imagine that the charge moves in a 2 dimensional plane. Its radial movements do not impact $e_{r'}$ significantly. If we simplify the situation such that $\frac{r'}{c}$ is approximated as constant, we can simplify the situation a little.

$$E = -\frac{q}{kc^2 r'} a_x\left(t - \frac{r'}{c}\right)$$

We divide by r' because we scale down the position vector by a factor of $1/r'$, and thus acceleration is scaled down by $1/r'$ as well.

We now see that there is a component of electric field which varies as the inverse of distance. Since energy carried in a field is the square of amplitude, we can say that the energy of the wave varies as the inverse square of distance. Thus, the flux through a section of a sphere will remain constant if we project that section outwards and keep the angle that it sweeps constant.

3 Interference and Diffraction

Say we have a number of sources n, all out of phase with their neighbour by ϕ, such that the amplitude R at a point is:

$$R = A\sum_n \cos(\omega t + n\phi)$$

We can actually solve this by extending each cosine to its complex counterpart. We make $cos(\omega t + n\phi) = e^{i(\omega t + n\phi)}$. For every term, the $e^{i\omega t}$ is constant so we ignore this. So we study the sum

$$\sum_n e^{i \cdot n\phi}$$

Geometrically, this will form a regular polygon with radius r such that $2r \sin \phi/2 = 1$. The distance from origin to endpoint is $2r \sin(n\phi/2)$. Using this, we see that the sum has magnitude $\frac{sin(n\phi/2)}{\sin(\phi/2)}$. So R has a magnitude of $A\frac{sin(n\phi/2)}{\sin(\phi/2)}$. Since intensity is the square of amplitude, the intensity will be:

$$I = I_0 \frac{\sin^2(n\phi/2)}{\sin^2(\phi/2)}$$

Where I_0 is the intensity due to only one source with no interference effects. We observe what happens near $\phi = 0$, using the fact that the $\lim_{x \to 0} \sin x = x$. With this, we see that $I = n^2 I_0$ when $\phi = 0$. If we assume that n is very large so that the top term varies much more widely than the bottom, I will hit 0 when $\sin^2(n\phi/2) = 0$, or, $n\phi/2 = \pi$, thus $\phi = 2\pi/n$. It hits a maximum next when $\sin^2 n\phi/2 = 1$, so $n\phi/2 = 3\pi/2$ and $\phi = 3\pi/n$. At this secondary maximum, $I = \frac{I_0}{\sin^2(\frac{3\pi}{2n})} = \frac{4I_0 n^2}{9\pi^2} = \frac{4}{9\pi^2} I_m$ (where I_m is the maximum intensity at $\phi = 0$), using the limit of $\sin(x) = x$ as $x \to 0$ since $\frac{3\pi}{2n}$ is usually quite small.

3.1 Different ϕ

We wonder what happens when the sources emit waves in phase, but the waves travel different distances. This happens when they exit at an angle θ. When this happens, their phase difference is $\frac{2\pi d \sin\theta}{\lambda}$ (if d is the separation between each slit). So now, we wish to find what θ will give us a minimum.

$$\begin{aligned}
\phi &= \frac{2\pi}{n} \\
\frac{2\pi}{n} &= \frac{2\pi d \sin\theta}{\lambda} \\
\lambda &= nd \sin\theta
\end{aligned}$$

We note that the path difference between the top and bottom slit is $(n-1)d\sin\theta$. Essentially, the restriction $nd\sin\theta = \lambda$ tells us that if we add one more source, it must have a path difference of λ. Why is there a minimum in this specific case? Because all our ϕ vectors add up to form a **circle**, and thus the magnitude of their sum is 0.

On the other hand, if we want to find where the **primary** maximum occurs, we set ϕ to some multiple of 2π.

$$\begin{aligned}
m \cdot 2\pi &= \frac{2\pi d \sin\theta}{\lambda} \\
m\lambda &= d\sin\theta
\end{aligned}$$

This means that the path difference between any consecutive sources must be a multiple of λ. This makes perfect sense intuitively.

3.2 $n \to \infty$

We wish to see what happens when the number of sources n approaches infinity, but the overall distance D from the first source to the last remains the same.

$$\begin{aligned}
I &= \lim_{n\to\infty} I_0 \frac{\sin^2(n\phi/2)}{\sin^2(\phi/2)} \\
&= \lim_{n\to\infty} I_0 \frac{\sin^2(n\frac{\Theta}{2n})}{\sin^2(\frac{\Theta}{2n})}
\end{aligned}$$

Where Θ is the phase difference from the first source to the last

$$= \lim_{n\to\infty} I_0 \frac{\sin^2(\Theta/2)}{\sin^2(\frac{\Theta}{2n})}$$
$$= 4n^2 I_0 \frac{\sin^2(\Theta/2)}{\Theta^2}$$

We see that $n^2 I_0$ is the maximum intensity at the center, which we call I_m

$$= 4I_m \frac{\sin^2(\Theta/2)}{\Theta^2}$$

We see that in this case, there is only **one** big maximum, since the denominator has Θ^2. This makes sense, because at no point will the condition $m\lambda = d\sin\theta$ be met, because d is infinitesimally small!

4 Diffraction Gratings

We can take a sheet of something (i.e. glass), and make a great deal of small, evenly spaced scratches on its surface that are very close together. If we shine a light through it, we will of course get a diffraction pattern. We make sure that the distance d is greater than λ so that there will be maxima of order greater than one observed. We remember that the maxima occurs where $m\lambda = d\sin\theta$. If we say the distance from one end of the grating to the other is D, we see $mn\lambda = D\sin\theta$. Now, we see the purpose of the diffraction grating – it is to separate light of different wavelengths. We use the Rayleigh criterion, which states that if we want light to be sufficiently 'separated', the maximum of one must occur at the minimum of the other. Now, if one maximum occurs at θ, we want the other wave's minimum to occur there.

$$mn\lambda_1 = D\sin\theta$$
$$\lambda_1 = \frac{D\sin\theta}{mn}$$

$$mn\lambda_2 + \lambda_2 = D\sin\theta$$

Why the $+\lambda_2$? We see that if we make the last source out of phase with the first by just one wavelength, the vector sum of all the phases is a circle again (recall the geometrical interpretation of adding).

$$\lambda_2 = \frac{D\sin\theta}{1+mn}$$

$$\Delta\lambda = D\sin\theta\left(\frac{1}{mn} - \frac{1}{1+mn}\right)$$

If we approximate the two wavelengths to be approximately the same (why else would we use a diffraction grating?), we see:

$$\frac{\Delta\lambda}{\lambda} = D\sin\theta \frac{\frac{1+mn-mn}{(1+mn)mn}}{\frac{D\sin\theta}{1+mn}}$$
$$= \frac{1}{mn}$$

4.1 Electric Field from a Uniformly Vibrating Sheet

We have an infinite sheet of material of charge density η. Each charge moves with respect to its average position as $x = x_0\cos\omega t$, or $x = x_0 e^{i\omega t}$. At a certain distance r, we see that the electric field due to that charge is:

$$E = \frac{q}{kc^2}\frac{\omega^2 x_0 e^{i\omega(t-r/c)}}{r}$$

This is not entirely correct, since we need to account for the fact that the charge may not be directly below us, and thus we would have to add an adjustment factor $\cos\theta$ for the angle that the charge is at w.r.t us.

But we imagine that we are so far away that this factor does not matter. We now have to integrate over every charge to find the total electric field.

$$E = \int_0^\infty \frac{q}{kc^2}\frac{\omega^2 x_0 e^{i\omega(t-r/c)}}{r}\eta 2\pi\rho\,\mathrm{d}\rho$$
$$= \frac{2\pi q\omega^2 x_0\eta}{kc^2}\int_0^\infty \frac{e^{i\omega(t-r/c)}}{r}\rho\,\mathrm{d}\rho$$

We see that $\rho^2 + z^2 = r^2$, if z is the distance of the normal from the observer to the plate. So $2\rho\,\mathrm{d}\rho = 2r\,\mathrm{d}r$.

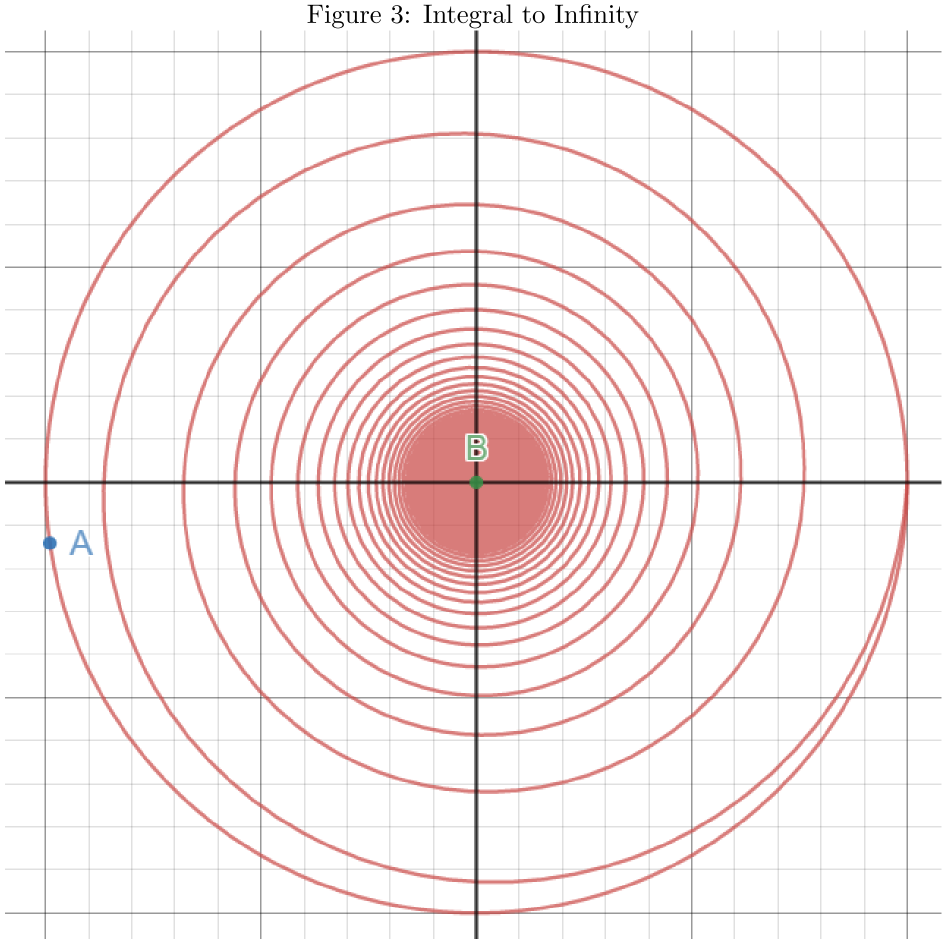
$$= \frac{2\pi q\omega^2 x_0\eta e^{i\omega t}}{kc^2}\int_z^\infty e^{-i\omega r/c}\,\mathrm{d}r$$

It is meaningless, mathematically, to take an integral from 0 to infinity of a cosine function! However, physically, we can have an answer.

$$\int_z^\infty e^{-i\omega r/c}\,\mathrm{d}r = -\frac{c}{i\omega}\left(e^{-i\infty} - e^{-i\omega z/c}\right)$$

But we have forgotten one thing. As we move farther out, the projection of the acceleration of the charges dies off (the **cosine factor** that we initially ignored)! So, if we were to graph $\alpha e^{-\omega r/c}$ in little increments of $\mathrm{d}r$, from $r = z$ to $r = \infty$, where α slowly decreased as we increased r, we'd get something like this:

Figure 3: Integral to Infinity



The starting point A is $e^{-\omega z/c}$, and the end point B is the $e^{-i\infty}$ (with an adjustment factor that dies out over time). Since this graph represents the sum of each little vector in increments of $\mathrm{d}r$, the vector $A \to B$

is the integral! Since this vector is just the negative of the position of A, it is simply $-e^{-\omega z/c}$. So the whole integral is just $-e^{-\omega z/c}$.

$$\begin{aligned}
E &= \frac{2\pi q\omega^2 x_0 \eta e^{i\omega t}}{kc^2} \cdot \left(\frac{ce^{-\omega z/c}}{i\omega}\right) \\
&= \frac{2\pi q\omega x_0 \eta e^{i\omega(t-z/c)}}{ikc} \\
&= \frac{2\pi q\eta}{kc} - \omega i x_0 e^{i\omega(t-z/c)} \\
&= -\frac{2\pi q\eta}{kc}\left[\text{velocity of charges at } t - z/c\right]
\end{aligned}$$

5 Origin of Index of Refraction

We have seen that light seems to travel slower by a factor of $\frac{1}{n}$ in a substance of index of refraction n. However, in Maxwell's equations, we notice that an electric field propagates at the speed of light **no matter what medium** it's in. So what must be happening is that the light merely *seems* to be travelling more slowly. We investigate why.

Say we have a source oscillating with amplitude $A = \cos\omega t$. We imagine it is an infinite plane of some material, so that the amplitude of the field does not vary with distance. We convert to complex form, so that $A = e^{i\omega t}$. We see that the electric field at a distance x is thus $E_0 e^{i\omega(t-\frac{x}{c})}$, where E_0 is the maximum magnitude of the electric field at z. However, if we have some material in the way of thickness Δx and index of refraction n, there is a time **delay** in the light of $\frac{n\Delta x}{c} - \frac{\Delta x}{c}$.

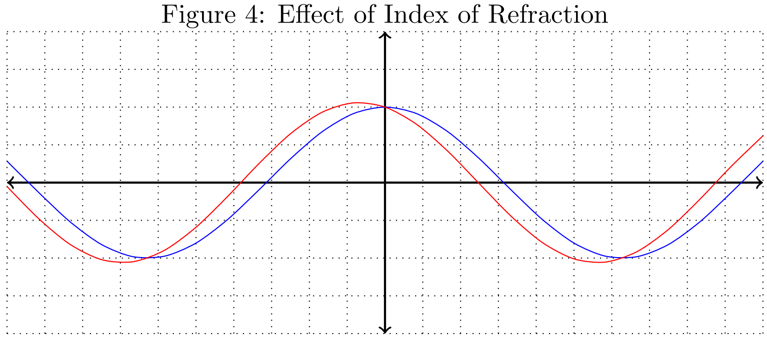
$$E(z) = E_0 e^{i\omega(t-x/c)} e^{-i\omega(n-1)\Delta x/c}$$

However, e^a can be approximated as $1 + a$ when $a \approx 0$. We see that the factor of $1/c$ the entire power is close to 0.

$$\begin{aligned}
&= E_0 e^{i\omega(t-x/c)}(1 - i\omega(n-1)\Delta x/c) \\
&= E_0 e^{i\omega(t-x/c)} - \frac{i\omega(n-1)\Delta x}{c} E_0 e^{i\omega(t-x/c)}
\end{aligned}$$

If we look at this expression carefully, we see very clearly what is going on. The first part is the light from the original source – let's say this is a cosine function. The second term is multiplied by $-i$. This means that it is a sine function with some small amplitude. The whole thing will look something like $E_0\cos\omega t - \beta E_0 \sin\omega t$, where β is very small.

Figure 4: Effect of Index of Refraction



The fact that amplitude is slightly higher in the shifted wave is merely due to the fact that our approximation $e^a \approx 1 + a$ is not 100% correct.

We now derive what this index of refraction really is. If we place our source at $z = 0$, and put the interfering object (say a sheet of glass of thickness Δx) at $z = 0$ as well, the atoms in the sheet will be influenced by the electric field. However, their motion will also impact their neighbours, who in turn will impact them! This is quite complicated, so we must assume that the motion of the electrons do not impact their neighbouring electrons significantly, and the dominant force here is still the electric field from the source. In materials of high charge density, this is not a good approximation, so this derivation only applies to materials with low charge density (and thus small n).

The atoms will encounter an electric field of $E_0 e^{i\omega t}$, and thus they will feel a force of $q_e E_0 e^{i\omega t}$. For this case, it is acceptable to model the electrons as balls attached on springs, so that they experience a restoring force proportional to their displacement. Thus they have a natural angular frequency. This is driven motion, which we have solved for before:

$$x = \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} e^{i\omega t}$$

Now, we can find their velocity and thus find the electric field they produce (since this sheet is 'infinite', we can use the equation we found before for the electric field at a distance z). The velocity is of course $\frac{i\omega q_e E_0}{m(\omega_0^2 - \omega^2)} e^{i\omega t}$. We plug this into the equation we found before:

$$E = -\frac{2\pi q_e \eta}{kc} \cdot \frac{i\omega q_e E_0}{m_e(\omega_0^2 - \omega^2)} e^{i\omega(t-x/c)}$$

We see that this field being produced by the sheet is of the same form as the expression $-\frac{i\omega(n-1)\Delta x}{c} E_0 e^{i\omega(t-x/c)}$ (which we said was the 'retardation' factor due to the sheet's index of refraction n). So we set the two to be equal to one another:

$$-\frac{i\omega(n-1)\Delta x}{c} E_0 e^{i\omega(t-x/c)} = -\frac{2\pi q_e \eta}{kc} \cdot \frac{i\omega q_e E_0}{m_e(\omega_0^2 - \omega^2)} e^{i\omega(t-x/c)}$$

$$(n-1)\Delta x = \frac{2\pi q_e^2 \eta}{m_e k(\omega_0^2 - \omega^2)}$$

We note that charge per volume η is simply equal to the charge per area multiplied by Δx.

$$n = 1 + \frac{2\pi q_e^2 N}{m_e k(\omega_0^2 - \omega^2)}$$

This is quite interesting – it tells us that n varies depending on ω, which is how prisms separate light into the rainbow! Also, it shows how n depends on the character of the material (it depends on charge density N, and the character of the electrons in the material ω_0).

5.1 Adjustments to Index of Refraction

We note that electrons must feel some sort of 'friction' force (of course, it is not actually friction, but some sort of force that acts against movement), otherwise it would continue oscillating forever even after the source was turned off. Also, the electrons of a given material do not all oscillate with the same natural frequency, instead there are m different natural angular frequencies, each with their own *gamma* coefficient. So now:

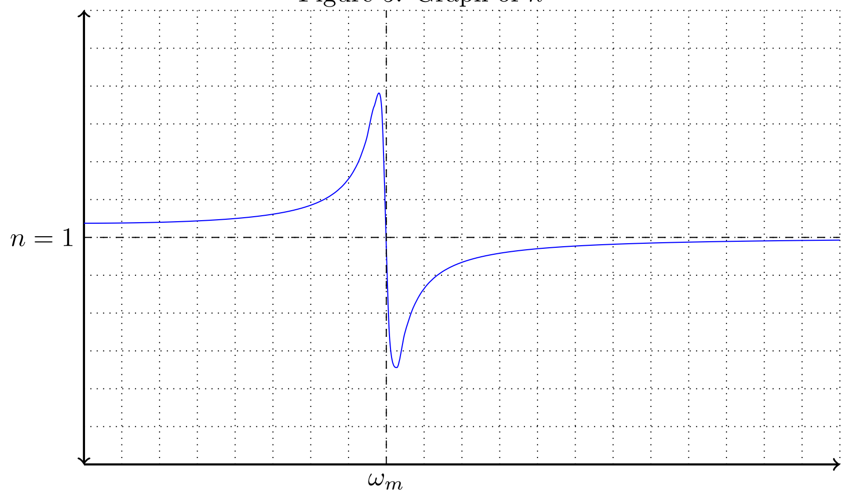
$$n = 1 + \frac{2\pi q_e^2}{m_e k} \sum_m \frac{N_m}{(\omega_m^2 - \omega^2 + i\gamma_m \omega)}$$

We convert this to complex form:

$$= 1 + \frac{2\pi q_e^2}{m_e k} \sum_m \frac{N_m}{(\omega_m^2 - \omega^2)^2 + \gamma_m^2 \omega^2} \left(\omega_m^2 - \omega^2 - i\gamma_m \omega\right)$$

We graph the real part of each term in the sum against ω:

Figure 5: Graph of n



For a material with $m > 1$, it will simply have this graph repeated with the major feature (at around ω_m) repeated for every ω_m.

Interestingly, we now have a complex index of refraction. If we say $n = n' - in''$. If we plug this back into the equation for the electric field after going through a material, we get:

$$\begin{aligned} E &= E_0 e^{i\omega(t-x/c)} e^{-i\omega(n'-in''-1)\Delta x/c} \\ &= E_0 e^{i\omega(t-x/c)} e^{-\omega n''\Delta x/c - i\omega(n'-1)\Delta x/c} \\ &= e^{-\omega n''\Delta x/c} \cdot E_0 e^{i\omega(t-x/c)} e^{-i\omega(n'-1)\Delta x/c} \end{aligned}$$

So we see that the original slowing down factor is still in there, but the magnitude is also decreased due to the $e^{-\omega n''\Delta x/c}$ factor. This explains the energy loss once light passes through a certain substance.

5.2 Energy of an Electric Wave

We have before postulated that the energy in a wave is proportional to the time average of its height squared so that $E \propto \overline{A^2}$. We now wish to find the constant of proportionality α. We do this by seeing that the total energy output of the source is equal to the sum of the energy of the transmitted wave and the work done by the wave on the material.

$$\alpha\overline{E_s^2} = \alpha\overline{(E_s + E_a)^2} + N\Delta z q_e \overline{E_s v}$$

We approximate $E_a^2 \approx 0$, since E_a can be assumed to be small compared to E_s.

$$-2\alpha\overline{E_s E_a} = N\Delta z q_e \overline{E_s v}$$

We recall that the electric field from the the sheet is $-\frac{2\pi q_e \eta}{kc} v[\text{at } t - z/c]$

$$N\Delta z q_e \overline{E_s v} = 2\alpha \overline{E_s[\text{at } z] \frac{2\pi q_e \eta}{kc} v[\text{at } t - z/c]}$$

We see that the product of $E_s v$ is not the same on the left and the right. On the left, we are looking at the surface of the material $z = 0$. On the right, we are z away. **However**, the average of the two over time is the same! So we can cancel them.

$$N\Delta z q_e = 4\frac{\alpha\pi q_e \eta}{kc}$$

We note $\eta = N\Delta z$

$$\frac{kc}{4\pi} = \alpha$$

Actually, k has always been defined by $k = 4\pi\epsilon_0$, where ϵ_0 is the permittivity of free space.

$$\alpha = \epsilon_0 c$$

Thus,

$$I = \epsilon_0 c\overline{E^2}$$

I is the intensity, or the power per area.

6 Rate of Energy Radiation

Say we have a charge oscillating with acceleration a in the z direction. At a distance r and angle θ from the z axis, the acceleration we see is $a' \sin\theta$ (it is delayed, so we call it a' rather than a). We have found before that the electric field is $-\frac{q_e a' \sin\theta}{kc^2 r'}$. So if we want to find the power per unit area of this field:

$$\begin{aligned} I &= \frac{k}{4\pi} c \frac{q_e^2 a'^2 \sin^2\theta}{k^2 c^4 r^2} \\ &= \frac{q_e^2 a'^2 \sin^2\theta}{4\pi k c^3 r^2} \end{aligned}$$

We use this now to find the power in a circle of infinitesimal thickness $r\,\mathrm{d}\theta$ and radius $r\sin\theta$, then integrate over θ to find the power radiated by the charge overall.

$$\begin{aligned} P &= \int_0^\pi \frac{q_e^2 a'^2 \sin^2\theta}{4\pi k c^3 r^2} 2\pi r \sin\theta r\,\mathrm{d}\theta \\ &= \int_0^\pi \frac{q_e^2 a'^2 \sin^3\theta}{8\pi\epsilon_0 c^3}\,\mathrm{d}\theta \\ &= \frac{q_e^2 a'^2}{8\pi\epsilon_0 c^3} \int_0^\pi \sin^3\theta\,\mathrm{d}\theta \\ &= \frac{q_e^2 a'^2}{6\pi\epsilon_0 c^3} \end{aligned}$$

We here note that a'^2 refers to **average** acceleration squared, since our formula for the intensity of a wave used the **average** of the electric field squared. If the charge is going simple harmonic motion, we know that the average of its acceleration squared is simply half the acceleration's amplitude squared. This is $\frac{\omega^4 x_0^2}{2}$. So:

$$P = \frac{q_e^2 \omega^4 x_0^2}{12\pi\epsilon_0 c^3}$$

So we have seen that an oscillating charge loses energy at a rate P. So now we want to find the charge's Q (ratio of stored energy to work done per radian).

$$Q = -\frac{E}{\frac{\mathrm{d}E}{\mathrm{d}\theta}}$$

We know that the charge oscillates with maximum velocity ωx_0. So its maximum kinetic energy is $m_e \omega^2 x_0^2/2$. Thus its total stored energy is simply $m_e \omega^2 x_0^2/2$ (when it achieves maximum velocity, it has 0 potential energy).

$$= -\frac{m_e \omega^2 x_0^2}{2\frac{\mathrm{d}E}{\mathrm{d}\theta}}$$

We see $\frac{\mathrm{d}E}{\mathrm{d}\theta} = \frac{\frac{\mathrm{d}E}{\mathrm{d}t}}{\frac{\mathrm{d}\theta}{\mathrm{d}t}} = \frac{P}{\omega}$

$$\begin{aligned} &= -\frac{m_e \omega^3 x_0^2}{\frac{q_e^2 \omega^4 x_0^2}{6\pi\epsilon_0 c^3}} \\ &= -\frac{6\pi m_e \epsilon_0 c^3}{q_e^2 \omega} \end{aligned}$$

We see that if the charge oscillates with a 'wavelength' λ, the light will have this same wavelength. This is because the second derivative of trigonometric function that describes the oscillation has the same period. So, we can say that $\lambda = \frac{2\pi c}{\omega}$.

$$= -\frac{3m_e\epsilon_0 c^2\lambda}{q_e^2}$$

Remembering that $1/Q = \gamma/w_0$, we can find γ. Also, since γ determines the width of the resonance curve, we can now find the width of an emitted spectral line for a freely vibrating atom.

$$\Delta\lambda = \frac{2\pi c}{\omega_A} - \frac{2\pi c}{\omega_B}$$

We assume $\omega_A \approx \omega_B$

$$= \frac{2\pi c\Delta\omega}{\omega^2}$$

Now, we recall that $\Delta\omega = \gamma = \frac{\omega}{Q}$ (the width of the resonance peak is given by γ).

$$\begin{aligned}
&= \frac{2\pi c}{Q\omega} \\
&= \frac{\lambda}{Q} \\
&= \frac{q_e^2}{3m_e\epsilon_0 c^2} \\
&= \frac{1.6^2 \cdot 10^{-38}}{3 \cdot 9.109 \cdot 10^{-31} \cdot 8.85 \cdot 10^{-12} \cdot 9 \cdot 10^{16}} \\
&\approx 1.18 \cdot 10^{-14} m
\end{aligned}$$

This closely matches experimental data.

6.1 Light Scattering

Say we have light incident on an atom so that the atom oscillates with:

$$\hat{x} = \frac{q_e\hat{E}_0}{m_e(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

But we estimate that γ is small.

$$\hat{x} = \frac{q_e\hat{E}_0}{m_e(\omega_0^2 - \omega^2)}$$

So now P is:

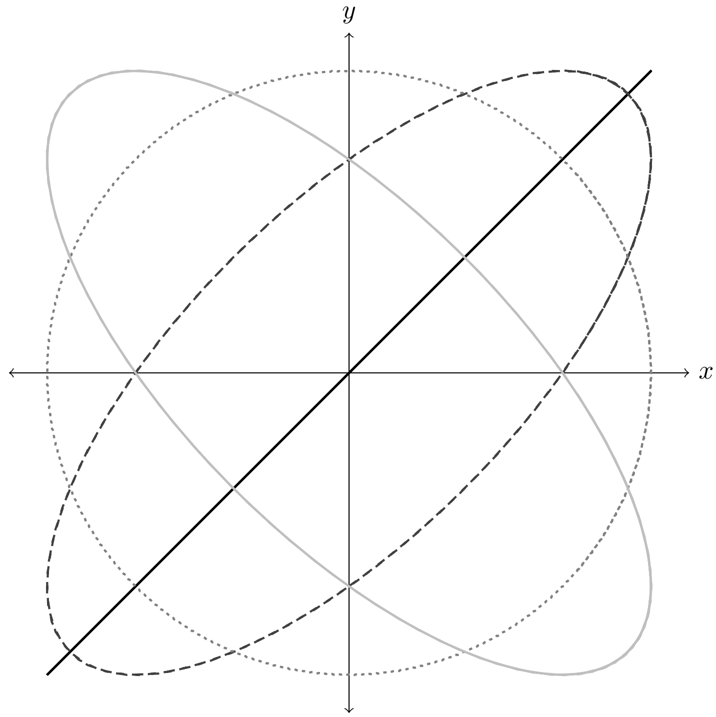
$$\begin{aligned}
P &= \frac{q_e^2\omega^4 q_e^2 E_0^2}{12\pi\epsilon_0 c^3 m_e^2(\omega_0^2 - \omega^2)^2} \\
&= \frac{1}{2}c\epsilon_0 E_0^2 \cdot \frac{q_e^4}{6\pi\epsilon_0^2 c^4 m_e^2} \cdot \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}
\end{aligned}$$

We see that this is the product of the incident intensity multiplied by some constant. So the power output of the atom is this product. We can see that the second part of the product $\sigma = \frac{q_e^4}{6\pi\epsilon_0^2 c^4 m_e^2} \cdot \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}$ is the 'area' of incident light that must 'hit' the atom in order to get the total amount of scattered light.

7 Polarization

We have so far only considered the oscillation of electric field in one direction. But, if it propagates in the z direction, the field can oscillate in the x and y directions. Now, if our source oscillates in the x and y directions, it is possible that the two are not always in phase. The below shows a few of the possible ways that the net electric field vector will evolve (as a parametric function of time).

Figure 6: Electric Field Vectors

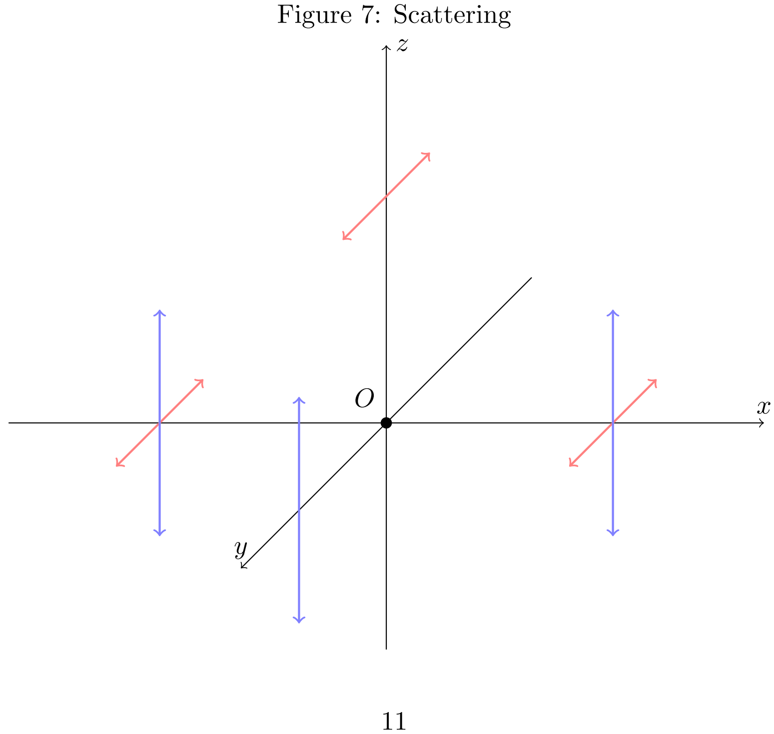


These patterns are the result of the x and y oscillations being out of phase. We note that the oscillations must have the same angular frequency for some electron orbiting an atom.

7.1 Polarization by Scattering

We see that if unpolarized light is scattered, there are specific viewing angles at 90 degrees to the direction of the light's movement that will yield polarized light.

Figure 7: Scattering



We see that if we have a particle at O, the particle will polarize light in the y and z directions if the direction of propagation is $+x$.

7.2 Birefringence

There are certain materials that have electrons that 'prefer' to oscillate in a certain direction. One such example might be cellophane, whose crystals are long and needle-like. Electrons prefer to oscillate in the direction of the needle. So for this substance, there are two different indices of refraction: one for electric fields vibrating along the cellophane's axis (which we call the optical axis), and one for electric fields vibrating perpendicular to it. Light polarized along the optical axis has a lower index of refraction than light polarized perpendicular to this axis. So, if we have light passing through cellophane, the resulting light will have a phase shift. If we had light entering such that

$$E_x = \sin \omega t$$
$$E_y = \sin \omega t$$

And the optical axis were along E_x, we would get light emerging with

$$E_x = \sin(\omega t + A)$$
$$E_y = \sin(\omega t + A + \Delta)$$

This difference in phase Δ causes the light to be differently polarized when it emerges. If $\Delta = \pi/2$, we see that originally linearly polarized light becomes circularly polarized (E_y becomes a cosine function). If $\Delta = \pi$, we see that linearly polarized light will rotate 90 degrees.

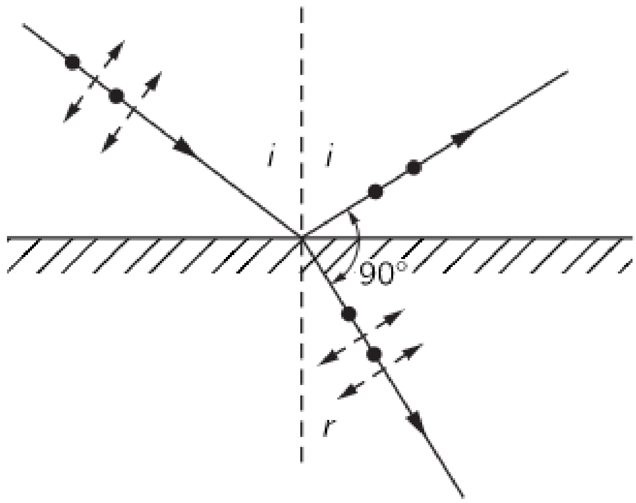
7.3 Polarizers

Polarizers work on a similar principal to birefringent substances, except instead of having a different index of refraction, they have a different index of absorption, so that they absorb light polarized in one direction extremely strongly, and leaves the light polarized in the other direction relatively untouched. We see that if polarized light enters a polarizer at an angle θ to its polarizing axis, it will leave with amplitude $\cos\theta$. Therefore it leaves with an energy $E_0 \cos^2\theta$.

7.4 Polarization by Reflection

We see that if unpolarized light hits a surface and the reflected and refracted beam make 90 degrees with one another, the reflected beam is linearly polarized in the plane of the surface. Why? If the refracted beam is unpolarized, we see that it has components of vibration which are parallel to the reflected light's direction of propagation, and it also has components of vibration which are perpendicular to the reflected light's propagation. Since the reflected beam of light is produced by vibrating charges in the surface, only the component **perpendicular** to the direction of propagation can exist in the reflected beam. This component happens to be along the plane of the material.

Figure 8: Brewster's Angle

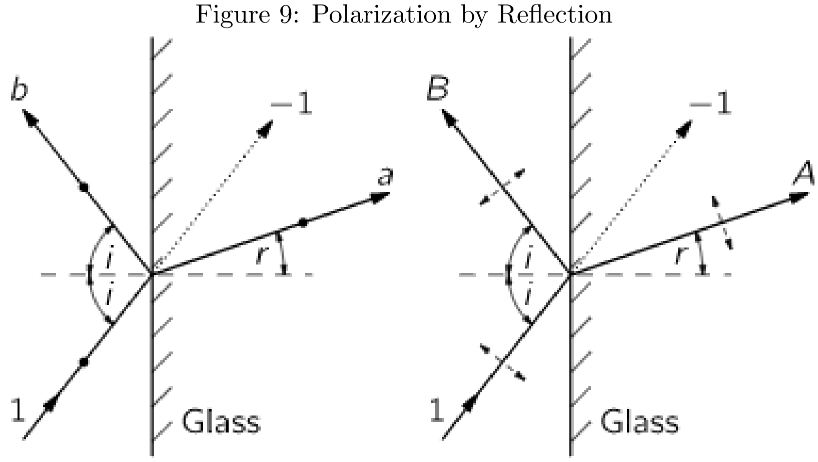


The angle at which incident light will produce purely linearly polarized light is called Brewster's angle. It is quite simple to find:

$$
\begin{aligned}
\pi/2 &= \pi/2 - \theta_B + \pi/2 - \arcsin\left(\frac{n_1}{n_2}\sin\theta_B\right) \\
\pi/2 - \theta_B &= \arcsin\left(\frac{n_1}{n_2}\sin\theta_B\right) \\
\sin(\pi/2 - \theta_B) &= \frac{n_1}{n_2}\sin\theta_B \\
\frac{\sin\theta_B}{\cos\theta_B} &= \frac{n_2}{n_1} \\
\tan\theta_B &= \frac{n_2}{n_1}
\end{aligned}
$$

We wish now to find the more general case of light hitting the surface at any angle, with two different cases of polarization.

Figure 9: Polarization by Reflection



We now observe the case when polarized light passes from a vacuum to glass. The first thing we might assume is that there is probably some ratio $b/a = B/A$ that characterizes the case on the left as well as that on the right. However, we must see that $b/a \neq B/A$, since the direction of polarization is different. In the case on the right, we see that B/A is undoubtedly smaller than b/a since the vibration in B can only take components normal to its propagation. Thus it picks up a portion of A proportional to $\cos(i+r)$. Now:

$$\frac{b}{a} = \frac{B}{A\cos(i+r)}$$

We also see now that the beam of a and A must **both** produce a field of amplitude -1 along the dotted line in the figure. We can say this because we do not observe a beam along this dotted line, so the oscillating charges in the glass must exactly cancel the incident electric field. We assume that along this dotted path, the field produced is related by some coefficient C to the field in the paths a and A. We see that:

$$-1 = C \cdot a$$
$$-1 = C \cdot A\cos(i-r)$$

We have a factor $\cos(i-r)$ because we take the direction of oscillation in A that is perpendicular to the dotted path. So:

$$1 = \frac{a}{A\cos(i-r)}$$
$$A\cos(i-r) = a$$

Thus

$$\frac{B}{b} = \frac{\cos(i+r)}{\cos(i-r)}$$

We can finally solve for both B and b individually by using conservation of energy. The refracted and reflected beams must have an energy that is equal to the incident beam. So:

$$1 = A^2 + B^2$$
$$1 = a^2 + b^2$$

With some substitution, we see:

$$|b|^2 = \frac{\sin^2(i-r)}{\sin^2(i+r)}$$
$$|B|^2 = \frac{\tan^2(i-r)}{\tan^2(i+r)}$$

These two b^2 and B^2 are the two coefficients of reflection for the two cases of polarization.

8 Relativistic Effects in Light

Before, we always neglected to include motion in the z direction, since if the line of sight is $\vec{z}$, motion along z will not change e_r much. However, what it **does** change is the retardation of e_r'. If we estimate that the x and y position of the charge do not vary very much, we can say that changes in the z position of the charge largely determine the retardation. If we call the apparent time τ, we can see that $\tau = t - \frac{R_0+z}{c}$. What this is saying is that if the time is $t = 5$ for the charge, the observer sees the charge at a smaller τ due to the retardation factor. So we can say that

$$E_x = -\frac{q}{kc^2R_0}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(\tau) = -\frac{q}{kc^2R_0}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x\left(t - \frac{R_0+z}{c}\right)$$

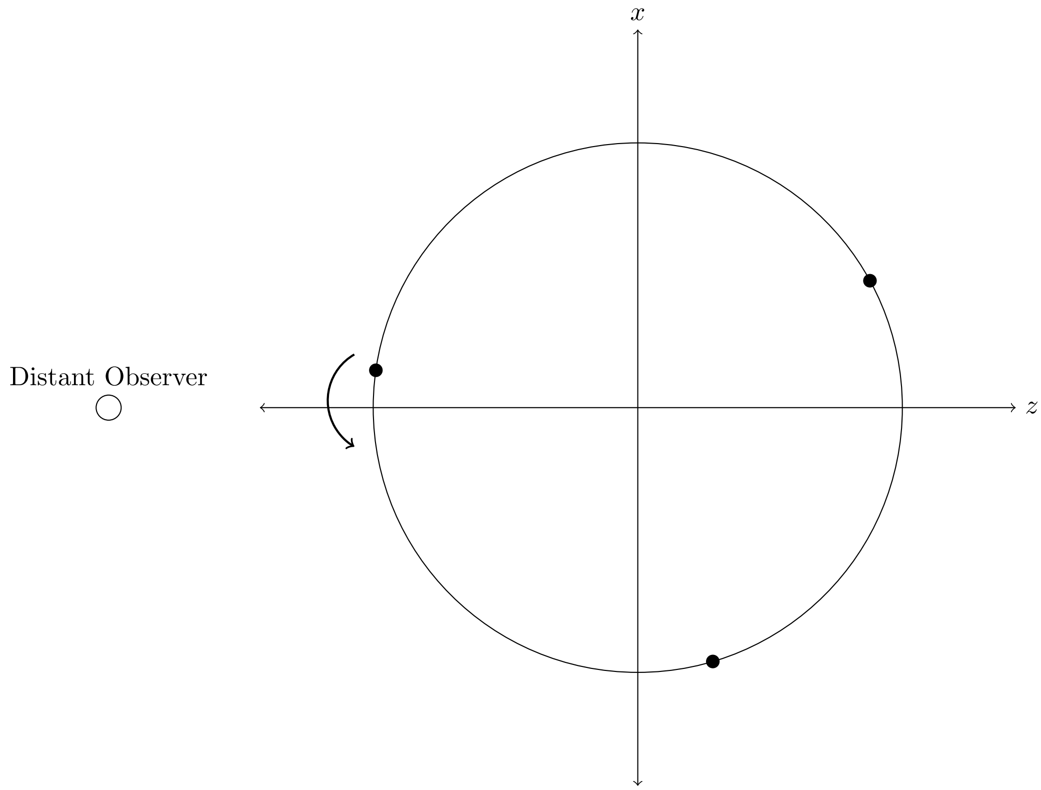
For now, we ignore the R_0/c factor since all this does is shift the origin of the time axis (R_0 is fixed). So really,

$$E_x = -\frac{q}{kc^2R_0}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t - z/c)$$

Here, we note that this simple notation is a little deceptive. We say that $\tau = t - z/c$. However, z is again a function of τ (since the delay is itself determined by the **delayed** z position). So this function is not as simple as it looks.

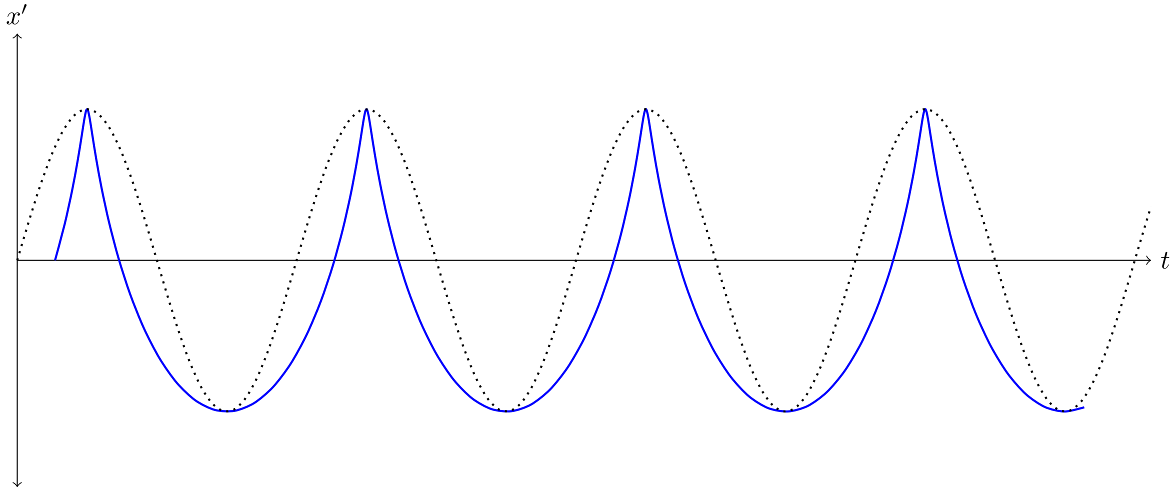
We analyze this visually. Imagine we have an electron moving at relativistic speeds in a circle:

Figure 10: Electron Path



If we say that the distance from the observer to the center of the circle is R_0, in general its distance from the observer will be around $R_0 + z$ (if the observer is very far). If the electron starts at the point farthest from the observer, its z position will be $r\cos(\omega t)$. Its x position will be $r\sin(\omega t)$. Now, the observer will only see the electron as being at $r\sin(\omega t)$ $\frac{R_0+z}{c}$ seconds **after** it was already there. So if we graph what the observer sees, we get something like this.

Figure 11: Graph of Motion



The blue line marks the apparent position of x (which we call x'), while the dotted line marks the **actual** position. Of course, we have neglected to include the effect of R_0/c here and thus when $z = 0$ there is no delay in the graph. The spacing between the dotted line and the blue line represents the delay in time caused by the variation in z. The blue line can actually be represented by a parametric system. We first see that the angular velocity ω is related to velocity by $\omega = v/r$. So:

$$\begin{cases} t = \tau + \frac{r}{c}\cos\left(\frac{v}{r}\tau\right) \\ x' = r\sin\left(\frac{v}{r}\tau\right) \end{cases} \qquad \tau \in (0, \infty)$$

By varying v, we can get different curve shapes – when $v = c$ (impossible), we get a perfect cycloid with infinitely sharp cusp.

8.1 Doppler Effect

We now analyze what happens if an oscillating charge moves towards an observer at a constant velocity v. Our new transform is $t = \tau + \frac{R_0 - v\tau}{c}$, or $\tau = \left(t - \frac{R_0}{c}\right) \cdot \frac{c}{c-v}$. Now, the new ω' is $\frac{c}{c-v}\omega = \frac{1}{1-v/c}\omega$. So if we have some charge oscillating at its natural frequency ω_0, and it moves towards an observer at speed v, the frequency of light received will be increased by a factor $\frac{1}{1-v/c}$ – but not quite. We have neglected relativistic effects. Although the charge is oscillating at ω_0 in its frame of reference, time is slowed since it's moving relative to the observer and thus the observer sees the charge oscillating at $\omega_0\sqrt{1 - v^2/c^2}$. So really, the observed frequency of the light emitted is

$$\omega' = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c}\omega$$

What if the observer is moving towards the charge at speed v? If we analyze this scenario from a perspective that is stationary relative to the charge, the observer covers an additional kv waves per second (where k is the number of radians per unit of space). However, from our stationary perspective, our seconds are shorter than the observer's perspective so frequency is increased (from the observer's point of view) by a factor of $1/\sqrt{1 - v^2/c^2}$. So now

$$\omega' = \frac{\omega + kv}{\sqrt{1 - v^2/c^2}}$$

But k is $1/\lambda = \omega/c$

$$= \frac{\omega + \frac{\omega v}{c}}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{1 + v/c}{\sqrt{1 - v^2/c^2}}\omega$$

The two expressions we have for a moving observer and a moving charge are actually equivalent, which is good because this preserves relativity. Notably, if we look at the expression for ω' with k again, it looks remarkably like a Lorentz transform. If we use the relationship $k = \omega/c$, we see now we have two transforms:

$$\omega' = \gamma(\omega + kv)$$

If we divide the above relationship by c, we get an expression for k':

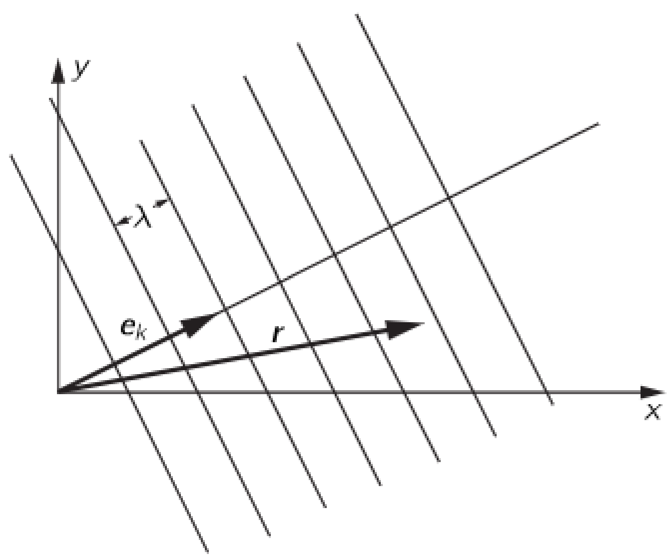
$$k' = \gamma\left(k + \frac{kv}{c}\right)$$
$$= \gamma\left(k + \frac{\omega v}{c^2}\right)$$

These are in identical form to the Lorentz transform, except k' is thought of as analogous to x' divided by c^2.

8.2 Applying Relativity to Waves

Say we have a wave travelling as so:

Figure 12: Travelling Wave



The equation of the wave at the origin is $\cos(\omega t)$. However, at $\vec{r}$, we are $k\vec{e_k} \cdot \vec{r}$ radians behind the origin, so the equation of the wave is $\cos(\omega t - k\vec{e_k} \cdot \vec{r})$. It is convenient to write a new vector $\vec{k} = k\vec{e_k}$ for simplicity, so that now the wave is $\cos\left(\omega t - \vec{k} \cdot \vec{r}\right) = \cos(\omega t - k_x x - k_y y - k_z z)$. Since k is the rate of change of phase with respect to position, k_x is larger than k by a factor $\sec\theta$ if θ is the angle between the direction of propagation and the x axis. Similar arguments apply for y and z. Now, we have four quantities ω, k_x, k_y, and k_z that transform like time and position.