Oscillation

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1 Introduction

The most simple case of oscillation is the harmonic oscillator – for example, an ideal spring system. The spring exerts a force $F = -kx$, giving us enough to find the equation of motion for a spring.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}x$$

We propose the solution:

$$x = A\cos\left(\sqrt{\frac{k}{m}}t + \Delta\right)$$

And we can verify by deriving twice:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}\cdot A\cos\left(\sqrt{\frac{k}{m}}t + \Delta\right)$$

We see now three things. The amplitude A of the oscillation can be anything and does not depend on the characteristics of the spring. The angular frequency (denoted by ω) is given by $\sqrt{\frac{k}{m}}$. It can also have a certain 'phase shift' given by Δ – this is if the oscillating mass is given an initial velocity. We see why this is by observing the x position of the mass is an analogue to the x position of a body undergoing uniform circular motion with angular velocity $\sqrt{\frac{k}{m}}$. Adding a Δ simply means 'fast forwarding' the motion so that the x position of the body at $t = 0$ is equal to the x position of another body with $\Delta = 0$ at a *later* time $t = \tau$. These three factors, amplitude A, angular frequency ω, and phase shift Δ define the motion of the spring completely.

2 Forced Oscillation

We have a theorem that $e^{i\omega t} = \cos\omega t + i\sin\omega t$. This is actually quite useful in analysis of differential equations, since the derivative of an exponential is found by simple multiplication. So oftentimes, we will propose a solution in complex form, then convert it to a real solution by taking the real part of the expression $\cos\omega t$.

For example, say we have a mass undergoing oscillation in a spring system. We apply an oscillating external force onto the object with an amplitude of F_0, an angular frequency of ω, and a phase shift of Δ. Now,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F_0}{m}\cos(\omega t + \Delta) - \frac{k}{m}x$$

We convert this to 'complex' form:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F_0}{m}e^{i(\omega t + \Delta)} - \frac{k}{m}x$$

We see that $e^{i(\omega t+\Delta)} = e^{i\Delta}e^{i\omega t}$. So we rewrite $F_0 e^{i(\omega t+\Delta)}$ as $\hat{F}e^{i\omega t}$. Now the phase shift is 'held' in $\hat{F}$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\hat{F}}{m}e^{i\omega t} - \frac{k}{m}x$$

We now postulate that x depends on ω, rather than $\frac{k}{m}$, so that:

$$x = x_0 e^{i(\omega t+\Delta)} = \hat{x}e^{i\omega t}$$

Now, deriving is easy because we have an exponential.

$$-\omega^2\hat{x}e^{i\omega t} = \frac{\hat{F}}{m}e^{i\omega t} - \frac{k}{m}\hat{x}e^{i\omega t}$$

$$-\omega^2\hat{x} = \frac{\hat{F}}{m} - \frac{k}{m}\hat{x}$$

We now note that $\frac{k}{m} = \omega_0^2$, the 'natural' angular frequency of the spring.

$$\hat{x}\big(\omega_0^2 - \omega^2\big) = \frac{\hat{F}}{m}$$

$$x_0 = \frac{\hat{F}}{m(\omega_0^2 - \omega^2)}$$

Now we may write out what x is:

$$x = \mathrm{Re}\left(\frac{\hat{F}e^{\omega t}}{m(\omega_0^2 - \omega^2)}\right)$$

$$x = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t + \Delta)$$

$$x = \frac{F}{m(\omega_0^2 - \omega^2)}$$

We now see that if ω exceeds ω_0^2, x oscillates opposite to the force. On the other hand, if ω is less than ω_0, the spring is dominant. We see why by looking at the net force on the spring.

$$\begin{aligned}
x &= \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t + \Delta) \\
F_{external} &= F_0\cos(\omega t + \Delta) \\
F_{spring} &= -k\frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t + \Delta) \\
F_{net} &= F_0\cos(\omega t + \Delta) - k\frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t + \Delta) \\
&= F_0\cos(\omega t + \Delta)\left(1 - \frac{k}{m(\omega_0^2 - \omega^2)}\right) \\
&= -F_0\cos(\omega t + \Delta)\cdot\frac{\omega^2}{\omega_0^2 - \omega^2}
\end{aligned}$$

We see now that F_{net} will always be opposite to the displacement, as required. Why? If $\omega > \omega_0$, we look what happens right after the mass hits equilibrium. The applied force is opposite to the displacement. The force the spring exerts is **also** opposite to the displacement. So net force is negative. On the other hand, if $\omega < \omega_0$, right after equilibrium, the applied force is the same direction as displacement. The force from the spring is opposite. However, in this case, just by doing some simple math, we can see that the magnitude of the force due to the spring will always be greater than the magnitude of applied force!

2.1 Friction

We now introduce friction. We approximate friction to be proportional to negative velocity.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F_0}{m}\cos(\omega t + \Delta) - \gamma\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{k}{m}x$$

We again convert to complex form:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\hat{F}}{m}e^{i\omega t} - \gamma\frac{\mathrm{d}x}{\mathrm{d}t} - \omega_0^2 x$$

We postulate that $x = \hat{x}e^{i\omega t}$ – but the phase shift of x need not equal the phase shift of F.

$$\begin{aligned}
-\omega^2\hat{x}e^{i\omega t} &= \frac{\hat{F}}{m}e^{i\omega t} - i\omega\gamma\hat{x}e^{i\omega t} - \omega_0^2\hat{x}e^{i\omega t} \\
-\omega^2\hat{x} &= \frac{\hat{F}}{m} - i\omega\gamma\hat{x} - \omega_0^2\hat{x} \\
\hat{x}\left(\omega_0^2 - \omega^2 + i\omega\gamma\right) &= \frac{\hat{F}}{m} \\
\hat{x} &= \frac{\hat{F}}{m(\omega_0^2 - \omega^2 + i\omega\gamma)}
\end{aligned}$$

If we now say that $\hat{x} = \rho e^{i\theta}F_0 e^{i\Delta} = \rho e^{i\theta}\hat{F}$, we find the magnitude of ρ by multiplying it by its conjugate and taking the square root.

$$\begin{aligned}
\rho &= \frac{1}{m}\sqrt{\frac{1}{(\omega_0^2 - \omega^2 + i\omega\gamma)(\omega_0^2 - \omega^2 - i\omega\gamma)}} \\
\rho &= \frac{1}{m}\sqrt{\frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}} \\
\rho^2 &= \frac{1}{m^2\left(\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2\right)}
\end{aligned}$$

Finding θ is also simple since $\frac{1}{\rho}e^{-i\theta} = m\left(\omega_0^2 - \omega^2 + i\omega\gamma\right)$

$$\begin{aligned}
e^{-i\theta} &= m\rho\left(\omega_0^2 - \omega^2 + i\omega\gamma\right) \\
\tan(-\theta) &= \frac{\omega\gamma}{\omega_0^2 - \omega^2} \\
\theta &= \arctan\frac{\omega\gamma}{\omega^2 - \omega_0^2}
\end{aligned}$$

We now make an interesting observation. Since the curve for ρ^2 and θ vary most near $\omega = \omega_0$, we see if we can find an approximation for this case.

$$\begin{aligned}
\rho^2 &= \frac{1}{m^2((\omega_0 - \omega)^2(\omega_0 + \omega)^2 + \omega^2\gamma^2)} \\
\rho^2 &= \frac{1}{m^2((\omega_0 - \omega)^2(2\omega_0)^2 + \omega_0^2\gamma^2)} \quad \text{using } \omega \approx \omega_0 \\
\rho^2 &= \frac{1}{4m^2\omega_0^2((\omega_0 - \omega)^2 + \gamma^2/4)}
\end{aligned}$$

We now see something interesting. The maximum height of this curve is clearly $\frac{1}{m^2\omega_0^2\gamma^2}$ at $\omega_0 = \omega$. We show that the width the curve at half this height is approximately γ (for small γ).

$$\begin{aligned}
\frac{1}{2m^2\omega_0^2\gamma^2} &= \frac{1}{4m^2\omega_0^2((\omega_0 - \omega)^2 + \gamma^2/4)} \\
2\gamma^2 &= 4\left(\left(\omega_0 - \omega\right)^2 + \frac{\gamma^2}{4}\right) \\
\gamma^2 &= 4\left(\omega_0 - \omega\right)^2 \\
\pm\frac{\gamma}{2} &= \omega_0 - \omega \\
\omega &= \omega_0 \pm \frac{\gamma}{2}
\end{aligned}$$

We see now that the width of the curve at half maximum height is γ!

2.2 Power

We rewrite the equation for the transient to find the power of the applied external force:

$$
\begin{aligned}
F &= m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma m\frac{\mathrm{d}x}{\mathrm{d}t} + kx \\
F\frac{\mathrm{d}x}{\mathrm{d}t} &= m\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t}kx + \gamma\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \\
P &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}kx^2\right) + \gamma m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2
\end{aligned}
$$

We see that the first two terms are kinetic energy and stored elastic potential energy, respectively. We call their sum 'store energy'. Some power is spent initially to get the stored energy to a certain level, but in the long term, it is **constant** over time. So power is purely spend on the last term.

$$
P = \gamma m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2
$$

If $x = x_0\cos(\omega t + \Delta)$:

$$
\begin{aligned}
\langle P\rangle &= \frac{1}{2}\gamma m\omega^2 x_0^2 \\
\langle P\rangle &= \langle x^2\rangle\gamma m\omega^2
\end{aligned}
$$

We now find the mean stored energy $\langle E\rangle$:

$$
\begin{aligned}
\langle E\rangle &= \frac{1}{2}m\langle(\mathrm{d}x/\mathrm{d}t)^2\rangle + \frac{1}{2}k\langle x^2\rangle \\
\langle E\rangle &= \frac{1}{2}m\cdot\frac{1}{2}\omega^2x_0^2 + \frac{1}{2}k\cdot\frac{1}{2}x_0^2 \\
\langle E\rangle &= \frac{1}{4}mx_0^2(\omega^2+\omega_0^2) \\
\langle E\rangle &= \frac{1}{2}m\langle x^2\rangle(\omega^2+\omega_0^2)
\end{aligned}
$$

Now there is a quantity Q defined as the ratio of stored energy to the work done per radian, which we will find:

$$
\begin{aligned}
Q &= \frac{\frac{1}{2}m\langle x\rangle(\omega^2+\omega_0^2)}{\langle x\rangle\gamma m\omega^2\cdot\frac{1}{\omega}} \\
&= \frac{\omega^2+\omega_0^2}{2\gamma\omega}
\end{aligned}
$$

A high Q tells us that we have a good oscillator since it loses very little energy to friction.

3 Transients

A transient event is a short-lived burst of energy in a system caused by a sudden change of state. One example is letting go of a spring in a system with friction. We study the change in energy in an infinitesimal interval of time

$$
\begin{aligned}
\mathrm{d}E &= -\frac{E}{Q}\omega\,\mathrm{d}t \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= -\frac{E\omega}{Q} \\
E &= E_0e^{-\frac{\omega}{Q}t}
\end{aligned}
$$

Since E is proportional to the square of displacement, we estimate that displacement decreases *half* as fast as energy ($\sqrt{e^{-\frac{\omega}{Q}t}} = e^{-\frac{\omega}{2Q}t}$). So we say that:

$$x = A_0 e^{-\frac{\omega}{2Q}t}\cos(\omega t)$$

This is merely a guess. We begin finding the real solution by saying:

$$x = Ae^{i\alpha t} \quad \text{where } \alpha \text{ and } A \text{ are complex numbers}$$

We now plug this into the equations of motion:

$$\begin{aligned}
\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= -\frac{k}{m}x - \gamma\frac{\mathrm{d}x}{\mathrm{d}t} \\
-\alpha^2 A e^{i\alpha t} &= -\omega_0^2 A e^{i\alpha t} - \gamma i\alpha A e^{i\alpha t} \\
-\alpha^2 &= -\omega_0^2 - \gamma\alpha i \\
0 &= -\alpha^2 + \alpha\gamma i + \omega_0^2 \\
\alpha &= \frac{-\gamma i \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{-2} \\
\alpha &= \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}
\end{aligned}$$

For brevity, we call $\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ ω_γ. So our solutions are:

$$\begin{aligned}
x &= A_0 e^{i\frac{i\gamma}{2} \pm \omega_\gamma t} \\
x &= A_0 e^{-\frac{\gamma}{2}t} \cdot e^{\pm i\omega_\gamma t}
\end{aligned}$$

Due to the properties of a linear differential equation, if two solutions are possible, the 'super set' solution is any linear combination of the two.

$$x = e^{-\gamma t/2}\left(Ae^{i\omega_\gamma t} + Be^{-i\omega_\gamma t}\right)$$

However, we see that x will not be real unless $B = A^*$

$$x = e^{-\gamma t/2}\left(Ae^{i\omega_\gamma t} + A^* e^{-i\omega_\gamma t}\right)$$

We now solve for A by plugging in initial conditions, initial displacement x_0 and initial velocity v_0:

$$\begin{aligned}
x_0 &= A + A^* \\
\mathrm{Re}(A) &= \frac{x_0}{2}
\end{aligned}$$

We now derive with respect to time to find velocity:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\gamma/2 e^{-\gamma t/2}\left(Ae^{i\omega_\gamma t} + A^* e^{-i\omega_\gamma t}\right) + e^{-\gamma t/2}\left(i\omega_\gamma A e^{i\omega_\gamma t} - i\omega_\gamma A^* e^{-i\omega_\gamma t}\right) \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= e^{-\gamma t/2}\left((i\omega_\gamma - \gamma/2)Ae^{i\omega_\gamma t} - (i\omega_\gamma + \gamma/2)A^* e^{-i\omega_\gamma t}\right) \\
v_0 &= Ai\omega_\gamma - A\gamma/2 - A^* i\omega_\gamma - A^*\gamma/2 \\
v_0 &= -2\,\mathrm{Im}(A)\omega_\gamma - \mathrm{Re}(A)\gamma \\
\mathrm{Im}(A) &= -\frac{v_0 + \gamma x_0/2}{2\omega_\gamma}
\end{aligned}$$

We have now found A, so we plug it back into the original equation:

$$x = e^{-\gamma t/2}\left(\left(x_0/2 - i\frac{v_0 + \gamma x_0/2}{2\omega_\gamma}\right)e^{i\omega_\gamma t} + \left(x_0/2 + i\frac{v_0 + \gamma x_0/2}{2\omega_\gamma}\right)e^{-i\omega_\gamma t}\right)$$

We use the fact that $Ae^{ix} + Ae^{-ix} = 2A\cos(x)$ and $-iBe^{ix} + iBe^{-ix} = 2B\sin(x)$

$$x = e^{-\gamma t/2}\left(x_0\cos(\omega_\gamma t) + \frac{v_0 + \gamma x_0/2}{\omega_\gamma}\sin(\omega_\gamma t)\right)$$

3.1 Large γ

However, in our solution for α, as assumed that $4\omega_0^2 \geq \gamma^2$. What if it isn't?

$$\alpha = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$
$$\alpha = \frac{i\gamma}{2} \pm i\sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

We plug this into $x = Ae^{i\alpha t}$. Also, for brevity we write $\sqrt{\gamma^2/4 - \omega_0^2} = \omega_\gamma$.

$$x = Ae^{i\left(\frac{i\gamma}{2} \pm i\omega_\gamma\right)t}$$
$$x = Ae^{(-\gamma/2 \mp \omega_\gamma)t}$$

Again, we accept both solutions.

$$x = Ae^{(-\gamma/2+\omega_\gamma)t} + Be^{(-\gamma/2-\omega_\gamma)t}$$

Again, we plug in initial conditions x_0 and v_0.

$$x_0 = A + B$$

We derive with respect to time:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(-\gamma/2+\omega_\gamma)e^{(-\gamma/2+\omega_\gamma)t} + B(-\gamma/2-\omega_\gamma)e^{(-\gamma/2-\omega_\gamma)t}$$
$$v_0 = A(-\gamma/2+\omega_\gamma) + B(-\gamma/2-\omega_\gamma)$$
$$v_0 = A(-\gamma/2+\omega_\gamma) + x_0(-\gamma/2-\omega_\gamma) - A(-\gamma/2-\omega_\gamma) \quad \text{since } B = x_0 - A$$
$$v_0 = 2A\omega_\gamma - \frac{\gamma x_0}{2} - x_0\omega_\gamma$$
$$A = \frac{v_0 + x_0\omega_\gamma + \gamma x_0/2}{2\omega_\gamma} \textbf{ and } B = x_0 - A$$

4 Electrical Circuits

There is an electrical analogue for the mechanics of this circuit, with an inductor of inductance L, capacitor of capacitance C, and resistor of resistance R:

$$V(t) = L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C}$$
$$\frac{V(t)}{L} = \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{LC}$$

We see that L is the analogue of mass (since in an inductor, the current doesn't 'want' to stop, it is almost like inertia). R of course acts as friction, $1/C$ acts as the 'stiffness' of the system (spring constant). Using the solution we found before, we see that:

$$\hat{q} = \frac{\hat{V}}{L(i\omega)^2 + R(i\omega) + 1/C}$$
$$= \frac{\hat{V}}{L(-\omega^2 + i\omega R/L + 1/LC)}$$

If we call $\frac{1}{LC} = \omega_0^2$ and call $\frac{R}{L} = \gamma$

$$= \frac{\hat{V}}{L(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

5 Superposition

In any linear differential equation, if we equate it to some function F that can be expressed as the sum of two other functions f and g, then the solution of the differential equation is the sum of the solutions for f and g:

$$F = \sum a_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n}$$
$$f + g = \sum a_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n}$$

We now solve for

$$f = \sum a_n \frac{\mathrm{d}^n x_f}{\mathrm{d}t^n}$$
$$\textbf{and}$$
$$g = \sum a_n \frac{\mathrm{d}^n x_g}{\mathrm{d}t^n}$$

We can now say that:

$$x = x_f + x_g$$

This is quite interesting. It means:

- If we can break down any force into a sum of simple, sinusodial forces, and solve for those forces, we can recombine their solutions to find the overall response of the system.
- It explains how we can tune the radio to different stations. If we have two stations, one broadcasting in a sinusodial function of frequency 40 hz, and the other at 50hz, the radio receives some signal that is a superposition of the two. If the two are functions f and g, the radio receives $f + g$. The response will be $x_f + x_g$. If we want to only listen to g, we will adjust the natural angular frequency w_0 of the system so that the amplitude of x_g is large and the amplitude of x_f is small. Now the overall response is *dominated* by x_g (with a small element of x_f). With a large γ, the difference in magnitude between x_g and x_f increases (recall how the shape of the 'maximum amplitude' varies as we increase γ - it decreases).

We now look at a special case. What if $F = 0$? Then we see that if we have a number N of solutions, a general solution can be expressed as any linear combination of them. N is called the number of degrees of freedom a system has. Also, if F is written as $F = f + g$, and $g = 0$, we see that the solution for the overall linear equation can be the sum of the solution for F and for 0! Physically, this means that if we have a spring with a force being applied, the solution can have a transient in it that dies out over time (since this transient is the solution to $g = 0$).