Portfolio Construction

Using Black-Litterman Model and Factors

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by

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Preface

I am delighted to present this final project, the culmination of my journey through the Certificate in Quantitative Finance (CQF) program. This report focuses on the implementation of the Black-Litterman model for portfolio construction, a subject deeply influenced by the valuable knowledge gained during my time in the CQF program.

I am sincerely grateful to Dr. Paul Wilmott, Professor Sébastien Lleo, Dr. Riaz Ahmad, Dr. Panos Parpas, Kannan Singaravelu, and Dr. Steve Phelps for their unwavering support, guidance, and insightful lessons on Quantitative Finance. Their expertise has been instrumental in shaping my understanding and approach to the complexities of this field.

In this paper, I explore the implementation of the basic Black-Litterman model, Mean-Variance Optimization, and the Maximize Sharpe Ratio optimization methodologies. While the report delves into these fundamental aspects, it does not incorporate investors' specific uncertainties or more intricate optimization techniques. This project serves as an initial step in my quantitative finance journey, and I aim to further develop my skills and contribute to the growth of this field.

I extend my gratitude to the CQF program, which is well-constructed and comprehensive. Among the numerous modules, the portfolio optimization and option pricing theory have been particularly enlightening and enjoyable. Portfolio construction, in particular, has emerged as one of my favourite subjects, and I am eager to witness its real-life applications.

As I embark on the next phase of my career in quantitative finance, I look forward to continued learning, growth, and the opportunity to apply the principles and methodologies explored in this report in practical, real-world scenarios.

Thank you for joining me on this academic and professional journey.

VAN HOANG BAO BACH
Bangkok, January 2024

Contents

Introduction

The Black-Litterman asset allocation model, devised by Fischer Black and Robert Litterman, stands out as a sophisticated method for constructing portfolios, adept at addressing issues like the creation of counterintuitive, highly concentrated portfolios, sensitivity to inputs, and the maximization of estimation errors.

Within the expansive investment universe, various options exist for constructing portfolios. Investors or fund managers may choose assets and assign weights using three primary categories: Heuristic Multi-factor Construction, Optimized Multi-factor Construction, and Risk-Based Multi-factor Construction as mentioned by Lin (2020). A prevalent method for developing multi-factor weightings involves a simple heuristic approach. A comprehensive factor score for any stock, derived heuristically, takes the form:

$$\alpha_i = 0.2 \times F_{1,i} + 0.2 \times F_{2,i} + 0.2 \times F_{3,i} + 0.2 \times F_{4,i} + 0.2 \times F_{5,i}$$

where $F_{j,i}$ denotes the factor exposure of each security i to the target factor j, and the values for each factor are arbitrarily chosen. This does not imply any specific method to assign these numbers.

In this model, investors select parameters to emphasize specific risk factors aligned with their beliefs. However, this method is highly subjective and biased, relying on the fund manager's opinions and lacking statistical evidence.

While the heuristic multi-factors model may not appeal to quantitative portfolio managers, the idea of using risk factors and parameters to evaluate portfolios holds promise.

This paper aims to summarize best practices for multi-factor portfolio construction using the Black-Litterman (BL) model. Part 1 presents the asset choices, analyzes returns and risks before their integration into the portfolio. Part 2 develops the allocation process using Modern Portfolio Theory and the Black-Litterman model, incorporating the level of confidence in investor views to control the magnitude of tilts caused by views.

1

Factor Data and Study

> To reduce risk it is necessary to avoid a portfolio whose securities are all highly correlated with each other. One hundred securities whose returns rise and fall in near unison afford little protection than the uncertain return of a single security.
>
> *Harry Markowitz*

This chapter introduces a method for selecting financial instruments to construct the portfolio and provides the rationale for choosing specific assets. The objective of this process is to choose uncorrelated assets, promoting diversification across various sectors of the capital market. The U.S. market is chosen for simplicity due to its high liquidity and data availability. However, this choice exposes the portfolio to market risk and lacks global representation. To illustrate, the steps outlined below can be replicated with different asset classes to construct a more well-rounded portfolio. In summary, the initial task in the portfolio construction process is to select assets that facilitate diversification within the U.S. capital market.

1.1. Factor Investing

Investors are increasingly adopting an alternative weighted indexing approach to traditional capital weighting. Factor Investing, also known as Smart Beta-based strategies, is experiencing rapid growth and gaining favour among investors.

A common method for creating multi-factor weightings, as mentioned earlier, is to apply a simple heuristic approach. The primary advantage of the heuristic weighting approach lies in its simplicity. It does not necessitate sophisticated optimization tools and avoids reliance on any single factor. However, this simplicity comes at a cost: higher tracking errors, suboptimal alpha (factor active return) generation, and inferior risk-adjusted performance (alpha).

Another approach aims to minimize risk using the mean-variance optimization method within a specified range. This is the optimized multi-factor approach. The objective function for minimal risk optimization could be expressed as follows:

$$\begin{aligned} \min_w \bar{\sigma}_p &= \frac{1}{2} w^\top \Sigma w \\ \text{s.t.} \quad & w^\top \mathbf{1} = 1 \end{aligned} \tag{1}$$

The vulnerability to modelling errors arises from computing the covariance matrix Σ, and as the number of assets in the portfolio increases, it becomes increasingly challenging to approximate the covariance matrix. The problem becomes more complex and often unsolvable when expanding the portfolio to include more assets.

Another widely used approach is the Risk-Based Multi-factor method. One of the most recognized models is the Fama-French 3-factor model Fama and French (1993). This model enhances the Capital Asset Pricing Model (CAPM) and incorporates three factors to explain a security's undiversified risk: market (Beta), size (SMB), and value (HML):

$$r = R_f + \beta(R_m - R_f) + b_s \times \text{SMB} + b_v \times \text{HML} + \alpha \tag{2}$$

The Fama-French model demonstrates the correlation between portfolio return and the risk factors of the market. Data for these factors, as well as the risk-free rate for the U.S. market, is well-documented on this website: Current Research Returns by Kenneth R. French. This data will be used later to evaluate the portfolio return against the benchmark.

1.2. Picking Asset Classes

The first task is to include different asset classes. Bonds, equities, and commodities are all considered for the portfolio since the goal is to diversify and capture most of the movements of the financial markets. Ideally, risks from different asset classes could be compensated by combining and hedging. For simplicity, I will only select assets from the U.S. market for the portfolio.

The second task is to incorporate factors such as the growth factor, value factor, quality factor, small/large market cap factors, and momentum, which are also taken into consideration. Inspired by findings from Fama and French (1993) and Fama and French (2012), I will select different stocks that are part of ETFs representing those factors. For each factor, I will pick 5 stocks that have the highest allocation in the ETFs holdings. From the daily return data, the technology industry has the highest growth against the market benchmark. Finally, I will use the Momentum ETF to proxily represent the momentum factor in the portfolio. Even though there are some correlations between asset returns, they are weakly correlated, as desired for the optimal result.

I will select 20 stocks, 2 bonds, 2 commodity ETFs, and 1 momentum ETF for the portfolio. The details of those assets are listed in **Table 1.1**. I also include the market benchmark and the risk-free rate for evaluating the portfolio. Analyzing the daily returns of these assets, there are some high correlations between tech stocks.

1.3. Filtering Highly Correlated Assets

For the best possible diversified portfolio, I will apply a filter to remove highly correlated assets. The goal of this paper is not only to implement the Black-Litterman model but also to select a widely diversified portfolio, i.e., less correlated assets. The correlation between assets is shown in **Figure 1.1**.

After the filtering process, I only keep 8 asset classes: Berkshire Hathaway Inc, UnitedHealth Group Incorporated, Exxon Mobil Corporation, Eli Lilly and Company, Mastercard Incorporated, PTC Inc, Atmos Energy Corporation, and United States Oil Fund LP. The result is listed in **Table 1.2**.

1.4. Asset Returns Analysis

From the summary statistics of the return data in **Table 1.3**, Exxon Mobil Corporation has the highest empirical return across the portfolio. PTC Inc has the lowest standard deviation, indicating this is a low-risk asset. Atmos Energy Corporation has a negative mean return, indicating that investors are losing money on average when investing in the company. Berkshire Hathaway Inc and Mastercard Incorporated returns move in sync with the S&P 500 index.

The data shown in **Figure 1.2** visualize the cumulative returns of each asset against the benchmark. Clearly, Exxon Mobil Corporation beat the market by a staggering $1,151\%$ at 31^{st} of July, 2023. Even though it is tempting to invest 100% of the wealth in only one stock, the risk of doing so will wipe out the returns from that strategy during market turbulence.

Figure 1.1: Asset Return Correlation Matrix

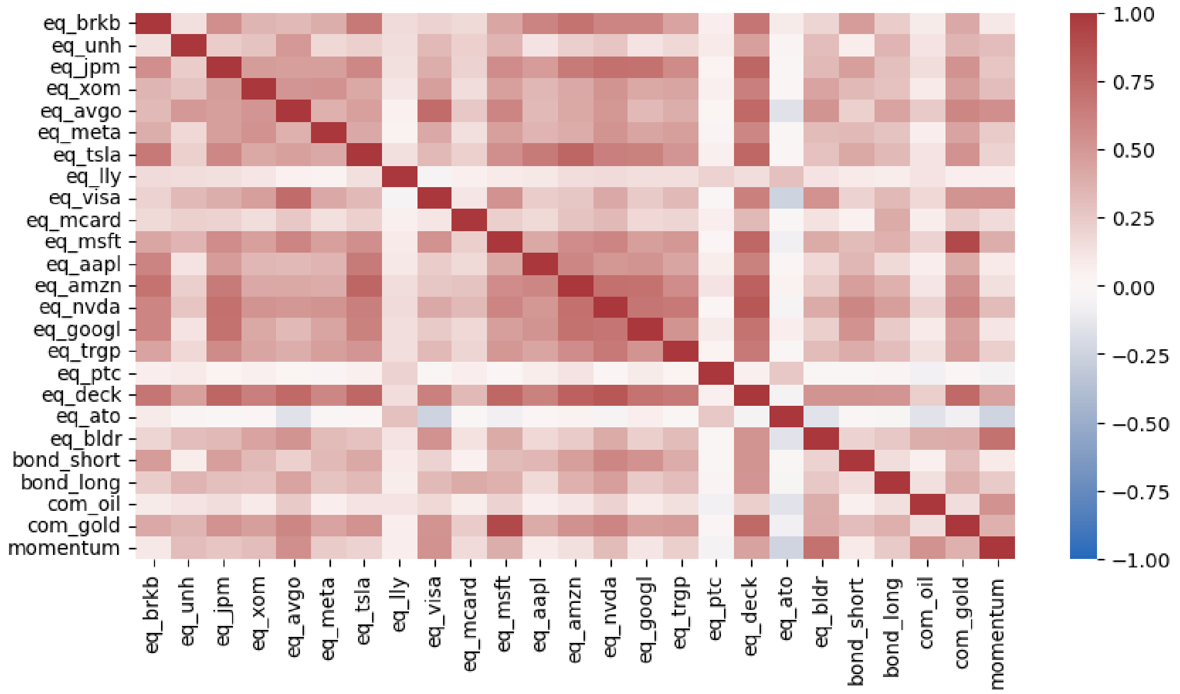


Table 1.1: List of Financial Instruments and Variable Names

Index	Name	Ticker	Variable	Factor
0	SPDR S&P 500 ETF Trust	SPY	sp500	Benchmark
1	Berkshire Hathaway Inc.	BRK.B	eq_brkb	Equity Value
2	UnitedHealth Group Incorporated	UNH	eq_unh	Equity Value
3	JPMorgan Chase & Co.	JPM	eq_jpm	Equity Value
4	Exxon Mobil Corporation	XOM	eq_xom	Equity Value
5	Broadcom Inc.	AVGO	eq_avgo	Equity Value
6	Meta Platforms, Inc.	META	eq_meta	Equity Growth
7	Tesla, Inc.	TSLA	eq_tsla	Equity Growth
8	Eli Lilly and Company	LLY	eq_lly	Equity Growth
9	Visa Inc.	V	eq_visa	Equity Growth
10	Mastercard Incorporated	MA	eq_mcard	Equity Growth
11	Microsoft Corporation	MSFT	eq_msft	Equity Large Cap
12	Apple Inc.	AAPL	eq_aapl	Equity Large Cap
13	Amazon.com, Inc.	AMZN	eq_amzn	Equity Large Cap
14	NVIDIA Corporation	NVDA	eq_nvda	Equity Large Cap
15	Alphabet Inc.	GOOGL	eq_googl	Equity Large Cap
16	Targa Resources Corp.	TRGP	eq_trgp	Equity Low Cap
17	PTC Inc.	PTC	eq_ptc	Equity Low Cap
18	Deckers Outdoor Corporation	DECK	eq_deck	Equity Low Cap
19	Atmos Energy Corporation	ATO	eq_ato	Equity Low Cap
20	Builders FirstSource, Inc.	BLDR	eq_bldr	Equity Low Cap
21	iShares Short Treasury Bond ETF	SHV	bond_short	Short-term Bond
22	iShares 20+ Year Treasury Bond ETF	TLT	bond_long	Long-term Bond
23	United States Oil Fund LP	USO	com_oil	Commodity
24	iShares Gold Trust	IAU	com_gold	Commodity
25	iShares MSCI USA Momentum Factor ETF	MTUM	momentum	Momentum Factor
26	Risk-free rate - Kenneth French Data	N/A	rf_rate	Risk-free Rate

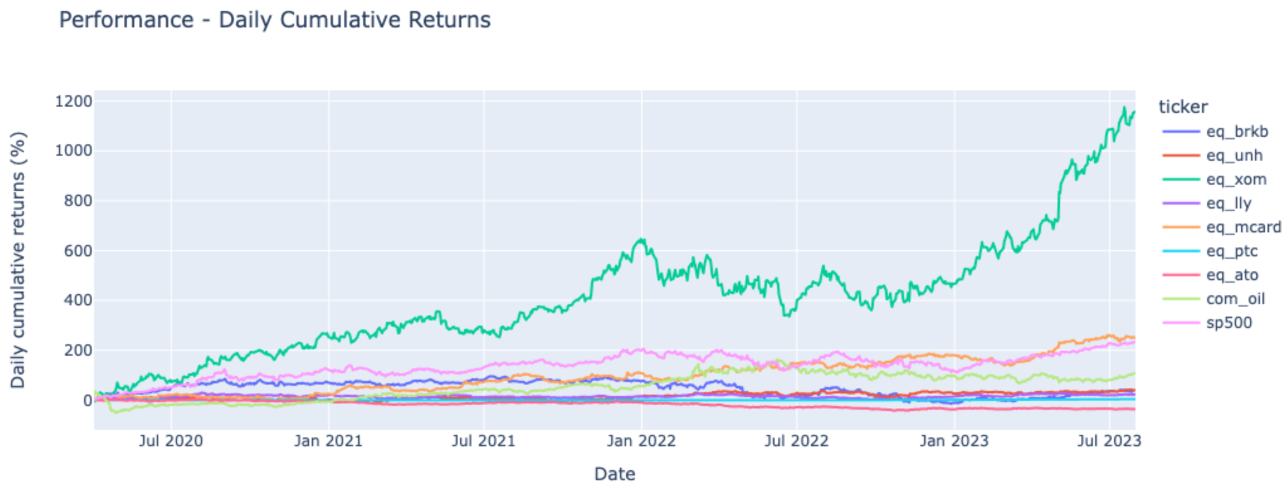
Table 1.2: Filtered Assets for Portfolio Construction

Index	Name	Ticker	Variable	Factor
1	Berkshire Hathaway Inc.	BRK.B	eq_brkb	Equity Value
2	UnitedHealth Group Incorporated	UNH	eq_unh	Equity Value
3	Exxon Mobil Corporation	XOM	eq_xom	Equity Value
4	Eli Lilly and Company	LLY	eq_lly	Equity Growth
5	Mastercard Incorporated	MA	eq_mcard	Equity Growth
6	PTC Inc.	PTC	eq_ptc	Equity Low Cap
7	Atmos Energy Corporation	ATO	eq_ato	Equity Low Cap
8	United States Oil Fund LP	USO	com_oil	Commodity

Table 1.3: Daily Returns Descriptive Statistics of Asset Classes

Variable	Count	Mean	Std	Min	25%	50%	75%	Max
eq_brkb	837.00000	0.00069	0.02383	-0.14049	-0.01231	0.00107	0.01310	0.13536
eq_unh	837.00000	0.00051	0.01495	-0.07380	-0.00746	0.00077	0.00881	0.10057
eq_xom	837.00000	0.00353	0.03188	-0.08979	-0.01470	0.00380	0.02034	0.17415
eq_lly	837.00000	0.00029	0.00958	-0.05334	-0.00467	0.00058	0.00574	0.03068
eq_mcard	837.00000	0.00168	0.01906	-0.09093	-0.00848	0.00105	0.01014	0.15680
eq_ptc	837.00000	0.00004	0.00017	-0.00082	0.00000	0.00000	0.00009	0.00118
eq_ato	837.00000	-0.00048	0.01039	-0.03418	-0.00763	-0.00079	0.00613	0.03847
com_oil	837.00000	0.00127	0.02770	-0.25067	-0.01180	0.00232	0.01614	0.16667
sp500	837.00000	0.00163	0.01965	-0.08006	-0.00913	0.00123	0.01347	0.10469

Figure 1.2: Daily Cumulative Returns (%)



2

Black-Litterman Implementation

> One of the things I like about doing science, the thing that is the most fun, is coming up with something that seems ridiculous when you first hear it but finally seems obvious when you're finished.
>
> *Fischer Black*

The next step in the portfolio optimization process is to allocate the weight to each asset in the portfolio. Consider the investment universe with N assets: r_f is the risk-free rate, $\mu \in \mathbb{R}^N$ the vector of expected returns and let $\Sigma \in \mathbb{R}^{N \times N}$ be the positive semidefinite sample covariance matrix. The goal is to find the optimal allocation for the weight vector w.

There are several methods to solve this problem. In this section, I will use the fixed weight allocation method, maximum Sharpe ratio method, and mean-variance optimization method to create different portfolios. In the end, I will compare the return between the different portfolio allocations and find the optimal portfolio for the next step.

2.1. Fixed weight portfolios

I will use two methods to allocate weight using a fixed weight vector. The first option is the **equally weighted portfolio**, i.e., to allocate equally to each asset. For the portfolio of 8 assets, the weight of each asset is 1/8 = 0.125. This is the easiest method to apply but lacks control over the portfolio risk and would not yield any significant return.

The other method is the **market capitalization weighted method**, in which we assign weights by using the percentage of market cap for each asset. This information is easily retrieved using publicly available information from Yahoo Finance or paid services like Bloomberg; however, there are some problems using the percentage of market cap to allocate the weight vector. The first problem is ETFs; as the proxies of those factors are produced by different fund providers, investors have different desires for those ETF providers. Meanwhile, many ETF providers provide similar ETFs that proxy the same factor, such as Blackrock, Fidelity, etc. In this case, using ETFs total net assets value to calculate the market capitalization-weighted portfolio would give a distorted result. Another problem is that using the market cap to weight factors is based on the assumption that the market is already in equilibrium. Although the "efficient market" assumption is generally used, there is also empirical evidence against the assumption.

2.2. Optimal weight allocation portfolios

Mean-variance optimization is based on Markowitz (1952) classic paper, which spearheaded the transformation of portfolio management from an art into a science. The key insight is that by combining assets with different expected returns and volatilities, one can decide on a mathematically optimal allocation.

2.2.1. Expected Returns Estimation

Mean-variance optimization requires knowledge of the expected returns. In practice, these are rather difficult to know with any certainty. Thus the best we can do is to come up with estimates, for example, by extrapolating historical data. This is the main flaw in mean-variance optimization – the optimization procedure is sound and provides strong mathematical guarantees, given the correct inputs.

In a search for a reasonable starting point for expected returns, He and Litterman (2002) (2003) explore several alternative forecasts: historical returns, equal "mean" returns for all assets, and risk-adjusted equal mean returns. They demonstrate that these alternative forecasts lead to extreme portfolios – when unconstrained, portfolios with large long and short positions; and, when subject to a long-only constraint, portfolios that are concentrated in a relatively small number of assets.

2.2.2. Risk Estimation

In addition to the expected returns, mean-variance optimization requires a risk model, some way of quantifying asset risk. The most commonly-used risk model is the covariance matrix, which describes asset volatilities and their co-dependence. This is important because one of the principles of diversification is that risk can be reduced by making many uncorrelated bets (correlation is just normalized covariance).

In many ways, the subject of risk models is far more important than that of expected returns because historical variance is generally a much more persistent statistic than mean historical returns. In fact, research by Kritzman et al. (2010) suggests that minimum variance portfolios, formed by optimizing without providing expected returns, actually perform much better out of sample.

The problem, however, is that in practice we do not have access to the covariance matrix (in the same way that we don't have access to expected returns) – the only thing we can do is to make estimates based on past data. The most straightforward approach is to just calculate the sample covariance matrix based on historical returns, but relatively recent (post-2000) research indicates that there are much more robust statistical estimators of the covariance matrix.

A simple version of a shrinkage estimator of the covariance matrix is represented by the Ledoit-Wolf shrinkage estimator, Ledoit and Wolf (2003b). One considers a convex combination of the empirical estimator F with some suitable chosen target S, i.e., the diagonal matrix. Subsequently, the mixing parameter δ is selected to maximize the expected accuracy of the shrunken estimator. The resulting regularized estimator $\delta F + (1 - \delta)S$ where δ is a number between 0 and 1, F is a highly structured estimator and S is the estimator with no structure. Shrinkage estimators have a long and successful history in statistics. The choice for S is obvious, given the context, it is the sample covariance matrix. Less obvious are the choice of the structured estimator, or F, and the shrinkage constant δ.

In Ledoit and Wolf (2003a), they used the constant correlation model as the shrinkage target. The model says that all the (pairwise) correlations are identical. The average of all the sample correlations is the estimator of the common constant correlation. This number together with the vector of sample variances implies our shrinkage target, denoted by F. The description of the shrinkage target is provided in the **Appendix A.1**.

The last piece of the puzzle is the shrinkage constant δ. As stated, any choice between 0 and 1 indicates the trade-off between F and S. Intuitively, there is an optimal shrinkage constant, denoted δ^* that minimizes the expected distance between the shrinkage estimator and the true covariance matrix. **Appendix A.2** derives the formula for estimating δ^* and the best estimation denoted $\hat{\delta}^*$. Shrinkage estimator of the covariance matrix Σ is now complete:

$$\hat{\Sigma}_{\text{Shrink}} = \hat{\delta}^* F + (1 - \hat{\delta}^*) S \tag{3}$$

This $\hat{\Sigma}_{\text{Shrink}}$ will be used for the optimizer and BL models.

Table 2.1: Shrinkage Estimation of the Covariance Matrix

Asset Class	BRK.B	UNH	XOM	LLY	MA	PTC	ATO	USO
BRK.B	0.000561	0.000047	0.000254	0.000036	0.000074	0.000000	0.000019	0.000050
UNH	0.000047	0.000229	0.000129	0.000021	0.000059	0.000000	0.000004	0.000047
XOM	0.000254	0.000129	0.000995	0.000033	0.000089	0.000000	0.000004	0.000080
LLY	0.000036	0.000021	0.000033	0.000101	0.000010	0.000000	0.000029	0.000032
MA	0.000074	0.000059	0.000089	0.000010	0.000364	0.000000	-0.000001	0.000039
PTC	0.000000	0.000000	0.000000	0.000000	0.000000	0.000013	0.000000	-0.000000
ATO	0.000019	0.000004	0.000004	0.000029	-0.000001	0.000000	0.000117	-0.000041
USO	0.000050	0.000047	0.000080	0.000032	0.000039	-0.000000	-0.000041	0.000754

To estimate the covariance matrix, I will apply the shrinkage estimation method proposed by Ledoit and Wolf (2003b). Fortunately, the Scikit-Learn library by Pedregosa et al. (2011) comes with the estimator built in. I will use this method to estimate Σ and feed it to the optimizers.

2.2.3. Mean-Variance Optimization

Mathematical optimization is a very difficult problem in general, particularly when we are dealing with complex objectives and constraints. However, convex optimization problems are a well-understood class of problems, which happen to be incredibly useful for finance.

Fortunately, portfolio optimization problems (with standard objectives and constraints) are convex. This allows us to immediately apply the vast body of theory as well as the refined solving routines – accordingly, the main difficulty is inputting our specific problem into a solver.

In this paper, I will find the weight allocation using the simple Mean-Variance Optimization to minimize the portfolio risk with only budget constraint:

$$\begin{aligned} &\underset{w}{\operatorname{argmax}}\ \mu^\top w - \frac{1}{2}\lambda w^\top \Sigma w \\ &\text{s.t.}\ \ w^\top \mathbf{1} = 1, \quad w \geq 0 \end{aligned} \tag{4}$$

Using the CVXPY package by Diamond and Boyd (2016), I can define the problem and find the optimal weight allocation using the sample returns and the shrinkage covariance matrix as the inputs.

2.2.4. Maximum Sharpe Ratio Portfolio

The Sharpe ratio, named after Nobel laureate William F. Sharpe, measures the risk-adjusted performance of a portfolio. It is calculated by subtracting the risk-free rate of return from a portfolio's expected return and dividing the result by the portfolio's standard deviation. Thus, a higher Sharpe ratio corresponds to more attractive risk-adjusted performance, as the portfolio's returns are higher relative to the portfolio's risk.

Our goal is to find an investment portfolio that maximizes the Sharpe ratio:

$$\begin{aligned} &\underset{w}{\operatorname{argmax}}\ \frac{\mu^\top w - r_f}{\sqrt{w^\top \Sigma w}} \\ &\text{s.t.}\ \ w^\top \mathbf{1} = 1, \quad w \geq 0 \end{aligned} \tag{5}$$

This problem poses significant challenges when employing the Lagrangian multiplier method utilized in the Mean-Variance Optimization (MVO) solution. Fortunately, Cornuéjols, Peña, and Tütüncü (2018) demonstrated a clever technique to reformulate it through a change of variables, transforming it into a simpler quadratic optimization problem.

An alternative approach involves utilizing the Monte Carlo method to randomly assign weight vectors while adhering to constraints, aiming to identify the portfolio with the highest Sharpe ratio. This process, however, demands substantial computing power and a certain degree of luck due to its reliance on a random number generator. Thanks to modern advancements in computer processing power, a task deemed impossible a decade ago can now be accomplished within mere seconds.

Thanks to the work of Virtanen et al. (2020), I can use the SciPy package to define the constraints and find the optimal weight allocation. The process is straightforward taking the sample returns and the shrinkage covariance matrix as the inputs.

2.3. The Black-Litterman model

The Black-Litterman model creates stable, mean-variance efficient portfolios, based on an investor's unique insights, which overcome the problem of input-sensitivity. According to Lee and Swaminathan (2000), the Black-Litterman model also "largely mitigates" the problem of estimation error-maximization by spreading the errors throughout the vector of expected returns.

2.3.1. The Prior

The Black-Litterman model uses "equilibrium" returns as a neutral starting point. Equilibrium returns are the set of returns that clear the market. The equilibrium returns are derived using a reverse optimization method in which the vector of implied excess equilibrium returns is extracted from known information using **Formula 6**:

$$\Pi = \lambda \Sigma w_{\text{mkt}} \tag{6}$$

where

Π	is the Implied Excess Equilibrium Return Vector (N x 1 column vector);
λ	is the risk aversion coefficient;
Σ	is the covariance matrix of excess returns (N x N matrix);
w_{mkt}	is the market capitalization weight (N x 1 column vector) of the assets.

The implied risk aversion coefficient λ for a portfolio can be estimated by dividing the expected excess return by the variance of the portfolio as stated in Grinold and Kahn (1999):

$$\lambda = \frac{\mathbb{E}(R_{\text{mkt}}) - r_f}{\sigma^2} \tag{7}$$

where

$\mathbb{E}(R_{\text{mkt}})$	is the expected market (or benchmark) total return,
r_f	is the risk-free rate,
$\sigma^2 = w_{\text{mkt}}^\top \Sigma w_{\text{mkt}}$	is the variance of the market (or benchmark) excess returns.

From the market capitalization of each asset, I can compute the weight ratio for the asset in the portfolio. This is w_{mkt} and used for the calculation of the implied equilibrium return of the portfolio.

Rearranging **Formula 7** and substituting μ for Π (representing the vector of Implied Excess Equilibrium Returns) leads to **Formula 8**, the solution to the unconstrained problem: $\underset{w}{\operatorname{argmax}}\ \mu^\top w - \frac{1}{2}\lambda w^\top \Sigma w$

$$w = (\lambda \Sigma)^{-1} \mu \tag{8}$$

Formula 8 is used to compute weight for the historical return of the portfolio using the excess return vector μ_{hist}. The other 2 portfolios are the Mean-Variance portfolio and the Max Sharpe Ratio portfolio using the proposed optimizer from the previous section. The portfolio information is as follows:

1. Historical return portfolio allocation: w_{hist}
2. Mean-Variance Optimization portfolio allocation: w_{MVO}
3. Max Sharpe Ratio portfolio allocation: w_{MSR}
4. Implied equilibrium return portfolio allocation: w_{Π}

Table 2.2 presents the sample annual mean returns and the implied equilibrium returns of 8 asset classes. The mean sample returns is used as the input for the MV optimizer and MSR optimizer to find the optimal weight allocation vectors. The risk-aversion (λ) for the MVO is derived from the **Formula 7**.

Table 2.2: Annual Expected Excess Return Vectors

Asset Class	Historical Return Vector μ_{hist}	Implied Equilibrium Return Vector Π
Berkshire Hathaway Inc.	17.31%	23.50%
UnitedHealth Group Inc.	12.84%	8.91%
Exxon Mobil Corporation	88.96%	27.15%
Eli Lilly and Company	7.30%	4.48%
Mastercard Incorporated	42.22%	10.42%
PTC Inc.	1.03%	0.04%
Atmos Energy Corporation	−12.14%	1.44%
United States Oil Fund LP	32.07%	4.91%

In **Table 2.3**, **Formula 8** is used to find the weight for the historical return approach and the implied equilibrium return approach. The other weight allocation schemes are computed using the historical returns with the optimizers introduced in the previous sections. From the table, one can find that asset allocation is heavily favouring high mean return assets.

The weight allocation using the historical returns shows an extreme approach. Since the optimizer has no constraints, it favours high-yield assets like Exxon Mobil stocks and Mastercard stocks. It penalizes low-yield assets like Berkshire Hathaway and UnitedHealth Group, despite the name and reputation of these companies. Interestingly, the same behaviour is shown in the weight allocation by MVO and MSR optimizers. The MVO finds a corner solution to put more than 60% of the wealth in Exxon Mobil stock. The MSR has a diversified approach by allocating wealth across the board and mitigating the risk as intended. The weight based on implied equilibrium returns is identical to the weight of the market capitalization since I have used the market capitalized weight allocation to compute the implied equilibrium return vector.

Table 2.3: Recommended Portfolio Weights

Asset Class	Weight Based on Historical w_{hist}	Weight Based on MVO w_{MVO}	Weight Based on MSR w_{MSR}	Weight Based on Implied Equilibrium Return Vector w_{Π}	Market Capitalization Weight w_{mkt}
Berkshire Hathaway Inc.	−20.97%	0.00%	0.00%	29.03%	29.03%
UnitedHealth Group Inc.	−21.23%	0.00%	0.00%	17.70%	17.70%
Exxon Mobil Corporation	82.60%	60.69%	36.32%	14.66%	14.66%
Eli Lilly and Company	68.69%	0.00%	8.37%	22.40%	22.40%
Mastercard Incorporated	95.51%	35.97%	42.50%	14.77%	14.77%
PTC Inc.	78.47%	0.00	0.00%	0.76%	0.76%
Atmos Energy Corporation	−108.13%	0.00%	0.00%	0.64%	0.64%
United States Oil Fund LP	21.44%	3.35%	12.79%	0.05%	0.05%

2.3.2. The Posterior - The Black Litterman Formula

Following the formula in Idzorek (2019), let introduce the Black-Litterman model. In this formula, K is used to represent the number of views and N is used to express the number of assets in the portfolio. The formula for BLM is:

$$\mathbb{E}(r) = \mu_{\text{BL}} = \left[(\tau\Sigma)^{-1} + P^{\top}\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P^{\top}\Omega^{-1}Q\right] \tag{9}$$

where

$\mathbb{E}(r) = \mu_{\text{BL}}$	is an $N \times 1$ vector of Black-Litterman expected excess returns, and it is also the Posterior;
Π	is the equilibrium excess return;
τ	is the scaling factor representing the uncertainty in the market equilibrium;
Σ	is an $N \times N$ covariance matrix of assets returns;
P	is a $K \times N$ matrix that has K views on N assets;
Ω	is the matrix that represents the confidence in each view;
Q	is a $K \times 1$ vector of expected returns of those K views.

A view portfolio may include one or more assets through non-zero elements in the corresponding elements in the P matrix.

2.3.3. Investor Views

Views reflect investors' information and beliefs about the further movement of the market. Views can be categorized as either Absolute or Relative views and either Asset Specific or Global views. The Black-Litterman model allows these views to be implemented into the model and reflected in the asset allocation for the portfolio.

Absolute views versus Relative views: An absolute view states the absolute level of expected excess return for an asset. For example, the expected excess return of asset A is 2%. In contrast, the relative view states the expected excess return outperforms or underperforms the other. In the case of relative views, all elements in a row sum to 0. For instance, the expected excess return of asset A will outperform asset B by 1%.

Asset Specific versus Global views: The Asset Specific view states the view for only one asset. However, the global view expresses views on a set of assets. For example, a portfolio consisting of asset A and asset B would outperform the portfolio consisting of asset C and asset D.

Below are two sample views expressed using input from my friend, a portfolio manager for UOB Asset Management.

- View 1: Berkshire will have an absolute excess return of 20%.
- View 2: Atmos Energy Corp will outperform the United States Oil Fund by 50 basis points.

View 1 is an example of an absolute view. From the final column of **Table 2.2**, the Implied Equilibrium return of Berkshire Hathaway Inc. is 23.5%, which is 300 basis points higher than the view of 20%.

Views 2 represents relative views. View 2 says that the return of Atmos Energy Corp will outperform the United States Oil Fund by 0.50%. To gauge whether View 2 will have a positive or negative effect on Atmos Energy Corp relative to the United States Oil Fund, it is necessary to evaluate the respective Implied Equilibrium returns of the two assets in the view. From **Table 2.2**, the Implied Equilibrium returns for Atmos Energy Corp and the United States Oil Fund are 1.44% and 4.91%, respectively, for a difference of 3.47%. The view of 0.5%, from View 2, is less than the 3.47% by which the return of Atmos Energy Corp exceeds the return of the United States Oil Fund; thus, one would expect the model to tilt the portfolio away from Atmos Energy Corp in favour of the United States Oil Fund. In general (and in the absence of constraints and additional views), if the view is less than the difference between the two Implied Equilibrium returns, the model tilts the portfolio toward the underperforming asset, as illustrated by View 2. Likewise, if the view is greater than the difference between the two Implied Equilibrium returns, the model tilts the portfolio toward the outperforming asset.

2.3.4. The view vector Q and the P matrix

In this paper, the number of views (k) is 2; thus, the View Vector (Q) is a 2×1 column vector. The uncertainty of the views results in a random, unknown, independent, normally distributed Error Term Vector (ϵ) with a mean of 0 and covariance matrix Ω. Thus, a view has the form $\epsilon + Q$.

$$Q + \epsilon = \begin{bmatrix} 0.2 \\ 0.005 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \tag{10}$$

The Error Term Vector (ϵ) does not directly enter the Black-Litterman formula. However, the variance of each error term (ω), which is the absolute difference from the error term's (ϵ) expected value of 0, does enter the formula. The variances of the error terms (ω) form Ω, where Ω is a diagonal covariance matrix with 0's in all of the off-diagonal positions. The off-diagonal elements of Ω are 0's because the model assumes that the views are independent of one another. The variances of the error terms (ω) represent the uncertainty of the views. The larger the variance of the error term (ω), the greater the uncertainty of the view.

Determining the individual variances of the error terms (ω) that constitute the diagonal elements of Ω is one of the most complicated aspects of the model.

The expressed views in column vector Q are matched to specific assets by Matrix P. Each expressed view results in a $1 \times N$ row vector. Thus, K views result in a $K \times N$ matrix. In the two-view example presented in Section 2.3.3, in which there are 8 assets, P is a 2×8 matrix.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{11}$$

The first row of Matrix P represents View 1, the absolute view. View 1 only involves one asset: Berkshire Hathaway Inc. Sequentially, Berkshire Hathaway Inc. is the 1^{st} asset in this eight-asset example, which corresponds with the "1" in the 1^{st} column of Row 1. View 2 is represented by Row 2. In the case of relative views, each row sums to 0. In Matrix P, the nominally outperforming assets receive positive weight, while the nominally underperforming assets receive negative weight.

Once Matrix P is defined, one can calculate the variance of each view portfolio. The variance of an individual view portfolio is $p_k \Sigma p_k^\top$ where p_k is a single $1 \times N$ row vector from Matrix P that corresponds to the k^{th} view and Σ is the covariance matrix of excess returns. The variances of the individual view portfolios are presented in **Table 2.4**.

Table 2.4: Variance of the View Portfolios

View	Formula	Variance
1	$p_1 \Sigma p_1^\top$	0.0561%
2	$p_2 \Sigma p_2^\top$	0.0954%

2.3.5. The uncertainty of the view

We use Ω to denote the uncertainty matrix of views. By assuming views are independent of each other, the variance-covariance and correlation between views are all zero, so that Ω is a diagonal matrix. The off-diagonal elements of Ω are all zero because the model assumes that views are independent of others. Although the assumption that views are independent might not be realistic because sometimes investors' views have logical induction from one view to the other, we would not discuss correlated views in this paper.

The covariance matrix of the error term (Ω) has the following form:

$$\Omega = \begin{bmatrix} \left(p_1 \Sigma p_1^\top\right) \times \tau & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \left(p_k \Sigma p_k^\top\right) \times \tau \end{bmatrix} \tag{12}$$

Conceptually, the Black-Litterman model is a complex, weighted average of the Implied Equilibrium Return Vector (Π) and the View Vector (Q), in which the relative weightings are a function of the scalar (τ) and the uncertainty of the views (Ω). Unfortunately, the scalar and the uncertainty in the views are the most abstract and difficult-to-specify parameters of the model. The greater the level of confidence (certainty) in the expressed views, the closer the new return vector will be to the views. If the investor is less confident in the expressed views, the new return vector should be closer to the Implied Equilibrium Return Vector (Π). Following the reasoning by Idzorek (2019), I assume $\tau = 1$ and use the variances from **Table 2.4** to compute Ω using **Formula 12**.

$$\Omega = \begin{bmatrix} 0.00056141 & 0 \\ 0 & 0.00095417 \end{bmatrix}$$

When the covariance matrix of the error term (Ω) is calculated using this method, the actual value of the scalar (τ) becomes irrelevant because only the ratio ω/τ enters the model. For example, changing the assumed value of the scalar (τ) from 1 to 15 dramatically changes the value of the diagonal elements of Ω, but the new Combined Return Vector $\mathbb{E}(r)$ is unaffected.

2.3.6. Calculating the New Combined Return Vector

All the inputs for the BLM are ready to compute the new combined return vectors. The New Recommended Weights ($\hat{w}$) are calculated by solving the unconstrained maximization problem, **Formula 8**. The covariance matrix estimated by the Ledoit-Wolf Shrinkage method (Σ_{shrink}) is presented in **Table 2.1**.

The New Weight Vector ($\hat{w}$) in column 4 of **Table 2.5** is based on the New Combined Return Vector $\mathbb{E}(r)$. One of the strongest features of the Black-Litterman model is illustrated in the final column of **Table 2.5**. Only the weights of the 3 assets for which views were expressed changed from their original market capitalization weights, and the directions of the changes are intuitive. No views were expressed on the rest of the portfolio, and their weights remain unchanged.

Table 2.5: Annual Returns and Resulting Portfolio Weights

Asset Class	New Combined Return Vector $\mathbb{E}(r)$	Implied Equilibrium Return Vector Π	Difference $\mathbb{E}(r) - \Pi$	New Weight $\hat{w}$	Market Cap Weight w_{mkt}	Difference $\hat{w} - w_{\text{mkt}}$
Berkshire Hathaway Inc.	21.72%	23.50%	−1.78%	26.06%	29.03%	−2.96%
UnitedHealth Group Inc.	8.68%	8.91%	−0.23%	17.70%	17.70%	0.00%
Exxon Mobil Corporation	26.22%	27.15%	−0.93%	14.66%	14.66%	0.00%
Eli Lilly and Company	4.37%	4.48%	−0.12%	22.40%	22.40%	0.00%
Mastercard Incorporated	10.11%	10.42%	−0.31%	14.77%	14.77%	0.00%
PTC Inc.	0.04%	0.04%	0.00%	0.76%	0.76%	0.00%
Atmos Energy Corporation	1.70%	1.44%	0.26%	2.61%	0.64%	1.97%
United States Oil Fund LP	3.15%	4.91%	−1.76%	−1.92%	0.05%	−1.97%

2.3.7. Different Types of Risk Aversion

Different level of risk aversions (λ) yield different result for the implementation of the Black Litterman model. I will use 3 different risk aversion (λ) to calculate the Black-Litterman weights under three different scenarios. Those three scenarios represent three types of investors:

1. Risk Seeking Investors: $\lambda_{\text{risky}} = 1$
2. Risk Neutral Investors: $\lambda_{\text{neutral}} = \lambda_{\text{mkt}} = 4.10$
3. Risk Averse Investors: $\lambda_{\text{riskless}} = 10$

As the name suggested, Risk Seeking Investors are interested in taking unnecessary risks in exchange for a gain in their reward. Risk Neutral Investors will have the same information about the market and apply the same method to compute the expected return for their portfolio. They will take the most "neutral" position and be happy with a similar return to the market expected returns. Lastly, Risk Averse Investors are conservative and take little to no risk in exchange for certainty in their return.

I will use this new value of risk aversions (λ) to compute a new combined return vector and calculate the new weight allocation for the portfolio. Denote $w_{\text{risky}}, w_{\text{neutral}}, w_{\text{riskless}}$ are the weight vectors of the new portfolios. These weight vectors are the solution of the MVO problem with constraints using the corresponding risk aversion value.

Table 2.6: Black-Litterman weights allocation for different levels of risk aversions

Asset Class	Risk Averse w_{riskless}	Risk Neutral w_{neutral}	Risk Seeking w_{risky}	Black Litterman $\hat{w}$
Berkshire Hathaway Inc.	0.00%	0.00%	0.00%	26.07%
UnitedHealth Group Inc.	0.00%	0.00%	0.00%	17.70%
Exxon Mobil Corporation	30.80%	60.69%	100.00%	14.66%
Eli Lilly and Company	9.12%	0.00%	0.00%	22.40%
Mastercard Incorporated	36.48%	35.97%	0.00%	14.77%
PTC Inc.	12.56%	0.00%	0.00%	0.76%
Atmos Energy Corporation	0.00%	0.00%	0.00%	2.61%
United States Oil Fund LP	11.05%	3.35%	0.00%	−1.91%

The results are presented in **Table 2.6**. Examining the weight allocation across different levels of risk aversions, there is a clear indication to invest in Exxon Mobil Corporation compared to the rest of the assets in the portfolio. In **Table 2.3**, the weight allocation based on historical returns tends to short the stocks from Berkshire Hathaway, United Health Group, and Atmos Energy Corporation. The constraints in the optimizer don't allow shorting these stocks, and therefore, they simply aren't allocated wealth. With a very interestingly high mean return and relatively low volatility, Exxon Mobil is the favoured asset among the 8 asset classes selected for this portfolio. Risk-Averse Investors would spread their wealth across different sectors of the capital market rather than taking a corner solution like Risk-Seeking Investors, who put 100% of their wealth in just one high-yield stock.

In contrast to other weight allocation schemes, the Black-Litterman approach distributes wealth across the portfolio and captures vital information conveyed by the views, even if Berkshire Hathaway's performance does not align with historical returns. The model showcases its ability to steer clear of corner solutions and effectively tackles the fundamental issue inherent in the Mean-Variance optimization method.

2.3.8. Performance Analysis

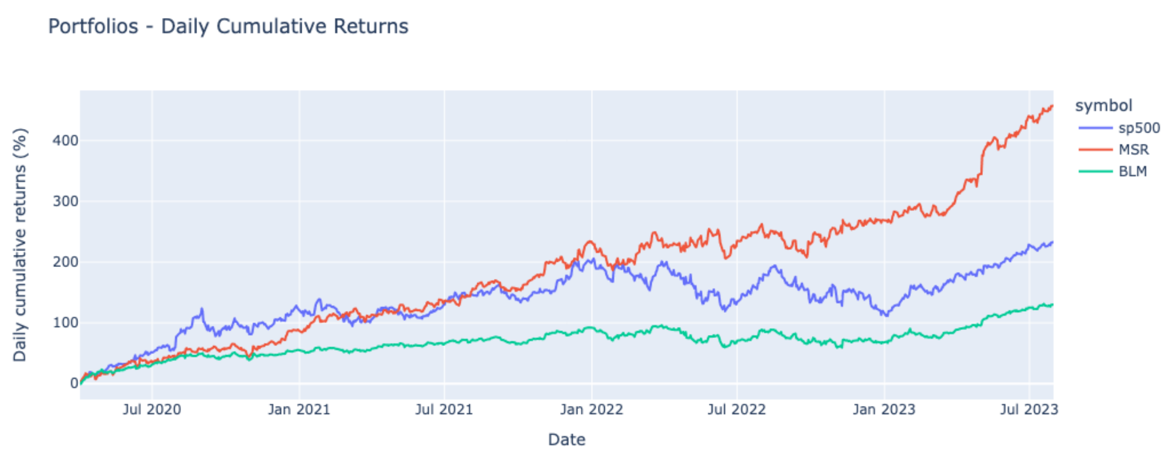
I will analyze the Max Sharpe Ratio portfolio and the Black-Litterman portfolio in comparison to the benchmark. The analysis is based on the portfolio's annual return, risk, and Sharpe ratio.

Table 2.7: Portfolio Annual Returns and Risks

Portfolios	Returns	Risk	Sharpe Ratio
Max Sharpe Ratio	54.98%	25.23%	2.13
Black-Litterman Portfolio	26.77%	18.15%	1.41
Benchmark (S&P 500)	41.10%	31.20%	1.28

The Max Sharpe Ratio (MSR) portfolio delivers the highest return with relatively low risk compared to the benchmark. Conversely, the Black-Litterman (BLM) portfolio exhibits the lowest risk and a higher Sharpe ratio than the benchmark. Despite the promising annual return showcased by the MSR portfolio, it is cautioned as an unreliable predictor of future results. The MSR relies solely on sample data to formulate the optimal strategy based on current information, lacking consideration for potential future scenarios. This is where the strength of the BLM portfolio becomes evident, as it integrates information from provided future data, adjusting weight allocations for a more robust portfolio with low risk, while maintaining a competitive return.

Figure 2.1: Portfolio Cumulative Returns (%)



Although the BLM portfolio did not outperform the benchmark, it has a relatively low risk during the sensitive period used for the portfolio construction process. The BLM is a robust contender, pending thorough testing of its robustness.

In summary, while the MSR portfolio exhibits superior performance within the sample data, its efficacy may shift with out-of-sample data. On the other hand, the BLM portfolio, despite trailing the benchmark and losing to the Max Sharpe Ratio portfolio by 200%, holds potential for practical out-of-sample performance, though rigorous testing and evidence are essential—especially during critical market regime changes or financial crises.

3 Conclusion

> If we've been a little more successful than other people, it is because we always realized that the school of life was always open, and if you were not learning more, you are falling behind.
>
> *Charlie Munger*

3.1. Discussion and Improvement

This paper elucidates the portfolio construction process, employing risk factors to select assets across well-diversified classes. Asset reduction is achieved through correlation analysis utilizing price data from publicly available databases such as Yahoo Finance. Addressing the issue of sample covariance matrix instability, the shrinkage estimator proposed by Ledoit and Wolf (2003b) is employed. The resulting covariance matrix is utilized as input for the Mean-Variance optimizer, Max Sharpe Ratios optimizer, and the Black Litterman model to determine the optimal weight allocation (w^*).

Implementation of the Black Litterman model involves forming a Combined Return Vector, yielding an intuitive and diversified portfolio allocation. The model's parameters, controlling the relative significance of equilibrium and view returns (scalar (τ) and view uncertainty (Ω)), pose challenges in the specification.

Key improvements to enhance the model's practicality include:

1. Incorporating the international market in the portfolio for further diversification. Introduction of international bonds and foreign exchange products can broaden exposure.
2. Considering the inclusion of periods during the Covid-19 crisis in the sample space, strategically identifying market states, and diversifying benchmarks and view strategies accordingly.
3. Exploring alternative optimization methods, such as minimum Value at Risk (VaR), and experimenting with constraints on the optimization process to assess robustness.
4. Investigating alternative approaches for systematic view generation, such as employing Machine Learning models or Deep Learning models for macroeconomic factor analysis.

An unexplored yet significant enhancement involves incorporating individual investor uncertainty into the model, and updating weight allocation based on this information, as demonstrated in Idzorek (2019). Advanced topics such as Index of Satisfaction, Tracking Error, and Stochastic control could further refine the Black Litterman model's functionality.

3.2. Conclusion

In summary, this paper underscores the importance of factor investing and diversification. It addresses issues with sample covariance and provides an elegant solution for robust covariance matrix estimation. The implementation of Maximum Sharpe Ratio and Mean-Variance optimizers, in conjunction with the Black-Litterman model, facilitates the determination of optimal portfolio weight allocation. The conclusion highlights that the Black-Litterman model effectively mitigates common Mean-Variance optimization weaknesses, delivering intuitive, diversified portfolios and maximizing benefits within the Markowitz paradigm.

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A

Technical Appendix

This appendix discusses some technical results that can be skipped at first reading.

A.1. Formula for Shrinkage Target

Let y_{it} denote the return on stock i during period t where $1 \leq i \leq N$, and $1 \leq t \leq T$. Stock returns are independent and identically distributed (i.i.d) over time and have finite fourth moments. The sample average of the returns of stock i is given by:

$$\bar{y}_i = T^{-1} \sum_{t=1}^{T} y_{it} \tag{A.1}$$

Let Σ denote the population (or true) covariance matrix and let S denote the sample covariance matrix. Typical entries of the matrices Σ and S are denoted by σ_{ij} and s_{ij}, respectively.

The population and sample correlations between the returns on stocks i and j are given by:

$$g_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \quad \text{and} \quad r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}} \tag{A.2}$$

The average population and sample correlations are given by:

$$\bar{g} = \frac{2}{(N-1)N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} g_{ij} \quad \text{and} \quad \bar{r} = \frac{2}{(N-1)N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} r_{ij} \tag{A.3}$$

Define the population constant correlation matrix Φ by means of the population variances and the average population correlation:

$$\Phi_{ii} = \sigma_{ii} \quad \text{and} \quad \Phi_{ij} = \bar{g}\sqrt{\sigma_{ii}\sigma_{jj}} \tag{A.4}$$

The constant correlation matrix F is defined by means of the sample variances and the average sample correlation:

$$f_{ii} = s_{ii} \quad \text{and} \quad f_{ij} = \bar{r}\sqrt{s_{ii}s_{ij}} \tag{A.5}$$

This matrix F is the shrinkage target.

A.2. Formula for Shrinkage Constant

Denote $\hat{\pi}$ is an estimator for π:

$$\hat{\pi} = \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{\pi}_{ij} \qquad \text{with} \qquad \hat{\pi}_{ij} = \frac{1}{T} \sum_{t=1}^{T} \left\{ (y_{it} - \bar{y}_{i} .)(y_{jt} - \bar{y}_{j} .) - s_{ij} \right\}^2$$

where π the sum of asymptotic variances of the entries of the sample covariance matrix scaled by $\sqrt{T}$: $\pi = \sum_{i=1}^{N} \sum_{j=1}^{N} \text{AsyVar}\left[\sqrt{T} s_{ij}\right]$

Denote ρ the sum of asymptotic covariances of the entries of the shrinkage target with the entries of the sample covariance matrix scaled by $\sqrt{T}$:

$$\begin{aligned} \rho &= \sum_{i=1}^{N} \sum_{j=1}^{N} \text{AsyCov}\left[\sqrt{T} f_{ij}, \sqrt{T} s_{ij}\right] \\ &= \sum_{i=1}^{N} \text{AsyVar}\left[\sqrt{T} s_{ii}\right] + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \text{AsyCov}\left[\sqrt{T} \bar{r} \sqrt{s_{ii} s_{jj}}, \sqrt{T} s_{ij}\right] \end{aligned}$$

On the diagonal, we know from Ledoit and Wolf (2003b) again that:

$$\text{AsyVar}\left[\sqrt{T} s_{ii}\right] = \hat{\pi}_{ii} = \frac{1}{T} \sum_{t=1}^{T} \left\{ (y_{it} - \bar{y}_{i\cdot})^2 - s_{ii} \right\}^2$$

On the off-diagonal, we have:

$$\text{AsyCov}\left[\sqrt{T}\, \bar{r} \sqrt{s_{ii} s_{jj}}, \sqrt{T} s_{ij}\right] = \text{AsyCov}\left[\sqrt{T}\, \bar{r} \sqrt{s_{ii} s_{jj}}, \sqrt{T} s_{ij}\right]$$

Since the estimation error in $\hat{r}$ is asymptotically negligible and by use of the delta method, any term $\text{AsyCov}\left[\sqrt{T}\, \bar{r} \sqrt{s_{ii} s_{jj}}, \sqrt{T} s_{ij}\right]$ can be consistently estimated by:

$$\frac{\hat{r}}{2} \left(\sqrt{\frac{s_{jj}}{s_{ii}}} \text{AsyCov}\left[\sqrt{T} s_{ii}, \sqrt{T} s_{ij}\right] + \sqrt{\frac{s_{ii}}{s_{jj}}} \text{AsyCov}\left[\sqrt{T} s_{jj}, \sqrt{T} s_{ij}\right] \right)$$

Standard theory implies that a consistent estimator for $\text{AsyCov}\left[\sqrt{T} s_{ii}, \sqrt{T} s_{ij}\right]$ is given by:

$$\hat{\vartheta}_{ii,ij} = \frac{1}{T} \sum_{t=1}^{T} \left\{ \left(y_{it} - \bar{y}_{i\cdot}\right)^2 - s_{ii} \right\} \left\{ \left(y_{it} - \bar{y}_{i\cdot}\right) \left(y_{jt} - \bar{y}_{j\cdot}\right) - s_{ij} \right\}$$

and that, analogously,a consistent estimator for $\text{AsyCov}\left[\sqrt{T} s_{jj}, \sqrt{T} s_{ij}\right]$ is given by:

$$\hat{\vartheta}_{jj,ij} = \frac{1}{T} \sum_{t=1}^{T} \left\{ (y_{jt} - \bar{y}_{j})^2 - s_{jj} \right\} \left\{ (y_{it} - \bar{y}_{i\cdot})(y_{jt} - \bar{y}_{j\cdot}) - s_{ij} \right\}$$

Collecting terms now yields a consistent estimator for ρ:

$$\hat{\rho} = \sum_{i=1}^{N} \hat{\pi}_{ii} + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{\hat{r}}{2} \left(\sqrt{\frac{s_{jj}}{s_{ii}}}\, \hat{y}_{ii,ij} + \sqrt{\frac{s_{ii}}{s_{jj}}}\, \hat{y}_{jj,ij} \right)$$

Third, a consistent estimator for γ is:

$$\hat{\gamma} = \sum_{i=1}^{N} \sum_{j=1}^{N} (f_{ij} - s_{ij})^2$$

Estimator for κ:

$$\hat{\kappa} = \frac{\hat{\pi} - \hat{\rho}}{\hat{\gamma}}$$

Estimator for shrinkage constant:

$$\hat{\delta}^* = \max\left\{0, \min\left\{\frac{\hat{\kappa}}{T}, 1\right\}\right\}$$