

Finding Available Parking Spaces Made Easy

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ABSTRACT

We discuss the problem of predicting the number of available parking spaces in a parking lot. The parking lot is modeled by a continuous-time Markov chain, following Caliskan, Barthels, Scheuermann, and Mauve. The parking lot regularly communicates the number of occupied spaces, capacity, arrival and parking rate through a vehicular ad-hoc network. The navigation system in the vehicle has to compute from this data the probability of an available parking space upon arrival. We derive a structural result that considerably simplifies the computation of the transition probabilities in the navigation system of the vehicle.

Categories and Subject Descriptors

H.4.3 [Information Systems]: Information Systems Applications—*Communication Applications*

General Terms

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1. INTRODUCTION

With the advance of sensing and communicating technology in vehicles, research in vehicular ad-hoc networks has been increasingly attracting interest in applications that

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may have a direct impact on everyday life. People driving vehicles in urban or metropolitan areas often experience that valuable resources such as gasoline and time are wasted in searching for available parking spaces. A recent Washington Post article [5] states that

“Hunting for parking produces more than frustration. Shoup studied a 15-block business district in Los Angeles and determined that cruising about 2.5 times around the block for the average of 3.3 minutes required to find a space added up to 950,000 excess miles traveled, 47,000 gallons of gas wasted and 730 tons of carbon dioxide produced in the course of a year.”

This prompts the question whether vehicular ad-hoc networks can help reduce the time in the search for available parking spaces.

For example, consider a parking lot of a mall or of an airport. When a driver approaches the parking lot, some questions can come to the driver's mind:

- (a) Will a parking space at a particular parking area be available when I get there in t minutes?
- (b) If there is no parking space available, how long do I need to wait until a parking space becomes available?

One important criterion to consider in finding answers to the above questions is the timing factor when the spot is available. Obviously the data gathered using the vehicular ad-hoc network are always from the past. Thus, a natural question is: How can one *predict* the availability of some parking space in a reasonably correct and efficient way? We propose an online scheme that answers this question with a very small communication overhead.

2. RELATED WORK

There have been numerous works studying the problem of finding available parking spaces using a vehicular ad-hoc network; those closely related to our work are [1, 2, 7–9].

In [8], Mathur *et al.* motivate the problem of finding available parking spaces, discuss research challenges and propose some possible solutions. They discuss a centralized and a distributed architecture for their solutions. In the centralized solution, some cars are equipped with ultrasonic sensors, which drive past the parking spaces to collect the occupancy data, and upload the data to the centralized database. The cars that need to park simply query the centralized

database. The idea of this centralized architecture is deployed and experimented with in [7]. The distributed architecture uses car-to-car communication and the authors in [8] address the importance of having such a distributed architecture.

A network model that consists of onboard units on vehicles and static road side units is proposed in [9] with an emphasis on secure communication between these units. In [2], the parking area is modeled as a grid, and schemes for information aggregation and dissemination over the grid are proposed.

Our work studies how to use the information collected by such vehicular ad-hoc networking systems, as the ones proposed above. In [1], Caliskan *et al.* introduced a simple and elegant parking lot model that we adopt here. We review this model and give some background in Section 3. We then show how to explicitly calculate the transition probabilities, which was claimed to be a difficult problem in [1]. Our solution is suitable for implementation in the navigation system of vehicles.

3. THE PARKING LOT MODEL

In this section, we review the parking lot model that was introduced in [1] and give the necessary background on continuous-time Markov chains.

Suppose that we are interested in parking at a parking lot with n parking spaces. Since we are only concerned with predicting whether or not the parking lot is full, it suffices to model the number of occupied parking spaces. For all times $t \geq 0$, let $X(t)$ denote a random variable with values in $\{0, 1, \dots, n\}$. We denote by $\Pr[X(t) = o]$ the probability that at time t there are precisely o parking spaces occupied. The availability of the parking spaces depends on the present occupancy, but not really on past occupancies. Therefore, one can model the changes of the parking spots by a continuous-time Markov chain. By definition, the stochastic process $\{X(t) | t \geq 0\}$ is a continuous-time Markov chain if and only if

$$\Pr[X(t+s) = u | X(s) = v, X(s_1) = w_1, \dots, X(s_n) = w_n] = \Pr[X(t+s) = u | X(s) = v] \quad (1)$$

holds for all $n \geq 0$ and for all occupancy numbers u, v, w_1, \dots, w_n in $\{0, 1, \dots, n\}$ and all nonnegative real numbers $s_1 < s_2 < \dots < s_n < s$ and t . This formalizes the notion that the future occupancy of the parking spots depends on the present occupancy but not on past occupancies. We will make a small excursion and introduce some terminology from the theory of continuous-time Markov chains.

The Markov chain is called homogeneous if the right-hand side of equation (1) does not depend on s . For simplicity, we will assume that our Markov chains are homogeneous. Furthermore, we will assume that the transition probabilities $\Pr[X(t+s) = u | X(s) = v]$ are right-continuous at $t = 0$, meaning that

$$\lim_{t \rightarrow 0+} \Pr[X(t+s) = u | X(s) = v] = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

This assumption ensures that the Markov chain has with probability 1 only a finite number of state changes in small time intervals.

Let us define the probability transition matrices

$$P(t) = (\Pr[X(t+s) = u - 1 | X(s) = v - 1])_{1 \leq u, v \leq n+1}$$

for all real numbers $t \geq 0$. Notice that the indices of matrices and vectors consistently range from 1 to $n+1$ in this paper.

Obviously, the behavior of the Markov chain is completely determined by the initial probability distribution on its states and the probability transition matrices. Some basic properties of the probability transition matrices are:

- (a) $P(t)$ is a stochastic matrix for all $t \geq 0$, that is, each row of a probability transition matrix sums to 1;
 - (b) $P(r+s) = P(r)P(s)$ holds for all real numbers $r, s \geq 0$;
 - (c) and $\lim_{h \rightarrow 0+} P(h) = P(0)$ is the identity matrix.
- Therefore, $\{P(t) | t \geq 0\}$ is a continuous semigroup (recall that a semigroup is a set that is equipped with a binary associative operation).

The continuity of this semigroup has the somewhat unexpected consequence that $P(t)$ is infinitely differentiable with respect to $t > 0$, see [6, p. 164]. Therefore,

$$Q = \lim_{h \rightarrow 0+} \frac{1}{h} (P(h) - I)$$

is a well-defined $(n+1) \times (n+1)$ matrix, called the *generator* or *transition matrix* of the Markov chain.

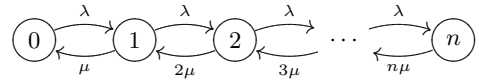
The transition matrix figures prominently in the Chapman-Kolmogorov differential equation

$$\begin{aligned} P(t)' &= \lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h} \\ &= \left(\lim_{h \rightarrow 0} \frac{P(h) - I}{h} \right) P(t) = QP(t) \end{aligned}$$

of the transition semigroup. It follows that $P(t) = e^{tQ}$ is the unique solution to this system of linear differential equations with initial condition $P(0) = I$. Thus, a continuous-time Markov chain is completely specified by its transition matrix Q and the probability distribution on the states at the time $t = 0$.

We now return from our excursion and detail the Markov-chain model of a parking lot that was introduced in [1]. The arrival of vehicles at the parking lot can be modeled by a Poisson process with arrival rate λ . It is assumed that each vehicle occupies a single parking space. Furthermore, it is assumed that the time a parking spot is occupied can be modeled by an exponential distribution with parameter μ . The parameter μ will be called the parking rate. If the parking lot is full (meaning it is occupied by n vehicles), then any arriving vehicle will be rejected.

Therefore, the state transitions of this parking lot model are given by



If we express this in the form of a transition matrix Q , we obtain

$$Q = \begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda + \mu) & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & & (n-1)\mu & -(\lambda + (n-1)\mu) & \lambda \\ & & & & n\mu & -n\mu \end{pmatrix} \quad (2)$$

In other words, $Q = (q_{ij})_{1 \leq i, j \leq n+1}$ is the tridiagonal $(n+1) \times (n+1)$ matrix with entries $q_{i,i+1} = \lambda$ for $1 \leq i \leq n$

on the superdiagonal, $q_{i+1,i} = i\mu$ for $1 \leq i \leq n$ on the subdiagonal, and $q_{ii} = -(\lambda + (i-1)\mu)$ for $1 \leq i \leq n$ and $q_{n+1,n+1} = -n\mu$ on the diagonal.

4. THE PARKING AVAILABILITY ALGORITHM

Currently, the navigation system of a vehicle can estimate the time of arrival at a destination. If the destination is, say, the parking lot of a mall or of an airport terminal, then it would be helpful to estimate the probability that a parking space will be available.

Suppose that the parking lot has n parking spaces, monitors the number o of occupied parking spaces, estimates the arrival rate λ , and the parking rate μ at time s . If the parking lot regularly disseminates these five parameters (s, n, o, λ, μ) in a vehicular ad-hoc network, then the navigation system can in principle estimate the probability of the availability of parking spot. Indeed, let $\pi(s)$ be the vector of length $n+1$ that has an entry 1 at position $o+1$ and zeros everywhere else. The vector $\pi(s)$ represents the probability distribution that the occupancy number is in state o with probability 1. If the navigation system estimates that the vehicle will arrive at this parking lot at time $t+s$, then the probability distribution of the occupancy number is given by $\pi(s+t) = \pi(s)e^{tQ}$. Therefore, the probability that the parking space is full at the time of arrival is $1 - \pi(s+t)_{n+1}$, since the $(n+1)$ -th component of the probability distribution $\pi(s+t)$ of the occupancy number is the probability that the parking lot is fully occupied at time $s+t$.

However, Caliskan, Barthels, Scheuermann and Mauve caution us that “Like evaluating the matrix exponential operator in general, calculating $\pi(s)e^{tQ}$ is non-trivial”. In this section, we solve the problem of evaluating the matrix exponential function e^{tQ} by investigating the structure of the matrix Q .

THEOREM 1. *The probability transition matrix $P(t)$ for the generator matrix Q given in (2) is given by*

$$P(t) = e^{tQ} = D^{-1}B^t e^{tE}BD,$$

where D and E are diagonal matrices and B an orthogonal matrix. These matrices will be explicitly defined in the proof below.

PROOF. Let us define the diagonal matrix

$$D = \text{diag}(d_1, d_2, \dots, d_{n+1})$$

with entries given by

$$d_k = \begin{cases} 1 & \text{if } k = 1, \\ \frac{1}{\sqrt{k-1}}d_{k-1}\sqrt{\lambda/\mu} & \text{if } k \in \{2, 3, \dots, n+1\}. \end{cases}$$

Then $R = DQD^{-1} = (r_{ij})_{1 \leq i, j \leq n+1}$ is triangular matrix with entries

$$\begin{aligned} r_{ii} &= d_i q_{ii} d_i^{-1} = q_{ii} & \text{for } 1 \leq i \leq n+1, \\ r_{i,i+1} &= d_i q_{i,i+1} d_{i+1}^{-1} = \sqrt{i\lambda\mu}, & \text{for } 1 \leq i \leq n, \\ r_{i+1,i} &= d_{i+1} q_{i+1,i} d_i^{-1} = \sqrt{i\lambda\mu} & \text{for } 1 \leq i \leq n. \end{aligned}$$

Therefore, R is a real symmetric triangular matrix with the same entries $a_i := r_{ii} = q_{ii}$ on the diagonal as Q and the entries $b_i := r_{i,i+1} = r_{i+1,i} = \sqrt{i\lambda\mu}$ on the super- and sub-

diagonals

$$R = DQD^{-1} = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \vdots & 0 & b_n & a_{n+1} \end{pmatrix}.$$

Our next goal is to explicitly construct an orthogonal matrix B with rows given by eigenvectors of R and a diagonal matrix E with the corresponding eigenvalues as diagonal entries such that $R = B^t E B$. This will allow us to write the generator matrix Q in the form $Q = D^{-1}B^t E B D$. It then follows that the probability transition matrix $P(t) = e^{tQ}$ can be factorized in the convenient form $P(t) = D^{-1}B^t e^{tE} B D$, as claimed.

We follow [4] and relate the matrix R to a sequence of orthogonal polynomials. One can then exploit the theory of orthogonal polynomials to explicitly construct eigenvectors of R .

We associate to the matrix R a family $\{p_k(x) \mid -1 \leq k \leq n+1\}$ of polynomials in $\mathbf{R}[x]$ such that $p_{-1}(x) = 0$, $p_0(x) = 1$, and

$$b_{k+1}p_{k+1}(x) = (x - a_{k+1})p_k(x) - b_k p_{k-1}(x) \quad (3)$$

for $0 \leq k \leq n$, where $b_0 = b_{n+1} = 1$. Then there exists a probability distribution $F(x)$ such that the polynomials p_k satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} p_j(x)p_k(x)dF(x) = 0$$

whenever the indices j and k are distinct. It follows from this orthogonality relation that the polynomial $p_k(x)$ has k distinct real roots. Furthermore, the roots of the polynomial $p_{k-1}(x)$ lie strictly between the roots of $p_k(x)$, a property known as interlacing.

The next property will make it clear why the polynomials $p_k(x)$ were introduced in the first place. Keeping in mind that $b_{n+1} = 1$, the recurrence (3) yields

$$xV(x) = RV(x) + p_{n+1}(x)e_{n+1},$$

where $V(x)$ is the vector $V(x) = (p_0(x), p_1(x), \dots, p_n(x))^t$ of polynomials and $e_{n+1} = (0, \dots, 0, 1)^t$. If we substitute any root λ of $p_{n+1}(x)$ into this equation, then $V(\lambda) \neq 0$, because of the interlacing property. Thus, we find that $V(\lambda)$ is an eigenvector to the eigenvalue λ , since

$$\lambda V(\lambda) = RV(\lambda) + p_{n+1}(\lambda)e_{n+1} = RV(\lambda)$$

holds. Let $\{\lambda_k \mid 1 \leq k \leq n+1\}$ be the set of roots of the polynomial $p_{n+1}(x)$. Since the roots are pairwise distinct, the $n+1$ vectors $V(\lambda_k)$ with $1 \leq k \leq n+1$ are $n+1$ eigenvectors of R . These eigenvectors of the real symmetric matrix R are orthogonal, since their eigenvalues are distinct.

Let $v_k = \frac{1}{\|V(\lambda_k)\|} V(\lambda_k)^t$ be the row-vectors obtained from $V(\lambda_k)$ by normalization to unit length and transposition. If we define $B = (v_k)_{1 \leq k \leq n+1}$, then this is an orthogonal matrix. Let $E = \text{diag}(\lambda_1, \dots, \lambda_{n+1})$. Then $R = B^t E B$, and it follows that $P(t) = e^{tQ} = D^{-1}B^t e^{tE} B D$, as claimed. \square

The previous theorem asserts in particular that the transition matrix Q given in (2) is diagonalizable. One can follow

the construction of the orthogonal matrix B given in the proof by calculating the polynomials $p_k(x)$ and determining the roots of $p_{n+1}(x)$, but one might as well use efficient algorithms to find eigenvalues and eigenvectors of a tridiagonal matrix.

The previous theorem can be used to calculate the probability distribution of the occupancy number at a given time. Let us assume a floating point implementation. We assume constant complexity for elementary floating point operations such as multiplication, exponentiation, forming square roots, and raising to a power.

COROLLARY 2. *Given the data (s, n, o, λ, μ) of a parking lot with n parking spaces that has arrival rate λ , parking rate μ , and o occupied parking spaces at time s , there exists an $O(n^2)$ algorithm that computes the probability distribution $\pi(s+t)$ of the occupancy number at time $s+t$ for any $t \geq 0$.*

PROOF. Given λ and μ , one can construct the matrices D and D^{-1} in linear time directly from the definition. Furthermore, the tridiagonal matrix Q can be constructed in sparse form with $O(n)$ operations. Therefore, the matrix $R = DQD^{-1}$ can be formed in $O(n)$ time as a product of sparse matrices. One can compute the matrices B^t , E , and B in $O(n^2)$ time by eigenvector and eigenvalue calculations using Dhillon's algorithm [3]. Given $E = (\lambda_1, \dots, \lambda_{n+1})$, the vector $e^{tE} = (e^{t\lambda_1}, \dots, e^{t\lambda_{n+1}})$ can be obtained in $O(n)$ time.

The probability distribution $\pi(s)$ of the occupancy number is given by the vector in \mathbf{R}^{n+1} that is 1 at position $o+1$ and zero otherwise, since the occupancy number has been observed at time s . Thus, we can calculate the probability distribution $\pi(s+t) = \pi(s)D^{-1}B^t e^{tE}BD$ using standard vector-matrix multiplication in $O(n^2)$ time. \square

The previous corollary allows one to answer the question (1) that we raised in the introduction. An answer to the question (2) of the introduction is given in the next proposition.

PROPOSITION 3. *Suppose that a parking lot with n parking space is occupied. If the parking rate is μ , then the expected time for a vehicle to leave the parking lot is $1/(n\mu)$.*

PROOF. For $1 \leq k \leq n$, let X_k be the random variables measuring the time until the vehicle in parking space k leaves the parking lot. These random variables are exponentially distributed with parameter μ . The time until the next parking space becomes available is thus given by $Y = \min\{X_1, \dots, X_n\}$, which is an exponentially distributed random variable with parameter $n\mu$. Therefore, $E[Y] = (n\mu)^{-1}$ is the expected time for the next vehicle to leave the parking lot. \square

5. CONCLUSIONS

We investigated the parking lot model that was introduced in [1]. This model aptly describes parking lots at malls and airports that have an admission control such as a gate and monitor the occupancy of the parking lot. If the admission is strictly enforced then the continuous-time Markov model studied here appears to be a good fit. We showed that communicating a minimal amount of data over a vehicular ad-hoc network is sufficient to make efficient prediction of future occupancies. Gathering data from various alternative parking locations could lead to an informed decision, saving gasoline and time.

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