The new scheme is described in Section 2, an example is given in Section 3, and the recommended design procedure is presented in Section 4. In Section 5 the AEWMA chart is compared with other control schemes. In Appendix A the approximation of the ARL of an AEWMA chart, based on a Markov chain approach, is briefly discussed. The numerical procedure used to design the AEWMA scheme is sketched in Appendix B.

2. THE ADAPTIVE EWMA SCHEME

The class of control charts that we consider attempts to track the current level of the process using a statistic of the type

$$x_t = x_{t-1} + \phi(e_t), \qquad x_0 = \eta_0, \tag{2}$$

where $e_t = y_t - x_{t-1}$ and $\phi(e_t)$ is a "score" function. An alarm is raised when $|x_t - \eta_0| > h$, where η_0 denotes the target value of the process mean and h is a suitable threshold. The h value is mainly determined by ensuring that the desired mean time between false alarms is large. Observe that when $y_t \neq x_{t-1}$, (2) can be rewritten in the form

$$x_t = (1 - w(e_t))x_{t-1} + w(e_t)y_t,$$

where $w(e) = \phi(e)/e$, that is, as an EWMA statistic with varying weights. Shewhart and EWMA charts can be obtained as special cases of (2) for $\phi(e) = e$ and $\phi(e) = \lambda e$. Many score functions could be used. To blend together the best features of EWMA and Shewhart charts, it seems reasonable to use scores within the envelope of the EWMA and Shewhart ϕ-functions and, in particular, to choose $\phi(\cdot)$ such that (a) $\phi(e)$ is monotone increasing in e; (b) $\phi(e) = -\phi(-e)$; (c) $\phi(e) \approx \lambda e$, when $|e|$ is small, for a suitable λ, $0 \le \lambda \le 1$; and (d) $(\phi(e)/e) \approx 1$, when $|e|$ is large. Observe that (c) and (d) have been introduced to obtain a control statistic that behaves like an EWMA chart when $|e_t|$ is small and like a Shewhart chart when $|e_t|$ is large. In addition, note that (d) may cure the inertia problem of the EWMA scheme because it leads to control statistics that can "jump" toward the observed value y_t in the worst-case situation.

In this article we show results for the following three score functions:

$$\phi_{hu}(e) = \begin{cases} e + (1-\lambda)k & \text{if } e < -k \\ \lambda e & \text{if } |e| \le k \\ e - (1-\lambda)k & \text{if } e > k, \end{cases} \tag{3}$$

$$\phi_{bs}(e) = \begin{cases} e\left(1 - (1-\lambda)(1-(e/k)^2)^2\right) & \text{if } |e| \le k \\ e & \text{otherwise,} \end{cases} \tag{4}$$

and

$$\phi_{cb}(e) = \begin{cases} e & \text{if } e \le -p_1 \\ -\tilde{\phi}_{cb}(-e) & \text{if } -p_1 < e < -p_0 \\ \lambda e & \text{if } |e| \le p_0 \\ \tilde{\phi}_{cb}(e) & \text{if } p_0 < e < p_1 \\ e & \text{if } e \ge p_1, \end{cases} \tag{5}$$

where $0 < \lambda \le 1$, $k \ge 0$, and $0 \le p_0 < p_1$ denote suitable constants and $\tilde{\phi}_{cb}(\cdot)$ is the cubic polynomial that makes $\phi_{cb}(\cdot)$ and its first derivative continuous, that is,

$$\tilde{\phi}_{cb}(e) = \lambda e + (1-\lambda)\left(\frac{e-p_0}{p_1-p_0}\right)^2 \times \left(2p_1 + p_0 - (p_0+p_1)\left(\frac{e-p_0}{p_1-p_0}\right)\right).$$

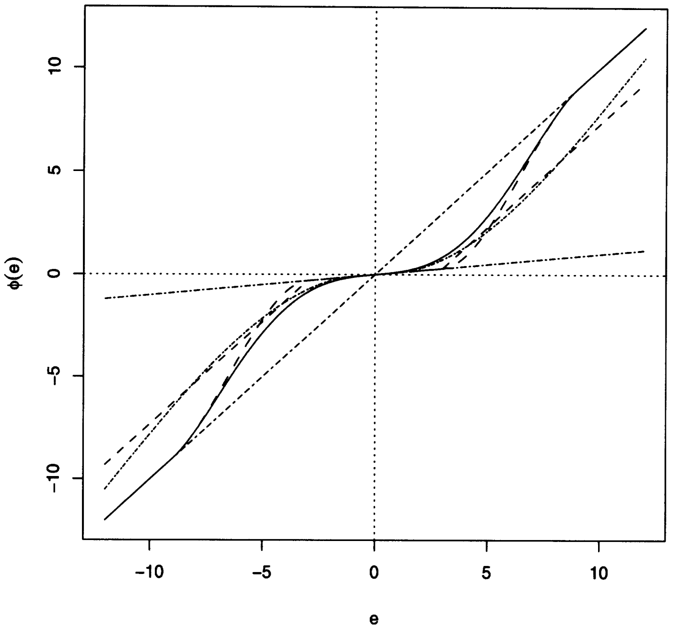

Figure 1. Comparisons of the $\phi(\cdot)$ Functions of Shewhart (– – -); EWMA with $\lambda = .1$ (- · – ·); and AEWMA Schemes $\phi_{hu}(\cdot)$ with $\lambda = .1$ and $k = 3$ (- - -); $\phi_{bs}(\cdot)$ with $\lambda = .1$ and $k = 9$ (—); $\phi^1_{cb}(\cdot)$ with $\lambda = .1$, $p_0 = 1$, and $p_1 = 18$ (- · · · -); and $\phi^2_{cb}(\cdot)$ with $\lambda = .1$, $p_0 = 3$, and $p_1 = 9$ (– – –).

The first two ϕ-functions are inspired by Huber's function (Huber 1981) and Tukey's bisquare function (Beaton and Tukey 1974). The third function has been suggested by an anonymous referee as another simple way to blend together the ϕ-functions of Shewhart and EWMA schemes. Because $\lim_{e\to\infty} \phi_{hu}(e)/e = 1$ but $\phi_{hu}(e) \neq e$ for every nonzero e, schemes based on (3) can almost (but not completely) ignore the past observations of the process. In contrast, schemes based on (4) and (5) completely discard the previous history when y_t is too far from x_{t-1}.

Figure 1 shows the ϕ-functions of the following schemes: Shewhart; an EWMA chart with $\lambda = .1$; a scheme based on $\phi_{hu}(\cdot)$ with $\lambda = .1$ and $k = 3$; a scheme based on $\phi_{bs}(\cdot)$ with $\lambda = .1$ and $k = 9$; a scheme based on $\phi_{cb}(\cdot)$ with $\lambda = .1, p_0 = 1$, and $p_1 = 18$ (denoted by ϕ^1_{cb}); and a scheme based on $\phi_{cb}(\cdot)$ with $\lambda = .1, p_0 = 3$, and $p_1 = 9$ (denoted by ϕ^2_{cb}). The figure illustrates how the suggested AEWMA charts update the control statistic like an EWMA scheme when the current observation y_t is close to x_{t-1} and like a Shewhart scheme when $|y_t - x_{t-1}|$ is large. In addition, note that for different choices of p_0 and p_1, $\phi_{cb}(e)$ can be close to the other two score functions for a wide range of values of e. This shows the greater flexibility of $\phi_{cb}(\cdot)$.

3. EXAMPLE

To illustrate an AEWMA control scheme, we use the capsules weights data given by Wetherill and Brown (1991, table 3.3). The data are a set of 50 measurements (in grams) taken every 30 seconds from a manufacturing process that is working in control. The target value is 5 g, and the standard deviation is $\sigma = 0.3$ g.

We simulate two different out-of-control situations, adding either 1σ or -3σ to the last 41 observations. Figure 2 shows a

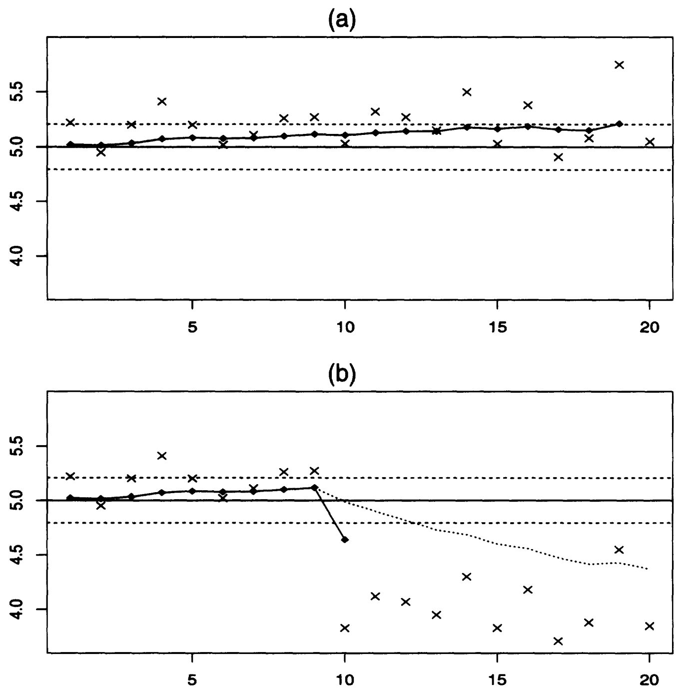

Figure 2. AEWMA Control Scheme Applied to the Capsule Weights. *(a) The value .3 (one standard deviation) is added to the last 41 observations. (b) The value .9 (three standard deviations) is subtracted from the last 41 observations. The original data are represented by X's. The solid line connects the AEWMA values. The dotted line in (b) connects the EWMA values with λ equal to .1. Dashed lines, in (a) and (b) are the control limits.*

plot of an AEWMA control statistic, together with the data. The scheme is based on $\phi_{hu}(\cdot)$ with $\lambda = .1$, $k = 3\sigma$, and $h = .6845\sigma$. The threshold h has been chosen so that the in-control ARL is equal to 500. In the $1\sigma/3\sigma$ shift cases, the scheme raises an alarm at the 10th/1st out-of-control observation; computations for the 3σ shift case are shown in Table 1. Observe in Figure 2(b) the jump of the control statistic at observation 10; the $w(e_t)$'s in Table 1 show that this jump is due to the adaptive nature of the scheme. In particular, y_{10} enters in the computation of x_{10} with a weight equal to 3.7λ.

Table 2 compares the delay in the detection of the change for three schemes: the AEWMA scheme previously considered and two EWMA schemes with the same in-control ARL suggested by Lucas and Saccucci (1990) to detect a 1σ shift and a 3σ shift

Table 1. Calculations of an AEWMA Scheme for the Capsule Weights *Data*

t	y_t	e_t	$\phi(e_t)$	$w(e_t)$	x_t
0					5.000
1	5.22	.220	.022	.10	5.022
2	4.95	−.072	−.007	.10	5.015
3	5.20	.185	.019	.10	5.033
4	5.41	.377	.038	.10	5.071
5	5.20	.129	.013	.10	5.084
6	5.02	−.064	−.006	.10	5.077
7	5.11	.032	.003	.10	5.081
8	5.26	.179	.018	.10	5.099
9	5.27	.171	.017	.10	5.116
10	3.83	−1.286	−.476	.37	4.640*

* Out-of-control signal.

NOTE: The process is in control for the first nine observations. 3σ has been subtracted from the original in-control 10th observation.

Table 2. Number of Out-of-Control Observations Required to Signal a Change for the Capsule Weights *Data*

Scheme	*1σ shift*	*3σ shift*
AEWMA based on $\phi_{hu}(\cdot)$ with $\lambda = .1$, $k = 3\sigma$, and $h = .6845\sigma$	10	1
EWMA with $\lambda = .12$ and $h = 2.8585\sigma\sqrt{\lambda/(2-\lambda)}$	10	3
EWMA with $\lambda = .70$ and $h = 3.0865\sigma\sqrt{\lambda/(2-\lambda)}$	20	1

NOTE: All schemes have an in-control ARL equal to 500. Lucas and Saccucci (1990) suggested the two EWMA schemes for detecting a 1σ shift and a 3σ shift.

in the process. Note that for both shifts, the single AEWMA scheme performs as the standard EWMA scheme designed to optimally detect that shift.

4. DESIGN OF AN AEWMA SCHEME

The usual design strategy is to find the scheme having minimum out-of-control ARL, for a specified shift μ, among the schemes with a desired in-control ARL. The problem is that the "optimal" scheme strongly depends on the specified magnitude of the shift. For example, according to Lucas and Saccucci (1990), if this strategy is used to design EWMA charts with an in-control ARL of 500, then the "optimal" value for λ goes from .05 when $\mu = .05$ to .95 when $\mu = 4$. As a consequence, the ARL of a chart designed for a small shift is quite different from that designed for a large shift.

Let θ be the parameters defining an AEWMA chart. For example, if $\phi_{hu}(\cdot)$ is used, then $\theta = (\lambda, h, k)$. To avoid the described flaw, we devise the following strategy:

1. Choose a desired in-control ARL, say B, and two values of the shift (e.g., a "small" shift, μ_1, and a "large" shift, μ_2).
2. For the specified in-control ARL, find the parameters θ^* having minimum ARL at μ_2; that is, θ^* is the solution of the following problem:

$$\begin{cases} \min_\theta \text{ARL}(\mu_2, \theta) \\ \text{subject to ARL}(0, \theta) = B, \end{cases}$$

where $\text{ARL}(\mu, \theta)$ denotes the ARL of a scheme with parameters equal to θ when the shift is μ.
3. Finally, choose a small positive constant α (e.g., $\alpha = .05$) and find the "optimal" θ as the solution of

$$\begin{cases} \min_\theta \text{ARL}(\mu_1, \theta) \\ \text{subject to ARL}(0, \theta) = B \\ \text{and ARL}(\mu_2, \theta) \le (1+\alpha)\text{ARL}(\mu_2, \theta^*); \end{cases} \tag{6}$$

that is, find the scheme with minimum ARL at μ_1 among those schemes for which the ARL at μ_2 is "nearly minimum."

This approach seems to produce AEWMA charts with reasonable performance for both small and large shifts. But the same approach is rather useless for the design of EWMA or CUSUM schemes. Indeed, when the foregoing strategy is applied to these control charts, the final θ is not substantially different from the intermediate θ^*, at least in terms of ARL. This