

Functional Analysis

lecture by

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Motivation

In linear algebra one mainly considers finite-dimensional vector spaces with additional structures like norm $\|\cdot\|$ or scalar product $\langle \cdot, \cdot \rangle$.

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional scalar product space and $A : V \rightarrow V$ a linear map, which is self-adjoint, that means for all $u, v \in V$:

$$\langle Au, v \rangle = \langle u, Av \rangle$$

Theorem (orthonormal eigenvector basis)

There exists an orthonormal eigenvector basis $(u_i)_{i \in \{1, \dots, n\}}$, that means with the eigenvalues $\lambda_i \in \mathbb{R}$:

$$\langle u_i, u_j \rangle = \delta_{ij} \qquad Au_i = \lambda_i u_i$$

In infinite dimensions the generalization is the *spectral theorem*.

First reformulate the result from linear algebra:

Let E_{λ_i} be the orthogonal projection operator on the eigenspace corresponding to λ_i . If this eigenspace is one dimensional, this means:

$$E_{\lambda_i} v = u_i \langle u_i, v \rangle = |u_i\rangle \langle u_i| v \rangle$$

Then one can write A as:

$$A = \sum_{i=1}^n \lambda_i E_{\lambda_i}$$

Theorem (spectral theorem)

Let $A \in L(H)$ be a self-adjoint (selbstadjungiert) operator, then it holds:

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

$\sigma(A) \subseteq \mathbb{R}$ is the spectrum of A and E_λ the projection-valued measure (Spektralmaß).

Applications typically are differential operators, for example:

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\Delta_{\mathbb{R}^3} : C_0^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3) \quad \text{linear operator}$$

Applications in more detail are studied in the lectures on partial differential equations I + II.

0 Basic Notions

Let E be a vector space (Vektorraum), for example the finite-dimensional vector space $E \simeq \mathbb{R}^3$. In the following list the later spaces are special cases of the previous ones:

- topological vector spaces
- metric spaces with a metric $d(.,.)$ (Polish spaces if complete)
- normed spaces with norm $\|.\|$ (Banach spaces if complete)
- scalar product spaces $\langle ., . \rangle$ (Hilbert spaces if complete)

Let \mathbb{K} be either \mathbb{R} or \mathbb{C} .

0.1 Definition (metric, ε -ball, Cauchy sequence, complete, Polish space)

A map $d : E \times E \rightarrow \mathbb{R}$ is called *metric*, if for all $x, y, z \in E$ holds:

- i) $d(x, y) = d(y, x)$ (symmetry)
- ii) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ (positive definiteness)
- iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

$B_\varepsilon(x) := \{z \in E \mid d(x, z) < \varepsilon\}$ is called ε -ball.

Consider the topology generated by $B_\varepsilon(x)$: A set $\Omega \subseteq E$ is open if and only if:

$$\forall_{x \in \Omega} \exists_{\varepsilon \in \mathbb{R}_{>0}} : B_\varepsilon(x) \subseteq \Omega$$

Completeness:

$(x_n)_{n \in \mathbb{N}}$ is a *Cauchy sequence* if and only if:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \exists_{N \in \mathbb{N}} \forall_{n, m \in \mathbb{N}_{>N}} : d(x_n, x_m) < \varepsilon$$

E is *complete* if and only if every Cauchy sequence has a limit.

A complete metric space is also called a *Polish space*.

0.2 Definition (norm, Banach space)

Let $(E, \|\cdot\|)$ be a *normed space*, i.e. a \mathbb{K} -vector space with a map $\|\cdot\| : E \rightarrow \mathbb{R}_{\geq 0}$ called *norm* with the following properties for $x, y \in E$ and $\lambda \in \mathbb{K}$:

- i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (positive definiteness)

ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity)

iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

Define the metric $d(x, y) := \|x - y\|$. A complete normed spaces is called *Banach space*.

Let $A : E \rightarrow F$ be a linear map between the Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$.

0.3 Definition (continuous, bounded)

A is *continuous* (stetig) if $A^{-1}(\Omega) \subseteq E$ is open for all open $\Omega \subseteq F$.

A is *bounded* (beschränkt) if there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq C \|u\|_E$$

0.4 Lemma (continuous \Leftrightarrow bounded)

A is continuous $\Leftrightarrow A$ is bounded.

(no proof)

0.5 Definition (dual space, sup-norm)

The *dual space* of E is the space of continuous linear mappings from E to \mathbb{K} :

$$E^* = L(E, \mathbb{K})$$

$L(E, F)$ is a vector space: For $A, B \in L(E, F)$, $\lambda, \mu \in \mathbb{K}$ and $u \in E$ define:

$$(\lambda A + \mu B)(u) := \lambda A(u) + \mu B(u)$$

Define also a norm on $L(E, F)$, which is called *sup-norm*:

$$\|A\| := \sup_{u \in E, \|u\|_E \leq 1} \|Au\|_F$$

0.6 Theorem

If F is complete, so is $L(E, F)$.

In particular E^* is a Banach space for every E .

(no proof)

1 The Hahn-Banach Theorem and Applications

As a preparation we need Zorn's lemma.

1.1 Definition (partial ordering, chain, upper bound, maximal)

Let A be a set and \leq a *partial ordering* (Halbordnung), i.e. for all $a, b, c \in A$:

- i) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)
- ii) $a \leq a$ (reflexivity)
- iii) $a \leq b \wedge b \leq a \Rightarrow a = b$ (antisymmetry)

Note: We do *not* demand that for all $a, b \in A$ holds:

$$(a \leq b) \vee (b \leq a)$$

This is a property of a ordering relation.

(A, \leq) is called *partially ordered set* (teilweise geordnete Menge).

A subset $K \subseteq A$ is called *chain* (Kette, total geordnete Teilmenge) if for all $x, y \in K$ holds:

$$(x \leq y) \vee (y \leq x)$$

An element $u \in A$ is called *upper bound* (obere Schranke) of $B \subseteq A$ if $x \leq u$ for all $x \in B$.

An element $m \in A$ is called *maximal* if $m \leq a \in A \Rightarrow m = a$.

1.2 Zorn's lemma

Let (A, \leq) be a partially ordered set in which every chain has an upper bound. Then there is a maximal element.

Proof

This follows from the axiom of choice, see e.g. Kowalsky: Linear algebra.

1.3 Definition (sublinear)

Let X be a *real* vector space (without topology) and $l : X \rightarrow \mathbb{R}$ linear. $p : X \rightarrow \mathbb{R}$ is called *sublinear* if for all $x, y \in X$ and $a \in \mathbb{R}_{>0}$:

- i) $p(ax) = ap(x)$
- ii) $p(x + y) \leq p(x) + p(y)$

A typical example is $p(x) = \|x\|$, but p does not need to be positive. Another example is any linear mapping.

1.4 Theorem (Hahn-Banach, real version, 1927/29)

Let X be a real vector space and $Y \subseteq X$ a subspace (Untervektorraum), $p : X \rightarrow \mathbb{R}$ sublinear and $l : Y \rightarrow \mathbb{R}$ linear with $l(y) \leq p(y)$ for all $y \in Y$.

Then there is a linear extension (Fortsetzung) $\tilde{l} : X \rightarrow \mathbb{R}$ of l to X , i.e. $\tilde{l}|_Y = l$, such that for all $x \in X$ holds:

$$\tilde{l}(x) \leq p(x)$$

Proof

- i) Assume $Y \subsetneq X$, since otherwise there is nothing to prove. Choose a vector $z \in X \setminus Y$. We want to extend l to the span of Y and $\langle z \rangle$. $\tilde{l}(z)$ needs to be prescribed. For all $y \in Y$ and $a \in \mathbb{R}$ holds:

$$\tilde{l}(y + az) \stackrel{\text{linearity}}{=} l(y) + a\tilde{l}(z) \stackrel{\text{demand}}{\leq} p(y + az)$$

If $a = 0$, the inequality is clear. By homogeneity assumptions, it is sufficient to consider the case $a = \pm 1$. We thus demand for all $y, y' \in Y$:

$$\begin{aligned} l(y) + \tilde{l}(z) &\leq p(y + z) \\ l(y') - \tilde{l}(z) &\leq p(y' - z) \end{aligned}$$

This is equivalent to:

$$l(y') - p(y' - z) \leq \tilde{l}(z) \leq p(y + z) - l(y)$$

We can choose $\tilde{l}(z)$ if and only if:

$$l(y') - p(y' - z) \leq p(y + z) - l(y)$$

(For example set $\tilde{l}(z) = \sup_{y' \in Y} l(y') - p(y' - z)$.)

$$\Leftrightarrow l(y') + l(y) \stackrel{\text{linearity}}{=} l(y' + y) \leq p(y + z) + p(y' - z)$$

Now prove this inequality:

From $y' + y \in Y$ follows that $l(y + y') \leq p(y + y')$ by hypothesis. Moreover, as p is sublinear, it follows:

$$p(y + z - z + y') \leq p(y' + z) + p(y' - z)$$

So the inequality is shown. Thus l can be extended to $Y + \langle z \rangle$.

ii) Consider all extensions:

$$A := \{(Z, l) \mid Y \subseteq Z \subseteq X \text{ subspace, } l : Z \rightarrow \mathbb{R} \text{ extension of } l_Y : Y \rightarrow \mathbb{R}\}$$

This set has a partial ordering \leq defined by $(Z, l) \leq (Z', l')$ if $Z \subseteq Z'$ and $l'|_Z = l$.

For an index set I (possibly infinite, uncountable) let $K = \{(Z_\nu, l_\nu) \mid \nu \in I\}$ be a chain, i.e. for all $(Z, l), (Z', l') \in K$:

$$((Z, l) \leq (Z', l')) \vee ((Z, l) \leq (Z, l))$$

Set $Z = \bigcup_{\nu \in I} Z_\nu$ and define $l : Z \rightarrow \mathbb{R}$ by $l|_{Z_\nu} = l_\nu$. (Thus suppose $u \in Z$, so there is a $\nu \in I$ with $u \in Z_\nu$. Set $l(u) := l_\nu(u)$. ν need not be unique. Suppose $u \in Z_{\nu'}$, then we know that either $Z_{\nu'} \subseteq Z_\nu$ and $l_\nu|_{Z_{\nu'}} = l_{\nu'}$ or $Z_\nu \subseteq Z_{\nu'}$ and $l_{\nu'}|_{Z_\nu} = l_\nu$. In both cases we have $l_\nu(u) = l_{\nu'}(u)$, thus $l(u)$ is well defined.)

This (Z, l) is an upper bound, because for all $\nu \in I$ we have $Z_\nu \subseteq Z = \bigcup_{\lambda \in I} Z_\lambda$ and l is an extension of l_ν .

With Zorn's Lemma follows, that there exists an maximal element (\tilde{Y}, \tilde{l}) .

Claim: $\tilde{Y} = X$

Proof: Otherwise there would be a vector $u \in X \setminus \tilde{Y}$, and \tilde{l} could be extended to $\tilde{Y} \oplus \langle u \rangle$, as shown in i), in contradiction to the maximality of \tilde{l} . Thus $(X = \tilde{Y}, \tilde{l})$ is the desired extension.

□_{1.4}

1.5 Theorem (Hahn-Banach, complex version)

Let X be a complex vector space and $Y \subseteq X$ a subspace. Before, we had $l(x) \leq p(x)$ as condition, which does not make sense in the complex case, since:

$$l(e^{i\varphi}x) = e^{i\varphi}l(x) \stackrel{\text{in general}}{\notin} \mathbb{R}$$

Let $p : X \rightarrow \mathbb{R}$ be a *seminorm*, i.e.:

- i) $p(ax) = |a|p(x)$ (homogeneity)
- ii) $p(x+y) \leq p(x) + p(y)$ (triangle inequality)

Let $l : Y \rightarrow \mathbb{C}$ be a linear functional with $|l(y)| \leq p(y)$ for all $y \in Y$.

Then l can be extended to X such that $|l(x)| \leq p(x)$ holds for all $x \in X$.

Proof

We also consider X as a real vector space. (u and iu are then linearly independent vectors.) Decompose l into its real and imaginary parts.

$$\begin{aligned} l(y) &= l_1(y) + i l_2(y) \\ l_1 &:= \operatorname{Re}(l(y)) \\ l_2 &:= \operatorname{Im}(l(y)) \end{aligned}$$

l_1 and l_2 are real-linear and:

$$l_1(\mathbf{i}y) = \operatorname{Re}(l(\mathbf{i}y)) = \operatorname{Re}(\mathbf{i}l(y)) = -\operatorname{Im}(l(y)) = -l_2(y)$$

Conversely, suppose that l_1 is real-linear. Then

$$l(x) := l_1(x) - \mathbf{i} \cdot l_1(\mathbf{i}x)$$

this is indeed a complex-linear function. We know that $|l(y)| \leq p(y)$ holds for all $y \in Y$.

$$\begin{aligned} l_1(y) &= \operatorname{Re}(l(y)) \leq |l(y)| \\ \Rightarrow \quad l_1(y) &\leq p(y) \end{aligned}$$

Theorem 1.4 yields an real-linear extension $\tilde{l}_1 : X \rightarrow \mathbb{R}$ such that $\tilde{l}_1(x) \leq p(x)$ for all $x \in X$. Set $\tilde{l}(x) = \tilde{l}_1(x) - \mathbf{i}\tilde{l}_1(\mathbf{i}x)$, so that $\tilde{l} : X \rightarrow \mathbb{C}$ is complex-linear.

Claim: $|\tilde{l}(x)| \leq p(x) \quad \forall x \in X$

Proof: Polar decomposition:

$$\begin{aligned} \tilde{l}(x) &= r e^{\mathbf{i}\varphi} \\ |\tilde{l}(x)| &= r = e^{-\mathbf{i}\varphi} \tilde{l}(x) \stackrel{\tilde{l} \text{ is complex-linear}}{=} \tilde{l}(e^{-\mathbf{i}\varphi} x) = \operatorname{Re}(\tilde{l}(e^{-\mathbf{i}\varphi} x)) = \\ &= \tilde{l}_1(e^{-\mathbf{i}\varphi} x) \leq p(e^{-\mathbf{i}\varphi} x) \stackrel{\text{homogeneity}}{=} p(x) \end{aligned}$$

□_{1.5}

Now to applications:

1.6 Theorem

Let $(X, \|\cdot\|)$ be a normed \mathbb{K} -space (real or complex), $Y \subseteq X$ a subspace. Let φ be a continuous linear functional from Y to \mathbb{K} , i.e. for all $y \in Y$ holds:

$$|\varphi(y)| \leq c \|y\|$$

Then φ can be continued to all of X with the same supnorm, i. e.:

$$\|\tilde{\varphi}\| := \sup_{x \in X, \|x\| \leq 1} |\varphi(x)| = \|\varphi\| := \sup_{y \in Y, \|y\| \leq 1} |\varphi(y)|$$

Proof

Apply the Hahn-Banach theorem with $\varphi := c \|x\|$.

□_{1.6}

1.7 Corollary

Let X be a normed space and $u_0 \in X$ with $\|u_0\| = 1$. Then there exists a linear functional $\varphi : X \rightarrow \mathbb{K}$ such that:

$$\varphi(u_0) = 1 \qquad \|\varphi\| = 1$$

Proof

Let $Y := \langle u_0 \rangle$ and define $\varphi_0 : \langle u_0 \rangle \rightarrow \mathbb{K}$ by $\varphi_0(u_0) = 1$. Extend φ_0 by the Hahn-Banach theorem 1.6. $\square_{1.7}$

The Hahn-Banach theorem also has a geometric formulation. Consider only the real case:
A set $K \subseteq X$ is called *convex* if for all $x, y \in K$ and $\tau \in [0, 1]$:

$$\tau x + (1 - \tau)y \in K$$

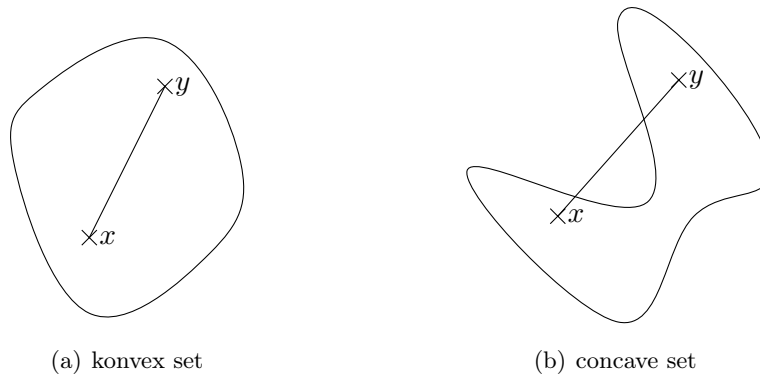


Figure 1.1: convexity

Geometric question:

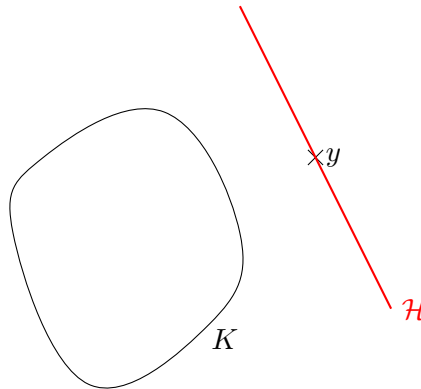


Figure 1.2: not intersecting hyperplane

Is there a hyperplane \mathcal{H} , which meets $y \notin K$, but does not intersect K . $\square_{1.7}$

1.8 Definition (interior point)

$x_0 \in K$ is an *interior point* (innerer Punkt) of K with respect to $u \in X$ if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ for all $t \in (-\varepsilon, \varepsilon)$.

$x_0 \in K$ is an *interior point* if for all $u \in X$ there is a $\varepsilon = \varepsilon(u) \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ for all $t \in (-\varepsilon, \varepsilon)$.

1.9 Theorem (geometric Hahn-Banach)

Let $K \neq \emptyset$ be convex and all points of K be interior points. Let $y \notin K$. Then there is a linear functional $l : X \rightarrow \mathbb{R}$ such that $l(x) < 1$ for all $x \in K$ and $l(y) = 1$.

$\mathcal{H} := \{x \in X \mid l(x) = 1\}$ defines a hyperplane. Now $y \in \mathcal{H}$ and $l|_K < 1$ mean that K lies in one half-space.

First introduce a suitable sublinear functional. Without loss of generality, assume $0 \in K$ (otherwise shift K).

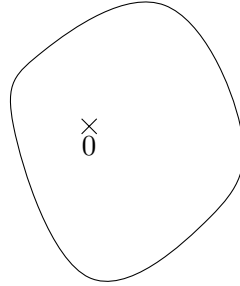


Figure 1.3: $0 \in K$

The functional $p : K \rightarrow \mathbb{R}_{\geq 0}$ with

$$p(x) := \inf \left\{ a \in \mathbb{R}_{>0} \mid \frac{x}{a} \in K \right\}$$

is called gauge (Eichung).

Since x is an interior point, we know that $\frac{x}{a} \in K$ if $a > 1 - \varepsilon(x)$.

p is even defined on all of X , because for $x \in X$, now $\tau x \in K$ if $|\tau|$ is sufficiently small, because $0 \in K$ is an interior point.

$$p(x) < 1 \quad \Leftrightarrow \quad x \in K$$

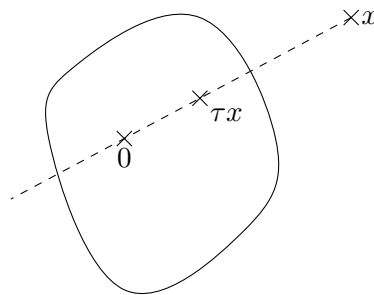


Figure 1.4: $x \notin K$, $\tau x \in K$

1.10 Lemma

p is sublinear.

Proof

The homogeneity is clear from the definition.

sub-additivity (triangle equation):

Take $x, y \in K$ and choose $a, b \in \mathbb{R}_{>0}$ such that $\frac{x}{a}, \frac{y}{b} \in K$. The convexity of K implies for all $\tau \in [0, 1]$:

$$\tau \frac{x}{a} + (1 - \tau) \frac{y}{b} \in K$$

Choose $\tau = \frac{a}{a+b}$, then holds $1 - \tau = \frac{b}{a+b}$, which gives:

$$\Rightarrow \frac{1}{a+b} (x+y) \in K$$

$$p(x+y) \leq a+b$$

Taking the infimum over a and b gives $p(x+y) \leq p(x) + p(y)$:

$$p(x+y) = \inf \underbrace{\left\{ c \in \mathbb{R}_{>0} \mid \frac{x+y}{c} \in K \right\}}_{\ni a+b} \leq a+b$$

$$\begin{aligned} p(x) = \inf \left\{ a \mid \frac{x}{a} \in K \right\} &\Rightarrow \forall_{\varepsilon > 0} \exists_{a \in \mathbb{R}_{>0}} : p(x) \geq a - \varepsilon \\ p(y) = \inf \left\{ b \mid \frac{y}{b} \in K \right\} &\Rightarrow \forall_{\varepsilon > 0} \exists_{b \in \mathbb{R}_{>0}} : p(y) \geq b - \varepsilon \end{aligned}$$

□_{1.10}

1.11 Lemma

$$p(x) < 1 \Leftrightarrow x \in K$$

Proof

If $x \notin K$ then $\frac{1}{a}x \notin K$ for all $0 < a < 1$ and so $p(x) \geq 1$.

For all $x \in K$ exists an $\varepsilon = \varepsilon(x) \in \mathbb{R}_{>0}$ with $(1+t)x \in K$ for all $t \in (-\varepsilon, \varepsilon)$.

$$\begin{aligned} &\Rightarrow \left(1 + \frac{\varepsilon}{2}\right)x \in K \\ &\Rightarrow p(x) \leq \frac{1}{1 + \frac{\varepsilon}{2}} < 1 \end{aligned}$$

□_{1.11}

Proof of Theorem 1.9

Introduce l on $\langle y \rangle$ by $l(y) = 1$. (Assume again that $0 \in K$ and so $y \neq 0$.)

Write $z = ay \in \langle y \rangle$ with $a \in \mathbb{R}$.

- If $a < 0$, then $l(z) = a \cdot l(y) = a < 0$ but $p(z) \geq 0$ and thus the inequality $l(z) \leq p(z)$ is trivially satisfied.
- If $a > 0$ it holds:

$$l(z) = a \underset{\Rightarrow p(y) \geq 1}{\overset{y \notin K}{\leq}} a \cdot p(y) \overset[\text{homogeneity}]{\text{positive}} p(ay) = p(z)$$

So for all $z \in \langle y \rangle$ holds $l(z) \leq p(z)$.

The Hahn-Banach Theorem yields an extension $l : X \rightarrow \mathbb{R}$ such that $l(x) \leq p(x)$ for all $x \in X$.

Therefore for all $x \in K$ we have:

$$l(x) \leq p(x) < 1$$

□_{1.9}

2 Normed Spaces

Let $(E, \|\cdot\|)$ be a normed space and let the open balls $B_\varepsilon(x) = \{y \mid \|x - y\| < \varepsilon\}$ generate the topology on E .

2.0.1 Definition (equivalent norms)

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent*, if there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1$$

2.0.2 Theorem

Equivalent norms give rise to the same topology.

(No proof)

2.0.3 Theorem

If E is finite dimensional, then any two norms on E are equivalent.

(No proof)

2.0.4 Constructions (Quotient space, Cartesian product)

Let $F \subseteq E$ be a *closed* subspace. Define the *quotient space* (Faktorraum) E/F as follows:

$$x \sim y \Leftrightarrow x - y \in F$$

defines an equivalence relation on E .

$$E/F := E/\sim$$

is a vector space.

$$\|u\|_{E/F} := \inf_{\substack{\hat{u} \in E \\ \hat{u} - u \in F}} \|\hat{u}\|_E$$

$(E/F, \|\cdot\|_{E/F})$ is a normed space. The closedness of F is essential:

Suppose $F \subseteq E$ is not closed. Then there exists an $x \in \overline{F} \setminus F$, thus there is a $(x_n)_{n \in \mathbb{N}}$, $x_n \in F$

with $x_n \rightarrow x$.

Let $[x] \in E/F$ be the equivalence class. Then $[x] \neq 0$, since $x \notin F$, but:

$$\|[x]\| = \inf_{\substack{\hat{x} \in E \\ \hat{x} - x \in F}} \|\hat{x}\| \stackrel{x - x_n \sim x}{\leq} \inf \|x - x_n\| = 0$$

If $\|\cdot\|_{E/F}$ was a norm, it would imply $[x] = 0$ and thus $x \in F$ in contradiction to $x \in \overline{F} \setminus F$.

Another construction is the *Cartesian product*: Let E and F be normed spaces.

$$E \times F := \{(u, v) \mid u \in E, v \in F\}$$

$$\|(u, v)\|_{E \times F} := \|u\|_E + \|v\|_F$$

is a norm on $E \times F$.

2.0.5 Definition (separable)

A normed space is called *separable*, if there is a countable dense subset, i.e. there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that every nonempty open subset of the space contains at least one element of the sequence.

2.0.6 Examples

The space ℓ^∞ of bounded sequences $(a_n)_{n \in \mathbb{N}}$, $a_n \in \mathbb{K}$ with $\|(a_n)_{n \in \mathbb{N}}\|_\infty := \sup_n |a_n|$ is a Banach space.

$$A := \left\{ (a_n)_{n \in \mathbb{N}} \mid a_{2n} = 0 \ \forall_{n \in \mathbb{N}} \right\} \subseteq \ell^\infty$$

is a closed subspace.

$$\ell^\infty / A \cong \left\{ (a_n) \mid a_{2n+1} = 0 \ \forall_{n \in \mathbb{N}} \right\}$$

$$d := \left\{ (a_n) \mid \exists_{N \in \mathbb{N}} \forall_{n \in \mathbb{N}_{>0}} a_n = 0 \right\} \subseteq \ell^\infty$$

is a subspace, but not closed in ℓ^∞ . Consider for example $(a_n = \frac{1}{n}) =: x \in \ell^\infty \setminus d$, $x_n \in d$ with $x_n = (a_{n_l})_{l \in \mathbb{N}}$ and:

$$a_{n_l} = \begin{cases} \frac{1}{l} & \text{if } l \leq n \\ 0 & \text{if } l > n \end{cases}$$

Then converges $x_n \rightarrow x \notin d$, and therefore d is not closed. The closure is:

$$\overline{d} = \left\{ (a_n) \mid a \xrightarrow{n \rightarrow \infty} 0 \right\}$$

ℓ^∞ is not separable.

2.0.7 Example

For $1 \leq p < \infty$ define

$$\ell^p = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

and the ℓ^p -norm:

$$\|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}$$

ℓ^p is a normed space (Hölder's inequality, Minkowski inequality) and also separable (see exercises).

2.0.8 Example

Let (Ω, μ) be a measure space (Maßraum).

$$\begin{aligned} L^p(\Omega) \quad (1 \leq p < \infty) \quad & \|f\|_p = \left(\int_{\Omega} |f(x)|^p d\mu \right)^{\frac{1}{p}} \\ L^{\infty}(\Omega) \quad & \|f\|_{\infty} = \sup_{\Omega} |f(x)| = \sup \{ L \in \mathbb{R} \mid \mu(f^{-1}([L, \infty))) > 0 \} \end{aligned}$$

2.1 Non-Compactness of the Unit Ball

Let $(E, \|\cdot\|)$ be a normed vector space.

$$K := \overline{B_1(0)} = \{x \in E \mid \|x\| \leq 1\}$$

If $\dim(E) < \infty$, K is compact by the Heine-Borel theorem.

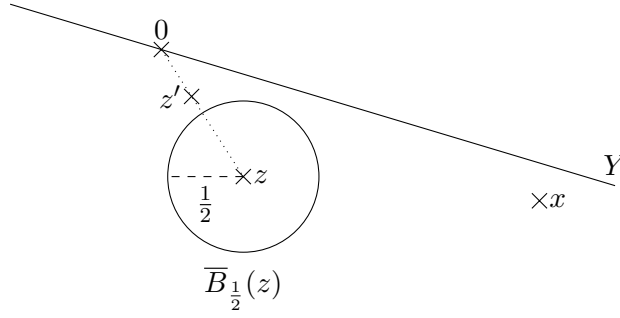
2.1.1 Theorem

If E is infinite-dimensional, then K is not sequentially compact (folgenkompakt), i.e. it is possible to construct a sequence (y_n) , $y_n \in K$, which has no convergent subsequence.

2.1.2 Lemma

Let $Y \subsetneq E$ be a proper (echter) closed subspace. Then there is a $z \in E \setminus Y$ with $\|z\| = 1$ such that holds:

$$\begin{aligned} \forall_{y \in Y} : \|z - y\| &> \frac{1}{2} \\ \Leftrightarrow \quad \overline{B_{\frac{1}{2}}(z)} \cap Y &= \emptyset \end{aligned}$$


 Figure 2.1: $\overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset$

Proof

Choose $x \in E \setminus Y \neq \emptyset$. As $E \setminus Y$ is open, there is a $\delta \in \mathbb{R}_{>0}$ with $B_\delta(x) \cap Y = \emptyset$. Thus we can define:

$$d := \inf_{y \in Y} \|x - y\| > 0$$

Choose $y_0 \in Y$ such that $\|x - y_0\| < 2d$. Set $z' = x - y_0$. Then $\|z'\| < 2d$ and $\|z' - y\| \geq d$ for all $y \in Y$. Thus $z := \frac{z'}{\|z'\|}$ has the desired properties. $\square_{2.1.2}$

Proof of Theorem 2.1.1

Choose inductively a sequence (y_n) : $y_1 \in K$ is arbitrary. $Y_1 := \langle y_1 \rangle$ is a one dimensional subspace, which is closed. Choose $y_2 \in K$ such that $\|y_2 - y\| > \frac{1}{2}$ for all $y \in Y_1$, which is possible according to Lemma 2.1.2.

Suppose y_1, \dots, y_n are given. $Y_n := \langle y_1, \dots, y_n \rangle$ is closed. So there exists a $y_{n+1} \in K$ such that for all $y \in Y_n$ holds:

$$\|y_{n+1} - y\| > \frac{1}{2}$$

This sequence has the following properties:

- $y_k \in K$
- For all $k, l \in \mathbb{N}$ with $k < l$ holds $\|y_l - y_k\| > \frac{1}{2}$, since $y_k \in Y_{l-1} = \langle y_1, \dots, y_{l-1} \rangle$ and we know by construction that $\|y_l - y\| > \frac{1}{2}$ for all $y \in Y_{l-1}$ so especially for $y_k \in Y_{l-1}$.

This implies that (y_k) has no convergent subsequence. $\square_{2.1.2}$

2.2 Spaces of linear Mappings, Dual Spaces

Let E, F be normed spaces.

$A : E \rightarrow F$ is continuous if and only if it is bounded, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq C \|u\|_E$$

Denote by $L(E, F)$ the normed space of all bounded linear maps from E to F and define:

$$\|A\| := \sup_{\|u\| \leq 1} \|Au\| = \sup_{\|u\|=1} \|Au\|$$

2.2.1 Lemma

If $B \in L(E, F)$ and $A \in L(F, G)$ then Scharz inequality or Kato inequality holds:

$$\begin{aligned}\|A \cdot B\| &\leq \|A\| \cdot \|B\| \\ \|Au\| &\leq \|A\| \cdot \|u\|\end{aligned}$$

(no proof)

2.2.2 Theorem and Definition (dual pairing)

If F is complete, so is $L(E, F)$.

Special case $F = \mathbb{R}$ and $\|x\|_{\mathbb{R}} = |x|$: $E^* := L(E, \mathbb{R})$ is the dual space.

For $\varphi \in E^*$ and $u \in E$

$$\varphi(u) = (\varphi, u)$$

is called *dual pairing* (duale Paarung).

$$(\cdot, \cdot) : E^* \times E \rightarrow \mathbb{R}$$

is a continuous bilinear map. For $u \in E$

$$(\cdot, u) : E^* \rightarrow \mathbb{R}$$

defines an element of $E^{**} = L(E^*, \mathbb{R})$. This gives rise to a linear mapping:

$$\iota : E \rightarrow E^{**}$$

(no proof)

2.2.3 Theorem

$\iota : E \hookrightarrow E^{**}$ is an isometric embedding of E into E^{**} .

Proof

For $u \in E$ holds:

$$\|\iota(u)\| := \sup_{\varphi \in E^*, \|\varphi\|=1} \|(\iota(u))(\varphi)\| = \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| \stackrel{?}{=} \|u\|$$

$$\|\varphi\| = \sup_{v \in E, \|v\|=1} |\varphi(v)|$$

$$\begin{aligned}\|\varphi(u)\| &\leq \|\varphi\| \cdot \|u\| \stackrel{\|\varphi\|=1}{=} \|u\| \\ \Rightarrow \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| &\leq \|u\|\end{aligned}$$

To prove $\|\iota(u)\| \geq \|u\|$ apply the Hahn-Banach theorem:

Let $l : \langle u \rangle \rightarrow \mathbb{R}$ be the linear map with $l(u) = \|u\|$, thus:

$$\|l\| = \sup_{v \in \langle u \rangle, \|v\|=1} (l(v)) = \sup \left(l \left(\pm \frac{u}{\|u\|} \right) \right) = 1$$

By the Hahn-Banach theorem we can extend l to

$$\tilde{l} : E \rightarrow \mathbb{R}$$

with $\|\tilde{l}\| = 1$ and then holds:

$$\sup_{\varphi \in E^*, \|\varphi\|=1} \varphi(u) \stackrel{\|\tilde{l}\|=1}{\geq} \tilde{l}(u) = \|u\|$$

Therefore ι is injective, because from $\iota(u) = 0$ follows $\|u\|_E = \|\iota(u)\| = 0$ and therefore $u = 0$. $\square_{2.2.3}$

2.2.4 Definition (reflexive)

A Banach space is called *reflexive* (reflexiv) if ι is bijective, i.e. $E \cong E^{**}$.

2.2.5 Example

Let ℓ_1 be the space of absolutely convergent functions with the norm:

$$\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty$$

Let $(\lambda_n) \in \ell_{\infty}$ be a bounded sequence and define $\Lambda \in \ell_1^*$:

$$\begin{aligned} \Lambda : \ell_1 &\rightarrow \mathbb{R} \\ \Lambda((a_n)) &= \sum_{n=1}^{\infty} \lambda_n a_n \end{aligned}$$

$$|\Lambda((a_n))| = \left| \sum_{n=1}^{\infty} \lambda_n a_n \right| \leq \sum_{n=1}^{\infty} |\lambda_n| \cdot |a_n| \leq \|(\lambda_n)\|_{\infty} \sum_{n=1}^{\infty} |a_n| = \|(\lambda_n)\|_{\infty} \cdot \|(a_n)\|_1 < \infty$$

Thus Λ is bounded and:

$$\|\Lambda\| = \sup_{n \in \mathbb{N}} |\lambda_n|$$

Claim: Every bounded linear functional on ℓ_1 is of this form, i.e. $\ell_1^* = \ell_{\infty}$.

Proof: Let $\Lambda \in \ell_1^*$. Choose $u_l \in \ell_1$ by $u_l = (0, \dots, 0, 1, 0, \dots)$ with a one at the l -th position.

Setting $\lambda_l := \Lambda(u_l)$ gives:

$$|\lambda_l| = |\Lambda(u_l)| \leq \underbrace{\|\Lambda\|}_{< \infty} \cdot \underbrace{\|u_l\|}_{=1} \leq \|\Lambda\| < \infty$$

So $(\lambda_l) \in \ell_\infty$.

Let (a_k) be a finite sequence, with only zeros for $k > K \in \mathbb{N}$. Then:

$$\Lambda((a_k)) = \Lambda\left(\sum_{k=1}^K a_k u_k\right) = \sum a_k \Lambda(u_k) = \sum \lambda_k a_k$$

Since the finite sequences are dense in ℓ_1 , the claim follows. \square_{Claim}

So $\ell_1^* = \ell_\infty$ and one could assume $\ell_\infty^* = \ell_1$, but this is not the case (see exercises).

Thus $\ell_1^{**} \neq \ell_1$, which means, that ℓ_1 is *not* reflexive.

2.3 Weak Convergence (Schwache Konvergenz)

Let E be a Banach space and (u_n) a sequence in E .

Normal convergence: $u_n \rightarrow u$ if and only if $\|u - u_n\| \xrightarrow{n \rightarrow \infty} 0$.

2.3.1 Definition (weak convergence, weak Cauchy sequence)

A sequence (u_n) in E *converges weakly* to u , written as $u_n \rightharpoonup u$, if for all $\varphi \in E^*$ the sequence $\varphi(u_n)$ converges to $\varphi(u)$, i.e. $\varphi(u_n) \rightarrow \varphi(u)$.

(u_n) is a *weak Cauchy sequence* if for all $\varphi \in E^*$ the sequence $\varphi(u_n)$ is a Cauchy sequence.

2.3.2 Theorem (Uniqueness of weak limit)

The weak limit is unique.

Proof

Let (u_n) be a sequence in E , which converges weakly to u and u' , i.e. for all $\varphi \in E^*$ holds:

$$\begin{aligned} \varphi(u_n) &\rightarrow \varphi(u) & \varphi(u_n) &\rightarrow \varphi(u') \\ \Rightarrow & 0 & \rightarrow \varphi(u - u') \end{aligned}$$

So $\varphi(u - u') = 0$ for all $\varphi \in E^*$.

Claim: $v := u - u' = 0$

Proof: Assume to the contrary that $v \neq 0$.

Choose $\varphi : \langle v \rangle \rightarrow \mathbb{R}$ with $\varphi(v) = 1$. By the Hahn-Banach theorem φ can be extended continuously to E .

Therefore there exists a $\varphi \in E^*$ with $\varphi(v) = 1$, which is a contradiction to $\varphi(v) = 0$.

$\square_{2.3.2}$

2.3.3 Theorem (convergence implies weak convergence)

Every convergent sequence converges weakly.

Proof

Suppose that $u_n \rightarrow u$. For $\varphi \in E^*$ follows:

$$|\varphi(u_n) - \varphi(u)| = |\varphi(u_n - u)| \leq \underbrace{\|\varphi\|}_{\in \mathbb{R}} \cdot \|u_n - u\| \rightarrow 0$$

$$\begin{aligned} \Rightarrow \quad \varphi(u_n) &\rightarrow \varphi(u) \\ \Rightarrow \quad u_n &\rightharpoonup u \end{aligned}$$

□_{2.3.3}**2.3.4 Example**

$E = \left\{ (a_n) \left| a_n \xrightarrow{n \rightarrow \infty} 0 \right. \right\} \subsetneq \ell_\infty$ with $\|(a_n)\| = \sup_n |a_n|$ is a Banach space.

Let $u_n = (0, \dots, 0, 1, 0, \dots)$ be the sequence with a one at the n -th position and zeros elsewhere. For $n \neq m$ we have:

$$\|u_n - u_m\| = \sup \{0, |1|, |-1|\} = 1$$

Thus (u_n) is *not* a Cauchy sequence. Every $\varphi \in E^*$ can be represented with $(\lambda_k) \in \ell_1$ as (see exercises):

$$\begin{aligned} \varphi((a_n)) &= \sum_k \lambda_k a_k \\ \|\varphi\| &= \sum_{k=1}^{\infty} |\lambda_k| < \infty \end{aligned}$$

$$\varphi(u_n) = \sum_{k=1}^{\infty} \lambda_k \delta_{kn} = \lambda_n \xrightarrow{n \rightarrow \infty} 0$$

From $(\lambda_n) \in \ell_1$ follows $\lambda_n \rightarrow 0$. This means that $u_k \rightharpoonup 0$.

This is used in the lectures on partial differential equations.

From $\mathcal{S}(u_n) \rightarrow \inf \mathcal{S}$ follows not necessarily $u_n \rightarrow u$, but $u_n \rightharpoonup u$.

Consider $A_n \in L(E, F)$.

- *norm convergence*: $A_n \rightarrow A$ in $L(E, F)$ means $\|A_n - A\| \rightarrow 0$.
- *strong convergence*: $A_n u \rightarrow Au$ in F for all $u \in E$.
- *weak convergence*: $A_n u \rightharpoonup Au$ for all $u \in E$, i.e. for all $\varphi \in F^*$ holds $\varphi(A_n u) \rightarrow \varphi(Au)$.

2.4 The Baire Category Theorem

Let E be a metric space (e.g. a normed space).

2.4.1 Definition (nowhere dense, set of first/second category)

A subset $A \subseteq E$ is called *nowhere dense* (nirgends dicht) if $\overline{A}^\circ = \emptyset$.

A is called *of first category* (or *meager*) if it can be written as a countable union of nowhere dense sets. Otherwise it is *of second category*.

Example

- $\mathbb{N} \subseteq \mathbb{R}$ is nowhere dense: $\overline{\mathbb{N}} = \mathbb{N}$, $\mathbb{N}^\circ = \emptyset$
- $\mathbb{Q} \subseteq \mathbb{R}$ is dense: $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Q}}^\circ = \mathbb{R}^\circ = \mathbb{R}$

2.4.2 Theorem (René Baire, 1899)

Let $E \neq \emptyset$ be a complete metric space (Polish space). Then E is of second category.

Proof

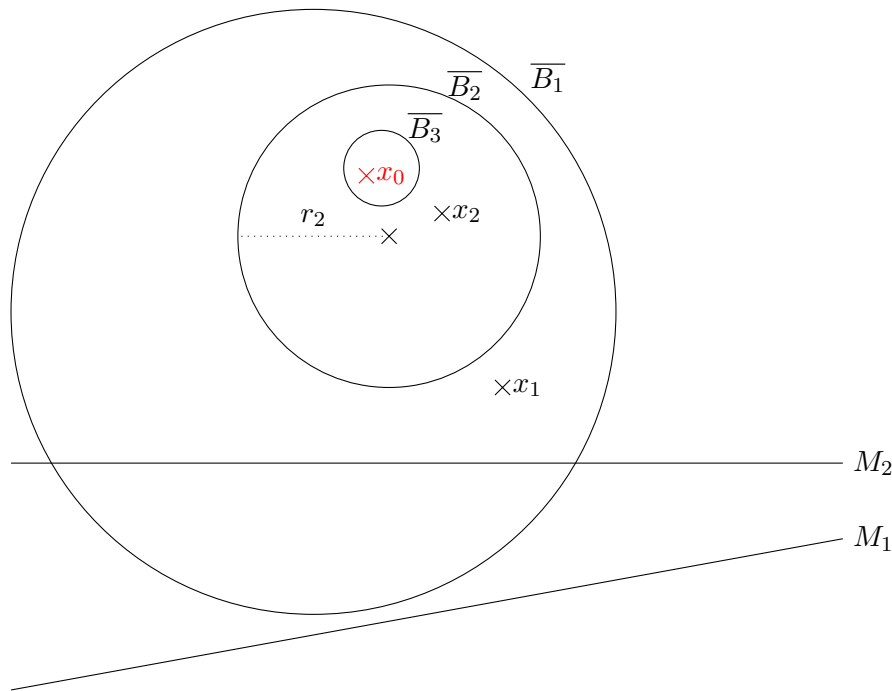


Figure 2.2: $B_n \cap M_n = \emptyset$

Assume in contrast that $E = \bigcup_{n \in \mathbb{N}} M_n$ and the sets M_n are nowhere dense. Without loss of generality assume that the M_n are closed, since otherwise one can replace M_n by $\overline{M_n}$.

We shall construct inductively balls $\overline{B_n} = \overline{B_{r_n}}(x_n)$ such that $\overline{B_{n+1}} \subseteq \overline{B_n}$, $r_n < 2^{-n}$ and $B_n \cap M_n = \emptyset$ for all n .

Then the points x_n form a Cauchy sequence, because for all $n < m \in \mathbb{N}$ we have $x_{n+1} \in B_n$

and so $\|x_n - x_{n+1}\| < r_n < 2^{-n}$:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_m\| \leq \dots \leq \\ &\leq 2^{-n} + 2^{-(n+1)} + \dots + 2^{-(m-1)} \leq 2^{-n} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \leq 2 \cdot 2^{-n} \end{aligned}$$

Since E is complete, $x_n \rightarrow x_0 \in E$ converges. Then $x_0 \in \overline{B_n}$ for all n , which implies $x_0 \notin M_n$ and thus the contradiction $x_0 \notin \bigcup_n M_n = E$ follows.

Construction of the balls $\overline{B_n}$:

M_1 is nowhere dense and therefore $B_1(0) \not\subseteq M_1$. So there exists a $x_1 \in B_1(0) \setminus M_1$. Since M_1 is closed, $B_1(0) \setminus M_1$ is open and therefore there exists a radius r_1 such that $B_{2r_1}(x_1)$ is contained in $B_1(0) \setminus M_1$ and thus $\overline{B_{r_1}(x_1)} \cap M_1 = \emptyset$.

Suppose $\overline{B_n}$ has been constructed. M_{n+1} is nowhere dense and closed and so there exists a $x_{n+1} \in \overline{B_n} \setminus M_{n+1}$ and $r_{n+1} < 2^{-(n+1)}$ such that $B_{2r_{n+1}}(x_{n+1}) \subseteq \overline{B_n} \setminus M_{n+1}$. Then follows $\overline{B_{r_{n+1}}(x_{n+1})} \cap M_{n+1} = \emptyset$. $\square_{2.4.2}$

2.4.3 Theorem (Uniform boundedness principle, Prinzip der gleichmäßigen Beschränktheit)

Let E be a Banach space and F a normed space. Let T_i be a sequence in $L(E, F)$ which is point-wise bounded, i.e. for all $u \in E$:

$$\sup_i \|T_i u\| \leq C(u) < \infty$$

Then sup-norms of T_i are bounded:

$$\sup_i \|T_i\| = \sup_i \sup_{\|u\|=1} \|T_i u\| \leq \tilde{C} < \infty$$

(Thus there exists a constant $C \in \mathbb{R}_{>0}$ such that $\|T_i u\| \leq C$ for all $i \in \mathbb{N}$ and for all $u \in E$ with $\|u\| = 1$.)

Proof

The sets $M_n = \{u \in E \mid \sup_i \|T_i u\| \leq n\}$ are closed by continuity of the $T_i \in L(E, F)$, i.e. for $u_k \rightarrow u$ converges $\|T_i u_k\| \xrightarrow{k \rightarrow \infty} \|T_i u\|$.

$E = \bigcup_n M_n$, because for any $u \in E$, $\sup_i \|T_i u\| < \infty$ and thus $u \in M_n$ for $n > \sup_i \|T_i u\|$.

If all the sets M_n had empty interior, we would get a contradiction to Baire's theorem.

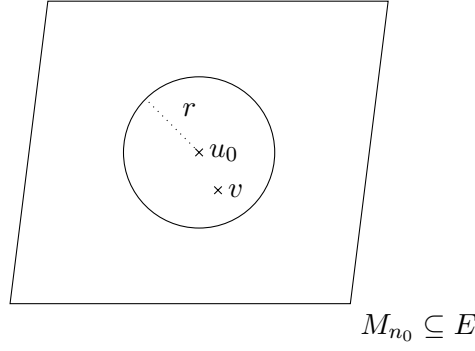
So there exists an $n_0 \in \mathbb{N}$ such that $\overset{\circ}{M_{n_0}} \neq \emptyset$ and thus there are $u_0 \in E$ and $r \in \mathbb{R}_{>0}$ such that $B_r(u_0) \subseteq M_{n_0}$.

For all $v \in B_r(u_0)$ we know that $\sup_i \|T_i v\| \leq n_0$ which is equivalent to:

$$\sup_{v \in B_r(u_0)} \|T_i v\| \leq n_0 \quad \forall_{i \in \mathbb{N}}$$

Let $w \in B_r(0)$ be arbitrary. Then $v := u_0 + w \in B_r(u_0)$.

$$T_i w \stackrel{T_i \text{ linear}}{=} T_i v - T_i u_0$$

Figure 2.3: $B_r(u_0) \subseteq M_{n_0}$

$$\|T_i w\| \leq \|T_i v\| + \|T_i u_0\| \leq n_0 + \sup_i \|T_i u_0\| < \infty$$

Here $\sup_i \|T_i u_0\| < \infty$, because the T_i are point-wise bounded.

$$\begin{aligned} \Rightarrow \|T_i w\| &\leq C \quad \forall w \in B_r(0) \\ \Rightarrow \|T_i \tilde{w}\| &\leq \tilde{C} = \frac{C}{r} \quad \forall \tilde{w} \in B_1(0) \end{aligned}$$

So $\|T_i\| \leq \tilde{C}$ for all $i \in \mathbb{N}$ and so $\|T_i\|$ is bounded. $\square_{2.4.3}$

2.4.4 Corollary

Let E be a normed space, not necessarily complete, and (u_n) a weak Cauchy sequence. Then $\|u_n\|$ is a bounded sequence.

Proof

$E^* = L(E, \mathbb{R})$ is a Banach space. For all $\varphi \in E^*$ we know that $\varphi(u_n)$ is a Cauchy sequence.

$$\Rightarrow |\varphi(u_n)| < C(\varphi)$$

Applying theorem 2.4.3 yields:

$$\begin{aligned} |\varphi(u_n)| &< C \quad \forall \varphi \text{ with } \|\varphi\|=1 \\ \Leftrightarrow \sup_{n \in \mathbb{N}} \sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(u_n)| &< C \end{aligned}$$

For any $v \in E$ we have

$$\sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(v)| = \|v\|$$

by the Hahn-Banach theorem:

- $|\varphi(v)| \leq \|\varphi\| \cdot \|v\| \stackrel{\|\varphi\|=1}{=} \|v\|$
- Choose $\varphi : \langle v \rangle \rightarrow \mathbb{R}$ with $\varphi(v) = \|v\|$ and so $\|\varphi\| = 1$. By the Hahn-Banach theorem we can extend φ to $\tilde{\varphi} : E \rightarrow \mathbb{R}$ such that $\|\tilde{\varphi}\| = 1$. Then $\tilde{\varphi}(v) = \|v\|$ and so $\sup_{\|\varphi\|=1} |\varphi(v)| \geq \|v\|$.

Thus we get $\sup_n \|u_n\| < C$. $\square_{2.4.4}$

2.4.5 Corollary and Definition (Banach-Steinhaus, equicontinuous, uniformly continuous)

Let E, F be Banach spaces and $T_i \in L(E, F)$.

If the (T_i) are point-wise bounded, then the T_i are *equicontinuous* (gleichgradig stetig).

Definition (uniformly continuous, equicontinuous)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function.

Continuity:

$$\forall_{x_0 \in \mathbb{R}} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

f is called *uniformly continuous* (gleichmäßig stetig) if:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad \|x - y\| < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon$$

Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a series of real-valued functions. (f_n) is called *equicontinuous* if:

$$\forall_{x_0 \in \mathbb{R}} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} \quad \forall_{n \in \mathbb{N}} : \quad \|x - x_0\| < \delta \quad \Rightarrow \quad \|f_n(x) - f_n(x_0)\| < \varepsilon$$

For a linear map $A \in L(E, F)$ holds:

$$\begin{aligned} \|Au\| &\leq \|A\| \|u\| \\ \|Au - Au_0\| &\leq \|A\| \|u - u_0\| \end{aligned}$$

Therefore choose $\delta = \frac{\varepsilon}{2\|A\|}$, i.e.:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad \|u\| < \delta \quad \Rightarrow \quad \|Au\| < \varepsilon$$

Proof

Since (T_i) is point-wise bounded there is a $C \in \mathbb{R}_{>0}$ such that for all $i \in \mathbb{N}$ holds $\|T_i\| \leq C$ due to the principle of uniform boundedness 2.4.3. So for all $i \in \mathbb{N}$ holds:

$$\|T_i u\| \leq \|T_i\| \|u\| \leq C \|u\|$$

Choose $\delta = \frac{\varepsilon}{2C}$ shows that the T_i is equicontinuous. □_{2.4.5}

In the following let E and F be Banach spaces.

2.4.6 Definition (open)

A (not necessarily linear) map $A : E \rightarrow F$ is called *open* if the image of every open set is open. (If there exists an inverse A^{-1} then “ A open” is equivalent to “ A^{-1} continuous”.)

Let A be linear and open. $B_1(0) \subseteq E$ is open, so $A(B_1(0)) \subseteq F$ is open.

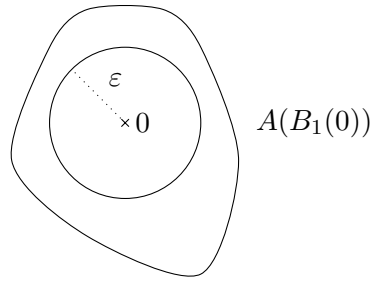
Since $0 \in A(B_1(0))$, there is a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(0) \subseteq A(B_1(0))$.

Due to the linearity holds in general:

$$B_\lambda(0) \subseteq A\left(B_{\frac{\lambda}{\varepsilon}}(0)\right)$$

In particular, A is surjective.

If A is additionally injective, then A is bijective and the openness means that A^{-1} is continuous.

Figure 2.4: $B_\varepsilon(0) \subseteq A(B_1(0))$ **2.4.7 Theorem** (Open mapping theorem, Prinzip der offenen Abbildung)

If $A \in L(E, F)$ is surjective, then A is open.

2.4.8 Corollary

If $A \in L(E, F)$ is bijective, then $A^{-1} \in L(F, E)$ is continuous.

Proof

Since A is surjective, from 2.4.7 follows that A is open, which means that A^{-1} is continuous. $\square_{2.4.8}$

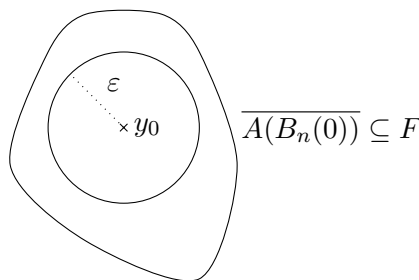
Proof of 2.4.7

Since A is surjective, $F = A(E)$. Since every element of E has a finite norm, we know:

$$E = \bigcup_{n \in \mathbb{N}} B_n(0)$$

$$\Rightarrow F = A\left(\bigcup_{n \in \mathbb{N}} B_n(0)\right) = \bigcup_{n \in \mathbb{N}} A(B_n(0))$$

According to Baire's theorem there is a $n \in \mathbb{N}$ such that $\overline{A(B_n(0))}^\circ \neq \emptyset$.

Figure 2.5: $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$

So there exists a $y_0 \in A(B_n(0))$ and a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$. Since A is surjective, there is a $x_0 \in B_n(0)$ with $y_0 = A(x_0)$.

$$\Rightarrow \overline{A(B_n(0) - x_0)} = \overline{A(B_n(0)) - y_0} = \overline{A(B_n(0))} - y_0 \supseteq B_\varepsilon(0)$$

If n' is large enough, then $B_n(-x_0) \subseteq B_{n'}(0)$ and so $\overline{A(B_{n'}(0))} \supseteq B_\varepsilon(0)$.

Since A is linear, we can rescale, i.e. there is a $c := \frac{\varepsilon}{n'} \in \mathbb{R}_{>0}$ such that for all $r \in \mathbb{R}_{>0}$ holds:

$$\overline{A(B_r(0))} \supseteq B_{cr}(0)$$

Now we show that every $u \in B_c(0)$ is the image of a $x \in B_2(0)$, i.e. $B_c(0) \subseteq A(B_2(0))$:

Ansatz as a series:

$$x = \sum_{j=1}^{\infty} x_j$$

Choose $x_1 \in B_1(0)$ with $\|u - Ax_1\| < \frac{c}{2}$, which is possible since $\overline{A(B_1(0))} \supseteq B_c(0)$.

Choose $x_2 \in B_2(0)$ with $\|u - Ax_1 - Ax_2\| < \frac{c}{4}$, which is possible since $u - Ax_1 \in B_{\frac{c}{2}}(0)$ and

$$\overline{A(B_{\frac{1}{2}}(0))} \subseteq B_{\frac{c}{2}}(0).$$

And so on choose $x_m \in B_{\frac{1}{2^m}}(0)$ with $\|u - \sum_{i=1}^m Ax_i\| < \frac{c}{2^m}$.

The series $\sum_{i=1}^{\infty} x_i$ converges, since:

$$\left\| \sum_{j=m}^M x_j \right\| \leq \sum_{j=m}^M \|x_j\| \leq \sum_{j=m}^M 2^{-j}$$

So the sequence of partial sums is a Cauchy sequence. Because E is complete, this sequence converges.

The continuity of A yields:

$$Ax = \sum_{j=1}^{\infty} Ax_j = u$$

So there exists a $x \in E$ with $\|x\| < 2$ and $Ax = u$.

□_{2.4.7}

$$\begin{aligned} \sum_{j=1}^n x_j &\xrightarrow{n \rightarrow \infty} x & \|x\| < 2 \\ \sum_{j=1}^n Ax_j &\xrightarrow{n \rightarrow \infty} u \\ &\parallel \\ A \left(\sum_{j=1}^n x_j \right) &\xrightarrow[\text{continuity of } A]{n \rightarrow \infty} Ax \end{aligned}$$

Definition (Graph)

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the *graph* is defined as:

$$\text{graph} f := \{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

For $A : E \rightarrow F$ the *graph* is:

$$\text{graph} A := \{(u, Au) \mid u \in E\} \subseteq E \times F$$

Here $E \times F$ is a product of normed spaces which has the norm:

$$\|(u, v)\| := \|u\|_E + \|v\|_F$$

Lemma

If A is continuous, then $\text{graph } A$ is closed.

Proof

Let $(u_n, Au_n) \in \text{graph } A$ be a Cauchy sequence in $E \times F$ for Banach spaces E and F , i.e. $u_n \rightarrow u$. Since A is continuous, it follows:

$$Au_n \rightarrow v := Au$$

Therefore $(u, v) \in \text{graph } (A)$ and so the graph is closed. □_{Lemma}

Consider the function:

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{x} \end{aligned}$$

f is not continuous, but $\text{graph } (f)$ is closed in $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.

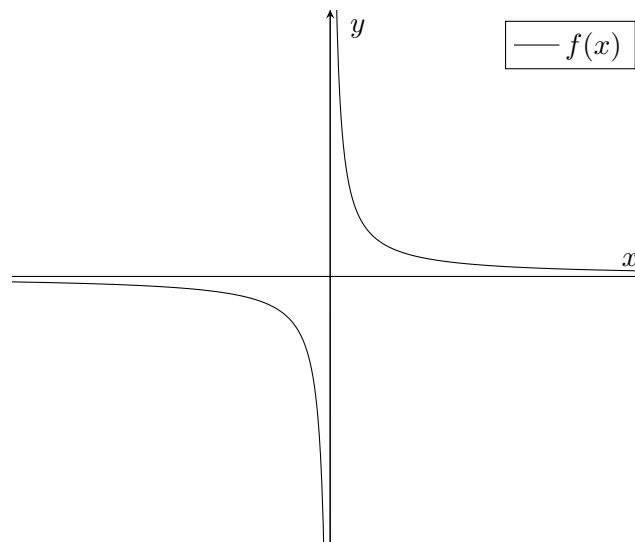


Figure 2.6: f is not continuous, but $\text{graph } f$ is closed.

2.4.9 Theorem (Closed graph theorem, Satz vom abgeschlossenen Graphen)

Suppose a linear map $A : E \rightarrow F$ between Banach spaces E and F has a closed graph. Then A is continuous.

$\text{graph } (A)$ closed means:

For all $u_n \in E$ with $u_n \rightarrow u$ and $Au_n \rightarrow v$, the point $(u, v) \in \text{graph } (A)$, i.e. $Au = v$.

A continuous means:

For all $u_n \in E$ with $u_n \rightarrow u$, the sequence $Au_n \rightarrow v$ converges and $Au = v$

Proof

On $E \times F$ we have the norm:

$$\|(u, v)\| := \|u\|_E + \|v\|_F$$

The graph

$$G := \{(u, Au) \mid u \in E\} \subseteq E \times F$$

is a subspace of $E \times F$, since for $\lambda \in \mathbb{R}$ and $u, \tilde{u} \in E$ holds:

$$\lambda(u, Au) + (\tilde{u}, A\tilde{u}) = (\lambda u + \tilde{u}, \lambda Au + A\tilde{u}) \stackrel{A \text{ linear}}{=} (\lambda u + \tilde{u}, A(\lambda u + \tilde{u})) \in G$$

So G is complete and therefore a Banach space, since we assumed it to be closed.

Define:

$$\begin{aligned} P : G &\rightarrow E \\ (u, Au) &\mapsto u \end{aligned}$$

$$\|(u, Au)\| = \|u\| + \|Au\| \geq \|u\| = \|P(u, Au)\|$$

So for all $w \in G$ holds $\|Pw\| \leq \|w\|$ and therefore $\|P\| \leq 1$. In particular, P is continuous. P is obviously surjective and it is also injective, since:

$$P^{-1}(u) = (u, Au)$$

Following the open mapping theorem, P^{-1} is continuous, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\|u\| + \|Au\| = \|(u, Au)\| = \|P^{-1}(u)\| \leq C \|u\|$$

Then follows:

$$\|Au\| \leq (C - 1) \|u\|$$

Therefore A is continuous. □_{2.4.9}

2.5 Neumann series

Let E be a Banach space and $A \in L(E, E) =: L(E)$.

When is A continuously invertible?

Remember that for $x \in \mathbb{K}$ with $|x| < 1$ holds:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is the geometric series.

Idea: $A = \mathbb{1} - B$ with $B \in L(E)$

$$\text{Ansatz: } A^{-1} := \sum_{n=0}^{\infty} B^n$$

This works indeed if $\|B\| < 1$.

2.5.1 Lemma and Definition (Neumann series)

The series

$$C := \sum_{n=0}^{\infty} B^n$$

is called Neumann series (Neumannsche Reihe).

If $\|B\| < 1$, then C defines an element of $L(E, E)$, i.e. the Neumann series converges absolutely.

Proof

Consider the partial sums:

$$S_n := \sum_{k=0}^n B^k$$

Since $L(E, E)$ is a Banach space, it is enough to show that S_n is a Cauchy series. Without loss of generality assume $m > n$:

$$\|S_n - S_m\| = \left\| \sum_{k=n}^m B^k \right\| \stackrel{\Delta \text{ inequality}}{\leq} \sum_{k=n}^m \|B^k\| \stackrel{\text{Schwarz}}{\leq} \sum_{k=n}^m \|B\|^k < c \|B\|^n \rightarrow 0$$

□_{2.5.1}

2.5.2 Theorem

$$C = (\mathbb{1} - B)^{-1}$$

Proof

$$(\mathbb{1} - B)C = (\mathbb{1} - B) \sum_{n=0}^{\infty} B^n = (\mathbb{1} + B + B^2 + \dots) - (B + B^2 + \dots) = \mathbb{1}$$

□_{2.5.2}

2.5.3 Theorem

The set of all continuously invertible mappings is open in $L(E)$.

Proof

Assume that $A \in L(E)$ is continuously invertible, i.e. A^{-1} exists and $A^{-1} \in L(E)$. Set:

$$\varepsilon = \frac{1}{2\|A^{-1}\|}$$

Let us show, that every element of $B_\varepsilon(A) \subseteq L(E)$ is continuously invertible:

Let $C \in B_\varepsilon(A)$, i.e. $\|A - C\| < \varepsilon$.

$$C = A - (A - C) = A(\mathbb{1} - \underbrace{A^{-1}(A - C)}_{=:B})$$

Then holds:

$$\|B\| \leq \|A^{-1}\| \cdot \|A - C\| = \|A^{-1}\| \cdot \frac{1}{2\|A^{-1}\|} = \frac{1}{2} < 1$$

Hence $\mathbb{1} - B$ is continuously invertible by the Neumann series and therefore

$$C^{-1} = (\mathbb{1} - B)^{-1} \cdot A^{-1}$$

is continuous.

□_{2.5.3}

3 Hilbert spaces

Definition (scalar product)

Let H be a real ($\mathbb{K} := \mathbb{R}$) or complex ($\mathbb{K} := \mathbb{C}$) vector space with *scalar product*:

$$\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{K}$$

- i) Positive definiteness: $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Rightarrow u = 0$.
- ii) Linear in the second and anti-linear in the first argument:

$$\langle \lambda u, v \rangle = \bar{\lambda} \langle u, v \rangle$$

- iii) Symmetry: $\overline{\langle u, v \rangle} = \langle u, v \rangle$

Define the corresponding norm:

$$\|u\| := \sqrt{\langle u, u \rangle}$$

3.0.1 Definition (Hilbert space)

A complete scalar product space is called *Hilbert space*.

The Schwarz inequality holds:

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

3.0.2 Lemma (parallelogram equality)

The parallelogram equality (Parallelogramm-Gleichung) is:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Proof

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ \|u - v\|^2 &= \langle u - v, u - v \rangle = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\ \Rightarrow \|u + v\|^2 + \|u - v\|^2 &= 2(\|u\|^2 + \|v\|^2) \end{aligned}$$

□_{3.0.2}

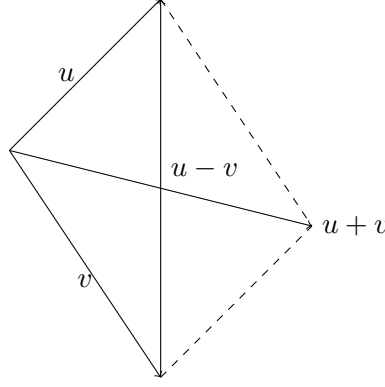


Figure 3.1: parallelogram

3.0.3 Definition (orthogonal, orthonormal)

- i) Vectors $u, v \in H$ are called *orthogonal*, symbolically $u \perp v$, if $\langle u, v \rangle = 0$.
- ii) Subspaces $M_1, M_2 \subseteq H$ are orthogonal, symbolically $M_1 \perp M_2$, if $\langle u, v \rangle = 0$ for all $u \in M_1$ and $v \in M_2$.
- iii) A family $(u_i)_{i \in I}$ of vectors $u_i \in H$ is called *orthonormal* if:

$$\langle u_i, u_j \rangle = \delta_{ij}$$

3.0.4 Theorem (Bessel's inequality)

Let $(u_i)_{1 \leq i \leq N}$ be an orthonormal family. Then for all $u \in H$ holds:

$$\begin{aligned} \|u\|^2 &= \sum_{i=1}^N \langle u_i, u \rangle^2 + \left\| u - \sum_{i=1}^N u_i \langle u_i, u \rangle \right\|^2 \\ \|u\|^2 &\geq \sum_{i=1}^N \langle u_i, u \rangle^2 \end{aligned}$$

Proof

$$\begin{aligned} \left\| u - \sum_{i=1}^N u_i \langle u_i, u \rangle \right\|^2 &= \left\langle u - \sum_{i=1}^N u_i \langle u_i, u \rangle, u - \sum_{j=1}^N u_j \langle u_j, u \rangle \right\rangle = \\ &= \langle u, u \rangle - \sum_{j=1}^N \langle u, u_j \rangle \langle u_j, u \rangle - \sum_{i=1}^N \overline{\langle u_i, u \rangle} \langle u_i, u \rangle + \sum_{i,j=1}^N \overline{\langle u_i, u \rangle} \langle u_j, u \rangle \underbrace{\langle u_i, u_j \rangle}_{=\delta_{ij}} = \\ &= \|u\|^2 - 2 \sum_{i=1}^N |\langle u_i, u \rangle|^2 + \sum_{i=1}^N |\langle u_i, u \rangle|^2 = \\ &= \|u\|^2 - \sum_{i=1}^N |\langle u_i, u \rangle|^2 \end{aligned}$$

□_{3.0.4}

Definition (Hilbert space isomorphism)

Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces.

A *Hilbert space isomorphism* is a mapping $U : H_1 \rightarrow H_2$ which is linear, bijective and isometric (isometrisch), i.e. for all $u, v \in H_1$:

$$\langle u, v \rangle_1 = \langle Uu, Uv \rangle_2$$

Definition (Direct sum)

Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces.

Define:

$$H := \{(u, v) \mid u \in H_1, v \in H_2\}$$

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$

$$\lambda(u, v) := (\lambda u, \lambda v)$$

$$\langle (u_1, v_1), (u_2, v_2) \rangle := \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$

This makes $H =: H_1 \oplus H_2$ a Hilbert space, called *direct sum* of H_1 and H_2 , which is sometimes called orthogonal due to:

$$\langle (u, 0), (0, v) \rangle = 0$$

3.0.5 Example

$$\ell_2 = \left\{ (a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{K}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

Define a scalar product:

$$\langle (a_n), (b_n) \rangle := \sum_{n=1}^{\infty} \bar{a}_n \cdot b_n$$

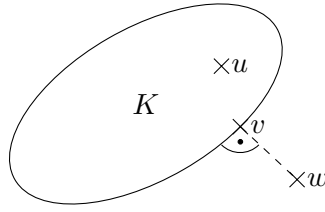
$$\langle (a_n), (a_n) \rangle = \sum_{n=1}^{\infty} |a_n|^2 = \|a_n\|_2^2$$

$(\ell^2, \|\cdot\|_2)$ is a Banach space. Thus $(\ell^2, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

3.1 Projection on closed convex subsets

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $K \subseteq H$ a closed convex subset.

$$u, v \in K \qquad w \in H \setminus K$$

Figure 3.2: $\|v - w\| = \inf_{u \in K} \|u - w\|$

We want to find a vector v such that $\|v - w\| = \inf_{u \in K} \|u - w\|$.

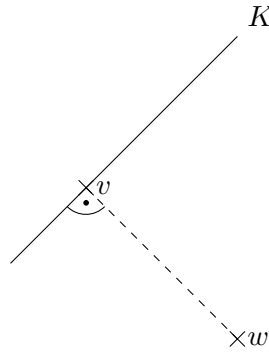
If K were compact, then choose minimizing sequence (Minimalfolge), i.e.:

$$\|u_i - w\| \rightarrow \inf_{u \in K} \|u - w\|$$

Choose a convergent subsequence $u_{i_l} \rightarrow v$. Then by continuity:

$$\|v - w\| = \lim_{i \rightarrow \infty} \|u_i - w\| = \inf_{u \in K} \|u - w\|$$

The main application are closed subspaces $K \subseteq H$.

Figure 3.3: $v - w \perp K$

In this case $v - w$ will be called orthogonal to K motivating the name *orthogonal projection*.

3.1.1 Theorem (Hilbert)

There is a unique $v \in K$ with:

$$\|v - w\| = \inf_{u \in K} \|u - w\|$$

Proof

Consider a minimizing sequence u_i :

$$\|u_i - w\| \rightarrow \inf_{u \in K} \|u - w\| =: d$$

We show that (u_i) is a Cauchy sequence:

$$\begin{aligned}
 \|u_i - u_j\|^2 &= \|(u_i - w) + (w - u_j)\|^2 = \\
 &\stackrel{3.0.2}{=} 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \|(u_i - w) - (w - u_j)\|^2 = \\
 &= 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \left\| -2\left(w - \frac{u_i + u_j}{2}\right) \right\|^2 = \\
 &= 2\left(\underbrace{\|u_i - w\|^2}_{\rightarrow d^2} + \underbrace{\|w - u_j\|^2}_{\rightarrow d^2} - 2\left\| \frac{u_i + u_j}{2} - w \right\|^2 \right)
 \end{aligned}$$

$$\|u_i - w\| \xrightarrow{i \rightarrow \infty} d = \inf_{u \in K} \|u - w\|$$

$$\|u_j - w\| \xrightarrow{j \rightarrow \infty} d = \inf_{u \in K} \|u - w\|$$

Since K is convex and $u_i, u_j \in K$, we know:

$$\frac{u_i + u_j}{2} \in K$$

$$\Rightarrow \left\| \frac{u_i + u_j}{2} - w \right\| \geq d$$

Thus:

$$\|u_i - u_j\|^2 \leq 2\left(\|u_i - w\|^2 + \|w - u_j\|^2 - 2d^2\right) \xrightarrow{i,j \rightarrow \infty} 2(d^2 + d^2 - 2d^2) = 0$$

So there exists a $N \in \mathbb{N}$ such that $\|u_i - u_j\| < \varepsilon$ for all $i, j > N$. Therefore (u_i) is a Cauchy sequence. Since H is complete, we know that $u_i \rightarrow u$ converges.

By continuity follows:

$$\|u - w\| = \lim_{i \rightarrow \infty} \|u_i - w\| = d$$

Uniqueness follows from the fact, that *every* minimizing sequence converges:

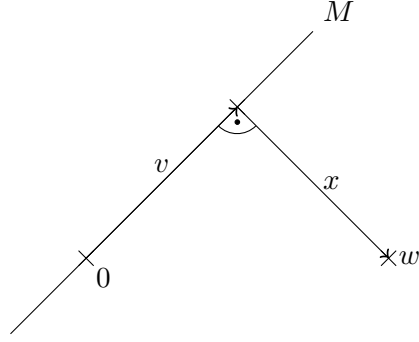
Let u, \tilde{u} be both minimizers, then the sequence $(u, \tilde{u}, u, \tilde{u}, \dots)$ is a minimizing sequence. Since it converges, $u = \tilde{u}$. $\square_{3.1.1}$

3.1.2 Corollary

Let $M \subseteq H$ be a closed subspace of H . Then a $w \in H$ can be decomposed uniquely in the form

$$w = v + x$$

with $v \in M$ and $x \in M^\perp$. We write $H = M \oplus M^\perp$.

Figure 3.4: $w = v + x$ **Proof**

Let $v \in M$ be as in Theorem 3.1.1.

$$\|v - w\| = \inf_{u \in M} \|u - w\|$$

Define $x := w - v$.

- H real: For $u \in M$ define $\tilde{u}(\tau) = v + \tau u$ with $\tau \in \mathbb{R}$.

$$\begin{aligned} \|\tilde{u} - w\|^2 &= \|x\|^2 + 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 \geq \|x\|^2 \\ 0 &\leq 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 =: f(\tau) \end{aligned}$$

$f(\tau)$ has a minimum at $\tau = 0$ and so $f'(0) = 0$.

$$\begin{aligned} f'(0) &= 2 \langle u, x \rangle \\ \Rightarrow 2 \langle u, x \rangle &= 0 \quad \forall_{u \in M} \end{aligned}$$

So $x \in M^\perp$.

- H complex: Define $\tilde{u}(\tau) = v + \tau u$, $\tau = re^{i\varphi} \in \mathbb{K}$ with $r \geq 0$.

$$\|\tilde{u} - w\|^2 = \|x\|^2 + 2\operatorname{Re} \left(re^{-i\varphi} \langle u, x \rangle \right) + r^2 \|u\|^2 =: f(r, \varphi)$$

This has a minimum at $r = 0$.

$$\begin{aligned} \Rightarrow 0 &= \partial_r f(0, \varphi) = 2\operatorname{Re} \left(e^{-i\varphi} \langle u, x \rangle \right) \\ \varphi \text{ arbitrary} \Rightarrow \langle u, x \rangle &= 0 \end{aligned}$$

So $x \in M^\perp$.

Uniqueness: Assume that $w = v_1 + x_1 = v_2 + x_2$ where $v_1, v_2 \in M$, $x_1, x_2 \in M^\perp$.

$$\underbrace{v_1 - v_2}_{\in M} = \underbrace{x_2 - x_1}_{\in M^\perp} \in M \cap M^\perp = \{0\}$$

Because from $u \in M \cap M^\perp$ follows $\langle u, u \rangle = 0$ and so $u = 0$.

□_{3.1.2}

For a Banach space E we have E, E^*, E^{**} and a natural injection $\iota : E \hookrightarrow E^{**}$.

For a Hilbert space H , suppose $u \in H$ and define:

$$\begin{aligned}\varphi &: H \rightarrow \mathbb{K} \\ \varphi(v) &:= \langle u, v \rangle\end{aligned}$$

φ is continuous, because:

$$|\varphi(v)| = |\langle u, v \rangle| \leq \|u\| \cdot \|v\| \leq C \|v\|$$

Now

$$\begin{aligned}\iota &: H \hookrightarrow H^* \\ \iota(u) &= \varphi\end{aligned}$$

is a linear mapping, which is injective.

3.1.3 Theorem (Fréchet-Riesz)

For any $\varphi \in H^*$ there is a unique $v \in H$ such that for all $x \in H$:

$$\varphi(x) = \langle v, x \rangle$$

In other words: $\iota : H \rightarrow H^*$ is a Banach space isomorphism.

Proof

Let $\varphi \in H^*$, without loss of generality $\varphi \neq 0$.

$$M := \ker \varphi \subseteq H$$

is a subspace. It is closed by continuity: For $u_n \in \ker \varphi$ with $u_n \rightarrow u$ holds:

$$\varphi(u) \stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} \varphi(u_n) = 0$$

So $u \in \ker \varphi$.

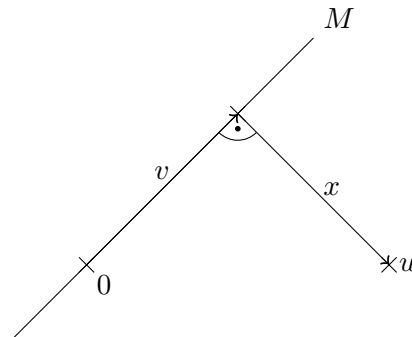


Figure 3.5: $u = v + x$

- M^\perp is a one-dimensional subspace of H :

$$M^\perp \neq \{0\}:$$

Since $\varphi \neq 0$ there exists a $u \in H$ with $\varphi(u) \neq 0$, thus $u \notin M$.

Now decompose $u = v + x$, $v \in M$, $x \in M^\perp \setminus \{0\}$.

M^\perp is one-dimensional: Take $u, v \in M^\perp$, $u, v \neq 0$, then $\varphi(u) \neq 0$ and $\varphi(v) \neq 0$.

$$\varphi(\varphi(v)u - \varphi(u)v) = 0$$

So $\varphi(v)u - \varphi(u)v \in M \cap M^\perp = \{0\}$. Thus $\varphi(v)u - \varphi(u)v = 0$, implying that u and v are linearly dependent.

- Choose $u \in M^\perp$ with $\varphi(u) = 1$, which is always possible by rescaling.

$$\begin{aligned} v &:= \frac{u}{\|u\|^2} \\ \Rightarrow \quad \varphi(v) &= \frac{1}{\|u\|^2} \underbrace{\varphi(u)}_{=1} = \frac{1}{\|u\|^2} \\ \langle v, v \rangle &= \frac{\langle u, u \rangle}{\|u\|^4} = \frac{1}{\|u\|^2} = \varphi(v) \end{aligned}$$

- This v has the desired properties:

For $x \in H$ decompose:

$$x = \underbrace{m}_{\in M} + \underbrace{\alpha v}_{\in M^\perp = \langle v \rangle}$$

$$\begin{aligned} \Rightarrow \quad \varphi(x) &= \underbrace{\varphi(m)}_{=0} + \alpha \varphi(v) = \alpha \langle v, v \rangle = \\ &= \langle v, \alpha v \rangle = \langle v, m + \alpha v \rangle = \langle v, x \rangle \end{aligned}$$

□_{3.1.3}

3.1.4 Theorem (Lax-Milgram)

Let H be a Hilbert space and $B : H \times H \rightarrow \mathbb{K}$ be a mapping with the following properties:

- i) $B(x, y)$ is linear in the second and anti-linear in the first argument.
- ii) $|B(x, y)| \leq C \|x\| \cdot \|y\|$ (continuity)
- iii) B is symmetric ($\overline{B(x, y)} = B(y, x)$) and positive definite, i.e. $B(x, x) \geq b \|x\|^2$ with $b \in \mathbb{R}_{>0}$.
- iii') $|B(x, x)| \geq b \|x\|^2$ with $b \in \mathbb{R}_{>0}$.

Then every $l \in H^*$ can be represented uniquely as:

$$l(y) = B(x, y) \quad \forall_{y \in H}$$

Proof

First the easy case iii):

We introduce a new scalar product $\langle \cdot, \cdot \rangle_B$ by:

$$\langle x, y \rangle_B := B(x, y)$$

Using ii) and iii) one sees that $\|\cdot\|_B$ is equivalent to $\|\cdot\|$, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C} \|x\| \leq \|x\|_B \leq C \|x\|$$

According to the Fréchet-Riesz theorem, there exists a unique $v \in H$ with

$$\varphi(x) = \langle v, x \rangle_B = B(v, x)$$

for all $x \in H$.

More difficult case iii'): Given $x \in H$,

$$B(x, \cdot) : H \rightarrow \mathbb{K}$$

is a linear bounded functional according to i) and ii), i.e. $B(x, \cdot) \in H^*$.

According to the Fréchet-Riesz theorem there exists a unique $z \in H$ such that $B(x, y) = \langle z, y \rangle$ for all $y \in H$. This yields a mapping:

$$\begin{aligned} \varphi : H &\rightarrow H \\ x &\mapsto z \end{aligned}$$

$$B(x, y) = \langle \varphi(x), y \rangle$$

- φ is linear, because both B and $\langle \cdot, \cdot \rangle$ are anti-linear in their first arguments.
- $\varphi(H) \subseteq H$ is closed:

$$\begin{aligned} b \|x\|^2 &\stackrel{\text{iii}'}{\leq} |B(x, x)| = |\langle z, x \rangle| \leq \|z\| \cdot \|x\| \\ b \|x\| &\leq \|z\| \end{aligned} \tag{3.1}$$

Let $z_n \in \varphi(H)$ be a sequence with $z_n \rightarrow z \in H$. Choose x_n such that $\varphi(x_n) = z_n$, i.e. $B(x_n, y) = \langle z_n, y \rangle$ for all $y \in H$.

Due to the anti-linearity in the first argument follows that:

$$B(x_n - x_m, y) = \langle z_n - z_m, y \rangle$$

(3.1) yields that $\|x_n - x_m\| \leq \|z_n - z_m\|$.

Hence (x_n) is a Cauchy sequence and so $x_n \rightarrow x \in H$ converges. Since B is continuous according to ii), we get:

$$\underbrace{B(x_n, y)}_{\rightarrow B(x, y)} = \underbrace{\langle z_n, y \rangle}_{\rightarrow \langle z, y \rangle}$$

This gives:

$$\begin{aligned} B(x, y) &= \langle z, y \rangle \\ \varphi(x) &= z \end{aligned}$$

Thus z is in $\varphi(H)$.

- $\varphi(H) = H$: Otherwise there would be a vector $y \in \varphi(H)^\perp \setminus \{0\}$ and thus for all $x \in H$ holds.

$$B(x, y) = \langle \varphi(x), y \rangle = 0$$

In particular for $x = y$ this gives:

$$\begin{aligned} 0 &= |B(y, y)| \geq b \|y\|^2 \\ \Rightarrow y &= 0 \end{aligned}$$

This is a contradiction and so $\varphi(H) = H$.

- φ is injective: Suppose there are $x, x' \in H$ with $\varphi(x) = \varphi(x')$. Then follows:

$$B(x - x', y) = \langle \underbrace{\varphi(x) - \varphi(x')}_{=0}, y \rangle = 0$$

Choose $y = x - x'$ so we get:

$$B(x - x', x - x') = 0$$

Since B is positive definite, it follows $x = x'$.

- Let $l \in H^*$. According to Fréchet-Riesz there exists a unique $z \in H$ with $l(y) = \langle z, y \rangle$ for all $y \in H$ and we have

$$\langle z, y \rangle = B(x, y)$$

for $x = \varphi^{-1}(z)$. So $l(y) = B(x, y)$.

□_{3.1.4}

3.1.5 Corollary

Every Hilbert space is reflexive.

Proof

Recall $\iota : H \hookrightarrow H^{**}$. H is *reflexive* if and only if ι is surjective, i.e. a Banach space isomorphism.

$$\begin{aligned} \tilde{\iota} : H &\rightarrow H^* \\ (\tilde{\iota}(u))(v) &= \langle u, v \rangle \end{aligned}$$

is bijective by Fréchet-Riesz. This holds also for $\bar{\iota} : H^* \rightarrow H^{**}$.

$$H \xrightarrow{\tilde{\iota}} H^* \xrightarrow{\bar{\iota}} H^{**}$$

So $\iota = \bar{\iota} \circ \tilde{\iota}$ is bijective as composition of bijective maps.

□_{3.1.5}

3.2 Orthonormal Bases in Separable Hilbert Spaces

3.2.1 Example

$$\ell_2 = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \right\}$$

with the scalar product

$$\langle (a_n), (b_n) \rangle := \sum_n \bar{a}_n b_n$$

is a Hilbert space.

Idea: Let H be an abstract Hilbert space. Choose an “orthonormal basis” (e_i) .

$$\begin{aligned} H \ni u &= \sum_{i=1}^{\infty} \lambda_i e_i \\ v &= \sum_{i=1}^{\infty} \nu_i e_i \end{aligned}$$

$$\langle u, v \rangle = \sum_{i,j=1}^{\infty} \langle \lambda_i e_i, \nu_j e_j \rangle = \sum_{i,j=1}^{\infty} \bar{\lambda}_i \nu_j \delta_{ij} = \sum_i \bar{\lambda}_i \nu_i$$

3.2.2 Definition (orthonormal system, Hilbert space basis, cardinality)

A system $(e_i)_{i \in J}$ is an *orthonormal system*, if $\langle e_i, e_j \rangle = \delta_{ij}$. The algebraic span is the vector space of *finite* linear combinations:

$$\langle (e_i) \rangle = \left\{ \sum_{i=1}^N \lambda_i e_i \mid N \in \mathbb{N}, \lambda_i \in \mathbb{K} \right\}$$

This is a subspace of H . Now the subspace $\overline{\langle (e_i) \rangle} \subseteq H$ is called *Hilbert space span* (Hilbertraumerzeugnis).

An orthonormal system (e_i) is called a *orthonormal Hilbert space basis* if $\overline{\langle (e_i) \rangle} = H$.

Two sets A and B have the same cardinality if there exists a bijective map $\varphi : A \rightarrow B$.

Theorem (Bernstein-Schröder)

A and B have the same cardinality if and only if there exists an injective map from A to B and an injective map from $B \rightarrow A$.

(no proof)

A typical application of the Lax-Milgram theorem is for $x \in \mathbb{R}^n$, given real-valued functions $V(x), f(x)$ and looking for $u(x)$ that solves:

$$-\Delta u(x) + V(x)u(x) = f(x)$$

Question: Is there a solution which “decays at infinity”?

1. Weak formulation:

Suppose we have a solution $u \in \mathcal{C}^2(\mathbb{R}^n)$

$$-\Delta u + Vu - f = 0$$

Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a test function.

$$0 = \int_{\mathbb{R}^n} (-\Delta u + Vu - f) \eta d^n x \stackrel{\text{integration by parts}}{=} \underbrace{\int_{\mathbb{R}^n} (\langle \nabla u, \nabla \eta \rangle + Vu\eta) d^n x}_{=: B(u, \eta)} - \underbrace{\int_{\mathbb{R}^n} f \eta d^n x}_{=: l(\eta)}$$

So for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ holds:

$$B(u, \eta) = l(\eta)$$

Definition: u is a *weak solution* of the equation $-\Delta u + Vu = f$ if for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ holds:

$$B(u, \eta) = l(\eta)$$

2. Choose the correct Hilbert space. The first idea is $L^2(\mathbb{R}^n)$ with the scalar product:

$$\langle u, v \rangle = \int_{\mathbb{R}^n} uv d^n x$$

$$u_n(x) := e^{-|x|^2} \sin(nx_1)$$

Then for all $n \in \mathbb{N}$ holds:

$$\|u_n\|_{L^2} \leq C$$

But $B(u_n, u_n) \xrightarrow{n \rightarrow \infty} \infty$ diverges. Thus B is *not* continuous.
Better choose instead:

$$\langle u, v \rangle = \int_{\mathbb{R}^n} (uv + \langle \nabla u, \nabla v \rangle) d^n x$$

The corresponding Hilbert space $H^{1,2}(\mathbb{R}^n)$ is a Sobolev space.

$$L^2(\mathbb{R}^3) \supseteq H^{1,2}(\mathbb{R}^3) \ni u$$

Assume for simplicity that $0 < \varepsilon \leq V \leq C < \infty$, then we get:

$$B(u, u) = \int_{\mathbb{R}^n} (|\nabla u|^2 + Vu^2) d^n x \leq \int_{\mathbb{R}^n} (|\nabla u|^2 + Cu^2) d^n x \leq (1 + C) \|u\|_{H^{1,2}}^2$$

$$|B(u, u)| \geq \int_{\mathbb{R}^n} (|\nabla u|^2 + \varepsilon u^2) d^n x \geq \min\{1, \varepsilon\} \|u\|_{H^{1,2}}^2$$

Thus the Lax-Milgram theorem applies and yields a unique weak solution and then a regularity theorem says that u is smooth.

Consider a matrix equation

$$Au = f$$

with $A \in \text{Symm}(\mathbb{R}^n)$ and $f \in \mathbb{R}^n$.

For a general existence and uniqueness result one needs that A is invertible or equivalently:

$$\forall_{u \in \mathbb{R}^n \setminus \{0\}} : Au \neq 0$$

This follows from the condition:

$$\forall_{u \in \mathbb{R}^n \setminus \{0\}} : \underbrace{\langle u, Au \rangle}_{=B(u,u)} \neq 0$$

In finite dimension this is equivalent to:

$$\forall_{u \in \mathbb{R}^n} : |B(u,u)| > b \|u\|^2$$

$(e_i)_{i \in I}$ is an orthonormal Hilbert space basis of H if

$$\langle e_i, e_j \rangle = \delta_{ij}$$

and:

$$\overline{\langle e_i \rangle} = H$$

3.2.3 Theorem

Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal system. Then the mapping

$$\begin{aligned} \ell_2 &\rightarrow \overline{\langle e_i \rangle}^{\text{closed}} \subseteq H \\ (\lambda_i) &\mapsto \sum_{i \in \mathbb{N}} \lambda_i e_i \end{aligned}$$

is a Hilbert space isomorphism.

Proof

The mapping is well-defined and isometric:

For $(\lambda_i) \in \ell_2$, i.e. $\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$ we construct:

$$u_N := \sum_{i=1}^N \lambda_i e_i \in H$$

Without loss of generality take $M < N$, then follows:

$$\|u_N - u_M\|^2 = \left\| \sum_{i=M}^N \lambda_i e_i \right\|^2 = \left\langle \sum_{i=M}^N \lambda_i e_i, \sum_{i=M}^N \lambda_i e_i \right\rangle = \sum_{i,j=M}^N \bar{\lambda}_i \lambda_j \underbrace{\langle e_i, e_j \rangle}_{=\delta_{ij}} = \sum_{i=M}^N |\lambda_i|^2$$

Thus u_N is a Cauchy sequence and converges since $\overline{\langle e_i \rangle}$ is complete as a closed subset of a complete space.

$$u := \lim_{N \rightarrow \infty} u_N = \sum_{i=1}^N \lambda_i e_i$$

$$\|u\|^2 = \lim_{N \rightarrow \infty} \|u_N\|^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^N |\lambda_i|^2 = \|(\lambda_i)\|_{\ell_2}$$

The mapping is also surjective:

Let $u \in \overline{\langle e_i \rangle}$ and $\varepsilon > 0$. So there exists a $v = \sum_{i=1}^N \lambda_i e_i \in \langle e_i \rangle$ with $\|v - u\| < \varepsilon$.

In other words there exists a finite $J \subseteq \mathbb{N}$ such that $d(\langle (e_i)_{i \in J} \rangle, u) < \varepsilon$. The vector which minimizes this distance is the orthogonal projection of u on $\langle (e_i)_{i \in J} \rangle$ since this is a finite-dimensional subspace, which is automatically closed.

$$u_J = \sum_{i \in J} e_i \langle e_i, u \rangle$$

Choose an increasing sequence $J_1 \subsetneq J_2 \subsetneq \dots$ of finite sets such that:

$$\|u_{J_k} - u\| \rightarrow 0 \quad \Rightarrow \quad u_{J_k} \rightarrow u$$

Thus u_{J_k} is bounded by a $C \in \mathbb{R}_{>0}$.

$$u_{J_k} = \sum_{i \in J_k} e_i \underbrace{\langle e_i, u \rangle}_{=\lambda_i}$$

$$C > \|u_{J_k}\| = \sum_{i \in J_k} |\lambda_i|^2$$

This gives:

$$\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$$

And so we get:

$$u = \sum_{i \in \mathbb{N}} \lambda_i e_i$$

□_{3.2.3}

3.2.4 Theorem (Existence of Hilbert space basis)

In every Hilbert space H exists an orthonormal Hilbert space basis.

Proof

Consider $(u_i)_{i \in I}$ with $I = H$ and $u_h = h$ for all $h \in H$. $(u_i)_{i \in I}$ is obviously a generating system of H . On the set

$$X := \left\{ \tilde{I} \subseteq I \mid (u_i)_{i \in \tilde{I}} \text{ is an orthonormal system} \right\}$$

defines „ \subseteq “ a partial ordering.

Let $U \subseteq X$ be a totally ordered subset and define:

$$I_U := \bigcup_{\tilde{I} \in U} \tilde{I} \subseteq I$$

I_U is an upper bound of U in X if $I_U \in X$. Assume $(u_i)_{i \in I_U}$ would not be orthonormal. Then there would exist $j, k \in I_U$ with $\langle u_j, u_k \rangle \neq \delta_{jk}$.

For $j = k$ would hold $\langle u_j, u_j \rangle \neq 1$, but j lies in $\tilde{I} \in U \subseteq X$ and therefor has to hold $\langle u_j, u_j \rangle = 1$. For $j \neq k$ we would get $\langle u_j, u_k \rangle \neq 0$. But j lies in $\tilde{I}_j \in U$ and k in $\tilde{I}_k \subseteq U$ and U is totally ordered, i.e. either holds $\tilde{I}_j \subseteq \tilde{I}_k$ or $\tilde{I}_k \subseteq \tilde{I}_j$.

Without loss of generality assume $\tilde{I}_j \subseteq \tilde{I}_k$ (otherwise exchange j and k). Then $j, k \in \tilde{I}_k \in U \subseteq X$ and hence $(u_i)_{i \in \tilde{I}_j}$ is an orthonormal system in contradiction to $\langle u_j, u_k \rangle \neq 0$. Therefore holds $I_U \in X$ and thus I_U is an upper bound of U .

Using Zorn's lemma we get a maximal element I_{\max} in X . Because $(u_i)_{i \in I_{\max}}$ is an orthonormal system and thus especially linearly independent, it suffices to show that this is an generating system of H .

Assume there exists a $i_0 \in I$ with $u_{i_0} \notin K := \overline{\langle (u_i)_{i \in I_{\max}} \rangle_{\text{alg.}}}$. Since $K \subseteq H$ is closed and convex, there is an unique projection v of u_{i_0} on K and thus $h := u_{i_0} - v \in K^\perp$. It holds $h = u_h$ with $h \in H = I$.

Because I_{\max} is maximal, holds then $I_{\max} \cup \{h\} \notin X$ and hence there is a $j \in I_{\max}$ with $\langle h, u_j \rangle \neq 0$, because $h = j$ cannot hold due to $h \notin I_{\max}$. This is a contradiction to $h \in K^\perp$ and thus holds $K = H$.

Therefore $(u_i)_{i \in I_{\max}}$ is an orthonormal Hilbert space basis of H . □_{3.2.4}

3.2.5 Theorem

Any two Hilbert space bases have the same cardinality (Mächtigkeit).

Proof

TODO: Proof from exercises

3.2.6 Theorem

If H is separable, then there exists a countable orthonormal Hilbert space basis $(e_i)_{i \in \mathbb{N}}$. Thus H is Hilbert space isomorphic to ℓ_2 .

Proof

Since H is separable, there is a countable dense subset $(x_i)_{i \in \mathbb{N}}$.

1. Arrange that the x_i are linearly independent:

Start with $n = 1$ and $k = 1$ set:

$$y_1 = x_1$$

If the y_1, \dots, y_{n-1}, x_k are linearly independent, we set $y_n = x_k$ and increase n and k by one.

If the y_1, \dots, y_{n-1}, x_k are linearly dependent, we only increase k by one.

Then the y_i are linearly independent and $\langle (y_i) \rangle = \langle (x_i) \rangle$.

2. Gram-Schmidt procedure for orthonormalization:

$$\begin{aligned} e_1 &:= y_1 \\ e_2 &:= \frac{y_2 - e_1 \langle y_2, e_1 \rangle}{\|y_2 - e_1 \langle y_2, e_1 \rangle\|} \\ e_n &:= \frac{y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n}{\|y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n\|} \end{aligned}$$

Since the y_i are linearly independent, $y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n$ is never zero.

Then by construction the e_i are orthonormal and $\langle e_i \rangle = \langle x_i \rangle \subseteq H$ is dense and so $(e_i)_{i \in \mathbb{N}}$ is a Hilbert space basis. $\square_{3.2.6}$

3.3 Weak Compactness of the Closed Unit Ball

For a Banach space E *weak convergence* for $(u_i)_{i \in \mathbb{N}}$ with $u_i \in E$ means:

$$u_n \rightharpoonup u \quad \Leftrightarrow \quad \forall_{\varphi \in E^*} : \varphi(u_n) \rightarrow \varphi(u)$$

In Hilbert spaces, we can identify H^* with H via the Fréchet-Riesz theorem.

3.3.1 Definition (weak (sequential) compactness)

$x_n \rightharpoonup x$ *converges weakly* if $\langle y, x_n \rangle \rightarrow \langle y, x \rangle$ converges for all $y \in H$.

Weak compactness is for us by definition the same as *weak sequential compactness* (schwache Folgenkompaktheit):

$K \subseteq H$ is *weakly compact* if every sequence (x_n) with $x_n \in K$ has a weakly convergent subsequence.

3.3.2 Proposition

Let H be *separable* and infinite-dimensional and let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal Hilbert space basis.

Then $e_n \rightharpoonup 0$ converges weakly.

Proof

Take $y \in H$ and expand it in the basis:

$$\begin{aligned} y &= \sum_{i=1}^{\infty} y_i e_i \\ y_i &= \langle e_i, y \rangle \end{aligned}$$

We know $(y_i)_{i \in \mathbb{N}} \in \ell_2$ and in particular $y_i \xrightarrow{i \rightarrow \infty} 0$, since the elements of an absolutely convergent series converge to zero. Therefore holds:

$$\langle y, e_n \rangle = \overline{y_n} \xrightarrow{n \rightarrow \infty} 0$$

Thus $e_n \rightharpoonup 0$ converges weakly. $\square_{3.3.2}$

3.3.3 Theorem (Weak Compactness of the Closed Unit Ball)

If H is *separable*, then the closed unit ball $\overline{B_1(0)} = \{u \mid \|u\| \leq 1\}$ is weakly compact.

Proof

Let (u_l) be a sequence with $u_l \in \overline{B_1(0)}$. Choose an orthonormal Hilbert space basis $(e_n)_{n \in \mathbb{N}}$.

$$u_l = \sum_{n=1}^{\infty} u_{ln} e_n \quad u_{ln} = \langle e_n, u_l \rangle \quad (u_{l,n})_{n \in \mathbb{N}} \in \ell_2$$

$$|u_{ln}| = |\langle e_n, u_l \rangle| \leq \underbrace{\|e_n\|}_{=1} \cdot \|u_l\| \leq 1$$

For $n = 1$: $(u_{l,1})_{l \in \mathbb{N}}$ is a bounded sequence of complex or real numbers. Therefore there exists a convergent subsequence of u_l , which we denote by $u_l^{(1)} \in H$. Then follows:

$$u_{l,1}^{(1)} = \langle e_1, u_l^{(1)} \rangle \xrightarrow{l \rightarrow \infty} v_1$$

For $n = 2$: Next we choose a subsequence $u_l^{(2)}$ of $u_l^{(1)}$ such that:

$$\langle e_2, u_l^{(2)} \rangle \xrightarrow{l \rightarrow \infty} v_2$$

Proceed inductively to obtain:

$$\langle e_n, u_l^{(n)} \rangle \rightarrow v_n$$

Then $w_l = u_l^{(l)} \in \overline{B_1(0)}$ for a sequence (w_l) in $\overline{B_1(0)}$.

Claim: $w_l \xrightarrow{l \rightarrow \infty} v := \sum_n v_n e_n$

Proof: We proceed as follows:

$$v_n = \lim_{l \rightarrow \infty} \langle e_n, u_l^{(n)} \rangle = \lim_{l \rightarrow \infty} \langle e_n, u_l^{(l)} \rangle = \lim_{l \rightarrow \infty} \langle e_n, w_l \rangle$$

This is because $u_l^{(l)} = u_{l'}^{(n)}$ for $l' \geq l$.

1. $(v_n) \in \ell_2$:

$$\sum_{n=1}^N |v_n|^2 = \sum_{n=1}^N \left| \lim_{l \rightarrow \infty} \langle e_n, w_l \rangle \right|^2 \stackrel{\text{finite sum}}{=} \lim_{l \rightarrow \infty} \sum_{n=1}^N |\langle e_n, w_l \rangle|^2$$

$\underbrace{\hspace{10em}}_{\substack{\text{Bessel's} \\ \leq \\ \text{inequality}}} \|w_l\|^2 \leq 1$

So we get for all $N \in \mathbb{N}$:

$$\sum_{n=1}^N |v_n|^2 \leq 1$$

And thus $(v_n) \in \ell_2$ and $v := \sum_{n=1}^{\infty} v_n e_n$ is well-defined and has $\|v\| \leq 1$.

2. $w_l \rightharpoonup v$, i.e. $\langle y, w_l - v \rangle \xrightarrow{l \rightarrow \infty} 0$ for all $y \in H$:

$$y = \sum_{n=1}^{\infty} y_n e_n$$

$$y_n = \langle e_n, y \rangle$$

$$y_{<} := \sum_{n \leq N} y_n e_n$$

$$y_{>} := \sum_{n > N} y_n e_n$$

$$\|y\|^2 = \|y_{<}\|^2 + \|y_{>}\|^2$$

$$\langle y, w_l - v \rangle = \sum_{n=1}^{\infty} y_n \langle e_n, w_l - v \rangle$$

Choose $N \in \mathbb{N}$ so large that

$$\|y_{>}\| = \left(\sum_{n > N} |y_n|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{4}$$

to get:

$$\begin{aligned} |\langle y, w_l - v \rangle| &\leq |\langle y_{<}, w_l - v \rangle| + |\langle y_{>}, w_l - v \rangle| \leq \\ &\leq \sum_{n=1}^N |y_n| |\langle e_n, w_l - v \rangle| + \underbrace{\|y_{>}\|}_{< \frac{\varepsilon}{4}} \cdot \underbrace{\|w_l - v\|}_{\leq 2} < \sum_{n=1}^N |y_n| |\langle e_n, w_l - v \rangle| + \frac{\varepsilon}{2} \end{aligned}$$

We know $|\langle e_n, w_l - v \rangle| \xrightarrow{l \rightarrow \infty} 0$ for each n . So we can choose $|\langle e_n, w_l - v \rangle| \leq \frac{\varepsilon}{2}$ for $n \leq N$ and for all $l > L(\varepsilon)$ for a sufficiently large $L(\varepsilon)$ and therefore:

$$|\langle y, w_l - v \rangle| \leq \varepsilon \quad \forall_{l > L(\varepsilon)}$$

Therefore $\langle y, w_l \rangle \rightarrow \langle y, v \rangle$ converges, which means $w_l \rightharpoonup v$. $\square_{3.3.3}$

The corresponding statement in Banach spaces is the *Banach-Alaoglu theorem*:

Banach proved it in 1932 for separable Banach spaces using diagonal sequences.

Alaoglu proved it in 1938 for any Banach space. The proof is based on Tychonov's theorem.

We have E , E^* , E^{**} and an injection $\iota : E \rightarrow E^{**}$.

Theorem (Banach-Alaoglu)

The closed unit ball in E^* is *weak*-sequentially compact*.

I.e. in simple terms:

If $\varphi_n \in \overline{B_1(0)} \subseteq E^*$, then there exists a subsequence φ_{n_l} such that $\varphi_{n_l}(u)$ converges for all $u \in E$.

Application: Consider

$$E = C^0(\mathbb{R}^n)$$

with the sup-norm:

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|$$

$$E^* = \{\text{regular Borel measures}\}$$

Suppose μ_n is a sequence of measures with $\|\mu_n\| \leq C$ for all $n \in \mathbb{N}$. Then there exists a measure μ such that $\mu_{n_l} \rightarrow \mu$ converges as a measure.

4 Operators on Hilbert spaces

Let H be a Hilbert space.

$$L(H) := L(H, H)$$

is the Banach space of bounded linear operators. (An linear map on an infinite dimensional space is usually called *linear operator*.) For $A \in L(H)$ define the norm:

$$\|A\| := \sup_{\|u\|=1} \|Au\|$$

4.0.1 Example

$H = L^2(\mathbb{R}, dx)$ with the Lebesgue measure dx .

$$\langle f, g \rangle = \int_{\mathbb{R}} \bar{f} g dx$$

$$A := \frac{d}{dx}$$

We would like to introduce this as an operator on H .

The inequality $\|Au\| \leq C \|u\|$ is violated even for $u \in C_0^\infty(\mathbb{R})$ for any constant $C \in \mathbb{R}$.

Namely consider

$$u_n(x) = \eta(x) \sin(nx)$$

with $\eta \in C_0^\infty(\mathbb{R})$ and $\eta|_{[-1,1]} = 1$. Then $\|u_n\| < \infty$ and $\|Au_n\| \xrightarrow{n \rightarrow \infty} \infty$.

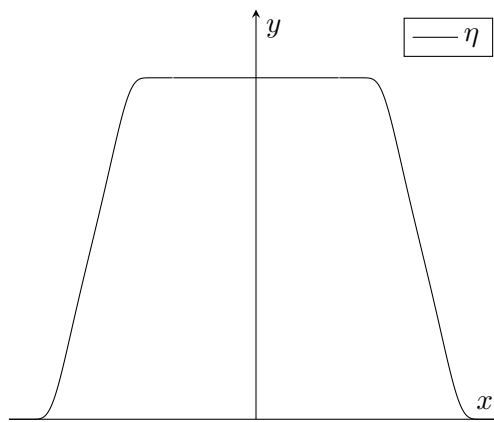


Figure 4.1: $\eta \in C_0^\infty(\mathbb{R})$ with $\eta|_{[-1,1]} = 1$

Moreover $\frac{d}{dx}f$ makes no sense for every vector f in H , because f does not need to be differentiable.

Way out: Define A only on a suitable subspace $\mathcal{D}(A)$ of H , called *domain* of definition. For example: Choose $\mathcal{D}(A) = C_0^\infty(\mathbb{R}) \subseteq H$ and:

$$A : \mathcal{D}(A) \xrightarrow{\text{linear}} H$$

$\mathcal{D}(A)$ is dense in H , i.e. $\overline{\mathcal{D}(A)} = H$.

4.0.2 Definition (linear operator, domain, bounded)

- i) Let $\mathcal{D} \subseteq H$ be a dense subspace. A linear map $A : \mathcal{D} \rightarrow H$ is called a *linear operator* on H with domain (of definition) \mathcal{D} .
- ii) A is called *bounded*, if there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in \mathcal{D}$ holds:

$$\|Au\| \leq C \|u\|$$

Otherwise A is called unbounded.

4.0.3 Lemma

If A is a bounded operator with dense domain $\mathcal{D} \subseteq H$, then it can be extended by continuity to a unique operator $A \in L(H)$.

Proof

Let $u \in H$, not necessarily in \mathcal{D} . Since $\overline{\mathcal{D}} = H$, there is a sequence (u_l) in \mathcal{D} with $u_l \rightarrow u$.

$$\|Au_i - Au_j\| = \|A(u_i - u_j)\| \leq C \cdot \|u_i - u_j\| \xrightarrow{i,j \rightarrow \infty} 0$$

Therefore we can set:

$$Au := \lim_{l \rightarrow \infty} Au_l$$

Since Au_l converges for any sequence $u_l \rightarrow u$, this is well-defined.

$$\|Au\| \leftarrow \|Au_i\| \leq C \|u_i\| \rightarrow C \|u\|$$

So there exists a C such that $\|Au\| \leq C \|u\|$ for all $u \in H$ and therefore $A \in L(H)$. $\square_{4.0.3}$

4.1 Isometric and unitary operators

4.1.1 Definition (isometric operator)

A operator $V : \mathcal{D}(V) \rightarrow H$ with dense domain $\mathcal{D}(V) \subseteq H$ is called *isometric* if for all $u \in \mathcal{D}(V)$ holds:

$$\langle Vu, Vu \rangle = \langle u, u \rangle$$

This operator is bounded, because:

$$\|Vu\| = \sqrt{\langle Vu, Vu \rangle} = \sqrt{\langle u, u \rangle} = \|u\| \stackrel{C:=1}{\leq} C \|u\|$$

Therefore we can extend it by continuity to H and

$$V : H \rightarrow H$$

is again isometric.

The “Hilbert hotel”

Consider $H = \ell_2$ and $(a_i) = (a_1, a_2, \dots) \in \ell_2$.

$$A(u_1, u_2, \dots) := (0, u_1, u_2, \dots)$$

A is isometric, but it is no bijection.

Suppose you have a hotel with an infinite number of rooms and an infinite number of guest, in every room one guest.

If a new guest arrives, just move the guest from room n to room $n + 1$ and the first room gets unoccupied, so the new guest can use it.

4.1.2 Proposition

For an isometric operator V the subspace $V(H) \subseteq H$ is closed.

Proof

Consider $y \in \overline{V(H)}$ and show $y \in V(H)$:

There exists a (y_n) with $y_n \in V(H)$ and $y_n \rightarrow y$ and a (x_n) with $V(x_n) = y_n$. Then holds:

$$\|x_i - x_j\| \stackrel{V \text{ isometric}}{=} \|V(x_i - x_j)\| = \|y_i - y_j\| \xrightarrow{i, j \rightarrow \infty} 0$$

Thus $x_i \rightarrow x$ converges. By continuity we get:

$$V(x) = \lim_{i \rightarrow \infty} V(x_i) = \lim_{i \rightarrow \infty} y_i = y$$

□_{4.1.2}

4.1.3 Definition (unitary operator)

If $V : H \rightarrow H$ is an isometric operator and $V(H) = H$, then V is called *unitary* (unitär).

4.2 The Closure of an Operator

Let E and F be Banach spaces and $A : \mathcal{D}(A) \subseteq E \rightarrow F$ be a densely defined linear operator.

$$\begin{aligned} \text{graph}(A) &:= \{(u, Au) \mid u \in \mathcal{D}(A)\} \subseteq E \times F \\ \overline{\text{graph}(A)} &\subseteq E \times F \end{aligned}$$

Try to realize this as the graph of a new operator \overline{A} .

$$\mathcal{D}(\overline{A}) := \text{pr}_1(\overline{\text{graph}A}) = \left\{ u \mid \exists_{v \in F} : (u, v) \in \overline{\text{graph}A} \right\}$$

For $u \in \mathcal{D}(\overline{A})$ and $(u, v) \in \overline{\text{graph}A}$ define:

$$\overline{A}u := v$$

v exists by definition of $\mathcal{D}(\overline{A})$. Is v unique?

Suppose $(u, v) \in \overline{\text{graph}A}$. Then there exists a sequence $(u_n, v_n) \in \text{graph}(A)$, with $(u_n, v_n) \rightarrow (u, v)$. Equivalently:

$$\forall_{n \in \mathbb{N}} \exists_{u_n \in \mathcal{D}(A)} : (u_n \rightarrow u) \wedge (Au_n \rightarrow v)$$

Then we set $\overline{A}u := v$.

Problem: There might be two different series (u_n) and (\tilde{u}_n) with $u_n \rightarrow u$, $\tilde{u}_n \rightarrow u$, $Au_n \rightarrow v$ and $A\tilde{u}_n \rightarrow \tilde{v} \neq v$.

4.2.1 Definition (closable operator)

A densely defined operator A is called closable (abschließbar) if $\overline{\text{graph}A}$ is the graph of an operator B .

B is called the *closure* of A , symbolically $B = \overline{A}$.

4.2.2 Definition (closed)

A is called *closed* if $\text{graph}A$ is a closed subset of $E \times F$.

4.2.3 Theorem (closed graph theorem)

Reformulation of 2.4.9:

If $\mathcal{D}(A) = E$, then A is closed if and only if A is bounded.

4.2.4 Example

Consider $E = C^0([0,1])$ with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$.

$$\mathcal{D}(A) = C^1([0,1]) \subseteq E$$

$$\begin{aligned} A : \mathcal{D}(A) &\rightarrow E \\ f &\mapsto f' \end{aligned}$$

A is a densely defined, unbounded operator. Is A closed?

Consider $(u, v) \in \overline{\text{graph} A}$, i.e. there exists a sequence $(u_n) \subseteq \mathcal{D}(A)$ with $u_n \rightarrow u$ and $Au_n \rightarrow v$. $u_n \rightarrow u$ means uniform convergence of $u_n \rightrightarrows u$, so u is continuous as a uniform limit of continuous functions.

$Au_n \rightarrow v$ means uniform convergence of $Au_n \rightrightarrows v$, so v is also continuous.

It follows that $u \in C^1$ and $u' = v$.

So $(u, v) \in \text{graph} A$ and therefore A is closed.

Consider $F := C^1([0, 1])$ with $\|u\| = \sup_{[0, 1]} |u| + \sup_{[0, 1]} |u'|$. This is a Banach space.

Remark

The closure of a closable operator is always closed.

This is obvious, because $\text{graph} \bar{A} := \overline{\text{graph} A}$, which is closed.

4.2.5 Theorem (Criterion for closable)

A is closable if and only if:

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \rightarrow 0) \wedge (Au_n \rightarrow v) \quad \Rightarrow \quad v = 0$$

Proof

“ \Rightarrow ”: Suppose A is closable. Thus there is an operator \bar{A} such that $\text{graph} \bar{A} = \overline{\text{graph} A}$.

Suppose that $u_n \in \mathcal{D}(A)$, $u_n \rightarrow 0$ and $Au_n \rightarrow v$. Then $(u_n, Au_n) \rightarrow (0, v) \in \overline{\text{graph} A} = \text{graph} \bar{A}$ and thus $v = \bar{A}(0) = 0$.

“ \Leftarrow ”: Suppose that the implication

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \rightarrow 0) \wedge (Au_n \rightarrow v) \quad \Rightarrow \quad v = 0$$

holds.

Define $\mathcal{D}(\bar{A})$ by: $u_n \in \mathcal{D}(A)$ with $u_n \rightarrow u$ and $Au_n \rightarrow v$. Then for $u \in \mathcal{D}(\bar{A})$ set $\bar{A}(u) = v$.

This is well-defined: Suppose $u_n, \tilde{u}_n \rightarrow u$, $Au_n \rightarrow v$ and $A\tilde{u}_n \rightarrow \tilde{v}$. Then $u_n - \tilde{u}_n \rightarrow 0$ and $A(u_n - \tilde{u}_n) \rightarrow v - \tilde{v}$. By assumption follows $v - \tilde{v} = 0$. □_{4.2.5}

4.3 The adjoint of a densely defined operator

Let $A : \mathcal{D}(A) \rightarrow H$ be a linear operator with $\overline{\mathcal{D}(\bar{A})} = H$.

In finite-dimensional linear algebra the definition of the adjoint A^* is:

$$\langle u, Av \rangle =: \langle A^*u, v \rangle \quad \forall_{u, v \in H}$$

Here it is more complicated, since in general $\mathcal{D}(A) \neq H$.

$$M := \left\{ (u, w) \in H \times H \mid \forall_{v \in \mathcal{D}(A)} : \langle u, Av \rangle = \langle w, v \rangle \right\}$$

Claim: M is the graph of a linear map A^* .

Proof: $M \neq \emptyset$ since $(0,0) \in M$.

- The image is unique: $u \rightarrow w$ is well-defined, as from $(u,w), (u,w') \in M$ follows for all $v \in \mathcal{D}(A)$:

$$\langle w - w', v \rangle = \langle u - u, Av \rangle = 0$$

Since $\mathcal{D}(A)$ is dense, $w - w' = 0$ follows.

- A^* is linear: For $(u,w), (u',w') \in M$ and $\lambda \in \mathbb{K}$ follows $(u + \lambda u', w + \lambda w') \in M$ which is obvious from the definition of M . \square_{Claim}

4.3.1 Theorem

A^* is closed.

Proof

TODO: Proof from exercises

4.3.2 Theorem

A^* is the maximal, i.e. not extensible, operator S with the property that for all $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(S)$:

$$\langle Au, v \rangle = \langle u, Sv \rangle$$

Proof

TODO: Proof from exercises

4.4 Symmetric and self-adjoint densely defined operators

4.4.1 Definition (symmetric, (essentially) self-adjoint)

- i) A is *symmetric* $:\Leftrightarrow \forall_{u,v \in \mathcal{D}(A)} : \langle Au, v \rangle = \langle u, Av \rangle$
- ii) A is *self-adjoint* $:\Leftrightarrow A^* = A$ (in particular, $\mathcal{D}(A^*) = \mathcal{D}(A)$)
- iii) A is *essentially self-adjoint* $:\Leftrightarrow \overline{A}$ is self-adjoint

For bounded A with $\mathcal{D}(A) = H$ all these notions coincide.

4.4.2 Example

Consider the operator $A := \Delta = \sum_{i=1}^n \partial_i^2$ on $L^2(\Omega)$ for a bounded open region $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{D}(A) = C_0^\infty(\Omega) \stackrel{\text{dense}}{\subseteq} L^2(\Omega)$.

– A is symmetric:

$$\langle Af, g \rangle \stackrel{\text{integration by parts}}{=} \langle f, Ag \rangle$$

– Adjoint of Δ on L^2 :

$$\int \mathrm{d}^n r (\Delta f) \cdot g = \int \mathrm{d}^n r f \cdot \underbrace{h}_{\in L^2}$$

Here $h := A^*g$. It is sufficient to consider $g \in H^{2,2}(\Omega)$ (Sobolev space). $\mathcal{D}(A^*) \supsetneq \mathcal{D}(A)$

4.4.3 Lemma

Let A be a symmetric operator. Then A is closable and \overline{A} and A^* are extensions of A and $\mathcal{D}(A) \stackrel{\text{i)}}{\subseteq} \mathcal{D}(\overline{A}) \stackrel{\text{ii)}}{\subseteq} \mathcal{D}(A^*)$.

Proof

Let $u_n \in \mathcal{D}(A)$ with $u_n \rightarrow 0$ and $Au_n \rightarrow w$.

$$\begin{aligned} \langle Au, v \rangle &= \langle u, Av \rangle \quad \forall_{u, v \in \mathcal{D}(A)} \\ \langle w, v \rangle &\leftarrow \langle Au_n, v \rangle = \langle u_n, Av \rangle \rightarrow \langle 0, Av \rangle = 0 \end{aligned}$$

Since this holds for all $v \in \mathcal{D}(A)$ now $w = 0$ follows. From the criterion 4.2.5 follows that A is closable.

i) is obvious from the definition of \overline{A} .

ii) Take $u \in \mathcal{D}(\overline{A})$. Then there is a sequence $u_n \in \mathcal{D}(A)$ with $u_n \rightarrow u$ and $Au_n \rightarrow \overline{A}u$. For all $v \in \mathcal{D}(A)$ holds:

$$\langle \overline{A}u, v \rangle \leftarrow \langle Au_n, v \rangle = \langle u_n, Av \rangle \rightarrow \langle u, Av \rangle$$

So $u \in \mathcal{D}(A^*)$ and $A^*u = \overline{A}u$.

□_{4.4.3}

„The smaller one chooses $\mathcal{D}(A)$, the larger becomes $\mathcal{D}(A^*)$.“

$$B \subseteq \mathcal{D}(A) \quad \Rightarrow \quad \mathcal{D}((A|_B)^*) \supseteq \mathcal{D}(A^*)$$

Difficulty: Construct $\mathcal{D}(A)$ such that $\mathcal{D}(A) = \mathcal{D}(A^*)$. (More on this later in the lecture.)

4.5 Heisenberg's uncertainty principle

In quantum mechanics:

The Hilbert space for one dimensional problems is usually $H = L^2(\mathbb{R})$.

The position operator is $x =: B$ and the momentum operator is $\frac{\hbar}{i} \frac{d}{dx} =: A$.

$$[A, B] := AB - BA = \frac{\hbar}{i} \mathbb{1}$$

4.5.1 Theorem (Winter-Wieland)

For two continuous operators A and B with $[A, B] = c \cdot \mathbb{1}$ and $B^n = B$ for all $n \in \mathbb{N}_{\geq 1}$, i.e. B is idempotent, follows $c = 0$.

Proof

Consider:

$$\begin{aligned} B^k AB^{n-k} &= B^k (AB) B^{n-k-1} = B^k (BA + c\mathbb{1}) B^{n-k-1} = B^{k+1} AB^{n-k-1} + cB^{n-1} \\ \Rightarrow cB^{n-1} &= B^k AB^{n-k} - B^{k+1} AB^{n-k-1} \end{aligned}$$

Sum this from $k = 0$ to $k = n - 1$:

$$ncB^{n-1} = \sum_{k=0}^{n-1} B^k AB^{n-k} - B^{k+1} AB^{n-k-1} \stackrel{\text{telescope sum}}{=} AB^n - B^n A$$

$$n|c| \|B^{n-1}\| = \|AB^n - B^n A\| \stackrel{\Delta\text{-inequality}}{\leq} \|AB^n\| + \|B^n A\| \leq (\|AB\| + \|BA\|) \cdot \|B^{n-1}\|$$

Since this must hold for all n either $c = 0$ or there exists a $n \in \mathbb{N}_{>1}$ with $\|B^{n-1}\| = 0$, i.e. $B^{n-1} = 0$. Since B is idempotent follows $B = 0$ and therefore $[A, B] = 0$ and also $c = 0$. $\square_{4.5.1}$

Consider $u \in \mathcal{D}(A)$ with $\|u\| = 1$, which represents a quantum mechanical state.

The expectation value of A in u is after the probabilistic interpretation:

$$E_u(A) := \langle u, Au \rangle$$

The “uncertainty”, i.e. the variance, is:

$$\Delta_u(A) := \|(A - E_u(A) \mathbb{1})u\|$$

4.5.2 Theorem (Heisenberg's uncertainty principle)

Let H be a \mathbb{C} -Hilbert space and $A : \mathcal{D}(A) \rightarrow H$, $B : \mathcal{D}(B) \rightarrow H$ be two symmetric operators with $\overline{\mathcal{D}(A)} = H = \overline{\mathcal{D}(B)}$. Assume for the image domains \mathcal{R} :

$$\mathcal{R}(A) \subseteq \mathcal{D}(B) \qquad \mathcal{R}(B) \subseteq \mathcal{D}(A)$$

So $[A, B]$ is well-defined on $\mathcal{D}(A) \cap \mathcal{D}(B)$.

Assume furthermore that $[A, B] = \frac{\hbar}{i} \mathbb{1}$ with $\hbar > 0$.

Then for all $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\|u\| = 1$ holds:

$$\Delta_u(A) \cdot \Delta_u(B) \geq \frac{\hbar}{2}$$

Proof

Replace A by $\tilde{A} := A - E_u(A) \cdot \mathbb{1}$ and $\tilde{B} := B - E_u(B) \cdot \mathbb{1}$. Then holds:

$$[\tilde{A}, \tilde{B}] = \frac{\hbar}{i} \mathbb{1}$$

$$\begin{aligned}\Delta_u(A) &= \|\tilde{A}u\| \\ \Delta_u(B) &= \|\tilde{B}u\|\end{aligned}$$

We have to show:

$$\Delta_u(A) \cdot \Delta_u(B) = \|\tilde{A}u\| \cdot \|\tilde{B}u\| \geq \frac{\hbar}{2}$$

$$\begin{aligned}\frac{\hbar}{2} &= \frac{\hbar}{2} \langle u, u \rangle = \frac{i}{2} \langle u, (\tilde{A}\tilde{B} - \tilde{B}\tilde{A})u \rangle \stackrel{\text{symmetry}}{=} \frac{i}{2} (\langle \tilde{A}u, \tilde{B}u \rangle - \langle \tilde{B}u, \tilde{A}u \rangle) = \\ &= -\text{Im}(\langle \tilde{A}u, \tilde{B}u \rangle) \stackrel{\text{Cauchy-Schwarz}}{\leq} \|\tilde{A}u\| \cdot \|\tilde{B}u\|\end{aligned}$$

□_{4.5.2}

4.6 Spectrum and resolvent

Let $A : \mathcal{D}(A) \rightarrow H$ be a closed, densely defined operator.

4.6.1 Definition (continuously invertible, resolvent, spectrum)

A is *continuously invertible* if and only if $A : \mathcal{D}(A) \rightarrow H$ is bijective and $A^{-1} : H \rightarrow \mathcal{D}(A)$ is continuous.

$$\varrho(A) := \{\lambda \in \mathbb{C} \mid (\lambda \mathbb{1} - A) \text{ is continuously invertible}\}$$

The *resolvent* is defined as

$$\mathcal{R}_\lambda(A) = (\lambda \mathbb{1} - A)^{-1}$$

and the *spectrum* of A as:

$$\sigma(A) = \mathbb{C} \setminus \varrho(A)$$

4.6.2 Lemma

$\varrho(A)$ is open and $\sigma(A)$ is closed.

Proof

For bounded operators cf. Theorem 2.5.3.

It's method works even for unbounded operators:

Take $\lambda, \mu \in \varrho(A)$.

$$\begin{aligned} (A - \mu) &= (A - \lambda) + (\lambda - \mu) = \\ &= \underbrace{(A - \lambda)}_{\text{continuously invertible}} \cdot \left(\mathbb{1} + (A - \lambda)^{-1} (\lambda - \mu) \right) \end{aligned}$$

$\mathbb{1} + (A - \lambda)^{-1} (\lambda - \mu)$ is continuously invertible using the Neumann series if:

$$|\lambda - \mu| < \frac{1}{\|(A - \lambda)^{-1}\|}$$

So $\varrho(A)$ is open and therefore the complement $\sigma(A)$ is closed. $\square_{4.6.2}$

4.6.3 Theorem (resolvent equation)

The map $\lambda \mapsto \mathcal{R}_\lambda(A)$ is complex analytic on $\varrho(A)$.

We have the *resolvent equation*:

$$\mathcal{R}_\lambda - \mathcal{R}_\mu = -(\lambda - \mu) \mathcal{R}_\lambda \cdot \mathcal{R}_\mu$$

Proof

Analogy with \mathbb{C} -numbers:

$$\begin{aligned} \frac{1}{\lambda - x} - \frac{1}{\mu - x} &= \frac{\mu - \lambda}{(\lambda - x)(\mu - x)} \\ (\mu - x) - (\lambda - x) &= \mu - \lambda \end{aligned}$$

Same thing for operators:

$$\begin{aligned} (\mu - A) - (\lambda - A) &= \mu - \lambda \\ \mathcal{R}_\mu^{-1} - \mathcal{R}_\lambda^{-1} &= \mu - \lambda \quad / \mathcal{R}_\mu \cdot \quad / \cdot \mathcal{R}_\lambda \\ \mathcal{R}_\lambda - \mathcal{R}_\mu &= (\mu - \lambda) \mathcal{R}_\mu \mathcal{R}_\lambda \\ \mathcal{R}_\lambda &= \mathcal{R}_\mu + (\mu - \lambda) \mathcal{R}_\mu \mathcal{R}_\lambda \end{aligned}$$

Assume $|\mu - \lambda| < \frac{1}{\|\mathcal{R}_\lambda\|}$.

$$\mathcal{R}_\mu = \mathcal{R}_\lambda (1 + (\mu - \lambda) \mathcal{R}_\lambda)^{-1} = \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-1)^n (\mu - \lambda)^n \mathcal{R}_\lambda$$

This series converges absolutely and so the map is analytic in $L(H)$. $\square_{4.6.3}$

Appendix

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Andreas Völklein

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