Functional Analysis

lecture by

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Motivation

In linear algebra one mainly considers finite-dimensional vector spaces with additional structures like norm $\|.\|$ or scalar product $\langle .,. \rangle$.

Let $(V, \langle .,. \rangle)$ be a finite-dimensional scalar product space and $A : V \to V$ a linear map, which is self-adjoint, that means for all $u,v \in V$:

$$\langle Au,v \rangle = \langle u,Av \rangle$$

Theorem (orthonormal eigenvector basis)

There exists an orthonormal eigenvector basis $(u_i)_{i\in\{1,\dots,n\}}$, that means with the eigenvalues $\lambda_i \in \mathbb{R}$:

$$\langle u_i,u_j \rangle = \delta_{ij} \qquad\qquad Au_i = \lambda_i u_i$$

In infinite dimensions the generalization is the *spectral theorem*.

First reformulate the result from linear algebra:
Let E_{λ_i} be the orthogonal projection operator on the eigenspace corresponding to λ_i. If this eigenspace is one dimensional, this means:

$$E_{\lambda_i} v = u_i \langle u_i,v \rangle = |u_i\rangle \langle u_i|v\rangle$$

Then one can write A as:

$$A = \sum_{i=1}^{n} \lambda_i E_{\lambda_i}$$

Theorem (spectral theorem)

Let $A \in L(H)$ be a self-adjoint (selbstadjungiert) operator, then it holds:

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E_\lambda$$

$\sigma(A) \subseteq \mathbb{R}$ is the spectrum of A and E_λ the projection-valued measure (Spektralmaß).

Applications typically are differential operators, for example:

$$\varDelta_{\mathbb{R}^3} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\varDelta_{\mathbb{R}^3} : C_0^\infty\left(\mathbb{R}^3\right) \to C^\infty\left(\mathbb{R}^3\right) \quad \text{linear operator}$$

Applications in more detail are studied in the lectures on partial differential equations I + II.

0 Basic Notions

Let E be a vector space (Vektorraum), for example the finite-dimensional vector space $E \simeq \mathbb{R}^3$. In the following list the later spaces are special cases of the previous ones:

- topological vector spaces
- metric spaces with a metric $d(.,.)$ (Polish spaces if complete)
- normed spaces with norm $\|.\|$ (Banach spaces if complete)
- scalar product spaces $\langle .,. \rangle$ (Hilbert spaces if complete)

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

0.1 Definition (metric, ε-ball, Cauchy sequence, complete, Polish space)

A map $d : E \times E \to \mathbb{R}$ is called *metric*, if for all $x,y,z \in E$ holds:

i) $d(x,y) = d(y,x)$ (symmetry)

ii) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$ (positive definiteness)

iii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

$B_\varepsilon(x) := \left\{ z \in E \middle| d(x,z) < \varepsilon \right\}$ is called *ε-ball*.
Consider the topology generated by $B_\varepsilon(x)$: A set $\Omega \subseteq E$ is open if and only if:

$$\underset{x \in \Omega}{\forall} \ \underset{\varepsilon \in \mathbb{R}_{>0}}{\exists} : B_\varepsilon(x) \subseteq \Omega$$

Completeness:
$(x_n)_{n \in \mathbb{N}}$ is a *Cauchy sequence* if and only if:

$$\underset{\varepsilon \in \mathbb{R}_{>0}}{\forall} \ \underset{N \in \mathbb{N}}{\exists} \ \underset{n,m \in \mathbb{N}_{>N}}{\forall} : d(x_n,x_m) < \varepsilon$$

E is *complete* if and only if every Cauchy sequence has a limit.
A complete metric space is also called a *Polish space*.

0.2 Definition (norm, Banach space)

Let $(E, \|.\|)$ be a *normed space*, i.e. a $\mathbb{K}$-vector space with a map $\|.\| : E \to \mathbb{R}_{\geq 0}$ called *norm* with the following properties for $x,y \in E$ and $\lambda \in \mathbb{K}$:

i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (positive definiteness)

ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity)

iii) $\|u+v\| \leq \|u\| + \|v\|$ (triangle inequality)

Define the metric $d(x,y) := \|x-y\|$. A complete normed spaces is called *Banach space*.

Let $A : E \to F$ be a linear map between the Banach spaces $(E, \|.\|_E)$ and $(F, \|.\|_F)$.

0.3 Definition (continuous, bounded)

A is *continuous* (stetig) if $A^{-1}(\Omega) \subseteq E$ is open for all open $\Omega \subseteq F$.
A is *bounded* (beschränkt) if there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq C\|u\|_E$$

0.4 Lemma (continuous $\Leftrightarrow$ bounded)

A is continuous $\Leftrightarrow$ A is bounded.

(no proof)

0.5 Definition (dual space, sup-norm)

The *dual space* of E is the space of continuous linear mappings from E to $\mathbb{K}$:

$$E^* = L(E,\mathbb{K})$$

$L(E,F)$ is a vector space: For $A,B \in L(E,F)$, $\lambda,\mu \in \mathbb{K}$ and $u \in E$ define:

$$(\lambda A + \mu B)(u) := \lambda A(u) + \mu B(u)$$

Define also a norm on $L(E,F)$, which is called *sup-norm*:

$$\|A\| := \sup_{u \in E, \|u\|_E \leq 1} \|Au\|_F$$

0.6 Theorem

If F is complete, so is $L(E,F)$.
In particular E^* is a Banach space for every E.

(no proof)

1 The Hahn-Banach Theorem and Applications

As a preparation we need Zorn's lemma.

1.1 Definition (partial ordering, chain, upper bound, maximal)

Let A be a set and $\leq$ a *partial ordering* (Halbordnung), i.e. for all a,b,$c \in A$:

i) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)

ii) $a \leq a$ (reflexivity)

iii) $a \leq b \wedge b \leq a \Rightarrow a = b$ (antisymmetry)

Note: We do *not* demand that for all a,$b \in A$ holds:

$$(a \leq b) \vee (b \leq a)$$

This is a property of a ordering relation.
$(A, \leq)$ is called *partially ordered set* (teilweise geordnete Menge).
A subset $K \subseteq A$ is called *chain* (Kette, total geordnete Teilmenge) if for all x,$y \in K$ holds:

$$(x \leq y) \vee (y \leq x)$$

An element $u \in A$ is called *upper bound* (obere Schranke) of $B \subseteq A$ if $x \leq u$ for all $x \in B$.
An element $m \in A$ is called *maximal* if $m \leq a \in A \Rightarrow m = a$.

1.2 Zorn's lemma

Let $(A, \leq)$ be a partially ordered set in which every chain has an upper bound. Then there is a maximal element.

Proof

This follows from the axiom of choice, see e.g. Kowalsky: Linear algebra.

1.3 Definition (sublinear)

Let X be a *real* vector space (without topology). $p : X \to \mathbb{R}$ is called *sublinear* if for all $x,y \in X$ and $a \in \mathbb{R}_{>0}$ holds:

i) $p(ax) = ap(x)$

ii) $p(x+y) \le p(x) + p(y)$

A typical example is $p(x) = \|x\|$, but p does not need to be positive. Another example is any linear mapping.

1.4 Theorem (Hahn-Banach, real version, 1927/29)

Let X be a real vector space and $Y \subseteq X$ a subspace (Untervektorraum), $p : X \to \mathbb{R}$ sublinear and $l : Y \to \mathbb{R}$ linear with $l(y) \le p(y)$ for all $y \in Y$.
Then there is a linear extension (Fortsetzung) $\tilde{l} : X \to \mathbb{R}$ of l to X, i.e. $\tilde{l}\big|_Y = l$, such that for all $x \in X$ holds:

$$\tilde{l}(x) \le p(x)$$

Proof

i) Assume $Y \subsetneq X$, since otherwise there is nothing to prove. Choose a vector $z \in X \setminus Y$. We want to extend l to the span of Y and $\langle z \rangle$. $\tilde{l}(z)$ needs to be prescribed. For all $y \in Y$ and $a \in \mathbb{R}$ holds:

$$\tilde{l}(y+az) \overset{\text{linearity}}{=} l(y) + a\tilde{l}(z) \overset{\text{demand}}{\le} p(y+az)$$

If $a = 0$, the inequality is clear. By homogeneity assumptions, it is sufficient to consider the case $a = \pm 1$. We thus demand for all $y,y' \in Y$:

$$l(y) + \tilde{l}(z) \le p(y+z)$$
$$l\left(y'\right) - \tilde{l}(z) \le p\left(y'-z\right)$$

This is equivalent to:

$$l\left(y'\right) - p\left(y'-z\right) \le \tilde{l}(z) \le p(y+z) - l(y)$$

We can choose $\tilde{l}(z)$ if and only if:

$$l\left(y'\right) - p\left(y'-z\right) \le p(y+z) - l(y)$$

(For example set $\tilde{l}(z) = \sup_{y' \in Y} l(y') - p(y'-z)$.)

$$\Leftrightarrow \quad l\left(y'\right) + l(y) \overset{\text{lineariy}}{=} l\left(y'+y\right) \le p(y+z) + p\left(y'-z\right)$$

Now prove this inequality:
From $y' + y \in Y$ follows that $l(y+y') \le p(y+y')$ by hypothesis. Moreover, as p is sublinear, it follows:

$$p\left(y+z-z+y'\right) \le p\left(y'+z\right) + p\left(y'-z\right)$$

So the inequality is shown. Thus l can be extended to $Y + \langle z \rangle$.

ii) Consider all extensions:

$$A := \{(Z,l) \,|Y \subseteq Z \subseteq X \text{ subspace, } l : Z \to \mathbb{R} \text{ extension of } l_Y : Y \to \mathbb{R}\}$$

This set has a partial ordering $\leq$ defined by $(Z,l) \leq (Z',l')$ if $Z \subseteq Z'$ and $l'\big|_Z = l$.
For an index set I (possibly infinite, uncountable) let $K = \{(Z_\nu,l_\nu) \,|\nu \in I\}$ be a chain, i.e. for all $(Z,l)\,,(Z',l') \in K$:

$$\left((Z,l) \leq \left(Z',l'\right)\right) \vee \left(\left(Z',l'\right) \leq (Z,l)\right)$$

Set $Z = \bigcup_{\nu\in I} Z_\nu$ and define $l : Z \to \mathbb{R}$ by $l\big|_{Z_\nu} = l_\nu$. (Thus suppose $u \in Z$, so there is a $\nu \in I$ with $u \in Z_\nu$. Set $l\,(u) := l_\nu\,(u)$. ν need not be unique. Suppose $u \in Z_{\nu'}$, then we know that either $Z_{\nu'} \subseteq Z_\nu$ and $l_\nu\big|_{Z_{\nu'}} = l_{\nu'}$ or $Z_\nu \subseteq Z_{\nu'}$ and $l_{\nu'}\big|_{Z_\nu} = l_\nu$. In both cases we have $l_\nu\,(u) = l_{\nu'}\,(u)$, thus $l\,(u)$ is well defined.)
This (Z,l) is an upper bound, because for all $\nu \in I$ we have $Z_\nu \subseteq Z = \bigcup_{\lambda\in I} Z_\lambda$ and l is an extension of l_ν.
With Zorn's Lemma follows, that there exists an maximal element $\left(\tilde{Y},\tilde{l}\right)$.

Claim: $\tilde{Y} = X$

Proof: Otherwise there would be a vector $u \in X \setminus \tilde{Y}$, and $\tilde{l}$ could be extended to $\tilde{Y} \oplus \langle u\rangle$, as shown in i), in contradiction to the maximality of $\tilde{l}$. Thus $\left(X = \tilde{Y},\tilde{l}\right)$ is the desired extension. $\square_{\text{Claim}}$

$\square_{1.4}$

1.5 Theorem (Hahn-Banach, complex version)

Let X be a complex vector space and $Y \subseteq X$ a subspace. Before, we had $l\,(x) \leq p\,(x)$ as condition, which does not make sense in the complex case, since:

$$l\left(e^{\mathbf{i}\varphi}x\right) = e^{\mathbf{i}\varphi}l\,(x) \overset{\text{in general}}{\notin} \mathbb{R}$$

Let $p : X \to \mathbb{R}$ be a *seminorm*, i.e.:

i) $p\,(ax) = |a|\,p\,(x)$ (homogeneity)

ii) $p\,(x+y) \leq p\,(x) + p\,(y)$ (triangle inequality)

Let $l : Y \to \mathbb{C}$ be a linear functional with $|l\,(y)| \leq p\,(y)$ for all $y \in Y$.
Then l can be extended to X such that $|l\,(x)| \leq p\,(x)$ holds for all $x \in X$.

Proof

We also consider X as a real vector space. (u and $\mathbf{i}u$ are then linearly independent vectors.) Decompose l into its real and imaginary parts.

$$l\,(y) = l_1\,(y) + \mathbf{i}l_2\,(y)$$
$$l_1 := \mathrm{Re}\,(l\,(y))$$
$$l_2 := \mathrm{Im}\,(l\,(y))$$

l_1 and l_2 are real-linear and:

$$l_1(\mathbf{i}y) = \mathrm{Re}(l(\mathbf{i}y)) = \mathrm{Re}(\mathbf{i}l(y)) = -\mathrm{Im}(l(y)) = -l_2(y)$$

Conversely, suppose that l_1 is real-linear. Then

$$l(x) := l_1(x) - \mathbf{i} \cdot l_1(\mathbf{i}x)$$

this is indeed a complex-linear function. We know that $|l(y)| \leq p(y)$ holds for all $y \in Y$.

$$\begin{aligned} l_1(y) = \mathrm{Re}(l(y)) &\leq |l(y)| \\ \Rightarrow \qquad l_1(y) &\leq p(y) \end{aligned}$$

Theorem 1.4 yields an real-linear extension $\tilde{l}_1 : X \to \mathbb{R}$ such that $\tilde{l}_1(x) \leq p(x)$ for all $x \in X$. Set $\tilde{l}(x) = \tilde{l}_1(x) - \mathbf{i}\,\tilde{l}_1(\mathbf{i}x)$, so that $\tilde{l} : X \to \mathbb{C}$ is complex-linear.

Claim: $\left|\tilde{l}(x)\right| \leq p(x)\ \forall_{x \in X}$

Proof: Polar decomposition:

$$\begin{aligned} \tilde{l}(x) &= re^{\mathbf{i}\varphi} \\ \left|\tilde{l}(x)\right| &= r = e^{-\mathbf{i}\varphi}\tilde{l}(x) \underset{\text{complex-linear}}{\overset{\tilde{l}\text{ is}}{=}} \tilde{l}\left(e^{-\mathbf{i}\varphi}x\right) = \mathrm{Re}\left(\tilde{l}\left(e^{-\mathbf{i}\varphi}x\right)\right) = \\ &= \tilde{l}_1\left(e^{-\mathbf{i}\varphi}x\right) \leq p\left(e^{-\mathbf{i}\varphi}x\right) \overset{\text{homogeneity}}{=} p(x) \end{aligned}$$

$\square_{\text{Claim}}$

$\square_{1.5}$

Now to applications:

1.6 Theorem

Let $(X, \|.\|)$ be a normed $\mathbb{K}$-space (real or complex), $Y \subseteq X$ a subspace. Let φ be a continuous linear functional from Y to $\mathbb{K}$, i.e. for all $y \in Y$ holds:

$$|\varphi(y)| \leq \|\varphi\| \cdot \|y\|$$

Then φ can be continued to all of X with the same supnorm, i. e.:

$$\|\tilde{\varphi}\| := \sup_{x \in X, \|x\| \leq 1} |\varphi(x)| = \|\varphi\| := \sup_{y \in Y, \|y\| \leq 1} |\varphi(y)|$$

Proof

Apply the Hahn-Banach theorem with $\tilde{\varphi} := \|\varphi\| \cdot \|x\|$. $\square_{1.6}$

1.7 Corollary

Let X be a normed space and $u_0 \in X$ with $\|u_0\| = 1$. Then there exists a linear functional $\varphi : X \to \mathbb{K}$ such that:

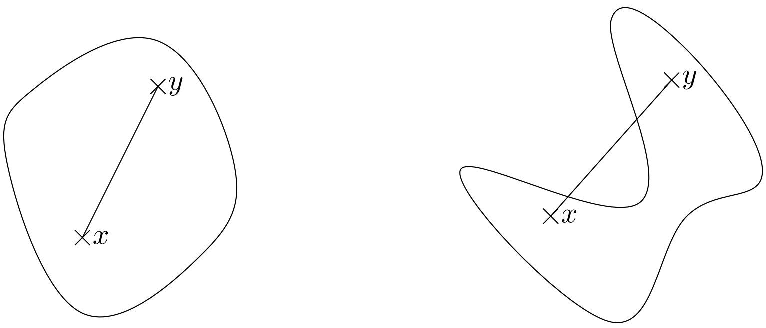
$$\varphi(u_0) = 1 \qquad\qquad \|\varphi\| = 1$$

Proof

Let $Y := \langle u_0 \rangle$ and define $\varphi_0 : \langle u_0 \rangle \to \mathbb{K}$ by $\varphi_0(u_0) = 1$. Extend φ_0 by the Hahn-Banach theorem 1.6. $\square_{1.7}$

The Hahn-Banach theorem also has a geometric formulation. Consider only the real case:
A set $K \subseteq X$ is called *convex* if for all $x, y \in K$ and $\tau \in [0,1]$:

$$\tau x + (1 - \tau) y \in K$$



(a) convex set (b) concave set

Figure 1.1: convexity

Geometric question:

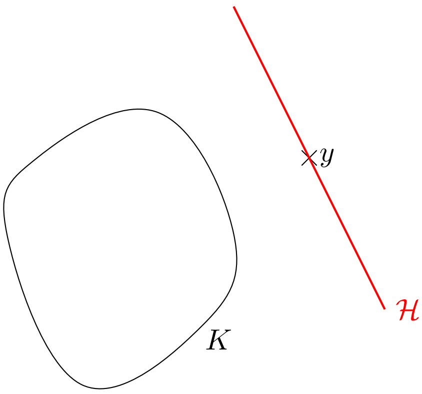


Figure 1.2: not intersecting hyperplane

Is there a hyperplane $\mathcal{H}$, which meets $y \notin K$, but does not intersect K?

1.8 Definition (interior point)

$x_0 \in K$ is an *interior point* (innerer Punkt) *of K with respect to $u \in X$* if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ for all $t \in (-\varepsilon, \varepsilon)$.
$x_0 \in K$ is an *interior point* if for all $u \in X$ there is a $\varepsilon = \varepsilon(u) \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ holds for all $t \in (-\varepsilon, \varepsilon)$.

1.9 Theorem (geometric Hahn-Banach)

Let $K \neq \emptyset$ be convex and all points of K be interior points. Let $y \notin K$. Then there is a linear functional $l : X \to \mathbb{R}$ such that $l(x) < 1$ for all $x \in K$ and $l(y) = 1$.
$\mathcal{H} := \left\{ x \in X \middle| l(x) = 1 \right\}$ defines a hyperplane. Now $y \in \mathcal{H}$ and $l\big|_K < 1$ mean that K lies in one half-space.

First introduce a suitable sublinear functional. Without loss of generality, assume $0 \in K$ (otherwise shift K).

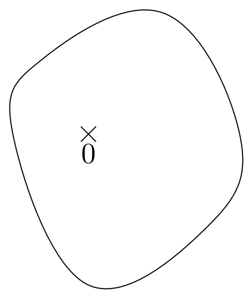


Figure 1.3: $0 \in K$

The functional $p : K \to \mathbb{R}_{\geq 0}$ with

$$p(x) := \inf \left\{ a \in \mathbb{R}_{>0} \middle| \frac{x}{a} \in K \right\}$$

is called gauge (Eichung).
Since x is an interior point, we know that $\frac{x}{a} \in K$ if $a > 1 - \varepsilon(x)$.
p is even defined on all of X, because for $x \in X$, now $\tau x \in K$ if $|\tau|$ is sufficiently small, because $0 \in K$ is an interior point.

$$p(x) < 1 \quad \Leftrightarrow \quad x \in K$$

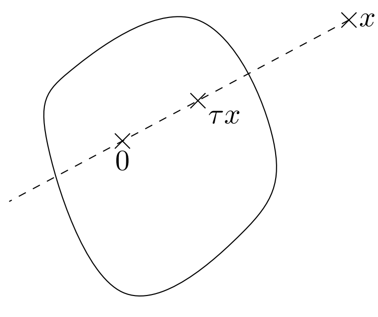


Figure 1.4: $x \notin K$, $\tau x \in K$

1.10 Lemma

p is sublinear.

Proof

The homogeneity is clear from the definition.

sub-additivity (triangle equation):
Take $x,y \in K$ and choose $a,b \in \mathbb{R}_{>0}$ such that $\frac{x}{a},\frac{y}{b} \in K$. The convexity of K implies for all $\tau \in [0,1]$:

$$\tau \frac{x}{a} + (1-\tau)\frac{y}{b} \in K$$

Choose $\tau = \dfrac{a}{a+b}$, then holds $1-\tau = \dfrac{b}{a+b}$, which gives:

$$\Rightarrow \quad \frac{1}{a+b}(x+y) \in K$$

$$p(x+y) \leq a+b$$

Taking the infimum over a and b gives $p(x+y) \leq p(x) + p(y)$:

$$p(x+y) = \inf \underbrace{\left\{ c \in \mathbb{R}_{>0} \middle| \frac{x+y}{c} \in K \right\}}_{\ni a+b} \leq a+b$$

$$\begin{aligned} p(x) &= \inf \left\{ a \middle| \frac{x}{a} \in K \right\} &\Rightarrow& \quad \underset{\varepsilon>0}{\forall} \underset{a\in\mathbb{R}_{>0}}{\exists} : p(x) \geq a - \varepsilon \\ p(y) &= \inf \left\{ b \middle| \frac{x}{b} \in K \right\} &\Rightarrow& \quad \underset{\varepsilon>0}{\forall} \underset{b\in\mathbb{R}_{>0}}{\exists} : p(y) \geq b - \varepsilon \end{aligned}$$

$\Box_{1.10}$

1.11 Lemma

$p(x) < 1 \Leftrightarrow x \in K$

Proof

If $x \notin K$ then $\frac{1}{a}x \notin K$ for all $0 < a < 1$ and so $p(x) \geq 1$.

For all $x \in K$ exists an $\varepsilon = \varepsilon(x) \in \mathbb{R}_{>0}$ with $(1+t)x \in K$ for all $t \in (-\varepsilon,\varepsilon)$.

$$\begin{aligned} &\Rightarrow \quad \left(1 + \frac{\varepsilon}{2}\right) x \in K \\ &\Rightarrow \quad p(x) \leq \frac{1}{1+\frac{\varepsilon}{2}} < 1 \end{aligned}$$

$\Box_{1.11}$

Proof of Theorem 1.9

Introduce l on $\langle y\rangle$ by $l(y) = 1$. (Assume again that $0 \in K$ and so $y \neq 0$.)
Write $z = ay \in \langle y\rangle$ with $a \in \mathbb{R}$.

- If $a < 0$, then $l(z) = a \cdot l(y) = a < 0$ but $p(z) \geq 0$ and thus the inequality $l(z) \leq p(z)$ is trivially satisfied.
- If $a > 0$ it holds:

$$l(z) = a \underset{\Rightarrow p(y)\geq 1}{\overset{y\notin K}{\leq}} a \cdot p(y) \underset{\text{homogeneity}}{\overset{\text{positive}}{=}} p(ay) = p(z)$$

So for all $z \in \langle y\rangle$ holds $l(z) \leq p(z)$.
The Hahn-Banach Theorem yields an extension $l : X \to \mathbb{R}$ such that $l(x) \leq p(x)$ for all $x \in X$.
Therefore for all $x \in K$ we have:

$$l(x) \leq p(x) < 1$$

$\square_{1.9}$

2 Normed Spaces

Let $(E, \|.\|)$ be a normed space and let the open balls $B_\varepsilon(x) = \left\{ y \middle| \|x - y\| < \varepsilon \right\}$ generate the topology on E.

2.0.1 Definition (equivalent norms)

Two norms $\|.\|_1$ and $\|.\|_2$ are *equivalent*, if there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_2$$

2.0.2 Theorem

Equivalent norms give rise to the same topology.

(No proof)

2.0.3 Theorem

If E is finite dimensional, then any two norms on E are equivalent.

(No proof)

2.0.4 Constructions (Quotient space, Cartesian product)

Let $F \subseteq E$ be a *closed* subspace. Define the *quotient space* (Faktorraum) ${}^{E}\!/\!_{F}$ as follows:

$$x \sim y :\Leftrightarrow x - y \in F$$

defines an equivalence relation on E.

$${}^{E}\!/\!_{F} := {}^{E}\!/\!_{\sim}$$

is a vector space.

$$\|u\|_{E/F} := \inf_{\substack{\hat{u} \in E \\ \hat{u} - u \in F}} \|\hat{u}\|_E$$

$\left({}^{E}\!/\!_{F}, \|.\|_{E/F} \right)$ is a normed space. The closedness of F is essential:
Suppose $F \subseteq E$ is not closed. Then there exists an $x \in \overline{F} \setminus F$, thus there is a $(x_n)_{n \in \mathbb{N}}$, $x_n \in F$

with $x_n \to x$.
Let $[x] \in {}^{E}\!/_{F}$ be the equivalence class. Then $[x] \neq 0$, since $x \notin F$, but:

$$\|[x]\| = \inf_{\substack{\hat{x}\in E\\ \hat{x}-x\in F}} \|\hat{x}\| \overset{x-x_n\sim x}{\leq} \inf \|x - x_n\| = 0$$

If $\|.\|_{E/F}$ was a norm, it would imply $[x] = 0$ and thus $x \in F$ in contradiction to $x \in \overline{F} \setminus F$.
Another construction is the *Cartesian product*: Let E and F be normed spaces.

$$E \times F := \left\{(u,v) \,\middle|\, u \in E, v \in F\right\}$$

$$\|(u,v)\|_{E\times F} := \|u\|_E + \|v\|_F$$

is a norm on $E \times F$.

2.0.5 Definition (separable)

A normed space is called *separable*, if there is a countable dense subset, i.e. there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that every nonempty open subset of the space contains at least one element of the sequence.

2.0.6 Examples

The space ℓ^∞ of bounded sequences $(a_n)_{n\in\mathbb{N}}$, $a_n \in \mathbb{K}$ with $\left\|(a_n)_{n\in\mathbb{N}}\right\|_\infty := \sup_n |a_n|$ is a Banach space.

$$A := \left\{(a_n)_{n\in\mathbb{N}} \,\middle|\, a_{2n} = 0 \underset{n\in\mathbb{N}}{\forall}\right\} \subseteq \ell^\infty$$

is a closed subspace.

$${}^{\ell^\infty}\!/_{A} \overset{\sim}{=} \left\{(a_n) \,\middle|\, a_{2n+1} = 0 \underset{n\in\mathbb{N}}{\forall}\right\}$$

$$d := \left\{(a_n) \,\middle|\, \underset{N\in\mathbb{N}}{\exists} \underset{n\in\mathbb{N}_{>N}}{\forall} a_n = 0\right\} \subseteq \ell^\infty$$

is a subspace, but not closed in ℓ^∞. Consider for example $\left(a_n = \frac{1}{n}\right) =: x \in \ell^\infty \setminus d$, $x_n \in d$ with $x_n = (a_{n_l})_{l\in\mathbb{N}}$ and:

$$a_{n_l} = \begin{cases} \frac{1}{l} & \text{if } l \leq n \\ 0 & \text{if } l > n \end{cases}$$

Then converges $x_n \to x \notin d$, and therefore d is not closed. The closure is:

$$\overline{d} = \left\{(a_n) \,\middle|\, a \xrightarrow{n\to\infty} 0\right\}$$

ℓ^∞ is not separable.

2.0.7 Example

For $1 \leq p < \infty$ define

$$\ell^p = \left\{ (a_n)_{n\in\mathbb{N}} \,\middle|\, \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

and the ℓ^p-norm:

$$\|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}$$

ℓ^p is a normed space (Hölder's inequality, Minkowski inequality) and also separable (see exercises).

2.0.8 Example

Let (Ω,μ) be a measure space (Maßraum).

$$\begin{array}{ll} L^p(\Omega) \;\; (1 \leq p < \infty) & \|f\|_p = \left(\int_\Omega |f(x)|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \\ L^\infty(\Omega) & \|f\|_\infty = \operatorname{supess}_\Omega |f(x)| = \sup \left\{ L \in \mathbb{R} \middle| \mu\left(f^{-1}([L,\infty))\right) > 0 \right\} \end{array}$$

2.1 Non-Compactness of the Unit Ball

Let $(E, \|.\|)$ be a normed vector space.

$$K := \overline{B_1(0)} = \left\{ x \in E \middle| \, \|x\| \leq 1 \right\}$$

If $\dim(E) < \infty$, K is compact by the Heine-Borel theorem.

2.1.1 Theorem

If E is infinite-dimensional, then K is not sequentially compact (folgenkompakt), i.e. it is possible to construct a sequence (y_n), $y_n \in K$, which has no convergent subsequence.

2.1.2 Lemma

Let $Y \subsetneq E$ be a proper (echter) closed subspace. Then there is a $z \in E \setminus Y$ with $\|z\| = 1$ such that holds:

$$\begin{aligned} &\underset{y\in Y}{\forall} : \; \|z - y\| > \frac{1}{2} \\ \Leftrightarrow \quad &\overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset \end{aligned}$$

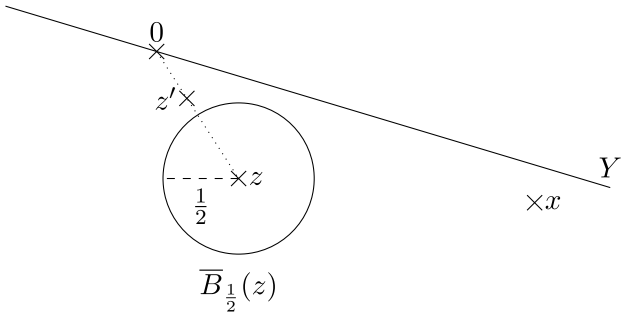


Figure 2.1: $\overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset$

Proof

Choose $x \in E \setminus Y \neq \emptyset$. As $E \setminus Y$ is open, there is a $\delta \in \mathbb{R}_{>0}$ with $B_\delta(x) \cap Y = \emptyset$. Thus we can define:

$$d := \inf_{y \in Y} \|x - y\| > 0$$

Choose $y_0 \in Y$ such that $\|x - y_0\| < 2d$. Set $z' = x - y_0$. Then $\|z'\| < 2d$ and $\|z' - y\| \geq d$ for all $y \in Y$. Thus $z := \frac{z'}{\|z'\|}$ has the desired properties. $\square_{2.1.2}$

Proof of Theorem 2.1.1

Choose inductively a sequence (y_n): $y_1 \in K$ is arbitrary. $Y_1 := \langle y_1 \rangle$ is a one dimensional subspace, which is closed. Choose $y_2 \in K$ such that $\|y_2 - y\| > \frac{1}{2}$ for all $y \in Y_1$, which is possible according to Lemma 2.1.2.
Suppose $y_1, \dots, y_n$ are given. $Y_n := \langle y_1, \dots, y_n \rangle$ is closed. So there exists a $y_{n+1} \in K$ such that for all $y \in Y_n$ holds:

$$\|y_{n+1} - y\| > \frac{1}{2}$$

This sequence has the following properties:

- $y_k \in K$
- For all $k,l \in \mathbb{N}$ with $k < l$ holds $\|y_l - y_k\| > \frac{1}{2}$, since $y_k \in Y_{l-1} = \langle y_1, \dots, y_{l-1} \rangle$ and we know by construction that $\|y_l - y\| > \frac{1}{2}$ for all $y \in Y_{l-1}$ so especially for $y_k \in Y_{l-1}$.

This implies that (y_k) has no convergent subspace. $\square_{2.1.1}$

2.2 Spaces of linear Mappings, Dual Spaces

Let E,F be normed spaces.
$A : E \to F$ is continuous if and only if it is bounded, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq C \|u\|_E$$

Denote by $L(E,F)$ the normed space of all bounded linear maps from E to F and define:

$$\|A\| := \sup_{\|u\| \leq 1} \|Au\| = \sup_{\|u\| = 1} \|Au\|$$

2.2.1 Lemma

If $B \in L(E,F)$ and $A \in L(F,G)$ then Schwarz inequality or Kato inequality holds:

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|$$
$$\|Au\| \leq \|A\| \cdot \|u\|$$

(no proof)

2.2.2 Theorem and Definition (dual pairing)

If F is complete, so is $L(E,F)$.
Special case $F = \mathbb{R}$ and $\|x\|_{\mathbb{R}} = |x|$: $E^* := L(E, \mathbb{R})$ is the dual space.

For $\varphi \in E^*$ and $u \in E$

$$\varphi(u) = (\varphi, u)$$

is called *dual pairing* (duale Paarung).

$$(.,.) : E^* \times E \to \mathbb{R}$$

is a continuous bilinear map. For $u \in E$

$$(.,u) : E^* \to \mathbb{R}$$

defines an element of $E^{**} = L(E^*,\mathbb{R})$. This gives rise to a linear mapping:

$$\iota : E \to E^{**}$$

(no proof)

2.2.3 Theorem

$\iota : E \hookrightarrow E^{**}$ is an isometric embedding of E into E^{**}.

Proof

For $u \in E$ holds:

$$\|\iota(u)\| := \sup_{\varphi \in E^*, \|\varphi\|=1} \|(\iota(u))(\varphi)\| = \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| \stackrel{?}{=} \|u\|$$

$$\|\varphi\| = \sup_{v \in E, \|v\|=1} |\varphi(v)|$$

$$\begin{aligned} & \|\varphi(u)\| \leq \|\varphi\| \cdot \|u\| \stackrel{\|\varphi\|=1}{=} \|u\| \\ \Rightarrow \quad \sup_{\varphi \in E^*, \|\varphi\|=1} & \|\varphi(u)\| \leq \|u\| \end{aligned}$$

To prove $\|\iota(u)\| \geq \|u\|$ apply the Hahn-Banach theorem:
Let $l : \langle u \rangle \to \mathbb{R}$ be the linear map with $l(u) = \|u\|$, thus:

$$\|l\| = \sup_{v \in \langle u \rangle, \|v\|=1} (l(v)) = \sup \left(l \left(\pm \frac{u}{\|u\|} \right) \right) = 1$$

By the Hahn-Banach theorem we can extend l to

$$\tilde{l} : E \to \mathbb{R}$$

with $\left\| \tilde{l} \right\| = 1$ and then holds:

$$\sup_{\varphi \in E^*, \|\varphi\|=1} \varphi(u) \overset{\|\tilde{l}\|=1}{\geq} \tilde{l}(u) = \|u\|$$

Therefore ι is injective, because from $\iota(u) = 0$ follows $\|u\|_E = \|\iota(u)\| = 0$ and therefore $u = 0$. $\square_{2.2.3}$

2.2.4 Definition (reflexive)

A Banach space is called *reflexive* (reflexiv) if ι is bijective, i.e. $E \overset{\sim}{=} E^{**}$.

2.2.5 Example

Let ℓ_1 be the space of absolutely convergent functions with the norm:

$$\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty$$

Let $(\lambda_n) \in \ell_\infty$ be a bounded sequence and define $\Lambda \in \ell_1^*$:

$$\Lambda : \ell_1 \to \mathbb{R}$$
$$\Lambda((a_n)) = \sum_{n=1}^{\infty} \lambda_n a_n$$

$$|\Lambda((a_n))| = \left| \sum_{n=1}^{\infty} \lambda_n a_n \right| \leq \sum_{n=1}^{\infty} |\lambda_n| \cdot |a_n| \leq \|(\lambda_n)\|_\infty \sum_{n=1}^{\infty} |a_n| = \|(\lambda_n)\|_\infty \cdot \|(a_n)\|_1 < \infty$$

Thus Λ is bounded and:

$$\|\Lambda\| = \sup_{n \in \mathbb{N}} |\lambda_n|$$

Claim: Every bounded linear functional on ℓ_1 is of this form, i.e. $\ell_1^* = \ell_\infty$.

Proof: Let $\Lambda \in \ell_1^*$. Choose $u_l \in \ell_1$ by $u_l = (0, \ldots, 0, 1, 0, \ldots)$ with a one at the l-th position. Setting $\lambda_l := \Lambda(u_l)$ gives:

$$|\lambda_l| = |\Lambda(u_l)| \leq \underbrace{\|\Lambda\|}_{<\infty} \cdot \underbrace{\|u_l\|}_{=1} \leq \|\Lambda\| < \infty$$

So $(\lambda_l) \in \ell_\infty$.
Let (a_k) be a finite sequence, with only zeros for $k > K \in \mathbb{N}$. Then:

$$\Lambda\left((a_k)\right) = \Lambda\left(\sum_{k=1}^{K} a_k u_k\right) = \sum a_k \Lambda\left(u_k\right) = \sum \lambda_k a_k$$

Since the finite sequences are dense in ℓ_1, the claim follows. $\square_{\text{Claim}}$

So $\ell_1^* = \ell_\infty$ and one could assume $\ell_\infty^* = \ell_1$, but this is not the case (see exercises).

Thus $\ell_1^{**} \neq \ell_1$, which means, that ℓ_1 is *not* reflexive.

2.3 Weak Convergence (Schwache Konvergenz)

Let E be a Banach space and (u_n) a sequence in E.

Normal convergence: $u_n \to u$ if and only if $\|u - u_n\| \xrightarrow{n\to\infty} 0$.

2.3.1 Definition (weak convergence, weak Cauchy sequence)

A sequence (u_n) in E *converges weakly* to u, written as $u_n \rightharpoonup u$, if for all $\varphi \in E^*$ the sequence $\varphi\left(u_n\right)$ converges to $\varphi\left(u\right)$, i.e. $\varphi\left(u_n\right) \to \varphi\left(u\right)$.

(u_n) is a *weak Cauchy sequence* if for all $\varphi \in E^*$ the sequence $\varphi\left(u_n\right)$ is a Cauchy sequence.

2.3.2 Theorem (Uniqueness of weak limit)

The weak limit is unique.

Proof

Let (u_n) be a sequence in E, which converges weakly to u and u', i.e. for all $\varphi \in E^*$ holds:

$$\varphi\left(u_n\right) \to \varphi\left(u\right) \qquad\qquad \varphi\left(u_n\right) \to \varphi\left(u'\right)$$

$$\Rightarrow \quad 0 = \varphi\left(u_n - u_n\right) \to \varphi\left(u - u'\right)$$

So $\varphi\left(u - u'\right) = 0$ for all $\varphi \in E^*$.

Claim: $v := u - u' = 0$

Proof: Assume to the contrary that $v \neq 0$.
Choose $\varphi : \langle v \rangle \to \mathbb{R}$ with $\varphi\left(v\right) = 1$. By the Hahn-Banach theorem φ can be extended continuously to E.
Therefore exists a $\varphi \in E^*$ with $\varphi\left(v\right) = 1$, which is a contradiction to $\varphi\left(v\right) = 0$. $\square_{\text{Claim}}$

$\square_{2.3.2}$

2.3.3 Theorem (convergence implies weak convergence)

Every convergent sequence converges weakly.

Proof

Suppose that $u_n \to u$. For $\varphi \in E^*$ follows:

$$|\varphi(u_n) - \varphi(u)| = |\varphi(u_n - u)| \leq \underbrace{\|\varphi\|}_{\in \mathbb{R}} \cdot \|u_n - u\| \to 0$$

$$\begin{aligned} \Rightarrow \quad & \varphi(u_n) \to \varphi(u) \\ \Rightarrow \quad & u_n \rightharpoonup u \end{aligned}$$

$\square_{2.3.3}$

2.3.4 Example

$E = \left\{ (a_n) \middle| a_n \xrightarrow{n\to\infty} 0 \right\} \subsetneq \ell_\infty$ with $\|(a_n)\| = \sup_n |a_n|$ is a Banach space.
Let $u_n = (0, \ldots, 0, 1, 0, \ldots)$ be the sequence with a one at the n-th position and zeros elsewhere.
For $n \neq m$ we have:

$$\|u_n - u_m\| = \sup\{0, |1|, |-1|\} = 1$$

Thus (u_n) is *not* a Cauchy sequence. Every $\varphi \in E^*$ can be represented with $(\lambda_k) \in \ell_1$ as (see exercises):

$$\begin{aligned} \varphi((a_n)) &= \sum_k \lambda_k a_k \\ \|\varphi\| &= \sum_{k=1}^{\infty} |\lambda_k| < \infty \end{aligned}$$

$$\varphi(u_n) = \sum_{k=1}^{\infty} \lambda_k \delta_{kn} = \lambda_n \xrightarrow{n\to\infty} 0$$

From $(\lambda_n) \in \ell_1$ follows $\lambda_n \to 0$. This means that $u_k \rightharpoonup 0$.
This is used in the lectures on partial differential equations.
From $\mathscr{S}(u_n) \to \inf \mathscr{S}$ follows not necessarily $u_n \to u$, but $u_n \rightharpoonup u$.

Consider $A_n \in L(E,F)$.

- *norm convergence*: $A_n \to A$ in $L(E,F)$ means $\|A_n - A\| \to 0$.
- *strong convergence*: $A_n u \to Au$ in F for all $u \in E$.
- *weak convergence*: $A_n u \rightharpoonup Au$ for all $u \in E$, i.e. for all $\varphi \in F^*$ holds $\varphi(A_n u) \to \varphi(Au)$.

2.4 The Baire Category Theorem

Let E be a metric space (e.g. a normed space).

2.4.1 Definition (nowhere dense, set of first/second category)

A subset $A \subseteq E$ is called *nowhere dense* (nirgends dicht) if $\overline{A}^\circ = \emptyset$.
A is called *of first category* (or *meager*) if it can be written as a countable union of nowhere dense sets. Otherwise it is *of second category*.

Example

- $\mathbb{N} \subseteq \mathbb{R}$ is nowhere dense: $\overline{\mathbb{N}} = \mathbb{N}$, $\mathbb{N}^\circ = \emptyset$
- $\mathbb{Q} \subseteq \mathbb{R}$ is dense: $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Q}}^\circ = \mathbb{R}^\circ = \mathbb{R}$

2.4.2 Theorem (René Baire, 1899)

Let $E \neq \emptyset$ be a complete metric space (Polish space). Then E is of second category.

Proof

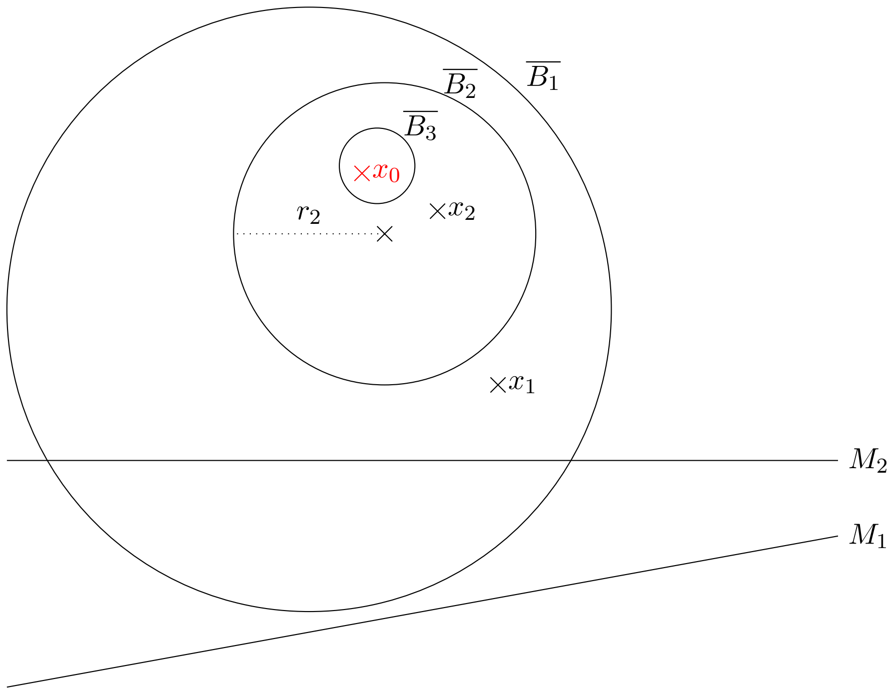


Figure 2.2: $B_n \cap M_n = \emptyset$

Assume in contrast that $E = \bigcup_{n \in \mathbb{N}} M_n$ and the sets M_n are nowhere dense. Without loss of generality assume that the M_n are closed, since otherwise one can replace M_n by $\overline{M_n}$.
We shall construct inductively balls $\overline{B_n} = \overline{B_{r_n}(x_n)}$ such that $\overline{B_{n+1}} \subseteq \overline{B_n}$, $r_n < 2^{-n}$ and $\overline{B_n} \cap M_n = \emptyset$ for all n.

Then the points x_n form a Cauchy sequence, because for all $n < m \in \mathbb{N}$ we have $x_{n+1} \in B_n$ and so $\|x_n - x_{n+1}\| < r_n < 2^{-n}$:

$$\|x_n - x_m\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_m\| \leq \ldots \leq$$

$$\leq 2^{-n} + 2^{-(n+1)} + \ldots + 2^{-(m-1)} \leq 2^{-n}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) \leq 2 \cdot 2^{-n}$$

Since E is complete, $x_n \to x_0 \in E$ converges. Then $x_0 \in \overline{B_n}$ for all n, which implies $x_0 \notin M_n$ and thus the contradiction $x_0 \notin \bigcup_n M_n = E$ follows.

Construction of the balls $\overline{B_n}$:
M_1 is nowhere dense and therefore $B_1(0) \not\subseteq M_1$. So there exists a $x_1 \in B_1(0) \setminus M_1$. Since M_1 is closed, $B_1(0) \setminus M_1$ is open and therefore there exists a radius r_1 such that $B_{2r_1}(x_1)$ is contained in $B_1(0) \setminus M_1$ and thus $\overline{B_{r_1}(x_1)} \cap M_1 = \emptyset$.

Suppose $\overline{B_n}$ has been constructed. M_{n+1} is nowhere dense and closed and so there exists a $x_{n+1} \in \overline{B_n} \setminus M_{n+1}$ and $r_{n+1} < 2^{-(n+1)}$ such that $B_{2r_{n+1}}(x_{n+1}) \subseteq \overline{B_n} \setminus M_{n+1}$. Then follows $\overline{B_{r_{n+1}}(x_{n+1})} \cap M_{n+1} = \emptyset$. $\square_{2.4.2}$

2.4.3 Theorem (Uniform boundedness principle, Prinzip der gleichmäßigen Beschränktheit)

Let E be a Banach space and F a normed space. Let T_i be a sequence in $L(E,F)$ which is point-wise bounded, i.e. for all $u \in E$:

$$\sup_i \|T_i u\| \leq C(u) < \infty$$

Then sup-norms of T_i are bounded:

$$\sup_i \|T_i\| = \sup_i \sup_{\|u\|=1} \|T_i u\| \leq \tilde{C} < \infty$$

(Thus there exists a constant $C \in \mathbb{R}_{>0}$ such that $\|T_i u\| \leq C$ for all $i \in \mathbb{N}$ and for all $u \in E$ with $\|u\| = 1$.)

Proof

The sets $M_n = \left\{u \in E \middle| \sup_i \|T_i u\| \leq n\right\}$ are closed by continuity of the $T_i \in L(E,F)$, i.e. for $u_k \to u$ converges $\|T_i u_k\| \xrightarrow{k\to\infty} \|T_i u\|$.
$E = \bigcup_n M_n$, because for any $u \in E$, $\sup_i \|T_i u\| < \infty$ and thus $u \in M_n$ for $n > \sup_i \|T_i u\|$.
If all the sets M_n had empty interior, we would get a contradiction to Baire's theorem.
So there exists an $n_0 \in \mathbb{N}$ such that $\overset{\circ}{M}_{n_0} \neq \emptyset$ and thus there are $u_0 \in E$ and $r \in \mathbb{R}_{>0}$ such that $B_r(u_0) \subseteq M_{n_0}$.

For all $v \in B_r(u_0)$ we know that $\sup_i \|T_i v\| \leq n_0$ which is equivalent to:

$$\sup_{v \in B_r(u_0)} \|T_i v\| \leq n_0 \qquad \forall_{i \in \mathbb{N}}$$

Let $w \in B_r(0)$ be arbitrary. Then $v := u_0 + w \in B_r(u_0)$.

$$T_i w \overset{T_i \text{ linear}}{=} T_i v - T_i u_0$$

$$\|T_i w\| \leq \|T_i v\| + \|T_i u_0\| \leq n_0 + \sup_i \|T_i u_0\| < \infty$$

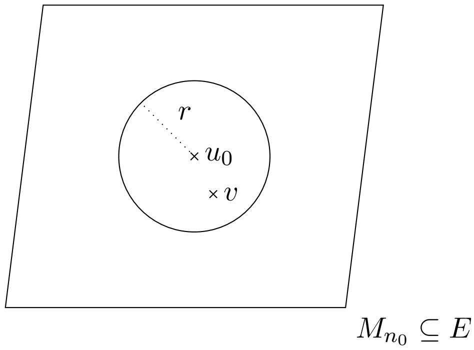


Figure 2.3: $B_r(u_0) \subseteq M_{n_0}$

Here $\sup_i \|T_i u_0\| < \infty$, because the T_i are point-wise bounded.

$$\Rightarrow \quad \|T_i w\| \leq C \qquad \underset{w \in B_r(0)}{\forall}$$

$$\Rightarrow \quad \|T_i \tilde{w}\| \leq \tilde{C} = \frac{C}{r} \qquad \underset{\tilde{w} \in \overline{B_1(0)}}{\forall}$$

So $\|T_i\| \leq \tilde{C}$ for all $i \in \mathbb{N}$ and so $\|T_i\|$ is bounded. $\square_{2.4.3}$

2.4.4 Corollary

Let E be a normed space, not necessarily complete, and (u_n) a weak Cauchy sequence. Then $\|u_n\|$ is a bounded sequence.

Proof

$E^* = L(E,\mathbb{R})$ is a Banach space after theorem 2.2.2, since $\mathbb{R}$ is complete. Now we can view every u_n as operator:

$$u_n : E^* \to \mathbb{R}$$
$$\varphi \mapsto \varphi(u_n)$$

So (u_n) is a sequence in $L(E^*,\mathbb{R})$. For all $\varphi \in E^*$ we know that $\varphi(u_n)$ is a Cauchy sequence and thus bounded:

$$\Rightarrow \quad |\varphi(u_n)| < C(\varphi)$$

Applying theorem 2.4.3 yields:

$$|\varphi(u_n)| < C \qquad \underset{\varphi \text{ with } \|\varphi\|=1}{\forall}$$

$$\Leftrightarrow \quad \sup_{n \in \mathbb{N}} \sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(u_n)| < C$$

For any $v \in E$ we have

$$\sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(v)| = \|v\|$$

by the Hahn-Banach theorem:

- $|\varphi(v)| \leq \|\varphi\| \cdot \|v\| \overset{\|\varphi\|=1}{=} \|v\|$
- Choose $\varphi : \langle v \rangle \to \mathbb{R}$ with $\varphi(v) = \|v\|$ and so $\|\varphi\| = 1$. By the Hahn-Banach theorem we can extend φ to $\tilde{\varphi} : E \to \mathbb{R}$ such that $\|\tilde{\varphi}\| = 1$. Then $\tilde{\varphi}(v) = \|v\|$ and so $\sup_{\|\varphi\|=1} |\varphi(v)| \geq \|v\|$.

Thus we get $\sup_n \|u_n\| < C$. $\square_{2.4.4}$

2.4.5 Corollary and Definition (Banach-Steinhaus, equicontinuous, uniformly continuous)

Let E,F be Banach spaces and $T_i \in L(E,F)$.

If the (T_i) are point-wise bounded, then the T_i are *equicontinuous* (gleichgradig stetig).

Definition (uniformly continuous, equicontinuous)

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function.

Continuity:

$$\underset{x_0 \in \mathbb{R}}{\forall} \; \underset{\varepsilon \in \mathbb{R}_{>0}}{\forall} \; \underset{\delta \in \mathbb{R}_{>0}}{\exists} : \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

f is called *uniformly continuous* (gleichmäßig stetig) if:

$$\underset{\varepsilon \in \mathbb{R}_{>0}}{\forall} \; \underset{\delta \in \mathbb{R}_{>0}}{\exists} : \quad \|x - y\| < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon$$

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a series of real-valued functions. (f_n) is called *equicontinuous* if:

$$\underset{x_0 \in \mathbb{R}}{\forall} \; \underset{\varepsilon \in \mathbb{R}_{>0}}{\forall} \; \underset{\delta \in \mathbb{R}_{>0}}{\exists} \; \underset{n \in \mathbb{N}}{\forall} : \quad \|x - x_0\| < \delta \quad \Rightarrow \quad \|f_n(x) - f_n(x_0)\| < \varepsilon$$

For a linear map $A \in L(E,F)$ holds:

$$\|Au\| \leq \|A\| \|u\|$$
$$\|Au - Au_0\| \leq \|A\| \|u - u_0\|$$

Therefore choose $\delta = \frac{\varepsilon}{2\|A\|}$, i.e.:

$$\underset{\varepsilon \in \mathbb{R}_{>0}}{\forall} \; \underset{\delta \in \mathbb{R}_{>0}}{\exists} : \quad \|u\| < \delta \quad \Rightarrow \quad \|Au\| < \varepsilon$$

Proof

Since (T_i) is point-wise bounded there is a $C \in \mathbb{R}_{>0}$ such that for all $i \in \mathbb{N}$ holds $\|T_i\| \leq C$ due to the principle of uniform boundedness 2.4.3. So for all $i \in \mathbb{N}$ holds:

$$\|T_i u\| \leq \|T_i\| \|u\| \leq C \|u\|$$

Choose $\delta = \frac{\varepsilon}{2C}$ shows that the T_i is equicontinuous. $\square_{2.4.5}$

In the following let E and F be Banach spaces.

2.4.6 Definition (open)

A (not necessarily linear) map $A : E \to F$ is called *open* if the image of every open set is open. (If there exists an inverse A^{-1} then "A open" is equivalent to "A^{-1} continuous".)

Let A be linear and open. $B_1(0) \subseteq E$ is open, so $A(B_1(0)) \subseteq F$ is open.
Since $0 \in A(B_1(0))$, there is a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(0) \subseteq A(B_1(0))$.

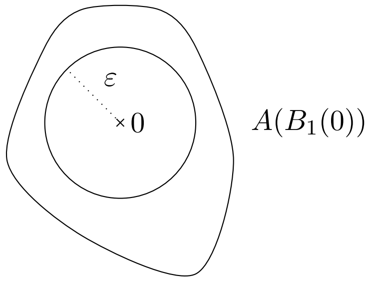


Figure 2.4: $B_\varepsilon(0) \subseteq A(B_1(0))$

Due to the linearity holds in general:

$$B_\lambda(0) \subseteq A\left(B_{\frac{\lambda}{\varepsilon}}(0)\right)$$

In particular, A is surjective.

If A is additionally injective, then A is bijective and the openness means that A^{-1} is continuous.

2.4.7 Theorem (Open mapping theorem, Prinzip der offenen Abbildung)

If $A \in L(E,F)$ is surjective, then A is open.

2.4.8 Corollary

If $A \in L(E,F)$ is bijective, then $A^{-1} \in L(F,E)$ is continuous.

Proof

A is open following 2.4.7, since A is surjective. This means that A^{-1} is continuous. $\square_{2.4.8}$

Proof of 2.4.7

Since A is surjective, $F = A(E)$. Since every element of E has a finite norm, we know:

$$\begin{aligned} & E = \bigcup_{n\in\mathbb{N}} B_n(0) \\ \Rightarrow\quad & F = A\left(\bigcup_{n\in\mathbb{N}} B_n(0)\right) = \bigcup_{n\in\mathbb{N}} A(B_n(0)) \end{aligned}$$

According to Baire's theorem there is a $n \in \mathbb{N}$ such that $\overline{A(B_n(0))}^\circ \neq \emptyset$.

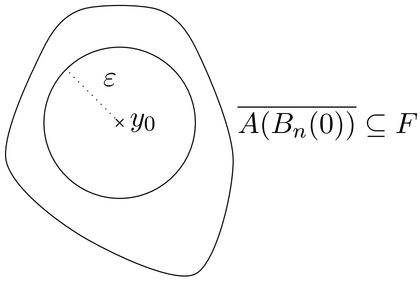


Figure 2.5: $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$

So there exists a $y_0 \in A(B_n(0))$ and a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$. Since A is surjective, there is a $x_0 \in B_n(0)$ with $y_0 = A(x_0)$.

$$\Rightarrow \quad \overline{A(B_n(0) - x_0)} = \overline{A(B_n(0)) - y_0} = \overline{A(B_n(0))} - y_0 \supseteq B_\varepsilon(y_0) - y_0 = B_\varepsilon(0)$$

If n' is large enough, then $B_n(-x_0) \subseteq B_{n'}(0)$ and so $\overline{A(B_{n'}(0))} \supseteq B_\varepsilon(0)$.
Since A is linear, we can rescale, i.e. there is a $c := \frac{\varepsilon}{n'} \in \mathbb{R}_{>0}$ such that for all $r \in \mathbb{R}_{<0}$ holds:

$$\overline{A(B_r(0))} \supseteq B_{cr}(0)$$

Now we show that every $u \in B_c(0)$ is the image of a $x \in B_2(0)$, i.e. $B_c(0) \subseteq A(B_2(0))$:
Ansatz as a series:

$$x = \sum_{j=1}^{\infty} x_j$$

Choose $x_1 \in B_1(0)$ with $\|u - Ax_1\| < \frac{c}{2}$, which is possible since $\overline{A(B_1(0))} \supseteq B_c(0)$.
Choose $x_2 \in B_2(0)$ with $\|u - Ax_1 - Ax_2\| < \frac{c}{4}$, which is possible since $u - Ax_1 \in B_{\frac{c}{2}}(0)$ and $\overline{A\left(B_{\frac{1}{2}}(0)\right)} \subseteq B_{\frac{c}{2}}(0)$.
And so on choose $x_m \in B_{\frac{1}{2^m}}(0)$ with $\|u - \sum_{i=1}^m Ax_i\| < \frac{c}{2^m}$.
The series $\sum_{i=1}^{\infty} x_i$ converges, since:

$$\left\| \sum_{j=m}^{M} x_j \right\| \leq \sum_{j=m}^{M} \|x_j\| \leq \sum_{j=m}^{M} 2^{-j}$$

So the sequence of partial sums is a Cauchy sequence. Because E is complete, this sequence converges.

The continuity of A yields:

$$Ax = \sum_{j=1}^{\infty} Ax_j = u$$

So there exists a $x \in E$ with $\|x\| < 2$ and $Ax = u$. $\square_{2.4.7}$

$$\sum_{j=1}^{n} x_j \xrightarrow{n \to \infty} x \qquad\qquad \|x\| < 2$$

$$\begin{array}{c} \sum_{j=1}^{n} Ax_j \xrightarrow{n\to\infty} u \\ \| \\ A\left(\sum_{j=1}^{n} x_j\right) \xrightarrow[\text{continuity of } A]{n\to\infty} Ax \end{array}$$

Definition (Graph)

For a function $f:\mathbb{R}\to\mathbb{R}$ the *graph* is defined as:

$$\operatorname{graph} f := \left\{(x,f(x))\,\middle|\,x\in\mathbb{R}\right\}\subseteq\mathbb{R}\times\mathbb{R}$$

For $A:E\to F$ the *graph* is:

$$\operatorname{graph} A := \left\{(u,Au)\,\middle|\,u\in E\right\}\subseteq E\times F$$

Here $E\times F$ is a product of normed spaces which has the norm:

$$\|(u,v)\| := \|u\|_E + \|v\|_F$$

Lemma

If A is continuous, then graph$A$ is closed.

Proof

Let $(u_n,Au_n)\in\operatorname{graph}A$ be a Cauchy sequence in $E\times F$ for Banach spaces E and F, i.e. $u_n\to u$. Since A is continuous, it follows:

$$Au_n\to v:=Au$$

Therefore $(u,v)\in\operatorname{graph}(A)$ and so the graph is closed. $\square_{\text{Lemma}}$

Consider the function:

$$\begin{aligned} f:\mathbb{R}\setminus\{0\}&\to\mathbb{R}\\ x&\mapsto\frac{1}{x}\end{aligned}$$

f is not continuous, but graph(f) is closed in $(\mathbb{R}\setminus\{0\})\times\mathbb{R}$.

2.4.9 Theorem (Closed graph theorem, Satz vom abgeschlossenen Graphen)

Suppose a linear map $A:E\to F$ between Banach spaces E and F has a closed graph. Then A is continuous.

graph(A) closed means:
For all $u_n\in E$ with $u_n\to u$ *and* $Au_n\to v$, the point $(u,v)\in\operatorname{graph}(A)$, i.e. $Au=v$.

A continuous means:
For all $u_n\in E$ with $u_n\to u$, the sequence $Au_n\to v$ converges and $Au=v$

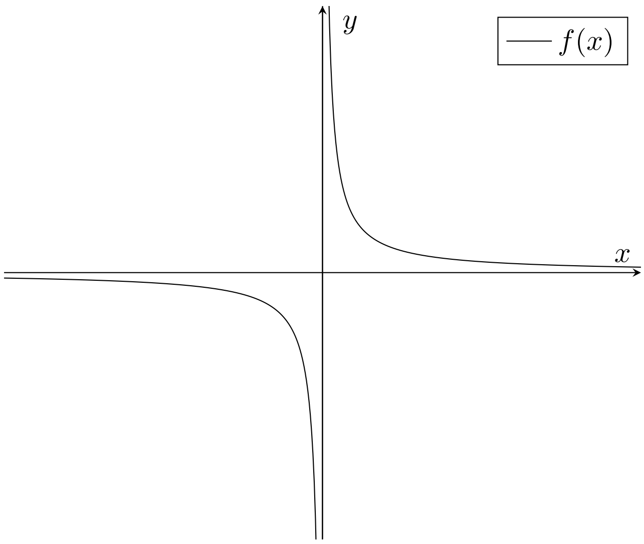


Figure 2.6: f is not continuous, but graph$f$ is closed.

Proof

On $E \times F$ we have the norm:

$$\|(u,v)\| := \|u\|_E + \|v\|_F$$

The graph

$$G := \left\{(u,Au) \,\middle|\, u \in E\right\} \subseteq E \times F$$

is a subspace of $E \times F$, since for $\lambda \in \mathbb{R}$ and $u,\tilde{u} \in E$ holds:

$$\lambda\left(u,Au\right) + \left(\tilde{u},A\tilde{u}\right) = \left(\lambda u + \tilde{u},\lambda Au + A\tilde{u}\right) \overset{A \text{ linear}}{=} \left(\lambda u + \tilde{u},A\left(\lambda u + \tilde{u}\right)\right) \in G$$

So G is complete and therefore a Banach space, since we assumed it to be closed.
Define:

$$\begin{aligned} P : G &\to E \\ (u,Au) &\mapsto u \end{aligned}$$

$$\|(u,Au)\| = \|u\| + \|Au\| \geq \|u\| = \|P\left(u,Au\right)\|$$

So for all $w \in G$ holds $\|Pw\| \leq \|w\|$ and therefore $\|P\| \leq 1$. In particular, P is continuous.
P is obviously surjective and it is also injective, since:

$$P^{-1}\left(u\right) = \left(u,Au\right)$$

Following the open mapping theorem, P^{-1} is continuous, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\|u\| + \|Au\| = \|(u,Au)\| = \left\|P^{-1}\left(u\right)\right\| \leq C\left\|u\right\|$$

Then follows:

$$\|Au\| \leq \left(C - 1\right)\|u\|$$

Therefore A is continuous. $\square_{2.4.9}$

2.5 Neumann series

Let E be a Banach space and $A \in L(E,E) =: L(E)$.

When is A continuously invertible?
Remember that for $x \in \mathbb{K}$ with $|x| < 1$ holds:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is the geometric series.
Idea: $A = \mathbb{1} - B$ with $B \in L(E)$

Ansatz: $A^{-1} := \sum_{n=0}^{\infty} B^n$

This works indeed if $\|B\| < 1$.

2.5.1 Lemma and Definition (Neumann series)

The series

$$C := \sum_{n=0}^{\infty} B^n$$

is called Neumann series (Neumannsche Reihe).

If $\|B\| < 1$, then C defines an element of $L(E,E)$, i.e. the Neumann series converges absolutely.

Proof

Consider the partial sums:

$$S_n := \sum_{k=0}^{n} B^k$$

Since $L(E,E)$ is a Banach space, it is enough to show that S_n is a Cauchy series. Without loss of generality assume $m > n$:

$$\|S_n - S_m\| = \left\| \sum_{k=n}^{m} B^k \right\| \overset{\Delta \text{ inequality}}{\leq} \sum_{k=n}^{m} \left\| B^k \right\| \overset{\text{Schwarz}}{\leq} \sum_{k=n}^{m} \|B\|^k < c \|B\|^n \to 0$$

$\square_{2.5.1}$

2.5.2 Theorem

$$C = (\mathbb{1} - B)^{-1}$$

Proof

$$(\mathbb{1} - B)\, C = (\mathbb{1} - B) \sum_{n=0}^{\infty} B^n = \left(\mathbb{1} + B + B^2 + \ldots\right) - \left(B + B^2 + \ldots\right) = \mathbb{1}$$

$\square_{2.5.2}$

2.5.3 Theorem

The set of all continuously invertible mappings is open in $L(E)$.

Proof

Assume that $A \in L(E)$ is continuously invertible, i.e. A^{-1} exists and $A^{-1} \in L(E)$. Set:

$$\varepsilon = \frac{1}{2\left\|A^{-1}\right\|}$$

Let us show, that every element of $B_\varepsilon(A) \subseteq L(E)$ is continuously invertible:
Let $C \in B_\varepsilon(A)$, i.e. $\|A - C\| < \varepsilon$.

$$C = A - (A - C) = A\big(\mathbb{1} - \underbrace{A^{-1}(A - C)}_{=:B}\big)$$

Then holds:

$$\|B\| \le \left\|A^{-1}\right\| \cdot \|A - C\| < \left\|A^{-1}\right\| \cdot \frac{1}{2\left\|A^{-1}\right\|} = \frac{1}{2} < 1$$

Hence $\mathbb{1} - B$ is continuously invertible by the Neumann series and therefore

$$C^{-1} = (\mathbb{1} - B)^{-1} \cdot A^{-1}$$

is continuous. $\square_{2.5.3}$

3 Hilbert spaces

Definition (scalar product)

Let H be a real ($\mathbb{K} := \mathbb{R}$) or complex ($\mathbb{K} := \mathbb{C}$) vector space with *scalar product*:

$$\langle .,. \rangle : H \times H \to \mathbb{K}$$

i) Positive definiteness: $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0 \Rightarrow u = 0$.

ii) Linear in the second and anti-linear in the first argument:

$$\langle \lambda u,v \rangle = \overline{\lambda} \langle u,v \rangle$$

iii) Symmetry: $\overline{\langle u,v \rangle} = \langle u,v \rangle$

Define the corresponding norm:

$$\|u\| := \sqrt{\langle u,u \rangle}$$

3.0.1 Definition (Hilbert space)

A complete scalar product space is called *Hilbert space*.

The Schwarz inequality holds:

$$|\langle u,v \rangle| \leq \|u\| \cdot \|v\|$$

3.0.2 Lemma (parallelogram equality)

The parallelogram equality (Parallelogramm-Gleichung) is:

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

Proof

$$\begin{aligned}
&\|u+v\|^2 = \langle u+v,u+v \rangle = \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle \\
&\|u-v\|^2 = \langle u-v,u-v \rangle = \langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle \\
\Rightarrow \quad &\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)
\end{aligned}$$

$\square_{3.0.2}$

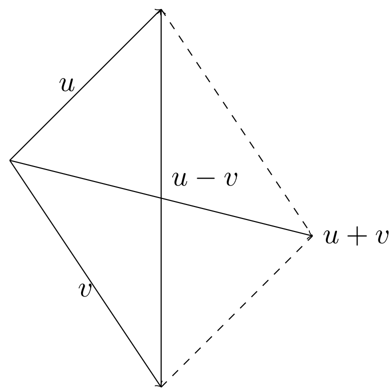


Figure 3.1: parallelogram

3.0.3 Definition (orthogonal, orthonormal)

i) Vectors $u,v \in H$ are called *orthogonal*, symbolically $u \perp v$, if $\langle u,v \rangle = 0$.

ii) Subspaces $M_1,M_2 \subseteq H$ are orthogonal, symbolically $M_1 \perp M_2$, if $\langle u,v \rangle = 0$ for all $u \in M_1$ and $v \in M_2$.

iii) A family $(u_i)_{i \in I}$ of vectors $u_i \in H$ is called *orthonormal* if:

$$\langle u_i,u_j \rangle = \delta_{ij}$$

3.0.4 Theorem (Bessel's inequality)

Let $(u_i)_{1 \leq i \leq N}$ be an orthonormal family. Then for all $u \in H$ holds:

$$\|u\|^2 = \sum_{i=1}^{N} \langle u_i,u \rangle^2 + \left\| u - \sum_{i=1}^{N} u_i \langle u_i,u \rangle \right\|^2$$

$$\|u\|^2 \geq \sum_{i=1}^{N} \langle u_i,u \rangle^2$$

Proof

$$\begin{aligned}
\left\| u - \sum_{i=1}^{N} u_i \langle u_i,u \rangle \right\|^2 &= \left\langle u - \sum_{i=1}^{N} u_i \langle u_i,u \rangle , u - \sum_{j=1}^{N} u_j \langle u_j,u \rangle \right\rangle = \\
&= \langle u,u \rangle - \sum_{j=1}^{N} \langle u,u_j \rangle \langle u_j,u \rangle - \sum_{i=1}^{N} \overline{\langle u_i,u \rangle} \langle u_i,u \rangle + \sum_{i,j=1}^{N} \overline{\langle u_i,u \rangle} \langle u_j,u \rangle \underbrace{\langle u_i,u_j \rangle}_{=\delta_{ij}} = \\
&= \|u\|^2 - 2 \sum_{i=1}^{N} |\langle u_i,u \rangle|^2 + \sum_{i=1}^{N} |\langle u_i,u \rangle|^2 = \\
&= \|u\|^2 - \sum_{i=1}^{N} |\langle u_i,u \rangle|^2
\end{aligned}$$

$\square_{3.0.4}$

Definition (Hilbert space isomorphism)

Let $(H_1, \langle .,. \rangle_1)$ and $(H_2, \langle .,. \rangle_2)$ be Hilbert spaces.

A *Hilbert space isomorphism* is a mapping $U : H_1 \to H_2$ which is linear, bijective and isometric (isometrisch), i.e. for all $u,v \in H_1$:

$$\langle u,v \rangle_1 = \langle Uu,Uv \rangle_2$$

Definition (Direct sum)

Let $(H_1, \langle .,. \rangle_1)$ and $(H_2, \langle .,. \rangle_2)$ be Hilbert spaces.

Define:

$$H := \left\{ (u,v) \,\middle|\, u \in H_1, v \in H_2 \right\}$$

$$\begin{aligned} (u_1,v_1) + (u_2,v_2) &:= (u_1 + u_2, v_1 + v_2) \\ \lambda\,(u,v) &:= (\lambda u, \lambda v) \\ \langle (u_1,v_1)\,,(u_2,v_2) \rangle &:= \langle u_1,u_2 \rangle + \langle v_1,v_2 \rangle \end{aligned}$$

This makes $H =: H_1 \oplus H_2$ a Hilbert space, called *direct sum* of H_1 and H_2, which is sometimes called orthogonal due to:

$$\langle (u,0)\,,(0,v) \rangle = 0$$

3.0.5 Example

$$\ell_2 = \left\{ (a_n)_{n\in\mathbb{N}} \,\middle|\, a_n \in \mathbb{K}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

Define a scalar product:

$$\langle (a_n)\,,(b_n) \rangle := \sum_{n=1}^{\infty} \overline{a}_n \cdot b_n$$

$$\langle (a_n)\,,(a_n) \rangle = \sum_{n=1}^{\infty} |a_n|^2 = \|a_n\|_2^2$$

$\left(\ell^2, \|.\|_2\right)$ is a Banach space. Thus $\left(\ell^2, \langle .,. \rangle\right)$ is a Hilbert space.

3.1 Projection on closed convex subsets

Let $(H, \langle .,. \rangle)$ be a Hilbert space and $K \subseteq H$ a closed convex subset.

$u,v \in K$ $\qquad\qquad$ $w \in H \setminus K$

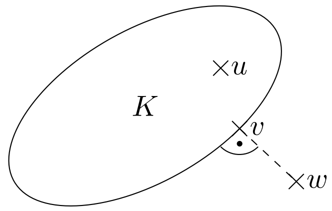


Figure 3.2: $\|v-w\| = \inf_{u \in K} \|u-w\|$

We want to find a vector v such that $\|v-w\| = \inf_{u \in K} \|u-w\|$.

If K were compact, then choose minimizing sequence (Minimalfolge), i.e.:

$$\|u_i - w\| \to \inf_{u \in K} \|u-w\|$$

Choose a convergent subsequence $u_{i_l} \to v$. Then by continuity:

$$\|v-w\| = \lim_{i\to\infty} \|u_i - w\| = \inf_{u \in K} \|u-w\|$$

The main application are closed subspaces $K \subseteq H$.

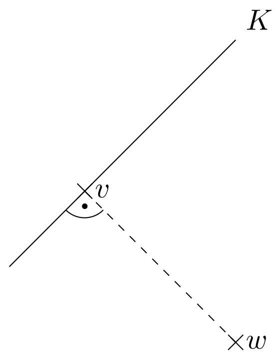


Figure 3.3: $v - w \perp K$

In this case $v-w$ will be called orthogonal to K motivating the name *orthogonal projection.*

3.1.1 Theorem (Hilbert)

There is a unique $v \in K$ with:

$$\|v-w\| = \inf_{u \in K} \|u-w\|$$

Proof

Consider a minimizing sequence u_i:

$$\|u_i - w\| \to \inf_{u \in K} \|u-w\| =: d$$

We show that (u_i) is a Cauchy sequence:

$$\begin{aligned}
\|u_i - u_j\|^2 &= \|(u_i - w) + (w - u_j)\|^2 = \\
&\overset{3.0.2}{=} 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \|(u_i - w) - (w - u_j)\|^2 = \\
&= 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \left\|-2\left(w - \frac{u_i + u_j}{2}\right)\right\|^2 = \\
&= 2\left(\underbrace{\|u_i - w\|^2}_{\to d^2} + \underbrace{\|w - u_j\|^2}_{\to d^2} - 2\left\|\frac{u_i + u_j}{2} - w\right\|^2\right)
\end{aligned}$$

$$\|u_i - w\| \xrightarrow{i\to\infty} d = \inf_{u\in K} \|u - w\|$$
$$\|u_j - w\| \xrightarrow{j\to\infty} d = \inf_{u\in K} \|u - w\|$$

Since K is convex and $u_i, u_j \in K$, we know:

$$\frac{u_i + u_j}{2} \in K$$

$$\Rightarrow \quad \left\|\frac{u_i + u_j}{2} - w\right\| \geq d$$

Thus:

$$\|u_i - u_j\|^2 \leq 2\left(\|u_i - w\|^2 + \|w - u_j\|^2 - 2d^2\right) \xrightarrow{i,j\to\infty} 2\left(d^2 + d^2 - 2d^2\right) = 0$$

So there exists a $N \in \mathbb{N}$ such that $\|u_i - u_j\| < \varepsilon$ for all $i,j > N$. Therefore (u_i) is a Cauchy sequence. Since H is complete, we know that $u_i \to u$ converges.
By continuity follows:

$$\|u - w\| = \lim_{i\to\infty} \|u_i - w\| = d$$

Uniqueness follows from the fact, that *every* minimizing sequence converges:
Let $u,\tilde{u}$ be both minimizers, then the sequence $(u,\tilde{u},u,\tilde{u},\ldots)$ is a minimizing sequence. Since it converges, $u = \tilde{u}$. $\square_{3.1.1}$

3.1.2 Corollary

Let $M \subseteq H$ be a closed subspace of H. Then a $w \in H$ can be decomposed uniquely in the form

$$w = v + x$$

with $v \in M$ and $x \in M^\perp$. We write $H = M \oplus M^\perp$.

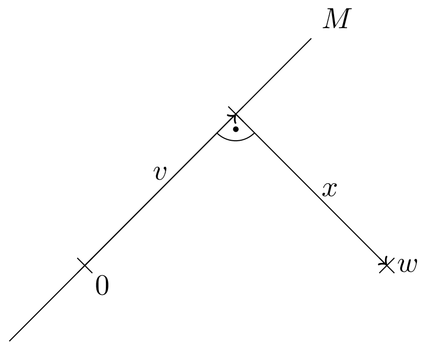


Figure 3.4: $w = v + x$

Proof

Let $v \in M$ be as in Theorem 3.1.1.

$$\|v - w\| = \inf_{u \in M} \|u - w\|$$

Define $x := w - v$.

- H real: For $u \in M$ define $\tilde{u}(\tau) = v + \tau u$ with $\tau \in \mathbb{R}$.

$$\begin{aligned} \|\tilde{u} - w\|^2 &= \|x\|^2 + 2\tau \langle u,x \rangle + \tau^2 \|u\|^2 \geq \|x\|^2 \\ 0 &\leq 2\tau \langle u,x \rangle + \tau^2 \|u\|^2 =: f(\tau) \end{aligned}$$

  $f(\tau)$ has a minimum at $\tau = 0$ and so $f'(0) = 0$.

$$\begin{aligned} & f'(0) = 2 \langle u,x \rangle \\ \Rightarrow \quad & 2 \langle u,x \rangle = 0 \quad \underset{u \in M}{\forall} \end{aligned}$$

  So $x \in M^\perp$.

- H complex: Define $\tilde{u}(\tau) = v + \tau u$, $\tau = re^{\mathbf{i}\varphi} \in \mathbb{K}$ with $r \geq 0$.

$$\|\tilde{u} - w\|^2 = \|x\|^2 + 2\mathrm{Re}\left(re^{-\mathbf{i}\varphi} \langle u,x \rangle\right) + r^2 \|u\|^2 =: f(r,\varphi)$$

  This has a minimum at $r = 0$.

$$\begin{aligned} \Rightarrow \quad & 0 = \partial_r f(0,\varphi) = 2\mathrm{Re}\left(e^{-\mathbf{i}\varphi} \langle u,x \rangle\right) \\ \overset{\varphi \text{ arbitrary}}{\Rightarrow} \quad & \langle u,x \rangle = 0 \end{aligned}$$

  So $x \in M^\perp$.

Uniqueness: Assume that $w = v_1 + x_1 = v_2 + x_2$ where $v_1,v_2 \in M$, $x_1,x_2 \in M^\perp$.

$$\underbrace{v_1 - v_2}_{\in M} = \underbrace{x_2 - x_1}_{\in M^\perp} \in M \cap M^\perp = \{0\}$$

Because from $u \in M \cap M^\perp$ follows $\langle u,u \rangle = 0$ and so $u = 0$. $\square_{3.1.2}$

For a Banach space E we have E,E^*,E^{**} and a natural injection $\iota : E \hookrightarrow E^{**}$.
For a Hilbert space H, suppose $u \in H$ and define:

$$\varphi : H \to \mathbb{K}$$
$$\varphi(v) := \langle u,v \rangle$$

φ is continuous, because:

$$|\varphi(v)| = |\langle u,v \rangle| \leq \|u\| \cdot \|v\| \leq C \|v\|$$

Now

$$\iota : H \hookrightarrow H^*$$
$$\iota(u) = \varphi$$

is a linear mapping, which is injective.

3.1.3 Theorem (Fréchet-Riesz)

For any $\varphi \in H^*$ there is a unique $v \in H$ such that for all $x \in H$:

$$\varphi(x) = \langle v,x \rangle$$

In other words: $\iota : H \to H^*$ is a Banach space isomorphism.

Proof

Let $\varphi \in H^*$, without loss of generality $\varphi \neq 0$.

$$M := \ker \varphi \subseteq H$$

is a subspace. It is closed by continuity: For $u_n \in \ker \varphi$ with $u_n \to u$ holds:

$$\varphi(u) \overset{\text{continuity}}{=} \lim_{n \to \infty} \varphi(u_n) = 0$$

So $u \in \ker \varphi$.

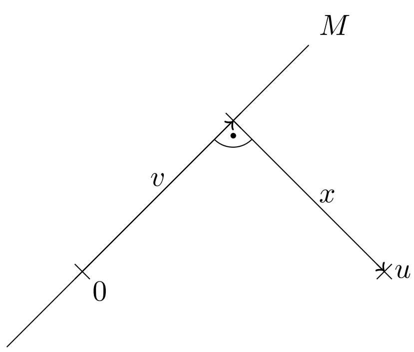


Figure 3.5: $u = v + x$

- $M^\perp$ is a one-dimensional subspace of H:
  $M^\perp \neq \{0\}$:
  Since $\varphi \neq 0$ there exists a $u \in H$ with $\varphi(u) \neq 0$, thus $u \notin M$.
  Now decompose $u = v + x$, $v \in M$, $x \in M^\perp \setminus \{0\}$.
  $M^\perp$ is one-dimensional: Take $u,v \in M^\perp$, $u,v \neq 0$, then $\varphi(u) \neq 0$ and $\varphi(v) \neq 0$.

$$\varphi(\varphi(v)\,u - \varphi(u)\,v) = 0$$

  So $\varphi(v)\,u - \varphi(u)\,v \in M \cap M^\perp = \{0\}$. Thus $\varphi(v)\,u - \varphi(u)\,v = 0$, implying that u and v are linearly dependent.

- Choose $u \in M^\perp$ with $\varphi(u) = 1$, which is always possible by rescaling.

$$\begin{aligned} v &:= \frac{u}{\|u\|^2} \\ \Rightarrow \quad \varphi(v) &= \frac{1}{\|u\|^2}\underbrace{\varphi(u)}_{=1} = \frac{1}{\|u\|^2} \\ \langle v,v\rangle &= \frac{\langle u,u\rangle}{\|u\|^4} = \frac{1}{\|u\|^2} = \varphi(v) \end{aligned}$$

- This v has the desired properties:
  For $x \in H$ decompose:

$$x = \underbrace{m}_{\in M} + \underbrace{\alpha v}_{\in M^\perp = \langle v\rangle}$$

$$\begin{aligned} \Rightarrow \quad \varphi(x) &= \underbrace{\varphi(m)}_{=0} + \alpha\varphi(v) = \alpha\,\langle v,v\rangle = \\ &= \langle v,\alpha v\rangle = \langle v,m+\alpha v\rangle = \langle v,x\rangle \end{aligned}$$

$\square_{3.1.3}$

3.1.4 Theorem (Lax-Milgram)

Let H be a Hilbert space and $B : H \times H \to \mathbb{K}$ be a mapping with the following properties:

i) $B(x,y)$ is linear in the second an anti-linear in the first argument.

ii) $|B(x,y)| \leq C\,\|x\| \cdot \|y\|$ (continuity)

iii) B is symmetric ($\overline{B(x,y)} = B(y,x)$) and positive definite, i.e. $B(x,x) \geq b\,\|x\|^2$ with $b \in \mathbb{R}_{>0}$.

iii') $|B(x,x)| \geq b\,\|x\|^2$ with $b \in \mathbb{R}_{>0}$.

Then every $l \in H^*$ can be represented uniquely as:

$$l(y) = B(x,y) \qquad \underset{y \in H}{\forall}$$

Proof

First the easy case iii):
We introduce a new scalar product $\langle .,. \rangle_B$ by:

$$\langle x,y \rangle_B := B(x,y)$$

Using ii) and iii) one sees that $\|.\|_B$ is equivalent to $\|.\|$, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C}\|x\| \leq \|x\|_B \leq C\|x\|$$

According to the Fréchet-Riesz theorem, there exists a unique $v \in H$ with

$$\varphi(x) = \langle v,x \rangle_B = B(v,x)$$

for all $x \in H$.

More difficult case iii'): Given $x \in H$,

$$B(x,.) : H \to \mathbb{K}$$

is a linear bounded functional according to i) and ii), i.e. $B(x,.) \in H^*$.
According to the Fréchet-Riesz theorem there exists a unique $z \in H$ such that $B(x,y) = \langle z,y \rangle$ for all $y \in H$. This yields a mapping:

$$\begin{aligned}\varphi : H &\to H \\ x &\mapsto z\end{aligned}$$

$$B(x,y) = \langle \varphi(x), y \rangle$$

- φ is linear, because both B and $\langle .,. \rangle$ are anti-linear in their first arguments.
- $\varphi(H) \subseteq H$ is closed:

$$\begin{aligned} b\|x\|^2 &\overset{\text{iii')}}{\leq} |B(x,x)| = |\langle z,x \rangle| \leq \|z\| \cdot \|x\| \\ b\|x\| &\leq \|z\| \end{aligned} \tag{3.1}$$

  Let $z_n \in \varphi(H)$ be a sequence with $z_n \to z \in H$. Choose x_n such that $\varphi(x_n) = z_n$, i.e. $B(x_n,y) = \langle z_n,y \rangle$ for all $y \in H$.
  Due to the anti-linearity in the first argument follows that:

$$B(x_n - x_m, y) = \langle z_n - z_m, y \rangle$$

  (3.1) yields that $\|x_n - x_m\| \leq \|z_n - z_m\|$.
  Hence (x_n) is a Cauchy sequence and so $x_n \to x \in H$ converges. Since B is continuous according to ii), we get:

$$\underbrace{B(x_n,y)}_{\to B(x,y)} = \underbrace{\langle z_n,y \rangle}_{\to \langle z,y \rangle}$$

  This gives:

$$\begin{aligned} B(x,y) &= \langle z,y \rangle \\ \varphi(x) &= z \end{aligned}$$

  Thus z is in $\varphi(H)$.

– $\varphi(H) = H$: Otherwise there would be a vector $y \in \varphi(H)^{\perp} \setminus \{0\}$ and thus for all $x \in H$ holds.

$$B(x,y) = \langle \varphi(x), y \rangle = 0$$

In particular for $x = y$ this gives:

$$\begin{aligned} & 0 = |B(y,y)| \geq b\,\|y\|^2 \\ \Rightarrow \quad & y = 0 \end{aligned}$$

This is a contradiction and so $\varphi(H) = H$.

– φ is injective: Suppose there are $x, x' \in H$ with $\varphi(x) = \varphi(x')$. Then follows:

$$B\left(x - x', y\right) = \left\langle \underbrace{\varphi(x) - \varphi\left(x'\right)}_{=0}, y \right\rangle = 0$$

Choose $y = x - x'$ so we get:

$$B\left(x - x', x - x'\right) = 0$$

Since B is positive definite, it follows $x = x'$.

– Let $l \in H^*$. According to Fréchet-Riesz there exists a unique $z \in H$ with $l(y) = \langle z, y \rangle$ for all $y \in H$ and we have

$$\langle z, y \rangle = B(x,y)$$

for $x = \varphi^{-1}(z)$. So $l(y) = B(x,y)$.

$\square_{3.1.4}$

3.1.5 Corollary

Every Hilbert space is reflexive.

Proof

Recall $\iota : H \hookrightarrow H^{**}$. H is *reflexive* if and only if ι is surjective, i.e. a Banach space isomorphism.

$$\begin{aligned} \tilde{\iota} : H \to H^* \\ (\tilde{\iota}(u))(v) = \langle u, v \rangle \end{aligned}$$

is bijective by Fréchet-Riesz. This holds also for $\bar{\iota} : H^* \to H^{**}$.

$$H \xrightarrow{\tilde{\iota}} H^* \xrightarrow{\bar{\iota}} H^{**}$$

So $\iota = \bar{\iota} \circ \tilde{\iota}$ is bijective as composition of bijective maps. $\square_{3.1.5}$

3.2 Orthonormal Bases in Separable Hilbert Spaces

3.2.1 Example

$$\ell_2 = \left\{ (a_n)_{n\in\mathbb{N}} \,\middle|\, \sum_{n\in\mathbb{N}} |a_n|^2 < \infty \right\}$$

with the scalar product

$$\langle (a_n), (b_n) \rangle := \sum_n \overline{a}_n b_n$$

is a Hilbert space.

Idea: Let H be an abstract Hilbert space. Choose an "orthonormal basis" (e_i).

$$H \ni u = \sum_{i=1}^{\infty} \lambda_i e_i$$
$$v = \sum_{i=1}^{\infty} \nu_i e_i$$

$$\langle u,v \rangle = \sum_{i,j=1}^{\infty} \langle \lambda_i e_i, \nu_j e_j \rangle = \sum_{i,j=1}^{\infty} \overline{\lambda_i} \nu_j \delta_{ij} = \sum_i \overline{\lambda_i} \nu_i$$

3.2.2 Definition (orthonormal system, Hilbert space basis, cardinality)

A system $(e_i)_{i\in J}$ is an *orthonormal system*, if $\langle e_i, e_j \rangle = \delta_{ij}$. The algebraic span is the vector space of *finite* linear combinations:

$$\langle (e_i) \rangle = \left\{ \sum_{i=1}^{N} \lambda_i e_i \middle| N \in \mathbb{N}, \lambda_i \in \mathbb{K} \right\}$$

This is a subspace of H. Now the subspace $\overline{\langle (e_i) \rangle} \subseteq H$ is called *Hilbert space span* (Hilbertraumerzeugnis).

An orthonormal system (e_i) is called a *orthonormal Hilbert space basis* if $\overline{\langle (e_i) \rangle} = H$.

Two sets A and B have the same cardinality if there exists an bijective map $\varphi : A \to B$.

Theorem (Bernstein-Schröder)

A and B have the same cardinality if and only if there exists an injective map from A to B and an injective map from $B \to A$.

(no proof)

A typical application of the Lax-Milgram theorem is for $x \in \mathbb{R}^n$, given real-valued functions $V(x)$, $f(x)$ and looking for $u(x)$ that solves:

$$-\Delta u(x) + V(x) u(x) = f(x)$$

Question: Is there a solution which "decays at infinity"?

1. Weak formulation:
   Suppose we have a solution $u \in \mathcal{C}^2(\mathbb{R}^n)$

   $$-\Delta u + Vu - f = 0$$

   Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a test function.

   $$0 = \int_{\mathbb{R}^n} (-\Delta u + Vu - f)\eta \mathrm{d}^n x \overset{\text{integration}}{\underset{\text{by parts}}{=}} \underbrace{\int_{\mathbb{R}^n} (\langle \nabla u, \nabla \eta \rangle + Vu\eta) \mathrm{d}^n x}_{=:B(u,\eta)} - \underbrace{\int_{\mathbb{R}^n} f\eta \mathrm{d}^n x}_{=l(\eta)}$$

   So for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ holds:

   $$B(u,\eta) = l(\eta)$$

   Definition: u is a *weak solution* of the equation $-\Delta u + Vu = f$ if for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ holds:

   $$B(u,\eta) = l(\eta)$$

2. Choose the correct Hilbert space. The first idea is $L^2(\mathbb{R}^n)$ with the scalar product:

   $$\langle u,v \rangle = \int_{\mathbb{R}^n} uv \mathrm{d}^n x$$

   $$u_n(x) := e^{-|x|^2} \sin(nx_1)$$

   Then for all $n \in \mathbb{N}$ holds:

   $$\|u_n\|_{L^2} \leq C$$

   But $B(u_n,u_n) \xrightarrow{n\to\infty} \infty$ diverges. Thus B is *not* continuous.
   Better choose instead:

   $$\langle u,v \rangle = \int_{\mathbb{R}^n} (uv + \langle \nabla u, \nabla v \rangle) \mathrm{d}^n x$$

   The corresponding Hilbert space $H^{1,2}(\mathbb{R}^n)$ is a Sobolev space.

   $$L^2(\mathbb{R}^3) \supseteq H^{1,2}(\mathbb{R}^3) \ni u$$

   Assume for simplicity that $0 < \varepsilon \leq V \leq C < \infty$, then we get:

   $$B(u,u) = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + Vu^2\right) \mathrm{d}^n x \leq \int_{\mathbb{R}^n} \left(|\nabla u|^2 + Cu^2\right) \mathrm{d}^n x \leq (1 + C) \|u\|_{H^{1,2}}^2$$

   $$|B(u,u)| \geq \int \left(|\nabla u|^2 + \varepsilon u^2\right) \geq \min\{1,\varepsilon\} \|u\|_{H^{1,2}}^2$$

   Thus the Lax-Milgram theorem applies and yields a unique weak solution and then a regularity theorem says that u is smooth.

Consider a matrix equation

$$Au = f$$

with $A \in \operatorname{Symm}(\mathbb{R}^n)$ and $f \in \mathbb{R}^n$.
For a general existence and uniqueness result one needs that A is invertible or equivalently:

$$\underset{u \in \mathbb{R}^n \setminus \{0\}}{\forall} : \; Au \neq 0$$

This follows from the condition:

$$\underset{u \in \mathbb{R}^n \setminus \{0\}}{\forall} : \; \underbrace{\langle u, Au \rangle}_{=B(u,u)} \neq 0$$

In finite dimension this is equivalent to:

$$\underset{u \in \mathbb{R}^n}{\forall} : \; |B(u,u)| > b \|u\|^2$$

$(e_i)_{i \in I}$ is an orthonormal Hilbert space basis of H if

$$\langle e_i, e_j \rangle = \delta_{ij}$$

and:

$$\overline{\langle e_i \rangle} = H$$

3.2.3 Theorem

Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal system. Then the mapping

$$\begin{aligned} \ell_2 &\to \overline{\langle e_i \rangle} \overset{\text{closed}}{\subseteq} H \\ (\lambda_i) &\mapsto \sum_{i \in \mathbb{N}} \lambda_i e_i \end{aligned}$$

is a Hilbert space isomorphism.

Proof

The mapping is well-defined and isometric:
For $(\lambda_i) \in \ell_2$, i.e. $\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$ we construct:

$$u_N := \sum_{i=1}^{N} \lambda_i e_i \in H$$

Without loss of generality take $M < N$, then follows:

$$\|u_N - u_M\|^2 = \left\| \sum_{i=M}^{N} \lambda_i e_i \right\|^2 = \left\langle \sum_{i=M}^{N} \lambda_i e_i, \sum_{i=M}^{N} \lambda_i e_i \right\rangle = \sum_{i,j=M}^{N} \overline{\lambda_i} \lambda_j \underbrace{\langle e_i, e_j \rangle}_{=\delta_{ij}} = \sum_{i=M}^{N} |\lambda_i|^2$$

Thus u_N is a Cauchy sequence and converges since $\overline{\langle e_i \rangle}$ is complete as a closed subset of a complete space.

$$u := \lim_{N\to\infty} u_N = \sum_{i=1}^{N} \lambda_i e_i$$

$$\|u\|^2 = \lim_{N\to\infty} \|u_N\|^2 = \lim_{N\to\infty} \sum_{i=1}^{N} |\lambda_i|^2 = \|(\lambda_i)\|_{\ell_2}$$

The mapping is also surjective:
Let $u \in \overline{\langle e_i \rangle}$ and $\varepsilon > 0$. So there exists a $v = \sum_{i=1}^{N} \lambda_i e_i \in \langle e_i \rangle$ with $\|v - u\| < \varepsilon$.
In other words there exists a finite $J \subseteq \mathbb{N}$ such that $d\left(\left\langle (e_i)_{i\in J} \right\rangle, u\right) < \varepsilon$. The vector which minimizes this distance is the orthogonal projection of u on $\left\langle (e_i)_{i\in J} \right\rangle$ since this is a finite-dimensional subspace, which is automatically closed.

$$u_J = \sum_{i\in J} e_i \langle e_i, u \rangle$$

Choose an increasing sequence $J_1 \subsetneq J_2 \subsetneq \ldots$ of finite sets such that:

$$\|u_{J_k} - u\| \to 0 \qquad\qquad \Rightarrow \quad u_{J_k} \to u$$

Thus u_{J_k} is bounded by a $C \in \mathbb{R}_{>0}$.

$$u_{J_k} = \sum_{i\in J_k} e_i \underbrace{\langle e_i, u \rangle}_{=\lambda_i}$$
$$C > \|u_{J_k}\| = \sum_{i\in J_k} |\lambda_i|^2$$

This gives:

$$\sum_{i\in\mathbb{N}} |\lambda_i|^2 < \infty$$

And so we get:

$$u = \sum_{i\in\mathbb{N}} \lambda_i e_i$$

$\square_{3.2.3}$

3.2.4 Theorem (Existence of Hilbert space basis)

In every Hilbert space H exists an orthonormal Hilbert space basis.

Proof

Consider $(u_i)_{i\in I}$ with $I = H$ and $u_h = h$ for all $h \in H$. $(u_i)_{i\in I}$ is obviously a generating system of H. On the set

$$X := \left\{ \tilde{I} \subseteq I \middle| \, (u_i)_{i\in\tilde{I}} \text{ is an orthonormal system} \right\}$$

defines „$\subseteq$“ a partial ordering.
Let $U \subseteq X$ be a totally ordered subset and define:

$$I_U := \bigcup_{\tilde{I} \in U} \tilde{I} \subseteq I$$

I_U is an upper bound of U in X if $I_U \in X$. Assume $(u_i)_{i \in I_U}$ would not be orthonormal. Then there would exist $j,k \in I_U$ with $\langle u_j,u_k \rangle \neq \delta_{jk}$.
For $j = k$ would hold $\langle u_j,u_j \rangle \neq 1$, but j lies in $\tilde{I} \in U \subseteq X$ and therefor has to hold$\langle u_j,u_j \rangle = 1$.
For $j \neq k$ we would get $\langle u_j,u_k \rangle \neq 0$. But j lies in $\tilde{I}_j \in U$ and k in $\tilde{I}_k \subseteq U$ and U is totally ordered, i.e. either holds $\tilde{I}_j \subseteq \tilde{I}_k$ or $\tilde{I}_k \subseteq \tilde{I}_j$.
Without loss of generality assume $\tilde{I}_j \subseteq \tilde{I}_k$ (otherwise exchange j and k). Then $j,k \in \tilde{I}_k \in U \subseteq X$ and hence $(u_i)_{i \in \tilde{I}_j}$ is an orthonormal system in contradiction to $\langle u_j,u_k \rangle \neq 0$. Therefore holds $I_U \in X$ and thus I_U is an upper bound of U.
Using Zorn's lemma we get a maximal element $I_{\max}$ in X. Because $(u_i)_{i \in I_{\max}}$ is an orthonormal system and thus especially linearly independent, it suffices to show that this is an generating system of H.
Assume there exists a $i_0 \in I$ with $u_{i_0} \notin K := \overline{\langle (u_i)_{i \in I_{\max}} \rangle_{\text{alg.}}}$. Since $K \subseteq H$ is closed and convex, there is an unique projection v of u_{i_0} on K and thus $h := u_{i_0} - v \in K^{\perp}$. It holds $h = u_h$ with $h \in H = I$.
Because $I_{\max}$ is maximal, holds then $I_{\max} \cup \{h\} \notin X$ and hence there is a $j \in I_{\max}$ with$\langle h,u_j \rangle \neq 0$, because $h = j$ cannot hold due to $h \notin I_{\max}$. This is a contradiction to $h \in K^{\perp}$ and thus holds $K = H$.
Therefore $(u_i)_{i \in I_{\max}}$ is an orthonormal Hilbert space basis of H. $\square_{3.2.4}$

3.2.5 Theorem

Let H be a Hilbert space.

i) For any $v \in H$ and for any orthonormal system $\{e_j \big| j \in J\}$, the set of elements $j \in J$ for which $\langle e_j,v \rangle = 0$ is finite or countable.

ii) Any two Hilbert space bases of H have the same cardinality (Mächtigkeit).

Proof

i) Consider $v \in J$. First we show that every $n \in \mathbb{N}$, the set $J_n := \left\{ j \in J \big| \langle e_j,v \rangle > \frac{1}{n} \right\}$ is finite. Indeed, by Bessel's inequality, for every finite number of elements $e_{j_1}, \ldots, e_{j_N}$ of the given orthonormal system, we have:

$$\sum_{k=1}^{N} |\langle e_{j_k},v \rangle|^2 \leq \|v\|^2$$

Now suppose that for some $n \in \mathbb{N}$, the set J_n were not finite. Then for any $N \in \mathbb{N}$ we could find elements $e_{j_1}, \ldots, e_{j_N}$ such that $\langle e_{j_k},v \rangle > \frac{1}{n}$ for all $k \in \{1, \ldots, N\}$. Hence, for these elements holds:

$$\sum_{k=1}^{N} |\langle e_{j_k},v \rangle|^2 > N \cdot \frac{1}{n}$$

Clearly these becomes larger than $\|v\|$ if we make N sufficiently large. Hence all the sets J_n must be finite. But then, we see that the set

$$\left\{ j \in J \middle| \langle e_j,v \rangle \neq 0 \right\} = \bigcup_{n \in \mathbb{N}} J_n$$

is a countable union of finite sets, and as such can be at most countable. $\square_{\text{i)}}$

ii) If H has is finite-dimensional, every Hilbert basis is a Hamel basis of H and thus the claim follows from linear algebra.
If H is infinite-dimensional, let $(e_i)_{i\in I}$ and $(b_j)_{j\in J}$ be two Hilbert bases of H. (I and J have infinitely many elements.)
For $x \in H = \overline{\langle (e_i)_{i\in I} \rangle} = \overline{\left\langle (b_j)_{j\in J} \right\rangle}$ define:

$$B_x := \left\{ j \in J \middle| \langle x,b_j \rangle \neq 0 \right\}$$

By i), the set B_x is at most countable for any $x \in H$. Next, let $j \in J$ be given. Since $\overline{\langle (e_i)_{i\in I} \rangle} = H$, we must have $\langle b_j,e_i \rangle \neq 0$ for some $i \in I$. Otherwise, $b_j \in \overline{\langle (e_i)_{i\in I} \rangle}^{\perp} = \{0\}$, which is not possible since $b_j \neq 0$. Therefore, we have $j \in B_{e_i}$ for some $i \in I$, and since $j \in J$ was arbitrary, it follows that $J \subseteq \bigcup_{i\in I} B_{e_i} \subseteq I \times \mathbb{N}$. Here the second inclusion uses that all the sets B_{e_j} are at most countable. It follows:

$$|J| \leq |I| \cdot |\mathbb{N}| = |I|$$

If we exchange the roles of I and J above, we also obtain $|I| \leq |J|$. By the Schröder-Bernstein theorem, we can combine both estimates to obtain that $|I| = |J|$. $\square_{\text{ii)}}$

$\square_{3.2.5}$

3.2.6 Theorem

If H is separable, then there exists a countable orthonormal Hilbert space basis $(e_i)_{i\in\mathbb{N}}$.
Thus H is Hilbert space isomorphic to ℓ_2.

Proof

Since H is separable, there is a countable dense subset $(x_i)_{i\in\mathbb{N}}$.

1. Arrange that the x_i are linearly independent:
   Start with $n = 1$ and $k = 1$ set:

   $$y_1 = x_1$$

   If the $y_1, \ldots ,y_{n-1},x_k$ are linearly independent, we set $y_n = x_k$ and increase n and k by one.
   If the $y_1, \ldots ,y_{n-1},x_k$ are linearly dependent, we only increase k by one.
   Then the y_i are linearly independent and $\langle (y_i) \rangle = \langle (x_i) \rangle$.

2. Gram-Schmidt procedure for orthonormalization:

   $$e_1 := y_1$$

$$e_2 := \frac{y_2 - e_1 \langle u_1, y_2 \rangle}{\|y_2 - e_1 \langle u_1, y_2 \rangle\|}$$
$$e_n := \frac{y_n - \Pr_{\langle e_1, \ldots, e_{n-1} \rangle} y_n}{\left\| y_n - \Pr_{\langle e_1, \ldots, e_{n-1} \rangle} y_n \right\|}$$

Since the y_i are linearly independent, $y_n - \Pr_{\langle e_1, \ldots, e_{n-1} \rangle} y_n$ is never zero.
Then by construction the e_i are orthonormal and $\langle e_i \rangle = \langle x_i \rangle \subseteq H$ is dense and so $(e_i)_{i\in\mathbb{N}}$ is a Hilbert space basis. $\square_{3.2.6}$

3.3 Weak Compactness of the Closed Unit Ball

For a Banach space E *weak convergence* for $(u_i)_{i\in\mathbb{N}}$ with $u_i \in E$ means:

$$u_n \rightharpoonup u \qquad \Leftrightarrow \qquad \underset{\varphi \in E^*}{\forall} : \varphi(u_n) \to \varphi(u)$$

In Hilbert spaces, we can identify H^* with H via the Fréchet-Riesz theorem.

3.3.1 Definition (weak (sequential) compactness)

$x_n \rightharpoonup x$ *converges weakly* if $\langle y, x_n \rangle \to \langle y, x \rangle$ converges for all $y \in H$.

Weak compactness is for us by definition the same as *weak sequential compactness* (schwache Folgenkompaktheit):
$K \subseteq H$ is *weakly compact* if every sequence (x_n) with $x_n \in K$ has a weakly convergent subsequence.

3.3.2 Proposition

Let H be *separable* and infinite-dimensional and let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal Hilbert space basis.

Then $e_n \rightharpoonup 0$ converges weakly.

Proof

Take $y \in H$ and expand it in the basis:

$$y = \sum_{i=1}^{\infty} y_i e_i$$
$$y_i = \langle e_i, y \rangle$$

We know $(y_i)_{i\in\mathbb{N}} \in \ell_2$ and in particular $y_i \xrightarrow{i\to\infty} 0$, since the elements of an absolutely convergent series converge to zero. Therefore holds:

$$\langle y, e_n \rangle = \overline{y_n} \xrightarrow{n\to\infty} 0$$

Thus $e_n \rightharpoonup 0$ converges weakly. $\square_{3.3.2}$

3.3.3 Theorem (Weak Compactness of the Closed Unit Ball)

If H is *separable*, then the closed unit ball $\overline{B_1(0)} = \left\{u \middle| \|u\| \leq 1\right\}$ is weakly compact.

Proof

Let (u_l) be a sequence with $u_l \in \overline{B_1(0)}$. Choose an orthonormal Hilbert space basis $(e_n)_{n\in\mathbb{N}}$.

$$u_l = \sum_{n=1}^{\infty} u_{ln} e_n \qquad u_{ln} = \langle e_n, u_l \rangle \qquad (u_{l,n})_{n\in\mathbb{N}} \in \ell_2$$

$$|u_{ln}| = |\langle e_n, u_l \rangle| \leq \underbrace{\|e_n\|}_{=1} \cdot \|u_l\| \leq 1$$

For $n = 1$: $(u_{l,1})_{l\in\mathbb{N}}$ is a bounded sequence of complex or real numbers. Therefore there exists a convergent subsequence of u_l, which we denote by $u_l^{(1)} \in H$. Then follows:

$$u_{l,1}^{(1)} = \left\langle e_1, u_l^{(1)} \right\rangle \xrightarrow{l\to\infty} v_1$$

For $n = 2$: Next we choose a subsequence $u_l^{(2)}$ of $u_l^{(1)}$ such that:

$$\left\langle e_2, u_l^{(2)} \right\rangle \xrightarrow{l\to\infty} v_2$$

Proceed inductively to obtain:

$$\left\langle e_n, u_l^{(n)} \right\rangle \to v_n$$

Then $w_l = u_l^{(l)} \in \overline{B_1(0)}$ for a sequence (w_l) in $\overline{B_1(0)}$.

Claim: $w_l \overset{l\to\infty}{\rightharpoonup} v := \sum_n v_n e_n$

Proof: We proceed as follows:

$$v_n = \lim_{l\to\infty} \left\langle e_n, u_l^{(n)} \right\rangle = \lim_{l\to\infty} \left\langle e_n, u_l^{(l)} \right\rangle = \lim_{l\to\infty} \langle e_n, w_l \rangle$$

This is because $u_l^{(l)} = u_{l'}^{(n)}$ for $l' \geq l$.

1. $(v_n) \in \ell_2$:

$$\sum_{n=1}^{N} |v_n|^2 = \sum_{n=1}^{N} \left| \lim_{l\to\infty} \langle e_n, w_l \rangle \right|^2 \overset{\text{finite sum}}{=} \lim_{l\to\infty} \underbrace{\sum_{n=1}^{N} |\langle e_n, w_l \rangle|^2}_{\underset{\text{inequality}}{\overset{\text{Bessel's}}{\leq}} \|w_l\|^2 \leq 1}$$

So we get for all $N \in \mathbb{N}$:

$$\sum_{n=1}^{N} |v_n|^2 \leq 1$$

And thus $(v_n) \in \ell_2$ and $v := \sum_{n=1}^{\infty} v_n e_n$ is well-defined and has $\|v\| \leq 1$.

2. $w_l \rightharpoonup v$, i.e. $\langle y,w_l - v\rangle \xrightarrow{l\to\infty} 0$ for all $y \in H$:

$$y = \sum_{n=1}^{\infty} y_n e_n$$

$$\begin{aligned} y_n &= \langle e_n,y\rangle \\ y_< &:= \sum_{n\le N} y_n e_n \\ y_> &:= \sum_{n> N} y_n e_n \\ \|y\|^2 &= \|y_<\|^2 + \|y_>\|^2 \end{aligned}$$

$$\langle y,w_l - v\rangle = \sum_{n=1}^{\infty} y_n \langle e_n,w_l - v\rangle$$

Choose $N \in \mathbb{N}$ so large that

$$\|y_>\| = \left(\sum_{n>N} |y_n|^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{4}$$

to get:

$$\begin{aligned} &|\langle y,w_l - v\rangle| \le |\langle y_<,w_l - v\rangle| + |\langle y_>,w_l - v\rangle| \le \\ &\qquad \le \sum_{n=1}^{N} |y_n| \, |\langle e_n,w_l - v\rangle| + \underbrace{\|y_>\|}_{<\frac{\varepsilon}{4}} \cdot \underbrace{\|w_l - v\|}_{\le 2} < \sum_{n=1}^{N} |y_n| \, |\langle e_n,w_l - v\rangle| + \frac{\varepsilon}{2} \end{aligned}$$

We know $|\langle e_n,w_l - v\rangle| \xrightarrow{l\to\infty} 0$ for each n. So we can choose $|\langle e_n,w_l - v\rangle| \le \frac{\varepsilon}{2}$ for $n \le N$ and for all $l > L(\varepsilon)$ for a sufficiently large $L(\varepsilon)$ and therefore:

$$|\langle y,w_l - v\rangle| \le \varepsilon \qquad \underset{l>L(\varepsilon)}{\forall}$$

Therefore $\langle y,w_l\rangle \to \langle y,v\rangle$ converges, which means $w_l \rightharpoonup v$. $\square_{\text{Claim}}$

$\square_{3.3.3}$

The corresponding statement in Banach spaces is the *Banach-Alaoglu theorem*:

Banach proved it in 1932 for separable Banach spaces using diagonal sequences.

Alaoglu proved it in 1938 for any Banach space. The proof is based on Tychonov's theorem.

We have E, E^*, E^{**} and an injection $\iota : E \to E^{**}$.

Theorem (Banach-Alaoglu)

The closed unit ball in E^* is *weak-$*$-sequentially compact.*

I.e. in simple terms:

If $\varphi_n \in \overline{B_1(0)} \subseteq E^*$, then there exists a subsequence φ_{n_l} such that $\varphi_{n_l}(u)$ converges for all $u \in E$.

Application: Consider

$$E = C^0(\mathbb{R}^n)$$

with the sup-norm:

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|$$

$$E^* = \{\text{regular Borel measures}\}$$

Suppose μ_n is a sequence of measures with $\|\mu_n\| \leq C$ for all $n \in \mathbb{N}$. Then there exists a measure μ such that $\mu_{n_l} \to \mu$ converges as a measure.

4 Operators on Hilbert spaces

Let H be a Hilbert space.

$$L(H) := L(H,H)$$

is the Banach space of bounded linear operators. (An linear map on an infinite dimensional space is usually called *linear operator*.) For $A \in L(H)$ define the norm:

$$|||A||| := \sup_{\|u\|=1} \|Au\|$$

4.0.1 Example

$H = L^2(\mathbb{R},\mathrm{d}x)$ with the Lebesgue measure $\mathrm{d}x$.

$$\langle f,g \rangle = \int_{\mathbb{R}} \overline{f} g \mathrm{d}x$$

$$A := \frac{\mathrm{d}}{\mathrm{d}x}$$

We would like to introduce this as an operator on H.
The inequality $\|Au\| \leq C\|u\|$ is violated even for $u \in C_0^\infty(\mathbb{R})$ for any constant $C \in \mathbb{R}$.
Namely consider

$$u_n(x) = \eta(x)\sin(nx)$$

with $\eta \in C_0^\infty(\mathbb{R})$ and $\eta\big|_{[-1,1]} = 1$. Then $\|u_n\| < \infty$ and $\|Au_n\| \xrightarrow{n\to\infty} \infty$.

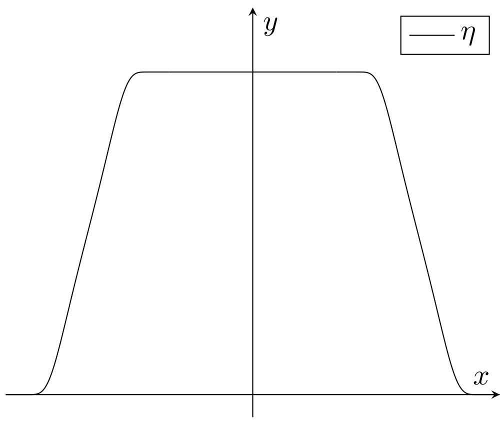


Figure 4.1: $\eta \in C_0^\infty(\mathbb{R})$ with $\eta\big|_{[-1,1]} = 1$

Moreover $\frac{\mathrm{d}}{\mathrm{d}x} f$ makes no sense for every vector f in H, because f does not need to be differentiable.

Way out: Define A only on a suitable subspace $\mathcal{D}(A)$ of H, called *domain* of definition.
For example: Choose $\mathcal{D}(A) = C_0^\infty(\mathbb{R}) \subseteq H$ and:

$$A : \mathcal{D}(A) \xrightarrow{\text{linear}} H$$

$\mathcal{D}(A)$ is dense in H, i.e. $\overline{\mathcal{D}(A)} = H$.

4.0.2 Definition (linear operator, domain, bounded)

i) Let $\mathcal{D} \subseteq H$ be a dense subspace. A linear map $A : \mathcal{D} \to H$ is called a *linear operator* on H with domain (of definition) $\mathcal{D}$.

ii) A is called *bounded*, if there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in \mathcal{D}$ holds:

$$\|Au\| \leq C \|u\|$$

Otherwise A is called unbounded.

4.0.3 Lemma

If A is a bounded operator with dense domain $\mathcal{D} \subseteq H$, then it can be extended by continuity to a unique operator $A \in L(H)$.

Proof

Let $u \in H$, not necessarily in $\mathcal{D}$. Since $\overline{\mathcal{D}} = H$, there is a sequence (u_l) in $\mathcal{D}$ with $u_l \to u$.

$$\|Au_i - Au_j\| = \|A(u_i - u_j)\| \leq C \cdot \|u_i - u_j\| \xrightarrow{i,j\to\infty} 0$$

Therefore we can set:

$$Au := \lim_{l\to\infty} Au_l$$

Since Au_l converges for any sequence $u_l \to u$, this is well-defined.

$$\|Au\| \leftarrow \|Au_i\| \leq C \|u_i\| \to C \|u\|$$

So there exists a C such that $\|Au\| \leq C \|u\|$ for all $u \in H$ and therefore $A \in L(H)$. $\square_{4.0.3}$

4.1 Isometric and unitary operators

4.1.1 Definition (isometric operator)

A operator $V : \mathcal{D}(V) \to H$ with dense domain $\mathcal{D}(V) \subseteq H$ is called *isometric* if for all $u \in \mathcal{D}(V)$ holds:

$$\langle Vu, Vu \rangle = \langle u, u \rangle$$

This operator is bounded, because:

$$\|Vu\| = \sqrt{\langle Vu,Vu\rangle} = \sqrt{\langle u,u\rangle} = \|u\| \overset{C:=1}{\leq} C\,\|u\|$$

Therefore we can extend it by continuity to H and

$$V : H \to H$$

is again isometric.

The "Hilbert hotel"

Consider $H = \ell_2$ and $(a_i) = (a_1,a_2,\ldots) \in \ell_2$.

$$A\,(u_1,u_2,\ldots) := (0,u_1,u_2,\ldots)$$

A is isometric, but it is no bijection.

Suppose you have a hotel with an infinite number of rooms and an infinite number of guest, in every room one guest.
If a new guest arrives, just move the guest from room n to room $n+1$ and the first room gets unoccupied, so the new guest can use it.

4.1.2 Proposition

For an isometric operator V the subspace $V\,(H) \subseteq H$ is closed.

Proof

Consider $y \in \overline{V\,(H)}$ and show $y \in V\,(H)$:
There exists a (y_n) with $y_n \in V\,(H)$ and $y_n \to y$ and a (x_n) with $V\,(x_n) = y_n$. Then holds:

$$\|x_i - x_j\| \overset{V \text{ isometric}}{=} \|V\,(x_i - x_j)\| = \|y_i - y_j\| \xrightarrow{i,j\to\infty} 0$$

Thus $x_i \to x$ converges. By continuity we get:

$$V\,(x) = \lim_{i\to\infty} V\,(x_i) = \lim_{i\to\infty} y_i = y$$

$\square_{4.1.2}$

4.1.3 Definition (unitary operator)

If $V : H \to H$ is an isometric operator and $V\,(H) = H$, then V is called *unitary* (unitär).

4.2 The Closure of an Operator

Let E and F be Banach spaces and $A : \mathcal{D}(A) \subseteq E \to F$ be a densely defined linear operator.

$$\begin{aligned} &\operatorname{graph}(A) := \left\{(u,Au) \,\middle|\, u \in \mathcal{D}(A)\right\} \subseteq E \times F \\ &\overline{\operatorname{graph}(A)} \subseteq E \times F \end{aligned}$$

Try to realize this as the graph of a new operator $\overline{A}$.

$$\mathcal{D}\left(\overline{A}\right) := \operatorname{pr}_1\left(\overline{\operatorname{graph} A}\right) = \left\{u \,\middle|\, \underset{v\in F}{\exists} : (u,v) \in \overline{\operatorname{graph} A}\right\}$$

For $u \in \mathcal{D}\left(\overline{A}\right)$ and $(u,v) \in \overline{\operatorname{graph} A}$ define:

$$\overline{A}u := v$$

v exists by definition of $\mathcal{D}\left(\overline{A}\right)$. Is v unique?
Suppose $(u,v) \in \overline{\operatorname{graph} A}$. Then there exists a sequence $(u_n,v_n) \in \operatorname{graph}(A)$, with $(u_n,v_n) \to (u,v)$. Equivalently:

$$\underset{n\in\mathbb{N}}{\forall} \; \underset{u_n\in\mathcal{D}(A)}{\exists} : (u_n \to u) \wedge (Au_n \to v)$$

Then we set $\overline{A}u := v$.

Problem: There might be two different series (u_n) and $(\tilde{u}_n)$ with $u_n \to u$, $\tilde{u}_n \to u$, $Au_n \to v$ and $A\tilde{u}_n \to \tilde{v} \neq v$.

4.2.1 Definition (closable operator)

A densely defined operator A is called closable (abschließbar) if $\overline{\operatorname{graph} A}$ is the graph of an operator B.
B is called the *closure* of A, symbolically $B = \overline{A}$.

4.2.2 Definition (closed)

A is called *closed* if $\operatorname{graph} A$ is a closed subset of $E \times F$.

4.2.3 Theorem (closed graph theorem)

Reformulation of 2.4.9:

If $\mathcal{D}(A) = E$, then A is closed if and only if A is bounded.

4.2.4 Example

Consider $E = C^0([0,1])$ with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$.

$$\mathcal{D}(A) = C^1([0,1]) \subseteq E$$

$$\begin{aligned} A : \mathcal{D}(A) &\to E \\ f &\mapsto f' \end{aligned}$$

A is a densely defined, unbounded operator. Is A closed?

Consider $(u,v) \in \overline{\text{graph} A}$, i.e. there exists a sequence $(u_n) \subseteq \mathcal{D}(A)$ with $u_n \to u$ and $Au_n \to v$.
$u_n \to u$ means uniform convergence of $u_n \rightrightarrows u$, so u is continuous as a uniform limit of continuous functions.
$Au_n \to u$ means uniform convergence of $Au_n \rightrightarrows v$, so v is also continuous.
It follows that $u \in C^1$ and $u' = v$.
So $(u,v) \in \text{graph} A$ and therefore A is closed.

Consider $F := C^1([0,1])$ with $\|u\| = \sup_{[0,1]} |u| + \sup_{[0,1]} |u'|$. This is a Banach space.

Remark

The closure of a closable operator is always closed.
This is obvious, because $\text{graph} \overline{A} \overset{\text{def.}}{=} \overline{\text{graph} A}$, which is closed.

4.2.5 Theorem (Criterion for closable)

A is closable if and only if:

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \to 0) \wedge (Au_n \to v) \quad \Rightarrow \quad v = 0$$

Proof

"$\Rightarrow$": Suppose A is closable. Thus there is an operator $\overline{A}$ such that $\text{graph} \overline{A} = \overline{\text{graph} A}$.
Suppose that $u_n \in \mathcal{D}(A)$, $u_n \to 0$ and $Au_n \to v$. Then $(u_n, Au_n) \to (0,v) \in \overline{\text{graph} A} = \text{graph} \overline{A}$ and thus $v = \overline{A}(0) = 0$.

"$\Leftarrow$": Suppose that the implication

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \to 0) \wedge (Au_n \to v) \quad \Rightarrow \quad v = 0$$

holds.
Define $\mathcal{D}(\overline{A})$ by: $u_n \in \mathcal{D}(A)$ with $u_n \to u$ and $Au_n \to v$. Then for $u \in \mathcal{D}(\overline{A})$ set $\overline{A}(u) = v$.
This is well-defined: Suppose $u_n, \tilde{u}_n \to u$, $Au_n \to v$ and $A\tilde{u}_n \to \tilde{v}$. Then $u_n - \tilde{u}_n \to 0$ and $A(u_n - \tilde{u}_n) \to v - \tilde{v}$. By assumption follows $v - \tilde{v} = 0$. $\square_{4.2.5}$

4.3 The adjoint of a densely defined operator

Let $A : \mathcal{D}(A) \to H$ be a linear operator with $\overline{\mathcal{D}(A)} = H$.
In finite-dimensional linear algebra the definition of the adjoint A^* is:

$$\langle u, Av \rangle =: \langle A^* u, v \rangle \qquad \underset{u,v \in H}{\forall}$$

Here it is more complicated, since in general $\mathcal{D}(A) \neq H$.

$$M := \left\{ (u,w) \in H \times H \,\middle|\, \underset{v \in \mathcal{D}(A)}{\forall} : \langle u, Av \rangle = \langle w, v \rangle \right\}$$

Claim: M is the graph of a linear map A^*.

Proof: $M \neq \emptyset$ since $(0,0) \in M$.

- The image is unique: $u \mapsto w$ is well-defined, as from $(u,w), (u,w') \in M$ follows for all $v \in \mathcal{D}(A)$:

$$\langle w - w', v \rangle = \langle u - u, Av \rangle = 0$$

  Since $\mathcal{D}(A)$ is dense, $w - w' = 0$ follows.

- A^* is linear: For $(u,w), (u',w') \in M$ and $\lambda \in \mathbb{K}$ follows $(u + \lambda u', w + \lambda w') \in M$, which is obvious from the definition of M. $\square_{\text{Claim}}$

4.3.1 Theorem

A^* is closed.

Proof

Let $x_n \in \mathcal{D}(A^*)$ converge to $x \in H$ and $A^* x_n \to y \in H$. For $z \in \mathcal{D}(A)$ holds:

$$\langle x, Az \rangle \overset{\langle .,. \rangle \text{ continuous}}{=} \lim_{n \to \infty} \langle x_n, Az \rangle = \lim_{n \to \infty} \langle A^* x_n, z \rangle \overset{\langle .,. \rangle \text{ continuous}}{=} \langle y, z \rangle$$

This shows $x \in \mathcal{D}(A^*)$ and $A^* x = y$, so A^* is closed. $\square_{4.3.1}$

4.3.2 Theorem

A^* is the maximal, i.e. not extensible, operator S with the property that for all $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(S)$:

$$\langle Au, v \rangle = \langle u, Sv \rangle$$

Proof

$$\begin{aligned}
\operatorname{graph}(S) &= \left\{ (v,w) \in \mathcal{D}(S) \times H \,\middle|\, Sv = w \right\} = \\
&= \left\{ (v,w) \in \mathcal{D}(S) \times H \,\middle|\, \underset{u \in \mathcal{D}(A)}{\forall} \langle Au, v \rangle = \langle u, w \rangle \right\} = \\
&= \left\{ (v,w) \in H \times H \,\middle|\, \underset{u \in \mathcal{D}(A)}{\forall} \langle v, Au \rangle = \langle w, u \rangle \right\} = \operatorname{graph}(A^*)
\end{aligned}$$

$\square_{4.3.2}$

4.4 Symmetric and self-adjoint densely defined operators

4.4.1 Definition (symmetric, (essentially) self-adjoint)

i) A is *symmetric* $:\Leftrightarrow \forall_{u,v\in\mathcal{D}(A)} : \langle Au,v\rangle = \langle u,Av\rangle$

ii) A is *self-adjoint* $:\Leftrightarrow A^* = A$ (in particular, $\mathcal{D}(A^*) = \mathcal{D}(A)$)

iii) A is *essentially self-adjoint* $:\Leftrightarrow \overline{A}$ is self-adjoint

For bounded A with $\mathcal{D}(A) = H$ all these notions coincide.

4.4.2 Example

Consider the operator $A := \Delta = \sum_{i=1}^{n} \partial_i^2$ on $L^2(\Omega)$ for a bounded open region $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{D}(A) = C_0^\infty(\Omega) \overset{\text{dense}}{\subseteq} L^2(\Omega)$.

- A is symmetric:

$$\langle Af,g\rangle \overset{\text{integration by parts}}{=} \langle f,Ag\rangle$$

- Adjoint of Δ on L^2:

$$\int \mathrm{d}^n r\,(\Delta f)\cdot g = \int \mathrm{d}^n r f \cdot \underbrace{h}_{\in L^2}$$

  Here $h := A^* g$. It is sufficient to consider $g \in H^{2,2}(\Omega)$ (Sobolev space). $\mathcal{D}(A^*) \supsetneq \mathcal{D}(A)$

4.4.3 Lemma

Let A be a symmetric operator. Then A is closable and $\overline{A}$ and A^* are extensions of A and $\mathcal{D}(A) \overset{\text{i)}}{\subseteq} \mathcal{D}\left(\overline{A}\right) \overset{\text{ii)}}{\subseteq} \mathcal{D}(A^*)$.

Proof

Let $u_n \in \mathcal{D}(A)$ with $u_n \to 0$ and $Au_n \to w$.

$$\langle Au,v\rangle = \langle u,Av\rangle \qquad \underset{u,v\in\mathcal{D}(A)}{\forall}$$

$$\langle w,v\rangle \leftarrow \langle Au_n,v\rangle = \langle u_n,Av\rangle \to \langle 0,Av\rangle = 0$$

Since this holds for all $v \in \mathcal{D}(A)$ now $w = 0$ follows. From the criterion 4.2.5 follows that A is closable.

i) is obvious from the definition of $\overline{A}$.

ii) Take $u \in \mathcal{D}\left(\overline{A}\right)$. Then there is a sequence $u_n \in \mathcal{D}(A)$ with $u_n \to u$ and $Au_n \to \overline{A}u$.
For all $v \in \mathcal{D}(A)$ holds:

$$\left\langle \overline{A}u,v\right\rangle \leftarrow \langle Au_n,v\rangle = \langle u_n,Av\rangle \to \langle u,Av\rangle$$

So $u \in \mathcal{D}(A^*)$ and $A^*u = \overline{A}u$.

$\square_{4.4.3}$

„The smaller one chooses $\mathcal{D}(A)$, the larger becomes $\mathcal{D}(A^*)$.“

$$B \subseteq \mathcal{D}(A) \quad \Rightarrow \quad \mathcal{D}\left(\left(A\big|_B\right)^*\right) \supseteq \mathcal{D}(A^*)$$

Difficulty: Construct $\mathcal{D}(A)$ such that $\mathcal{D}(A) = \mathcal{D}(A^*)$. (More on this later in the lecture.)

4.5 Heisenberg's uncertainty principle

In quantum mechanics:
The Hilbert space for one dimensional problems is usually $H = L^2(\mathbb{R})$.
The position operator is $x =: B$ and the momentum operator is $\frac{\hbar}{\mathbf{i}}\frac{\mathrm{d}}{\mathrm{d}x} =: A$.

$$[A,B] := AB - BA = \frac{\hbar}{\mathbf{i}}\mathbb{1}$$

4.5.1 Theorem (Winter-Wieland)

For two continuous operators A and B with $[A,B] = c \cdot \mathbb{1}$ and $B^n = B$ for all $n \in \mathbb{N}_{\geq 1}$, i.e. B is idempotent, follows $c = 0$.

Proof

Consider:

$$B^k A B^{n-k} = B^k (AB) B^{n-k-1} = B^k (BA + c\mathbb{1}) B^{n-k-1} = B^{k+1} A B^{n-k-1} + cB^{n-1}$$

$$\Rightarrow \quad cB^{n-1} = B^k A B^{n-k} - B^{k+1} A B^{n-k-1}$$

Sum this from $k = 0$ to $k = n-1$:

$$ncB^{n-1} = \sum_{k=0}^{n-1} B^k A B^{n-k} - B^{k+1} A B^{n-k-1} \overset{\text{telescope}}{\underset{\text{sum}}{=}} AB^n - B^n A$$

$$n\,|c|\, \left\|\left\|B^{n-1}\right\|\right\| = \left\|\left\|AB^n - B^n A\right\|\right\| \overset{\Delta\text{-inequality}}{\leq} \|AB^n\| + \|B^n A\| \leq (\|AB\| + \|BA\|) \cdot \left\|B^{n-1}\right\|$$

Since this must hold for all n either $c = 0$ or there exists a $n \in \mathbb{N}_{>1}$ with $\left\|B^{n-1}\right\| = 0$, i.e. $B^{n-1} = 0$. Since B is idempotent follows $B = 0$ and therefore $[A,B] = 0$ and also $c = 0$. $\square_{4.5.1}$

Consider $u \in \mathcal{D}(A)$ with $\|u\| = 1$, which represents a quantum mechanical state.
The expectation value of A in u is after the probabilistic interpretation:

$$E_u(A) := \langle u,Au \rangle$$

The "uncertainty", i.e. the variance, is:

$$\Delta_u(A) := \|(A - E_u(A)\,\mathbb{1})\,u\|$$

4.5.2 Theorem (Heisenberg's uncertainty principle)

Let H be a $\mathbb{C}$-Hilbert space and $A : \mathcal{D}(A) \to H$, $B : \mathcal{D}(B) \to H$ be two symmetric operators with $\overline{\mathcal{D}(A)} = H = \overline{\mathcal{D}(B)}$. Assume for the image domains $\mathcal{R}$:

$$\mathcal{R}(A) \subseteq \mathcal{D}(B) \qquad\qquad \mathcal{R}(B) \subseteq \mathcal{D}(A)$$

So $[A,B]$ is well-defined on $\mathcal{D}(A) \cap \mathcal{D}(B)$.
Assume furthermore that $[A,B] = \frac{\hbar}{\mathbf{i}}\mathbb{1}$ with $\hbar > 0$.
Then for all $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\|u\| = 1$ holds:

$$\Delta_u(A) \cdot \Delta_u(B) \geq \frac{\hbar}{2}$$

Proof

Replace A by $\tilde{A} := A - E_u(A) \cdot \mathbb{1}$ and $\tilde{B} := B - E_u(B) \cdot \mathbb{1}$. Then holds:

$$\left[\tilde{A},\tilde{B}\right] = \frac{\hbar}{\mathbf{i}}\mathbb{1}$$

$$\Delta_u(A) = \left\|\tilde{A}u\right\|$$

$$\Delta_u(B) = \left\|\tilde{B}u\right\|$$

We have to show:

$$\Delta_u(A) \cdot \Delta_u(B) = \left\|\tilde{A}u\right\| \cdot \left\|\tilde{B}u\right\| \geq \frac{\hbar}{2}$$

$$\begin{aligned}\frac{\hbar}{2} &= \frac{\hbar}{2}\langle u,u\rangle = \frac{\mathbf{i}}{2}\left\langle u, \left(\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\right)u\right\rangle \overset{\text{symmetry}}{=} \frac{\mathbf{i}}{2}\left(\left\langle \tilde{A}u,\tilde{B}u\right\rangle - \left\langle \tilde{B}u,\tilde{A}u\right\rangle\right) = \\ &= -\mathrm{Im}\left(\left\langle \tilde{A}u,\tilde{B}u\right\rangle\right) \overset{\text{Cauchy-Schwarz}}{\leq} \left\|\tilde{A}u\right\| \cdot \left\|\tilde{B}u\right\|\end{aligned}$$

$\square_{4.5.2}$

4.6 Spectrum and resolvent

Let $A : \mathcal{D}(A) \to H$ be a closed, densely defined operator.

4.6.1 Definition (continuously invertible, resolvent, spectrum)

A is *continuously invertible* if and only if $A : \mathcal{D}(A) \to H$ is bijective and $A^{-1} : H \to \mathcal{D}(A)$ is continuous.

$$\varrho(A) := \left\{\lambda \in \mathbb{K}\middle| (\lambda\mathbb{1} - A) \text{ is continously invertible}\right\}$$

The *resolvent* (Resolvente) is defined for $\lambda \in \varrho(A)$ as

$$\mathcal{R}_\lambda(A) = (\lambda\mathbb{1} - A)^{-1} \in L(H)$$

and the *spectrum* of A as:

$$\sigma(A) = \mathbb{K} \setminus \varrho(A)$$

4.6.2 Lemma

$\varrho(A)$ is open and $\sigma(A)$ is closed.

Proof

For bounded operators cf. Theorem 2.5.3.

It's method works even for unbounded operators:

Take $\lambda,\mu \in \varrho(A)$.

$$
\begin{aligned}
(A-\mu) &= (A-\lambda) + (\lambda-\mu) = \\
&= \underbrace{(A-\lambda)}_{\text{continuously invertible}} \cdot \Big(\mathbb{1} + (A-\lambda)^{-1}(\lambda-\mu)\Big)
\end{aligned}
$$

$\mathbb{1} + (A-\lambda)^{-1}(\lambda-\mu)$ is continuously invertible using the Neumann series if:

$$|\lambda-\mu| < \frac{1}{\left|\!\left|\!\left|(A-\lambda)^{-1}\right|\!\right|\!\right|}$$

So $\varrho(A)$ is open and therefore the complement $\sigma(A)$ is closed. $\square_{4.6.2}$

4.6.3 Theorem (resolvent equation)

The map $\lambda \mapsto \mathcal{R}_\lambda(A)$ is complex analytic on $\varrho(A)$.

We have the *resolvent equation* (Resolventengleichung):

$$\mathcal{R}_\lambda - \mathcal{R}_\mu = -(\lambda-\mu)\,\mathcal{R}_\lambda \cdot \mathcal{R}_\mu$$

Proof

Analogy with $\mathbb{C}$-numbers:

$$
\begin{aligned}
\frac{1}{\lambda-x} - \frac{1}{\mu-x} &= \frac{\mu-\lambda}{(\lambda-x)(\mu-x)} \\
(\mu-x)-(\lambda-x) &= \mu-\lambda
\end{aligned}
$$

Same thing for operators:

$$
\begin{aligned}
(\mu-A)-(\lambda-A) &= \mu-\lambda \\
\mathcal{R}_\mu^{-1} - \mathcal{R}_\lambda^{-1} &= \mu-\lambda \qquad /\mathcal{R}_\mu\cdot \quad /\cdot\mathcal{R}_\lambda \\
\mathcal{R}_\lambda - \mathcal{R}_\mu &= (\mu-\lambda)\,\mathcal{R}_\mu\mathcal{R}_\lambda \\
\mathcal{R}_\lambda &= \mathcal{R}_\mu + (\mu-\lambda)\,\mathcal{R}_\mu\mathcal{R}_\lambda
\end{aligned}
$$

Assume $|\mu-\lambda| < \frac{1}{|||\mathcal{R}_\lambda|||}$.

$$\mathcal{R}_\mu = \mathcal{R}_\lambda\left(1+(\mu-\lambda)\,\mathcal{R}_\lambda\right)^{-1} = \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-1)^n (\mu-\lambda)^n\,\mathcal{R}_\lambda$$

This series converges absolutely and so the map is analytic in $L(H)$. $\square_{4.6.3}$

5 Compact Operators

Let E and F be Banach spaces and $A \in L(E,F)$.

Remember: There exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\| \leq C \|u\|$$

A maps bounded sets in E to bounded sets in F.

But: Bounded sets are not precompact in general.

5.1 Definition (compact operator)

A is called *compact* operator if and only if A maps bounded sets to relatively compact sets, i.e. the closure is compact.
(In complete spaces relatively compact is equivalent to precompact.)

5.2 Example (integral operator)

Let $E = \left(C^0([0,1]), \|.\|_\infty\right)$ and consider an integral kernel $K \in C^0([0,1] \times [0,1])$, $K : E \to E$.

$$(K\varphi)(x) := \int_0^1 K(x,y)\,\varphi(y)\,\mathrm{d}y$$

$$\begin{aligned} |(K\varphi)(x)| &\leq \sup_y |K(x,y)|\, \|\varphi\| \qquad / \sup_x \\ \|K\varphi\| &\leq C\|\varphi\| \end{aligned}$$

So $K \in L(E)$. Furthermore the integral kernel K is continuous and defined on a compact set. Therefore K is uniformly continuous after the Heine-Cantor theorem.

$$\underset{\varepsilon\in\mathbb{R}_{>0}}{\forall}\ \underset{\delta\in\mathbb{R}_{>0}}{\exists} : \left|K(x,y) - K\left(x',y\right)\right| < \varepsilon \qquad \underset{|x-x'|<\delta,\ y\in[0,1]}{\forall}$$

$$\left|(K\varphi)(x) - (K\varphi)\left(x'\right)\right| = \left|\int_0^1 \left(K(x,y) - K\left(x',y\right)\right)\varphi(y)\,\mathrm{d}y\right| \leq \varepsilon \|\varphi\|_\infty$$

Let now $B := B_M(0)$ with $M \in \mathbb{R}_{>0}$. Then $K(B) \subseteq E$.

- uniformly bounded ($\|\varphi\| < CM$)
- uniformly continuous

The Arzelà-Ascoli theorem yields, that $K(B)$ is precompact and so K is a compact operator.

5.3 Theorem

Let H be a Hilbert space.
A compact operator $A : H \to H$ maps weakly convergent sequences to convergent sequences.

Proof

Let $x_n \rightharpoonup x$, then (x_n) is bounded, i.e. there is a $C \in \mathbb{R}_{>0}$ such that $\|x_n\| < C$ for all $n \in \mathbb{N}$. Define $y_n := Ax_n$. For all $z \in H$ holds:

$$\langle z, y_n - y \rangle = \langle z, A(x_n - x) \rangle = \langle A^* z, x_n - x \rangle \to 0$$

Therefore $y_n \rightharpoonup y$ converges weakly. Because A is compact, every subsequence of y_n contains a convergent subsequence with limes $\tilde{y}$. For $z = \tilde{y} - y$ converges:

$$0 \leftarrow \langle z, y_n - y \rangle \to \langle \tilde{y} - y, \tilde{y} - y \rangle = \|\tilde{y} - y\|$$

Therefore $\tilde{y} = y$.
Since this holds for every subsequence of y_n follows $y_n \to y$. $\square_{5.3}$

5.4 Lemma

Consider operators $A, B : E \to F$.

i) If A and B are compact, so are $A + B$ and λA for all $\lambda \in \mathbb{K}$.

ii) If $A : E \to F$ is compact (continuous) and $B : F \to E$ continuous (compact), than $B \circ A$ is compact.
(In particular A^n is compact for $A : E \to E$.)

iii) The compact operators form a closed subspace of $L(E, F)$.

Proof

i) is obvious. $\square_{\text{i)}}$

ii) follows, since a continuous operator is bounded. $\square_{\text{ii)}}$

iii) Let (x_n) be bounded and T_k a convergent sequence of compact operators. By diagonal choice get a subsequence, also written x_n, such that $T_k x_n$ converges for all $k \in \mathbb{N}$.

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq \underbrace{\|Tx_n - T_k x_n\|}_{\leq \|T - T_k\| \cdot \|x_n\|} + \|T_k x_n - T_k x_m\| + \underbrace{\|T_k x_m - Tx_m\|}_{\leq \|T - T_k\| \cdot \|x_m\|} \leq \\ &\leq \|T - T_k\| \cdot \|x_n\| + \|T_k x_n - T_k x_m\| + \|T - T_k\| \cdot \|x_m\| \xrightarrow{n,m,k \to \infty} 0 \end{aligned}$$

$\square_{5.4}$

5.5 Lemma (Fredholm operator)

Let $A : E \to E$ be compact and define $T := \mathbb{1} - A$. T is called *Fredholm operator*.

i) $\ker(T)$ is finite-dimensional.

ii) There exists a $i \in \mathbb{N}$ such that $\ker\left(T^k\right) = \ker\left(T^i\right)$ for all $k \in \mathbb{N}_{>i}$.

iii) The image of T is closed.

Proof

i) $\ker(T) =: Z = \left\{u \middle| u = Au\right\}$. Since $Z \cap B_1(0)$ is bounded

$$A\left(Z \cap B_1(0)\right) = Z \cap B_1(0)$$

is precompact and therefore Z is finite-dimensional. $\square_{\text{i)}}$

ii) Define $N_i := \ker\left(T^i\right)$, which are closed subspaces of E, since the T^i are continuous. Suppose the claim is wrong, then $N_j \subsetneq N_{j+1} \subsetneq \ldots$, so in particular all N_j are proper subspaces. Choose $y_j \in N_j$ with:

$$\|y_j\| = 1 \qquad\qquad d\left(y_j, N_{j-1}\right) > \frac{1}{2}$$

This is possible after Lemma 2.1.2.
For all $m < n$ holds:

$$Ay_n - Ay_m = y_n - \underbrace{T_{y_n} - y_m + T_{y_m}}_{\in N_{n-1}}$$

Therefore follows:

$$\|Ay_n - Ay_m\| > \frac{1}{2}$$

So (Ay_n) has no accumulation value in contradiction to the compactness of A. $\square_{\text{ii)}}$

iii) Let $y_k \in \operatorname{im}(T)$ with $y_k \to y$ and $y_k = Tx_k$. We want to show $y \in \operatorname{im}(T)$. Define:

$$d_k := d\left(x_k, \ker(T)\right) = \inf_{z \in \ker(T)} \|x_k - z\|$$

Claim: (d_k) is bounded. Equivalently $(D_k) = |\max\{1, d_k\}|$ is bounded.

Proof: Choose $z_k \in \ker(T)$, $w_k := x_k - z_k$ with $\|w_k\| < 2d_k$ and $Tw_k = y_k$.
Assume D_k is unbounded. Since y_k is convergent and thus bounded, follows:

$$T\left(\frac{w_k}{D_k}\right) = \frac{y_k}{D_k} \xrightarrow{k \to \infty} 0$$

Now consider $u_k := \frac{w_k}{D_k}$. We know $\|u_k\| < 2$ and $T(u_k) \to 0$.
Thus $u_k - Au_k \to 0$. Since A is compact, every subsequence of Au_k has a convergent subsequence, and therefore $u_k \to 0$ converges.
The continuity of T gives:

$$T(u) = \lim_{k \to \infty} T(u_k) = 0$$

So $u \in \ker(T)$.
On the other hand we have for all $z \in \ker(T)$:

$$\|w_k - z\| \geq D_k$$
$$\Rightarrow \quad \left\| u_k - \frac{z}{D_k} \right\| \geq 1$$

Since T is a subspace this means, that for all $z \in \ker(T)$ holds:

$$\|u_k - z\| \geq 1$$

This is a contradiction to $u \in \ker(T)$. $\square_{\text{Claim}}$

So u_k is bounded and $T(w_k) = T(x_k) = y_k \to y$. So we get:

$$w_k - Aw_k \to y$$

Since A is compact Aw_k converges and with this follows, that $w_k \to w$ also converges.
By continuity we get:

$$T(w) = \lim_{k\to\infty} T(w_k) = y$$

So $w \in \operatorname{im}(T)$. $\square_{5.5}$

5.6 Theorem (Fredholm Alternative)

Let $A : E \to E$ be compact and define $T := \mathbb{1} - A$.
If the kernel $\ker(T) = \{0\}$ is trivial, then T is continuously invertible.

Proof

$\ker(T) = \{0\}$ means, that T is injective. We only need to show, that T is surjective, because then T is invertible and 2.4.7 yields then, that T is open and therefore T^{-1} continuous.

$\operatorname{im}(T)$ is closed following 5.5 iii).
$\operatorname{im}(T) = E$, since otherwise $T(E) \subsetneq E$. Then the injectivity implies for all $k \in \mathbb{N}$:

$$T^{k+1}(E) \subsetneq \underbrace{T^k(E)}_{=E_k}$$

E_k is closed for all $k \in \mathbb{N}$:

$$E_k = (\mathbb{1} - A)^k (E) = \left(\mathbb{1} + \underbrace{\sum_{l=1}^{k} (-1)^l \binom{k}{l} A^l}_{A:=A_k}\right)(E)$$

Now A_k is compact, as the compact operators form a (closed) ideal subalgebra $\mathrm{CP}(E)$.
Choose $x_k \in E_k$ with $\|x_k\| = 1$ and $d(x_k, E_k) > \frac{1}{2}$, which is possible after Lemma 2.1.2. Then holds for all $m < n$:

$$Ax_m - Ax_n = x_m - \underbrace{Tx_m - x_n + Tx_n}_{\in H_{m+1}}$$

$$\Rightarrow \quad \|Ax_m - Ax_n\| > \frac{1}{2}$$

This is a contradiction to the compactness of A.
Therefore T is surjective and the theorem follows. $\square_{5.6}$

5.7 Theorem (Riesz-Schauder)

Let $A \in L(H)$ be compact.

i) $\sigma(A)$ consists of a a finite or countable set of complex numbers and 0 is the only possible accumulation point.

ii) Every $0 \neq \lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity, i.e. $\ker(A-\lambda)$ is finite-dimensional. That means, there exists a $i \in \mathbb{N}$ such that for all $k > i$ holds:

$$\ker(A-\lambda)^k = \ker(A-\lambda)^i$$

One says also that the Jordan chains are finite.

Proof

ii) is an immediate consequence of the Lemmas 5.5 and 5.6. (Divide A by λ.)

i) Assume $\lambda_n \neq 0$ are pairwise different eigenvalues. Choose eigenvectors $x_n \in H$ such that:

$$Ax_n = \lambda_n x_n$$

$$Y_n := \langle x_1, \ldots, x_n \rangle$$

Since the eigenvalues are pairwise different $Y_n \subsetneq Y_{n+1}$ must hold, because the x_k are linearly independent.
Assume $Y_n \subsetneq H$, since otherwise H would be finite-dimensional and therefore $\sigma(A)$ a finite set.
So following Lemma 2.1.2 we can choose $y_n \in Y_n$ with $\|y_n\| = 1$ and:

$$d(y_n, Y_{n+1}) > \frac{1}{2}$$

Since $y_n \in Y_n$ one can find $\alpha_j \in \mathbb{K}$ such that:

$$y_n = \sum_j \alpha_j x_j$$

Then follows:

$$(A-\lambda_n)\, y_n = \sum_{i=1}^{n-1} (\lambda_j - \lambda_n)\, \alpha_j x_j =: \tilde{y}_n \in Y_{n-1}$$

For all $n > m$ holds:

$$Ay_n - Ay_m = \lambda_n y_n - \underbrace{\tilde{y}_n - Ay_m}_{\in Y_{n-1}}$$

So we get:

$$\|Ay_n - Ay_m\| \geq \frac{|\lambda_n|}{2}$$

But (Ay_n) is precompact and thus for all $\delta \in \mathbb{R}_{>0}$ exist only finitely many λ_n with $|\lambda_n| > \delta$. Therefore 0 is the only accumulation point and $\sigma(A)$ is a countable union of finite sets and thus countable. $\Box_{5.7}$

Jordan decomposition:

$$A = \begin{pmatrix} \lambda_1 & & & & & & 0 \\ 1 & \ddots & & & & & \\ & 1 & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & 1 & \ddots & & \\ & & & & 1 & \lambda_2 & \\ 0 & & & & & & \ddots \end{pmatrix}$$

$$\lambda_1 - A = \begin{pmatrix} 0 & & & & & & 0 \\ -1 & \ddots & & & & & \\ & -1 & 0 & & & & \\ & & & -\lambda_2 & & & \\ & & & -1 & \ddots & & \\ & & & & -1 & -\lambda_2 & \\ 0 & & & & & & \ddots \end{pmatrix}$$

So the first block is nilpotent. If it has k dimensions this means:

$$(\lambda_1 - A)^k = \begin{pmatrix} 0 & & 0 \\ & * & \\ 0 & & * \end{pmatrix}$$

So k is the length of the Jordan chain.

5.8 Theorem

Let $A \in L(H)$ be compact and H be a separable Hilbert space.
Then A can be approximated in $L(H)$ by operators of finite rank.

Proof

Choose a countable orthonormal Hilbert basis $(\varphi_j)_{j\in\mathbb{N}}$ of H, which is possible, since H is separable. Define:

$$\lambda_n := \sup_{\psi \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp, \|\psi\|=1} \|A\psi\|$$

Since A is bounded, this supremum exists. Obviously $\lambda_1 \geq \lambda_2 \geq \dots$. Thus $\lambda_n \searrow \lambda \geq 0$.

Claim: $\lambda = 0$

Proof: Choose $\psi_n \in \langle \varphi_1, \ldots, \varphi_n \rangle^\perp$ with $\|\psi_n\| = 1$ and $\|A\psi_n\| \geq \frac{\lambda}{2}$ which is possible after Lemma 2.1.2, since $\langle \varphi_1, \ldots, \varphi_n \rangle$ is a proper closed subspace of H. Write:

$$\psi_n = \sum_{j=1}^{\infty} \nu_j \varphi_j = (\nu_1, \nu_2, \ldots)$$

Due to $\psi_n \in \langle \varphi_1, \ldots, \varphi_n \rangle^\perp$ follows:

$$\psi_n = (0, \ldots 0, \nu_{n+1}, \nu_{n+2}, \ldots)$$

For $u \in H$ holds:

$$\langle u, \psi_n \rangle = \sum_{j=n+1}^{\infty} \nu_j \cdot \overline{u}_j \underset{\text{inequality}}{\overset{\text{Schwarz}}{\leq}} \underbrace{\left(\sum_{j=n+1}^{\infty} |\nu_j|^2 \right)^{\frac{1}{2}}}_{=\|\psi_n\|} \cdot \left(\sum_{j=n+1}^{\infty} |u_j|^2 \right)^{\frac{1}{2}} \xrightarrow{n \to \infty} 0$$

So by construction $\psi_n \rightharpoonup 0$. Therefore $A\psi_n \to 0$ and thus $\|A\lambda_n\| \to 0$.
On the other hand we have $\|A\psi_n\| \geq \frac{\lambda}{2}$ and so $\lambda = 0$. $\square_{\text{Claim}}$

Let P_n be the orthogonal projection on $\langle \varphi_1, \ldots, \varphi_n \rangle$.

$$P_n u = \sum_{j=1}^{n} \varphi_j \langle \varphi_j, u \rangle$$

AP_n is an operator of finite rank $r \leq n$, since $\operatorname{rank}(P_n) = n$.

Claim: $AP_n \xrightarrow{n \to \infty} A$ in $L(H)$.

Proof: Consider:

$$|||A - AP_n||| = \sup_{u \in H, \, \|u\| = 1} \|A(\mathbb{1} - P_n) u\|$$

$(\mathbb{1} - P_n) u \in \langle \varphi_1, \ldots, \varphi_n \rangle^\perp$ and $\|(\mathbb{1} - P_n) u\| \leq \|u\| = 1$. $(\mathbb{1} - P_n = P_{\langle \varphi_1, \ldots, \varphi_n \rangle^\perp})$
Thus we get:

$$|||A - AP_n||| \leq \sup_{v \in \langle \varphi_1, \ldots, \varphi_n \rangle^\perp, \, \|v\| \leq 1} \|Av\| = \lambda_n \xrightarrow{n \to \infty} 0$$

$\square_{\text{Claim}}$

$\square_{5.8}$

5.9 Lemma

Let $A \in L(H)$ be compact and symmetric. (This implies that A is bounded and self-adjoint.) Then $\sigma(A) \subseteq \mathbb{R}$ and if u is an eigenvector, $Au = \lambda u$, then its orthogonal is invariant under A.

Proof

For $\lambda \in \sigma(A)$ holds $\ker(A-\lambda) \neq \{0\}$. Thus there exists a $u \in \ker(\lambda - A) \setminus \{0\}$.

$$\lambda \langle u,u \rangle = \langle u,Au \rangle = \langle Au,u \rangle = \overline{\lambda} \langle u,u \rangle$$

Since $\|u\| \neq 0$ follows $\lambda = \overline{\lambda}$, which means that $\lambda \in \mathbb{R}$.
For $v \in \langle u \rangle^{\perp}$ holds:

$$\langle Av,u \rangle = \langle v,Au \rangle = \lambda \langle v,u \rangle = 0$$

Therefore $Av \in \langle u \rangle^{\perp}$. $\square_{5.9}$

5.10 Theorem (Hilbert-Schmidt)

Let $A \in L(H)$ be a symmetric compact operator on the separable Hilbert space H.
Then there exists an orthonormal Hilbert space basis of eigenvectors $(u_n)_{n\in\mathbb{N}}$, so with the eigenvalues $\lambda_n \in \mathbb{R}$ holds:

$$Au_n = \lambda_n u_n$$

Proof

$\sigma(A)$ is countable and therefore we can write $\sigma(A) \setminus \{0\} = \{\lambda_1,\lambda_2,\ldots\} \subseteq \mathbb{R}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. $\ker(\lambda_j - A)$ is finite-dimensional. So we choose a (finite) orthonormal basis of the eigenspace. Taking these eigenvectors for all eigenvalues, we obtain a countable orthonormal system $(u_n)_{n\in\mathbb{N}}$.

$$M := \overline{\langle u_n \rangle} \overset{\text{closed}}{\subseteq} H$$

$M^{\perp}$ is an invariant subspace of H under A, i.e.:

$$\tilde{A} := A\big|_{M^{\perp}} : M^{\perp} \to M^{\perp}$$

This is again symmetric and compact. We know that $\sigma\left(\tilde{A}\right) = \{0\}$.

Question: Why is $\tilde{A} = 0$?
This is not true for a general operator, e.g.:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad\qquad \sigma(A) = \{0\}$$

Answer: If A is symmetric and $\sigma(A) = \{0\}$, then one can show $A = 0$ using the following theorem 5.12:
From $\sigma\left(\tilde{A}\right) = 0$ follows $r\left(\tilde{A}\right) = 0$ and since $\tilde{A}$ is self-adjoint theorem 5.12 gives $\left\|\tilde{A}\right\| = 0$ and thus $\tilde{A} = 0$. In other words $A\big|_{M^{\perp}} = 0$.
Now choose an orthonormal Hilbert basis $(v_n)_{n\in\mathbb{N}_{\leq N}}$ of $M^{\perp}$ for an $N \in \mathbb{N}\cup\{\infty\}$. Therefore $\{u_n\} \cup \{v_n\}$ is the desired orthonormal Hilbert basis of H. $\square_{5.10}$

5.11 Definition (spectral radius)

Let $A : \mathcal{D}(A) \subset H \to H$ be a densely defined operator. Then the *spectral radius* $r(A)$ of A is defined by:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

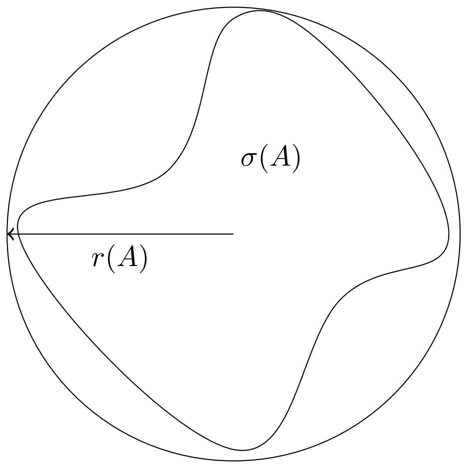


Figure 5.1: $\sigma(A) \subseteq \overline{B_{r(A)}(0)}$

5.12 Theorem

For $A \in L(H)$ holds:

$$r(A) = \limsup_{n \to \infty} |||A^n|||^{\frac{1}{n}}$$

If A is symmetric, then:

$$r(A) = |||A|||$$

Proof

Recall for a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

with a_n,$z \in \mathbb{K}$ the root test (Wurzelkriterium):

- If

$$\limsup_{n \to \infty} |a_n z^n|^{\frac{1}{n}} =: c < 1$$

then $|a_n z^n| < c^n$ and therefore is

$$\sum_{n=0}^{\infty} c^n$$

a convergent dominating sequence. Thus $\sum_{n=0}^{\infty} a_n z^n$ converges as well.

– If

$$\limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} =: c > 1$$

then $|a_n z^n| > c^n > 1$ for an infinite number of n. Therefore $a_n z^n$ does *not* converge to zero, which implies that $\sum_{n=0}^{\infty} a_n z^n$ does not converge as well.

– If

$$\limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} = 1$$

no conclusion is possible.

$$\limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} = |z| \cdot \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

The Radius of convergence (Konvergenzradius) is thus defined by:

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

If $|z| < R$ the sum converges absolutely and if $|z| > R$ the sum diverges.
In our setting for $A = 0$ is nothing to prove. For $\lambda \in \varrho(A) \setminus \{0\}$ we make a formal expansion:

$$\mathcal{R}_\lambda = (\lambda - A)^{-1} = \frac{1}{\lambda}\left(\mathbb{1} - \frac{A}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty} A^n \cdot \left(\frac{1}{\lambda}\right)^n$$

This is a power series in $\frac{1}{\lambda}$, but the coefficients are operators.

$$R := \frac{1}{\limsup_{n\to\infty} |||A^n|||^{\frac{1}{n}}}$$

For $\frac{1}{|\lambda|} < R$

$$\left|\left|\left|\sum_{n=0}^{\infty} A^n \left(\frac{1}{\lambda}\right)^n\right|\right|\right| \le \sum_{n=0}^{\infty} |||A^n||| \frac{1}{\lambda^n}$$

converges absolutely and so

$$\sum_{n=0}^{\infty} A^n \left(\frac{1}{\lambda}\right)^n$$

converges in $L(H)$. Thus the resolvent

$$\mathcal{R}_\lambda = (\lambda - A)^{-1}$$

exists and $\sigma(A) \subseteq \overline{B_{\frac{1}{R}}(0)}$, i.e.:

$$r(A) \leq \frac{1}{R} = \limsup_{n \to \infty} \left|\left|\left| A^n \right|\right|\right|^{\frac{1}{n}}$$

If $\frac{1}{|\lambda|} > R$

$$\left|\left|\left| \sum_{n=0}^{\infty} A^n \left(\frac{1}{\lambda}\right)^n \right|\right|\right|$$

diverges.

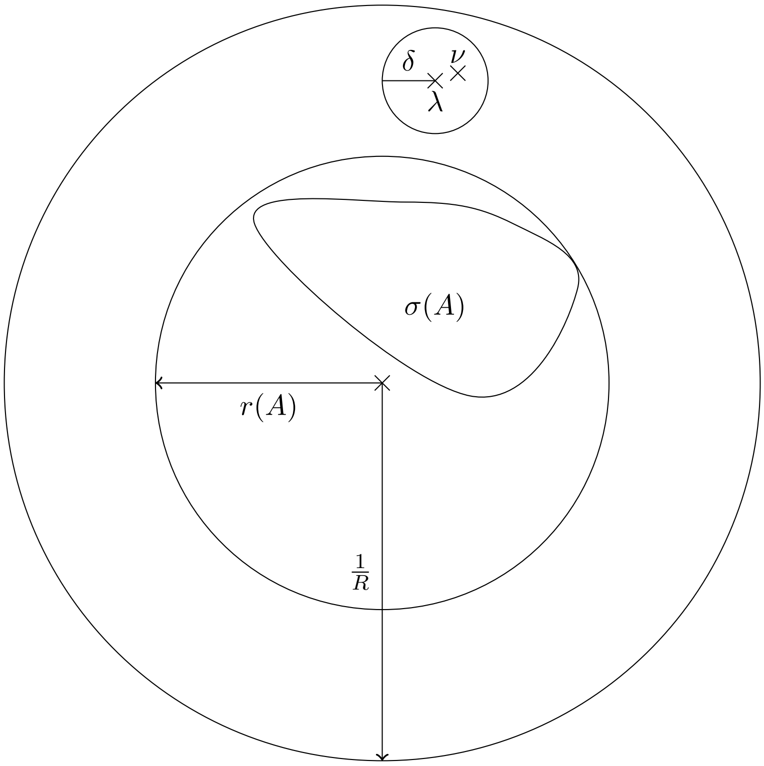


Figure 5.2: $\frac{1}{R} > r(A)$?

Why is r not smaller than $\frac{1}{R}$?
Assume that $r < \frac{1}{R}$ and choose λ with $r < |\lambda| < \frac{1}{R}$. Then $\mathcal{R}_\lambda$ exists and is analytic. Consider a $\nu \in B_\delta(\lambda)$.

$$\begin{aligned} \mathcal{R}_\nu &= (\nu - A)^{-1} = ((\nu - \lambda) + (\lambda - A))^{-1} = \\ &= (((\nu - \lambda)\mathcal{R}_\lambda + \mathbb{1})(\lambda - A))^{-1} = \\ &= \mathcal{R}_\lambda (\mathbb{1} + (\nu - \lambda)\mathcal{R}_\lambda)^{-1} = \\ &= \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-(\nu - \lambda))^n \mathcal{R}_\lambda^n \end{aligned}$$

For $|\nu - \lambda| < \delta := \frac{1}{|||\mathcal{R}_\lambda|||}$ the Neumann series converges.
Thus $\mathcal{R}_\lambda$ can be expanded locally in a power series, i.e. $\mathcal{R}_\lambda$ is complex analytic or holomorphic.

Furthermore for $|\lambda| > \frac{1}{R}$ holds:

$$\mathcal{R}_\lambda = \sum_{n=0}^{\infty} A^n \frac{1}{\lambda^{n+1}}$$

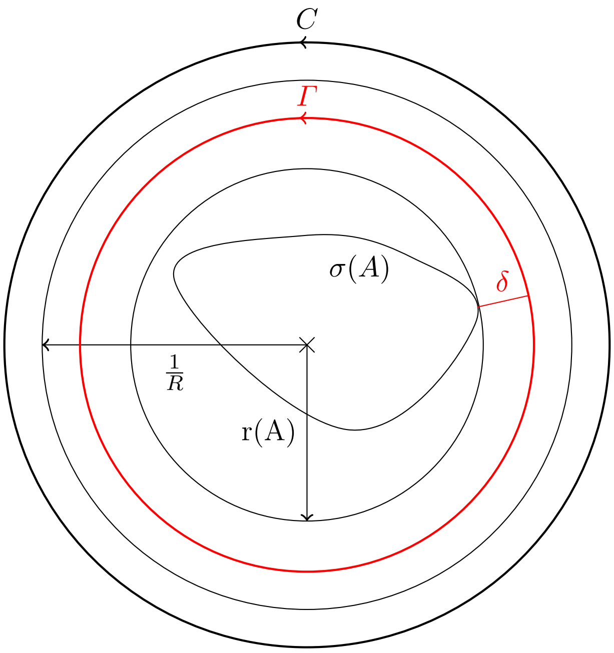


Figure 5.3: Contours $\varGamma$ and C for integration

Integrate along the contour C:

$$\frac{1}{2\pi\mathbf{i}} \oint_C \lambda^n \mathcal{R}_\lambda \mathrm{d}\lambda = \sum_{k=0}^{\infty} A^k \underbrace{\frac{1}{2\pi\mathbf{i}} \oint_C \frac{\lambda^n}{\lambda^{k+1}} \mathrm{d}\lambda}_{=:I}$$

Since the geometric series converges absolutely, the summation and the integration can be interchanged. The residue theorem gives:

$$I = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Therefore we get:

$$\frac{1}{2\pi\mathbf{i}} \oint_C \lambda^n \mathcal{R}_\lambda \mathrm{d}\lambda = A^n$$

Choose $\varGamma = \partial B_{r+\delta}(0)$. We know, that $\mathcal{R}_\lambda$ is holomorphic outside $\varGamma$. Thus we may continuously deform the contour to obtain:

$$\frac{1}{2\pi\mathbf{i}} \oint_\varGamma \lambda^n \mathcal{R}_\lambda \mathrm{d}\lambda = A^n$$

Thus we have:

$$|||A^n||| = \left|\left|\left|\frac{1}{2\pi\mathbf{i}} \oint_\varGamma \lambda^n \mathcal{R}_\lambda \mathrm{d}\lambda\right|\right|\right| \leq C\,(r+\delta)^n\,(r+\delta)$$

$$C := \frac{1}{2\pi} \sup_{\lambda \in \Gamma} |||\mathcal{R}_\lambda|||$$

$$\Rightarrow \quad |||A^n|||^{\frac{1}{n}} \leq (r+\delta) \left(C^{\frac{1}{n}} (r+\delta)^{\frac{1}{n}} \right) \xrightarrow{n\to\infty} r+\delta$$

Therefore:

$$\limsup_{n\to\infty} |||A^n|||^{\frac{1}{n}} \leq r+\delta$$

Since δ is arbitrary, it follows that:

$$\frac{1}{R} = \limsup_{n\to\infty} |||A^n|||^{\frac{1}{n}} = r$$

We even conclude:

$$|||A^n|||^{\frac{1}{n}} \xrightarrow{n\to\infty} r(A)$$

Assume that A is *symmetric* (to show $|||A^n|||^{\frac{1}{n}} = |||A|||$). The Schwarz inequality gives:

$$\left|\left|\left| A^2 \right|\right|\right| \leq |||A||| \cdot |||A||| = |||A|||^2$$

$$|||A|||^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \left\langle u, Au^2 \right\rangle \leq \sup_{\|u\|=1} \underbrace{\|u\|}_{=1} \cdot \left\| A^2 u \right\|$$

Iteratively for $n \in \mathbb{N}$:

$$\left|\left|\left| A^{2^n} \right|\right|\right| = |||A|||^{2^n}$$

For arbitrary $m \in \mathbb{N}$ the Schwarz inequality gives:

$$|||A^m||| \leq |||A|||^m$$

Choose n such that $2^n > m$. THen:

$$\begin{aligned} & |||A|||^{2^n} = \left|\left|\left| A^{2^n} \right|\right|\right| = \left|\left|\left| A^m \cdot A^{2^n - m} \right|\right|\right| \leq |||A^m||| \cdot |||A|||^{2^n - m} \\ \Rightarrow \quad & |||A|||^m \leq |||A|||^m \end{aligned}$$

$\square_{5.12}$

5.13 Ritz method

Let $A \in L(H)$ be a symmetric compact operator on the separable Hilbert space H.
From the Hilbert-Schmidt theorem 5.10 we know that there exists an orthonormal eigenvalue basis (u_n) of H.

$$Au_n = \lambda_n u_n$$

We now want to construct the u_n:

Consider the "expectation value" functional:

$$\begin{aligned} S : H &\to \mathbb{R} \\ u &\mapsto \langle u, Au \rangle \end{aligned}$$

This is well defined, since:

$$\overline{S(u)} = \overline{\langle u, Au \rangle} = \langle Au, u \rangle = \langle u, Au \rangle = S(u)$$

S is bounded, because:

$$|S(u)| = |\langle u, Au \rangle| \leq |||A||| \cdot \|u\|^2 \overset{\|u\| \leq 1}{\leq} |||A|||$$

Maximize $|S(u)|$ on $\left\{ u \in H \middle| \|u\| = 1 \right\}$:
Choose a maximizing sequence (u_n) with $\|u_n\| = 1$ and:

$$|S(u_n)| \xrightarrow{n \to \infty} \sup_{\|u\|=1} |S(u)|$$

Since $\overline{B_1(0)}$ is weakly compact, there is a subsequence u_{k_l}, which converges weakly $u_{k_l} \rightharpoonup u$.
Since A is compact, the sequence

$$v_{k_l} := Au_{k_l} \to v$$

converges and $Au = v$. As a consequence:

$$S(u_{k_l}) = \langle u_{k_l}, Au_{k_l} \rangle = \langle u_{k_l}, v_{k_l} \rangle = \underbrace{\langle u_{k_l}, v \rangle}_{\to \langle u, v \rangle} + \langle u_{k_l}, v_{k_l} - v \rangle \xrightarrow{l \to \infty} \langle u, v \rangle = \langle u, Au \rangle = S(u)$$

This follows, because:

$$|\langle u_{k_l}, v_{k_l} - v \rangle| \leq \underbrace{\|u_{k_l}\|}_{=1} \cdot \underbrace{\|v_{k_l} - v\|}_{\to 0} \xrightarrow{l \to \infty} 0$$

Thus S is weakly continuous, i.e. for any $u_k \rightharpoonup u$ converges $S(u_k) \to S(u)$.
Because (u_n) is a maximizing sequence, we get:

$$|S(u)| = \sup_{\|\tilde{u}\|=1} |S(\tilde{u})|$$

Therefore u is the desired maximizer.

- u is on the unit sphere:
  The simple approach

  $$\|u\|^2 \neq \lim_{l \to \infty} \|u_{k_l}\|^2$$

  does not work, because u_{k_l} only converges weakly.
  Example:
  If (e_l) is an orthonormal Hilbert basis in a separable Hilbert space, then $e_l \rightharpoonup 0$, but:

  $$\lim_{l \to \infty} \|e_l\| = 1 \neq 0 = \|0\|$$

But it holds:

$$\|u\|^2 = \lim_{l\to\infty} |\langle u, u_{k_l}\rangle| \leq \lim_{l\to\infty} \|u_{k_l}\| \cdot \|u\| = \|u\|$$
$$\Rightarrow \quad \|u\| \leq 1$$

Assume $\|u\| < 1$, then the vector $\hat{u} := \frac{u}{\|u\|}$ would satisfy the equation:

$$|S(\hat{u})| = |\langle \hat{u}, A\hat{u}\rangle| = \frac{1}{\|u\|^2} |\langle u, Au\rangle| = \frac{1}{\|u\|^2} \sup_{\|v\|=1} |S(v)| \overset{\|u\|<1}{>} \sup_{\|v\|=1} |S(v)|$$

This is a contradiction. Therefore u is in fact a unit vector.

– u is an eigenvector corresponding to the eigenvalue $\lambda = \langle u, Au\rangle \in \mathbb{R}$: Consider the variation for $v \in H$:

$$\tilde{u}(\tau) = u + \tau v$$

$$S\left(\frac{\tilde{u}}{\|\tilde{u}\|}\right) = \frac{\langle \tilde{u}, A\tilde{u}\rangle}{\langle \tilde{u}, \tilde{u}\rangle} = \frac{\langle u + \tau v, A(u + \tau v)\rangle}{\langle u + \tau v, u + \tau v\rangle}$$

This is called *Rayleigh quotient*. We know that $S(\tilde{u}(\tau))$ is extremal at $\tau = 0$:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau} S(\tilde{u}(\tau))\bigg|_{\tau=0} = \\
&= \frac{\langle u, Av\rangle + \langle v, Au\rangle + 2\tau\langle v, v\rangle}{\langle u + \tau v, u + \tau v\rangle} - \frac{\langle u + \tau v, A(u + \tau v)\rangle}{\langle u + \tau v, u + \tau v\rangle^2} \cdot (\langle v, u\rangle + \langle u, v\rangle + \tau\langle v, v\rangle)\bigg|_{\tau=0} = \\
&\overset{A \text{ symmetric}}{=} 2\frac{\mathrm{Re}(\langle v, Au\rangle)}{\langle u, u\rangle} - 2\mathrm{Re}(\langle v, u\rangle)\frac{\langle u, Au\rangle}{\langle u, u\rangle^2} = \\
&\underset{\langle u,u\rangle=1}{\overset{\lambda=\langle u,Au\rangle}{=}} 2(\mathrm{Re}(\langle v, Au\rangle) - \lambda\mathrm{Re}(\langle v, u\rangle)) = 2\mathrm{Re}(\langle v, (A-\lambda)u\rangle)
\end{aligned}$$

Set $v = e^{\mathbf{i}\varphi} w$ for any $\varphi \in \mathbb{R}$ and $w \in H$. So:

$$0 = \mathrm{Re}(\langle v, (A-\lambda)u\rangle) = \mathrm{Re}\left(e^{-\mathbf{i}\varphi}\langle w, (A-\lambda)u\rangle\right) \qquad \underset{\varphi\in\mathbb{R}}{\forall}$$

$$\Rightarrow \quad \langle w, (A-\lambda)u\rangle = 0 \qquad \underset{w\in H}{\forall}$$
$$(A-\lambda)u = 0$$
$$Au = \lambda u$$

– It holds $|\lambda| = |||A|||$:
There is no point ν in the spectrum of A with $|\nu| > |\lambda|$, because otherwise for all $v \in H$ with $Av = \nu v$ follows:

$$\frac{|\langle v, Av\rangle|}{\langle v, v\rangle} = |\nu| > |\lambda| = |\langle u, Au\rangle| = \sup_{w\in H} \frac{|\langle w, Aw\rangle|}{\langle w, w\rangle}$$

This is a contradiction. Thus we get:

$$|\lambda| = \sup_{\nu\in\sigma(A)} |\nu| \overset{\text{by definition}}{=} r(A) \overset{5.12}{=} |||A|||$$

Thus we have *constructed* a $u \in H$ with $\|u\| = 1$, $Au = \lambda u$ and $|\lambda| = |||A|||$. Now one can proceed inductively:

$$H_1 := \langle u \rangle^{\perp}$$

$$A\big|_{H_1} : H_1 \to H_1$$

(We saw that H_1 is invariant under A.)
Repeat the above procedure to maximize $|\langle v,Av \rangle|$ on $H_1 \cap \left\{ v \in H \big| \, \|v\| = 1 \right\}$. This gives u_1 with $\|u\|_1 = 1$, $Au_1 = \lambda_1 u_1$ and:

$$|\lambda_1| = \left|\left|\left| A\big|_{H_1} \right|\right|\right| \leq \left|\left|\left| A\big|_H \right|\right|\right| = |\lambda|$$

Now set $H_2 = \langle u,u_1 \rangle^{\perp}$ and proceed inductively.
This gives a sequence $u_0 := u,\, u_1,\, u_2,\, \dots$ of orthonormal eigenvectors, i.e. $Au_j = \lambda_j u_j$, with decreasing eigenvalues$|\lambda_j|$.
These (u_j) are an orthonormal basis. (Proof as in Theorem 5.10) $\square_{5.13}$

Ritz, Galerkin: Finite element method
Example: Helium molecule wave function in $H = L^2\left(\mathbb{R}^3,\mathbb{C}\right)$

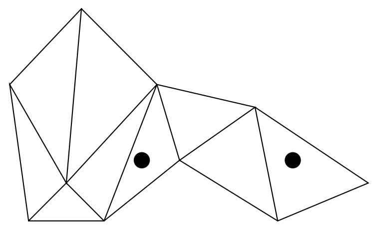


Figure 5.4: finite lattice for numerical approximation

$$A = -\frac{\hbar^2}{2m}\Delta - \frac{ze^2}{\|x - x_1\|} - \frac{ze^2}{\|x - x_2\|}$$

Now minimize

$$\frac{\langle u,Au \rangle}{\langle u,u \rangle}$$

on a finite subspace of H.

6 A few (technical) results

6.1 Dini's theorem

Let E be a metric space and $f_n : E \to \mathbb{R}$ a sequence of real valued functions.

6.1.1 Definition (point-wise/uniform convergence)

f_n *converges point-wise* to f if $f_n(x) \to f(x)$ converges for all $x \in E$, i.e.:

$$\underset{x\in E}{\forall}\ \underset{\varepsilon\in\mathbb{R}_{>0}}{\forall}\ \underset{N(\varepsilon,x)}{\exists}\ \underset{n\in\mathbb{N}_{\geq N}}{\forall} : |f_n(x) - f(x)| < \varepsilon$$

f_n *converges uniformly* to f, in symbols $f_n \rightrightarrows f$, if for all $\varepsilon \in \mathbb{R}_{>0}$ exists a $N(\varepsilon)$ such that for all $n \geq N$ and all $x \in E$ holds:

$$|f_n(x) - f(x)| < \varepsilon$$

With quantifiers this is:

$$\underset{\varepsilon\in\mathbb{R}_{>0}}{\forall}\ \underset{N(\varepsilon)}{\exists}\ \underset{n\in\mathbb{N}_{\geq N}}{\forall}\ \underset{x\in E}{\forall} : |f_n(x) - f(x)| < \varepsilon$$

6.1.2 Theorem

If (f_n) is a sequence of continuous functions with $f_n \rightrightarrows f$, then f is also continuous.
This is not true in general for point wise convergence:

$$f_n(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2}\left(1 - \frac{1}{n}\right) \\ 0 & \text{for } x \geq \frac{1}{2} \\ n(1-2x) & \text{for } \frac{1}{2}\left(1-\frac{1}{n}\right) < x < \frac{1}{2} \end{cases}$$

$f_n \to f$ converges pointwise to:

$$f(x) = \begin{cases} 1 & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{cases}$$

This f is *not* continuous.

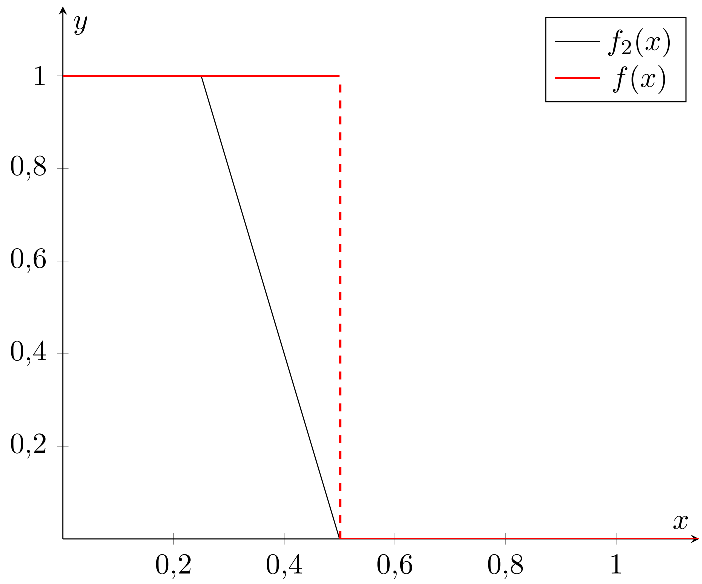


Figure 6.1: $f_n(x)$ is continuous, but not $f(x)$

Proof

Show that for all $x \in E$ the ε-δ-criterion is satisfied:
Since $f_n \rightrightarrows f$ converges uniformly, there is a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ and all $x \in E$ holds:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

Because the f_n are continuous, there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $y \in B_\delta(x)$ holds:

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$$

Now follows for all $y \in B_\delta(x)$:

$$|f(y) - f(x)| \leq \underbrace{|f(y) - f_N(y)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(y) - f_N(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f(x)|}_{<\frac{\varepsilon}{3}} < \varepsilon$$

Therefore f is continuous. $\square_{6.1.2}$

6.1.3 Definition (monotonically increasing/decreasing)

The sequence of functions (f_n), $f_n : E \to \mathbb{R}$ is called *monotonically increasing (decreasing)* if for all $x \in E$ the real sequence $f_n(x)$ is monotonically increasing (decreasing).

6.1.4 Theorem (Dini)

Let E be a *compact* metric space, (f_n) monotone and $f_n \to f$.
If f_n and f are continuous, then the convergence $f_n \rightrightarrows f$ is uniform.

Proof

Without loss of generality we assume (f_n) is a monotonically increasing sequence (otherwise consider $-f_n$), i.e. $f_n(x) \leq f_{n+1}(x)$ for all $x \in E$ and all $n \in \mathbb{N}$.
Given $\varepsilon > 0$ we want to show:

$$\underset{N\in\mathbb{N}}{\exists} \; \underset{x\in E}{\forall} \; \underset{\in\mathbb{N}_{\geq N}}{\forall} : |f(x) - f_n(x)| < \varepsilon$$

For any $x \in E$ there exists an $N(x)$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}_{\geq N}$ (point-wise convergence). Since both $f_{N(x)}$ and f are continuous functions, there exists a neighborhood $U(x) = B_{\delta(x)}(x)$ of x such that for all $z \in U(x)$ holds:

$$\left|f_{N(x)}(z) - f_{N(x)}(x)\right| \leq \frac{\varepsilon}{4}$$
$$|f(z) - f(x)| \leq \frac{\varepsilon}{4}$$

Then follows:

$$\left|f_{N(x)}(z) - f(z)\right| \leq \underbrace{\left|f_{N(x)}(z) - f_{N(x)}(x)\right|}_{\leq\frac{\varepsilon}{4}} + \underbrace{\left|f_{N(x)}(x) - f(x)\right|}_{<\frac{\varepsilon}{2}} + \underbrace{|f(x) - f(z)|}_{\leq\frac{\varepsilon}{4}} < \varepsilon$$

Since $f_n(z)$ is monotonically increasing, it follows that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in B_{\delta(x)}(x)$.
Now use a standard compactness argument: Since E is compact, it can be covered by a finite number of these balls $B_{\delta(x_1)}(x_1), \ldots, B_{\delta(x_n)}(x_n)$. Define:

$$N = \max\{N(x_1), \ldots, N(x_n)\}$$

So for all $n \in \mathbb{N}_{\geq N}$ holds:

$$|f_n(x) - f(x)| < \varepsilon$$

$\square_{6.1.4}$

6.2 Stone-Weierstraß theorem

We follow the nice (since constructive) proof by Bernstein.

6.2.1 Definition (polynomials)

Let $E = C^0([0,1])$ be the Banach space of real valued functions with norm:

$$\|f\| = \sup_{x\in[0,1]} |f(x)|$$

$\mathcal{P}([0,1])$ are the *real polynomials*, i.e. for $f \in \mathcal{P}([0,1])$ there are $a_j \in \mathbb{R}$ such that:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

Clearly $\mathcal{P}([0,1]) \subseteq C^0([0,1])$ forms a subspace.

We want to show that $\mathcal{P}([0,1])$ is dense in $C^0([0,1])$.

6.2.2 Lemma

For $x \in [0,1]$ holds:

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k} = 1$$

Proof

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k} = (x+1-x)^n = 1$$

$\square_{6.2.2}$

6.2.3 Lemma

For $x \in [0,1]$ holds:

$$\sum_{k=0}^{n} (nx-k)^2 \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$$

Obviously holds

$$(nx-k)^2 \leq 4n^2$$

and therefore:

$$\sum_{k=0}^{n} (nx-k)^2 \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k} \leq 4n^2 \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k} = 4n^2$$

Proof

It holds:

$$\begin{aligned}
\sum_{k=0}^{n} k \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k} &= \sum_{k=0}^{n} k \frac{n!}{k!\,(n-k)!} x^k (1-x)^{n-k} = \\
&= 0 + \sum_{k=1}^{n} \frac{n \cdot (n-1)!}{(k-1)!\,(n-k)!} x^k (1-x)^{n-k} = \\
&= n \sum_{k=1}^{n} \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right) x^k (1-x)^{n-k} = \\
&\overset{j:=k-1}{=} n \sum_{j=0}^{n-1} \left(\begin{array}{c} n-1 \\ j \end{array} \right) x^{j+1} (1-x)^{n-j-1} =
\end{aligned}$$

$$= nx \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{(n-1)-j} = nx (x+1-x)^{n-1} = nx$$

Similarly one gets:

$$\sum_{k=0}^{n} k(k-1) \binom{n}{k} x^k (1-x)^{n-k} = n(n-1) \sum_{k=2} \binom{n-2}{k-2} x^k (1-x)^{n-k} = n(n-1)x^2$$

Together this gives:

$$\begin{aligned} \sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^{n} \left(n^2x^2 - 2nxk + k^2\right) \binom{n}{k} x^k (1-x)^{n-k} = \\ &= \sum_{k=0}^{n} \left(n^2x^2 - 2nxk + k(k-1) + k\right) \binom{n}{k} x^k (1-x)^{n-k} = \\ &= n^2x^2 - 2nx \cdot nx + n(n-1)x^2 + nx = \\ &= -n^2x^2 + n^2x^2 - nx^2 + nx = nx(1-x) \end{aligned}$$

$\square_{6.2.3}$

A more elegant method is to use derivatives:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} &= (x+y)^n \\ \sum_{k=0}^{n} k \binom{n}{k} x^k y^{n-k} &= x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right) \\ \sum_{k=0}^{n} k^2 \binom{n}{k} x^k y^{n-k} &= \left(x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \right)^2 \left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right) \end{aligned}$$

6.2.4 Definition

For $f \in C^0([0,1])$ define:

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

6.2.5 Theorem (Bernstein)

For any $f \in C^0([0,1], \mathbb{R})$, $B_n f \rightrightarrows f$ converges uniformly.

Example: $f(x) = 10x \cdot e^{-3x} + \frac{1}{5} \cos\left((4x)^2\right)$

$$\begin{aligned} B_4 f(x) &\approx 0{,}2 \cdot (1-x)^4 + 5{,}2 \cdot x \cdot (1-x)^3 + 5{,}9 \cdot x^2 \cdot (1-x)^2 + 2{,}4 \cdot x^3 \cdot (1-x) + 0{,}3 \cdot x^4 \\ B_{10} f(x) &\approx 0{,}2 \cdot (1-x)^{10} + 9{,}4 \cdot x \cdot (1-x)^9 + 56{,}6 \cdot x^2 \cdot (1-x)^8 + 149{,}5 \cdot x^3 \cdot (1-x)^7 + \\ &\quad + 217{,}9 \cdot x^4 \cdot (1-x)^6 + 248{,}2 \cdot x^5 \cdot (1-x)^5 + 244{,}7 \cdot x^6 \cdot (1-x)^4 + \\ &\quad + 103{,}2 \cdot x^7 \cdot (1-x)^3 + 26{,}5 \cdot x^8 \cdot (1-x)^2 + 7{,}9 \cdot x^9 \cdot (1-x) + 0{,}3 \cdot x^{10} \end{aligned}$$

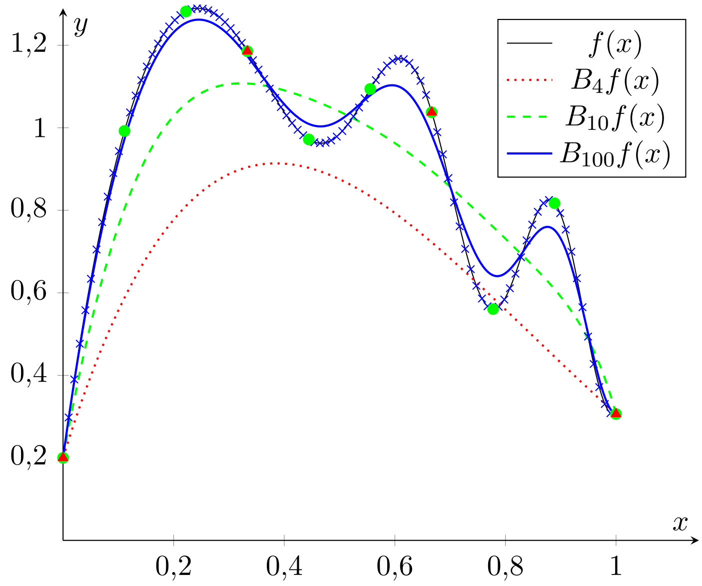


Figure 6.2: Approximation of $f(x)$ by $B_n f(x)$

Proof

Without loss of generality assume $f \neq 0$ (otherwise $B_n f = 0 = f$).

$$M := \|f\| > 0$$

Consider an arbitrary $\varepsilon \in \mathbb{R}_{>0}$. f is continuous on the compact set $[0,1]$ and thus uniformly continuous, i.e. there exists a $\delta \in \mathbb{R}_{>0}$ such that:

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2}$$

Choose $\mathbb{N} \ni N \geq \frac{M}{\varepsilon \delta^2}$.

Claim: $|B_n f(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ and all $n \geq N$.

Proof: It holds:

$$f(x) = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k}$$
$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$(B_n f - f)(x) = \sum_{k=0}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x)\right) \binom{n}{k} x^k (1-x)^{n-k}$$

Define:

$$A := \left\{ k \,\middle|\, \left|\frac{k}{n} - x\right| < \delta \right\} \qquad B := \left\{ k \,\middle|\, \left|\frac{k}{n} - x\right| \geq \delta \right\}$$

We have:

$$\sum_{k\in A}\underbrace{\left|f\left(\frac{k}{n}\right)-f(x)\right|}_{<\frac{\varepsilon}{2}}\binom{n}{k}x^k(1-x)^{n-k}<\frac{\varepsilon}{2}\sum_{k\in A}\binom{n}{k}x^k(1-x)^{n-k}\le\frac{\varepsilon}{2}$$

$$\begin{aligned}
&\sum_{k\in B}\underbrace{\left|f\left(\frac{k}{n}\right)-f(x)\right|}_{\le 2\|f\|=2M}\binom{n}{k}x^k(1-x)^{n-k}\le\\
&\le 2M\sum_{k\in B}\binom{n}{k}x^k(1-x)^{n-k}\le\\
&\underset{\Rightarrow n^2\delta^2\le(k-nx)^2}{\overset{k\in B}{\le}}\frac{2M}{n^2\delta^2}\sum_{k=0}^{n}\underbrace{(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k}}_{\le\frac{n}{4}}\le\\
&\overset{n\ge N}{\le}\frac{M}{2n\delta^2}\le\frac{M}{2\frac{M}{\varepsilon\delta^2}\delta^2}=\frac{\varepsilon}{2}
\end{aligned}$$

Therefore holds for all $x\in[0{,}1]$.

$$|B_nf(x)-f(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$\square_{\text{Claim}}$

Therefore $B_nf\rightrightarrows f$ converges uniformly. $\square_{6.2.5}$

Now generalize: Let E be a compact metric space. $C^0(E{,}\mathbb{R})$ with

$$\|f\|=\sup_{x\in E}|f(x)|$$

is a Banach space. Moreover, it is an algebra with the point-wise multiplication:

$$(f\cdot g)(x):=f(x)\cdot g(x)$$

The multiplication is continuous:

$$\|f\cdot g\|\le\|f\|\cdot\|g\|$$

In summary $\left(C^0(E{,}\mathbb{R}),\|.\|,+,\cdot\right)$ is a *Banach algebra*.

6.2.6 Theorem (Weierstraß)

The polynomials are dense in $C^0([0{,}1]{,}\mathbb{R})$.

Proof

For any $f\in C^0([0{,}1]{,}\mathbb{R})$, $B_nf\rightrightarrows f$ converges uniformly and since the B_nf are polynomials, these are dense. $\square_{6.2.6}$

6.2.7 Theorem (Stone-Weierstraß)

Let $\mathcal{A} \subseteq C^0(E,\mathbb{R})$ be a subalgebra with the following properties:

1. $\mathcal{A}$ contains $f = 1$ and so by scalar multiplication all the constant functions.
2. $\mathcal{A}$ separates the points of E, i.e. for all $x,y \in E$ with $x \neq y$ there exists a $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Then $\mathcal{A}$ is dense in $C^0(E,\mathbb{R})$.

Proof

i) There is a sequence of polynomials u_n on $[0,1]$ such that $u_n \rightrightarrows f$ with $f(t) = \sqrt{t}$. This follows immediately from theorem 6.2.6.

ii) If $f \in \mathcal{A}$, then $|f|$ defined by $|f|(x) := |f(x)|$ is in the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$:
For $f \in \mathcal{A}$ define:

$$a := \|f\| = \max_{x\in E} |f(x)|$$

$$\Rightarrow \quad \frac{f^2(x)}{a^2} \in [0,1]$$

Then converges:

$$u_n\left(\frac{f^2(x)}{a^2}\right) \xrightarrow{n\to\infty} \sqrt{\frac{f^2(x)}{a^2}} = \frac{|f(x)|}{a}$$

The functions $u_n\left(\frac{f^2}{a^2}\right)$ lie in $\mathcal{A}$, since these are a polynomials of f and thus again elements of the algebra $\mathcal{A}$. Moreover $u_n\left(\frac{f^2}{a^2}\right)$ converges uniformly to $\frac{|f|}{a}$, because for a given $\varepsilon \in \mathbb{R}_{>0}$ exists a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ and all $t \in [0,1]$ holds:

$$\left|u_n(t) - \sqrt{t}\right| < \varepsilon$$

Then follows with $t = \frac{f^2(x)}{a^2}$:

$$\left|u_n\left(\frac{f^2(x)}{a^2}\right) - \frac{|f|}{a}\right| < \varepsilon$$

Thus $\frac{|f|}{a} \in \overline{\mathcal{A}}$ and therefore also $|f| \in \overline{\mathcal{A}}$.

iii) For $f,g \in \overline{\mathcal{A}}$ also $\min(f,g)$ and $\max(f,g)$ (defined point-wise) are again in $\overline{\mathcal{A}}$:

$$\min(f,g) = \frac{1}{2}(f + g - |f - g|)$$
$$\max(f,g) = \frac{1}{2}(f + g + |f - g|)$$

Choose $f_n,g_n \in \mathcal{A}$ such that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. By ii) follows $|f_n - g_n| \in \overline{\mathcal{A}}$ and $|f_n - g_n| \rightrightarrows |f - g|$. Therefore holds:

$$\overline{\mathcal{A}} \ni \min(f_n,g_n) \rightrightarrows \min(f,g) \in \overline{\mathcal{A}}$$

Similarly the claim follows for max.

iv) For all $x,y \in E$ with $x \neq y$ and $\alpha,\beta \in \mathbb{R}$ exists a $f \in \mathcal{A}$ such that $f(x) = \alpha$ and $f(y) = \beta$:
For $\alpha = \beta$ we choose $f = \alpha$ as constant function.
For $\alpha \neq \beta$ there exists, since $\mathcal{A}$ separates points of E, a $g \in \mathcal{A}$ with $g(x) \neq g(y)$. Set $f = c_0 + c_1 g$ and choose:

$$\begin{aligned} \alpha &= c_0 + c_1 g(x) \\ \beta &= c_0 + c_1 g(y) \\ \Rightarrow \quad c_1 &= \frac{\alpha - \beta}{g(x) - g(y)} \\ \Rightarrow \quad c_0 &= \alpha - \frac{\alpha - \beta}{g(x) - g(y)} g(x) = \frac{\alpha g(x) - \alpha g(y) - \alpha g(x) + \beta g(x)}{g(x) - g(y)} = \\ &= \frac{\beta g(x) - \alpha g(y)}{g(x) - g(y)} \end{aligned}$$

v) For all $f \in C^0$, $x \in E$ and $\varepsilon \in \mathbb{R}_{>0}$ there is a $g \in \overline{\mathcal{A}}$ such that

$$g(x) = f(x)$$

and for all $y \in \overline{\mathcal{A}}$ holds:

$$g(y) \leq f(y) + \varepsilon$$

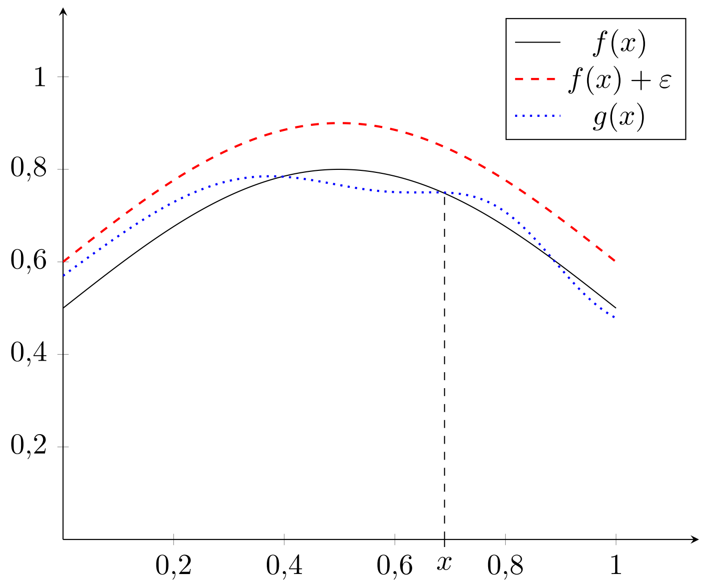


Figure 6.3: $g(x) \leq f(x) + \varepsilon$

To show this, choose for any $z \in E$ a $h_z \in \overline{\mathcal{A}}$ with $h_z(x) = f(x)$ and $h_z(z) \leq f(z) + \frac{\varepsilon}{2}$, which is possible after iv).
Since h_z is continuous, there is a neighborhood U_z of z such that $h_z \leq f + \varepsilon$ on U_z.

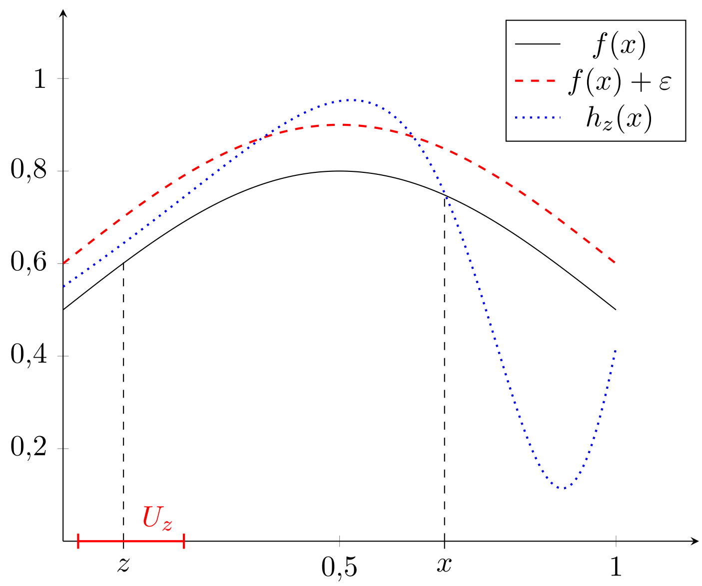


Figure 6.4: $h_z \leq f + \varepsilon$ on U_z

Since E is compact, we can cover it by a finite number of such neighborhoods $U_{z_1}, \ldots, U_{z_N}$. Define:

$$g := \min \{h_{z_1}, \ldots, h_{z_N}\} \in \overline{\mathcal{A}}$$

It holds $g(x) = f(x)$, because $h_{z_i}(x) = f(x)$. We also know:

$$g\big|_{U_j} \leq h_{z_j}\big|_{U_j} \leq f + \varepsilon$$

vi) $\overline{\mathcal{A}} = C^0$: Denote the function g constructed in step v) by g_x.

$$g_x(x) = f(x)$$
$$g_x \leq f + \varepsilon$$

By continuity of g_x there exists a neighborhood U_x of x such that $g_x \geq f - \varepsilon$ on U_x. By compactness we can cover E by a finite number of such neighborhoods $U_{x_1}, \ldots, U_{x_k}$ and define:

$$g := \max \{g_{x_1}, \ldots, g_{x_k}\}$$

Then follows:

$$f - \varepsilon \leq g \leq f + \varepsilon$$
$$\|f - g\| < \varepsilon$$

$\square_{6.2.7}$

Counterexample in the complex case:

$$E = [0,1] \times [0,1] \subseteq \mathbb{C}$$

Consider the set $\mathcal{A} = \mathcal{P}(z)$ of polynomials in z.

- The constant functions are in $\mathcal{A}$.
- $\mathcal{A}$ separates points:
  If $z_1 \neq z_2$ take $f(z) = z$ then $f(z_1) \neq f(z_2)$.

$$\overline{\mathcal{A}} = ?$$

By Morera's theorem we get:

$$\overline{\mathcal{A}} = \left\{ f \in C^0\left([0,1]^2\right) \middle| \left. f\right|_{(0,1)^2} \text{ is holomorphic} \right\} \neq C^0\left([0,1]^2\right)$$

For example $f(x + \mathbf{i}y) = x - \mathbf{i}y$. We have $f \in C^0\left([0,1]^2\right)$, but $f \notin \overline{\mathcal{A}}$.

6.2.8 Theorem (Stone-Weierstraß, complex version)

Let $\mathcal{A} \subseteq C^0(E,\mathbb{C})$ be a subalgebra with the properties 1. and 2. from theorem 6.2.7 and additionally:

3. $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$

Then $\mathcal{A}$ is dense in $C^0(E,\mathbb{C})$.

Proof

Consider the algebras:

$$\begin{aligned} \operatorname{Re}(\mathcal{A}) &= \left\{ f + \overline{f} \middle| f \in \mathcal{A} \right\} \subseteq \mathcal{A} \\ \operatorname{Im}(\mathcal{A}) &= \left\{ \frac{1}{\mathbf{i}}\left(f - \overline{f}\right) \middle| f \in \mathcal{A} \right\} \subseteq \mathcal{A} \end{aligned}$$

These are subalgebras of $C^0(E,\mathbb{R})$. By the real Stone-Weierstraß theorem we get:

$$\overline{\operatorname{Re}(\mathcal{A})} = \overline{\operatorname{Im}(\mathcal{A})} = C^0(E,\mathbb{R})$$

For given $f \in C^0(E,\mathbb{C})$ approximate $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$. $\square_{6.2.8}$

6.3 Arzelà-Ascoli theorem

Let K be a compact metric space and E a Banach space.

$C^0(K,E)$ is the Banach space of continuous functions $f : K \to E$ with norm:

$$\|f\| := \sup_{x \in K} \|f(x)\|_E$$

Let $\mathcal{F} \subseteq C^0(K,E)$ be a subset. Is $\mathcal{F}$ compact?

6.3.1 Definition (relatively compact)

A subset A of a metric space is called *relatively compact*, if $\overline{A}$ is compact.

6.3.2 Definition (equicontinuous)

A family $\mathcal{F} \subseteq C^0(K,E)$ is called *equicontinuous* (gleichgradig stetig) if for all $x \in K$ and all $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $y \in B_\delta(x)$ and for all $f \in \mathcal{F}$ holds:

$$\|f(x) - f(y)\| < \varepsilon$$

(Thus δ is independent of $f \in \mathcal{F}$.)

6.3.3 Theorem (Arzelà-Ascoli)

$\mathcal{F} \subseteq C^0(K,E)$ is relatively compact if and only if the following two conditions holds:

i) $\mathcal{F}$ is equicontinuous.

ii) For every $x \in K$ the set

$$\mathcal{F}(x) := \{f(x) \,|\, f \in \mathcal{F}\}$$

is relatively compact in E.

Proof

„$\Rightarrow$“: Assume that $\mathcal{F} \subseteq C^0(K,E)$ is relatively compact.

i) Assume that $\mathcal{F}$ is *not* equicontinuous. Then there exists an $\varepsilon \in \mathbb{R}_{>0}$ and sequences $x_n \in K$, $f_n \in \mathcal{F}$ and $y_n \in B_{\frac{1}{n}}(x_n)$ such that:

$$\|f_n(x_n) - f_n(y_n)\| \geq \varepsilon$$

After choosing subsequences (with the same notation), we can arrange:

$x_n \to x$	$y_n \to x$	(use that K is compact)
$f_n \to f$		(use that $\mathcal{F}$ is relatively compact)

This means that there is a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{>N}$ holds for all $y \in K$:

$$\|f_n(y) - f(y)\| < \frac{\varepsilon}{3}$$

(Since convergence in $C^0(K,E)$ is the same as uniform convergence $f_n \rightrightarrows f$.)
Since f is continuous there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $y \in B_\delta(x)$:

$$\|f(x) - f(y)\| < \frac{\varepsilon}{3}$$

With this we get:

$$\|f_n(x) - f_n(y)\| \leq \underbrace{\|f_n(x) - f(x)\|}_{<\frac{\varepsilon}{3}} + \underbrace{\|f(x) - f(y)\|}_{<\frac{\varepsilon}{3}} + \underbrace{\|f(y) - f_n(y)\|}_{<\frac{\varepsilon}{3}} < \varepsilon$$

This is a contradiction to $\|f_n(x_n) - f_n(y_n)\| \geq \varepsilon$. $\square_{\text{i)}}$

ii) Consider $y_n \in \mathcal{F}(x) \subseteq E$ (to show that y_n has a convergent subsequence in E).
Then there are functions $f_n \in \mathcal{F}$ with $f_n(x) = y_n$. Since $\mathcal{F}$ is relatively compact, a subsequence is a Cauchy sequence in $C^0(K,E)$, i.e. $\left\|f_{n_l} \to f_{n_{l'}}\right\| \xrightarrow{l,l'\to\infty} 0$.

$$\left\|f_{n_l} - f_{n_{l'}}\right\| = \sup_{z\in K} \left\|f_{n_l}(z) - f_{n_{l'}}(z)\right\|_E \geq \left\|f_{n_l}(x) - f_{n_{l'}}(x)\right\|_E = \left\|y_{n_l} - y_{n_{l'}}\right\|$$

Therefore we get+:

$$\left\|y_{n_l} - y_{n_{l'}}\right\| \xrightarrow{l,l'\to\infty} 0$$

Thus (y_{n_l}) is a Cauchy sequence in E. $\square_{\text{ii)}}$

„$\Leftarrow$“: Let (f_l) be a sequence in $\mathcal{F}$ and show that a subsequence (g_l) converges in $C^0(K,E)$:
Since K is compact, there is a countable dense subset $\{x_1,x_2,\ldots\} \subseteq K$. Since $\mathcal{F}(x_1)$ is relatively compact, there is a subsequence $f_l^{(1)} \in \mathcal{F}$ of (f_l) such that $f_l^{(1)}(x_1)$ converges in E. Since $\mathcal{F}(x_2)$ is relatively compact, there is a subsequence $f_l^{(2)}$ of $f_l^{(1)}$ such that $f_l^{(2)}(x_2)$ converges.
Inductively choose a subsequence $\left(f_l^{(n+1)}\right)$ of $\left(f_l^{(n)}\right)$ such that $f_l^{(n+1)}(x_{n+1})$ converges in E.
Take the diagonal sequence $g_l := f_l^{(l)}$. This is for $l \geq n$ a subsequence of $f_l^{(n)}$, so for all $n \in \mathbb{N}$ converges $g_l(x_n) \xrightarrow{l\to\infty} y_n$.

Claim: g_n is a Cauchy sequence in $C^0(K,E)$, i.e. for all $\varepsilon \in \mathbb{R}_{>0}$ exists a $N \in \mathbb{N}$ such that for all $n,m \in \mathbb{N}_{>N}$ and all $x \in K$ holds:

$$|g_n(x) - g_m(x)| \leq \varepsilon$$

Proof: Since $\mathcal{F}$ is equicontinuous, for all $x \in E$ exists a $\delta \in \mathbb{R}_{>0}$ such that for all $z,z' \in B_{\delta(x)}(x)$ and all $f \in \mathcal{F}$ holds:

$$\left\|f(z) - f(z')\right\| < \frac{\varepsilon}{3}$$

We cover K by a finite number of such balls $B_1,\ldots,B_L$. In every Ball B_l there is at least one point of $\{x_1,x_2,\ldots\}$. We choose such a point $\xi_l \in B_l$. Since $(g_n(\xi_l))$ converges for every $l \in \{1,\ldots,L\}$ we can choose a $N \in \mathbb{N}$ such that for all $l \in \{1,\ldots,L\}$ and all $m,n \in \mathbb{N}_{>N}$ holds:

$$\|g_n(\xi_l) - g_m(\xi_l)\| < \frac{\varepsilon}{3}$$

For every $x \in K$ exists a $l \in \{1,\ldots,L\}$ with $x \in B_l$.

$$\|g_n(x) - g_m(x)\| \leq \underbrace{\|g_n(x) - g_n(\xi_l)\|}_{<\frac{\varepsilon}{3}} + \underbrace{\|g_n(\xi_l) - g_m(\xi_l)\|}_{<\frac{\varepsilon}{3}} + \underbrace{\|g_m(\xi_l) - g_m(x)\|}_{<\frac{\varepsilon}{3}}$$

$\square_{\text{Claim}}$

Therefore the subsequence (g_l) for (f_l) converges in $C^0(K,E)$, since $C^0(K,E)$ is complete, because E is a Banach space. $\square_{6.3.3}$

Application to integral operators

Let $K \subseteq \mathbb{R}^n$ be compact. Consider an integral operator $A : C^0(K,\mathbb{R}) \to C^0(K,\mathbb{R})$, i.e.:

$$(Af)(x) = \int_K A(x,y)\, f(y)\, \mathrm{d}^n y$$

$\mathcal{F} := A\left(C^0(K,\mathbb{R})\right)$ is equicontinuous provided that $A(.,y)$ is continuous.

6.4 The Riesz representation theorem

Let K again be a compact metric space. $E = C^0(K,\mathbb{R})$ with the sup-norm is a Banach space.
Question: What is E^*?
Consider $l \in E^*$, i.e.

$$l : E \to \mathbb{R}$$

and for all $f \in C^0(K)$ holds:

$$|l(f)| \leq C \|f\|$$

This means f is bounded or equivalently continuous.

6.4.1 Examples

Consider $K = [0,1] \subseteq \mathbb{R}$. For any $\varphi \in L^1([0,1])$, the functional

$$l(f) := \int_0^1 \varphi(x) f(x)\,\mathrm{d}x$$

is linear and bounded:

$$|l(f)| \leq \int_0^1 |\varphi(x)| \cdot |f(x)|\,\mathrm{d}x \leq \underbrace{\sup_{x\in[0,1]} |f|}_{=\|f\|} \cdot \underbrace{\int_0^1 |\varphi(x)|\,\mathrm{d}x}_{=\|\varphi\|_{L^1}}$$

It is convenient to identify $l \in E^*$ with the function $\varphi \in L^1$. We have represented l by an L^1-function φ.
This can also be written as a *signed measure* (signiertes Maß):

$$\mathrm{d}\mu := \varphi(x)\,\mathrm{d}x$$

But not every $l \in E^*$ can be represented in this form.

Example

$$l(f) := f\left(\frac{1}{2}\right)$$

is bounded:

$$|l(f)| = \left|f\left(\frac{1}{2}\right)\right| \leq \sup_{[0,1]} |f| = \|f\|$$

It can be represented by the Dirac measure:

$$l(f) = \int_0^1 f(x)\,\delta\left(x - \frac{1}{2}\right)\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}\mu$$

Here $\delta(x)$ is the δ-Distribution. $\mu = \delta_{\frac{1}{2}}$ is the Dirac measure.

$$\delta_{x_0}(\Omega) = \begin{cases} 1 & \text{if } x_0 \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

6.4.2 Definition (bounded, positive, regular measure)

Let $X \neq \emptyset$ be a set. A σ-algebra $\mathcal{M}$ over X is a set of subsets of X such that holds:

i) $\emptyset \in \mathcal{M}$

ii) $A \in \mathcal{M} \;\Rightarrow\; \complement A := X \setminus A \in \mathcal{M}$

iii) For a countable family $(A_j)_{j\in\mathbb{N}}$ holds:

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

The elements of $\mathcal{M}$ are called *measurable sets* (messbare Mengen).

Let K be a compact metric space. Denote by $\mathfrak{M}$ the *Borel algebra*, i.e. the smallest σ-algebra over K, which contains all open and therefore all closed subsets of K.

A *bounded (signed) measure* is a mapping

$$\mu : \mathfrak{M} \to \mathbb{R}$$

(not $\mu : \mathfrak{M} \to \mathbb{R}^+ \cup \{0\} \cup \{\infty\}$ as before in measure theory) with the following properties:

- The empty set measures zero:

$$\mu(\emptyset) = 0$$

- σ-additivity: For $M_j \in \mathfrak{M}$ with $M_i \cap M_j = \emptyset$ for all $i \neq j$ holds:

$$\mu\left(\dot{\bigcup_{j=1}^{\infty}} M_j\right) = \sum_{j=1}^{\infty} \mu(M_j)$$

μ is *positive*, if $\mu(M) \geq 0$ for all $M \in \mathfrak{M}$.
μ is *regular*, if for all $A \in \mathfrak{M}$ holds:

$$\mu(A) = \sup_{\substack{B\subseteq A\\ B \text{ compact}}} \mu(B) = \inf_{\substack{\Omega\supseteq A\\ \Omega \text{ open}}} \mu(\Omega)$$

Example

The Lebesgue measure $\mathrm{d}^n x$ restricted to the Borel algebra on $[0,1]^n$ is a bounded, positive and regular measure.

6.4.3 Theorem (Riesz representation theorem)

Consider $l \in C^0(K,\mathbb{R})^*$. Then there is a unique bounded regular Borel measure μ (i.e. a measure on the Borel algebra $\mathfrak{M}$) such that for all $f \in C^0(K,\mathbb{R})$ holds:

$$l(f) = \int_K f \mathrm{d}\mu$$

Here we only prove the case $K = [0,1]$. (We also need it for $K = [0,1]^2$.)

How can one construct positive regular Borel measures on $[0,1]$?

Lebesgue-Stieltjes integral

Let $\alpha : [0,1] \to \mathbb{R}$ be monotonically increasing (not necessarily continuous).

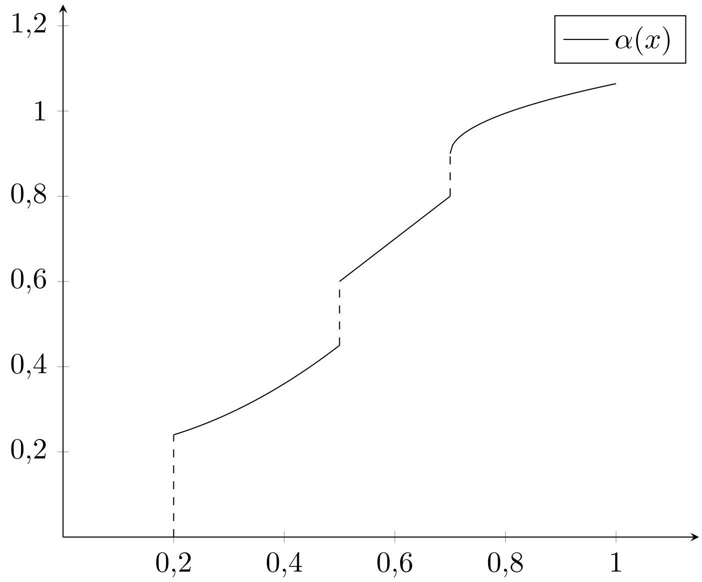


Figure 6.5: α is monotonically increasing, but not continuous

The two one-sided limits

$$\lim_{x \nearrow x_0} \alpha(x), \ \lim_{x \searrow x_0} \alpha(x)$$

exist. In general:

$$\lim_{x \nearrow x_0} \alpha(x) \leq \alpha(x_0) \leq \lim_{x \searrow x_0} \alpha(x)$$

But equality does not need to hold. Define:

$$\mu((a,b)) := \lim_{x \nearrow b} \alpha(x) - \lim_{x \searrow a} \alpha(x)$$

By σ-additivity, this measure can be extended to a positive regular bounded Borel measure. (This can be proven exactly as for the Lebesgue integral.) The corresponding integral

$$\int_0^1 f \mathrm{d}\mu$$

is called Lebesgue-Stieltjes integral. If $\alpha(x) = x + c$, the Lebesgue-Stieltjes integral reduces to the Lebesgue integral

6.4.4 Example

Let $\alpha \in C^1([0,1])$ be monotonically increasing. Then holds:

$$\mu((a,b)) = \alpha(b) - \alpha(a) = \int_a^b \alpha'(x) \, \mathrm{d}x = \int_0^1 \chi_{(a,b)} \alpha'(x) \, \mathrm{d}x$$

The corresponding Lebesgue-Stieltjes integral is:

$$\int f \mathrm{d}\mu = \int_0^1 f(x) \cdot \alpha'(x) \, \mathrm{d}x$$

The following short notation is used in general:

$$\mathrm{d}\mu = \alpha'(x) \, \mathrm{d}x$$
$$\mathrm{d}\mu = \mathrm{d}\alpha$$

If $\alpha \in C^1([0,1])$ is not monotone, we can still set:

$$\int_0^1 f \mathrm{d}\mu := \int_0^1 f \cdot \alpha'(x) \, \mathrm{d}x$$

$\mathrm{d}\mu$ is a signed measure.

In order to extend the Lebesgue-Stieltjes construction to functions α, which are *not* monotone (such as to obtain signed measures), we need to assume, that α has bounded variation.

6.4.5 **Definition** (total variation)

Let $\alpha : [0,1] \to \mathbb{R}$ be a function (not necessarily continuous).
The *total variation* (Totalvariation) is defined by:

$$(\mathrm{TV}(\alpha))(x) := \sup_{\substack{N \in \mathbb{N} \\ 0 = x_0 < \ldots < x_N = x}} \sum_{i=1}^{N} |\alpha(x_1) - \alpha(x_{i-1})| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

α is of *bounded variation* (beschränkte Totalvariation), $\alpha \in \mathcal{BV}([0,1])$, if $(\mathrm{TV}(f))(1) < \infty$.

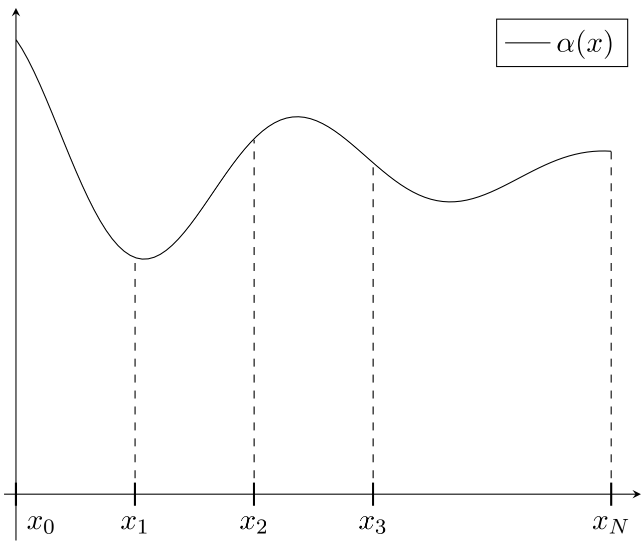


Figure 6.6: total variation of α

Note: If α is monotonically increasing, then holds:

$$(\mathrm{TV}(\alpha))(x) = \alpha(x) - \alpha(0) < \infty$$

Thus every monotonically function has bounded variation.

But there are even continuous functions, which have unbounded variation, e.g. for large enough $p \in \mathbb{R}_{>0}$:

$$\alpha(x) = x \sin\left(\frac{1}{x^p}\right)$$

For $\alpha \in C^1([0,1])$ holds:

$$\mathrm{TV}(\alpha)(x) = \int_0^x \left|\alpha'(\tau)\right| \mathrm{d}\tau$$

Lemma (Properties of the total variation)

$\mathrm{TV}(\alpha)(x)$ is monotonically increasing and:

$$\mathrm{TV}(\alpha)(0) = 0$$

$\mathrm{TV}(\alpha)(x) \pm \alpha(x)$ is also monotonically increasing.

Proof

Assume that $y \in \mathbb{R}_{>x}$.

$$\mathrm{TV}(\alpha)(y) = \sup_{\substack{N \in \mathbb{N} \\ 0=x_0<\ldots<x_N=y}} \sum_{i=1}^{N} |\alpha(x_i) - \alpha(x_{i-1})| \geq \sup_{\substack{N \in \mathbb{N}_{\geq 2} \\ 0=x_0<\ldots<x_{N-1}=x<x_N=y}} \sum_{i=1}^{N} |\alpha(x_i) - \alpha(x_{i-1})| \geq$$
$$\geq \sup_{\substack{N \in \mathbb{N}_{\geq 2} \\ 0=x_0<\ldots<x_{N-1}=x<x_N=y}} \sum_{i=1}^{N-1} |\alpha(x_i) - \alpha(x_{i-1})| = \mathrm{TV}(\alpha)(x)$$

$$\mathrm{TV}(\alpha)(x) \pm \alpha(x) = \pm\alpha(0) + \sup_{\substack{N \in \mathbb{N} \\ 0=x_0<\ldots<x_N=x}} \sum_{i=1}^{N} \underbrace{|\alpha(x_i) - \alpha(x_{i-1})| \pm (\alpha(x_i) - \alpha(x_{i-1}))}_{\geq 0}$$

Just as before this implies that

$$\mathrm{TV}(\alpha)(x) \pm \alpha(x)$$

is monotonically increasing. $\square_{6.4.5}$

Suppose that $f \in \mathcal{BV}([0,1])$. Then the functions

$$f_+ = \frac{1}{2}(\mathrm{TV}(f) + f)$$
$$f_- = \frac{1}{2}(\mathrm{TV}(f) - f)$$

are monotonically increasing and:

$$f = f_+ - f_-$$

Let $\mathrm{d}\mu_\pm = \mathrm{d}f_\pm$ be the bounded positive regular Borel measures of the corresponding Lebesgue-Stieltjes integrals. Then

$$\mu := \mu_+ - \mu_-$$

defines a bounded regular Borel measure with the property:

$$\begin{aligned}\mu\left((a,b)\right) &= \mu_+\left((a,b)\right) - \mu_-\left((a,b)\right) = \lim_{x\nearrow b} f_+(x) - \lim_{x\searrow a} f_+(x) - \lim_{x\nearrow b} f_-(x) + \lim_{x\searrow a} f_-(x) = \\ &= \lim_{x\nearrow b} f(x) - \lim_{x\searrow a} f(x)\end{aligned}$$

6.4.6 Example

Consider the Heaviside function:

$$f := \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

$\mathrm{d}\mu := \mathrm{d}f$ has the form $\mu = \delta_{\frac{1}{2}}$.

Proof of Theorem 6.4.3 in the case $K = [0,1]$

$\mathcal{PC}([0,1])$ are the piecewise continuous functions, i.e. for all $f \in \mathcal{PC}([0,1])$ exists a $N \in \mathbb{N}$ and points $0 = x_0 < \ldots < x_N = 1$ such that $f\big|_{(x_{i-1},x_i)}$ is continuous and has a continuous continuation to $[x_{i-1},x_i]$ for all $i \in \{1,\ldots,N\}$.
On $\mathcal{PC}$ we introduce the norm:

$$\|f\| = \sup_{x\in[0,1]} |f(x)|$$

This makes $\mathcal{PC}([0,1])$ a Banach space.

$$C^0([0,1]) \subseteq \mathcal{PC}([0,1])$$

is a subspace, which is closed, since it is complete.
Consider $l \in C^0([0,1])^*$, i.e.

$$l : C^0([0,1]) \to \mathbb{R}$$

with:

$$|l(f)| \leq C\,\|f\|_{C^0}$$

According to the Hahn-Banach theorem, there is an extension

$$\tilde{l} : \mathcal{PC}([0,1]) \to \mathbb{R}$$

with $\tilde{l}\big|_{C^0} = l$ and $|l(f)| \leq C\,\|f\|_{\mathcal{PC}([0,1])}$. Define $\alpha : [0,1] \to \mathbb{R}$ by:

$$\alpha(x) := \begin{cases} \tilde{l}\left(\chi_{[0,x)}\right) & \text{if } x < 1 \\ \tilde{l}\left(\chi_{[0,1]}\right) & \text{if } x = 1 \end{cases}$$

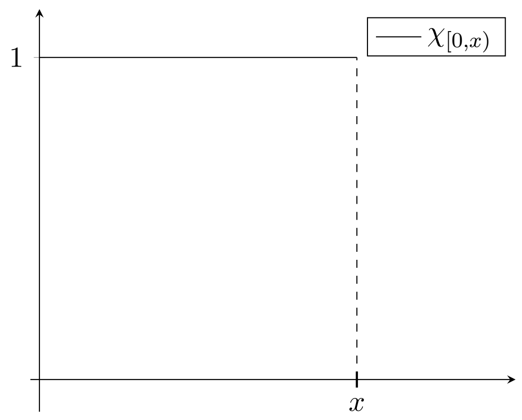


Figure 6.7: $\chi_{[0,x)}$

$l\left(\chi_{[0,x)}\right)$ is ill-defined, because $\chi_{[0,x)}$ is *not* continuous.

$\tilde{l}\left(\chi_{[0,x)}\right)$ is well-defined, because $\chi_{[0,x)}$ is piecewise-continuous.

– α has bounded variation: Consider:

$$0 = x_0 < \ldots < x_N = 1$$

We need to show:

$$\sum_{i=1}^{N} |\alpha(x_i) - \alpha(x_{i-1})| < C$$

C has to be independent of N and the (x_i).
Define $s_i \in \{\pm 1\}$ by:

$$s_i := \begin{cases} +1 & \text{if } \alpha(x_i) - \alpha(x_{i-1}) \geq 0 \\ -1 & \text{if } \alpha(x_i) - \alpha(x_{i-1}) < 0 \end{cases}$$

Then holds:

$$\sum_{i=1}^{N} |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^{N} s_i\left(\alpha(x_i) - \alpha(x_{i-1})\right) = \tilde{l}\left(\sum_{i=1}^{N-1} s_i \chi_{[x_{i-1},x_i)} + s_N \chi_{[x_{N-1},1]}\right)$$

Since $\tilde{l}$ is bounded by construction, we know:

$$\sum_{i=1}^{N} |\alpha(x_i) - \alpha(x_{i-1})| \leq \left|\tilde{l}\left(\sum_{i=1}^{N-1} s_i \chi_{[x_{i-1},x_i)} + s_N \chi_{[x_{N-1},1]}\right)\right| \leq$$
$$\leq C \left\|\sum_{i=1}^{N-1} s_i \chi_{[x_{i-1},x_i)} + s_N \chi_{[x_{N-1},1]}\right\| = C$$

Therefore we have $\alpha \in \mathcal{BV}([0,1])$.

– Consider $\mathrm{d}\mu := \mathrm{d}\alpha_+ - \mathrm{d}\alpha_-$ for the corresponding bounded regular Borel measure, where $\alpha = \alpha_+ - \alpha_-$ and $\alpha_\pm$ are monotonically increasing.

Claim: For all $f \in C^0([0,1])$ holds:

$$l(f) = \int_0^1 f \mathrm{d}\mu$$

Proof: Consider $f \in C^0([0,1])$. Set:

$$f_n(x) := \begin{cases} \sum_{i=1}^n f\left(\frac{i}{n}\right) \cdot \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right)} & \text{if } x < 1 \\ f(1) & \text{if } x = 1 \end{cases}$$

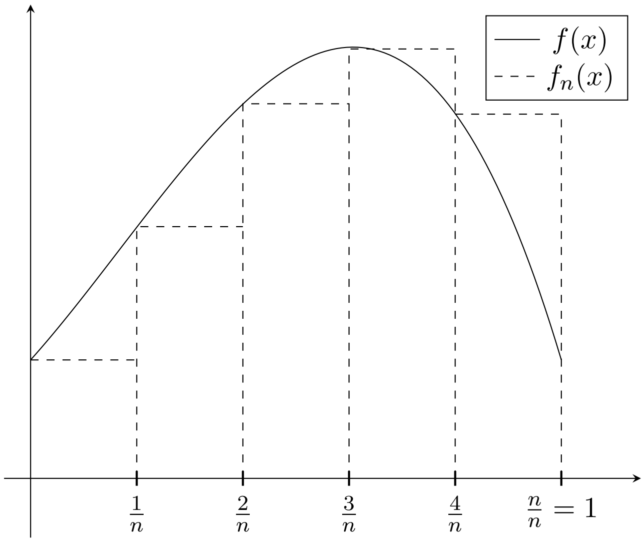


Figure 6.8: Approximation of f by $f\left(\frac{i}{n}\right)$ for $n = 5$

Since f_n is uniformly continuous, i.e. $f_n \rightrightarrows f$, we get:

$$l(f) = \tilde{l}(f) = \tilde{l}\left(\lim_{n\to\infty} f_n\right) \overset{\tilde{l} \text{ continuous}}{=} \lim_{n\to\infty} \tilde{l}(f_n) =$$
$$\overset{\text{by construction}}{=} \lim_{n\to\infty} \int_0^1 f_n \mathrm{d}\mu \overset{(*)}{=} \int_0^1 \lim_{n\to\infty} f_n \mathrm{d}\mu = \int_0^1 f \mathrm{d}\mu$$

For $(*)$ consider:

$$\left| \int_0^1 (f_n - f) \,\mathrm{d}\mu \right| \le \underbrace{\sup |f - f_n|}_{\to 0} \cdot \underbrace{\mathrm{TV}(\alpha)(1)}_{<\infty} \xrightarrow{n\to\infty} 0$$

$\square_{\text{Claim}}$

$\square_{6.4.3}$

Remarks

– Our proof only works in the case $K = [a,b] \subseteq \mathbb{R}$. (see Reed, Simon: Appendix “The Riesz-Markov Theorem”)

- In general dimension the idea would be:

$$\mu(\Omega) := \tilde{l}(\chi_\Omega)$$

But how to extend l? So choose $f_n \to \chi_\Omega$ and define:

$$\mu(\Omega) := \lim_{n\to\infty} l(f_n)$$

(see Rudin: *Real and complex analysis*)

- Total variation of a bounded Borel measure:

$$|\mu|(\Omega) := \sup_{\substack{N\in\mathbb{N} \\ \Omega_1,\ldots,\Omega_N \\ \text{with } \Omega_1 \dot\cup \ldots \dot\cup \Omega_N = \Omega}} \sum_{i=1}^{N} |\mu(\Omega_i)|$$

$|\mu|$ is a positive bounded Borel measure. (see Rudin)
Then we can write:

$$\left|\int_K (f - f_n)\,\mathrm{d}\mu\right| \leq \int_K |f - f_n| \cdot \mathrm{d}|\mu| \leq \sup_K |f - f_n| \cdot |\mu|(K)$$

7 The Spectral Theorem for symmetric bounded operators

Let $A \in L(H)$ be symmetric and H be a separable Hilbert space. Let $p(A)$ be a polynomial in A, for example the characteristic polynomial for $A \in L(\mathbb{C}^N)$ with $p(A) = 0$.
Extend this idea to functions $f(A)$ with $f \in C^0(\sigma(A))$. (Stone-Weierstraß)
Then for

$$\langle u, f(A)\, u\rangle =: l(f)$$

holds $l \in C^0(\sigma(A))^*$. Using the Riesz representation theorem we can write:

$$\langle u, f(A)\, u\rangle = \int_{\sigma(A)} f(\lambda)\, \mathrm{d}\mu_u(\lambda)$$

$$\mathrm{d}\mu_u(\lambda) = \langle u, \mathrm{d}E_\lambda u\rangle$$

$\mathrm{d}E_\lambda$ is the so-called *spectral measure*. Then holds the spectral theorem:

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E_\lambda$$

7.1 The Spectrum of symmetric bounded operators

Let $A \in L(H)$ be symmetric, i.e. $\langle u, Av\rangle = \langle Au, v\rangle$ for all $u,v \in H$. The resolvent set is:

$$\begin{aligned} \varrho(A) &= \left\{\lambda \in \mathbb{C} \,\middle|\, (\lambda - A) \text{ has a continuous inverse}\right\} \\ \sigma(A) &= \mathbb{C} \setminus \varrho(A) \end{aligned}$$

$\varrho(A) \subseteq \mathbb{C}$ is open and so the spectrum $\sigma(A) \subseteq \mathbb{C}$ is closed. The spectral radius is:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = |||A|||$$

Warning

Consider $\lambda \in \sigma(A)$, i.e. $\lambda - A$ has no continuous inverse. This does not mean $\ker(\lambda - A)$ is non-trivial. Thus λ does *not* need to be an eigenvalue!

7.1.1 Theorem

Let $A \in L(H)$ be self-adjoint. Then $\sigma(A) \subseteq \mathbb{R}$.

Proof

Consider $\lambda = \alpha + \mathbf{i}\beta$ with $\alpha,\beta \in \mathbb{R}$ and $\beta \neq 0$. We need to show that $\lambda - A$ has a continuous inverse. Introduce the following bilinear form:

$$B(x,y) = \left\langle x, \left(A - \overline{\lambda}\right) y\right\rangle = \langle (A-\lambda)x,y\rangle$$

This bilinear form satisfies the assumptions of the Lax-Milgram theorem:

i) The sesquilinearity is clear, since the scalar product is sesquilinear.

ii) B is bounded:

$$\left|\left\langle x, \left(A - \overline{\lambda}\right) y\right\rangle\right| \leq \|x\| \cdot \underbrace{\left\|A - \overline{\lambda}\right\|}_{\leq \|A\| + |\lambda|} \cdot \|y\| \leq C\|x\|\|y\|$$

iii) B is bounded from below, i.e. there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that for all $x \in H$ holds:

$$|B(x,x)| \geq \varepsilon \|x\|^2$$

We know:

$$B(x,x) = \left\langle x, \left(A - \overline{\lambda}\right) x\right\rangle = \underbrace{\langle x,Ax\rangle}_{\text{real}} - \underbrace{\mathrm{Re}\left(\lambda\langle x,x\rangle\right)}_{\text{real}} - \underbrace{\mathbf{i}\mathrm{Im}\left(\lambda\langle x,x\rangle\right)}_{\text{imaginary}}$$

$$|B(x,x)| \geq |\mathrm{Im}\left(\lambda\langle x,x\rangle\right)| = |\beta| \cdot \|x\|^2$$

Set $\varepsilon := |\beta| \neq 0$.

The Lax-Milgram theorem yields that the linear functional $l(x) = \langle z,x\rangle$ can be represented as

$$l(x) = B(y,x)$$

with a unique $y = y(z) \in H$. Thus we get for all $x \in H$:

$$\begin{aligned} \langle z,x\rangle &= \langle (A-\lambda)y,x\rangle \\ \Rightarrow \quad z &= (A-\lambda)y \end{aligned}$$

Therefore, for all $z \in H$ exists a unique $y \in H$ su ch that $(A-\lambda)y = x$. Thus $A - \lambda$ is invertible. The inverse $(A-\lambda)^{-1}$ is continuous due to the open mapping theorem (see Corollary 2.4.8). $\square_{7.1.1}$

7.1.2 Theorem

It holds $\sigma(A) \subseteq [a,b]$ and $a,b \in \sigma(A)$ with:

$$a := \inf_{\|u\|=1} \langle u,Au\rangle$$
$$b := \sup_{\|u\|=1} \langle u,Au\rangle$$

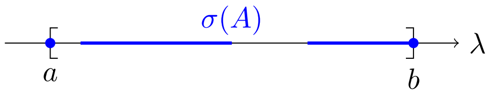


Figure 7.1: $\sigma(A) \subseteq [a,b]$ and $a,b \in \sigma(A)$

Proof

For $\lambda \in \mathbb{R}_{<a}$ holds:

$$\langle x, (A-\lambda)\,x\rangle = \langle x,Ax\rangle - \lambda\,\|x\|^2 \geq a\,\|x\|^2 - \lambda\,\|x\|^2 = \underbrace{(a-\lambda)}_{>0}\|x\|^2$$

Thus

$$\langle .,.\rangle_A := \langle ., (A-\lambda)\,.\rangle$$

is a scalar product on H. The corresponding norm

$$\|u\|_A := \sqrt{\langle u,u\rangle_A}$$

is equivalent to the norm $\|.\|$, because it holds:

$$(a-\lambda)\,\|u\|^2 \leq \|u\|_A = \langle u, (A-\lambda)\,u\rangle \leq (|||A||| - \lambda)\,\|u\|^2$$

For $u \in H$ and $l(w) := \langle u,w\rangle$ is $l \in H^*$. According to the Fréchet-Riesz theorem 3.1.3 (for the scalar product $\langle .,.\rangle_A$) there is a unique vector $v \in H$, such that for all $w \in H$ holds:

$$l(w) = \langle v,w\rangle_A$$

Thus we get for all $w \in H$:

$$\langle u,w\rangle = l(w) = \langle v,w\rangle_A = \langle v, (A-\lambda)\,w\rangle \overset{A-\lambda \text{ symmetric}}{=} \langle (A-\lambda)\,v,w\rangle$$

$$\Rightarrow \quad u = (A-\lambda)\,v$$

Thus there exists a

$$\varphi : H \to H$$
$$u \mapsto v$$

such that $u = (A - \lambda)\varphi(u)$, i.e. $A - \lambda \in L(H)$ is surjective. φ is linear and bounded according to the open mapping theorem 2.4.8. Thus we have

$$\varphi = (A - \lambda)^{-1} \in L(H)$$

and therefore $\lambda \in \varrho(A)$.
Applying the same argument to the operator $(-A)$, one sees that $(b,\infty) \subseteq \varrho(A)$.
Therefore holds $\sigma(A) \subseteq [a,b]$.

Only prove that $b \in \sigma(A)$. For $a \in \sigma(A)$ consider similarly the operator $-A$. Furthermore replace $A \to A - a$ to get $\sigma(A) \subseteq [0,b]$. We know:

$$|||A||| = r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \sup_{\lambda \in \sigma(A)} \lambda = \sup \sigma(A)$$

As a consequence we get $|||A||| \leq b$. On the other hand we have:

$$b = \sup_{\|u\|=1} \langle u, Au\rangle \leq \sup_{\|u\|=1} \|Au\| \cdot \underbrace{\|u\|}_{=1} = |||A|||$$

Thus we have $b = |||A||| = r(A)$, especially b is a limit point of the spectrum of A. Since $\sigma(A)$ is closed, it follows that $b \in \sigma(A)$. $\square_{7.1.2}$

7.2 The continuous functional calculus

7.2.1 Theorem (continuous functions of operators)

Let $A \in L(H)$ be symmetric. Then there is a unique mapping $\Phi : C^0(\sigma(A), \mathbb{C}) \to L(H)$ (remember $\sigma(A) \subseteq [a,b]$) with the following properties:

i) Φ is an involutive algebra homomorphism, i.e.:
   - Φ is linear.
   - $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$
   - $\Phi(\overline{f}) = (\Phi(f))^*$ (involution)

ii) Φ is continuous:

$$\|\Phi(f)\|_{L(H)} \leq C \|f\|_\infty$$

iii) If $f(t) = t$, then $\Phi(f) = A$.

iv) If $Au = \lambda u$, i.e. $u \in H$ is an eigenvector of A, then $\Phi(f)u = f(\lambda)u$.

v) If $f \geq 0$, then $\Phi(f) \geq 0$, meaning that $\Phi(f)$ is a positive semi-definite operator, i.e. $\langle u, \Phi(f)u\rangle \geq 0$ for all $u \in H$.

vi) $\sigma(\Phi(f)) = f(\sigma(A))$ (spectral mapping theorem (spektraler Abbildungssatz))

vii) $\|\Phi(f)\|_{L(H)} = \|f\|_\infty$

Often we just write $\Phi(f) = f(A)$.

What if $f(t) = p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0$ is a polynomial?

$$\Phi(t) \overset{\text{iii)}}{=} A$$

From i) follows:

$$\Phi(1) = \Phi(1 \cdot 1) = \Phi(1) \cdot \Phi(1)$$

Therefore we get:

$$\Phi(1) = \mathbb{1}$$

Now follows:

$$\begin{aligned} \Phi\left(t^2\right) &= \Phi(t \cdot t) = \Phi(t) \cdot \Phi(t) = A \cdot A = A^2 \\ \Phi\left(t^l\right) &= A^l \\ \Phi(p) &= p(A) = a_n A^n + a_{n-1} A^{n-1} + \ldots + a_0 \mathbb{1} \end{aligned}$$

7.2.2 Lemma (spectral mapping theorem for polynomials)

For a complex polynomial $p \in \mathbb{P}_{\mathbb{C}}$ holds:

$$\sigma(p(A)) = p(\sigma(A))$$

Proof

- If $p = c \in \mathbb{C}$ is constant, then the lemma is trivial:

  $$p(\sigma(A)) = c = \sigma(c\mathbb{1}) = \sigma(p(A))$$

  So further on let p be not constant.
- $p(\sigma(A)) \subseteq \sigma(p(A))$: For $\lambda \in \sigma(A)$ and $z \in \mathbb{C}$ yields the fundamental theorem of algebra:

  $$p(z) - p(\lambda) = (z - \lambda) q(z)$$

  Here $q(z)$ is a new polynomial with $\deg(q) = \deg(p) - 1$. This also holds if we set $z = A$:

  $$p(A) - p(\lambda) = (A - \lambda) q(A)$$

  Assume $p(\lambda) \in \varrho(p(A))$, i.e. $p(A) - p(\lambda)$ has a bounded inverse. Then holds:

  $$\begin{aligned} \mathbb{1} &= (p(A) - p(\lambda)) \cdot (p(A) - p(\lambda))^{-1} = (A - \lambda) \cdot q(A) \cdot (p(A) - p(\lambda))^{-1} \\ \Rightarrow \quad (A - \lambda)^{-1} &= \underbrace{q(A)}_{\in L(H)} \cdot \underbrace{(p(A) - p(\lambda))^{-1}}_{\in L(H)} \in L(H) \end{aligned}$$

  This gives $\lambda \in \varrho(A)$ in contradiction to $\lambda \in \sigma(A)$ and so $p(\lambda) \in \sigma(p(A))$.

– $\sigma(p(A)) \subseteq p(\sigma(A))$: Consider $\mu \in \sigma(p(A))$ and set $n := \deg(p)$. Using the fundamental theorem of algebra we get:

$$q(z) := p(z) - \mu = a(z - \lambda_1) \cdot \ldots \cdot (z - \lambda_n)$$
$$q(A) := p(A) - \mu = a(A - \lambda_1) \cdot \ldots \cdot (A - \lambda_n)$$

If all the operators $A - \lambda_i$ had a continuous inverse, then this would hold also for their product in contradiction to the assumption $\mu \in \sigma(p(A))$. Thus one of the λ_i is in the spectrum of A. Because one of the linear factors vanishes, follows:

$$\begin{aligned} & 0 = q(\lambda_i) = p(\lambda_i) - \mu \\ \Rightarrow \quad & \mu = p(\lambda_i) \in p(\sigma(A)) \end{aligned}$$

$\square_{7.2.2}$

Let $p \in \mathbb{P}_{\mathbb{C}}$ be a complex polynomial.

$$(p(A))^* = \overline{p}(A)$$

Thus $p(A)$ is not symmetric.

7.2.3 Definition (normal operator)

$A \in L(H)$ is called *normal*, if $[A,A^*] = 0$.

7.2.4 Theorem

For a normal $A \in L(H)$ holds $r(A) = |||A|||$.

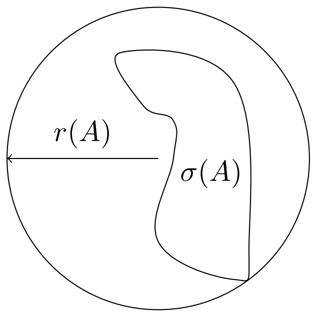


Figure 7.2: $r(A) = |||A|||$

Proof

We already proved for a general $A \in L(H)$:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{n \to \infty} |||A^n|||^{\frac{1}{n}} \tag{7.1}$$

For symmetric operators, we know furthermore:

$$r(A) = |||A||| = \sup_{\|u\|=1} |\langle u,Au \rangle| \tag{7.2}$$

For *normal* operators, we conclude the following: A^*A is symmetric and thus:

$$\begin{aligned}|||A|||^2 &= \sup_{\|u\|=1} \|Au\|^2 = \sup_{\|u\|=1} \langle Au, Au\rangle = \sup_{\|u\|=1} \langle u, A^*Au\rangle \overset{(7.2)}{=} |||A^*A||| = \\ &\overset{(7.2)}{=} r\left(A^*A\right) \overset{(7.1)}{=} \lim_{n\to\infty} |||(A^*A)^n|||^{\frac{1}{n}}\end{aligned}$$

$$(A^*A)^n = \underbrace{A^*A \cdot A^*A \cdot \ldots \cdot A^*A}_{n\text{-times}} \overset{A \text{ normal}}{=} (A^*)^n \cdot A^n$$

With

$$|||A|||^2 = \sup_{\|u\|=1} \langle Au, Au\rangle = \sup_{\|u\|=1} \langle u, A^*Au\rangle \overset{A \text{ normal}}{=} \sup_{\|u\|=1} \langle u, AA^*u\rangle = \sup_{\|u\|=1} \langle A^*u, A^*u\rangle = |||A^*|||^2$$

we get:

$$|||(A^*A)^n||| \leq |||(A^*)^n||| \cdot |||A^n||| = |||A^n|||^2$$

It follows:

$$|||A|||^2 = \lim_{n\to\infty} |||(A^*A)^n|||^{\frac{1}{n}} \leq \lim_{n\to\infty} \left(|||A^n|||^2\right)^{\frac{1}{n}} \leq |||A|||^2$$

This gives:

$$|||A|||^2 = \lim_{n\to\infty} \left(|||A^n|||^{\frac{1}{n}}\right)^2 = \left(\lim_{n\to\infty} |||A^n|||^{\frac{1}{n}}\right)^2 = \left(r\left(A\right)\right)^2$$

$$\Rightarrow \quad r\left(A\right) = |||A|||$$

$\square_{7.2.4}$

7.2.5 Lemma

Let $A \in L\left(H\right)$ be symmetric and $p \in \mathbb{P}_{\mathbb{C}}$ a complex polynomial. Then holds:

$$|||p\left(A\right)||| = \sup_{\lambda\in\sigma(A)} |p\left(\lambda\right)|$$

Proof

$p\left(A\right)$ is normal and thus, according to Theorem 7.2.4 holds:

$$|||p\left(A\right)||| = \sup_{\mu\in\sigma(p(A))} |\mu| \overset{7.2.2}{=} \sup_{\lambda\in\sigma(A)} |p\left(\lambda\right)|$$

$\square_{7.2.5}$

Proof of theorem 7.2.1

- For complex polynomials, we set $\varPhi(p) = p(A)$. Then holds:

$$|||\varPhi(p)||| = |||p(A)||| = r(p(A)) = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = \|p\|_{C^0(\sigma(A),\mathbb{C})}$$

  Thus $\varPhi : \mathbb{P}_{\mathbb{C}} \to L(H)$ is an isometry. ($\mathbb{P}_{\mathbb{C}} \subseteq C^0(\sigma(A),\mathbb{C})$)
  Remark: If we had considered $C^0([a,b],\mathbb{C})$ with

$$a = \inf_{\|u\|=1} \langle u,Au\rangle$$
$$b = \sup_{\|u\|=1} \langle u,Au\rangle$$

  then we would only have an inequality:

$$|||\varPhi(p)||| \le \|p\|_{C^0([a,b])}$$

- Moreover holds:

$$\varPhi(p \cdot q) = (p \cdot q)(A) = p(A) \cdot q(A) = \varPhi(p) \cdot \varPhi(q)$$
$$(\varPhi(p))^* = \varPhi(\overline{p})$$

- Using the Stone-Weierstraß approximation theorem, $\varPhi$ uniquely extends to an isometry:

$$\varPhi : C^0(\sigma(A),\mathbb{C}) \to L(H)$$

  This yields i), ii), iii), vii).
- More specifically, consider $f \in C^0(\sigma(A),\mathbb{C})$. Then there exist $p_n \in \mathbb{P}_{\mathbb{C}}$ such that $p_n \rightrightarrows f$ on $\sigma(A)$. ($K = \sigma(A)$ is a compact metric space.) This means:

$$\|p_n - f\|_{C^0(\sigma(A),\mathbb{C})} = \sup_{z \in \sigma(A)} |p_n(z) - f(z)| \xrightarrow{n\to\infty} 0$$

$$|||\varPhi(p_n) - \varPhi(p_m)||| \overset{\text{isometry}}{=} \|p_n - p_m\| \xrightarrow{n,m\to\infty} 0$$

  Thus the operators $\varPhi(p_n)$ form a Cauchy sequence in $L(H)$ and since $L(H)$ is a Banach space, this sequence converges to:

$$\varPhi(f) := \lim_{n\to\infty} \varPhi(p_n)$$

iv) For $Au = \lambda u$ holds:

$$\varPhi(f)u = \lim_{n\to\infty} \varPhi(p_n)u = \lim_{n\to\infty} p_n(A)u = \lim_{n\to\infty} p_n(\lambda)u = f(\lambda)u$$

vi) Now we prove the spectral mapping theorem:
„$\subseteq$“: Assume $\mu \in \sigma(f(A))$, but $\mu \notin f(\sigma(A))$. Then holds $f - \mu \neq 0$ on $\sigma(A)$ and we can invert:

$$\frac{1}{f-\mu} \in C^0(\sigma(A),\mathbb{C})$$

Now follows:

$$\mathbb{1} = \varPhi(1) = \varPhi\left(\frac{1}{f-\mu}(f-\mu)\right) = \underbrace{\varPhi\left(\frac{1}{f-\mu}\right)}_{\in L(H)} \cdot \underbrace{\varPhi(f-\mu)}_{=f(A)-\mu\mathbb{1}}$$

So $f(A) - \mu\mathbb{1}$ has a bounded inverse in contradiction to the assumption $\mu \in \sigma(f(A))$.
„$\supseteq$“: Consider $\lambda \in \sigma(A)$. Choose polynomials $p_n \in \mathbb{P}_{\mathbb{C}}$ with $p_n \rightrightarrows f$. Then converges in $L(H)$:

$$p_n(A) - p_n(\lambda)\mathbb{1} \xrightarrow{n\to\infty} f(A) - f(\lambda)\mathbb{1}$$

Assume that $f(\lambda) \notin \sigma(f(A))$. Then $f(A) - f(\lambda)\mathbb{1}$ has a bounded inverse.
According to Theorem 2.5.3, the invertible operators are open in $L(H)$. Therefore there exists a $\delta \in \mathbb{R}_{>0}$ such that B has a bounded inverse for all $B \in B_\delta(f(A) - f(\lambda)\mathbb{1})$. In particular, the operators $p_n(A) - p_n(\lambda)\mathbb{1}$ have a bounded inverse for sufficiently large n. This is a contradiction to the spectral mapping theorem for polynomials 7.2.2.

v) Claim: $f \geq 0 \quad \Rightarrow \quad \varPhi(f) \geq 0$
Let $f \in C^0(\sigma(A),\mathbb{R})$ be real-valued and $f \geq 0$. Then $g := \sqrt{f} \in C^0(\sigma(A),\mathbb{R})$ and $f = g^2$.

$$\langle u,\varPhi(f)u\rangle = \langle u,\varPhi(g^2)u\rangle = \langle u,\varPhi(g)\varPhi(g)u\rangle = \langle \varPhi(\overline{g})u,\varPhi(g)u\rangle = \langle \varPhi(g)u,\varPhi(g)u\rangle \geq 0$$

$\square_{7.2.1}$

$\chi_\varOmega(A)$ would be the projector onto the invariant subspace corresponding to the spectrum in $\varOmega$. Formally we can compute:

$$(\chi_\varOmega(A))^* = \overline{\chi_\varOmega}(A) = \chi_\varOmega(A)$$
$$\chi_\varOmega(A)\chi_\varOmega(A) = \chi_\varOmega^2(A) = \chi_\varOmega(A)$$

This motivates, why we would like to form $f(A)$ for a bounded Borel function f on $\sigma(A)$.

7.3 Spectral Measures

Let $A \in L(H)$ be symmetric. Choose a $u \in H$ (fixed).

$$\varPhi_u : C^0(\sigma(A),\mathbb{R}) \to \mathbb{R} \subseteq \mathbb{C}$$
$$f \mapsto \langle u,\varPhi(f)u\rangle$$

$$|\varPhi_u(f)| = |\langle u,\varPhi(f)u\rangle| \leq |||\varPhi(f)||| \cdot \|u\|^2 = \|f\|_{C^0(\sigma(A),\mathbb{R})} \cdot \|u\|^2$$

Thus ϕ_u is a bounded linear functional on $C^0(\sigma(A),\mathbb{R})$. According to the Riesz representation theorem there exists a unique regular bounded Borel measure μ_u such that:

$$\langle u,f(A)u\rangle = \int_{\sigma(A)} f(\lambda)\,\mathrm{d}\mu_u(\lambda)$$

The measure μ_u is even positive, because if $f \geq 0$, set $g = \sqrt{f}$ to get:

$$\int_{\sigma(A)} f(\lambda)\, \mathrm{d}\mu_u(\lambda) = \langle u, f(A)\, u\rangle = \langle g(A)\, u, g(A)\, u\rangle \geq 0 \qquad \underset{f\in C^0(\sigma(A),\mathbb{R}),\ f\geq 0}{\forall}$$

Hence by approximation follows $\mu_u(\Omega) \geq 0$ for all Borel sets $\Omega \subseteq \sigma(A)$. So μ_u is a positive measure.
The resulting integral can be defined for a more general class of functions.
A *Borel function* f is a function, which is measurable for the Borel algebra, i.e. $f^{-1}(\Omega)$ is a Borel function for all open $\Omega \subseteq \mathbb{C}$.
We use the following notation: $\mathfrak{M}$ is the set of all Borel sets in $\sigma(A)$.
$\mathcal{B}(\sigma(A), \mathbb{R}) = L^\infty(\mathrm{d}\mu_u)$ are the bounded Borel functions on $\sigma(A)$. We always assume:

$$\sup_{\sigma(A)} |f| < \infty$$

We define:

$$\begin{aligned} \phi_u : \mathcal{B}(\sigma(A), \mathbb{R}) &\to \mathbb{R} \\ \phi_u(f) &:= \int_{\sigma(A)} f(\lambda)\, \mathrm{d}\mu_u(\lambda) \end{aligned}$$

7.3.1 Lemma

$$|\phi_u(f)| \leq \|f\|_{L^\infty} \cdot \|u\|^2$$

Proof

For $f \in \mathcal{B}(\sigma(A), \mathbb{R})$ choose $\varphi_n \in C^0(\sigma(A), \mathbb{R})$ such that $\varphi_n \to f$ converges point-wise and $\|\varphi_n\|_\infty \leq \|f\|_\infty$. (Approximate f by step-functions and then approximate the step functions by continuous functions.)
Due to $|\varphi_n| \leq C$ and

$$\int_{\sigma(A)} C \mathrm{d}\mu_u = C\mu_u(\sigma(A)) = C\langle u, \Phi(1)\, u\rangle = C\langle u, \mathbb{1}u\rangle = C\|u\|^2 < \infty$$

we can use the dominated convergence theorem:

$$\begin{aligned} \left|\int_{\sigma(A)} f \mathrm{d}\mu_u\right| \overset{\text{dominated}}{\underset{\text{convergence}}{=}} \lim_{n\to\infty} \left|\int_{\sigma(A)}^{\infty} \varphi_n \mathrm{d}\mu_n\right| &= \lim_{n\to\infty} |\langle u, \Phi(\varphi_n)\, u\rangle| \leq \\ \leq \lim_{n\to\infty} \|u\|^2 \cdot \|\Phi(\varphi_n)\| = \lim_{n\to\infty} \|u\|^2 \cdot \|\varphi_n\| &\leq \|f\| \cdot \|u\|^2 \end{aligned}$$

$\square_{7.3.1}$

Define using the Fréchet-Riesz theorem the unique Operator $\Phi(f)$ by:

$$\Phi_u(f) := \langle u, \Phi(f)\, u\rangle$$

By polarization we get:

$$B_f(u,v) = \Phi_{\frac{u+v}{2}}(f) - \Phi_{\frac{u-v}{2}}(f) - \mathbf{i}\Phi_{\frac{u+\mathbf{i}v}{2}}(f) + \mathbf{i}\Phi_{\frac{u-\mathbf{i}v}{2}}(f)$$

Alternatively define for $f \in C^0(\sigma(A),\mathbb{C})$:

$$\Phi_{u,v}(f) := \langle u, \Phi(f)\, v\rangle = \int_{\sigma(A)} f(\lambda)\, \mathrm{d}\mu_{u,v}(\lambda)$$

$$B_f(u,v) := \int_{\sigma(A)} f(\lambda)\, \mathrm{d}\mu_{u,v}(\lambda)$$

$\mathrm{d}\mu_{u,v}$ is a only a *complex-valued,* bounded, regular Borel measure.

7.3.2 Lemma

$B_f(u,v)$ is a *sesquilinear form*, i.e. linear in the second and anti-linear in the first argument, and it holds:

$$|B_f(u,v)| \le \|f\| \cdot \|u\| \cdot \|v\|$$

Proof

This follows from the polarization formula and Lemma 7.3.1. $\square_{7.3.2}$

7.3.3 Theorem

Let B be a bounded sesquilinear form, i.e.:

$$|B(u,v)| \le C \cdot \|u\| \cdot \|v\| \qquad \underset{u,v\in H}{\forall}$$

Then there is a unique operator $D \in L(H)$ with $|||D||| \le C$ such that:

$$B(u,v) = \langle u, Dv\rangle$$

Proof

For $v \in H$ the map

$$\psi := \overline{B(.,v)}$$

is a bounded linear form. According to the Fréchet-Riesz theorem 3.1.3 there exists a $w \in H$ such that for all $u \in H$ holds:

$$\psi(u) = \langle w, u\rangle$$

Then follows:

$$B(u,v) = \overline{\langle w,u\rangle} = \langle u,w\rangle$$

Thus D is uniquely determined by $Dv = w$. So $D : H \to H$ is linear and bounded by the open mapping principle 2.4.7, i.e. $D \in L(H)$ and for all $v \in H$ holds:

$$B(u,v) = \langle u, Dv \rangle$$

Choose $u = Dv$ to get:

$$\begin{aligned} B(Dv,v) &= \langle Dv, Dv \rangle = \|Dv\|^2 \\ &\leq C \cdot \|Dv\| \cdot \|v\| \end{aligned}$$

Therefore we have for all $v \in H$:

$$\begin{aligned} \|Dv\| &\leq C \cdot \|v\| \\ |||D||| &\leq C \end{aligned}$$

$\square_{7.3.3}$

We conclude: For $f \in \mathcal{B}(\sigma(A), \mathbb{C})$ we construct $B_f(u,v)$. Then there exists a $\Phi(f) \in L(H)$ such that for all $u,v \in H$ holds:

$$\langle u, \Phi(f)\, v \rangle = B_f(u,v)$$

So $\Phi : \mathcal{B}(\sigma(A), \mathbb{C}) \to L(H)$ gives a functional calculus on $\mathcal{B}(\sigma(A), \mathbb{C})$, i.e. we can calculate $f(A)$ for an arbitrary Borel function.

7.3.4 Theorem (Spectral theorem in functional calculus form)

Let $A \in L(H)$ be symmetric. Then there is a unique mapping $\Phi : \mathcal{B}(\sigma(A)) \to L(H)$ with the following properties:

i) Φ is an involutive algebra homomorphism, i.e.:

$$\begin{aligned} \Phi(f) \cdot \Phi(g) &= \Phi(f \cdot g) \\ \Phi(f)^* &= \Phi\left(\overline{f}\right) \end{aligned}$$

If $f \in C^0(\sigma(A), \mathbb{C})$, then $\Phi(f)$ agrees with the corresponding operator of the continuous functional calculus.

ii) $|||\Phi(f)||| \leq \|f\|_\infty$

iii) If $f_n \to f$ converges point-wise and it holds $\|f_n\|_\infty < C$, then $\Phi(f_n) \to \Phi(f)$ converges strongly, i.e. for all $u \in H$ converges in H:

$$\Phi(f_n)\, u \to \Phi(f)\, u$$

iv) From $Au = \lambda u$ follows:

$$\Phi(f)\, u = f(\lambda)\, u$$

v) If $f \geq 0$ holds, then $\Phi(f) \geq 0$ is positive semidefinite.

vi) If $B \in L(H)$ commutes with A, i.e. $[A,B] = AB - BA = 0$, then $[B, \Phi(f)] = 0$. We write also $f(A) = \Phi(f)$.

Note: There is no spectral mapping theorem.

Proof

i) Prove the homomorphism property by approximation:
First step: Assume $f \in C^0(\sigma(A),\mathbb{C})$ and $g \in \mathcal{B}(\sigma(A),\mathbb{C})$. Then there exists a series $g_n \in C^0$ such that $g_n \to g$ converges point-wise and $\|g_n\|_\infty < C$. Then follows the point-wise convergence:

$$f g_n \to f g$$

We use the notation:

$$\begin{aligned} & \phi_{u,v}(h) := \langle u,\Phi(h)\, v\rangle \\ \Rightarrow \qquad & \phi_{u,u}(h) = \phi_u(h) \end{aligned}$$

Since μ_u is a regular bounded Borel measure, we can apply the dominated convergence theorem:

$$\begin{aligned} \phi_{u,u}(f \cdot g) &\overset{\text{Definition}}{=} \int_{\sigma(A)} f \cdot g \mathrm{d}\mu_u \underset{\text{convergence}}{\overset{\text{dominated}}{=}} \lim_{n\to\infty} \int_{\sigma(A)} f \cdot g_n \mathrm{d}\mu_u = \lim_{n\to\infty} \phi_{u,u}(f,g_n) = \\ &= \lim_{n\to\infty} \langle u,\Phi(f \cdot g_n)\, u\rangle = \lim_{n\to\infty} \langle u,f(A) \cdot g_n(A)\, u\rangle = \\ &= \lim_{n\to\infty} \langle (f(A))^*\, u,g_n(A)\, u\rangle = \lim_{n\to\infty} \phi_{(f(A))^* u,u}(g_n) \end{aligned}$$

We know for all $u \in H$ using dominated convergence (see above):

$$\phi_{u,u}(g_n) \to \phi_{u,u}(g)$$

By polarization follows for all $u,v \in H$:

$$\phi_{v,u}(g_n) \to \phi_{v,u}(g)$$

This gives:

$$\phi_{u,u}(f \cdot g) = \lim_{n\to\infty} \phi_{(f(A))^* u,u}(g_n) = \phi_{(f(A))^* u,u}(g) = \langle (f(A))^*\, u,\Phi(g)\, u\rangle$$

$$\Rightarrow \quad \langle u,\Phi(f \cdot g)\, u\rangle = \langle u,f(A) \cdot g(A)\, u\rangle$$

Polarization yields:

$$\Phi(fg) = \Phi(f) \cdot \Phi(g)$$

Second Step: Consider $f,g \in \mathcal{B}$. We choose $f_n \in C^0$ with $f_n \to f$ and $\|f_n\| < C$. Then $f_n \cdot g \to f \cdot g$ converges point-wise.

$$\begin{aligned} \langle u,\Phi(f \cdot g)\, u\rangle &\underset{\text{convergence}}{\overset{\text{dominated}}{=}} \lim_{n\to\infty} \langle u,\Phi(f_n \cdot g)\, u\rangle \overset{\text{First step}}{=} \lim_{n\to\infty} \langle u,\Phi(f_n) \cdot \Phi(g)\, u\rangle = \\ &= \lim_{n\to\infty} \phi_{u,g(A)u}(f_n) = \phi_{u,g(A)u}(f) = \langle u,f(A)\, g(A)\, u\rangle \end{aligned}$$

$$\Rightarrow \qquad \langle u, (\Phi(fg) - \Phi(f)\, \Phi(g))\, u\rangle = 0 \qquad \underset{u\in H}{\forall}$$

By polarization follows:

$$\Phi(fg) = \Phi(f)\, \Phi(g)$$

The involution property follows similarly. $\square_{\text{i)}}$

iii) Claim: From point-wise convergence $f_n \to f$ and $\|f_n\| < C$ follows strong convergence $f_n(A) \to f(A)$.

a) From the dominated convergence theorem it is clear that holds:

$$\phi_u(f_n) \to \phi_u(f)$$
$$\langle u, f_n(A) u\rangle \to \langle u, f(A) u\rangle$$

Polarization gives for all $u, v \in H$:

$$\langle u, f_n(A) v\rangle \to \langle u, f(A) v\rangle$$

In other words for all $v \in H$ holds:

$$f_n(A) v \rightharpoonup f(A) v$$

b) It holds:

$$\begin{aligned}\|f_n(A) v\|^2 &= \langle f_n(A) v, f_n(A) v\rangle = \langle v, (f_n(A))^* f_n(A) v\rangle = \\ &= \left\langle v, \overline{f_n}(A) f_n(A) v\right\rangle = \left\langle v, |f_n(A)|^2 v\right\rangle \underset{\text{convergence}}{\overset{\text{dominated}}{\to}} \left\langle v, |f|^2(A) v\right\rangle = \\ &= \left\langle v, \overline{f}(A) f(A) v\right\rangle = \langle f(A) v, f(A) v\rangle = \|f(A) v\|^2\end{aligned}$$

c) Now apply the following general Lemma:

Lemma: $u_n \rightharpoonup u$ and $\|u_n\| \to \|u\|$ imply $u_n \to u$.

Proof:

$$\begin{aligned}\|u - u_n\| &= \langle u - u_n, u - u_n\rangle = \\ &= \|u\|^2 - 2\mathrm{Re}\underbrace{\langle u, u_n\rangle}_{\substack{\to\langle u,u\rangle \\ \text{because } u \rightharpoonup u_n}} + \underbrace{\|u_n\|^2}_{\substack{\to\|u\|^2 \\ \text{because } \|u_n\|\to\|u\|}} \to \|u\|^2 - 2\|u\|^2 + \|u\|^2 = 0\end{aligned}$$

$\square_{\text{Lemma}}$

d) This gives:

$$f_n(A) v \to f(A) v$$

$\square_{\text{iii)}}$

ii) Claim: $\|f(A)\| \le \|f\|_\infty$ for $f \in \mathcal{B}$.
Choose $f_n \in C^0$ which converge point-wise to f and $\|f_n\|_\infty < \|f\|$.

$$\|f(A) u\| \overset{\text{iii)}}{=} \lim_{n\to\infty} \|f_n(A) u\| \le \lim_{n\to\infty} \underbrace{|||f_n(A)|||}_{=\|f_n\|_\infty} \cdot \|u\| = \lim_{n\to\infty} \|f_n\|_\infty \cdot \|u\| = \|f\|_\infty \cdot \|u\|$$

$$\Rightarrow \qquad |||f(A)||| \le \|f\|_\infty$$

$\square_{\text{ii)}}$

iv) - vi) follow immediately by approximation.

$\square_{7.3.4}$

7.3.5 Remark

So far we considered Borel measures on $\sigma(A) \subseteq \mathbb{R}$. These measures can be extended to Borel measures on $\mathbb{R}$ by defining for a Borel set $\Omega \in \mathfrak{M}(\mathbb{R})$:

$$\mu(\Omega) := \mu(\Omega \cap \sigma(A))$$

$\Omega \cap \sigma(A)$ is a Borel set of $\sigma(A)$, since $\sigma(A)$ is closed.

Now let $M \subseteq \mathfrak{M}(\mathbb{R})$ be a Borel set. $f(A)$ is well defined for any $f \in \mathcal{B}(\mathbb{R})$. With the characteristic function χ_M of M define:

$$E_M := \chi_M(A)$$

Then we get:

$$\begin{aligned} E_M^* &= \overline{\chi_M}(A) = \chi_M(A) = E_M \\ E_M^2 &= \chi_M(A) \cdot \chi_M(A) = (\chi_M \cdot \chi_M)(A) = \chi_M(A) = E_M \end{aligned}$$

Thus E_M is symmetric and idempotent, in other words E_M is a projection operator.

The mapping $M \mapsto E_M$ is the spectral measure.

7.3.6 Definition (projection operator, spectral measure)

$P \in L(H)$ is a *projection operator* if $P^2 = P = P^*$.
An operator-valued *spectral measure* E is a mapping

$$\begin{aligned} E : \mathfrak{M}(\mathbb{R}^n) &\to L(H) \\ M &\mapsto E_M := E(M) \end{aligned}$$

with the following properties:

i) E_M is a projection operator for all $M \in \mathfrak{M}$.

ii) $E_\emptyset = 0$, $E_{\mathbb{R}^n} = \mathbb{1}$

iii) For $M = \dot{\bigcup}_{n=1}^{\infty} M_n$ the operator E_M is the strong limit of the partial sums $\sum\limits_{n=1}^{k} E_{M_n}$:

$$E_M = \operatorname*{s-lim}_{k\to\infty} \sum_{n=1}^{k} E_{M_n}$$

This means that for all $u \in H$ holds:

$$E_M u = \sum_{n=1}^{\infty} (E_{M_n} u)$$

The series does not necessarily converge in the operator norm!

iv) $E_M \cdot E_N = E_{M\cap N}$

v) For all $u \in H$, the mapping $M \mapsto \langle u, E_M u\rangle \in \mathbb{R}$ is a (real) bounded regular Borel measure.

$\operatorname{supp}(E)$ is the complement of the largest open set Ω with $E_\Omega = 0$, which exists due to the σ-additivity.
E is called a *compact* spectral measure if $\operatorname{supp}(E)$ is compact.

7.3.7 Theorem

Let $A \in L(H)$ be symmetric. Then the mapping

$$E : M \mapsto \chi_M(A)$$

is a spectral measure on $\mathbb{R}$ with $\operatorname{supp}(E) \subseteq \sigma(A)$.

Proof

We have to show the properties from the definition 7.3.6.
i) is clear.

$$\chi_\emptyset(A) = 0(A) = 0$$
$$\chi_\mathbb{R}(A) = \Phi(1) = \mathbb{1}$$

So ii) is shown.
iv) follows from:

$$\chi_M(A) \cdot \chi_N(A) = (\chi_M \cdot \chi_N)(A) = \chi_{M \cap N}(A)$$

For v) consider:

$$\langle u, E_M u \rangle = \langle u, \chi_M(A) u \rangle = \phi_u(\chi_M) = \int \chi_M \mathrm{d}\mu_u = \mu_u(M)$$

It remains to show iii) and $\operatorname{supp}(E) \subseteq \sigma(A)$.
For the later consider $\Omega \subseteq \varrho(A)$:

$$E_\Omega = \chi_\Omega(A) = \Phi(\chi_\Omega) \overset{\text{extension to } \mathcal{B}(\mathbb{R})}{=} \Phi\left(\chi_\Omega \chi_{\sigma(A)}\right) = \Phi\left(\chi_{\Omega \cap \sigma(A)}\right) = \Phi(0) = 0$$

Now show iii): From

$$M = \dot{\bigcup_{j=1}^{\infty}} M_j$$

follows with ponit-wise convergence:

$$\chi_M = \sum_{j=1}^{\infty} \chi_{M_j}$$

Theorem 7.3.4 iii) yields:

$$\operatorname*{s-lim}_{n \to \infty} \sum_{j=1}^{n} \underbrace{\chi_{M_j}(A)}_{=E_{M_j}} = \underbrace{\chi_M(A)}_{=E_M}$$

$\square_{7.3.7}$

Notation

$M \mapsto E_M$ is the spectral measure, which is projection operator valued.
$M \mapsto \langle u, E_M u \rangle = \mu_u(M) = \mu_{u,u}(M)$ is the real, bounded, regular Borel measure.
$M \mapsto \langle u, E_M v \rangle = \mu_{u,v}(M)$ is the complex, bounded, regular Borel measure.
Consider the integral:

$$\int_{\mathbb{R}} f(\lambda)\, \mathrm{d}\mu_u(\lambda)$$

$$\begin{aligned} \mathrm{d}\mu_u(\lambda) &= \mathrm{d}\langle u, E_\lambda u \rangle \\ \mathrm{d}\mu_{u,v}(\lambda) &= \mathrm{d}\langle u, E_\lambda v \rangle \end{aligned}$$

7.3.8 Lemma

Let E be a spectral measure on $\mathbb{R}^n$ and $M \in \mathfrak{M}(\mathbb{R}^n)$. Then holds for all $u,v \in H$:

$$\mathrm{d}\langle u, E_\lambda E_M v \rangle = \chi_M(\lambda)\, \mathrm{d}\langle u, E_\lambda v \rangle = \mathrm{d}\langle E_M u, E_\lambda v \rangle$$

Proof

For all $f \in \mathcal{B}(\mathbb{R}^n)$ we have to show:

$$\int_{\mathbb{R}^n} f(\lambda)\, \mathrm{d}\langle u, E_\lambda E_M v \rangle = \int_{\mathbb{R}^n} f(\lambda) \cdot \chi_M(\lambda)\, \mathrm{d}\langle u, E_\lambda v \rangle$$

By approximation, it suffices to show for all $\Omega \in \mathfrak{M}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \chi_\Omega(\lambda)\, \mathrm{d}\langle u, E_\lambda E_M v \rangle = \int_{\mathbb{R}^n} \chi_\Omega(\lambda)\, \chi_M(\lambda)\, \mathrm{d}\langle u, E_\lambda v \rangle$$

Since $\int \chi_M(x)\, \mathrm{d}\mu(x) = \mu(M)$, we get:

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_\Omega(\lambda)\, \mathrm{d}\langle u, E_\lambda E_M v \rangle = \langle u, E_\Omega E_M v \rangle &\overset{\text{property iv)}}{=} \langle u, E_{\Omega \cap M} v \rangle = \\ &= \int_{\mathbb{R}^n} \chi_{\Omega \cap M} \langle u, \mathrm{d}E_\lambda v \rangle = \int_{\mathbb{R}^n} \chi_\Omega \chi_M \langle u, \mathrm{d}E_\lambda v \rangle \end{aligned}$$

$\square_{7.3.8}$

We write:

$$\int_{\mathbb{R}^n} f(\lambda)\, \mathrm{d}\langle u, E_\lambda v \rangle =: \left\langle u, \left(\int_{\mathbb{R}^n} f(\lambda)\, \mathrm{d}E_\lambda \right) v \right\rangle$$

We will use this to define integration in $L(H)$.

7.3.9 Theorem

Let E be a spectral measure on $\mathbb{R}^n$ and $f \in \mathcal{B}(\mathbb{R}^n)$. Then the relations

$$\int f(\lambda)\,\mathrm{d}\langle u,E_\lambda v\rangle = \langle u,Av\rangle \qquad \underset{u,v\in H}{\forall}$$

define a unique normal operator $A \in L(H)$, which we also denote by:

$$A = \int f(\lambda)\,\mathrm{d}E_\lambda$$

Moreover:

$$A^* = \int \overline{f(\lambda)}\mathrm{d}E_\lambda$$

Proof

We define a bilinear form $B : H \times H \to \mathbb{C}$ by:

$$B(u,v) = \int_{\mathbb{R}^n} f(\lambda)\,\mathrm{d}\langle u,E_\lambda v\rangle$$

Then we have:

$$|B(u,u)| \le \int_{\mathbb{R}^n} |f(\lambda)| \underbrace{\mathrm{d}\langle u,E_\lambda u\rangle}_{\text{positive measure}} \le \|f\|_\infty \cdot \left\langle u, \underbrace{E_{\mathbb{R}^n}}_{=\mathbb{1}} u \right\rangle = \|f\|_\infty \cdot \|u\|^2$$

Polarization and estimation yields:

$$|B(u,v)| \le \|f\|_\infty \|u\| \cdot \|v\|$$

Thus by the Fréchet-Riesz theorem, there is a unique $A \in L(H)$ with:

$$B(u,v) = \langle u,Av\rangle$$

$$\begin{aligned} \langle u,Av\rangle &= \int f(\lambda)\,\mathrm{d}\langle u,E_\lambda v\rangle \\ \langle u,A^*v\rangle = \langle v,Au\rangle &= \int \overline{f(\lambda)}\mathrm{d}\langle u,E_\lambda v\rangle \\ \Rightarrow \qquad A^* &= \int \overline{f(\lambda)}\mathrm{d}E_\lambda \end{aligned}$$

$\square_{7.3.9}$

7.3.10 Theorem

Let E be a spectral measure on $\mathbb{R}^n$ and $f,g \in \mathcal{B}(\mathbb{R}^n)$. Then holds:

$$\left(\int_{\mathbb{R}^n} f(\lambda)\,\mathrm{d}E_\lambda\right)\left(\int_{\mathbb{R}^n} g(\lambda')\,\mathrm{d}E_{\lambda'}\right) = \int_{\mathbb{R}^n} f(\lambda)\,g(\lambda)\,\mathrm{d}E_\lambda$$

Proof

By approximation it suffices to consider the case $g = \chi_M$ for $M \in \mathfrak{M}(\mathbb{R}^n)$.

$$A := \int_{\mathbb{R}^n} f(\lambda)\,\mathrm{d}E_\lambda \qquad\qquad E_M = \int_{\mathbb{R}^n} \chi_M \mathrm{d}E_\lambda$$

For all $u,v \in H$ holds:

$$\begin{aligned}\langle u, A \cdot E_M v\rangle &= \int_{\mathbb{R}^n} f(\lambda)\,\mathrm{d}\langle u, E_\lambda E_M v\rangle \overset{(7.3.8)}{=} \int_{\mathbb{R}^n} f(\lambda)\,\chi_M(\lambda)\,\mathrm{d}\langle u, E_\lambda v\rangle = \\ &= \left\langle u, \int_{\mathbb{R}^n} (f \cdot \chi_M)(\lambda)\,\mathrm{d}E_\lambda v\right\rangle\end{aligned}$$

$$\Rightarrow \qquad A \cdot E_M = \int_{\mathbb{R}^n} f \cdot \chi_M \mathrm{d}E_\lambda$$

$\square_{7.3.10}$

Physicists write:

$$E_\lambda \cdot E_\mu = \delta_{\lambda-\mu} E_\lambda$$

This follows, because E_λ is idempotent and for $\lambda \neq \mu$ holds:

$$E_\lambda E_\mu = E_{\{\lambda\}} \cdot E_{\{\mu\}} = E_{\{\lambda\}\cap\{\mu\}} = E_\emptyset = 0$$

7.3.11 Theorem (spectral decomposition of a bounded symmetric operator)

There is a one-to-one correspondence between bounded symmetric operators $A \in L(H)$ and compact spectral measures E on $\mathbb{R}$ by:

$$A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda$$

This means for a given A with corresponding spectral measure $E_M = \chi_M(A)$ holds this equation. Conversely, if E is a compact spectral measure, then this equation defines a bounded symmetric Operator and $E_M = \chi_M(A)$.

Moreover holds:

i) $f(A) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda$

ii) $\sigma(A) = \mathrm{supp}(E)$

Proof

For a given A, let $E_M = \chi_M(A)$ be the corresponding spectral measure. Then holds for all $u,v \in H$ by construction:

$$\langle u, f(A)\,v\rangle = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\langle u, E_\lambda v\rangle$$

By the definition of $\int f(\lambda)\,\mathrm{d}E_\lambda$ follows:

$$f(A) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda$$

For the polynomial $f(\lambda) = \lambda$, i.e. $f(A) = A$, this gives:

$$A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda$$

If E is a compact spectral measure, $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda$ defines a normal operator with:

$$\left(\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda\right)^* = \int_{\mathbb{R}} \overline{f(\lambda)}\mathrm{d}E_\lambda$$

The compatibility with the spectral calculus follows from theorem 7.3.10.

Thus it remains to show $\sigma(A) \subseteq \operatorname{supp}(E)$. Consider $\mu \notin \operatorname{supp}(E)$. We want to show $\mu \in \varrho(A)$. Define the following bounded real function:

$$\begin{aligned} g(\lambda) &:= \frac{1}{\lambda - \mu}\chi_{\operatorname{supp}(E)} \\ f(\lambda) &:= \lambda - \mu \end{aligned}$$

$$B := \int_{\mathbb{R}} g \mathrm{d}E_\lambda \in L(H)$$

is a well-defined integral.

$$\begin{aligned} &\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda = A - \mu\mathbb{1} \\ &(A - \mu\mathbb{1})\,B = \left(\int_{\mathbb{R}} f(\lambda')\,\mathrm{d}E_{\lambda'}\right)\left(\int_{\mathbb{R}} g(\lambda)\,\mathrm{d}E_\lambda\right) = \int_{\mathbb{R}} f\cdot g \mathrm{d}E_\lambda = \\ &\qquad = \int_{\mathbb{R}} \chi_{\operatorname{supp}(E)} \underbrace{\mathrm{d}E_\lambda}_{=0 \text{ outside of } \operatorname{supp}(E)} = \int_{\mathbb{R}} \mathrm{d}E_\lambda = \mathbb{1} \end{aligned}$$

Thus $B = (A - \mu\mathbb{1})^{-1}$ and therefore $\mu \in \varrho(A)$. $\square_{7.3.11}$

7.3.12 Corollary

For $f \in \mathcal{B}(\mathbb{R})$ holds:

$$|||f(A)||| = \sup_{\sigma(A)} \operatorname{ess} |f|$$

Proof

„$\leq$“ was already proved in theorem 7.3.4 ii).

To prove equality, we first note that $f(A)$ is a normal operator, because it holds:

$$f(A) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda \qquad\qquad (f(A))^* = \int_{\mathbb{R}} \overline{f(\lambda)}\mathrm{d}E_\lambda$$

$$\begin{aligned} f(A)\cdot(f(A))^* &= \left(\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda\right)\left(\int_{\mathbb{R}} \overline{f(\lambda)}\mathrm{d}E_\lambda\right) = \\ &= \int_{\mathbb{R}} f(\lambda)\,\overline{f(\lambda)}\mathrm{d}E_\lambda = \int_{\mathbb{R}} \overline{f(\lambda)} f(\lambda)\,\mathrm{d}E_\lambda = (f(A))^* f(A) \end{aligned}$$

For a normal operator B holds:

$$|||B||| = r(B) = \sup_{x\in\sigma(B)} |x|$$

Now follows by theorem 7.3.11 ii):

$$|||f(A)||| = \sup_{x\in\sigma(f(A))} |x| = \sup(\operatorname{supp}(f(E))) = \sup_{\lambda\in\operatorname{supp}(E)} \operatorname{ess} |f(\lambda)|$$

$\square_{7.3.12}$

7.4 Simple Examples

7.4.1 Example: finite dimensions

Consider $H = \mathbb{C}^n$ and a symmetric operator $A \in L(\mathbb{C}^n)$. Choose an orthonormal eigenvector basis such that A has the matrix representation:

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The eigenvalues $\lambda_i \in \mathbb{R}$ are real, but there can be degeneracies, i.e. $\lambda_i = \lambda_j$ for some $i \neq j$.

$$A^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$$

Similarly we can compute polynomials of A.
The Stone-Weierstraß approximation yields for $f \in C^0(\sigma(A), \mathbb{C})$:

$$f(A) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix}$$

Since the spectrum

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$$

is a finite set, we have $C^0(\sigma(A)) = \mathcal{B}(\sigma(A))$. The spectral measure for $\Omega \subseteq \mathbb{C}$ is:

$$E_\Omega := \chi_\Omega(A) = \begin{pmatrix} \chi_\Omega(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \chi_\Omega(\lambda_n) \end{pmatrix}$$

Thus E_Ω is the projection operator on the eigenspaces, for which the eigenvalues λ lie in Ω.

$$\int f(\lambda)\,\mathrm{d}E_\lambda = \sum_{j=1}^{n} f(\lambda_j)\, E_{\{\lambda_j\}}$$

More specifically, let u_j be an orthonormal eigenvector basis, $Au_j = \lambda_j u_j$ and $\langle u_i, u_j \rangle = \delta_{ij}$. Then for any $v \in \mathbb{C}^n$ let $u_1^{(\lambda)}, \ldots, u_\mu^{(\lambda)}$ be all eigenvectors with the eigenvalue λ, i.e. $Au_k^{(\lambda)} = \lambda u_k^{(\lambda)}$, so

$$E_{\{\lambda\}} v = \sum_{k=1}^{\mu} u_k^{(\lambda)} \left\langle u_k^{(\lambda)}, v \right\rangle$$

is the projection on the eigenspace $\langle u^{(k)} \rangle$.

7.4.2 Example: compact operator

Let H be an infinite-dimensional Hilbert space and $A \in L(H)$ be symmetric and compact. According to the Hilbert-Schmidt theorem, there is an orthonormal eigenvector basis (u_n), i.e.:

$$Au_n = \lambda_n u_n$$

Then $\lambda_n \to 0$, because A is compact. The λ_n have finite-dimensional eigenspaces.

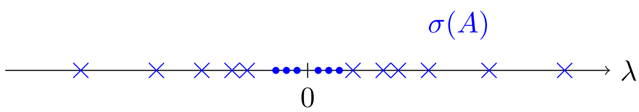


Figure 7.3: $\sigma(A)$ has only zero as limit point

$$A^2 u_n = \lambda_n^2 u_n$$
$$p(A)\, u_n = p(\lambda_n)\, u_n$$

This holds for any polynomial p. The Stone-Weierstraß approximation yields for $f \in C^0(\sigma(A))$:

$$f(A)\, u_n = f(\lambda_n)$$

The Riesz representation theorem gives

$$f(A)\, u_n = f(\lambda_n)$$

for all $f \in \mathcal{B}(\sigma(A))$ or even $f \in \mathcal{B}(\mathbb{R})$. Then follows:

$$E_\Omega u_n := \chi_\Omega(A)\, u_n = \chi_\Omega(\lambda_n)\, u_n$$

Thus E_Ω is the projection operator to all eigenspaces whose eigenvalues λ lie in Ω. But $E_{(-\varepsilon,\varepsilon)}$ has infinite rank for all $\varepsilon > 0$.

$$A = \sum_{\lambda \in \sigma(A)} \lambda E_{\{\lambda\}}$$

$$A_N := \sum_{\substack{\lambda\in\sigma(A)\\ |\lambda|>\frac{1}{N}}} \lambda E_{\{\lambda\}}$$

is a finite-dimensional approximation of A (cf. 5.8) in the sense:

$$\|A - A_N\| \xrightarrow{N\to\infty} 0$$

More precisely we have:

$$\|A - A_N\| \leq \frac{1}{N}$$

Now consider:

$$\mathbb{1} = \sum_{\lambda\in\sigma(A)} E_{\{\lambda\}}$$

$$E_N := \sum_{\substack{\lambda\in\sigma(A)\\ |\lambda|>\frac{1}{N}}} E_{\{\lambda\}}$$

This converges strongly, but it does not converge in the operator norm:

$$\|E - E_N\| = \left\|E_{\left[-\frac{1}{N},\frac{1}{N}\right]}\right\| = 1$$

7.4.3 Example: continuous spectrum

Consider the Hilbert space $H = L^2(\mathbb{R})$ and the function:

$$g(t) := \begin{cases} t & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

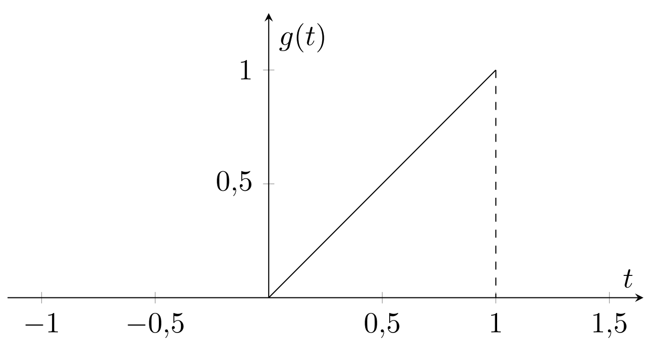


Figure 7.4: Plot of $g(t)$

$A \in L(H)$ defined by

$$(Au)(t) := g(t) \cdot u(t) = (T_g \cdot u)(t)$$

for $u \in H$ is a multiplication operator. From $|g(t)| \leq 1$ follows $\|A\| \leq 1$. As before we get:

$$A^2 = T_{g^2}$$

$$p(A) = T_{p(g)} \qquad \underset{\text{polynomial } p}{\forall}$$
$$f(A) = T_{f(g)} \qquad \underset{f \in \mathcal{B}(\mathbb{R})}{\forall}$$

Therefore we get:

$$E_\Omega = T_{\chi_\Omega(g)}$$

$$\begin{aligned}(\chi_\Omega(g))(t) &= \begin{cases} 1 & \text{if } g(t) \in \Omega \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{g^{-1}(\Omega)}\end{aligned}$$

In general for multiplication operators holds:

$$E_\Omega = T_{\chi_\Omega(g)} = T_{\chi_{g^{-1}(\Omega)}}$$

For $\Omega = (a,b) \subseteq (0,1)$ we get $g^{-1}(\Omega) = \Omega$ and thus $E_\Omega u = \chi_\Omega \cdot u$. If on the other hand $\Omega = \{0\}$, then holds:

$$g^{-1}(\Omega) = \mathbb{R} \setminus (0,1) = (-\infty,0] \cup [1,\infty)$$

Thus we get:

$$E_{\{0\}} u = \chi_{\mathbb{R}\setminus(0,1)} u$$

The spectrum of A is $\sigma(A) = [0,1]$. (Remember that the spectrum is always closed!)

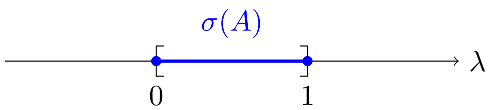


Figure 7.5: Continuous spectrum $\sigma(A)$ of A

Zero is an eigenvalue corresponding to an infinite-dimensional eigenspace, $Au = 0$ for $u\big|_{[0,1]} = 0$. Any $\lambda \in (0,1]$ is *not* an eigenvalue:

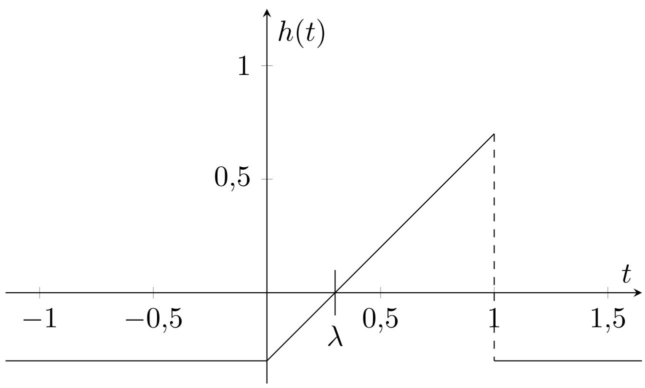


Figure 7.6: Plot of $g(t) - \lambda$

$$(A-\lambda)\,u = T_{g-\lambda}u$$

$$h := g-\lambda$$

$$\begin{aligned} & h(x)\cdot u(x) = 0 \\ \Leftrightarrow \quad & u = 0 \quad \underset{x\in\mathbb{R},\ h(x)\neq 0}{\forall} \\ \Leftrightarrow \quad & u = 0 \quad \text{almost everywhere} \\ \Leftrightarrow \quad & u = 0 \in L^2(\mathbb{R}) \end{aligned}$$

Thus the eigenvalue equation only has the trivial solution.

7.4.4 Example

Consider $H = L^2(\mathbb{R})$ and the multiplication operator $A = T_g$ for $g \in C_0^0(\mathbb{R})$. Then follows $E_\Omega = T_{g^{-1}(\Omega)}$ as before and $\sigma(A) = g(\mathbb{R})$.

That $\lambda \in \sigma(A)$ is an eigenvalue is equivalent to $g^{-1}(\{\lambda\})$ is a set of strictly positive Borel measure.

7.5 Essential and discrete spectrum

Let $A \in L(H)$ be symmetric. (The definitions are similar for normal operators or for unbounded self-adjoint operators). Let E be the corresponding spectral measure.

7.5.1 Definition (essential and discrete spectrum)

The essential spectrum $\sigma_{\text{ess}}(A)$ contains all $\lambda \in \mathbb{C}$ for which $\operatorname{rg}\left(E_{B_\varepsilon(\lambda)}\right) = \infty$ for all $\varepsilon \in \mathbb{R}_{>0}$.

The discrete spectrum $\sigma_{\text{disc}}(A)$ contains all $\lambda \in \sigma(A)$ for which exists a $\varepsilon \in \mathbb{R}_{>0}$ such that the rank of $E_{B_\varepsilon(\lambda)}$ is finite.

Note: $\lambda \in \sigma_{\text{ess}}(A)$ implies $\lambda \in \operatorname{supp}(E) = \sigma(A)$. Thus $\sigma(A) = \sigma_{\text{ess}}(A) \dot\cup \sigma_{\text{disc}}(A)$.

7.5.2 Example

Let A be a compact symmetric operator of infinite rank.

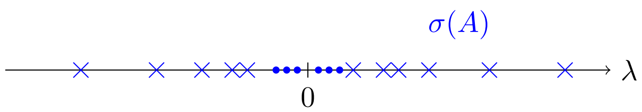


Figure 7.7: $\sigma(A)$ has only zero as limit point

Here we have:

$$\sigma_{\text{disc}} = \sigma(A) \setminus \{0\} \qquad\qquad \sigma_{\text{ess}} = \{0\}$$

7.5.3 Theorem (condition for discrete spectrum)

$\lambda \in \sigma_{\text{disc}}(A)$ holds if and only if both of the following conditions are satisfied:

i) λ is an isolated point of $\sigma(A)$, i.e. there exists a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(\lambda) \cap \sigma(A) = \{\lambda\}$.

ii) λ is an eigenvalue of finite multiplicity, i.e. $\ker(A - \lambda)$ is finite-dimensional.

Proof

„$\Leftarrow$“: If i) and ii) hold, then for an appropriately chosen $\varepsilon \in \mathbb{R}_{>0}$

$$E_{B_\varepsilon(\lambda)} = E_{\{\lambda\}}$$

is the projection operator on the finite-dimensional eigenspace.

„$\Rightarrow$“: Consider $\lambda \in \sigma_{\text{disc}}(A)$.

i) Choose $\varepsilon \in \mathbb{R}_{>0}$ such that $E_{B_\varepsilon(\lambda)}$ has finite rank.

$$J := E_{B_\varepsilon(\lambda)}(H)$$

is a finite-dimensional subspace of H. For $u \in J$ holds:

$$Au = AE_{B_\varepsilon(\lambda)}u = E_{B_\varepsilon(\lambda)}Au$$

Therefore follows $Au \in J$ and thus $A\big|_J : J \to J$ is a symmetric operator on a finite-dimensional Hilbert space. Diagonalize as in linear algebra:

$$\sigma\left(A\big|_J\right) = \{\lambda_1, \ldots, \lambda_n\} = \sigma(A) \cap B_\varepsilon(\lambda)$$

The λ_i lie discrete and thus are isolated.

ii) follows, because the eigenspace of A is the same as that of $A\big|_J$, which is finite-dimensional.

$\square_{7.5.3}$

7.5.4 Theorem (Weyl criterion)

i) $\lambda \in \sigma(A)$ holds if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in H such that for all $n \in \mathbb{N}$ holds $\|u_n\| = 1$ and:

$$(A - \lambda)\, u_n \xrightarrow{n\to\infty} 0$$

One also says, that λ is an *approximate eigenvalue*, because this can also be expressed as follows: For any $\varepsilon \in \mathbb{R}_{>0}$ there exists a $u \in H$ with $\|u\| = 1$ and $\|(A - \lambda)\, u\| \leq \varepsilon$.

ii) $\lambda \in \sigma_{\text{ess}}(A)$ holds if and only if the (u_n) from above can be chosen as an orthonormal basis.

Proof

i) For $\lambda \in \varrho(A)$ the operator $A - \lambda$ is continuously invertible, i.e. $(A-\lambda)^{-1} \in L(H)$. So for all $u \in H$ holds:

$$\left\|(A-\lambda)^{-1} u\right\| \leq C \|u\|$$

Since $A - \lambda$ is bijective, this is equivalent to:

$$\|v\| \leq C \|(A-\lambda) v\| \qquad \underset{v \in H}{\forall}$$

This gives:

$$\|(A-\lambda) v\| \geq \frac{1}{C} \|v\|$$
$$\|(A-\lambda) u_n\| \geq \frac{1}{C} \|u_n\| = \frac{1}{C}$$

Thus $(A-\lambda) u_n$ cannot converge to zero and thus λ is no approximate eigenvalue.
For $\lambda \in \sigma(A)$ the operator $(A-\lambda)$ has no bounded inverse. Then either $(A-\lambda)$ has a non-trivial kernel, i.e. there exists a $u \in H$ with $\|u\| = 1$ and:

$$(A-\lambda) u = 0$$

In this case one can choose $u_n := u$.
If on the other hand $(A-\lambda)$ is injective, but has no bounded inverse, then exists a sequence (u_n) with $\|(A-\lambda) u_n\| \leq \frac{1}{n} \|u_n\|$. This means that λ is an approximate eigenvalue.

ii) This follows directly from theorem 7.5.3.

$\square_{7.5.4}$

7.6 The Stone Formula

Let $A \in L(H)$ be symmetric, so we have $\sigma(A) \subseteq \mathbb{R}$. Thus for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the resolvent

$$R_\lambda := (A-\lambda)^{-1} \in L(H)$$

exists.

Figure 7.8: $\lambda \notin \mathbb{R}$

$$A = \int_{\mathbb{R}} \mu \cdot \mathrm{d}E_\mu \qquad\qquad R_\lambda = \int_{\mathbb{R}} \frac{1}{\mu-\lambda} \mathrm{d}E_\mu$$

$\frac{1}{\mu-\lambda} \in \mathcal{B}(\mathbb{R})$ holds, because the pole is away from the real axis.

$$(A-\lambda) R_\lambda = \left(\int_{\mathbb{R}} (\mu-\lambda) \, \mathrm{d}E_\mu\right)\left(\int_{\mathbb{R}} \frac{1}{\mu-\lambda} \mathrm{d}E_\mu\right) = \int_{\mathbb{R}} \frac{\mu-\lambda}{\mu-\lambda} \mathrm{d}E_\mu = \int_{\mathbb{R}} \mathrm{d}E_\mu = E_{\mathbb{R}} = \mathbb{1}$$

7.6.1 Theorem

For $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_{>0}$ holds:

$$\frac{1}{2\pi\mathbf{i}}\int_a^b \left(R_{\lambda+\mathbf{i}\varepsilon} - R_{\lambda-\mathbf{i}\varepsilon}\right)\mathrm{d}\lambda = \frac{1}{2}\left(E_{(a,b)} + E_{[a,b]}\right) = \int_a^b \frac{1}{\mu-\lambda}\mathrm{d}E_\mu$$

This is a convenient method for computing the spectral measure or the projection operator on eigenspaces.

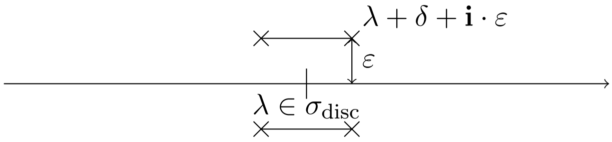


Figure 7.9: Calculating the spectral measure for a $\lambda \in \sigma_{\text{disc}}$

$$\underset{\delta\searrow 0}{\text{s-lim}}\ \underset{\varepsilon\searrow 0}{\text{s-lim}}\ \frac{1}{2\pi\mathbf{i}}\int_{\lambda-\delta}^{\lambda+\delta}\left(R_{\mu+\mathbf{i}\varepsilon} - R_{\mu-\mathbf{i}\varepsilon}\right)\mathrm{d}\mu = E_{\{\lambda\}}$$

Proof

Let $a < b \in \mathbb{R}$ be given.

$$\phi_\varepsilon(\mu) := \frac{1}{2\pi\mathbf{i}}\int_a^b \left(\frac{1}{\mu-\lambda-\mathbf{i}\varepsilon} - \frac{1}{\mu-\lambda+\mathbf{i}\varepsilon}\right)\mathrm{d}\lambda$$

Then holds $\phi_\varepsilon : \mathbb{R} \to \mathbb{C}$ and:

$$\phi_\varepsilon(A) = \int_{\mathbb{R}} \phi_\varepsilon(\mu)\,\mathrm{d}E_\mu = \frac{1}{2\pi\mathbf{i}}\int_a^b\int_{\mathbb{R}}\left(\underbrace{\frac{\mathrm{d}E_\mu}{\mu-\lambda-\mathbf{i}\varepsilon}}_{=R_{\lambda+\mathbf{i}\varepsilon}} - \underbrace{\frac{\mathrm{d}E_\mu}{\mu-\lambda+\mathbf{i}\varepsilon}}_{=R_{\lambda-\mathbf{i}\varepsilon}}\right)\mathrm{d}\lambda =$$

$$= \frac{1}{2\pi\mathbf{i}}\int_a^b\left(R_{\lambda+\mathbf{i}\varepsilon} - R_{\lambda-\mathbf{i}\varepsilon}\right)\mathrm{d}\lambda$$

Now analyze the limit $\varepsilon \to 0$.

$$\phi_\varepsilon(\mu) = \frac{-1}{2\pi\mathbf{i}}\left(\ln(\lambda-\mu+\mathbf{i}\varepsilon) - \ln(\lambda-\mu-\mathbf{i}\varepsilon)\right)\Big|_{\lambda=a}^{\lambda=b}$$

The logarithm is cut at the negative real axis.

$$\ln(z) = \ln(|z|) + \mathbf{i}\arg(z) \qquad\qquad z = |z|\,e^{\mathbf{i}\arg(z)}$$

The argument of z lies in the range $(-\pi,\pi)$.

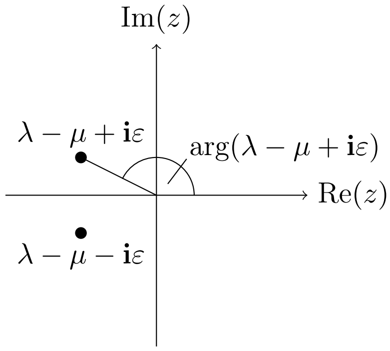


Figure 7.10: $-\pi < \arg(z) < \pi$

Thus we get:

$$\lim_{\varepsilon \searrow 0} \left(\ln\left(\lambda - \mu + \mathbf{i}\varepsilon\right) - \ln\left(\lambda - \mu - \mathbf{i}\varepsilon\right)\right) = \begin{cases} 0 & \text{if } \lambda - \mu > 0 \\ \pi\mathbf{i} & \text{if } \lambda - \mu = 0 \\ 2\pi\mathbf{i} & \text{if } \lambda - \mu < 0 \end{cases}$$

Then follows:

$$\phi(\mu) := \lim_{\varepsilon \searrow 0} \phi_\varepsilon(\mu) = \frac{-1}{2\pi\mathbf{i}} \begin{cases} 0 & \text{if } \mu \notin [a,b] \\ -\pi\mathbf{i} & \text{if } \mu \in \{a,b\} \\ -2\pi\mathbf{i} & \text{if } \mu \in (a,b) \end{cases} = \begin{cases} 0 & \text{if } \mu \notin [a,b] \\ \frac{1}{2} & \text{if } \mu \in \{a,b\} \\ 1 & \text{if } \mu \in (a,b) \end{cases}$$

Thus $\phi_\varepsilon(\mu) \to \phi(\mu)$ converges point-wise.

Idea:

$$\phi_\varepsilon(A) \to \phi(A) = \frac{1}{2}\left(E_{[a,b]} + E_{(a,b)}\right)$$

But how does this converge?
Consider weak convergence:

$$\langle u, \phi_\varepsilon(A) u\rangle = \int_{\mathbb{R}} \phi_\varepsilon(\mu) \underbrace{\mathrm{d}\langle u, E_\mu u\rangle}_{=\mathrm{d}\mu_u = \mathrm{d}\mu_{u,u}}$$

$\mathrm{d}\mu_u$ is a bounded regular real Borel measure. From $|\phi(\mu)| \leq 1$ follows for small enough $\varepsilon \in \mathbb{R}_{>0}$ now $|\phi_\varepsilon(\mu)| \leq 2$. Because our Borel measure is bounded, 2 is an integrable function, i.e. $2 \in L^1(\mathbb{R}, \mathrm{d}\mu_u)$. Therefore we can use the bounded convergence theorem to get:

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \phi_\varepsilon(\mu)\, \mathrm{d}\langle u, E_\mu u\rangle = \int_{\mathbb{R}} \phi(\mu)\, \mathrm{d}\langle u, E_\mu u\rangle = \langle u, \phi_u(A) u\rangle$$

What about strong convergence?
We want to show for all $u \in H$ the convergence $\phi_\varepsilon(A) u \to \phi(A) u$ in H, or equivalently:

$$\begin{aligned} & (\phi_\varepsilon - \phi)(A) u \to 0 \\ \Leftrightarrow \quad & \|(\phi_\varepsilon - \phi)(A) u\| \to 0 \end{aligned}$$

$$\|(\phi_\varepsilon - \phi)(A) u\|^2 = \langle(\phi_\varepsilon - \phi)(A) u, (\phi_\varepsilon - \phi)(A) u\rangle = \langle u, ((\phi_\varepsilon - \phi)(A))^* (\phi_\varepsilon - \phi)(A) u\rangle =$$

$$= \left\langle u, \left(\overline{\phi}_\varepsilon - \overline{\phi}\right)(A)\left(\phi_\varepsilon - \phi\right)(A)\, u\right\rangle = \left\langle u, \left|\phi_\varepsilon - \phi\right|^2 (A)\, u\right\rangle =$$
$$= \int_{\mathbb{R}} \underbrace{\left|\phi_\varepsilon - \phi\right|^2(\mu)}_{\to 0 \text{ point-wise}} \underbrace{\mathrm{d}\left\langle u, E_\mu u\right\rangle}_{\substack{\text{point-wise regular} \\ \text{Borel measure}}} \xrightarrow[\substack{\text{dominated} \\ \text{convergence}}]{\varepsilon \searrow 0} 0$$

Therefore it converges strongly. $\square_{7.6.1}$

8 Spectral Theorem for bounded normal operators

$A \in L(H)$ is normal if it commutes with its adjoint, i.e. $[A,A^*] = 0$. Before we considered symmetric $A \in L(H)$. Then for a complex valued function f the operator $f(A)$ is normal, but in general not symmetric, because:

$$(f(A))^* = \overline{f}(A) \overset{\text{in general}}{\neq} f(A)$$

$$f(A) \cdot (f(A))^* = \left(f \cdot \overline{f}\right)(A) = \left(\overline{f} \cdot f\right)(A) = (f(A))^* \cdot f(A)$$

The basic idea is:

$$\frac{1}{2}(A + A^*) =: B \qquad\qquad \frac{1}{2\mathbf{i}}(A - A^*) =: C$$

$A = B + \mathbf{i}C$, B and C are symmetric and $[B,C] = 0$.

8.1 Theorem

Let H be a complex separable Hilbert space, $A_i \in L(H)$ for $i \in \{1,\ldots,n\}$ be symmetric operators, which commute pair wise, i.e. $[A_i,A_j] = 0$ for all $i,j \in \{1,\ldots,n\}$ and

$$K := \prod_{i=1}^{n} \underbrace{\left[-\left|\left|\left|A_i\right|\right|\right|, \left|\left|\left|A_i\right|\right|\right|\right]}_{\supseteq \sigma(A_i)} \subseteq \mathbb{R}^n$$

be compact. Then there is a mapping

$$\Phi : C^0(K,\mathbb{C}) \to L(H)$$

(notation: $\Phi(f) = f(A_1,\ldots,A_n)$) with the following properties:

i) Φ is an involutive algebra homomorphism.

ii) $\left|\left|\left|\Phi(f)\right|\right|\right| \leq \|f\|_\infty = \sup_K |f|$

iii) $\Phi(\mathrm{pr}_i) = A_i$ for the projection maps:

$$\begin{aligned} \mathrm{pr}_i &: \mathbb{R}^n \to \mathbb{R} \\ (x_1,\ldots,x_n) &\mapsto x_i \end{aligned}$$

Proof

Let E_i be the spectral measure of the operator A_i.

$$E_i(M) = \chi_M(A_i)$$

Let $M \subseteq K$ be a cube, i.e. $M = M_1 \times \ldots \times M_n$. Define:

$$\chi_M(A_1,\ldots,A_n) := \chi_{M_1}(A_1) \cdot \ldots \cdot \chi_{M_n}(A_{M_n})$$

- Now holds $\left[\chi_{M_i}(A_i), \chi_{M_j}(A_j)\right] = 0$, because from

  $$[A_i,A_j] = 0$$

  follows via induction for any polynomials p,q:

  $$[p(A_i), q(A_j)] = 0$$

  With the Stone-Weierstraß and the Riesz representation theorem follows for all Borel functions $f,g \in \mathcal{B}(\mathbb{R})$:

  $$[f(A_i), g(A_j)] = 0$$

- $\chi_M(A_1,\ldots,A_n)$ is a projection operator.

  $$\begin{aligned}(\chi_M(A_1,\ldots,A_n))^* &= \overline{\chi_{M_n}}(A_n) \cdot \ldots \cdot \overline{\chi_{M_1}}(A_1) = \\ &= \chi_{M_1}(A_1) \cdot \ldots \cdot \chi_{M_n}(A_n) = \chi_M(A_1,\ldots,A_n)\end{aligned}$$

  $$\begin{aligned}&\chi_M(A_1,\ldots,A_n) \cdot \chi_{M'}(A_1,\ldots,A_n) \\ &\qquad = \chi_{M_1}(A_1) \cdot \ldots \cdot \chi_{M_n}(A_n) \cdot \chi_{M_1'}(A_1) \cdot \ldots \cdot \chi_{M_n'}(A_n) = \\ &\qquad = \chi_{M_1 \cap M_1'}(A_1) \cdot \ldots \cdot \chi_{M_n \cap M_n'}(A_n) = \chi_{M \cap M'}(A_1,\ldots,A_n)\end{aligned}$$

- Let $f = \sum_\alpha a_\alpha \chi_{M_\alpha}$ be a step function, meaning that the M_α are disjoint cubes and $a_\alpha \in \mathbb{C}$. Define:

  $$\varPhi(f) = \sum_\alpha a_\alpha \chi_{M_\alpha}(A_1,\ldots,A_n)$$

  Claim: This definition is well-defined, i.e. it does not depend on the decomposition of f into cells.

  Proof: Suppose we have:

  $$f = \sum_{\alpha=1}^{N} a_\alpha \chi_{M_\alpha} = \sum_{\beta=1}^{\tilde{N}} \tilde{a}_\beta \chi_{\tilde{M}_\beta}$$

  Choose a joint refinement. In fact, it suffices to consider the case that $\tilde{M}_\beta$ is already a refinement of M_α. Thus $M_\alpha = \dot{\bigcup}_{\beta \in I_\alpha} M_\beta$ and the I_α form a partition of $\left\{1,\ldots,\tilde{N}\right\}$. Using the properties of the E_i, a direct computation gives:

  $$\chi_{M_\alpha} = \sum_{\beta \in I_\alpha} \chi_{\tilde{M}_\alpha}$$

  Substitute this in the formula for f and reoreder the sums, to the that the definition is well-defined. $\square_{\text{Claim}}$

- Verify the properties i) and ii) for step functions: By direct computation follows:

$$(\Phi(f))^* = \Phi\left(\overline{f}\right)$$

$$\Phi(f)\cdot\Phi(g) = \left(\sum_\alpha a_\alpha \chi_{M_\alpha}\right)\left(\sum_\beta a_\beta \chi_{M_\beta}\right) \overset{\text{as above}}{=} \sum_{\alpha,\beta} a_\alpha b_\beta \chi_{M_\alpha \cap M_\beta} = \Phi(f\cdot g)$$

$$\|\Phi(f)\| = \left\|\sum_\alpha a_\alpha \chi_{M_\alpha}\right\| \leq \left(\max_\alpha |a_\alpha|\right)\cdot \underbrace{\left\|\sum_\alpha \chi_{M_\alpha}\right\|}_{\leq 1} \leq \|f\|_\infty$$

- Now consider $f \in C^0(K,\mathbb{C})$. There is a sequence of step functions f_k such that $f_k \rightrightarrows f$ converges uniformly.

$$\|\Phi(f_k) - \Phi(f_l)\| = \Phi(f_k - f_l) \overset{\text{ii)}}{\leq} \sup|f_k - f_l| \xrightarrow{k,l\to\infty} 0$$

  Since H is complete, $\Phi(f_k)$ converges in $L(H)$ and we define $\Phi(f) := \lim_{k\to\infty}\phi(f_k)$. Then the properties i) and ii) remain true by continuity.

- Compute $\Phi(\mathrm{pr}_i)$. For this let $f_k = \sum_\alpha a_\alpha \chi_{M_\alpha}$ be a step function with $f_k(x) \rightrightarrows x$ and set $\mathrm{pr}_i^k(x) = f_k(x_i)$, which implies $\mathrm{pr}_i^k \rightrightarrows \mathrm{pr}_i$.

$$\begin{aligned}\Phi\left(\mathrm{pr}_i^k\right) &= \sum_\alpha a_\alpha \chi_{\mathbb{R}\times\ldots\times \underbrace{M_\alpha}_{i\text{-th position}} \times\ldots\times\mathbb{R}}(A_1,\ldots,A_n) = \\ &= \prod_{j\neq i}\underbrace{\chi_{\mathbb{R}}(A_j)}_{=\mathbb{1}}\cdot\sum_\alpha a_\alpha \chi_{M_\alpha}(A_i) = \chi_{f_k}(A_i) \xrightarrow{\text{in } L(H)} A_i\end{aligned}$$

$\square_{8.1}$

We know $\operatorname{supp}(\chi(A_j)) = \sigma(A_j) \subseteq [-|||A_j|||, |||A_j|||]$.

$$K := [-|||A_1|||, |||A_1|||] \times [-|||A_2|||, |||A_2|||] \subseteq \mathbb{C}$$

Goal: Construct a spectral measure $\chi_M(A_1,A_2)$ on K.

- For $M = M_1 \times M_2$ ("cubes") we set:

$$\chi_{M_1\times M_2}(A_1,A_2) = \chi_{M_1}(A)\cdot\chi_{M_2}(A_2)$$

- For step functions

$$f = \sum_{\alpha=1}^N c_\alpha \chi_{M_1^\alpha \times M_2^\alpha}$$

  we set:

$$\Phi(f) = \sum_{\alpha=1}^N c_\alpha \chi_{M_1^\alpha\times M_2^\alpha}(A_1,A_2)$$

- For $f \in C^0(K)$ we choose step functions f_n such that $f_n \rightrightarrows f$ converges on K.

$$\Phi(f) := \lim_{n\to\infty}\Phi(f_n)$$

  This convergence is in $L(H)$.

8.2 Theorem

Now let $A \in L(H)$ be normal, i.e. $[A,A^*] = 0$, and define the symmetric bounded operators:

$$A_1 := \frac{1}{2}(A + A^*) \qquad\qquad A_2 := \frac{1}{2\mathbf{i}}(A - A^*)$$

Then follows $A = A_1 + \mathbf{i}A_2$ and $[A_1,A_2] = 0$, which implies $[\chi_{M_1}(A_1),\chi_{M_2}(A_2)] = 0$ for all sets $M_1,M_2 \subseteq \mathbb{R}$.

$$K := [-|||A_1|||, |||A_1|||] \times [-|||A_2|||, |||A_2|||] \subseteq \mathbb{C}$$

Then there exists exactly one map

$$\Phi : C^0(K,\mathbb{C}) \to L(H)$$

with the following properties:

i) Φ is an involutive algebra homomorphism.

ii) $|||\Phi(f)||| \leq \|f\|_\infty$

iii) $f(z) = z$ for $z \in K$ already implies $\Phi(f) = A$.

iv) $Au = \lambda u$ implies $\Phi(f)u = f(\lambda)u$

v) If f is real-valued, then $\Phi(f)$ is symmetric.

vi) $f \geq 0$ implies $\Phi(f) \geq 0$.

vii) For a $T \in L(H)$ with $[T,A] = [T,A^*] = 0$ follows for all $f \in C^0$:

$$[T,\Phi(f)] = 0$$

Proof

$$\begin{aligned}\mathrm{pr}_1(x_1,x_2) &= x_1\\ \Phi(\mathrm{pr}_1) &= A_1\end{aligned}$$

Choose step functions f_n of one variable, such that $f_n(x) \rightrightarrows x$ on $[-|||A_1|||, |||A_1|||]$. Then the functions

$$g_n(x_1,x_2) := f_n(x_1)$$

converge uniformly to pr_1 on K.

$$\Phi(g_n) = \sum_{\alpha=1}^{N} c_\alpha \underbrace{\chi_{M_1^\alpha \times [-|||A_2|||,|||A_2|||]}}_{=\chi_{M_1^\alpha}(A_1)\cdot \underbrace{\chi_{[-|||A_2|||,|||A_2|||]}(A_2)}_{=1}} = \sum_{\alpha=1}^{N} c_\alpha \chi_{M_1^\alpha}(A_1) = f_n(A_1) \to A_1$$

This converges follows from the functional calculus for *a symmetric operator*.

Choose Φ as in Theorem 8.1 for the commuting operators A_1 and A_2. Then i), ii) and v) follow immediately.

vi) For $f \geq 0$ there exists a $g \in C^0(K,\mathbb{R})$ with $f = g^2$.

$$\langle u,\phi(f)u\rangle = \langle u,\phi(g)\cdot\phi(g)u\rangle = \langle \phi(g)u,\phi(g)u\rangle \geq 0$$

vii) From $[T,A_1] = 0 = [T,A_2]$ follows:

$$[T,\chi_M(A_1)] = 0 = [T,\chi_M(A_2)]$$

This gives by approximation

$$[T,\chi_M(A_1,A_2)] = 0$$

for all $M \subseteq \mathbb{R}^2 \cong \mathbb{C}$.

iii) From $f(z) = z$ follows $\Phi(f) = A$.

$$\begin{aligned} z &= x_1 + \mathbf{i}x_2 \\ f(x_1,x_2) &= x_1 + \mathbf{i}x_2 \end{aligned}$$

$$\Rightarrow \qquad \Phi(f) = \Phi(\mathrm{pr}_1) + \mathbf{i}\Phi(\mathrm{pr}_2) = A_1 + \mathbf{i}A_2 = A$$

iv) We want to show $Au = \lambda u$ implies $\Phi(f)u = f(\lambda)u$. Consider $u \in H$ with $Au = \lambda u$.

Claim: $A^*u = \overline{\lambda}u$

Proof: It holds:

$$A(A^*u) = A^*Au = A^*\lambda u = \lambda(A^*u)$$

Thus A^* maps the eigenspace $\ker(A-\lambda)$ to itself, which implies:

$$A^*u - \overline{\lambda}u \in \ker(A-\lambda)$$

For $v \in \ker(A-\lambda)$ holds:

$$\left\langle v,\left(A^* - \overline{\lambda}\right)u\right\rangle = \langle (A-\lambda)v,u\rangle = 0$$

Thus we get $\left(A^* - \overline{\lambda}\right)u \in \ker(A-\lambda) \cap (\ker(A-\lambda))^{\perp} = \{0\}$. Now we have:

$$\left(A^* - \overline{\lambda}\right)u = 0$$

$\square_{\text{Claim}}$

So we have:

$$A_1u = \lambda_1 u \qquad A_2u = \lambda_2 u \qquad \lambda = \lambda_1 + \mathbf{i}\lambda_2$$

So $\Phi(p)u = p(\lambda)u$ holds for all polynomials p. The Stone-Weierstraß theorem in two dimensions gives the result.

$\square_{8.2}$

Now apply the Riesz representation theorem to extend the functional calculus to bounded Borel functions $\mathcal{B}(K)$.

8.3 Theorem

Let $A \in L(H)$ be normal. Then there exists a map

$$\Phi : \mathcal{B}(K,\mathbb{C}) \to L(H)$$

with the following properties:

i) Φ is an involutive algebra homomorphism.

ii) $|||\Phi(f)||| \leq \|f\|_{L^\infty(K)}$

iii) For $f \in C^0$, $\Phi(f)$ coincides with the continuous functional calculus.

iv) For point-wise converging $f_n \to f$ with $\|f\|_\infty < C$ converges $\Phi(f_n) \to \Phi(f)$ strongly.

v) $Au = \lambda u$ implies $\Phi(f)\,u = f(\lambda)\,u$

vi) If f is real-valued, then $\Phi(f)$ is symmetric.
$f \geq 0$ and implies $\Phi(f) \geq 0$.

vii) For a $T \in L(H)$ with $[T,A] = [T,A^*] = 0$ follows for all $f \in C^0$:

$$[T,\Phi(f)] = 0$$

Proof

The proof is the same as for the symmetric case. $\square_{8.3}$

8.4 Theorem (spectral theorem for bounded normal operators)

There is a one-to-one correspondence between bounded normal operators on H and compact spectral measures via:

$$A = \int_{\mathbb{R}^2 \cong \mathbb{C}} \lambda \mathrm{d}E_\lambda$$

Moreover holds:

i) $f(A) = \Phi(f) = \int_{\mathbb{R}^2} f(\lambda)\,\mathrm{d}E_\lambda$

ii) $\sigma(A) = \mathrm{supp}(E) \subseteq \mathbb{R}^2 \cong \mathbb{C}$

Proof

The proof is just as in the symmetric case, except for the property ii).

„$\mathrm{supp}(E) \supseteq \sigma(A)$“: Consider $\mu \notin \mathrm{supp}(E)$. Then

$$g(\lambda) := \frac{1}{\lambda - \mu} \cdot \chi_{\mathrm{supp}(E)}$$

is a bounded Borel function, since $|g(\lambda)| \leq \frac{1}{\varepsilon}$, where $B_\varepsilon(\mu) \cap \mathrm{supp}(E) = \emptyset$ and:

$$g(A) \cdot (A - \mu) = \int_{\mathbb{R}^2} \frac{\lambda - \mu}{\lambda - \mu} \mathrm{d}E_\lambda = \mathbb{1}$$

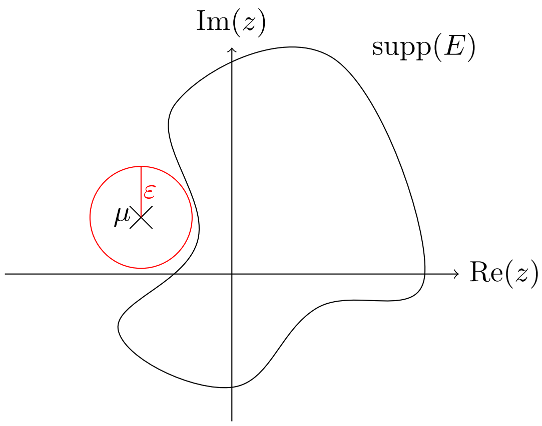


Figure 8.1: $B_\varepsilon(\mu) \cap \operatorname{supp}(E) = \emptyset$

Hence $(A - \lambda)$ has a bounded inverse and therefore $\lambda \notin \sigma(A)$.

„$\operatorname{supp}(E) \subseteq \sigma(A)$“: For $\mu_0 \in \varrho(A)$ we show $\mu_0 \notin \operatorname{supp}(E)$.
Since $\varrho(A)$ is open, there exists a $\varepsilon \in \mathbb{R}_{>0}$ with $B_\varepsilon(\mu_0) \subseteq \varrho(A)$.

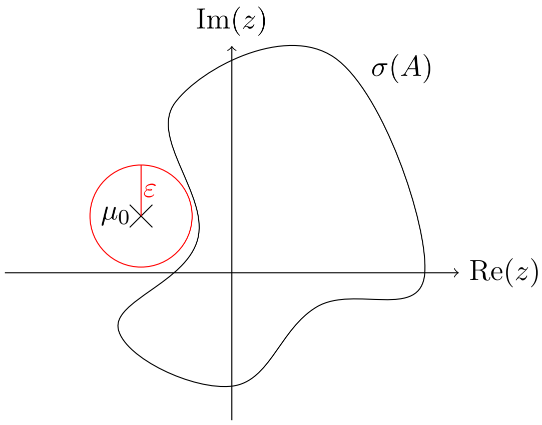


Figure 8.2: $B_\varepsilon(\mu_0) \subseteq \varrho(A)$

Lemma 8.5 states: Let $B \in L(H)$ be an operator with bounded inverse and $B_n \in L(H)$ a sequence with $B_n \to B$ in $L(H)$. Then B_n^{-1} exists for large enough n and $B_n^{-1} \to B$ converges in $L(H)$.
In particular, for $\mu_n \xrightarrow{n\to\infty} \mu_0$ converges also $(A - \mu_n)^{-1} \to (A - \mu_0)^{-1}$ in $L(H)$.
Consider now $\mu \in B_r(\mu_0)$ for any $r \in \mathbb{R}_{>0}$ and define:

$$B := (A - \mu) \cdot (A^* - \overline{\mu}) = \int |\lambda - \mu|^2 \, \mathrm{d}E_\lambda$$

Now choose a $\delta \in \mathbb{R}_{>0}$ to get:

$$\begin{aligned} B + \delta &= \int \left(|\lambda - \mu|^2 + \delta \right) \mathrm{d}E_\lambda \\ \Rightarrow \qquad (B + \delta)^{-1} &= \int \frac{1}{|\lambda - \mu|^2 + \delta} \mathrm{d}E_\lambda \in L(H) \end{aligned}$$

Similarly follows:

$$B^p = \int |\lambda - \mu|^{2p} \,\mathrm{d}E_\lambda$$
$$(B+\delta)^{-p} = \int \left(|\lambda - \mu|^2 + \delta\right)^{-p} \mathrm{d}E_\lambda$$

For $u \in H$ with $\|u\| = 1$ holds:

$$\left\langle u, (B+\delta)^{-p}\, u \right\rangle = \int_{\mathbb{R}^2} \frac{1}{\left(|\lambda - \mu|^2 + \delta\right)^p} \mathrm{d}\left\langle u, E_\lambda u\right\rangle$$

$\mathrm{d}\left\langle u, E_\lambda u\right\rangle$ is a point-wise bounded Borel measure.

$$\left|\left\langle u, (B+\delta)^{-p}\, u \right\rangle\right| \leq \underbrace{\|u\|^2}_{=1} \cdot \left\|(B+\delta)^{-1}(B+\delta)^{-(p-1)}\right\| \leq$$
$$\leq \ldots \leq \left\|\left\|(B+\delta)^{-1}\right\|\right\|^p \overset{\text{choose } r<\varepsilon}{\leq} \left\|\left\|B^{-1}\right\|\right\|^p$$

$$\Rightarrow \qquad \liminf_{\delta} \left|\left\langle u, (B+\delta)^{-p}\, u \right\rangle\right| \leq \left\|B^{-1}\right\|^p$$

Remember Fatou's lemma:

$$\int \liminf_{\delta} f_\delta \leq \liminf_{\delta} \int f_\delta$$

holds if $\lim_{\delta \searrow 0} f_\delta$ exists point-wise. (cf. RUDIN: *Real and complex analysis*)

Applying Fatou's lemma gives:

$$\int_{\mathbb{R}^2} \liminf_{\delta} \frac{1}{\left(|\lambda - \mu|^2 + \delta\right)^p} \mathrm{d}\left\langle u, E_\lambda u\right\rangle = \int_{\mathbb{R}^2} \frac{1}{|\lambda - \mu|^{2p}} \mathrm{d}\left\langle u, E_\lambda u\right\rangle \leq$$
$$\leq \liminf_{\delta} \int_{\mathbb{R}^2} \frac{1}{\left(|\lambda - \mu|^2 + \delta\right)^p} \mathrm{d}\left\langle u, E_\lambda u\right\rangle \leq \left\|\left\|B^{-1}\right\|\right\|^p$$

Thus we get:

$$\left(\int \frac{1}{|\lambda - \mu|^{2p}} \mathrm{d}\left\langle u, E_\lambda u\right\rangle\right)^{\frac{1}{p}} \leq \left\|\left\|B^{-1}\right\|\right\|$$

In other words, setting $g(\lambda) = \frac{1}{|\lambda - \mu|^2}$, we know for all $p \in \mathbb{N}_{\geq 1}$ and all $\mu \in B_{\frac{\varepsilon}{2}}(\mu_0)$:

$$\|g\|_{L^p(\mathrm{d}\langle u, E_\lambda u\rangle)} \leq \left\|\left\|B^{-1}\right\|\right\|$$

This implies that there exists an $\varepsilon' \in \mathbb{R}_{>0}$ such that $B_{\varepsilon'}(\mu_0)$ is a set with measure zero with respect to $\mathrm{d}\left\langle u, E_\lambda u\right\rangle$, since otherwise:

$$\left(\int \frac{1}{|\lambda - \mu|^{2p}} \mathrm{d}\left\langle u, E_\lambda u\right\rangle\right)^{\frac{1}{p}} \geq \inf_{\lambda \in B_{\varepsilon'}(\mu_0)} \frac{1}{|\lambda - \mu|^2} \cdot \underbrace{\left(\left\langle u, \mathrm{d}E_{B_{\varepsilon'}(\mu_0)} u\right\rangle\right)^{\frac{1}{p}}}_{>0} \xrightarrow{p \to \infty} \inf_{\lambda \in B_{\varepsilon'}(\mu_0)} \frac{1}{|\mu - \lambda|^2}$$

Since u is arbitrary (and ε' can be chosen uniformly in u) it follows that $E_{B_{\varepsilon'}(\mu_0)} = 0$ and thus $\mu_0 \notin \operatorname{supp}(E)$. $\square_{8.4}$

8.5 Lemma

Let $B \in L(H)$ be an operator with bounded inverse and $B_n \in L(H)$ a sequence with $B_n \to B$ in $L(H)$. Then B_n^{-1} exists for large enough n and $B_n^{-1} \to B$ converges in $L(H)$.

Proof

Use the Neumann series:

$$B_n^{-1} = (B + (B_n - B))^{-1} = \left(\mathbb{1} + B^{-1}(B_n - B)\right) B^{-1} = \sum_{k=0}^{\infty} \left(-B^{-1}(B_n - B)\right)^k B^{-1}$$

This converges absolutely, if $|||B_n - B|||$ is sufficiently small. Therefore holds:

$$\left|\left|\left|B_n^{-1} - B^{-1}\right|\right|\right| \leq \sum_{k=1}^{\infty} \left|\left|\left|B^{-1}\right|\right|\right|^{k+1} \cdot |||B_n - B|||^k \xrightarrow{|||B_n \to B|||} 0$$

$\square_{\text{Lemma}}$

8.6 Theorem

Let $A \in L(H)$ be normal and E the corresponding spectral measure. Then holds for all $\varepsilon \in \mathbb{R}_{>0}$:

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad E_{B_\varepsilon(\lambda)} \neq 0$$

Proof

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad \lambda \in \operatorname{supp}(E) \quad \overset{\text{definition of supp}(E)}{\Leftrightarrow} \quad E_{B_\varepsilon(\lambda)} \neq 0$$

$\square_{8.6}$

8.7 Theorem (spectral mapping theorem for normal operators)

Let $A \in L(H)$ be normal and $f \in C^0(\sigma(A), \mathbb{C})$. Then $\sigma(f(A)) = f(\sigma(A))$.

Note: This is not true in general for $f \in \mathcal{B}(\sigma(A), \mathbb{C})$.

Proof

i) „$\sigma(f(A)) \subseteq f(\sigma(A))$“: Since $\sigma(A)$ is compact and f continuous and therefore maps compact sets to compact sets, follows:

$$f(\sigma(A)) = \overline{f(\sigma(A))}$$

We show more generally:

$$\sigma(f(A)) \subseteq \overline{f(\sigma(A))}$$

for any *Borel* function $f \in \mathcal{B}(\sigma(A))$. Consider $\mu \notin \overline{f(\sigma(A))}$ and set:

$$g(\lambda) = \frac{1}{f(\lambda) - \mu} \cdot \chi_{\sigma(A)}$$

This is a bounded Borel function. Thus follows:

$$g(A) \cdot (f(A) - \mu) = \int_{\mathbb{R}^2} \frac{f(\lambda) - \mu}{f(\lambda) - \mu} \chi_{\sigma(A)} \mathrm{d}E_\lambda \overset{\sigma(A)=\mathrm{supp}(E)}{=} \mathbb{1}$$

Hence $f(A) - \mu$ has a bounded inverse $g(A)$ and thus $\mu \in \varrho(f(A))$, i.e. $\mu \notin \sigma(f(A))$. $\square_{\mathrm{i})}$

ii) „$f(\sigma(A)) \subseteq \sigma(f(A))$“: Consider $\mu \in \sigma(A)$ and show $f(\mu) \in \sigma(f(A))$.
From $\sigma(A) = \mathrm{supp}(E)$ follows for all $\varepsilon \in \mathbb{R}_{>0}$:

$$E_{B_\varepsilon(\mu)} \neq 0$$

Thus we may choose $u \neq 0$ with:

$$E_{B_\varepsilon(\mu)} u = u$$

Then holds:

$$\begin{aligned}
\|(f(A) - f(\mu)) u\|^2 &= \langle (f(A) - f(\mu)) u, (f(A) - f(\mu)) u \rangle = \\
&= \left\langle u, \left(\overline{f}(A) - \overline{f}(\mu)\right)(f(A) - f(\mu)) u \right\rangle = \\
&= \int_{\mathbb{R}^2} |f(\lambda) - f(\mu)|^2 \,\mathrm{d}\langle u, E_\lambda u \rangle = \\
&= \int_{\mathbb{R}^2} |f(\lambda) - f(\mu)|^2 \,\mathrm{d}\left\langle E_{B_\varepsilon(\mu)} u, E_\lambda E_{B_\varepsilon(\mu)} u \right\rangle = \\
&= \int_{B_\varepsilon(\mu)} |f(\lambda) - f(\mu)|^2 \,\mathrm{d}\langle u, E_\lambda u \rangle \leq \\
&\leq \sup_{B_\varepsilon(\mu)} |f(\lambda) - f(\mu)|^2 \int_{\mathbb{R}^2} \mathrm{d}\langle u, E_\lambda u \rangle \leq \\
&\leq \sup_{B_\varepsilon(\mu)} |f(\lambda) - f(\mu)|^2 \|u\|^2
\end{aligned}$$

Since f is continuous, there exists a sequence $u_n \in H$ with $\|u_n\| = 1$ such that holds:

$$\|(f(A) - f(\mu)) u_n\| \to 0$$

Hence $f(A) - f(\mu)$ has no bounded inverse and therefore follows $\mu \in \sigma(f(A))$.

$\square_{8.7}$

8.8 Corollary

For a normal $A \in L(H)$ and a $f \in C^0(\sigma(A))$ holds:

$$|||f(A)||| = \|f\|_{L^\infty(\sigma(A))}$$

Proof

From $(f(A))^* = \overline{f}(A)$ follows:

$$(f(A))^* f(A) = |f|^2(A) = f(A)(f(A))^*$$

Hence the operator $f(A)$ is normal.

$$\begin{aligned}|||f(A)||| = r(f(A)) = \sup\left\{|\mu| \,\middle|\, \mu \in \sigma(f(A))\right\} = \\ = \sup\left\{|\mu| \,\middle|\, \mu \in f(\sigma(A))\right\} = \sup\left\{|f(\lambda)| \,\middle|\, \lambda \in \sigma(A)\right\} = \|f\|_{L^\infty(\sigma(A))}\end{aligned}$$

$\square_{8.8}$

Thus the mapping

$$\varPhi : C^0(\sigma(A),\mathbb{C}) \to L(H)$$

is preserving the norm. Be careful to remember that

$$\varPhi : C^0\left(\mathbb{R}^2,\mathbb{C}\right) \to L(H)$$

is *not* preserving the norm. Instead holds:

$$|||f(A)||| \leq \|f\|_{L^\infty(\mathbb{R})}$$

8.9 Theorem

Let $A \in L(H)$ be normal and E the corresponding spectral measure. Then μ is an eigenvalue of A if and only if $E_{\{\mu\}} \neq 0$.

Proof

„$\Leftarrow$“: Assume that $E_{\{\mu\}} \neq 0$. Now choose a vector $u \neq 0$ with $E_{\{\mu\}}u = u$. Then holds:

$$\begin{aligned}\|(A-\mu)u\|^2 = \int_{\mathbb{R}^2} |\lambda-\mu|^2 \,\mathrm{d}\langle u,E_\lambda u\rangle = \int_{\mathbb{R}^2} |\lambda-\mu|^2 \,\mathrm{d}\left\langle E_{\{\mu\}}u,E_\lambda E_{\{\mu\}}u\right\rangle = \\ = \int_{\mathbb{R}^2} \underbrace{|\lambda-\mu|^2 \chi_{\{\mu\}}(\lambda)}_{=0} \,\mathrm{d}\langle u,E_\lambda u\rangle = 0\end{aligned}$$

„$\Rightarrow$“: Let u be an eigenvector.

$$Au = \mu u$$

Then holds for all $f \in \mathcal{B}\left(\mathbb{R}^2\right)$ after theorem 8.3 v):

$$f(A)u = f(\mu)u$$

Choose $f = \chi_{\{\mu\}}$ to get:

$$f(A) = \chi_{\{\mu\}}(A) = E_{\{\mu\}}$$

$$\Rightarrow \qquad E_{\{\mu\}}u = u$$

Hence follows $E_{\{\mu\}} \neq 0$. $\square_{8.9}$

9 Cyclic vectors, the spectral theorem in its multiplicative form

Let $A \in L(H)$ be normal.

9.1 Definition (cyclic vector)

A vector $u \in H$ is called *cyclic* (with respect to A) if holds:

$$\overline{\left\{f(A)\,u \middle| f \in C^0(\sigma(A),\mathbb{C})\right\}} = H$$

9.2 Theorem

Let $u \in H$ be a cyclic vector. Then there exists a unitary operator

$$\mathcal{U} : H \to L^2\big(\sigma(A), \underbrace{\mathrm{d}\langle u, E_\lambda u\rangle}_{=\mathrm{d}\mu_u}\big)$$

such that for $f \in L^2(\sigma(A), \mathrm{d}\langle u, E_\lambda u\rangle)$ and $g(\lambda) = \lambda$ holds:

$$\mathcal{U} A \mathcal{U}^{-1} f = g \cdot f$$

Proof

$$\alpha(f(A)\,u) + \beta(g(A)\,u) = (\alpha f + \beta g)(A)\,u$$

$$\Rightarrow \qquad I_u := \left\{f(A)\,u \middle| f \in C^0(\sigma(A),\mathbb{C})\right\} = \left\langle f(A)\,u \middle| f \in C^0(\sigma(A),\mathbb{C})\right\rangle$$

By assumption, I_u is dense in H. Define

$$\mathcal{U} : I_u \to L^2(\sigma(A), \mathrm{d}\mu_u)$$

by:

$$\mathcal{U}(f(A)\,u) = f$$

This is well-defined and an isometry, because:

$$\langle f(A)u, f(A)u\rangle = \int |f(\lambda)|^2 \underbrace{\mathrm{d}\langle u, E_\lambda u\rangle}_{=\mathrm{d}\mu_u} = \langle f, f\rangle_{L^2(\sigma(A), \mathrm{d}\mu_u)}$$

Moreover, the image of $\mathcal{U}$ is $C^0(\sigma(A), \mathbb{C})$ and this is dense in $L^2(\sigma(A), \mathrm{d}\mu_u)$. Therefore $\mathcal{U}$ can be uniquely extended by continuity to an unitary operator:

$$\mathcal{U} : H = \overline{I_u} \to \overline{C^0(\sigma(A), \mathbb{C})} = L^2(\sigma(A), \mathrm{d}\mu_u)$$

Compute now $\mathcal{U}A\mathcal{U}^{-1}$:

$$\mathcal{U}(f(A)u) = f$$

$$\mathcal{U}A\mathcal{U}^{-1}f = \mathcal{U}\underbrace{A}_{=g(A)}(f(A)u) = \mathcal{U}((g \cdot f)(A)u) = g \cdot f$$

Using a density argument one shows that this holds for any $f \in L^2$. $\square_{9.2}$

9.3 Examples

1. Let H be finite-dimensional and A symmetric with simple eigenvalues $\lambda_1, \ldots, \lambda_n$. In an eigenvector basis holds:

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

For $v = (1, 0, \ldots, 0)^{\mathrm{T}}$ follows:

$$f(A)v = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} v = f(\lambda_1)v$$

Therefore this v is not cyclic. Choose $u = (1, \ldots, 1)^{\mathrm{T}}$ to get:

$$f(A)u = \begin{pmatrix} f(\lambda_1) \\ \vdots \\ f(\lambda_n) \end{pmatrix}$$

Since $\lambda_i \neq \lambda_j$ holds for $i \neq j$, there are $f_i \in C^0(\sigma(A))$ such that $f_i(\lambda_i) = 1$ and $f_i(\lambda_j) = 0$ for $i \neq j$. With this holds $f_i(A)u = e_i$. Therefore holds:

$$\left\{ f(A)u \middle| f \in C^0 \right\} = H$$

2. Let A be as in 1., but with the degeneracy $\lambda_1 = \lambda_2$ and $u = (u_1, \ldots, u_n)^{\mathrm{T}}$. Then follows

$$f(A)u = \begin{pmatrix} f(\lambda_1)u_1 \\ \vdots \\ f(\lambda_n)u_n \end{pmatrix}$$

and the vector $v = (v_1,v_2,0,\ldots,0)^{\mathrm{T}}$ with

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \nparallel \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is not in:

$$\left\{ f(A)\, u \middle| f \in C^0 \right\}$$

Hence there is no cyclic vector.

Question: What can we do if there is a cyclic vector?

9.4 Lemma

Let $A \in L(H)$ be normal and A symmetric. Then there exists an orthogonal decomposition

$$H = \bigoplus_{j \in J} H_j$$

with a finite or countable J and to every $j \in J$ there is a cyclic vector $u_j \in H_j$, i.e.:

$$H_j = \overline{\left\{ f(A)\, u_i \middle| f \in C^0(\sigma(A),\mathbb{C}) \right\}}$$

Proof

Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal Hilbert basis. Choose $u_1 = e_1$ and define:

$$H_1 := \overline{\left\{ f(A)\, u_1 \middle| f \in C^0 \right\}} \subseteq H$$

If $H_1 = H$, we are done. Otherwise, let $i_0 \in \mathbb{N}$ be the smallest number with $e_{i_0} \notin H_1$ and set:

$$u_2 := e_{i_0} - \mathrm{pr}_{H_1}(e_{i_0}) = \mathrm{pr}_{H_1^\perp}(e_{i_0})$$
$$H_2 := \overline{\left\{ f(A)\, u_2 \middle| f \in C^0 \right\}} \subseteq H$$

For $H = \langle H_1,H_2 \rangle$ we stop the procedure. Otherwise choose i_1 as the smallest number such that $e_{i_1} \notin \langle H_1,H_2 \rangle$, and so on.
Proceeding inductively, we obtain that $J = \left\{ i_k \middle| k \in \mathbb{N} \right\}$ is finite or countable and for $j \in J$ we have:

$$H_j = \overline{\left\{ f(A)\, u_j \middle| f \in C^0 \right\}}$$

- $H_i \perp H_j$ for $i \neq j$:

  $$\langle f(A)\, u_i, g(A)\, u_j \rangle = \Big\langle \underbrace{(\overline{g} \cdot f)(A)\, u_i}_{\in H_i}, u_j \Big\rangle \overset{u_j \in H_i^\perp}{=} 0$$

  The result follows by using that $\{f(A)\, u_i\}$ and $\{g(A)\, u_j\}$ are dense in H_i respectively H_j.

- The H_i generate a dense subset of H: By construction we have:

  $$e_{i_k} \in \langle H_1,H_2,\ldots,H_{k+2} \rangle$$

  Since $i_k \geq k$ holds, every basis vector e_i is contained in $\langle H_1,H_2,\ldots,H_{i+2} \rangle$. Hence the algebraic span of the (e_i) is contained in the span of the $(H_i)_{i\in J}$.

$\square_{9.4}$

9.5 Theorem (spectral theorem in its multiplicative form)

Let $A \in L(H)$ be normal and H separable. Then there is a σ-compact measure space Ω with a finite measure μ and a unitary operator

$$\mathcal{U} : H \to L^2(\Omega, \mathrm{d}\mu)$$

such that for $g \in L^\infty(\Omega, \mu)$ holds:

$$\mathcal{U} A \mathcal{U}^{-1} f = g \cdot f$$

Proof

Choose an orthogonal decomposition

$$H = \bigoplus_{i \in J} H_i$$

with cyclic $u_i \in H_i$. The subspaces

$$H_i = \overline{\left\{ f(A)\, u_i \middle| f \in C^0 \right\}}$$

are invariant under A, i.e. $A_i := A\big|_{H_i} : H_i \to H_i$. Now we rescale u_i to get $\|u_i\| = 2^{-i}$.

$$\begin{aligned} \mathcal{U}_i : H_i &\to L^2\big(\sigma(A), \underbrace{\mathrm{d}\langle u_i, E_\lambda u_i\rangle}_{=\mathrm{d}\mu_{u_i}}\big) \\ f(A)\, u_i &\mapsto f \end{aligned}$$

This is just as before in theorem 9.2 unitary and for $g_i(\lambda) = \lambda$ holds:

$$\mathcal{U}_i A_i \mathcal{U}_i^{-1} f_i = g_i f_i$$

Now define:

$$\Omega := \sigma(A) \times J \qquad\qquad \Omega_i = \sigma(A) \times \{i\}$$

Thus holds:

$$\Omega = \dot{\bigcup_{i \in J}} \Omega_i$$

Define a measure:

$$\begin{gathered} \mu : \Omega_i \to \mathbb{R}_0^+ \\ \mu(U \times \{i\}) := \mu_{u_i}(U) \end{gathered}$$

Extend μ by σ-additivity to a unique measure on Ω. For $U \subseteq \Omega$ we write with appropriate $U_i \subseteq \Omega_i$:

$$U = \dot{\bigcup_{i \in I}} U_i$$

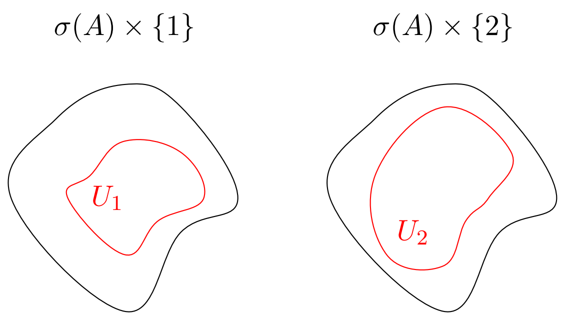


Figure 9.1: $U = \dot{\bigcup\limits_{i\in I}} U_i$

Define $\mu(U) := \sum\limits_{i\in J} \mu(U_i)$.

$$\mu(\Omega_i) = \mu_{u_i}(\sigma(A)) = \langle u_i, \underbrace{E_{\sigma(A)}}_{=\mathbb{1}} u_i\rangle = \|u_i\|^2 = 2^{-2i}$$

$$\mu(\Omega) = \sum_{i\in J} \mu(\Omega_i) = \sum_{i\in J} 2^{-2i} \leq 1$$

Thus μ is a bounded Borel measure.

$$\mathcal{U} := \bigoplus_{i\in J} \mathcal{U}_i : H \to L^2(\Omega, \mathrm{d}\mu)$$

is unitary.

$$\begin{array}{ccc} L^2(\Omega,\mathrm{d}\mu) & = & \bigoplus\limits_{i\in J} L^2(\Omega_i,\mathrm{d}\mu_i) \\ \mathcal{U}\uparrow & & \uparrow\mathcal{U}_i \\ H & = & \bigoplus\limits_{i\in J} H_i \end{array}$$

$$\left(\mathcal{U}A\mathcal{U}^{-1}\right) f = \bigoplus_{i\in J} g_i \underbrace{f_i}_{\in L^2(\Omega_i,\mathrm{d}\mu_i)}$$

Here $g_i(\{\lambda\}\times\{i\}) = \lambda$. Now

$$g := \bigoplus_{i\in J} g_i$$

is a bounded function:

$$\|g\|_{L^\infty} \leq \sup_{\lambda\in\sigma(A)} |\lambda| = r(A)$$

$\square_{9.5}$

9.6 The pure point spectrum and the absolutely continuous spectrum

Let $A \in L(H)$ be symmetric and H separable. Then

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E_\lambda$$

gave the decomposition:

$$\sigma(A) = \sigma_{\text{disc}}(A) \dot{\cup} \sigma_{\text{ess}}(A)$$

The spectral theorem in its multiplicative form gives another decomposition of the spectrum. There exists a operator

$$\mathcal{U} : H \to L^2(\Omega, \mathrm{d}\mu)$$

with $\mathcal{U}A\mathcal{U}^{-1}$ is the operator of multiplication by $g \in L^\infty(\Omega, \mathrm{d}\mu)$ and $\mathrm{d}\mu$ is a positive finite Borel measure on $\Omega = \sigma(A) \times J$. Since the spectrum is compact, it holds $\sigma(A) \subseteq [a,b] \subseteq \mathbb{R}$.

On Ω we also have the Lebesgue measure $\mathrm{d}x$. According to the Raden-Nikodym theorem (that we use without proof), $\mathrm{d}\mu$ can be decomposed as:

$$\mathrm{d}\mu = \mathrm{d}\mu_{\text{pp}} + \mathrm{d}\mu_{\text{ac}} + \mathrm{d}\mu_{\text{sing}}$$

$\mathrm{d}\mu_{\text{pp}}$ is the *pure point*, $\mathrm{d}\mu_{\text{ac}}$ the *absolutely continuous* and $\mathrm{d}\mu_{\text{sing}}$ the *singular* measure. It holds

$$\mathrm{d}\mu_{\text{ac}} = f(x)\,\mathrm{d}x$$

for a $f \in L^2(\Omega, \mathrm{d}x)$. $\mathrm{d}\mu_{\text{pp}}$ is a weighted counting measure, i.e. there is a countable set K and $c_j \in \mathbb{R}_{>0}$ for $j \in K$ with:

$$\mathrm{d}\mu_{\text{pp}}(\Omega) = \sum_{j \in K} c_j \delta_{x_j}$$
$$\sum_{j \in K} c_j < \infty$$

This gives rise to a decomposition of the Hilbert spaces.

$$L^2(\Omega, \mathrm{d}\mu) = L^2(\Omega, \mathrm{d}\mu_{\text{pp}}) \oplus L^2(\Omega, \mathrm{d}\mu_{\text{ac}}) \oplus L^2(\Omega, \mathrm{d}\mu_{\text{sing}})$$

Applying $\mathcal{U}^{-1}$ gives the corresponding decomposition:

$$H = H_{\text{pp}} + H_{\text{ac}} + H_{\text{sing}}$$

$$\begin{aligned}
A\big|_{H_{\text{pp}}} &: H_{\text{pp}} \to H_{\text{pp}} & \sigma_{\text{pp}}(A) &:= \sigma\left(A|_{H_{\text{pp}}}\right) \\
A\big|_{H_{\text{ac}}} &: H_{\text{ac}} \to H_{\text{ac}} & \sigma_{\text{ac}}(A) &:= \sigma\left(A|_{H_{\text{ac}}}\right) \\
A\big|_{H_{\text{sing}}} &: H_{\text{sing}} \to H_{\text{sing}} & \sigma_{\text{sing}}(A) &:= \sigma\left(A|_{H_{\text{sing}}}\right)
\end{aligned}$$

10 The Spectral Theorem for Unbounded Self-Adjoint Operators

Let $A : \mathcal{D}(A) \to H$ be a densely defined linear operator with domain of definition $\mathcal{D}(A) \overset{\text{dense}}{\subseteq} H$.

Recall:

- A is *symmetric* if $\langle u,Av \rangle = \langle Au,v \rangle$ for all $u,v \in \mathcal{D}(A)$. (also called *formally self-adjoint*)
- A is *self-adjoint* if $A^* = A$, or equivalently:

$$\left(\underset{u \in \mathcal{D}(A)}{\forall} : \langle Au,v \rangle = \langle u,w \rangle \right) \qquad \Rightarrow \qquad \big((w \in \mathcal{D}(A)) \wedge (Av = w) \big)$$

10.1 Theorem (The basic criterion for self-adjointness)

Let A be a symmetric operator with dense domain of definition $\mathcal{D}(A)$. Then the following statements are equivalent.

i) A is self-adjoint.

ii) A is closed and $\ker(A^* \pm \mathbf{i}) = \{0\}$ (for $+$ and $-$).

iii) $\operatorname{im}(A \pm \mathbf{i}) = H$ (for $+$ and $-$)

Proof

„i) $\Rightarrow$ ii)“: Let A be self-adjoint, i.e. $A = A^*$. Since A^* is always closed, it follows that A is closed. Let $\varphi \in \mathcal{D}(A^*) = \mathcal{D}(A)$ be in the kernel of $A^* \pm \mathbf{i}$, i.e. $\mp\mathbf{i}\varphi = A^*\varphi = A\varphi$. Then follows:

$$\mp\mathbf{i} \langle \varphi,\varphi \rangle = \langle \varphi,A\varphi \rangle = \langle A\varphi,\varphi \rangle = \pm\mathbf{i} \langle \varphi,\varphi \rangle$$

This shows $\|\varphi\| = 0$ and thus $\varphi = 0$.

„ii) $\Rightarrow$ iii)“: Let A be closed and $\ker(A \pm \mathbf{i}) = \{0\}$ be trivial.

- $\operatorname{im}(A \pm \mathbf{i})$ is dense in H. Assume conversely that there exists a $u \neq 0$ in $(\operatorname{im}(A \pm \mathbf{i}))^\perp$. Then follows for all $v \in \mathcal{D}(A)$:

$$0 = \langle (A \pm \mathbf{i})\, v,u \rangle$$

So $u \in \mathcal{D}((A \pm \mathbf{i})^*) = \mathcal{D}(A^*)$ and $(A^* \mp \mathbf{i})\, u = 0$ in contradiction to $\ker(A^* \mp \mathbf{i}) = \{0\}$.

– $\operatorname{im}(A \pm \mathbf{i})$ is closed in H. Let $\psi \in \overline{\operatorname{im}(A \pm \mathbf{i})}$ lie in the closure of the image. Then there exist $\varphi_n \in \mathcal{D}(A)$ such that:

$$(A \pm \mathbf{i})\,\varphi_n \to \psi$$

For any $\varphi \in \mathcal{D}(A)$ holds:

$$\|(A \pm \mathbf{i})\,\varphi\|^2 = \langle (A \pm \mathbf{i})\,\varphi, (A \pm \mathbf{i})\,\varphi\rangle = \|A\varphi\|^2 + \|\varphi\|^2 \pm \mathbf{i}\underbrace{(\langle A\varphi,\varphi\rangle - \langle \varphi,A\varphi\rangle)}_{=0,\text{ since }A\text{is symmetric}}$$

Especially for $\varphi = \varphi_n - \varphi_m$ holds:

$$\underbrace{\|A(\varphi_n - \varphi_m)\|^2}_{\geq 0} + \underbrace{\|\varphi_n - \varphi_m\|^2}_{\geq 0} = \|(A \pm \mathbf{i})(\varphi_n - \varphi_m)\|^2 \xrightarrow[(A\pm\mathbf{i})\varphi_n \to \psi]{n,m\to\infty} 0$$

It follows:

$$\begin{aligned} \|\varphi_n - \varphi_m\| &\to 0 & \qquad \varphi_n &\to \varphi \\ \|A\varphi_n - A\varphi_m\| &\to 0 & \qquad A\varphi_n &\to \psi \mp \mathbf{i}\varphi \end{aligned}$$

Thus $(\varphi_n, A\varphi_n)$ is a Cauchy sequence in $\operatorname{graph}(A) \subseteq H \times H$.
Since A is closed, which means by definition that $\operatorname{graph}(A)$ is closed in $H \times H$, the limit point $(\varphi, \psi \mp \mathbf{i}\varphi)$ is in $\operatorname{graph}(A)$. Then follows $\varphi \in \mathcal{D}(A)$ and $A\varphi = \psi \mp \mathbf{i}\varphi$, i.e. $\psi \in \operatorname{im}(A \pm \mathbf{i})$.

„iii) ⇒ i)“: Assume that $\operatorname{im}(A \pm \mathbf{i}) = H$. Consider $\varphi \in \mathcal{D}(A^*)$. Since $\operatorname{im}(A \pm \mathbf{i}) = H$, there is a $u \in \mathcal{D}(A)$ such that $(A \pm \mathbf{i})\,u = (A^* \pm \mathbf{i})\,\varphi$. From $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ (always true for symmetric operators) follows $\varphi - u \in \mathcal{D}(A^*)$ and:

$$(A^* \pm \mathbf{i})(\varphi - u) = 0$$

Consider $w \in \ker(A^* \pm \mathbf{i}) \setminus \{0\}$. Then holds for all $\xi \in \mathcal{D}(A)$:

$$\begin{aligned} \langle (A^* \pm \mathbf{i})\,w, \xi\rangle &= 0 \\ \langle w, (A \mp \mathbf{i})\,\xi\rangle &= 0 \end{aligned}$$

Using assumption $\operatorname{im}(A \mp \mathbf{i}) = H$ one can choose ξ such that $(A \mp \mathbf{i})\,\xi = w$, which means $\langle w,w\rangle = 0$, i.e. $w = 0$. Thus holds:

$$\ker(A^* \pm \mathbf{i}) = \{0\}$$

This gives $\varphi = u \in \mathcal{D}(A)$, which implies $\varphi \in \mathcal{D}(A^*)$ and thus A is self-adjoint. $\square_{10.1}$

10.2 Unbounded Multiplication Operators

Let (Ω,μ) be a measure space with a σ-finite measure μ. (For example, Ω is a σ-compact topological space and μ a positive Borel measure on Ω.)
$H = L^2(\Omega, \mathrm{d}\mu)$ is our Hilbert space. Let $g : \Omega \to \mathbb{R}$ be measurable (and finite almost everywhere). We want to introduce T_g:

$$T_g f = g \cdot f$$

For $g \in L^\infty(\Omega, \mathrm{d}\mu)$, T_g is a bounded symmetric operator. Suppose g is unbounded. What is $\mathcal{D}(T_g)$? How to choose $\mathcal{D}(T_g)$ such that T_g becomes self-adjoint?

Lemma

Define:

$$\mathcal{D}(T_g) = \left\{ f \in L^2(\Omega,\mathrm{d}\mu) \,\middle|\, g \cdot f \in L^2(\Omega,\mathrm{d}\mu) \right\} \subseteq L^2(\Omega,\mathrm{d}\mu)$$

Then $T_g : \mathcal{D}(T_g) \to L^2(\Omega,\mathrm{d}\mu)$ is self-adjoint and $\sigma_{\mathrm{ess}}(T_g) = g(\Omega)$.

Proof

T_g is symmetric:

$$\begin{aligned}\langle T_g f,h\rangle &= \int_\Omega \overline{(T_g f)} h \mathrm{d}\mu = \int_\Omega \overline{g(x) \cdot f(x)} h(x) \,\mathrm{d}\mu(x) = \int_\Omega g(x) \cdot \overline{f}(x) \, h(x) \,\mathrm{d}\mu(x) = \\ &= \int_\Omega \overline{f(x)} g(x) \, h(x) \,\mathrm{d}\mu(x) = \langle f, T_g h\rangle\end{aligned}$$

T_g is self-adjoint: For $\psi \in \mathcal{D}(T_g^*)$ we show $\psi \in \mathcal{D}(T_g)$. This is equivalent to the existence of a $v \in H$ such that for all $u \in \mathcal{D}(T_g)$ holds

$$\langle T_g u, \psi\rangle = \langle u, v\rangle$$

and we have $v = T_g^* \psi$. Now we write

$$\Omega = \bigcup_N K_N$$

with $K_N \subseteq K_{N+1}$ having finite measure and set:

$$\chi_N(x) = \begin{cases} 1 & \text{if } |g(x)| \leq N \text{ and } x \in K_N \\ 0 & \text{otherwise} \end{cases}$$

So $\chi_N(x) \nearrow 1$ converges monotonously and it holds:

$$\begin{aligned}\left\|T_g^* \psi\right\|_{L^2}^2 &= \int_\Omega \left|\left(T_g^* \psi\right)(x)\right|^2 \mathrm{d}\mu(x) \overset{\text{monotone}}{\underset{\text{convergence}}{=}} \lim_{N\to\infty} \int_\Omega \chi_N(x) \left|\left(T_g^* \psi\right)(x)\right|^2 \mathrm{d}\mu(x) = \\ &= \lim_{N\to\infty} \left\|\chi_N T_g^* \psi\right\|^2\end{aligned}$$

$$\begin{aligned}\Rightarrow \qquad \left\|T_g^* \psi\right\|_{L^2} &= \lim_{N\to\infty} \left\|\chi_N T_g^* \psi\right\|_{L^2} = \lim_{N\to\infty} \sup_{\|\varphi\|=1} \left|\left\langle \varphi, \chi_N T_g^* \psi\right\rangle\right| = \\ &\overset{\star}{=} \lim_{N\to\infty} \sup_{\|\varphi\|=1} \left|\left\langle T_g \chi_N \varphi, \psi\right\rangle\right|\end{aligned}$$

In $\star$ we used that $\chi_N \varphi$ is in $\mathcal{D}(T_g)$. This is really the case, since for $\chi_N \varphi \in L^2(\Omega,\mathrm{d}\mu)$ holds:

$$T_g \chi_N \varphi = \underbrace{g \cdot \chi_N}_{\text{is bounded}} \varphi = T_{g\cdot\chi_N} \varphi \in L^2(\Omega,\mathrm{d}\mu)$$

Since the function $g \cdot \chi_N$ is bounded, the multiplication operator $T_{g\cdot\chi_N}$ is bounded and thus follows:

$$\begin{aligned}\infty > \left\|T_g^* \psi\right\| &= \lim_{N\to\infty} \sup_{\|\varphi\|=1} \left|\left\langle \varphi, \chi_N \cdot g \cdot \psi\right\rangle\right| = \lim_{N\to\infty} \left\|\chi_N \cdot g \cdot \psi\right\| = \\ &= \lim_{N\to\infty} \int_\Omega \chi_N(x) \left|g\psi\right|^2(x) \,\mathrm{d}\mu(x) \overset{\text{monotone}}{\underset{\text{convergence}}{=}} \int_\Omega \left|(g\psi)(x)\right|^2 \mathrm{d}\mu(x)\end{aligned}$$

So we have $g\psi \in L^2(\Omega,\mathrm{d}\mu)$ and thus $\psi \in \mathcal{D}(T_g)$ holds by definition of $\mathcal{D}(T_g)$.

We omit the proof that $\sigma_{\mathrm{ess}}(T_g) = g(\Omega)$. $\square_{10.2}$

10.3 Theorem (The Spectral Theorem in its Multiplicative Form)

Let $A : \mathcal{D}(H) \overset{\text{dense}}{\subseteq} H \to H$ be a self-adjoint operator and H separable. Then there is a finite measure space (M,μ), a unitary operator $\mathcal{U} : H \to L^2(M,\mathrm{d}\mu)$ and a measurable function $f : M \to \mathbb{R}$ such that holds:

a) $\psi \in \mathcal{D}(A) \quad \Leftrightarrow \quad f \cdot \mathcal{U}\psi \in L^2(M,\mathrm{d}\mu)$

b) $\varphi \in \mathcal{U}(\mathcal{D}(A))$ implies $\mathcal{U}A\mathcal{U}^{-1}\varphi = f \cdot \varphi = T_f \cdot \varphi$.

Thus A is unitarily equivalent to the multiplication T_f on $L^2(M,\mathrm{d}\mu)$ and as chosen in 10.2:

$$\mathcal{U}(\mathcal{D}(A)) = \mathcal{D}(T_f) = \left\{\phi \in L^2 \middle| f \cdot \phi \in L^2(M,\mathrm{d}\mu)\right\}$$

Proof

According to our basic criterion 10.1, the mapping

$$A \pm \mathbf{i} : \mathcal{D}(A) \to H$$

is surjective (by property iii)) and injective (by property ii)), noting:

$$\{0\} = \ker(A^* \pm \mathbf{i}) = \ker(A \pm \mathbf{i})$$

So $A \pm \mathbf{i}$ is bijective and thus the inverse $(A \pm \mathbf{i})^{-1} : H \to \mathcal{D}(A) \subseteq H$ exists.
The operators $(A \pm \mathbf{i})^{-1}$ are bounded, because for all $u \in \mathcal{D}(A)$ holds (cf. proof of 10.1):

$$\|(A + \mathbf{i})\,u\|^2 = \|Au\|^2 + \|u\|^2$$

Thus for $v := (A + \mathbf{i})\,u$ follows:

$$\left\|(A + \mathbf{i})^{-1} v\right\| \leq \|v\|$$
$$\left|\left|\left|(A + \mathbf{i})^{-1}\right|\right|\right| \leq 1$$

The operators $(A \pm \mathbf{i})^{-1}$ are *normal*: The resolvent identity gives:

$$(A + \mathbf{i})^{-1} - (A - \mathbf{i})^{-1} = -2\mathbf{i} \cdot (A + \mathbf{i})^{-1} \cdot (A - \mathbf{i})^{-1}$$
$$(A - \mathbf{i})^{-1} - (A + \mathbf{i})^{-1} = +2\mathbf{i} \cdot (A - \mathbf{i})^{-1} \cdot (A + \mathbf{i})^{-1}$$

Together this yields:

$$\left[(A + \mathbf{i})^{-1}, (A - \mathbf{i})^{-1}\right] = 0$$

Let us compute $\left((A + \mathbf{i})^{-1}\right)^*$. For $u,v \in \mathcal{D}(A)$ holds:

$$\langle (A - \mathbf{i})\,u, v\rangle \overset{A \text{ symmetric}}{=} \langle u, (A + \mathbf{i})\,v\rangle$$
$$\quad\|\qquad\qquad\qquad\qquad\|$$
$$\Big\langle \underbrace{(A - \mathbf{i})\,u}_{=\psi}, (A + \mathbf{i})^{-1} \underbrace{(A + \mathbf{i})\,v}_{=\varphi} \Big\rangle = \Big\langle (A - \mathbf{i})^{-1} \underbrace{(A - \mathbf{i})\,u}_{=\psi}, \underbrace{(A + \mathbf{i})\,v}_{=\varphi} \Big\rangle$$

$$\left\langle \psi, (A+\mathbf{i})^{-1}\phi \right\rangle = \left\langle (A-\mathbf{i})^{-1}\psi, \phi \right\rangle$$

Since $(A-\mathbf{i})$ and $(A+\mathbf{i})$ are surjective, this holds for all $\psi,\phi \in H$ and thus follows:

$$\left((A+\mathbf{i})^{-1}\right)^* = (A-\mathbf{i})^{-1}$$

$$\Rightarrow \qquad \left[(A+\mathbf{i})^{-1}, \left((A+\mathbf{i})^{-1}\right)^*\right] = 0$$

So $(A+\mathbf{i})^{-1}$ is normal and we can apply the spectral theorem in its multiplicative form to the operator $(A+\mathbf{i})^{-1}$. This gives:

$$\mathcal{U} : H \to L^2(M,\mathrm{d}\mu)$$

μ is a bounded positive Borel measure on the σ-compact topological space M.

$$M = \sigma\left((A+\mathbf{i})^{-1}\right) \times J$$

And for $\varphi \in L^2(M,\mathrm{d}\mu)$ holds

$$\left(\mathcal{U}(A+\mathbf{i})^{-1}\mathcal{U}^{-1}\right)\varphi = g\cdot\varphi$$

with a $g \in L^\infty(M,\mathrm{d}\mu)$.
Moreover, since $(A+\mathbf{i})^{-1}$ is injective, the function g is non-zero almost everywhere: Assume conversely that there exists a $\Omega \subseteq M$ with $\mu(\Omega) \neq 0$ and $g\big|_\Omega = 0$. Then $\varphi := \chi_\Omega$ is a non-zero vector in $L^2(M,\mathrm{d}\mu)$ with $g\cdot\varphi \neq 0$.

$$\|\varphi\|^2 = \int_M \chi_\Omega^2 \mathrm{d}\mu = \mu(\Omega) > 0$$

Thus $\mathcal{U}^{-1}\varphi$ is a non-trivial vector in the kernel of $(A+\mathbf{i})^{-1}$, which is a contradiction to the injectivity of A.

a) Set $f = \frac{1}{g} - \mathbf{i}$. This function is measurable and finite almost everywhere.
„$\Rightarrow$“: Since $(A+\mathbf{i})^{-1} : H \to \mathcal{D}(A)$ is bijective, a $\psi \in \mathcal{D}(A)$ can be written uniquely as:

$$\psi = (A+\mathbf{i})^{-1}\phi$$

$$\Rightarrow \qquad \mathcal{U}\psi = \mathcal{U}(A+\mathbf{i})^{-1}\phi = \underbrace{\mathcal{U}(A+\mathbf{i})^{-1}\mathcal{U}^{-1}}_{=T_g}\mathcal{U}\phi = g\mathcal{U}\phi$$

$$f\mathcal{U}\psi = fg\mathcal{U}\phi = \underbrace{(1-\mathbf{i}g)}_{\in L^\infty(M,\mathrm{d}\mu)} \cdot \underbrace{\mathcal{U}\phi}_{\in L^2(M,\mathrm{d}\mu)} \in L^2(M,\mathrm{d}\mu)$$

„$\Leftarrow$“: Assume $f\mathcal{U}\psi \in L^2(M,\mathrm{d}\mu)$, which implies $(f+\mathbf{i})\mathcal{U}\psi \in L^2(M,\mathrm{d}\mu)$. Now there exists a $\phi \in H$ such that holds:

$$\mathcal{U}\phi = (f+\mathbf{i})\mathcal{U}\psi \qquad /\cdot g$$
$$g\mathcal{U}\phi = g(f+\mathbf{i})\mathcal{U}\psi = \mathcal{U}\psi$$
$$\Rightarrow \qquad \psi = \underbrace{\mathcal{U}^{-1}g\mathcal{U}}_{=(A+\mathbf{i})^{-1}}\phi = (A+\mathbf{i})^{-1}\phi$$

Since $(A+\mathbf{i})^{-1} : H \to \mathcal{D}(A)$ is bijective, $\psi \in \mathcal{D}(A)$ follows.

b) We need to show for all $\varphi \in \mathcal{U}(\mathcal{D}(A))$:

$$\mathcal{U}A\mathcal{U}^{-1}\varphi = f\varphi$$

Write $\psi \in \mathcal{D}(A)$ as $\psi = (A+\mathbf{i})^{-1}\varphi$ to get just as in a) „$\Rightarrow$“:

$$\begin{aligned}
\mathcal{U}\psi &= g\mathcal{U}\varphi \\
\mathcal{U}\varphi &= \frac{1}{g}\mathcal{U}\psi \\
\mathcal{U}(A+\mathbf{i})\psi &= \frac{1}{g}\mathcal{U}\psi \\
\mathcal{U}A\psi &= \frac{1}{g}\mathcal{U}\psi - \mathbf{i}\mathcal{U}\psi = \left(\frac{1}{g} - \mathbf{i}\right)\mathcal{U}\psi = f\mathcal{U}\psi \\
\mathcal{U}A\mathcal{U}^{-1}\chi &\overset{\chi=\mathcal{U}\psi}{=} f\cdot\chi
\end{aligned}$$

Finally we show that f is real-valued. For all $\psi \in \mathcal{D}(A)$ holds, because A is symmetric:

$$\begin{aligned}
0 &= \operatorname{Im}(\langle\psi,A\psi\rangle) = \operatorname{Im}\left(\langle\psi,\mathcal{U}^{-1}f\mathcal{U}\psi\rangle\right) \overset{\mathcal{U}\text{ unitary}}{=} \operatorname{Im}(\langle\mathcal{U}\psi,f\mathcal{U}\psi\rangle) = \\
&= \int_M \operatorname{Im}(f(x))\cdot|(\mathcal{U}\psi)(x)|^2\,\mathrm{d}\mu(x)
\end{aligned}$$

Since $\mathcal{U}\psi$ can be any L^2-function χ (just choose $\psi = \mathcal{U}^{-1}\chi$), it follows that $\operatorname{Im}(f) = 0$ almost everywhere. $\square_{10.3}$

Connection to the Cayley transformation

The operators

$$\begin{aligned}
V &:= (A+\mathbf{i})(A-\mathbf{i})^{-1} \\
V^* &= (A+\mathbf{i})^{-1}(A-\mathbf{i})
\end{aligned}$$

are unitary, because it holds:

$$\begin{aligned}
V\cdot V^* &= (A+\mathbf{i})(A-\mathbf{i})^{-1}(A+\mathbf{i})^{-1}(A-\mathbf{i}) = \\
&= (A+\mathbf{i})(A+\mathbf{i})^{-1}(A-\mathbf{i})^{-1}(A-\mathbf{i}) = \mathbb{1}
\end{aligned}$$

We worked here with $(A-\mathbf{i})^{-1}$.

10.4 The unbounded Functional Calculus, Projection-valued Spectral measures

Goal: Suppose E_λ is a spectral measure on $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$.

$$f(A) = \int f(\lambda)\,\mathrm{d}E_\lambda$$

So far we had $f \in \mathcal{B}(\mathbb{K})$. This gave us a bounded linear operator. We want to calculate

$$f(A) = \int f(\lambda)\,\mathrm{d}E_\lambda$$

for any Borel function f, possibly unbounded. Then $f(A)$ is a possibly unbounded operator. What is $\mathcal{D}(A)$ and what is $\mathcal{D}(A^*)$?

$$\mathcal{D}(A) = \left\{ u \in H \middle| \int_{\mathbb{K}} |f(\lambda)|^2 \, \mathrm{d}\langle u, E_\lambda u\rangle < \infty \right\}$$

Example

$$\left(UAU^{-1}\right) f = gf$$

and $g : \Omega \to \mathbb{R}$ is measurable.

$$\begin{aligned}\mathcal{D}(A) &= \left\{U^{-1}\varphi \middle| \varphi \in L^2(\Omega,\mathrm{d}\mu) \wedge gf \in L^2(\Omega,\mathrm{d}\mu)\right\} = \\ &= U^{-1}\mathcal{D}\left(UAU^{-1}\right)U\end{aligned}$$

The spectral calculus yields:

$$UA^2U^{-1} = \left(UAU^{-1}\right)^2 = g^2$$

$$\Rightarrow \qquad \mathcal{D}\left(A^2\right) = \left\{U^{-1}\varphi \middle| \varphi \in L^2(\Omega,\mathrm{d}\mu) \wedge g^2 f \in L^2(\Omega,\mathrm{d}\mu)\right\}$$

So the domain of definition changes.

10.4.1 Theorem (The spectral theorem in functional calculus form)

Let $A : \mathcal{D}(A) \subseteq H \to H$ be self-adjoint. Then there is a unique mapping

$$\Phi : \mathcal{B}(\mathbb{R}) \to L(H)$$

such that the following holds:

i) Φ is an involutive algebra homomorphism.

ii) $\|\Phi(f)\|_{L(H)} \leq \|f\|_\infty$

iii) Let $g_n \in \mathcal{B}(\mathbb{R})$ be the elements of a sequence such that $g_n \to g$ converges point-wise and $|g_n(x)| \leq |x|$ holds. Then for every $\psi \in \mathcal{D}(A)$ converges:

$$\Phi(g_n)\psi \to \Phi(g)\psi$$

iv) If $g_n \to g$ converges point-wise with $|g_n(x)| < C$, then holds for all $\psi \in H$ converges:

$$\Phi(g_n)\psi \to \Phi(g)\psi$$

v) For $A\psi = \lambda\psi$ follows $\Phi(f)\psi = f(\lambda)\psi$

vi) For $h \geq 0$ holds $\Phi(h) \geq 0$.

Proof

After a unitary transformation with the operator U from the spectral theorem in its multiplicative form, we can assume $H = L^2(M,\mathrm{d}\mu)$ and:

$$\begin{aligned}\mathcal{D}(A) &= \left\{\varphi \in L^2(M,\mathrm{d}\mu) \,\middle|\, g\varphi \in L^2(M,\mathrm{d}\mu)\right\} \\ A\varphi &= g\varphi \\ (\Phi(f)\varphi)(x) &= f(g(x)) \cdot \varphi(x)\end{aligned}$$

Since $f(g) \in L^\infty$ holds, define for any $\varphi \in L^2$:

$$\Phi(f)\varphi := f(g)\varphi \in L^2$$

This defines an operator in $L(H)$.
The properties i) and ii) are obvious. iii) and iv) follow from dominated convergence:

iii) It holds:

$$\begin{aligned}\Phi(f_n)\varphi &= f_n(g) \cdot \varphi \\ \Phi(f)\varphi &= f(g)\varphi \\ f_n(g) &\xrightarrow{\text{point-wise}} f(g)\end{aligned}$$

By assumption holds $|f_n(g)| \leq |g|$ and by our formula for $\mathcal{D}(A)$ follows for all $\varphi \in \mathcal{D}(A)$:

$$|f_n(g)\varphi|, |f(g)\varphi| \leq |g| \cdot |\varphi| \in L^2$$

iv) follows similarly and v) and vi) are obvious.

Uniqueness of Φ: Let $K \subseteq \mathbb{R}$ be compact and $\varphi \in L^2(K,\mathrm{d}\mu)$. Then holds:

$$\Phi(g \cdot \chi_K)\varphi = \underbrace{\Phi(g)}_{=A} \cdot \Phi(\chi_K)\varphi = A\Phi(\chi_K)\varphi$$

On K we can approximate g using Stone-Weierstraß. Then choose a sequence $K_1 \subseteq K_2 \subseteq \ldots$ of compact K_n with $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}$. Now take the limit $n \to \infty$ and use property iii) to get $\Phi(g)\varphi = A\varphi$, which shows the uniqueness of Φ. $\square_{10.4.1}$

Now we write $\Phi(f) =: f(A)$. We can again introduce the spectral measure:

$$E_\Omega := \Phi(\chi_\Omega) = \chi_\Omega(A)$$

After a unitary transformation holds:

$$E_\Omega\varphi = \chi_\Omega(g) \cdot \varphi$$

This shows:

$$\begin{aligned}E_\Omega^* &= E_\Omega = E_\Omega^2 \\ E_U \cdot E_V &= E_{U\cap V}\end{aligned}$$

$$\begin{aligned}\langle\varphi, E_\Omega\varphi\rangle &= \int_{\mathbb{R}} |\varphi|^2 \chi_\Omega(g)\,\mathrm{d}\mu \\ \langle\varphi, f(A)\varphi\rangle &= \int_{\mathbb{R}} |\varphi|^2 f(g)\,\mathrm{d}\mu = \int_{\mathbb{R}} f\,\mathrm{d}\langle\varphi, E_\lambda\varphi\rangle\end{aligned}$$

10.4.2 Theorem

There is a one-to-one correspondence between self-adjoint operators and projection-valued spectral measures (not necessarily with compact support) given by:

$$A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda$$
$$\mathcal{D}(A) = \left\{ u \in H \middle| \int_{\mathbb{R}} \lambda^2 \mathrm{d}\langle u, E_\lambda u\rangle < \infty \right\}$$

Moreover holds:

i) $f(A) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_\lambda$ holds for all bounded Borel functions f.

ii) If f is an unbounded Borel function, we set:

$$\mathcal{D}_f = \left\{ u \in H \middle| \int_{\mathbb{R}} |f|^2 \,\mathrm{d}\langle u, E_\lambda u\rangle < \infty \right\}$$

The set $\mathcal{D}_f \subseteq H$ is dense and

$$B := \int_{\mathbb{R}} f \mathrm{d}E_\lambda : \mathcal{D}_f \to H$$

is a densely defined closed operator with:

$$B^* = \int_{\mathbb{R}} \overline{f} \mathrm{d}E_\lambda : \mathcal{D}_f \to H$$

(In particular, if f is real-valued, the operator B is again self-adjoint.)

Proof

- $\mathcal{D}_f$ is dense in H: After a unitary transformation we identify H with $L^2(M,\mathrm{d}\mu)$ and define:

$$\mathcal{D}_f = \left\{ \varphi \in L^2(M,\mathrm{d}\mu) \middle| \int |f(g)|^2 \cdot |\varphi|^2 \,\mathrm{d}\mu < \infty \right\}$$

(Recall $f(A) = f(g)$.) For $\psi \in L^2(M,\mathrm{d}\mu)$, we want to show $\psi \in \overline{\mathcal{D}_f}$. To this end we set:

$$\psi_n(x) := \begin{cases} \psi(x) & \text{if } |f(g(x))| \le n \\ 0 & \text{otherwise} \end{cases}$$

Then holds:

$$\int |f(g)|^2 \cdot |\psi_n|^2 \,\mathrm{d}\mu \le n^2 \int |\psi_n|^2 \,\mathrm{d}\mu \le n^2 \int |\psi|^2 \,\mathrm{d}\mu < \infty$$

Hence follows $\psi_n \in \mathcal{D}_f$. Obviously $\psi_n \to \psi$ converges point-wise and it holds:

$$|\psi_n| \le |\psi| \in L^2(M,\mathrm{d}\mu)$$

Thus dominated convergence yields $\psi_n \to \psi$ in $L^2(M,\mathrm{d}\mu)$.

- Next, $B\varphi = f(g)\varphi$ with

$$\mathcal{D}(B) = \left\{ \varphi \in L^2 \middle| f(g)\varphi \in L^2 \right\}$$

is an unbounded multiplication operator. Its adjoint can be computed as in section 10.2.

$\square_{10.4.2}$

11 Examples, Construction of Self-Adjoint extensions

The (interesting) operator $H = -\Delta_{\mathbb{R}^3} + V(x)$ requires Sobolev spaces and Fourier transform. This is discussed in the lecture partial differential equations I.

Here we only consider more simple, one-dimensional examples.

11.1 Example

Consider $A = \mathbf{i}\frac{\mathrm{d}}{\mathrm{d}x}$ on $H = L^2(\mathbb{R},\mathrm{d}x)$ with domain of definition:

$$\mathcal{D}(A) = C_0^\infty(\mathbb{R})$$

– A is symmetric: For $\psi,\phi \in C_0^\infty(\mathbb{R})$ holds:

$$\begin{aligned}\langle \psi, A\phi\rangle &= \int_{\mathbb{R}} \overline{\psi(x)}\mathbf{i}\left(\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)\right)\mathrm{d}x = \\ &\underset{\text{by parts}}{\overset{\text{integration}}{=}} \underbrace{\overline{\psi(x)}\cdot\mathbf{i}\phi(x)\,\Big|_{-\infty}^{\infty}}_{=0,\ \text{(compact support)}} \int_{\mathbb{R}}(-\mathbf{i})\left(\frac{\mathrm{d}}{\mathrm{d}x}\overline{\psi(x)}\right)\phi(x)\,\mathrm{d}x = \\ &= \int_{\mathbb{R}}\overline{\left(\mathbf{i}\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\right)}\phi(x)\,\mathrm{d}x = \langle A\psi,\phi\rangle\end{aligned}$$

– A is *not* self-adjoint: If A were self-adjoint, the following computation would hold:

$$\underset{u\in\mathcal{D}(A)}{\forall} : \langle Au,v\rangle = \langle u,w\rangle \quad \Rightarrow \quad (v \in \mathcal{D}(A)) \wedge (Av = w)$$

Any $v \in C_0^1(\mathbb{R}) \setminus C_0^\infty(\mathbb{R})$ is a counter example.
We could even satisfy the condition on the left by choosing $v \in C^1(\mathbb{R})$. (We need no decay assumption, since it suffices that one function has compact support). Thus follows:

$$\mathcal{D}(A^*) \subseteq C^1(\mathbb{R})$$

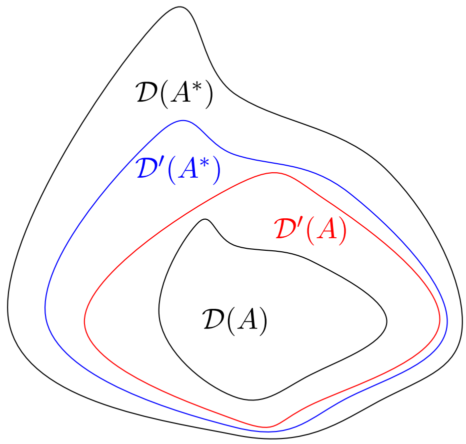


Figure 11.1: The large $\mathcal{D}'(A) \supseteq \mathcal{D}(A)$, the smaller is $\mathcal{D}'(A^*) \subseteq \mathcal{D}(A^*)$.

– $A : \mathcal{D}(A) \to H$ is essentially self-adjoint: This means that $\overline{A}$ with $\operatorname{graph}(\overline{A}) := \overline{\operatorname{graph}(A)}$ is self-adjoint.
  According to the basic criterion for self-adjointness (Theorem 10.1), we know:

$$A \text{ self-adjoint} \quad \Leftrightarrow \quad \operatorname{im}(A \pm \mathbf{i}) = H$$

  Therefore, for essential self-adjointness it suffices to show that $(A \pm \mathbf{i})(C_0^\infty(\mathbb{R})) \subseteq H = L^2$ is dense.

  Claim: For all $v \in H$ there exists a $u \in H$ such that $(u,v) \in \overline{\operatorname{graph}(A \pm \mathbf{i})}$.
  (In other words, $\overline{A} \pm \mathbf{i}$ is surjective.)

  Proof: Since $(A \pm \mathbf{i})(C_0^\infty) \subseteq H$ is dense, there exists a sequence of $u_n \in C_0^\infty$ such that with $w_n := Au_n$ converges:

$$(A \pm \mathbf{i})\, u_n = w_n \pm \mathbf{i} u_n \to v$$

  The estimates from the proof of the basic criterion imply:

$$w_n = Au_n \to w \qquad\qquad u_n \to u$$

  This yields that $(u_n,w_n) \to (u,w)$ converges. From $(u_n,w_n) \in \operatorname{graph}(A)$ follows $(u,w) \in \overline{\operatorname{graph}(A)}$. $\square_{\text{Claim}}$

  Claim: $(A \pm \mathbf{i})(C_0^\infty(\mathbb{R}))$ is dense in L^2.

  Proof: The vectors in the image of $A \pm \mathbf{i}$ are of the form:

$$\mathbf{i}\frac{\mathrm{d}}{\mathrm{d}x}u \pm \mathbf{i}u =: v$$

  From $u \in C_0^\infty$ follows $v \in C_0^\infty$. Multiply by $e^{\mp x}$ and integrate by parts to get:

$$\int_{-\infty}^{\infty} e^{\mp x} v(x)\,\mathrm{d}x = \mathbf{i}\int_{-\infty}^{\infty} e^{\mp x}\left(\left(\frac{\mathrm{d}}{\mathrm{d}x} \pm 1\right) u(x)\right)\mathrm{d}x =$$
$$\overset{\text{integrate}}{\underset{\text{by parts}}{=}} -\mathbf{i}\int_{-\infty}^{\infty} u(x)\left(\left(\frac{\mathrm{d}}{\mathrm{d}x} \pm 1\right) e^{\mp x}\right)\mathrm{d}x =$$

$$= -\mathbf{i}\int_{-\infty}^{\infty} u\left(x\right)\underbrace{\left(\mp e^{\mp x} \pm e^{\mp x}\right)}_{=0}\mathrm{d}x = 0$$

Thus the functions in the image of $A \pm \mathbf{i}$ satisfy the condition:

$$\int_{-\infty}^{\infty} e^{\mp x} v\left(x\right)\mathrm{d}x = 0$$

Conversely, if a function $v\left(x\right)$ satisfies this condition for $+$ and $-$, then

$$u\left(x\right) := \int_{-\infty}^{x} e^{\mp t} v\left(t\right)\mathrm{d}t$$

is in $C_0^\infty\left(\mathbb{R}\right)$ and $\left(A \pm \mathbf{i}\right)u = v$.
Now we need to show:

$$\overline{\left\{v \in C_0^\infty\left(\mathbb{R}\right)\middle| \int e^{\pm x} v\left(x\right)\mathrm{d}x = 0\right\}} = H$$

Since $C_0^\infty\left(\mathbb{R}\right)$ is dense in H, we only need to prove that $\psi \in C_0^\infty\left(\mathbb{R}\right)$ is an element of the left set. We look for $v_n \in C_0^\infty\left(\mathbb{R}\right)$ with

$$\int e^{\pm x} v_n\left(x\right)\mathrm{d}x = 0$$

such that $v_n \to \psi$ converges in L^2.
Choose $\eta \in C_0^\infty\left([0,1]\right)$ and use the ansatz:

$$v_n = \psi + c_+\eta\left(x - L\right) + c_-\eta\left(x + L\right)$$

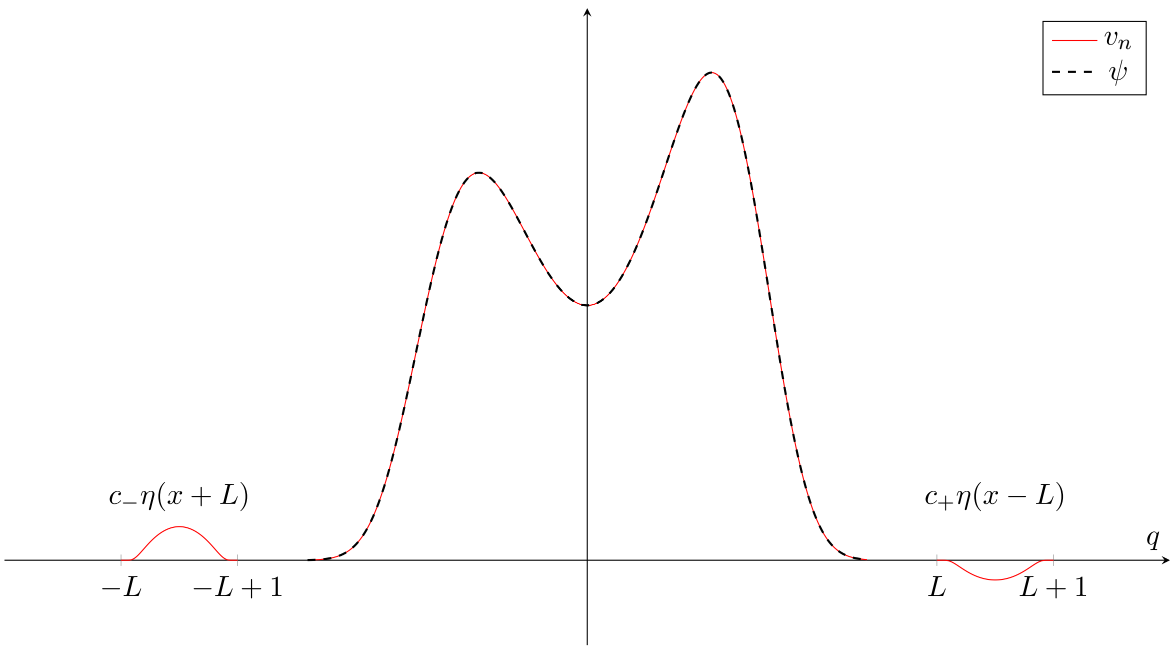


Figure 11.2: Approximation of ψ with the v_n

Then holds:

$$\begin{aligned}0 &\overset{!}{=} \int_{-\infty}^{\infty} e^{\pm x} v_n(x)\,\mathrm{d}x = \\ &= \int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x + c_+ \underbrace{\int_{-\infty}^{\infty} e^{\pm x}\eta(x-L)\,\mathrm{d}x}_{\sim e^{\pm L}} + c_- \underbrace{\int_{-\infty}^{\infty} e^{\pm x}\eta(x+L)\,\mathrm{d}x}_{\sim e^{\mp L}}\end{aligned}$$

We have two conditions and two free parameters. One sees that c_+, c_- are proportional to e^{-L}. Thus $v_n \to \psi$ converges in L^2. $\square_{\text{Claim}}$

Thus $\overline{A}$ with $\mathcal{D}\left(\overline{A}\right)$ (which can be described in detail) is self-adjoint.

$$\overline{A} = \int_{-\infty}^{\infty} \lambda \mathrm{d}E_\lambda \qquad \text{spectral theorem}$$

11.2 Example

On the Hilbert space $H = L^2([0,1], \mathrm{d}x)$ consider the operator $A := \frac{\mathrm{d}}{\mathrm{d}x}$ with $\mathcal{D}(A) = C_0^\infty((0,1))$.

a) A is *not* essentially self-adjoint. Just as in the previous example, A being essentially self-adjoint is equivalent to

$$(A \pm \mathbf{i})\left(C_0^\infty((0,1))\right) \subseteq H$$

being dense, or equivalently

$$M := \left\{ v \in C_0^\infty((0,1)) \,\middle|\, 0 = \int_0^1 e^{\pm x} v(x)\,\mathrm{d}x \right\} \subseteq H$$

being dense. For $\psi(x) = e(x) \in H$ holds for all $v \in M$:

$$\langle \psi, v \rangle = \int_0^1 \psi(x) v(x)\,\mathrm{d}x = \int_0^1 e^x v(x)\,\mathrm{d}x = 0$$

Therefore holds $0 \neq \psi \in M^\perp$ and M is *not* dense in H.

b) For $f \in C_0^\infty([0,1])$ and $n \in \mathbb{Z}$ define:

$$c_n := \int_0^1 f(x)\, e^{2\pi \mathbf{i} n x} \mathrm{d}x$$

This gives rise to a unitary transformation (Plancherel theorem):

$$\begin{aligned} U : L^2([0,1]) &\to \ell_2 \\ f &\mapsto (c_n)_{n\in\mathbb{Z}} \end{aligned}$$

$$\int_0^1 |f|^2\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} |c_n|^2$$

$$\hat{A}(c_n) = \left(U \mathbf{i} \frac{\mathrm{d}}{\mathrm{d}x} U^{-1}\right)(c_n) = (-2\pi n c_n)_n$$

$\hat{A}$ is a multiplication operator with:

$$\mathcal{D}\left(\hat{A}\right) = \left\{(c_n)_n \in \ell^2 \middle| (nc_n)_n \in \ell^2\right\} \subseteq \ell^2$$

Then

$$\hat{A} : \mathcal{D}\left(\hat{A}\right) \to \ell^2$$

is self-adjoint. Thus

$$A : \mathcal{D}(A) := U^{-1}\mathcal{D}\left(\hat{A}\right) \to L^2$$

is self-adjoint.

11.3 Example

Consider $H = L^2(\mathbb{R},\mathrm{d}x)$, $A = \mathbf{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and $T = T_g$ with a real valued g.

$$(A+T)\psi(x) = \mathbf{i}\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) + g(x)\psi(x)$$

How to choose $\mathcal{D}(A+T)$ in order to make the operator self-adjoint?

There are two solutions:

- Friedrichs extension (by K. O. Friedrichs) for semi-bounded operators.
- Katos's method

11.4 Theorem (Kato-Rellich)

Let $A : \mathcal{D}(A) \to H$ be self-adjoint and T symmetric with $\mathcal{D}(T) \supseteq \mathcal{D}(A)$. Moreover, assume that there are constants $a,b \in \mathbb{R}_{\geq 0}$ with $b < 1$ such that for all $u \in \mathcal{D}(A)$ holds:

$$\|Tu\|^2 \leq a^2\|u\|^2 + b^2\|Au\|^2 \tag{11.1}$$

Then $A+T$ with

$$\mathcal{D}(A+T) = \mathcal{D}(A)$$

is self-adjoint.

T is *relatively bounded with respect to* A.

Proof

The inequality (11.1) implies:

$$\|Tu\| \leq a\|u\| + b\|Au\|$$

For $u \in \mathcal{D}(A)$ holds:

$$Au = (A+T)u - Tu$$

$$\begin{aligned}\|Au\| &\leq \|(A+T)\,u\| + \|Tu\| \leq \\ &\leq \|(A+T)\,u\| + a\,\|u\| + b\,\|Au\|\end{aligned}$$

This gives:

$$\|Au\| \leq \frac{1}{1-b}\left(\|(A+T)\,u\| + a\,\|u\|\right) \tag{11.2}$$

- $(A+T)$ with $\mathcal{D}(A+T) := \mathcal{D}(A)$ is closed: Choose $u_n \in \mathcal{D}(A)$ such that $u_n \to u$ and $(A+T)\,u_n \to w$ converge. We want to show $u \in \mathcal{D}(A)$ and $(A+T)\,u = w$. (11.2) implies:

$$\|A(u_n - u_m)\| \leq \frac{1}{1-b}\underbrace{\|(A+T)(u_n-u_m)\|}_{\to 0} + \frac{a}{1-b}\underbrace{\|u_n - u_m\|}_{\to 0}$$

  This gives $A(u_n - u_m) \to 0$ and thus $Au_n \to v$. Since A is self-adjoint, it is closed, implying that $u \in \mathcal{D}(A)$ and $Au = v$.
- It remains to be showed that $\frac{A+T}{c} \pm \mathbf{i}$ is surjective for any $c \in \mathbb{R}_{>0}$. This is equivalent to $A + T \pm \mathbf{i}c$ being surjective. Since A is self-adjoint, we know that

$$A \pm \mathbf{i}c : \mathcal{D}(A) \to H$$

  is bijective with:

$$(A \pm \mathbf{i}c)^{-1} : H \to \mathcal{D}(A)$$

  This gives:

$$(A + T + \mathbf{i}c) = \underbrace{\left(T(A+\mathbf{i}c)^{-1} + \mathbb{1}\right)}_{\text{to show that this is invertible}}\underbrace{(A+\mathbf{i}c)}_{\text{invertible}}$$

  We show that $\left\|T(A+\mathbf{i}c)^{-1}\right\| < 1$. Then $\mathbb{1} + T(A+\mathbf{i}c)^{-1}$ has a bounded inverse in terms of the Neumann series.
  For $u \in H$ define $v := (A+\mathbf{i}c)^{-1}\,u \in \mathcal{D}(A)$, so it holds:

$$u = (A+\mathbf{i}c)\,v$$

$$\|u\|^2 = \|Av\|^2 + c^2\,\|v\|^2$$

$$\|v\|^2 \leq \frac{1}{c^2}\,\|u\|^2 \tag{11.3}$$

$$\|Av\|^2 \leq \|u\|^2 \tag{11.4}$$

  We get:

$$\begin{aligned}\left\|T(A+\mathbf{i}c)^{-1}\,u\right\|^2 &= \|Tv\|^2 \leq a^2\,\|v\|^2 + b^2\,\|Av\|^2 \leq \\ &\overset{(11.4)}{\leq} a^2\,\|v\|^2 + b^2\,\|u\|^2 \leq \\ &\overset{(11.3)}{\leq} \frac{a^2}{c^2}\,\|u\|^2 + b^2\,\|u\|^2 = \left(\frac{a^2}{c^2} + b^2\right)\|u\|^2\end{aligned}$$

  By choosing c sufficiently large, we can arrange that with $\tilde{c} < 1$ holds for all $u \in H$:

$$\left\|T(A+\mathbf{i}c)^{-1}\,u\right\|^2 \leq \tilde{c}\,\|u\|^2$$

  This gives:

$$\left\|T(A+\mathbf{i}c)^{-1}\right\| < 1$$

$\square_{11.4}$

Back to example 11.3

$A = -\mathbf{i}\frac{\mathrm{d}}{\mathrm{d}x}$ is self-adjoint with $\mathcal{D}(A)$ being the domain of definition of the closure of $A : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ and $T = T_g$.

If *Kato's condition* is fulfilled, i.e. for all $u \in \mathcal{D}(A)$ the inequality

$$\|T_g u\|^2 \leq a^2 \|u\|^2 + b^2 \|Au\|^2$$

with $a,b \in \mathbb{R}_{>0}$ and $b < 1$ holds, then $A + T$ is also self-adjoint.

For which g is Kato's condition satisfied?

$$\|Au\|^2 = \int_{-\infty}^{\infty} \left|u'(x)\right|^2 \mathrm{d}x$$

(Let us assume $u \in C_0^\infty$.)

$$\begin{aligned}|u(x) - u(y)| &= \left|\int_x^y 1 \cdot u'(t)\,\mathrm{d}t\right| \overset{\text{Schwarz}}{\leq} \left(\int_x^y 1^2 \mathrm{d}t\right)^{\frac{1}{2}} \cdot \left(\int_x^y \left|u'(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \leq \\ &\leq |x-y|^{\frac{1}{2}} \cdot \|Au\|\end{aligned}$$

Moreover, the mean value theorem (Mittelwertungleichung) gives for all $a \in \mathbb{R}$ the existence of a $y \in \left[a - \frac{1}{2}, a + \frac{1}{2}\right]$ such that holds:

$$|u(y)| \leq \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} |u(\tau)|\,\mathrm{d}\tau \overset{\text{Schwarz}}{\leq} \underbrace{\left(\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} 1^2 \mathrm{d}t\right)^{\frac{1}{2}}}_{=1} \cdot \left(\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} |u(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \leq \|u\|_{L^2}$$

This gives:

$$|u(x)| \leq |u(y)| + |u(x) - u(y)| \leq \|u\| + \|Au\|$$

Consider now different cases:

1. case: g is bounded, i.e. $\|g\|_\infty \leq c \in \mathbb{R}_{\geq 0}$. Then holds:

$$\begin{aligned} & \|T_g u\|^2 = \int_{\mathbb{R}} |g(x)|^2 |u(x)|^2 \mathrm{d}x \leq c^2 \|u\|^2 \\ \Rightarrow \qquad & \|T_g u\| \leq c\|u\| \end{aligned}$$

   Thus Kato's condition is satisfied with $b = 0$.

2. case: g is not bounded and $\|g\|_{L^2} < 1$. Then holds:

$$\begin{aligned} & \|T_g u\|^2 = \int_{\mathbb{R}} |g(x)|^2 |u(x)|^2 \mathrm{d}x \leq \sup_{x \in \mathbb{R}} |u(x)|^2 \cdot \|g\|_{L^2}^2 \\ \Rightarrow \qquad & \|T_g u\| \leq (\|u\| + \|Au\|)\|g\|_{L^2} = \|g\|_{L^2} \cdot \|u\| + \|g\|_{L^2} \cdot \|Au\| \end{aligned}$$

   Kato's condition is again satisfied.

3. case: $g \in L^2(\mathbb{R})$, but no bound on $\|g\|_{L^2}$. Decompose $g = g_1 + g_2$:

$$\begin{aligned} g_1^{(L)} &:= g \cdot \chi_{[-L,L]} \in L^\infty \\ g_2^{(L)} &:= g - g_1 \end{aligned}$$

From the dominated convergence theorem follows:

$$\left\| g_2^{(L)} \right\|_{L^2} \xrightarrow{L\to\infty} 0$$

Thus there exists a $L \in \mathbb{R}_{>0}$ with $\left\| g_2^{(L)} \right\| < 1$. Combining case 1 for $g_1^{(L)}$ and case 2 for $g_2^{(L)}$ shows that $A + T_g$ is again self-adjoint.

11.5 Example

Consider the operator

$$H = -\Delta_{\mathbb{R}^3} + V$$

on $L^2(\mathbb{R}^3)$ with:

$$V(x) = \begin{cases} \frac{c}{\|x\|} & \text{Coulomb potential} \\ c \cdot \frac{e^{-\|x\|}}{\|x\|} & \text{Yukawa potential} \end{cases}$$

The goal is to find $\mathcal{D}(H)$ such that H is self-adjoint.
Consider the "unperturbed operator" $-\Delta_{\mathbb{R}^3}$ on $L^2(\mathbb{R}^3)$ and use a Fourier transformation

$$\hat{A} := U(-\Delta_{\mathbb{R}^3})U^{-1}f = T_g f$$

with:

$$(T_g f)(k) = \|k\|^2 f(k)$$

Define:

$$\begin{aligned} \mathcal{D}\left(\hat{A}\right) &:= \left\{ f \in L^2(\mathbb{R}^3) \mid \|k\|^2 f(k) \in L^2(\mathbb{R}^3) \right\} \\ \mathcal{D}(-\Delta_{\mathbb{R}^3}) &:= U^{-1}\left(\mathcal{D}\left(\hat{A}\right)\right) = W^{2,2}(\mathbb{R}) \end{aligned}$$

Here $W^{k,p}(\mathbb{R})$ is a Sobolov space and the special case $W^{k,2}(\mathbb{R})$ is also a Hilbert space. The norm of $W^{2,2}(\mathbb{R})$ is:

$$\|f\|^2_{W^{2,2}} = \int \left(|f|^2 + \|\nabla f\|^2 + \left|\nabla^2 f\right|^2 \right)(x)\, \mathrm{d}^3 x$$

Functions in $W^{2,2}$ are only weakly differentiable. With elliptic estimates follows:

$$\|u\|_{W^{2,2}} \leq (1+\varepsilon)\|\Delta u\|^2 + c\|u\|^2$$

Also the *Sobolov inequality* and the *Sobolov embedding theorem* holds:

$$\|u\|_{L^{2p}} \leq \varepsilon\|\Delta u\|^2 + c(\varepsilon)\|u\|^2$$

Kato's condition is:

$$\|Vu\|_{L^2}^2 \leq a^2 \|u\|^2 + b^2 \|\Delta u\|^2$$

With

$$\frac{1}{p} + \frac{1}{q} = 1$$

holds:

$$\begin{aligned}\|Vu\|_{L^2}^2 ? \int_{\mathbb{R}^3} |V(x)|^2 |u(x)|^2 \mathrm{d}^3x \leq \|V\|_{2q} \cdot \underbrace{\|u\|_{2p}}_{\text{Sobolev inequality}} \leq \\ \leq \|V\|_{2q} \left(\varepsilon \|\Delta u\|^2 + c(\varepsilon) \|u\|^2\right)\end{aligned}$$

Now holds $b := \varepsilon \|V\|_{2q}^2 < 1$ for sufficiently small ε, provided that $\|V\|_{L^{2q}} < \infty$. This is satisfied for the Yukawa potential, but for the Coulomb potential one must work a bit harder.

Appendix

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Andreas Völklein

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