

Functional Analysis

lecture by

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Motivation

In linear algebra one mainly considers finite-dimensional vector spaces with additional structures like norm $\|\cdot\|$ or scalar product $\langle \cdot, \cdot \rangle$.

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional scalar product space and $A : V \rightarrow V$ a linear map, which is self-adjoint, that means for all $u, v \in V$:

$$\langle Au, v \rangle = \langle u, Av \rangle$$

Theorem (orthonormal eigenvector basis)

There exists an orthonormal eigenvector basis $(u_i)_{i \in \{1, \dots, n\}}$, that means with the eigenvalues $\lambda_i \in \mathbb{R}$:

$$\langle u_i, u_j \rangle = \delta_{ij} \qquad Au_i = \lambda_i u_i$$

In infinite dimensions the generalization is the *spectral theorem*.

First reformulate the result from linear algebra:

Let E_{λ_i} be the orthogonal projection operator on the eigenspace corresponding to λ_i . If this eigenspace is one dimensional, this means:

$$E_{\lambda_i} v = u_i \langle u_i, v \rangle = |u_i\rangle \langle u_i| v$$

Then one can write A as:

$$A = \sum_{i=1}^n \lambda_i E_{\lambda_i}$$

Theorem (spectral theorem)

Let $A \in L(H)$ be a self-adjoint (selbstadjungiert) operator, then it holds:

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

$\sigma(A) \subseteq \mathbb{R}$ is the spectrum of A and E_λ the projection-valued measure (Spektralmaß).

Applications typically are differential operators, for example:

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\Delta_{\mathbb{R}^3} : C_0^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3) \quad \text{linear operator}$$

Applications in more detail are studied in the lectures on partial differential equations I + II.

0 Basic Notions

Let E be a vector space (Vektorraum), for example the finite-dimensional vector space $E \simeq \mathbb{R}^3$. In the following list the later spaces are special cases of the previous ones:

- topological vector spaces
- metric spaces with a metric $d(.,.)$ (Polish spaces if complete)
- normed spaces with norm $\|.\|$ (Banach spaces if complete)
- scalar product spaces $\langle .,.\rangle$ (Hilbert spaces if complete)

Let \mathbb{K} be either \mathbb{R} or \mathbb{C} .

0.1 Definition (metric, ε -ball, Cauchy sequence, complete, Polish space)

A map $d : E \times E \rightarrow \mathbb{R}$ is called *metric*, if for all $x, y, z \in E$ holds:

- i) $d(x, y) = d(y, x)$ (symmetry)
- ii) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ (positive definiteness)
- iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

$B_\varepsilon(x) := \{z \in E \mid d(x, z) < \varepsilon\}$ is called ε -ball.

Consider the topology generated by $B_\varepsilon(x)$: A set $\Omega \subseteq E$ is open if and only if:

$$\forall_{x \in \Omega} \exists_{\varepsilon \in \mathbb{R}_{>0}} : B_\varepsilon(x) \subseteq \Omega$$

Completeness:

$(x_n)_{n \in \mathbb{N}}$ is a *Cauchy sequence* if and only if:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \exists_{N \in \mathbb{N}} \forall_{n, m \in \mathbb{N}_{>N}} : d(x_n, x_m) < \varepsilon$$

E is *complete* if and only if every Cauchy sequence has a limit.

A complete metric space is also called a *Polish space*.

0.2 Definition (norm, Banach space)

Let $(E, \|\cdot\|)$ be a *normed space*, i.e. a \mathbb{K} -vector space with a map $\|\cdot\| : E \rightarrow \mathbb{R}_{\geq 0}$ called *norm* with the following properties for $x, y \in E$ and $\lambda \in \mathbb{K}$:

- i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (positive definiteness)

ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity)

iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

Define the metric $d(x, y) := \|x - y\|$. A complete normed spaces is called *Banach space*.

Let $A : E \rightarrow F$ be a linear map between the Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$.

0.3 Definition (continuous, bounded)

A is *continuous* (stetig) if $A^{-1}(\Omega) \subseteq E$ is open for all open $\Omega \subseteq F$.

A is *bounded* (beschränkt) if there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq C \|u\|_E$$

0.4 Lemma (continuous \Leftrightarrow bounded)

A is continuous $\Leftrightarrow A$ is bounded.

(no proof)

0.5 Definition (dual space, sup-norm)

The *dual space* of E is the space of continuous linear mappings from E to \mathbb{K} :

$$E^* = L(E, \mathbb{K})$$

$L(E, F)$ is a vector space: For $A, B \in L(E, F)$, $\lambda, \mu \in \mathbb{K}$ and $u \in E$ define:

$$(\lambda A + \mu B)(u) := \lambda A(u) + \mu B(u)$$

Define also a norm on $L(E, F)$, which is called *sup-norm*:

$$\|A\| := \sup_{u \in E, \|u\|_E \leq 1} \|Au\|_F$$

0.6 Theorem

If F is complete, so is $L(E, F)$.

In particular E^* is a Banach space for every E .

(no proof)

1 The Hahn-Banach Theorem and Applications

As a preparation we need Zorn's lemma.

1.1 Definition (partial ordering, chain, upper bound, maximal)

Let A be a set and \leq a *partial ordering* (Halbordnung), i.e. for all $a, b, c \in A$:

- i) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)
- ii) $a \leq a$ (reflexivity)
- iii) $a \leq b \wedge b \leq a \Rightarrow a = b$ (antisymmetry)

Note: We do *not* demand that for all $a, b \in A$ holds:

$$(a \leq b) \vee (b \leq a)$$

This is a property of a ordering relation.

(A, \leq) is called *partially ordered set* (teilweise geordnete Menge).

A subset $K \subseteq A$ is called *chain* (Kette, total geordnete Teilmenge) if for all $x, y \in K$ holds:

$$(x \leq y) \vee (y \leq x)$$

An element $u \in A$ is called *upper bound* (obere Schranke) of $B \subseteq A$ if $x \leq u$ for all $x \in B$.

An element $m \in A$ is called *maximal* if $m \leq a \in A \Rightarrow m = a$.

1.2 Zorn's lemma

Let (A, \leq) be a partially ordered set in which every chain has an upper bound. Then there is a maximal element.

Proof

This follows from the axiom of choice, see e.g. Kowalsky: Linear algebra.

1.3 Definition (sublinear)

Let X be a *real* vector space (without topology) and $l : X \rightarrow \mathbb{R}$ linear. $p : X \rightarrow \mathbb{R}$ is called *sublinear* if for all $x, y \in X$ and $a \in \mathbb{R}_{>0}$:

- i) $p(ax) = ap(x)$
- ii) $p(x + y) \leq p(x) + p(y)$

A typical example is $p(x) = \|x\|$, but p does not need to be positive. Another example is any linear mapping.

1.4 Theorem (Hahn-Banach, real version, 1927/29)

Let X be a real vector space and $Y \subseteq X$ a subspace (Untervektorraum), $p : X \rightarrow \mathbb{R}$ sublinear and $l : Y \rightarrow \mathbb{R}$ linear with $l(y) \leq p(y)$ for all $y \in Y$.

Then there is a linear extension (Fortsetzung) $\tilde{l} : X \rightarrow \mathbb{R}$ of l to X , i.e. $\tilde{l}|_Y = l$, such that for all $x \in X$ holds:

$$\tilde{l}(x) \leq p(x)$$

Proof

- i) Assume $Y \subsetneq X$, since otherwise there is nothing to prove. Choose a vector $z \in X \setminus Y$. We want to extend l to the span of Y and $\langle z \rangle$. $\tilde{l}(z)$ needs to be prescribed. For all $y \in Y$ and $a \in \mathbb{R}$ holds:

$$\tilde{l}(y + az) \stackrel{\text{linearity}}{=} l(y) + a\tilde{l}(z) \stackrel{\text{demand}}{\leq} p(y + az)$$

If $a = 0$, the inequality is clear. By homogeneity assumptions, it is sufficient to consider the case $a = \pm 1$. We thus demand for all $y, y' \in Y$:

$$\begin{aligned} l(y) + \tilde{l}(z) &\leq p(y + z) \\ l(y') - \tilde{l}(z) &\leq p(y' - z) \end{aligned}$$

This is equivalent to:

$$l(y') - p(y' - z) \leq \tilde{l}(z) \leq p(y + z) - l(y)$$

We can choose $\tilde{l}(z)$ if and only if:

$$l(y') - p(y' - z) \leq p(y + z) - l(y)$$

(For example set $\tilde{l}(z) = \sup_{y' \in Y} l(y') - p(y' - z)$.)

$$\Leftrightarrow l(y') + l(y) \stackrel{\text{linearity}}{=} l(y' + y) \leq p(y + z) + p(y' - z)$$

Now prove this inequality:

From $y' + y \in Y$ follows that $l(y + y') \leq p(y + y')$ by hypothesis. Moreover, as p is sublinear, it follows:

$$p(y + z - z + y') \leq p(y' + z) + p(y' - z)$$

So the inequality is shown. Thus l can be extended to $Y + \langle z \rangle$.

ii) Consider all extensions:

$$A := \{(Z, l) \mid Y \subseteq Z \subseteq X \text{ subspace, } l : Z \rightarrow \mathbb{R} \text{ extension of } l_Y : Y \rightarrow \mathbb{R}\}$$

This set has a partial ordering \leq defined by $(Z, l) \leq (Z', l')$ if $Z \subseteq Z'$ and $l'|_Z = l$.

For an index set I (possibly infinite, uncountable) let $K = \{(Z_\nu, l_\nu) \mid \nu \in I\}$ be a chain, i.e. for all $(Z, l), (Z', l') \in K$:

$$((Z, l) \leq (Z', l')) \vee ((Z', l') \leq (Z, l))$$

Set $Z = \bigcup_{\nu \in I} Z_\nu$ and define $l : Z \rightarrow \mathbb{R}$ by $l|_{Z_\nu} = l_\nu$. (Thus suppose $u \in Z$, so there is a $\nu \in I$ with $u \in Z_\nu$. Set $l(u) := l_\nu(u)$. ν need not be unique. Suppose $u \in Z_{\nu'}$, then we know that either $Z_{\nu'} \subseteq Z_\nu$ and $l_\nu|_{Z_{\nu'}} = l_{\nu'}$ or $Z_\nu \subseteq Z_{\nu'}$ and $l_{\nu'}|_{Z_\nu} = l_\nu$. In both cases we have $l_\nu(u) = l_{\nu'}(u)$, thus $l(u)$ is well defined.)

This (Z, l) is an upper bound, because for all $\nu \in I$ we have $Z_\nu \subseteq Z = \bigcup_{\lambda \in I} Z_\lambda$ and l is an extension of l_ν .

With Zorn's Lemma follows, that there exists an maximal element (\tilde{Y}, \tilde{l}) .

Claim: $\tilde{Y} = X$

Proof: Otherwise there would be a vector $u \in X \setminus \tilde{Y}$, and \tilde{l} could be extended to $\tilde{Y} \oplus \langle u \rangle$, as shown in i), in contradiction to the maximality of \tilde{l} . Thus $(X = \tilde{Y}, \tilde{l})$ is the desired extension. \square_{Claim}

$\square_{1.4}$

1.5 Theorem (Hahn-Banach, complex version)

Let X be a complex vector space and $Y \subseteq X$ a subspace. Before, we had $l(x) \leq p(x)$ as condition, which does not make sense in the complex case, since:

$$l(e^{i\varphi}x) = e^{i\varphi}l(x) \stackrel{\text{in general}}{\notin} \mathbb{R}$$

Let $p : X \rightarrow \mathbb{R}$ be a *seminorm*, i.e.:

- i) $p(ax) = |a|p(x)$ (homogeneity)
- ii) $p(x+y) \leq p(x) + p(y)$ (triangle inequality)

Let $l : Y \rightarrow \mathbb{C}$ be a linear functional with $|l(y)| \leq p(y)$ for all $y \in Y$.

Then l can be extended to X such that $|l(x)| \leq p(x)$ holds for all $x \in X$.

Proof

We also consider X as a real vector space. (u and iu are then linearly independent vectors.) Decompose l into its real and imaginary parts.

$$\begin{aligned} l(y) &= l_1(y) + i l_2(y) \\ l_1 &:= \operatorname{Re}(l(y)) \\ l_2 &:= \operatorname{Im}(l(y)) \end{aligned}$$

l_1 and l_2 are real-linear and:

$$l_1(\mathbf{i}y) = \operatorname{Re}(l(\mathbf{i}y)) = \operatorname{Re}(\mathbf{i}l(y)) = -\operatorname{Im}(l(y)) = -l_2(y)$$

Conversely, suppose that l_1 is real-linear. Then

$$l(x) := l_1(x) - \mathbf{i} \cdot l_1(\mathbf{i}x)$$

this is indeed a complex-linear function. We know that $|l(y)| \leq p(y)$ holds for all $y \in Y$.

$$\begin{aligned} l_1(y) &= \operatorname{Re}(l(y)) \leq |l(y)| \\ \Rightarrow \quad l_1(y) &\leq p(y) \end{aligned}$$

Theorem 1.4 yields an real-linear extension $\tilde{l}_1 : X \rightarrow \mathbb{R}$ such that $\tilde{l}_1(x) \leq p(x)$ for all $x \in X$. Set $\tilde{l}(x) = \tilde{l}_1(x) - \mathbf{i}\tilde{l}_1(\mathbf{i}x)$, so that $\tilde{l} : X \rightarrow \mathbb{C}$ is complex-linear.

Claim: $|\tilde{l}(x)| \leq p(x) \quad \forall x \in X$

Proof: Polar decomposition:

$$\begin{aligned} \tilde{l}(x) &= r e^{\mathbf{i}\varphi} \\ |\tilde{l}(x)| &= r = e^{-\mathbf{i}\varphi} \tilde{l}(x) \stackrel{\tilde{l} \text{ is complex-linear}}{=} \tilde{l}(e^{-\mathbf{i}\varphi} x) = \operatorname{Re}(\tilde{l}(e^{-\mathbf{i}\varphi} x)) = \\ &= \tilde{l}_1(e^{-\mathbf{i}\varphi} x) \leq p(e^{-\mathbf{i}\varphi} x) \stackrel{\text{homogeneity}}{=} p(x) \end{aligned}$$

□_{Claim}

□_{1.5}

Now to applications:

1.6 Theorem

Let $(X, \|\cdot\|)$ be a normed \mathbb{K} -space (real or complex), $Y \subseteq X$ a subspace. Let φ be a continuous linear functional from Y to \mathbb{K} , i.e. for all $y \in Y$ holds:

$$|\varphi(y)| \leq \|\varphi\| \cdot \|y\|$$

Then φ can be continued to all of X with the same supnorm, i. e.:

$$\|\tilde{\varphi}\| := \sup_{x \in X, \|x\| \leq 1} |\varphi(x)| = \|\varphi\| := \sup_{y \in Y, \|y\| \leq 1} |\varphi(y)|$$

Proof

Apply the Hahn-Banach theorem with $\tilde{\varphi} := \|\varphi\| \cdot \|x\|$.

□_{1.6}

1.7 Corollary

Let X be a normed space and $u_0 \in X$ with $\|u_0\| = 1$. Then there exists a linear functional $\varphi : X \rightarrow \mathbb{K}$ such that:

$$\varphi(u_0) = 1 \qquad \|\varphi\| = 1$$

Proof

Let $Y := \langle u_0 \rangle$ and define $\varphi_0 : \langle u_0 \rangle \rightarrow \mathbb{K}$ by $\varphi_0(u_0) = 1$. Extend φ_0 by the Hahn-Banach theorem 1.6. $\square_{1.7}$

The Hahn-Banach theorem also has a geometric formulation. Consider only the real case:
A set $K \subseteq X$ is called *convex* if for all $x, y \in K$ and $\tau \in [0, 1]$:

$$\tau x + (1 - \tau) y \in K$$

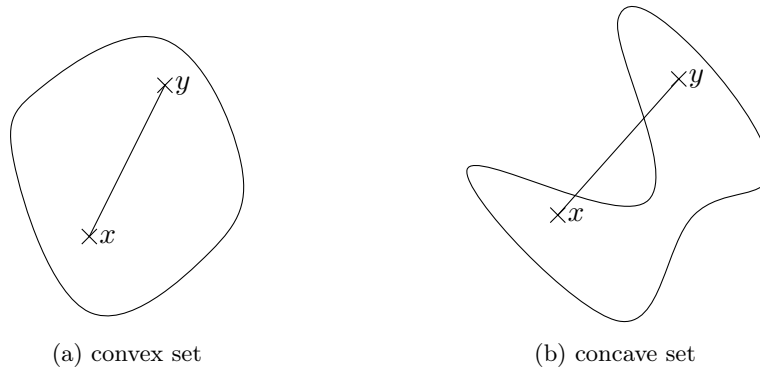


Figure 1.1: convexity

Geometric question:

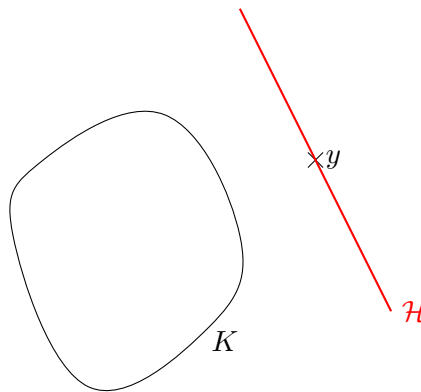


Figure 1.2: not intersecting hyperplane

Is there a hyperplane \mathcal{H} , which meets $y \notin K$, but does not intersect K ?

1.8 Definition (interior point)

$x_0 \in K$ is an *interior point* (innerer Punkt) of K with respect to $u \in X$ if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ for all $t \in (-\varepsilon, \varepsilon)$.

$x_0 \in K$ is an *interior point* if for all $u \in X$ there is a $\varepsilon = \varepsilon(u) \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ for all $t \in (-\varepsilon, \varepsilon)$.

1.9 Theorem (geometric Hahn-Banach)

Let $K \neq \emptyset$ be convex and all points of K be interior points. Let $y \notin K$. Then there is a linear functional $l : X \rightarrow \mathbb{R}$ such that $l(x) < 1$ for all $x \in K$ and $l(y) = 1$.

$\mathcal{H} := \{x \in X \mid l(x) = 1\}$ defines a hyperplane. Now $y \in \mathcal{H}$ and $l|_K < 1$ mean that K lies in one half-space.

First introduce a suitable sublinear functional. Without loss of generality, assume $0 \in K$ (otherwise shift K).

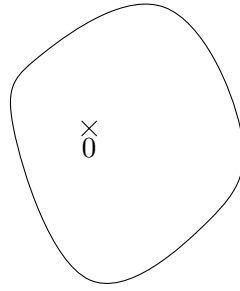


Figure 1.3: $0 \in K$

The functional $p : K \rightarrow \mathbb{R}_{\geq 0}$ with

$$p(x) := \inf \left\{ a \in \mathbb{R}_{>0} \mid \frac{x}{a} \in K \right\}$$

is called gauge (Eichung).

Since x is an interior point, we know that $\frac{x}{a} \in K$ if $a > 1 - \varepsilon(x)$.

p is even defined on all of X , because for $x \in X$, now $\tau x \in K$ if $|\tau|$ is sufficiently small, because $0 \in K$ is an interior point.

$$p(x) < 1 \quad \Leftrightarrow \quad x \in K$$

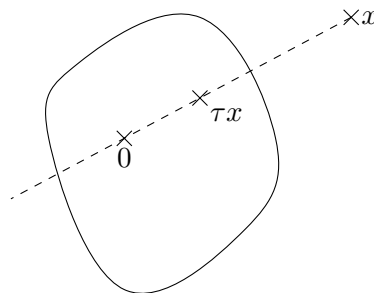


Figure 1.4: $x \notin K$, $\tau x \in K$

1.10 Lemma

p is sublinear.

Proof

The homogeneity is clear from the definition.

sub-additivity (triangle equation):

Take $x, y \in K$ and choose $a, b \in \mathbb{R}_{>0}$ such that $\frac{x}{a}, \frac{y}{b} \in K$. The convexity of K implies for all $\tau \in [0, 1]$:

$$\tau \frac{x}{a} + (1 - \tau) \frac{y}{b} \in K$$

Choose $\tau = \frac{a}{a+b}$, then holds $1 - \tau = \frac{b}{a+b}$, which gives:

$$\Rightarrow \frac{1}{a+b} (x+y) \in K$$

$$p(x+y) \leq a+b$$

Taking the infimum over a and b gives $p(x+y) \leq p(x) + p(y)$:

$$p(x+y) = \inf \underbrace{\left\{ c \in \mathbb{R}_{>0} \mid \frac{x+y}{c} \in K \right\}}_{\ni a+b} \leq a+b$$

$$\begin{aligned} p(x) = \inf \left\{ a \mid \frac{x}{a} \in K \right\} &\Rightarrow \forall_{\varepsilon > 0} \exists_{a \in \mathbb{R}_{>0}} : p(x) \geq a - \varepsilon \\ p(y) = \inf \left\{ b \mid \frac{y}{b} \in K \right\} &\Rightarrow \forall_{\varepsilon > 0} \exists_{b \in \mathbb{R}_{>0}} : p(y) \geq b - \varepsilon \end{aligned}$$

□_{1.10}

1.11 Lemma

$$p(x) < 1 \Leftrightarrow x \in K$$

Proof

If $x \notin K$ then $\frac{1}{a}x \notin K$ for all $0 < a < 1$ and so $p(x) \geq 1$.

For all $x \in K$ exists an $\varepsilon = \varepsilon(x) \in \mathbb{R}_{>0}$ with $(1+t)x \in K$ for all $t \in (-\varepsilon, \varepsilon)$.

$$\begin{aligned} &\Rightarrow \left(1 + \frac{\varepsilon}{2}\right)x \in K \\ &\Rightarrow p(x) \leq \frac{1}{1 + \frac{\varepsilon}{2}} < 1 \end{aligned}$$

□_{1.11}

Proof of Theorem 1.9

Introduce l on $\langle y \rangle$ by $l(y) = 1$. (Assume again that $0 \in K$ and so $y \neq 0$.)

Write $z = ay \in \langle y \rangle$ with $a \in \mathbb{R}$.

- If $a < 0$, then $l(z) = a \cdot l(y) = a < 0$ but $p(z) \geq 0$ and thus the inequality $l(z) \leq p(z)$ is trivially satisfied.
- If $a > 0$ it holds:

$$l(z) = a \underset{\Rightarrow p(y) \geq 1}{\overset{y \notin K}{\leq}} a \cdot p(y) \overset[\text{homogeneity}]{\text{positive}} p(ay) = p(z)$$

So for all $z \in \langle y \rangle$ holds $l(z) \leq p(z)$.

The Hahn-Banach Theorem yields an extension $l : X \rightarrow \mathbb{R}$ such that $l(x) \leq p(x)$ for all $x \in X$.

Therefore for all $x \in K$ we have:

$$l(x) \leq p(x) < 1$$

□_{1.9}

2 Normed Spaces

Let $(E, \|\cdot\|)$ be a normed space and let the open balls $B_\varepsilon(x) = \{y \mid \|x - y\| < \varepsilon\}$ generate the topology on E .

2.0.1 Definition (equivalent norms)

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent*, if there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1$$

2.0.2 Theorem

Equivalent norms give rise to the same topology.

(No proof)

2.0.3 Theorem

If E is finite dimensional, then any two norms on E are equivalent.

(No proof)

2.0.4 Constructions (Quotient space, Cartesian product)

Let $F \subseteq E$ be a *closed* subspace. Define the *quotient space* (Faktorraum) E/F as follows:

$$x \sim y \Leftrightarrow x - y \in F$$

defines an equivalence relation on E .

$$E/F := E/\sim$$

is a vector space.

$$\|u\|_{E/F} := \inf_{\substack{\hat{u} \in E \\ \hat{u} - u \in F}} \|\hat{u}\|_E$$

$(E/F, \|\cdot\|_{E/F})$ is a normed space. The closedness of F is essential:

Suppose $F \subseteq E$ is not closed. Then there exists an $x \in \overline{F} \setminus F$, thus there is a $(x_n)_{n \in \mathbb{N}}$, $x_n \in F$

with $x_n \rightarrow x$.

Let $[x] \in E/F$ be the equivalence class. Then $[x] \neq 0$, since $x \notin F$, but:

$$\|[x]\| = \inf_{\substack{\hat{x} \in E \\ \hat{x} - x \in F}} \|\hat{x}\| \stackrel{x - x_n \sim x}{\leq} \inf \|x - x_n\| = 0$$

If $\|\cdot\|_{E/F}$ was a norm, it would imply $[x] = 0$ and thus $x \in F$ in contradiction to $x \in \overline{F} \setminus F$.

Another construction is the *Cartesian product*: Let E and F be normed spaces.

$$E \times F := \{(u, v) \mid u \in E, v \in F\}$$

$$\|(u, v)\|_{E \times F} := \|u\|_E + \|v\|_F$$

is a norm on $E \times F$.

2.0.5 Definition (separable)

A normed space is called *separable*, if there is a countable dense subset, i.e. there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that every nonempty open subset of the space contains at least one element of the sequence.

2.0.6 Examples

The space ℓ^∞ of bounded sequences $(a_n)_{n \in \mathbb{N}}$, $a_n \in \mathbb{K}$ with $\|(a_n)_{n \in \mathbb{N}}\|_\infty := \sup_n |a_n|$ is a Banach space.

$$A := \left\{ (a_n)_{n \in \mathbb{N}} \mid a_{2n} = 0 \ \forall_{n \in \mathbb{N}} \right\} \subseteq \ell^\infty$$

is a closed subspace.

$$\ell^\infty / A \cong \left\{ (a_n) \mid a_{2n+1} = 0 \ \forall_{n \in \mathbb{N}} \right\}$$

$$d := \left\{ (a_n) \mid \exists_{N \in \mathbb{N}} \forall_{n \in \mathbb{N}_{>N}} a_n = 0 \right\} \subseteq \ell^\infty$$

is a subspace, but not closed in ℓ^∞ . Consider for example $(a_n = \frac{1}{n}) =: x \in \ell^\infty \setminus d$, $x_n \in d$ with $x_n = (a_{n_l})_{l \in \mathbb{N}}$ and:

$$a_{n_l} = \begin{cases} \frac{1}{l} & \text{if } l \leq n \\ 0 & \text{if } l > n \end{cases}$$

Then converges $x_n \rightarrow x \notin d$, and therefore d is not closed. The closure is:

$$\overline{d} = \left\{ (a_n) \mid a \xrightarrow{n \rightarrow \infty} 0 \right\}$$

ℓ^∞ is not separable.

2.0.7 Example

For $1 \leq p < \infty$ define

$$\ell^p = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

and the ℓ^p -norm:

$$\|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}$$

ℓ^p is a normed space (Hölder's inequality, Minkowski inequality) and also separable (see exercises).

2.0.8 Example

Let (Ω, μ) be a measure space (Maßraum).

$$\begin{aligned} L^p(\Omega) \quad (1 \leq p < \infty) \quad & \|f\|_p = \left(\int_{\Omega} |f(x)|^p d\mu \right)^{\frac{1}{p}} \\ L^{\infty}(\Omega) \quad & \|f\|_{\infty} = \sup_{\Omega} |f(x)| = \sup \{ L \in \mathbb{R} \mid \mu(f^{-1}([L, \infty))) > 0 \} \end{aligned}$$

2.1 Non-Compactness of the Unit Ball

Let $(E, \|\cdot\|)$ be a normed vector space.

$$K := \overline{B_1(0)} = \{x \in E \mid \|x\| \leq 1\}$$

If $\dim(E) < \infty$, K is compact by the Heine-Borel theorem.

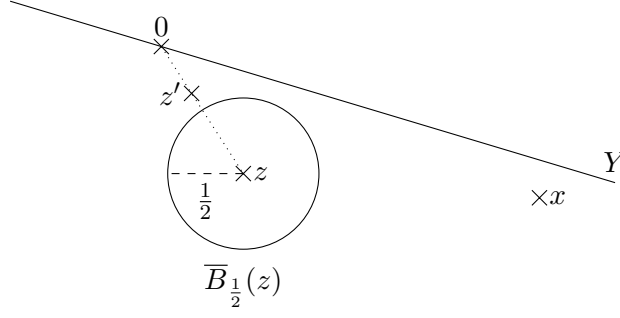
2.1.1 Theorem

If E is infinite-dimensional, then K is not sequentially compact (folgenkompakt), i.e. it is possible to construct a sequence (y_n) , $y_n \in K$, which has no convergent subsequence.

2.1.2 Lemma

Let $Y \subsetneq E$ be a proper (echter) closed subspace. Then there is a $z \in E \setminus Y$ with $\|z\| = 1$ such that holds:

$$\begin{aligned} & \forall_{y \in Y} : \|z - y\| > \frac{1}{2} \\ \Leftrightarrow & \overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset \end{aligned}$$

Figure 2.1: $\overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset$ **Proof**

Choose $x \in E \setminus Y \neq \emptyset$. As $E \setminus Y$ is open, there is a $\delta \in \mathbb{R}_{>0}$ with $B_\delta(x) \cap Y = \emptyset$. Thus we can define:

$$d := \inf_{y \in Y} \|x - y\| > 0$$

Choose $y_0 \in Y$ such that $\|x - y_0\| < 2d$. Set $z' = x - y_0$. Then $\|z'\| < 2d$ and $\|z' - y\| \geq d$ for all $y \in Y$. Thus $z := \frac{z'}{\|z'\|}$ has the desired properties. $\square_{2.1.2}$

Proof of Theorem 2.1.1

Choose inductively a sequence (y_n) : $y_1 \in K$ is arbitrary. $Y_1 := \langle y_1 \rangle$ is a one dimensional subspace, which is closed. Choose $y_2 \in K$ such that $\|y_2 - y\| > \frac{1}{2}$ for all $y \in Y_1$, which is possible according to Lemma 2.1.2.

Suppose y_1, \dots, y_n are given. $Y_n := \langle y_1, \dots, y_n \rangle$ is closed. So there exists a $y_{n+1} \in K$ such that for all $y \in Y_n$ holds:

$$\|y_{n+1} - y\| > \frac{1}{2}$$

This sequence has the following properties:

- $y_k \in K$
- For all $k, l \in \mathbb{N}$ with $k < l$ holds $\|y_l - y_k\| > \frac{1}{2}$, since $y_k \in Y_{l-1} = \langle y_1, \dots, y_{l-1} \rangle$ and we know by construction that $\|y_l - y\| > \frac{1}{2}$ for all $y \in Y_{l-1}$ so especially for $y_k \in Y_{l-1}$.

This implies that (y_k) has no convergent subspace. $\square_{2.1.1}$

2.2 Spaces of linear Mappings, Dual Spaces

Let E, F be normed spaces.

$A : E \rightarrow F$ is continuous if and only if it is bounded, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq C \|u\|_E$$

Denote by $L(E, F)$ the normed space of all bounded linear maps from E to F and define:

$$\|A\| := \sup_{\|u\| \leq 1} \|Au\| = \sup_{\|u\|=1} \|Au\|$$

2.2.1 Lemma

If $B \in L(E, F)$ and $A \in L(F, G)$ then Schwarz inequality or Kato inequality holds:

$$\begin{aligned}\|A \cdot B\| &\leq \|A\| \cdot \|B\| \\ \|Au\| &\leq \|A\| \cdot \|u\|\end{aligned}$$

(no proof)

2.2.2 Theorem and Definition (dual pairing)

If F is complete, so is $L(E, F)$.

Special case $F = \mathbb{R}$ and $\|x\|_{\mathbb{R}} = |x|$: $E^* := L(E, \mathbb{R})$ is the dual space.

For $\varphi \in E^*$ and $u \in E$

$$\varphi(u) = (\varphi, u)$$

is called *dual pairing* (duale Paarung).

$$(\cdot, \cdot) : E^* \times E \rightarrow \mathbb{R}$$

is a continuous bilinear map. For $u \in E$

$$(\cdot, u) : E^* \rightarrow \mathbb{R}$$

defines an element of $E^{**} = L(E^*, \mathbb{R})$. This gives rise to a linear mapping:

$$\iota : E \rightarrow E^{**}$$

(no proof)

2.2.3 Theorem

$\iota : E \hookrightarrow E^{**}$ is an isometric embedding of E into E^{**} .

Proof

For $u \in E$ holds:

$$\|\iota(u)\| := \sup_{\varphi \in E^*, \|\varphi\|=1} \|(\iota(u))(\varphi)\| = \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| \stackrel{?}{=} \|u\|$$

$$\|\varphi\| = \sup_{v \in E, \|v\|=1} |\varphi(v)|$$

$$\begin{aligned}\|\varphi(u)\| &\leq \|\varphi\| \cdot \|u\| \stackrel{\|\varphi\|=1}{=} \|u\| \\ \Rightarrow \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| &\leq \|u\|\end{aligned}$$

To prove $\|\iota(u)\| \geq \|u\|$ apply the Hahn-Banach theorem:

Let $l : \langle u \rangle \rightarrow \mathbb{R}$ be the linear map with $l(u) = \|u\|$, thus:

$$\|l\| = \sup_{v \in \langle u \rangle, \|v\|=1} (l(v)) = \sup \left(l \left(\pm \frac{u}{\|u\|} \right) \right) = 1$$

By the Hahn-Banach theorem we can extend l to

$$\tilde{l} : E \rightarrow \mathbb{R}$$

with $\|\tilde{l}\| = 1$ and then holds:

$$\sup_{\varphi \in E^*, \|\varphi\|=1} \varphi(u) \stackrel{\|\tilde{l}\|=1}{\geq} \tilde{l}(u) = \|u\|$$

Therefore ι is injective, because from $\iota(u) = 0$ follows $\|u\|_E = \|\iota(u)\| = 0$ and therefore $u = 0$. $\square_{2.2.3}$

2.2.4 Definition (reflexive)

A Banach space is called *reflexive* (reflexiv) if ι is bijective, i.e. $E \cong E^{**}$.

2.2.5 Example

Let ℓ_1 be the space of absolutely convergent functions with the norm:

$$\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty$$

Let $(\lambda_n) \in \ell_{\infty}$ be a bounded sequence and define $\Lambda \in \ell_1^*$:

$$\begin{aligned} \Lambda : \ell_1 &\rightarrow \mathbb{R} \\ \Lambda((a_n)) &= \sum_{n=1}^{\infty} \lambda_n a_n \end{aligned}$$

$$|\Lambda((a_n))| = \left| \sum_{n=1}^{\infty} \lambda_n a_n \right| \leq \sum_{n=1}^{\infty} |\lambda_n| \cdot |a_n| \leq \|(\lambda_n)\|_{\infty} \sum_{n=1}^{\infty} |a_n| = \|(\lambda_n)\|_{\infty} \cdot \|(a_n)\|_1 < \infty$$

Thus Λ is bounded and:

$$\|\Lambda\| = \sup_{n \in \mathbb{N}} |\lambda_n|$$

Claim: Every bounded linear functional on ℓ_1 is of this form, i.e. $\ell_1^* = \ell_{\infty}$.

Proof: Let $\Lambda \in \ell_1^*$. Choose $u_l \in \ell_1$ by $u_l = (0, \dots, 0, 1, 0, \dots)$ with a one at the l -th position.

Setting $\lambda_l := \Lambda(u_l)$ gives:

$$|\lambda_l| = |\Lambda(u_l)| \leq \underbrace{\|\Lambda\|}_{< \infty} \cdot \underbrace{\|u_l\|}_{=1} \leq \|\Lambda\| < \infty$$

So $(\lambda_l) \in \ell_\infty$.

Let (a_k) be a finite sequence, with only zeros for $k > K \in \mathbb{N}$. Then:

$$\Lambda((a_k)) = \Lambda\left(\sum_{k=1}^K a_k u_k\right) = \sum a_k \Lambda(u_k) = \sum \lambda_k a_k$$

Since the finite sequences are dense in ℓ_1 , the claim follows. \square_{Claim}

So $\ell_1^* = \ell_\infty$ and one could assume $\ell_\infty^* = \ell_1$, but this is not the case (see exercises).

Thus $\ell_1^{**} \neq \ell_1$, which means, that ℓ_1 is *not* reflexive.

2.3 Weak Convergence (Schwache Konvergenz)

Let E be a Banach space and (u_n) a sequence in E .

Normal convergence: $u_n \rightarrow u$ if and only if $\|u - u_n\| \xrightarrow{n \rightarrow \infty} 0$.

2.3.1 Definition (weak convergence, weak Cauchy sequence)

A sequence (u_n) in E *converges weakly* to u , written as $u_n \rightharpoonup u$, if for all $\varphi \in E^*$ the sequence $\varphi(u_n)$ converges to $\varphi(u)$, i.e. $\varphi(u_n) \rightarrow \varphi(u)$.

(u_n) is a *weak Cauchy sequence* if for all $\varphi \in E^*$ the sequence $\varphi(u_n)$ is a Cauchy sequence.

2.3.2 Theorem (Uniqueness of weak limit)

The weak limit is unique.

Proof

Let (u_n) be a sequence in E , which converges weakly to u and u' , i.e. for all $\varphi \in E^*$ holds:

$$\begin{aligned} \varphi(u_n) &\rightarrow \varphi(u) & \varphi(u_n) &\rightarrow \varphi(u') \\ \Rightarrow \quad 0 &= \varphi(u_n - u_n) \rightarrow \varphi(u - u') \end{aligned}$$

So $\varphi(u - u') = 0$ for all $\varphi \in E^*$.

Claim: $v := u - u' = 0$

Proof: Assume to the contrary that $v \neq 0$.

Choose $\varphi : \langle v \rangle \rightarrow \mathbb{R}$ with $\varphi(v) = 1$. By the Hahn-Banach theorem φ can be extended continuously to E .

Therefore exists a $\varphi \in E^*$ with $\varphi(v) = 1$, which is a contradiction to $\varphi(v) = 0$. \square_{Claim}

$\square_{2.3.2}$

2.3.3 Theorem (convergence implies weak convergence)

Every convergent sequence converges weakly.

Proof

Suppose that $u_n \rightarrow u$. For $\varphi \in E^*$ follows:

$$|\varphi(u_n) - \varphi(u)| = |\varphi(u_n - u)| \leq \underbrace{\|\varphi\|}_{\in \mathbb{R}} \cdot \|u_n - u\| \rightarrow 0$$

$$\begin{aligned} \Rightarrow \quad & \varphi(u_n) \rightarrow \varphi(u) \\ \Rightarrow \quad & u_n \rightarrow u \end{aligned}$$

□_{2.3.3}**2.3.4 Example**

$E = \left\{ (a_n) \left| a_n \xrightarrow{n \rightarrow \infty} 0 \right. \right\} \subsetneq \ell_\infty$ with $\|(a_n)\| = \sup_n |a_n|$ is a Banach space.

Let $u_n = (0, \dots, 0, 1, 0, \dots)$ be the sequence with a one at the n -th position and zeros elsewhere. For $n \neq m$ we have:

$$\|u_n - u_m\| = \sup \{0, |1|, |-1|\} = 1$$

Thus (u_n) is *not* a Cauchy sequence. Every $\varphi \in E^*$ can be represented with $(\lambda_k) \in \ell_1$ as (see exercises):

$$\begin{aligned} \varphi((a_n)) &= \sum_k \lambda_k a_k \\ \|\varphi\| &= \sum_{k=1}^{\infty} |\lambda_k| < \infty \end{aligned}$$

$$\varphi(u_n) = \sum_{k=1}^{\infty} \lambda_k \delta_{kn} = \lambda_n \xrightarrow{n \rightarrow \infty} 0$$

From $(\lambda_n) \in \ell_1$ follows $\lambda_n \rightarrow 0$. This means that $u_k \rightarrow 0$.

This is used in the lectures on partial differential equations.

From $\mathcal{S}(u_n) \rightarrow \inf \mathcal{S}$ follows not necessarily $u_n \rightarrow u$, but $u_n \rightarrow u$.

Consider $A_n \in L(E, F)$.

- *norm convergence*: $A_n \rightarrow A$ in $L(E, F)$ means $\|A_n - A\| \rightarrow 0$.
- *strong convergence*: $A_n u \rightarrow Au$ in F for all $u \in E$.
- *weak convergence*: $A_n u \rightarrow Au$ for all $u \in E$, i.e. for all $\varphi \in F^*$ holds $\varphi(A_n u) \rightarrow \varphi(Au)$.

2.4 The Baire Category Theorem

Let E be a metric space (e.g. a normed space).

2.4.1 Definition (nowhere dense, set of first/second category)

A subset $A \subseteq E$ is called *nowhere dense* (nirgends dicht) if $\overline{A}^\circ = \emptyset$.

A is called *of first category* (or *meager*) if it can be written as a countable union of nowhere dense sets. Otherwise it is *of second category*.

Example

- $\mathbb{N} \subseteq \mathbb{R}$ is nowhere dense: $\overline{\mathbb{N}} = \mathbb{N}$, $\mathbb{N}^\circ = \emptyset$
- $\mathbb{Q} \subseteq \mathbb{R}$ is dense: $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Q}}^\circ = \mathbb{R}^\circ = \mathbb{R}$

2.4.2 Theorem (René Baire, 1899)

Let $E \neq \emptyset$ be a complete metric space (Polish space). Then E is of second category.

Proof

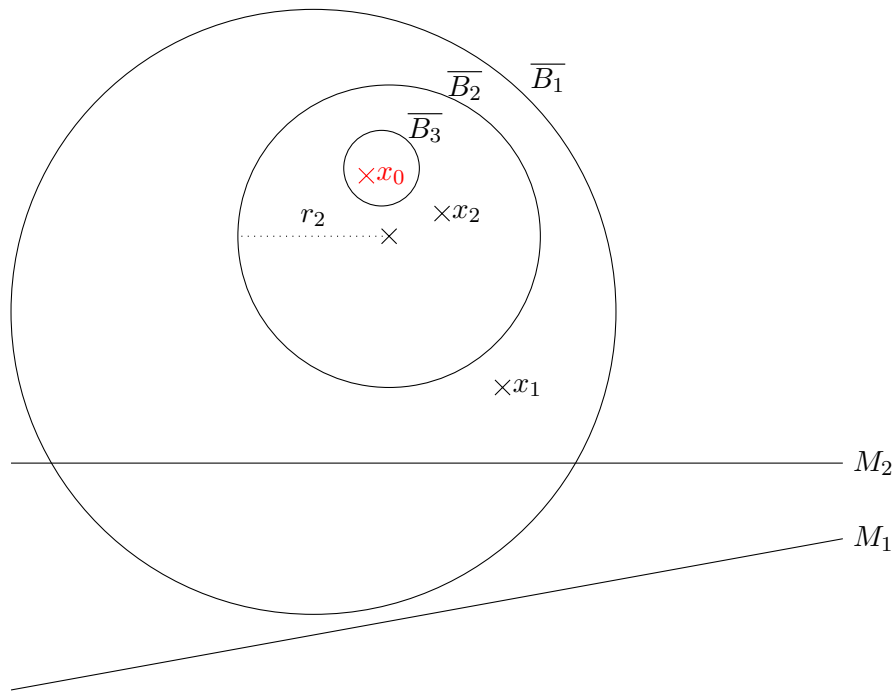


Figure 2.2: $B_n \cap M_n = \emptyset$

Assume in contrast that $E = \bigcup_{n \in \mathbb{N}} M_n$ and the sets M_n are nowhere dense. Without loss of generality assume that the M_n are closed, since otherwise one can replace M_n by $\overline{M_n}$.

We shall construct inductively balls $\overline{B_n} = \overline{B_{r_n}(x_n)}$ such that $\overline{B_{n+1}} \subseteq \overline{B_n}$, $r_n < 2^{-n}$ and $\overline{B_n} \cap M_n = \emptyset$ for all n .

Then the points x_n form a Cauchy sequence, because for all $n < m \in \mathbb{N}$ we have $x_{n+1} \in B_n$ and so $\|x_n - x_{n+1}\| < r_n < 2^{-n}$:

$$\|x_n - x_m\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_m\| \leq \dots \leq$$

$$\leq 2^{-n} + 2^{-(n+1)} + \dots + 2^{-(m-1)} \leq 2^{-n} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) \leq 2 \cdot 2^{-n}$$

Since E is complete, $x_n \rightarrow x_0 \in E$ converges. Then $x_0 \in \overline{B_n}$ for all n , which implies $x_0 \notin M_n$ and thus the contradiction $x_0 \notin \bigcup_n M_n = E$ follows.

Construction of the balls $\overline{B_n}$:

M_1 is nowhere dense and therefore $B_1(0) \not\subseteq M_1$. So there exists a $x_1 \in B_1(0) \setminus M_1$. Since M_1 is closed, $B_1(0) \setminus M_1$ is open and therefore there exists a radius r_1 such that $B_{2r_1}(x_1)$ is contained in $B_1(0) \setminus M_1$ and thus $\overline{B_{r_1}(x_1)} \cap M_1 = \emptyset$.

Suppose $\overline{B_n}$ has been constructed. M_{n+1} is nowhere dense and closed and so there exists a $x_{n+1} \in \overline{B_n} \setminus M_{n+1}$ and $r_{n+1} < 2^{-(n+1)}$ such that $B_{2r_{n+1}}(x_{n+1}) \subseteq \overline{B_n} \setminus M_{n+1}$. Then follows $\overline{B_{r_{n+1}}(x_{n+1})} \cap M_{n+1} = \emptyset$. $\square_{2.4.2}$

2.4.3 Theorem (Uniform boundedness principle, Prinzip der gleichmäßigen Beschränktheit)

Let E be a Banach space and F a normed space. Let T_i be a sequence in $L(E, F)$ which is point-wise bounded, i.e. for all $u \in E$:

$$\sup_i \|T_i u\| \leq C(u) < \infty$$

Then sup-norms of T_i are bounded:

$$\sup_i \|T_i\| = \sup_i \sup_{\|u\|=1} \|T_i u\| \leq \tilde{C} < \infty$$

(Thus there exists a constant $C \in \mathbb{R}_{>0}$ such that $\|T_i u\| \leq C$ for all $i \in \mathbb{N}$ and for all $u \in E$ with $\|u\| = 1$.)

Proof

The sets $M_n = \{u \in E \mid \sup_i \|T_i u\| \leq n\}$ are closed by continuity of the $T_i \in L(E, F)$, i.e. for $u_k \rightarrow u$ converges $\|T_i u_k\| \xrightarrow{k \rightarrow \infty} \|T_i u\|$.

$E = \bigcup_n M_n$, because for any $u \in E$, $\sup_i \|T_i u\| < \infty$ and thus $u \in M_n$ for $n > \sup_i \|T_i u\|$.

If all the sets M_n had empty interior, we would get a contradiction to Baire's theorem.

So there exists an $n_0 \in \mathbb{N}$ such that $M_{n_0} \neq \emptyset$ and thus there are $u_0 \in E$ and $r \in \mathbb{R}_{>0}$ such that $B_r(u_0) \subseteq M_{n_0}$.

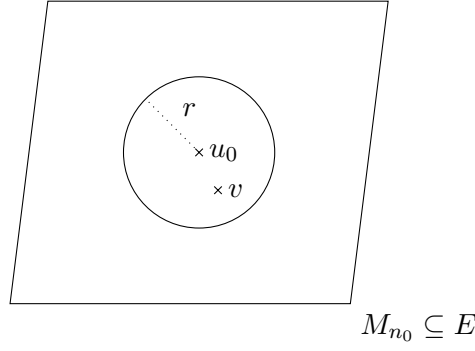
For all $v \in B_r(u_0)$ we know that $\sup_i \|T_i v\| \leq n_0$ which is equivalent to:

$$\sup_{v \in B_r(u_0)} \|T_i v\| \leq n_0 \quad \forall_{i \in \mathbb{N}}$$

Let $w \in B_r(0)$ be arbitrary. Then $v := u_0 + w \in B_r(u_0)$.

$$T_i w \stackrel{T_i \text{ linear}}{=} T_i v - T_i u_0$$

$$\|T_i w\| \leq \|T_i v\| + \|T_i u_0\| \leq n_0 + \sup_i \|T_i u_0\| < \infty$$

Figure 2.3: $B_r(u_0) \subseteq M_{n_0}$

Here $\sup_i \|T_i u_0\| < \infty$, because the T_i are point-wise bounded.

$$\begin{aligned} \Rightarrow \quad \|T_i w\| &\leq C && \forall w \in B_r(0) \\ \Rightarrow \quad \|T_i \tilde{w}\| &\leq \tilde{C} = \frac{C}{r} && \forall \tilde{w} \in \overline{B_1(0)} \end{aligned}$$

So $\|T_i\| \leq \tilde{C}$ for all $i \in \mathbb{N}$ and so $\|T_i\|$ is bounded. $\square_{2.4.3}$

2.4.4 Corollary

Let E be a normed space, not necessarily complete, and (u_n) a weak Cauchy sequence. Then $\|u_n\|$ is a bounded sequence.

Proof

$E^* = L(E, \mathbb{R})$ is a Banach space after theorem 2.2.2, since \mathbb{R} is complete. Now we can view every u_n as operator:

$$\begin{aligned} u_n : E^* &\rightarrow \mathbb{R} \\ \varphi &\mapsto \varphi(u_n) \end{aligned}$$

So (u_n) is a sequence in $L(E^*, \mathbb{R})$. For all $\varphi \in E^*$ we know that $\varphi(u_n)$ is a Cauchy sequence and thus bounded:

$$\Rightarrow \quad |\varphi(u_n)| < C(\varphi)$$

Applying theorem 2.4.3 yields:

$$\begin{aligned} |\varphi(u_n)| &< C && \forall \varphi \text{ with } \|\varphi\|=1 \\ \Leftrightarrow \quad \sup_{n \in \mathbb{N}} \sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(u_n)| &< C \end{aligned}$$

For any $v \in E$ we have

$$\sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(v)| = \|v\|$$

by the Hahn-Banach theorem:

- $|\varphi(v)| \leq \|\varphi\| \cdot \|v\| \stackrel{\|\varphi\|=1}{=} \|v\|$
- Choose $\varphi : \langle v \rangle \rightarrow \mathbb{R}$ with $\varphi(v) = \|v\|$ and so $\|\varphi\| = 1$. By the Hahn-Banach theorem we can extend φ to $\tilde{\varphi} : E \rightarrow \mathbb{R}$ such that $\|\tilde{\varphi}\| = 1$. Then $\tilde{\varphi}(v) = \|v\|$ and so $\sup_{\|\varphi\|=1} |\varphi(v)| \geq \|v\|$.

Thus we get $\sup_n \|u_n\| < C$.

□_{2.4.4}

2.4.5 Corollary and Definition (Banach-Steinhaus, equicontinuous, uniformly continuous)

Let E, F be Banach spaces and $T_i \in L(E, F)$.

If the (T_i) are point-wise bounded, then the T_i are *equicontinuous* (gleichgradig stetig).

Definition (uniformly continuous, equicontinuous)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function.

Continuity:

$$\forall_{x_0 \in \mathbb{R}} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

f is called *uniformly continuous* (gleichmäßig stetig) if:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad \|x - y\| < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon$$

Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a series of real-valued functions. (f_n) is called *equicontinuous* if:

$$\forall_{x_0 \in \mathbb{R}} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} \quad \forall_{n \in \mathbb{N}} : \quad \|x - x_0\| < \delta \quad \Rightarrow \quad \|f_n(x) - f_n(x_0)\| < \varepsilon$$

For a linear map $A \in L(E, F)$ holds:

$$\begin{aligned} \|Au\| &\leq \|A\| \|u\| \\ \|Au - Au_0\| &\leq \|A\| \|u - u_0\| \end{aligned}$$

Therefore choose $\delta = \frac{\varepsilon}{2\|A\|}$, i.e.:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad \|u\| < \delta \quad \Rightarrow \quad \|Au\| < \varepsilon$$

Proof

Since (T_i) is point-wise bounded there is a $C \in \mathbb{R}_{>0}$ such that for all $i \in \mathbb{N}$ holds $\|T_i\| \leq C$ due to the principle of uniform boundedness 2.4.3. So for all $i \in \mathbb{N}$ holds:

$$\|T_i u\| \leq \|T_i\| \|u\| \leq C \|u\|$$

Choose $\delta = \frac{\varepsilon}{2C}$ shows that the T_i is equicontinuous.

□_{2.4.5}

In the following let E and F be Banach spaces.

2.4.6 Definition (open)

A (not necessarily linear) map $A : E \rightarrow F$ is called *open* if the image of every open set is open. (If there exists an inverse A^{-1} then “ A open” is equivalent to “ A^{-1} continuous”.)

Let A be linear and open. $B_1(0) \subseteq E$ is open, so $A(B_1(0)) \subseteq F$ is open. Since $0 \in A(B_1(0))$, there is a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(0) \subseteq A(B_1(0))$.

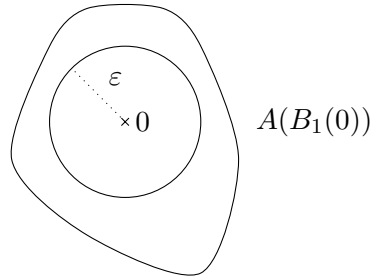


Figure 2.4: $B_\varepsilon(0) \subseteq A(B_1(0))$

Due to the linearity holds in general:

$$B_\lambda(0) \subseteq A\left(B_{\frac{\lambda}{\varepsilon}}(0)\right)$$

In particular, A is surjective.

If A is additionally injective, then A is bijective and the openness means that A^{-1} is continuous.

2.4.7 Theorem (Open mapping theorem, Prinzip der offenen Abbildung)

If $A \in L(E, F)$ is surjective, then A is open.

2.4.8 Corollary

If $A \in L(E, F)$ is bijective, then $A^{-1} \in L(F, E)$ is continuous.

Proof

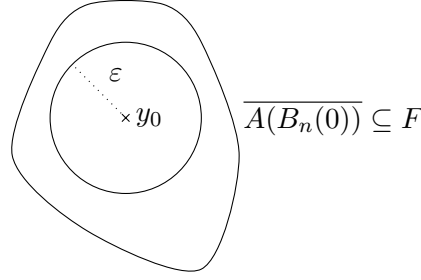
A is open following 2.4.7, since A is surjective. This means that A^{-1} is continuous. $\square_{2.4.8}$

Proof of 2.4.7

Since A is surjective, $F = A(E)$. Since every element of E has a finite norm, we know:

$$\begin{aligned} E &= \bigcup_{n \in \mathbb{N}} B_n(0) \\ \Rightarrow F &= A\left(\bigcup_{n \in \mathbb{N}} B_n(0)\right) = \bigcup_{n \in \mathbb{N}} A(B_n(0)) \end{aligned}$$

According to Baire's theorem there is a $n \in \mathbb{N}$ such that $\overline{A(B_n(0))}^\circ \neq \emptyset$.

Figure 2.5: $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$

So there exists a $y_0 \in A(B_n(0))$ and a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$. Since A is surjective, there is a $x_0 \in B_n(0)$ with $y_0 = A(x_0)$.

$$\Rightarrow \overline{A(B_n(0) - x_0)} = \overline{A(B_n(0)) - y_0} = \overline{A(B_n(0))} - y_0 \supseteq B_\varepsilon(y_0) - y_0 = B_\varepsilon(0)$$

If n' is large enough, then $B_n(-x_0) \subseteq B_{n'}(0)$ and so $\overline{A(B_{n'}(0))} \supseteq B_\varepsilon(0)$.

Since A is linear, we can rescale, i.e. there is a $c := \frac{\varepsilon}{n'} \in \mathbb{R}_{>0}$ such that for all $r \in \mathbb{R}_{>0}$ holds:

$$\overline{A(B_r(0))} \supseteq B_{cr}(0)$$

Now we show that every $u \in B_c(0)$ is the image of a $x \in B_2(0)$, i.e. $B_c(0) \subseteq A(B_2(0))$:

Ansatz as a series:

$$x = \sum_{j=1}^{\infty} x_j$$

Choose $x_1 \in B_1(0)$ with $\|u - Ax_1\| < \frac{c}{2}$, which is possible since $\overline{A(B_1(0))} \supseteq B_c(0)$.

Choose $x_2 \in B_2(0)$ with $\|u - Ax_1 - Ax_2\| < \frac{c}{4}$, which is possible since $u - Ax_1 \in B_{\frac{c}{2}}(0)$ and

$$\overline{A\left(B_{\frac{1}{2}}(0)\right)} \subseteq B_{\frac{c}{2}}(0).$$

And so on choose $x_m \in B_{\frac{1}{2^m}}(0)$ with $\|u - \sum_{i=1}^m Ax_i\| < \frac{c}{2^m}$.

The series $\sum_{i=1}^{\infty} x_i$ converges, since:

$$\left\| \sum_{j=m}^M x_j \right\| \leq \sum_{j=m}^M \|x_j\| \leq \sum_{j=m}^M 2^{-j}$$

So the sequence of partial sums is a Cauchy sequence. Because E is complete, this sequence converges.

The continuity of A yields:

$$Ax = \sum_{j=1}^{\infty} Ax_j = u$$

So there exists a $x \in E$ with $\|x\| < 2$ and $Ax = u$.

□_{2.4.7}

$$\sum_{j=1}^n x_j \xrightarrow{n \rightarrow \infty} x \qquad \|x\| < 2$$

$$\begin{aligned} & \sum_{j=1}^n Ax_j \xrightarrow{n \rightarrow \infty} u \\ & \parallel \\ & A \left(\sum_{j=1}^n x_j \right) \xrightarrow[\text{continuity of } A]{n \rightarrow \infty} Ax \end{aligned}$$

Definition (Graph)

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the *graph* is defined as:

$$\text{graph} f := \{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

For $A : E \rightarrow F$ the *graph* is:

$$\text{graph} A := \{(u, Au) \mid u \in E\} \subseteq E \times F$$

Here $E \times F$ is a product of normed spaces which has the norm:

$$\|(u, v)\| := \|u\|_E + \|v\|_F$$

Lemma

If A is continuous, then $\text{graph} A$ is closed.

Proof

Let $(u_n, Au_n) \in \text{graph} A$ be a Cauchy sequence in $E \times F$ for Banach spaces E and F , i.e. $u_n \rightarrow u$. Since A is continuous, it follows:

$$Au_n \rightarrow v := Au$$

Therefore $(u, v) \in \text{graph}(A)$ and so the graph is closed. □ Lemma

Consider the function:

$$\begin{aligned} f : \mathbb{R} \setminus \{0\} &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{x} \end{aligned}$$

f is not continuous, but $\text{graph}(f)$ is closed in $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.

2.4.9 Theorem (Closed graph theorem, Satz vom abgeschlossenen Graphen)

Suppose a linear map $A : E \rightarrow F$ between Banach spaces E and F has a closed graph. Then A is continuous.

$\text{graph}(A)$ closed means:

For all $u_n \in E$ with $u_n \rightarrow u$ and $Au_n \rightarrow v$, the point $(u, v) \in \text{graph}(A)$, i.e. $Au = v$.

A continuous means:

For all $u_n \in E$ with $u_n \rightarrow u$, the sequence $Au_n \rightarrow v$ converges and $Au = v$

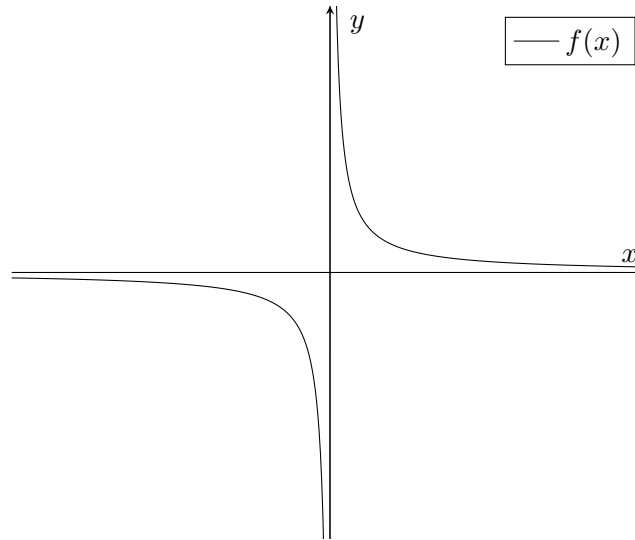


Figure 2.6: f is not continuous, but $\text{graph } f$ is closed.

Proof

On $E \times F$ we have the norm:

$$\|(u, v)\| := \|u\|_E + \|v\|_F$$

The graph

$$G := \{(u, Au) \mid u \in E\} \subseteq E \times F$$

is a subspace of $E \times F$, since for $\lambda \in \mathbb{R}$ and $u, \tilde{u} \in E$ holds:

$$\lambda(u, Au) + (\tilde{u}, A\tilde{u}) = (\lambda u + \tilde{u}, \lambda Au + A\tilde{u}) \stackrel{A \text{ linear}}{=} (\lambda u + \tilde{u}, A(\lambda u + \tilde{u})) \in G$$

So G is complete and therefore a Banach space, since we assumed it to be closed.

Define:

$$\begin{aligned} P : G &\rightarrow E \\ (u, Au) &\mapsto u \end{aligned}$$

$$\|(u, Au)\| = \|u\| + \|Au\| \geq \|u\| = \|P(u, Au)\|$$

So for all $w \in G$ holds $\|Pw\| \leq \|w\|$ and therefore $\|P\| \leq 1$. In particular, P is continuous.

P is obviously surjective and it is also injective, since:

$$P^{-1}(u) = (u, Au)$$

Following the open mapping theorem, P^{-1} is continuous, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\|u\| + \|Au\| = \|(u, Au)\| = \|P^{-1}(u)\| \leq C \|u\|$$

Then follows:

$$\|Au\| \leq (C - 1) \|u\|$$

Therefore A is continuous.

□_{2.4.9}

2.5 Neumann series

Let E be a Banach space and $A \in L(E, E) =: L(E)$.

When is A continuously invertible?

Remember that for $x \in \mathbb{K}$ with $|x| < 1$ holds:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is the geometric series.

Idea: $A = \mathbb{1} - B$ with $B \in L(E)$

$$\text{Ansatz: } A^{-1} := \sum_{n=0}^{\infty} B^n$$

This works indeed if $\|B\| < 1$.

2.5.1 Lemma and Definition (Neumann series)

The series

$$C := \sum_{n=0}^{\infty} B^n$$

is called Neumann series (Neumannsche Reihe).

If $\|B\| < 1$, then C defines an element of $L(E, E)$, i.e. the Neumann series converges absolutely.

Proof

Consider the partial sums:

$$S_n := \sum_{k=0}^n B^k$$

Since $L(E, E)$ is a Banach space, it is enough to show that S_n is a Cauchy series. Without loss of generality assume $m > n$:

$$\|S_n - S_m\| = \left\| \sum_{k=n}^m B^k \right\| \stackrel{\Delta \text{ inequality}}{\leq} \sum_{k=n}^m \|B^k\| \stackrel{\text{Schwarz}}{\leq} \sum_{k=n}^m \|B\|^k < c \|B\|^n \rightarrow 0$$

□_{2.5.1}

2.5.2 Theorem

$$C = (\mathbb{1} - B)^{-1}$$

Proof

$$(\mathbb{1} - B)C = (\mathbb{1} - B) \sum_{n=0}^{\infty} B^n = (\mathbb{1} + B + B^2 + \dots) - (B + B^2 + \dots) = \mathbb{1}$$

□_{2.5.2}**2.5.3 Theorem**

The set of all continuously invertible mappings is open in $L(E)$.

Proof

Assume that $A \in L(E)$ is continuously invertible, i.e. A^{-1} exists and $A^{-1} \in L(E)$. Set:

$$\varepsilon = \frac{1}{2\|A^{-1}\|}$$

Let us show, that every element of $B_\varepsilon(A) \subseteq L(E)$ is continuously invertible:

Let $C \in B_\varepsilon(A)$, i.e. $\|A - C\| < \varepsilon$.

$$C = A - (A - C) = A(\mathbb{1} - \underbrace{A^{-1}(A - C)}_{=:B})$$

Then holds:

$$\|B\| \leq \|A^{-1}\| \cdot \|A - C\| < \|A^{-1}\| \cdot \frac{1}{2\|A^{-1}\|} = \frac{1}{2} < 1$$

Hence $\mathbb{1} - B$ is continuously invertible by the Neumann series and therefore

$$C^{-1} = (\mathbb{1} - B)^{-1} \cdot A^{-1}$$

is continuous.

□_{2.5.3}

3 Hilbert spaces

Definition (scalar product)

Let H be a real ($\mathbb{K} := \mathbb{R}$) or complex ($\mathbb{K} := \mathbb{C}$) vector space with *scalar product*:

$$\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{K}$$

- i) Positive definiteness: $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Rightarrow u = 0$.
- ii) Linear in the second and anti-linear in the first argument:

$$\langle \lambda u, v \rangle = \bar{\lambda} \langle u, v \rangle$$

- iii) Symmetry: $\overline{\langle u, v \rangle} = \langle u, v \rangle$

Define the corresponding norm:

$$\|u\| := \sqrt{\langle u, u \rangle}$$

3.0.1 Definition (Hilbert space)

A complete scalar product space is called *Hilbert space*.

The Schwarz inequality holds:

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

3.0.2 Lemma (parallelogram equality)

The parallelogram equality (Parallelogramm-Gleichung) is:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Proof

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ \|u - v\|^2 &= \langle u - v, u - v \rangle = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\ \Rightarrow \|u + v\|^2 + \|u - v\|^2 &= 2(\|u\|^2 + \|v\|^2) \end{aligned}$$

□_{3.0.2}

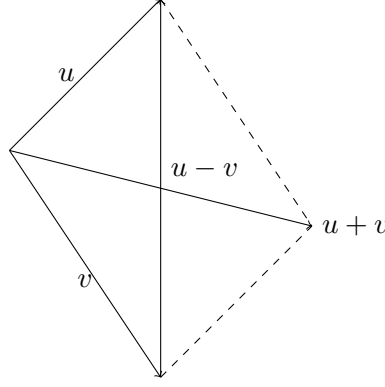


Figure 3.1: parallelogram

3.0.3 Definition (orthogonal, orthonormal)

- i) Vectors $u, v \in H$ are called *orthogonal*, symbolically $u \perp v$, if $\langle u, v \rangle = 0$.
- ii) Subspaces $M_1, M_2 \subseteq H$ are orthogonal, symbolically $M_1 \perp M_2$, if $\langle u, v \rangle = 0$ for all $u \in M_1$ and $v \in M_2$.
- iii) A family $(u_i)_{i \in I}$ of vectors $u_i \in H$ is called *orthonormal* if:

$$\langle u_i, u_j \rangle = \delta_{ij}$$

3.0.4 Theorem (Bessel's inequality)

Let $(u_i)_{1 \leq i \leq N}$ be an orthonormal family. Then for all $u \in H$ holds:

$$\begin{aligned} \|u\|^2 &= \sum_{i=1}^N \langle u_i, u \rangle^2 + \left\| u - \sum_{i=1}^N u_i \langle u_i, u \rangle \right\|^2 \\ \|u\|^2 &\geq \sum_{i=1}^N \langle u_i, u \rangle^2 \end{aligned}$$

Proof

$$\begin{aligned} \left\| u - \sum_{i=1}^N u_i \langle u_i, u \rangle \right\|^2 &= \left\langle u - \sum_{i=1}^N u_i \langle u_i, u \rangle, u - \sum_{j=1}^N u_j \langle u_j, u \rangle \right\rangle = \\ &= \langle u, u \rangle - \sum_{j=1}^N \langle u, u_j \rangle \langle u_j, u \rangle - \sum_{i=1}^N \overline{\langle u_i, u \rangle} \langle u_i, u \rangle + \sum_{i,j=1}^N \overline{\langle u_i, u \rangle} \langle u_j, u \rangle \underbrace{\langle u_i, u_j \rangle}_{=\delta_{ij}} = \\ &= \|u\|^2 - 2 \sum_{i=1}^N |\langle u_i, u \rangle|^2 + \sum_{i=1}^N |\langle u_i, u \rangle|^2 = \\ &= \|u\|^2 - \sum_{i=1}^N |\langle u_i, u \rangle|^2 \end{aligned}$$

□_{3.0.4}

Definition (Hilbert space isomorphism)

Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces.

A *Hilbert space isomorphism* is a mapping $U : H_1 \rightarrow H_2$ which is linear, bijective and isometric (isometrisch), i.e. for all $u, v \in H_1$:

$$\langle u, v \rangle_1 = \langle Uu, Uv \rangle_2$$

Definition (Direct sum)

Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be Hilbert spaces.

Define:

$$H := \{(u, v) \mid u \in H_1, v \in H_2\}$$

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$

$$\lambda(u, v) := (\lambda u, \lambda v)$$

$$\langle (u_1, v_1), (u_2, v_2) \rangle := \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$

This makes $H =: H_1 \oplus H_2$ a Hilbert space, called *direct sum* of H_1 and H_2 , which is sometimes called orthogonal due to:

$$\langle (u, 0), (0, v) \rangle = 0$$

3.0.5 Example

$$\ell_2 = \left\{ (a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{K}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

Define a scalar product:

$$\langle (a_n), (b_n) \rangle := \sum_{n=1}^{\infty} \bar{a}_n \cdot b_n$$

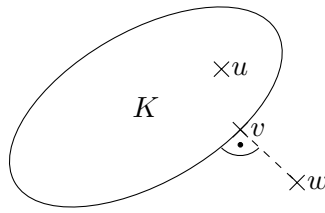
$$\langle (a_n), (a_n) \rangle = \sum_{n=1}^{\infty} |a_n|^2 = \|a_n\|_2^2$$

$(\ell^2, \|\cdot\|_2)$ is a Banach space. Thus $(\ell^2, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

3.1 Projection on closed convex subsets

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $K \subseteq H$ a closed convex subset.

$$u, v \in K \qquad w \in H \setminus K$$

Figure 3.2: $\|v - w\| = \inf_{u \in K} \|u - w\|$

We want to find a vector v such that $\|v - w\| = \inf_{u \in K} \|u - w\|$.

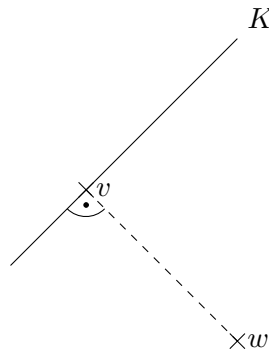
If K were compact, then choose minimizing sequence (Minimalfolge), i.e.:

$$\|u_i - w\| \rightarrow \inf_{u \in K} \|u - w\|$$

Choose a convergent subsequence $u_{i_l} \rightarrow v$. Then by continuity:

$$\|v - w\| = \lim_{i \rightarrow \infty} \|u_i - w\| = \inf_{u \in K} \|u - w\|$$

The main application are closed subspaces $K \subseteq H$.

Figure 3.3: $v - w \perp K$

In this case $v - w$ will be called orthogonal to K motivating the name *orthogonal projection*.

3.1.1 Theorem (Hilbert)

There is a unique $v \in K$ with:

$$\|v - w\| = \inf_{u \in K} \|u - w\|$$

Proof

Consider a minimizing sequence u_i :

$$\|u_i - w\| \rightarrow \inf_{u \in K} \|u - w\| =: d$$

We show that (u_i) is a Cauchy sequence:

$$\begin{aligned}
 \|u_i - u_j\|^2 &= \|(u_i - w) + (w - u_j)\|^2 = \\
 &\stackrel{3.0.2}{=} 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \|(u_i - w) - (w - u_j)\|^2 = \\
 &= 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \left\| -2\left(w - \frac{u_i + u_j}{2}\right) \right\|^2 = \\
 &= 2\left(\underbrace{\|u_i - w\|^2}_{\rightarrow d^2} + \underbrace{\|w - u_j\|^2}_{\rightarrow d^2} - 2\left\| \frac{u_i + u_j}{2} - w \right\|^2 \right)
 \end{aligned}$$

$$\|u_i - w\| \xrightarrow{i \rightarrow \infty} d = \inf_{u \in K} \|u - w\|$$

$$\|u_j - w\| \xrightarrow{j \rightarrow \infty} d = \inf_{u \in K} \|u - w\|$$

Since K is convex and $u_i, u_j \in K$, we know:

$$\frac{u_i + u_j}{2} \in K$$

$$\Rightarrow \left\| \frac{u_i + u_j}{2} - w \right\| \geq d$$

Thus:

$$\|u_i - u_j\|^2 \leq 2\left(\|u_i - w\|^2 + \|w - u_j\|^2 - 2d^2\right) \xrightarrow{i,j \rightarrow \infty} 2(d^2 + d^2 - 2d^2) = 0$$

So there exists a $N \in \mathbb{N}$ such that $\|u_i - u_j\| < \varepsilon$ for all $i, j > N$. Therefore (u_i) is a Cauchy sequence. Since H is complete, we know that $u_i \rightarrow u$ converges.

By continuity follows:

$$\|u - w\| = \lim_{i \rightarrow \infty} \|u_i - w\| = d$$

Uniqueness follows from the fact, that *every* minimizing sequence converges:

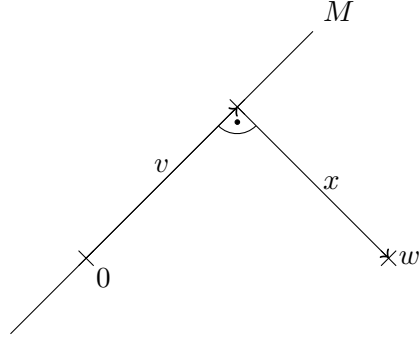
Let u, \tilde{u} be both minimizers, then the sequence $(u, \tilde{u}, u, \tilde{u}, \dots)$ is a minimizing sequence. Since it converges, $u = \tilde{u}$. $\square_{3.1.1}$

3.1.2 Corollary

Let $M \subseteq H$ be a closed subspace of H . Then a $w \in H$ can be decomposed uniquely in the form

$$w = v + x$$

with $v \in M$ and $x \in M^\perp$. We write $H = M \oplus M^\perp$.

Figure 3.4: $w = v + x$ **Proof**

Let $v \in M$ be as in Theorem 3.1.1.

$$\|v - w\| = \inf_{u \in M} \|u - w\|$$

Define $x := w - v$.

- H real: For $u \in M$ define $\tilde{u}(\tau) = v + \tau u$ with $\tau \in \mathbb{R}$.

$$\begin{aligned} \|\tilde{u} - w\|^2 &= \|x\|^2 + 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 \geq \|x\|^2 \\ 0 &\leq 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 =: f(\tau) \end{aligned}$$

$f(\tau)$ has a minimum at $\tau = 0$ and so $f'(0) = 0$.

$$\begin{aligned} f'(0) &= 2 \langle u, x \rangle \\ \Rightarrow 2 \langle u, x \rangle &= 0 \quad \forall_{u \in M} \end{aligned}$$

So $x \in M^\perp$.

- H complex: Define $\tilde{u}(\tau) = v + \tau u$, $\tau = re^{i\varphi} \in \mathbb{K}$ with $r \geq 0$.

$$\|\tilde{u} - w\|^2 = \|x\|^2 + 2\operatorname{Re} \left(re^{-i\varphi} \langle u, x \rangle \right) + r^2 \|u\|^2 =: f(r, \varphi)$$

This has a minimum at $r = 0$.

$$\begin{aligned} \Rightarrow 0 &= \partial_r f(0, \varphi) = 2\operatorname{Re} \left(e^{-i\varphi} \langle u, x \rangle \right) \\ \varphi \text{ arbitrary} \Rightarrow \langle u, x \rangle &= 0 \end{aligned}$$

So $x \in M^\perp$.

Uniqueness: Assume that $w = v_1 + x_1 = v_2 + x_2$ where $v_1, v_2 \in M$, $x_1, x_2 \in M^\perp$.

$$\underbrace{v_1 - v_2}_{\in M} = \underbrace{x_2 - x_1}_{\in M^\perp} \in M \cap M^\perp = \{0\}$$

Because from $u \in M \cap M^\perp$ follows $\langle u, u \rangle = 0$ and so $u = 0$.

□_{3.1.2}

For a Banach space E we have E, E^*, E^{**} and a natural injection $\iota : E \hookrightarrow E^{**}$.

For a Hilbert space H , suppose $u \in H$ and define:

$$\begin{aligned}\varphi &: H \rightarrow \mathbb{K} \\ \varphi(v) &:= \langle u, v \rangle\end{aligned}$$

φ is continuous, because:

$$|\varphi(v)| = |\langle u, v \rangle| \leq \|u\| \cdot \|v\| \leq C \|v\|$$

Now

$$\begin{aligned}\iota &: H \hookrightarrow H^* \\ \iota(u) &= \varphi\end{aligned}$$

is a linear mapping, which is injective.

3.1.3 Theorem (Fréchet-Riesz)

For any $\varphi \in H^*$ there is a unique $v \in H$ such that for all $x \in H$:

$$\varphi(x) = \langle v, x \rangle$$

In other words: $\iota : H \rightarrow H^*$ is a Banach space isomorphism.

Proof

Let $\varphi \in H^*$, without loss of generality $\varphi \neq 0$.

$$M := \ker \varphi \subseteq H$$

is a subspace. It is closed by continuity: For $u_n \in \ker \varphi$ with $u_n \rightarrow u$ holds:

$$\varphi(u) \stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} \varphi(u_n) = 0$$

So $u \in \ker \varphi$.

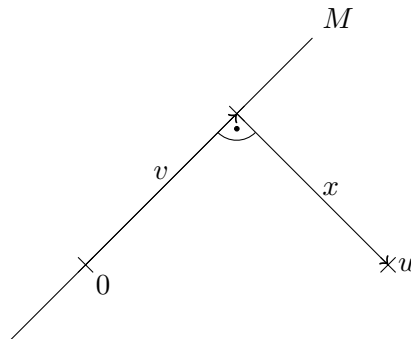


Figure 3.5: $u = v + x$

- M^\perp is a one-dimensional subspace of H :

$$M^\perp \neq \{0\}:$$

Since $\varphi \neq 0$ there exists a $u \in H$ with $\varphi(u) \neq 0$, thus $u \notin M$.

Now decompose $u = v + x$, $v \in M$, $x \in M^\perp \setminus \{0\}$.

M^\perp is one-dimensional: Take $u, v \in M^\perp$, $u, v \neq 0$, then $\varphi(u) \neq 0$ and $\varphi(v) \neq 0$.

$$\varphi(\varphi(v)u - \varphi(u)v) = 0$$

So $\varphi(v)u - \varphi(u)v \in M \cap M^\perp = \{0\}$. Thus $\varphi(v)u - \varphi(u)v = 0$, implying that u and v are linearly dependent.

- Choose $u \in M^\perp$ with $\varphi(u) = 1$, which is always possible by rescaling.

$$\begin{aligned} v &:= \frac{u}{\|u\|^2} \\ \Rightarrow \quad \varphi(v) &= \frac{1}{\|u\|^2} \underbrace{\varphi(u)}_{=1} = \frac{1}{\|u\|^2} \\ \langle v, v \rangle &= \frac{\langle u, u \rangle}{\|u\|^4} = \frac{1}{\|u\|^2} = \varphi(v) \end{aligned}$$

- This v has the desired properties:

For $x \in H$ decompose:

$$x = \underbrace{m}_{\in M} + \underbrace{\alpha v}_{\in M^\perp = \langle v \rangle}$$

$$\begin{aligned} \Rightarrow \quad \varphi(x) &= \underbrace{\varphi(m)}_{=0} + \alpha \varphi(v) = \alpha \langle v, v \rangle = \\ &= \langle v, \alpha v \rangle = \langle v, m + \alpha v \rangle = \langle v, x \rangle \end{aligned}$$

□_{3.1.3}

3.1.4 Theorem (Lax-Milgram)

Let H be a Hilbert space and $B : H \times H \rightarrow \mathbb{K}$ be a mapping with the following properties:

- $B(x, y)$ is linear in the second and anti-linear in the first argument.
- $|B(x, y)| \leq C \|x\| \cdot \|y\|$ (continuity)
- B is symmetric ($\overline{B(x, y)} = B(y, x)$) and positive definite, i.e. $B(x, x) \geq b \|x\|^2$ with $b \in \mathbb{R}_{>0}$.
- iii') $|B(x, x)| \geq b \|x\|^2$ with $b \in \mathbb{R}_{>0}$.

Then every $l \in H^*$ can be represented uniquely as:

$$l(y) = B(x, y) \quad \forall_{y \in H}$$

Proof

First the easy case iii):

We introduce a new scalar product $\langle \cdot, \cdot \rangle_B$ by:

$$\langle x, y \rangle_B := B(x, y)$$

Using ii) and iii) one sees that $\|\cdot\|_B$ is equivalent to $\|\cdot\|$, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C} \|x\| \leq \|x\|_B \leq C \|x\|$$

According to the Fréchet-Riesz theorem, there exists a unique $v \in H$ with

$$\varphi(x) = \langle v, x \rangle_B = B(v, x)$$

for all $x \in H$.

More difficult case iii'): Given $x \in H$,

$$B(x, \cdot) : H \rightarrow \mathbb{K}$$

is a linear bounded functional according to i) and ii), i.e. $B(x, \cdot) \in H^*$.

According to the Fréchet-Riesz theorem there exists a unique $z \in H$ such that $B(x, y) = \langle z, y \rangle$ for all $y \in H$. This yields a mapping:

$$\begin{aligned} \varphi : H &\rightarrow H \\ x &\mapsto z \end{aligned}$$

$$B(x, y) = \langle \varphi(x), y \rangle$$

- φ is linear, because both B and $\langle \cdot, \cdot \rangle$ are anti-linear in their first arguments.
- $\varphi(H) \subseteq H$ is closed:

$$\begin{aligned} b \|x\|^2 &\stackrel{\text{iii}'}{\leq} |B(x, x)| = |\langle z, x \rangle| \leq \|z\| \cdot \|x\| \\ b \|x\| &\leq \|z\| \end{aligned} \tag{3.1}$$

Let $z_n \in \varphi(H)$ be a sequence with $z_n \rightarrow z \in H$. Choose x_n such that $\varphi(x_n) = z_n$, i.e. $B(x_n, y) = \langle z_n, y \rangle$ for all $y \in H$.

Due to the anti-linearity in the first argument follows that:

$$B(x_n - x_m, y) = \langle z_n - z_m, y \rangle$$

(3.1) yields that $\|x_n - x_m\| \leq \|z_n - z_m\|$.

Hence (x_n) is a Cauchy sequence and so $x_n \rightarrow x \in H$ converges. Since B is continuous according to ii), we get:

$$\underbrace{B(x_n, y)}_{\rightarrow B(x, y)} = \underbrace{\langle z_n, y \rangle}_{\rightarrow \langle z, y \rangle}$$

This gives:

$$\begin{aligned} B(x, y) &= \langle z, y \rangle \\ \varphi(x) &= z \end{aligned}$$

Thus z is in $\varphi(H)$.

- $\varphi(H) = H$: Otherwise there would be a vector $y \in \varphi(H)^\perp \setminus \{0\}$ and thus for all $x \in H$ holds.

$$B(x, y) = \langle \varphi(x), y \rangle = 0$$

In particular for $x = y$ this gives:

$$\begin{aligned} 0 &= |B(y, y)| \geq b \|y\|^2 \\ \Rightarrow y &= 0 \end{aligned}$$

This is a contradiction and so $\varphi(H) = H$.

- φ is injective: Suppose there are $x, x' \in H$ with $\varphi(x) = \varphi(x')$. Then follows:

$$B(x - x', y) = \langle \underbrace{\varphi(x) - \varphi(x')}_{=0}, y \rangle = 0$$

Choose $y = x - x'$ so we get:

$$B(x - x', x - x') = 0$$

Since B is positive definite, it follows $x = x'$.

- Let $l \in H^*$. According to Fréchet-Riesz there exists a unique $z \in H$ with $l(y) = \langle z, y \rangle$ for all $y \in H$ and we have

$$\langle z, y \rangle = B(x, y)$$

for $x = \varphi^{-1}(z)$. So $l(y) = B(x, y)$.

□_{3.1.4}

3.1.5 Corollary

Every Hilbert space is reflexive.

Proof

Recall $\iota : H \hookrightarrow H^{**}$. H is *reflexive* if and only if ι is surjective, i.e. a Banach space isomorphism.

$$\begin{aligned} \tilde{\iota} : H &\rightarrow H^* \\ (\tilde{\iota}(u))(v) &= \langle u, v \rangle \end{aligned}$$

is bijective by Fréchet-Riesz. This holds also for $\bar{\iota} : H^* \rightarrow H^{**}$.

$$H \xrightarrow{\tilde{\iota}} H^* \xrightarrow{\bar{\iota}} H^{**}$$

So $\iota = \bar{\iota} \circ \tilde{\iota}$ is bijective as composition of bijective maps.

□_{3.1.5}

3.2 Orthonormal Bases in Separable Hilbert Spaces

3.2.1 Example

$$\ell_2 = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \right\}$$

with the scalar product

$$\langle (a_n), (b_n) \rangle := \sum_n \bar{a}_n b_n$$

is a Hilbert space.

Idea: Let H be an abstract Hilbert space. Choose an “orthonormal basis” (e_i) .

$$\begin{aligned} H \ni u &= \sum_{i=1}^{\infty} \lambda_i e_i \\ v &= \sum_{i=1}^{\infty} \nu_i e_i \end{aligned}$$

$$\langle u, v \rangle = \sum_{i,j=1}^{\infty} \langle \lambda_i e_i, \nu_j e_j \rangle = \sum_{i,j=1}^{\infty} \bar{\lambda}_i \nu_j \delta_{ij} = \sum_i \bar{\lambda}_i \nu_i$$

3.2.2 Definition (orthonormal system, Hilbert space basis, cardinality)

A system $(e_i)_{i \in J}$ is an *orthonormal system*, if $\langle e_i, e_j \rangle = \delta_{ij}$. The algebraic span is the vector space of *finite* linear combinations:

$$\langle (e_i) \rangle = \left\{ \sum_{i=1}^N \lambda_i e_i \mid N \in \mathbb{N}, \lambda_i \in \mathbb{K} \right\}$$

This is a subspace of H . Now the subspace $\overline{\langle (e_i) \rangle} \subseteq H$ is called *Hilbert space span* (Hilbertraumzeugnis).

An orthonormal system (e_i) is called a *orthonormal Hilbert space basis* if $\overline{\langle (e_i) \rangle} = H$.

Two sets A and B have the same cardinality if there exists an bijective map $\varphi : A \rightarrow B$.

Theorem (Bernstein-Schröder)

A and B have the same cardinality if and only if there exists an injective map from A to B and an injective map from $B \rightarrow A$.

(no proof)

A typical application of the Lax-Milgram theorem is for $x \in \mathbb{R}^n$, given real-valued functions $V(x), f(x)$ and looking for $u(x)$ that solves:

$$-\Delta u(x) + V(x)u(x) = f(x)$$

Question: Is there a solution which “decays at infinity”?

1. Weak formulation:

Suppose we have a solution $u \in \mathcal{C}^2(\mathbb{R}^n)$

$$-\Delta u + Vu - f = 0$$

Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a test function.

$$0 = \int_{\mathbb{R}^n} (-\Delta u + Vu - f) \eta d^n x \stackrel{\text{integration by parts}}{=} \underbrace{\int_{\mathbb{R}^n} (\langle \nabla u, \nabla \eta \rangle + Vu\eta) d^n x}_{=: B(u, \eta)} - \underbrace{\int_{\mathbb{R}^n} f \eta d^n x}_{=: l(\eta)}$$

So for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ holds:

$$B(u, \eta) = l(\eta)$$

Definition: u is a *weak solution* of the equation $-\Delta u + Vu = f$ if for all $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ holds:

$$B(u, \eta) = l(\eta)$$

2. Choose the correct Hilbert space. The first idea is $L^2(\mathbb{R}^n)$ with the scalar product:

$$\langle u, v \rangle = \int_{\mathbb{R}^n} uv d^n x$$

$$u_n(x) := e^{-|x|^2} \sin(nx_1)$$

Then for all $n \in \mathbb{N}$ holds:

$$\|u_n\|_{L^2} \leq C$$

But $B(u_n, u_n) \xrightarrow{n \rightarrow \infty} \infty$ diverges. Thus B is *not* continuous.
Better choose instead:

$$\langle u, v \rangle = \int_{\mathbb{R}^n} (uv + \langle \nabla u, \nabla v \rangle) d^n x$$

The corresponding Hilbert space $H^{1,2}(\mathbb{R}^n)$ is a Sobolev space.

$$L^2(\mathbb{R}^3) \supseteq H^{1,2}(\mathbb{R}^3) \ni u$$

Assume for simplicity that $0 < \varepsilon \leq V \leq C < \infty$, then we get:

$$B(u, u) = \int_{\mathbb{R}^n} (|\nabla u|^2 + Vu^2) d^n x \leq \int_{\mathbb{R}^n} (|\nabla u|^2 + Cu^2) d^n x \leq (1 + C) \|u\|_{H^{1,2}}^2$$

$$|B(u, u)| \geq \int_{\mathbb{R}^n} (|\nabla u|^2 + \varepsilon u^2) d^n x \geq \min\{1, \varepsilon\} \|u\|_{H^{1,2}}^2$$

Thus the Lax-Milgram theorem applies and yields a unique weak solution and then a regularity theorem says that u is smooth.

Consider a matrix equation

$$Au = f$$

with $A \in \text{Symm}(\mathbb{R}^n)$ and $f \in \mathbb{R}^n$.

For a general existence and uniqueness result one needs that A is invertible or equivalently:

$$\forall_{u \in \mathbb{R}^n \setminus \{0\}} : Au \neq 0$$

This follows from the condition:

$$\forall_{u \in \mathbb{R}^n \setminus \{0\}} : \underbrace{\langle u, Au \rangle}_{=B(u,u)} \neq 0$$

In finite dimension this is equivalent to:

$$\forall_{u \in \mathbb{R}^n} : |B(u,u)| > b \|u\|^2$$

$(e_i)_{i \in I}$ is an orthonormal Hilbert space basis of H if

$$\langle e_i, e_j \rangle = \delta_{ij}$$

and:

$$\overline{\langle e_i \rangle} = H$$

3.2.3 Theorem

Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal system. Then the mapping

$$\begin{aligned} \ell_2 &\rightarrow \overline{\langle e_i \rangle}^{\text{closed}} \subseteq H \\ (\lambda_i) &\mapsto \sum_{i \in \mathbb{N}} \lambda_i e_i \end{aligned}$$

is a Hilbert space isomorphism.

Proof

The mapping is well-defined and isometric:

For $(\lambda_i) \in \ell_2$, i.e. $\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$ we construct:

$$u_N := \sum_{i=1}^N \lambda_i e_i \in H$$

Without loss of generality take $M < N$, then follows:

$$\|u_N - u_M\|^2 = \left\| \sum_{i=M}^N \lambda_i e_i \right\|^2 = \left\langle \sum_{i=M}^N \lambda_i e_i, \sum_{i=M}^N \lambda_i e_i \right\rangle = \sum_{i,j=M}^N \bar{\lambda}_i \lambda_j \underbrace{\langle e_i, e_j \rangle}_{=\delta_{ij}} = \sum_{i=M}^N |\lambda_i|^2$$

Thus u_N is a Cauchy sequence and converges since $\overline{\langle e_i \rangle}$ is complete as a closed subset of a complete space.

$$u := \lim_{N \rightarrow \infty} u_N = \sum_{i=1}^N \lambda_i e_i$$

$$\|u\|^2 = \lim_{N \rightarrow \infty} \|u_N\|^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^N |\lambda_i|^2 = \|(\lambda_i)\|_{\ell_2}$$

The mapping is also surjective:

Let $u \in \overline{\langle e_i \rangle}$ and $\varepsilon > 0$. So there exists a $v = \sum_{i=1}^N \lambda_i e_i \in \langle e_i \rangle$ with $\|v - u\| < \varepsilon$.

In other words there exists a finite $J \subseteq \mathbb{N}$ such that $d(\langle (e_i)_{i \in J} \rangle, u) < \varepsilon$. The vector which minimizes this distance is the orthogonal projection of u on $\langle (e_i)_{i \in J} \rangle$ since this is a finite-dimensional subspace, which is automatically closed.

$$u_J = \sum_{i \in J} e_i \langle e_i, u \rangle$$

Choose an increasing sequence $J_1 \subsetneq J_2 \subsetneq \dots$ of finite sets such that:

$$\|u_{J_k} - u\| \rightarrow 0 \quad \Rightarrow \quad u_{J_k} \rightarrow u$$

Thus u_{J_k} is bounded by a $C \in \mathbb{R}_{>0}$.

$$\begin{aligned} u_{J_k} &= \sum_{i \in J_k} e_i \underbrace{\langle e_i, u \rangle}_{=\lambda_i} \\ C > \|u_{J_k}\| &= \sum_{i \in J_k} |\lambda_i|^2 \end{aligned}$$

This gives:

$$\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$$

And so we get:

$$u = \sum_{i \in \mathbb{N}} \lambda_i e_i$$

□_{3.2.3}

3.2.4 Theorem (Existence of Hilbert space basis)

In every Hilbert space H exists an orthonormal Hilbert space basis.

Proof

Consider $(u_i)_{i \in I}$ with $I = H$ and $u_h = h$ for all $h \in H$. $(u_i)_{i \in I}$ is obviously a generating system of H . On the set

$$X := \left\{ \tilde{I} \subseteq I \mid (u_i)_{i \in \tilde{I}} \text{ is an orthonormal system} \right\}$$

defines „ \subseteq “ a partial ordering.

Let $U \subseteq X$ be a totally ordered subset and define:

$$I_U := \bigcup_{\tilde{I} \in U} \tilde{I} \subseteq I$$

I_U is an upper bound of U in X if $I_U \in X$. Assume $(u_i)_{i \in I_U}$ would not be orthonormal. Then there would exist $j, k \in I_U$ with $\langle u_j, u_k \rangle \neq \delta_{jk}$.

For $j = k$ would hold $\langle u_j, u_j \rangle \neq 1$, but j lies in $\tilde{I} \in U \subseteq X$ and therefor has to hold $\langle u_j, u_j \rangle = 1$. For $j \neq k$ we would get $\langle u_j, u_k \rangle \neq 0$. But j lies in $\tilde{I}_j \in U$ and k in $\tilde{I}_k \subseteq U$ and U is totally ordered, i.e. either holds $\tilde{I}_j \subseteq \tilde{I}_k$ or $\tilde{I}_k \subseteq \tilde{I}_j$.

Without loss of generality assume $\tilde{I}_j \subseteq \tilde{I}_k$ (otherwise exchange j and k). Then $j, k \in \tilde{I}_k \in U \subseteq X$ and hence $(u_i)_{i \in \tilde{I}_j}$ is an orthonormal system in contradiction to $\langle u_j, u_k \rangle \neq 0$. Therefore holds $I_U \in X$ and thus I_U is an upper bound of U .

Using Zorn's lemma we get a maximal element I_{\max} in X . Because $(u_i)_{i \in I_{\max}}$ is an orthonormal system and thus especially linearly independent, it suffices to show that this is an generating system of H .

Assume there exists a $i_0 \in I$ with $u_{i_0} \notin K := \overline{\langle (u_i)_{i \in I_{\max}} \rangle_{\text{alg.}}}$. Since $K \subseteq H$ is closed and convex, there is an unique projection v of u_{i_0} on K and thus $h := u_{i_0} - v \in K^\perp$. It holds $h = u_h$ with $h \in H = I$.

Because I_{\max} is maximal, holds then $I_{\max} \cup \{h\} \notin X$ and hence there is a $j \in I_{\max}$ with $\langle h, u_j \rangle \neq 0$, because $h = j$ cannot hold due to $h \notin I_{\max}$. This is a contradiction to $h \in K^\perp$ and thus holds $K = H$.

Therefore $(u_i)_{i \in I_{\max}}$ is an orthonormal Hilbert space basis of H . □_{3.2.4}

3.2.5 Theorem

Let H be a Hilbert space.

- i) For any $v \in H$ and for any orthonormal system $\{e_j | j \in J\}$, the set of elements $j \in J$ for which $\langle e_j, v \rangle = 0$ is finite or countable.
- ii) Any two Hilbert space bases of H have the same cardinality (Mächtigkeit).

Proof

- i) Consider $v \in J$. First we show that every $n \in \mathbb{N}$, the set $J_n := \{j \in J | \langle e_j, v \rangle > \frac{1}{n}\}$ is finite. Indeed, by Bessel's inequality, for every finite number of elements e_{j_1}, \dots, e_{j_N} of the given orthonormal system, we have:

$$\sum_{k=1}^N |\langle e_{j_k}, v \rangle|^2 \leq \|v\|^2$$

Now suppose that for some $n \in \mathbb{N}$, the set J_n were not finite. Then for any $N \in \mathbb{N}$ we could find elements e_{j_1}, \dots, e_{j_N} such that $\langle e_{j_k}, v \rangle > \frac{1}{n}$ for all $k \in \{1, \dots, N\}$. Hence, for these elements holds:

$$\sum_{k=1}^N |\langle e_{j_k}, v \rangle|^2 > N \cdot \frac{1}{n}$$

Clearly these becomes larger than $\|v\|$ if we make N sufficiently large. Hence all the sets J_n must be finite. But then, we see that the set

$$\{j \in J \mid \langle e_j, v \rangle \neq 0\} = \bigcup_{n \in \mathbb{N}} J_n$$

is a countable union of finite sets, and as such can be at most countable. \square_{i}

- ii) If H has is finite-dimensional, every Hilbert basis is a Hamel basis of H and thus the claim follows from linear algebra.

If H is infinite-dimensional, let $(e_i)_{i \in I}$ and $(b_j)_{j \in J}$ be two Hilbert bases of H . (I and J have infinitely many elements.)

For $x \in H = \overline{\langle (e_i)_{i \in I} \rangle} = \overline{\langle (b_j)_{j \in J} \rangle}$ define:

$$B_x := \{j \in J \mid \langle x, b_j \rangle \neq 0\}$$

By i), the set B_x is at most countable for any $x \in H$. Next, let $j \in J$ be given. Since $\overline{\langle (e_i)_{i \in I} \rangle} = H$, we must have $\langle b_j, e_i \rangle \neq 0$ for some $i \in I$. Otherwise, $b_j \in \overline{\langle (e_i)_{i \in I} \rangle}^\perp = \{0\}$, which is not possible since $b_j \neq 0$. Therefore, we have $j \in B_{e_i}$ for some $i \in I$, and since $j \in J$ was arbitrary, it follows that $J \subseteq \bigcup_{i \in I} B_{e_i} \subseteq I \times \mathbb{N}$. Here the second inclusion uses that all the sets B_{e_j} are at most countable. It follows:

$$|J| \leq |I| \cdot |\mathbb{N}| = |I|$$

If we exchange the roles of I and J above, we also obtain $|I| \leq |J|$. By the Schröder-Bernstein theorem, we can combine both estimates to obtain that $|I| = |J|$. \square_{ii}

$\square_{3.2.5}$

3.2.6 Theorem

If H is separable, then there exists a countable orthonormal Hilbert space basis $(e_i)_{i \in \mathbb{N}}$. Thus H is Hilbert space isomorphic to ℓ_2 .

Proof

Since H is separable, there is a countable dense subset $(x_i)_{i \in \mathbb{N}}$.

1. Arrange that the x_i are linearly independent:
Start with $n = 1$ and $k = 1$ set:

$$y_1 = x_1$$

If the y_1, \dots, y_{n-1}, x_k are linearly independent, we set $y_n = x_k$ and increase n and k by one.

If the y_1, \dots, y_{n-1}, x_k are linearly dependent, we only increase k by one.

Then the y_i are linearly independent and $\langle (y_i) \rangle = \langle (x_i) \rangle$.

2. Gram-Schmidt procedure for orthonormalization:

$$e_1 := y_1$$

$$e_2 := \frac{y_2 - e_1 \langle u_1, y_2 \rangle}{\|y_2 - e_1 \langle u_1, y_2 \rangle\|}$$

$$e_n := \frac{y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n}{\|y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n\|}$$

Since the y_i are linearly independent, $y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n$ is never zero.

Then by construction the e_i are orthonormal and $\langle e_i \rangle = \langle x_i \rangle \subseteq H$ is dense and so $(e_i)_{i \in \mathbb{N}}$ is a Hilbert space basis. $\square_{3.2.6}$

3.3 Weak Compactness of the Closed Unit Ball

For a Banach space E *weak convergence* for $(u_i)_{i \in \mathbb{N}}$ with $u_i \in E$ means:

$$u_n \rightharpoonup u \quad \Leftrightarrow \quad \forall_{\varphi \in E^*} : \varphi(u_n) \rightarrow \varphi(u)$$

In Hilbert spaces, we can identify H^* with H via the Fréchet-Riesz theorem.

3.3.1 Definition (weak (sequential) compactness)

$x_n \rightharpoonup x$ *converges weakly* if $\langle y, x_n \rangle \rightarrow \langle y, x \rangle$ converges for all $y \in H$.

Weak compactness is for us by definition the same as *weak sequential compactness* (schwache Folgenkompaktheit):

$K \subseteq H$ is *weakly compact* if every sequence (x_n) with $x_n \in K$ has a weakly convergent subsequence.

3.3.2 Proposition

Let H be *separable* and infinite-dimensional and let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal Hilbert space basis.

Then $e_n \rightharpoonup 0$ converges weakly.

Proof

Take $y \in H$ and expand it in the basis:

$$y = \sum_{i=1}^{\infty} y_i e_i$$

$$y_i = \langle e_i, y \rangle$$

We know $(y_i)_{i \in \mathbb{N}} \in \ell_2$ and in particular $y_i \xrightarrow{i \rightarrow \infty} 0$, since the elements of an absolutely convergent series converge to zero. Therefore holds:

$$\langle y, e_n \rangle = \overline{y_n} \xrightarrow{n \rightarrow \infty} 0$$

Thus $e_n \rightharpoonup 0$ converges weakly. $\square_{3.3.2}$

3.3.3 Theorem (Weak Compactness of the Closed Unit Ball)

If H is *separable*, then the closed unit ball $\overline{B_1(0)} = \{u \mid \|u\| \leq 1\}$ is weakly compact.

Proof

Let (u_l) be a sequence with $u_l \in \overline{B_1(0)}$. Choose an orthonormal Hilbert space basis $(e_n)_{n \in \mathbb{N}}$.

$$u_l = \sum_{n=1}^{\infty} u_{ln} e_n \quad u_{ln} = \langle e_n, u_l \rangle \quad (u_{l,n})_{n \in \mathbb{N}} \in \ell_2$$

$$|u_{ln}| = |\langle e_n, u_l \rangle| \leq \underbrace{\|e_n\|}_{=1} \cdot \|u_l\| \leq 1$$

For $n = 1$: $(u_{l,1})_{l \in \mathbb{N}}$ is a bounded sequence of complex or real numbers. Therefore there exists a convergent subsequence of u_l , which we denote by $u_l^{(1)} \in H$. Then follows:

$$u_{l,1}^{(1)} = \langle e_1, u_l^{(1)} \rangle \xrightarrow{l \rightarrow \infty} v_1$$

For $n = 2$: Next we choose a subsequence $u_l^{(2)}$ of $u_l^{(1)}$ such that:

$$\langle e_2, u_l^{(2)} \rangle \xrightarrow{l \rightarrow \infty} v_2$$

Proceed inductively to obtain:

$$\langle e_n, u_l^{(n)} \rangle \rightarrow v_n$$

Then $w_l = u_l^{(l)} \in \overline{B_1(0)}$ for a sequence (w_l) in $\overline{B_1(0)}$.

Claim: $w_l \xrightarrow{l \rightarrow \infty} v := \sum_n v_n e_n$

Proof: We proceed as follows:

$$v_n = \lim_{l \rightarrow \infty} \langle e_n, u_l^{(n)} \rangle = \lim_{l \rightarrow \infty} \langle e_n, u_l^{(l)} \rangle = \lim_{l \rightarrow \infty} \langle e_n, w_l \rangle$$

This is because $u_l^{(l)} = u_{l'}^{(n)}$ for $l' \geq l$.

1. $(v_n) \in \ell_2$:

$$\sum_{n=1}^N |v_n|^2 = \sum_{n=1}^N \left| \lim_{l \rightarrow \infty} \langle e_n, w_l \rangle \right|^2 \stackrel{\text{finite sum}}{=} \lim_{l \rightarrow \infty} \sum_{n=1}^N |\langle e_n, w_l \rangle|^2$$

$\underbrace{\hspace{10em}}_{\substack{\text{Bessel's} \\ \leq \\ \text{inequality}}} \|w_l\|^2 \leq 1$

So we get for all $N \in \mathbb{N}$:

$$\sum_{n=1}^N |v_n|^2 \leq 1$$

And thus $(v_n) \in \ell_2$ and $v := \sum_{n=1}^{\infty} v_n e_n$ is well-defined and has $\|v\| \leq 1$.

2. $w_l \rightarrow v$, i.e. $\langle y, w_l - v \rangle \xrightarrow{l \rightarrow \infty} 0$ for all $y \in H$:

$$y = \sum_{n=1}^{\infty} y_n e_n$$

$$y_n = \langle e_n, y \rangle$$

$$y_{<} := \sum_{n \leq N} y_n e_n$$

$$y_{>} := \sum_{n > N} y_n e_n$$

$$\|y\|^2 = \|y_{<}\|^2 + \|y_{>}\|^2$$

$$\langle y, w_l - v \rangle = \sum_{n=1}^{\infty} y_n \langle e_n, w_l - v \rangle$$

Choose $N \in \mathbb{N}$ so large that

$$\|y_{>}\| = \left(\sum_{n > N} |y_n|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{4}$$

to get:

$$\begin{aligned} |\langle y, w_l - v \rangle| &\leq |\langle y_{<}, w_l - v \rangle| + |\langle y_{>}, w_l - v \rangle| \leq \\ &\leq \sum_{n=1}^N |y_n| |\langle e_n, w_l - v \rangle| + \underbrace{\|y_{>}\|}_{< \frac{\varepsilon}{4}} \cdot \underbrace{\|w_l - v\|}_{\leq 2} < \sum_{n=1}^N |y_n| |\langle e_n, w_l - v \rangle| + \frac{\varepsilon}{2} \end{aligned}$$

We know $|\langle e_n, w_l - v \rangle| \xrightarrow{l \rightarrow \infty} 0$ for each n . So we can choose $|\langle e_n, w_l - v \rangle| \leq \frac{\varepsilon}{2}$ for $n \leq N$ and for all $l > L(\varepsilon)$ for a sufficiently large $L(\varepsilon)$ and therefore:

$$|\langle y, w_l - v \rangle| \leq \varepsilon \quad \forall_{l > L(\varepsilon)}$$

Therefore $\langle y, w_l \rangle \rightarrow \langle y, v \rangle$ converges, which means $w_l \rightarrow v$.

□_{Claim}

□_{3.3.3}

The corresponding statement in Banach spaces is the *Banach-Alaoglu theorem*:

Banach proved it in 1932 for separable Banach spaces using diagonal sequences.

Alaoglu proved it in 1938 for any Banach space. The proof is based on Tychonov's theorem.

We have E , E^* , E^{**} and an injection $\iota : E \rightarrow E^{**}$.

Theorem (Banach-Alaoglu)

The closed unit ball in E^* is *weak*-sequentially compact*.

I.e. in simple terms:

If $\varphi_n \in \overline{B_1(0)} \subseteq E^*$, then there exists a subsequence φ_{n_l} such that $\varphi_{n_l}(u)$ converges for all $u \in E$.

Application: Consider

$$E = C^0(\mathbb{R}^n)$$

with the sup-norm:

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|$$

$$E^* = \{\text{regular Borel measures}\}$$

Suppose μ_n is a sequence of measures with $\|\mu_n\| \leq C$ for all $n \in \mathbb{N}$. Then there exists a measure μ such that $\mu_{n_l} \rightarrow \mu$ converges as a measure.

4 Operators on Hilbert spaces

Let H be a Hilbert space.

$$L(H) := L(H, H)$$

is the Banach space of bounded linear operators. (An linear map on an infinite dimensional space is usually called *linear operator*.) For $A \in L(H)$ define the norm:

$$\|A\| := \sup_{\|u\|=1} \|Au\|$$

4.0.1 Example

$H = L^2(\mathbb{R}, dx)$ with the Lebesgue measure dx .

$$\langle f, g \rangle = \int_{\mathbb{R}} \bar{f} g dx$$

$$A := \frac{d}{dx}$$

We would like to introduce this as an operator on H .

The inequality $\|Au\| \leq C \|u\|$ is violated even for $u \in C_0^\infty(\mathbb{R})$ for any constant $C \in \mathbb{R}$. Namely consider

$$u_n(x) = \eta(x) \sin(nx)$$

with $\eta \in C_0^\infty(\mathbb{R})$ and $\eta|_{[-1,1]} = 1$. Then $\|u_n\| < \infty$ and $\|Au_n\| \xrightarrow{n \rightarrow \infty} \infty$.

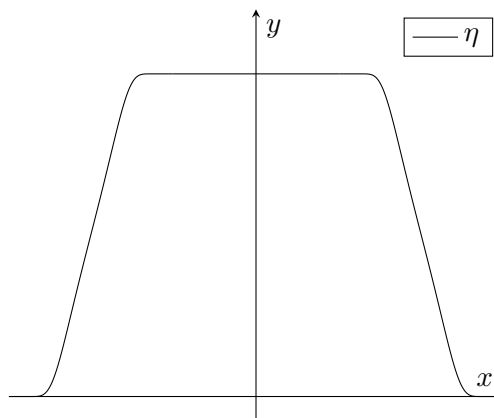


Figure 4.1: $\eta \in C_0^\infty(\mathbb{R})$ with $\eta|_{[-1,1]} = 1$

Moreover $\frac{d}{dx}f$ makes no sense for every vector f in H , because f does not need to be differentiable.

Way out: Define A only on a suitable subspace $\mathcal{D}(A)$ of H , called *domain* of definition. For example: Choose $\mathcal{D}(A) = C_0^\infty(\mathbb{R}) \subseteq H$ and:

$$A : \mathcal{D}(A) \xrightarrow{\text{linear}} H$$

$\mathcal{D}(A)$ is dense in H , i.e. $\overline{\mathcal{D}(A)} = H$.

4.0.2 Definition (linear operator, domain, bounded)

- i) Let $\mathcal{D} \subseteq H$ be a dense subspace. A linear map $A : \mathcal{D} \rightarrow H$ is called a *linear operator* on H with domain (of definition) \mathcal{D} .
- ii) A is called *bounded*, if there exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in \mathcal{D}$ holds:

$$\|Au\| \leq C \|u\|$$

Otherwise A is called unbounded.

4.0.3 Lemma

If A is a bounded operator with dense domain $\mathcal{D} \subseteq H$, then it can be extended by continuity to a unique operator $A \in L(H)$.

Proof

Let $u \in H$, not necessarily in \mathcal{D} . Since $\overline{\mathcal{D}} = H$, there is a sequence (u_l) in \mathcal{D} with $u_l \rightarrow u$.

$$\|Au_i - Au_j\| = \|A(u_i - u_j)\| \leq C \cdot \|u_i - u_j\| \xrightarrow{i,j \rightarrow \infty} 0$$

Therefore we can set:

$$Au := \lim_{l \rightarrow \infty} Au_l$$

Since Au_l converges for any sequence $u_l \rightarrow u$, this is well-defined.

$$\|Au\| \leftarrow \|Au_i\| \leq C \|u_i\| \rightarrow C \|u\|$$

So there exists a C such that $\|Au\| \leq C \|u\|$ for all $u \in H$ and therefore $A \in L(H)$. $\square_{4.0.3}$

4.1 Isometric and unitary operators

4.1.1 Definition (isometric operator)

A operator $V : \mathcal{D}(V) \rightarrow H$ with dense domain $\mathcal{D}(V) \subseteq H$ is called *isometric* if for all $u \in \mathcal{D}(V)$ holds:

$$\langle Vu, Vu \rangle = \langle u, u \rangle$$

This operator is bounded, because:

$$\|Vu\| = \sqrt{\langle Vu, Vu \rangle} = \sqrt{\langle u, u \rangle} = \|u\| \stackrel{C:=1}{\leq} C \|u\|$$

Therefore we can extend it by continuity to H and

$$V : H \rightarrow H$$

is again isometric.

The “Hilbert hotel”

Consider $H = \ell_2$ and $(a_i) = (a_1, a_2, \dots) \in \ell_2$.

$$A(u_1, u_2, \dots) := (0, u_1, u_2, \dots)$$

A is isometric, but it is no bijection.

Suppose you have a hotel with an infinite number of rooms and an infinite number of guest, in every room one guest.

If a new guest arrives, just move the guest from room n to room $n + 1$ and the first room gets unoccupied, so the new guest can use it.

4.1.2 Proposition

For an isometric operator V the subspace $V(H) \subseteq H$ is closed.

Proof

Consider $y \in \overline{V(H)}$ and show $y \in V(H)$:

There exists a (y_n) with $y_n \in V(H)$ and $y_n \rightarrow y$ and a (x_n) with $V(x_n) = y_n$. Then holds:

$$\|x_i - x_j\| \stackrel{V \text{ isometric}}{=} \|V(x_i - x_j)\| = \|y_i - y_j\| \xrightarrow{i, j \rightarrow \infty} 0$$

Thus $x_i \rightarrow x$ converges. By continuity we get:

$$V(x) = \lim_{i \rightarrow \infty} V(x_i) = \lim_{i \rightarrow \infty} y_i = y$$

□_{4.1.2}

4.1.3 Definition (unitary operator)

If $V : H \rightarrow H$ is an isometric operator and $V(H) = H$, then V is called *unitary* (unitär).

4.2 The Closure of an Operator

Let E and F be Banach spaces and $A : \mathcal{D}(A) \subseteq E \rightarrow F$ be a densely defined linear operator.

$$\begin{aligned} \text{graph}(A) &:= \{(u, Au) \mid u \in \mathcal{D}(A)\} \subseteq E \times F \\ \overline{\text{graph}(A)} &\subseteq E \times F \end{aligned}$$

Try to realize this as the graph of a new operator \bar{A} .

$$\mathcal{D}(\bar{A}) := \text{pr}_1(\overline{\text{graph}A}) = \left\{ u \mid \exists_{v \in F} : (u, v) \in \overline{\text{graph}A} \right\}$$

For $u \in \mathcal{D}(\bar{A})$ and $(u, v) \in \overline{\text{graph}A}$ define:

$$\bar{A}u := v$$

v exists by definition of $\mathcal{D}(\bar{A})$. Is v unique?

Suppose $(u, v) \in \overline{\text{graph}A}$. Then there exists a sequence $(u_n, v_n) \in \text{graph}(A)$, with $(u_n, v_n) \rightarrow (u, v)$. Equivalently:

$$\forall_{n \in \mathbb{N}} \exists_{u_n \in \mathcal{D}(A)} : (u_n \rightarrow u) \wedge (Au_n \rightarrow v)$$

Then we set $\bar{A}u := v$.

Problem: There might be two different series (u_n) and (\tilde{u}_n) with $u_n \rightarrow u$, $\tilde{u}_n \rightarrow u$, $Au_n \rightarrow v$ and $A\tilde{u}_n \rightarrow \tilde{v} \neq v$.

4.2.1 Definition (closable operator)

A densely defined operator A is called closable (abschließbar) if $\overline{\text{graph}A}$ is the graph of an operator B .

B is called the *closure* of A , symbolically $B = \bar{A}$.

4.2.2 Definition (closed)

A is called *closed* if $\text{graph}A$ is a closed subset of $E \times F$.

4.2.3 Theorem (closed graph theorem)

Reformulation of 2.4.9:

If $\mathcal{D}(A) = E$, then A is closed if and only if A is bounded.

4.2.4 Example

Consider $E = C^0([0,1])$ with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$.

$$\mathcal{D}(A) = C^1([0,1]) \subseteq E$$

$$\begin{aligned} A : \mathcal{D}(A) &\rightarrow E \\ f &\mapsto f' \end{aligned}$$

A is a densely defined, unbounded operator. Is A closed?

Consider $(u, v) \in \overline{\text{graph} A}$, i.e. there exists a sequence $(u_n) \subseteq \mathcal{D}(A)$ with $u_n \rightarrow u$ and $Au_n \rightarrow v$. $u_n \rightarrow u$ means uniform convergence of $u_n \rightrightarrows u$, so u is continuous as a uniform limit of continuous functions.

$Au_n \rightarrow v$ means uniform convergence of $Au_n \rightrightarrows v$, so v is also continuous.

It follows that $u \in C^1$ and $u' = v$.

So $(u, v) \in \text{graph} A$ and therefore A is closed.

Consider $F := C^1([0, 1])$ with $\|u\| = \sup_{[0, 1]} |u| + \sup_{[0, 1]} |u'|$. This is a Banach space.

Remark

The closure of a closable operator is always closed.

This is obvious, because $\text{graph} \bar{A} \stackrel{\text{def.}}{=} \overline{\text{graph} A}$, which is closed.

4.2.5 Theorem (Criterion for closable)

A is closable if and only if:

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \rightarrow 0) \wedge (Au_n \rightarrow v) \quad \Rightarrow \quad v = 0$$

Proof

“ \Rightarrow ”: Suppose A is closable. Thus there is an operator \bar{A} such that $\text{graph} \bar{A} = \overline{\text{graph} A}$.

Suppose that $u_n \in \mathcal{D}(A)$, $u_n \rightarrow 0$ and $Au_n \rightarrow v$. Then $(u_n, Au_n) \rightarrow (0, v) \in \text{graph} \bar{A} = \overline{\text{graph} A}$ and thus $v = \bar{A}(0) = 0$.

“ \Leftarrow ”: Suppose that the implication

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \rightarrow 0) \wedge (Au_n \rightarrow v) \quad \Rightarrow \quad v = 0$$

holds.

Define $\mathcal{D}(\bar{A})$ by: $u_n \in \mathcal{D}(A)$ with $u_n \rightarrow u$ and $Au_n \rightarrow v$. Then for $u \in \mathcal{D}(\bar{A})$ set $\bar{A}(u) = v$.

This is well-defined: Suppose $u_n, \tilde{u}_n \rightarrow u$, $Au_n \rightarrow v$ and $A\tilde{u}_n \rightarrow \tilde{v}$. Then $u_n - \tilde{u}_n \rightarrow 0$ and $A(u_n - \tilde{u}_n) \rightarrow v - \tilde{v}$. By assumption follows $v - \tilde{v} = 0$. $\square_{4.2.5}$

4.3 The adjoint of a densely defined operator

Let $A : \mathcal{D}(A) \rightarrow H$ be a linear operator with $\overline{\mathcal{D}(A)} = H$.

In finite-dimensional linear algebra the definition of the adjoint A^* is:

$$\langle u, Av \rangle =: \langle A^*u, v \rangle \quad \forall_{u, v \in H}$$

Here it is more complicated, since in general $\mathcal{D}(A) \neq H$.

$$M := \left\{ (u, w) \in H \times H \mid \forall_{v \in \mathcal{D}(A)} : \langle u, Av \rangle = \langle w, v \rangle \right\}$$

Claim: M is the graph of a linear map A^* .

Proof: $M \neq \emptyset$ since $(0,0) \in M$.

- The image is unique: $u \mapsto w$ is well-defined, as from $(u,w), (u,w') \in M$ follows for all $v \in \mathcal{D}(A)$:

$$\langle w - w', v \rangle = \langle u - u, Av \rangle = 0$$

Since $\mathcal{D}(A)$ is dense, $w - w' = 0$ follows.

- A^* is linear: For $(u,w), (u',w') \in M$ and $\lambda \in \mathbb{K}$ follows $(u + \lambda u', w + \lambda w') \in M$, which is obvious from the definition of M . \square_{Claim}

4.3.1 Theorem

A^* is closed.

Proof

Let $x_n \in \mathcal{D}(A^*)$ converge to $x \in H$ and $A^*x_n \rightarrow y \in H$. For $z \in \mathcal{D}(A)$ holds:

$$\langle x, Az \rangle \stackrel{\langle \cdot, \cdot \rangle \text{ continuous}}{=} \lim_{n \rightarrow \infty} \langle x_n, Az \rangle = \lim_{n \rightarrow \infty} \langle A^*x_n, z \rangle \stackrel{\langle \cdot, \cdot \rangle \text{ continuous}}{=} \langle y, z \rangle$$

This shows $x \in \mathcal{D}(A^*)$ and $A^*x = y$, so A^* is closed. $\square_{4.3.1}$

4.3.2 Theorem

A^* is the maximal, i.e. not extensible, operator S with the property that for all $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(S)$:

$$\langle Au, v \rangle = \langle u, Sv \rangle$$

Proof

$$\begin{aligned} \text{graph}(S) &= \left\{ (v, w) \in \mathcal{D}(S) \times H \mid Sv = w \right\} = \\ &= \left\{ (v, w) \in \mathcal{D}(S) \times H \mid \forall_{u \in \mathcal{D}(A)} \langle Au, v \rangle = \langle u, w \rangle \right\} = \\ &= \left\{ (v, w) \in H \times H \mid \forall_{u \in \mathcal{D}(A)} \langle v, Au \rangle = \langle w, u \rangle \right\} = \text{graph}(A^*) \end{aligned}$$

$\square_{4.3.2}$

4.4 Symmetric and self-adjoint densely defined operators

4.4.1 Definition (symmetric, (essentially) self-adjoint)

- i) A is *symmetric* : $\Leftrightarrow \forall_{u,v \in \mathcal{D}(A)} : \langle Au, v \rangle = \langle u, Av \rangle$
- ii) A is *self-adjoint* : $\Leftrightarrow A^* = A$ (in particular, $\mathcal{D}(A^*) = \mathcal{D}(A)$)
- iii) A is *essentially self-adjoint* : $\Leftrightarrow \overline{A}$ is self-adjoint

For bounded A with $\mathcal{D}(A) = H$ all these notions coincide.

4.4.2 Example

Consider the operator $A := \Delta = \sum_{i=1}^n \partial_i^2$ on $L^2(\Omega)$ for a bounded open region $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{D}(A) = C_0^\infty(\Omega) \stackrel{\text{dense}}{\subseteq} L^2(\Omega)$.

- A is symmetric:

$$\langle Af, g \rangle \stackrel{\text{integration by parts}}{=} \langle f, Ag \rangle$$

- Adjoint of Δ on L^2 :

$$\int d^n r (\Delta f) \cdot g = \int d^n r f \cdot \underbrace{h}_{\in L^2}$$

Here $h := A^*g$. It is sufficient to consider $g \in H^{2,2}(\Omega)$ (Sobolev space). $\mathcal{D}(A^*) \supsetneq \mathcal{D}(A)$

4.4.3 Lemma

Let A be a symmetric operator. Then A is closable and \overline{A} and A^* are extensions of A and $\mathcal{D}(A) \stackrel{\text{i)}}{\subseteq} \mathcal{D}(\overline{A}) \stackrel{\text{ii)}}{\subseteq} \mathcal{D}(A^*)$.

Proof

Let $u_n \in \mathcal{D}(A)$ with $u_n \rightarrow 0$ and $Au_n \rightarrow w$.

$$\begin{aligned} \langle Au, v \rangle &= \langle u, Av \rangle \quad \forall_{u,v \in \mathcal{D}(A)} \\ \langle w, v \rangle &\leftarrow \langle Au_n, v \rangle = \langle u_n, Av \rangle \rightarrow \langle 0, Av \rangle = 0 \end{aligned}$$

Since this holds for all $v \in \mathcal{D}(A)$ now $w = 0$ follows. From the criterion 4.2.5 follows that A is closable.

- i) is obvious from the definition of \overline{A} .
- ii) Take $u \in \mathcal{D}(\overline{A})$. Then there is a sequence $u_n \in \mathcal{D}(A)$ with $u_n \rightarrow u$ and $Au_n \rightarrow \overline{A}u$. For all $v \in \mathcal{D}(A)$ holds:

$$\langle \overline{A}u, v \rangle \leftarrow \langle Au_n, v \rangle = \langle u_n, Av \rangle \rightarrow \langle u, Av \rangle$$

So $u \in \mathcal{D}(A^*)$ and $A^*u = \overline{A}u$.

□_{4.4.3}

„The smaller one chooses $\mathcal{D}(A)$, the larger becomes $\mathcal{D}(A^*)$.“

$$B \subseteq \mathcal{D}(A) \quad \Rightarrow \quad \mathcal{D}((A|_B)^*) \supseteq \mathcal{D}(A^*)$$

Difficulty: Construct $\mathcal{D}(A)$ such that $\mathcal{D}(A) = \mathcal{D}(A^*)$. (More on this later in the lecture.)

4.5 Heisenberg's uncertainty principle

In quantum mechanics:

The Hilbert space for one dimensional problems is usually $H = L^2(\mathbb{R})$.

The position operator is $x =: B$ and the momentum operator is $\frac{\hbar}{i} \frac{d}{dx} =: A$.

$$[A, B] := AB - BA = \frac{\hbar}{i} \mathbb{1}$$

4.5.1 Theorem (Winter-Wieland)

For two continuous operators A and B with $[A, B] = c \cdot \mathbb{1}$ and $B^n = B$ for all $n \in \mathbb{N}_{\geq 1}$, i.e. B is idempotent, follows $c = 0$.

Proof

Consider:

$$B^k AB^{n-k} = B^k (AB) B^{n-k-1} = B^k (BA + c\mathbb{1}) B^{n-k-1} = B^{k+1} AB^{n-k-1} + cB^{n-1}$$

$$\Rightarrow \quad cB^{n-1} = B^k AB^{n-k} - B^{k+1} AB^{n-k-1}$$

Sum this from $k = 0$ to $k = n - 1$:

$$ncB^{n-1} = \sum_{k=0}^{n-1} B^k AB^{n-k} - B^{k+1} AB^{n-k-1} \stackrel{\text{telescope sum}}{=} AB^n - B^n A$$

$$n|c| \|B^{n-1}\| = \|AB^n - B^n A\| \stackrel{\Delta\text{-inequality}}{\leq} \|AB^n\| + \|B^n A\| \leq (\|AB\| + \|BA\|) \cdot \|B^{n-1}\|$$

Since this must hold for all n either $c = 0$ or there exists a $n \in \mathbb{N}_{>1}$ with $\|B^{n-1}\| = 0$, i.e. $B^{n-1} = 0$. Since B is idempotent follows $B = 0$ and therefore $[A, B] = 0$ and also $c = 0$. □_{4.5.1}

Consider $u \in \mathcal{D}(A)$ with $\|u\| = 1$, which represents a quantum mechanical state.

The expectation value of A in u is after the probabilistic interpretation:

$$E_u(A) := \langle u, Au \rangle$$

The “uncertainty”, i.e. the variance, is:

$$\Delta_u(A) := \|(A - E_u(A) \mathbb{1}) u\|$$

4.5.2 Theorem (Heisenberg's uncertainty principle)

Let H be a \mathbb{C} -Hilbert space and $A : \mathcal{D}(A) \rightarrow H$, $B : \mathcal{D}(B) \rightarrow H$ be two symmetric operators with $\overline{\mathcal{D}(A)} = H = \overline{\mathcal{D}(B)}$. Assume for the image domains \mathcal{R} :

$$\mathcal{R}(A) \subseteq \mathcal{D}(B) \qquad \mathcal{R}(B) \subseteq \mathcal{D}(A)$$

So $[A, B]$ is well-defined on $\mathcal{D}(A) \cap \mathcal{D}(B)$.

Assume furthermore that $[A, B] = \frac{\hbar}{i} \mathbb{1}$ with $\hbar > 0$.

Then for all $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $\|u\| = 1$ holds:

$$\Delta_u(A) \cdot \Delta_u(B) \geq \frac{\hbar}{2}$$

Proof

Replace A by $\tilde{A} := A - E_u(A) \cdot \mathbb{1}$ and $\tilde{B} := B - E_u(B) \cdot \mathbb{1}$. Then holds:

$$[\tilde{A}, \tilde{B}] = \frac{\hbar}{i} \mathbb{1}$$

$$\Delta_u(A) = \|\tilde{A}u\|$$

$$\Delta_u(B) = \|\tilde{B}u\|$$

We have to show:

$$\Delta_u(A) \cdot \Delta_u(B) = \|\tilde{A}u\| \cdot \|\tilde{B}u\| \geq \frac{\hbar}{2}$$

$$\begin{aligned} \frac{\hbar}{2} &= \frac{\hbar}{2} \langle u, u \rangle = \frac{i}{2} \left\langle u, \left(\tilde{A}\tilde{B} - \tilde{B}\tilde{A} \right) u \right\rangle \stackrel{\text{symmetry}}{=} \frac{i}{2} \left(\langle \tilde{A}u, \tilde{B}u \rangle - \langle \tilde{B}u, \tilde{A}u \rangle \right) = \\ &= -\text{Im} \left(\langle \tilde{A}u, \tilde{B}u \rangle \right) \stackrel{\text{Cauchy-Schwarz}}{\leq} \|\tilde{A}u\| \cdot \|\tilde{B}u\| \end{aligned}$$

□_{4.5.2}

4.6 Spectrum and resolvent

Let $A : \mathcal{D}(A) \rightarrow H$ be a closed, densely defined operator.

4.6.1 Definition (continuously invertible, resolvent, spectrum)

A is *continuously invertible* if and only if $A : \mathcal{D}(A) \rightarrow H$ is bijective and $A^{-1} : H \rightarrow \mathcal{D}(A)$ is continuous.

$$\varrho(A) := \{ \lambda \in \mathbb{K} \mid (\lambda \mathbb{1} - A) \text{ is continuously invertible} \}$$

The *resolvent* (Resolvente) is defined for $\lambda \in \varrho(A)$ as

$$\mathcal{R}_\lambda(A) = (\lambda \mathbb{1} - A)^{-1} \in L(H)$$

and the *spectrum* of A as:

$$\sigma(A) = \mathbb{K} \setminus \varrho(A)$$

4.6.2 Lemma

$\varrho(A)$ is open and $\sigma(A)$ is closed.

Proof

For bounded operators cf. Theorem 2.5.3.

It's method works even for unbounded operators:

Take $\lambda, \mu \in \varrho(A)$.

$$\begin{aligned} (A - \mu) &= (A - \lambda) + (\lambda - \mu) = \\ &= \underbrace{(A - \lambda)}_{\text{continuously invertible}} \cdot \left(\mathbb{1} + (A - \lambda)^{-1} (\lambda - \mu) \right) \end{aligned}$$

$\mathbb{1} + (A - \lambda)^{-1} (\lambda - \mu)$ is continuously invertible using the Neumann series if:

$$|\lambda - \mu| < \frac{1}{\| (A - \lambda)^{-1} \|}$$

So $\varrho(A)$ is open and therefore the complement $\sigma(A)$ is closed. $\square_{4.6.2}$

4.6.3 Theorem (resolvent equation)

The map $\lambda \mapsto \mathcal{R}_\lambda(A)$ is complex analytic on $\varrho(A)$.

We have the *resolvent equation* (Resolventengleichung):

$$\mathcal{R}_\lambda - \mathcal{R}_\mu = -(\lambda - \mu) \mathcal{R}_\lambda \cdot \mathcal{R}_\mu$$

Proof

Analogy with \mathbb{C} -numbers:

$$\begin{aligned} \frac{1}{\lambda - x} - \frac{1}{\mu - x} &= \frac{\mu - \lambda}{(\lambda - x)(\mu - x)} \\ (\mu - x) - (\lambda - x) &= \mu - \lambda \end{aligned}$$

Same thing for operators:

$$\begin{aligned} (\mu - A) - (\lambda - A) &= \mu - \lambda \\ \mathcal{R}_\mu^{-1} - \mathcal{R}_\lambda^{-1} &= \mu - \lambda \quad / \mathcal{R}_\mu \cdot \quad / \cdot \mathcal{R}_\lambda \\ \mathcal{R}_\lambda - \mathcal{R}_\mu &= (\mu - \lambda) \mathcal{R}_\mu \mathcal{R}_\lambda \\ \mathcal{R}_\lambda &= \mathcal{R}_\mu + (\mu - \lambda) \mathcal{R}_\mu \mathcal{R}_\lambda \end{aligned}$$

Assume $|\mu - \lambda| < \frac{1}{\| \mathcal{R}_\lambda \|}$.

$$\mathcal{R}_\mu = \mathcal{R}_\lambda (1 + (\mu - \lambda) \mathcal{R}_\lambda)^{-1} = \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-1)^n (\mu - \lambda)^n \mathcal{R}_\lambda$$

This series converges absolutely and so the map is analytic in $L(H)$. $\square_{4.6.3}$

5 Compact Operators

Let E and F be Banach spaces and $A \in L(E, F)$.

Remember: There exists a $C \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\| \leq C \|u\|$$

A maps bounded sets in E to bounded sets in F .

But: Bounded sets are not precompact in general.

5.1 Definition (compact operator)

A is called *compact* operator if and only if A maps bounded sets to relatively compact sets, i.e. the closure is compact.

(In complete spaces relatively compact is equivalent to precompact.)

5.2 Example (integral operator)

Let $E = (C^0([0,1]), \|\cdot\|_\infty)$ and consider an integral kernel $K \in C^0([0,1] \times [0,1])$, $K : E \rightarrow E$.

$$(K\varphi)(x) := \int_0^1 K(x,y) \varphi(y) dy$$

$$\begin{aligned} |(K\varphi)(x)| &\leq \sup_y |K(x,y)| \|\varphi\| & / \sup_x \\ \|K\varphi\| &\leq C \|\varphi\| \end{aligned}$$

So $K \in L(E)$. Furthermore the integral kernel K is continuous and defined on a compact set. Therefore K is uniformly continuous after the Heine-Cantor theorem.

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \exists_{\delta \in \mathbb{R}_{>0}} : |K(x,y) - K(x',y)| < \varepsilon \quad \forall_{|x-x'| < \delta, y \in [0,1]}$$

$$|(K\varphi)(x) - (K\varphi)(x')| = \left| \int_0^1 (K(x,y) - K(x',y)) \varphi(y) dy \right| \leq \varepsilon \|\varphi\|_\infty$$

Let now $B := B_M(0)$ with $M \in \mathbb{R}_{>0}$. Then $K(B) \subseteq E$.

- uniformly bounded ($\|\varphi\| < CM$)
- uniformly continuous

The Arzelà-Ascoli theorem yields, that $K(B)$ is precompact and so K is a compact operator.

5.3 Theorem

Let H be a Hilbert space.

A compact operator $A : H \rightarrow H$ maps weakly convergent sequences to convergent sequences.

Proof

Let $x_n \rightharpoonup x$, then (x_n) is bounded, i.e. there is a $C \in \mathbb{R}_{>0}$ such that $\|x_n\| < C$ for all $n \in \mathbb{N}$. Define $y_n := Ax_n$. For all $z \in H$ holds:

$$\langle z, y_n - y \rangle = \langle z, A(x_n - x) \rangle = \langle A^* z, x_n - x \rangle \rightarrow 0$$

Therefore $y_n \rightharpoonup y$ converges weakly. Because A is compact, every subsequence of y_n contains a convergent subsequence with limit \tilde{y} . For $z = \tilde{y} - y$ converges:

$$0 \leftarrow \langle z, y_n - y \rangle \rightarrow \langle \tilde{y} - y, \tilde{y} - y \rangle = \|\tilde{y} - y\|$$

Therefore $\tilde{y} = y$.

Since this holds for every subsequence of y_n follows $y_n \rightarrow y$. □_{5.3}

5.4 Lemma

Consider operators $A, B : E \rightarrow F$.

- i) If A and B are compact, so are $A + B$ and λA for all $\lambda \in \mathbb{K}$.
- ii) If $A : E \rightarrow F$ is compact (continuous) and $B : F \rightarrow E$ continuous (compact), then $B \circ A$ is compact.
(In particular A^n is compact for $A : E \rightarrow E$.)
- iii) The compact operators form a closed subspace of $L(E, F)$.

Proof

- i) is obvious. □_i
- ii) follows, since a continuous operator is bounded. □_{ii}
- iii) Let (x_n) be bounded and T_k a convergent sequence of compact operators. By diagonal choice get a subsequence, also written x_n , such that $T_k x_n$ converges for all $k \in \mathbb{N}$.

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq \underbrace{\|Tx_n - T_k x_n\|}_{\leq \|T - T_k\| \cdot \|x_n\|} + \|T_k x_n - T_k x_m\| + \underbrace{\|T_k x_m - Tx_m\|}_{\leq \|T - T_k\| \cdot \|x_m\|} \leq \\ &\leq \|T - T_k\| \cdot \|x_n\| + \|T_k x_n - T_k x_m\| + \|T - T_k\| \cdot \|x_m\| \xrightarrow{n, m, k \rightarrow \infty} 0 \end{aligned}$$

□_{5.4}

5.5 Lemma (Fredholm operator)

Let $A : E \rightarrow E$ be compact and define $T := \mathbb{1} - A$. T is called *Fredholm operator*.

- i) $\ker(T)$ is finite-dimensional.
- ii) There exists a $i \in \mathbb{N}$ such that $\ker(T^k) = \ker(T^i)$ for all $k \in \mathbb{N}_{>i}$.
- iii) The image of T is closed.

Proof

- i) $\ker(T) =: Z = \{u \mid u = Au\}$. Since $Z \cap B_1(0)$ is bounded

$$A(Z \cap B_1(0)) = Z \cap B_1(0)$$

is precompact and therefore Z is finite-dimensional. $\square_{\text{i)}$

- ii) Define $N_i := \ker(T^i)$, which are closed subspaces of E , since the T^i are continuous. Suppose the claim is wrong, then $N_j \subsetneq N_{j+1} \subsetneq \dots$, so in particular all N_j are proper subspaces. Choose $y_j \in N_j$ with:

$$\|y_j\| = 1 \qquad d(y_j, N_{j-1}) > \frac{1}{2}$$

This is possible after Lemma 2.1.2.

For all $m < n$ holds:

$$Ay_n - Ay_m = y_n - \underbrace{T_{y_n} - y_m + T_{y_m}}_{\in N_{n-1}}$$

Therefore follows:

$$\|Ay_n - Ay_m\| > \frac{1}{2}$$

So (Ay_n) has no accumulation value in contradiction to the compactness of A . $\square_{\text{ii)}$

- iii) Let $y_k \in \text{im}(T)$ with $y_k \rightarrow y$ and $y_k = Tx_k$. We want to show $y \in \text{im}(T)$. Define:

$$d_k := d(x_k, \ker(T)) = \inf_{z \in \ker(T)} \|x_k - z\|$$

Claim: (d_k) is bounded. Equivalently $(D_k) = |\max\{1, d_k\}|$ is bounded.

Proof: Choose $z_k \in \ker(T)$, $w_k := x_k - z_k$ with $\|w_k\| < 2d_k$ and $Tw_k = y_k$.

Assume D_k is unbounded. Since y_k is convergent and thus bounded, follows:

$$T\left(\frac{w_k}{D_k}\right) = \frac{y_k}{D_k} \xrightarrow{k \rightarrow \infty} 0$$

Now consider $u_k := \frac{w_k}{D_k}$. We know $\|u_k\| < 2$ and $T(u_k) \rightarrow 0$.

Thus $u_k - Au_k \rightarrow 0$. Since A is compact, every subsequence of Au_k has a convergent subsequence, and therefore $u_k \rightarrow 0$ converges.

The continuity of T gives:

$$T(u) = \lim_{k \rightarrow \infty} T(u_k) = 0$$

So $u \in \ker(T)$.

On the other hand we have for all $z \in \ker(T)$:

$$\begin{aligned} \|w_k - z\| &\geq D_k \\ \Rightarrow \left\| u_k - \frac{z}{D_k} \right\| &\geq 1 \end{aligned}$$

Since T is a subspace this means, that for all $z \in \ker(T)$ holds:

$$\|u_k - z\| \geq 1$$

This is a contradiction to $u \in \ker(T)$.

□_{Claim}

So u_k is bounded and $T(w_k) = T(x_k) = y_k \rightarrow y$. So we get:

$$w_k - Aw_k \rightarrow y$$

Since A is compact Aw_k converges and with this follows, that $w_k \rightarrow w$ also converges. By continuity we get:

$$T(w) = \lim_{k \rightarrow \infty} T(w_k) = y$$

So $w \in \operatorname{im}(T)$.

□_{5.5}

5.6 Theorem (Fredholm Alternative)

Let $A : E \rightarrow E$ be compact and define $T := \mathbb{1} - A$.

If the kernel $\ker(T) = \{0\}$ is trivial, then T is continuously invertible.

Proof

$\ker(T) = \{0\}$ means, that T is injective. We only need to show, that T is surjective, because then T is invertible and 2.4.7 yields then, that T is open and therefore T^{-1} continuous.

$\operatorname{im}(T)$ is closed following 5.5 iii).

$\operatorname{im}(T) = E$, since otherwise $T(E) \subsetneq E$. Then the injectivity implies for all $k \in \mathbb{N}$:

$$T^{k+1}(E) \subsetneq \underbrace{T^k(E)}_{=E_k}$$

E_k is closed for all $k \in \mathbb{N}$:

$$E_k = (\mathbb{1} - A)^k(E) = \left(\mathbb{1} + \underbrace{\sum_{l=1}^k (-1)^l \binom{k}{l} A^l}_{A := A_k} \right)(E)$$

Now A_k is compact, as the compact operators form a (closed) ideal subalgebra $\operatorname{CP}(E)$.

Choose $x_k \in E_k$ with $\|x_k\| = 1$ and $d(x_k, E_k) > \frac{1}{2}$, which is possible after Lemma 2.1.2. Then holds for all $m < n$:

$$Ax_m - Ax_n = x_m - \underbrace{Tx_m - x_n + Tx_n}_{\in H_{m+1}}$$

$$\Rightarrow \|Ax_m - Ax_n\| > \frac{1}{2}$$

This is a contradiction to the compactness of A .

Therefore T is surjective and the theorem follows. $\square_{5.6}$

5.7 Theorem (Riesz-Schauder)

Let $A \in L(H)$ be compact.

- i) $\sigma(A)$ consists of a finite or countable set of complex numbers and 0 is the only possible accumulation point.
- ii) Every $0 \neq \lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity, i.e. $\ker(A - \lambda)$ is finite-dimensional. That means, there exists a $i \in \mathbb{N}$ such that for all $k > i$ holds:

$$\ker(A - \lambda)^k = \ker(A - \lambda)^i$$

One says also that the Jordan chains are finite.

Proof

- ii) is an immediate consequence of the Lemmas 5.5 and 5.6. (Divide A by λ .)
- i) Assume $\lambda_n \neq 0$ are pairwise different eigenvalues. Choose eigenvectors $x_n \in H$ such that:

$$Ax_n = \lambda_n x_n$$

$$Y_n := \langle x_1, \dots, x_n \rangle$$

Since the eigenvalues are pairwise different $Y_n \subsetneq Y_{n+1}$ must hold, because the x_k are linearly independent.

Assume $Y_n \subsetneq H$, since otherwise H would be finite-dimensional and therefore $\sigma(A)$ a finite set.

So following Lemma 2.1.2 we can choose $y_n \in Y_n$ with $\|y_n\| = 1$ and:

$$d(y_n, Y_{n+1}) > \frac{1}{2}$$

Since $y_n \in Y_n$ one can find $\alpha_j \in \mathbb{K}$ such that:

$$y_n = \sum_j \alpha_j x_j$$

Then follows:

$$(A - \lambda_n) y_n = \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) \alpha_j x_j =: \tilde{y}_n \in Y_{n-1}$$

For all $n > m$ holds:

$$Ay_n - Ay_m = \lambda_n y_n - \underbrace{\tilde{y}_n - Ay_m}_{\in Y_{n-1}}$$

So we get:

$$\|Ay_n - Ay_m\| \geq \frac{|\lambda_n|}{2}$$

But (Ay_n) is precompact and thus for all $\delta \in \mathbb{R}_{>0}$ exist only finitely many λ_n with $|\lambda_n| > \delta$. Therefore 0 is the only accumulation point and $\sigma(A)$ is a countable union of finite sets and thus countable. $\square_{5.7}$

Jordan decomposition:

$$A = \begin{pmatrix} \lambda_1 & & & & 0 \\ & 1 & \ddots & & \\ & & 1 & \lambda_1 & \\ & & & \lambda_2 & \\ & & & & 1 & \ddots \\ & & & & & 1 & \lambda_2 \\ 0 & & & & & & \ddots \end{pmatrix}$$

$$\lambda_1 - A = \begin{pmatrix} 0 & & & & 0 \\ -1 & \ddots & & & \\ & -1 & 0 & & \\ & & & -\lambda_2 & \\ & & & -1 & \ddots \\ & & & & -1 & -\lambda_2 \\ 0 & & & & & \ddots \end{pmatrix}$$

So the first block is nilpotent. If it has k dimensions this means:

$$(\lambda_1 - A)^k = \begin{pmatrix} 0 & 0 \\ & * \\ 0 & * \end{pmatrix}$$

So k is the length of the Jordan chain.

5.8 Theorem

Let $A \in L(H)$ be compact and H be a separable Hilbert space. Then A can be approximated in $L(H)$ by operators of finite rank.

Proof

Choose a countable orthonormal Hilbert basis $(\varphi_j)_{j \in \mathbb{N}}$ of H , which is possible, since H is separable. Define:

$$\lambda_n := \sup_{\psi \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp, \|\psi\|=1} \|A\psi\|$$

Since A is bounded, this supremum exists. Obviously $\lambda_1 \geq \lambda_2 \geq \dots$. Thus $\lambda_n \searrow \lambda \geq 0$.

Claim: $\lambda = 0$

Proof: Choose $\psi_n \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp$ with $\|\psi_n\| = 1$ and $\|A\psi_n\| \geq \frac{\lambda}{2}$ which is possible after Lemma 2.1.2, since $\langle \varphi_1, \dots, \varphi_n \rangle$ is a proper closed subspace of H . Write:

$$\psi_n = \sum_{j=1}^{\infty} \nu_j \varphi_j = (\nu_1, \nu_2, \dots)$$

Due to $\psi_n \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp$ follows:

$$\psi_n = (0, \dots, 0, \nu_{n+1}, \nu_{n+2}, \dots)$$

For $u \in H$ holds:

$$\langle u, \psi_n \rangle = \sum_{j=n+1}^{\infty} \nu_j \cdot \bar{u}_j \stackrel{\substack{\text{Schwarz} \\ \text{inequality}}}{\leq} \underbrace{\left(\sum_{j=n+1}^{\infty} |\nu_j|^2 \right)^{\frac{1}{2}}}_{=\|\psi_n\|} \cdot \left(\sum_{j=n+1}^{\infty} |u_j|^2 \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$$

So by construction $\psi_n \rightarrow 0$. Therefore $A\psi_n \rightarrow 0$ and thus $\|A\lambda_n\| \rightarrow 0$.

On the other hand we have $\|A\psi_n\| \geq \frac{\lambda}{2}$ and so $\lambda = 0$. □_{Claim}

Let P_n be the orthogonal projection on $\langle \varphi_1, \dots, \varphi_n \rangle$.

$$P_n u = \sum_{j=1}^n \varphi_j \langle \varphi_j, u \rangle$$

AP_n is an operator of finite rank $r \leq n$, since $\text{rank}(P_n) = n$.

Claim: $AP_n \xrightarrow{n \rightarrow \infty} A$ in $L(H)$.

Proof: Consider:

$$\|A - AP_n\| = \sup_{u \in H, \|u\|=1} \|A(\mathbb{1} - P_n)u\|$$

$(\mathbb{1} - P_n)u \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp$ and $\|(\mathbb{1} - P_n)u\| \leq \|u\| = 1$. ($\mathbb{1} - P_n = P_{\langle \varphi_1, \dots, \varphi_n \rangle^\perp}$)

Thus we get:

$$\|A - AP_n\| \leq \sup_{v \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp, \|v\| \leq 1} \|Av\| = \lambda_n \xrightarrow{n \rightarrow \infty} 0$$

□_{Claim}

□_{5.8}

5.9 Lemma

Let $A \in L(H)$ be compact and symmetric. (This implies that A is bounded and self-adjoint.) Then $\sigma(A) \subseteq \mathbb{R}$ and if u is an eigenvector, $Au = \lambda u$, then its orthogonal is invariant under A .

Proof

For $\lambda \in \sigma(A)$ holds $\ker(A - \lambda) \neq \{0\}$. Thus there exists a $u \in \ker(A - \lambda) \setminus \{0\}$.

$$\lambda \langle u, u \rangle = \langle u, Au \rangle = \langle Au, u \rangle = \bar{\lambda} \langle u, u \rangle$$

Since $\|u\| \neq 0$ follows $\lambda = \bar{\lambda}$, which means that $\lambda \in \mathbb{R}$.

For $v \in \langle u \rangle^\perp$ holds:

$$\langle Av, u \rangle = \langle v, Au \rangle = \lambda \langle v, u \rangle = 0$$

Therefore $Av \in \langle u \rangle^\perp$.

□_{5.9}**5.10 Theorem (Hilbert-Schmidt)**

Let $A \in L(H)$ be a symmetric compact operator on the separable Hilbert space H .

Then there exists an orthonormal Hilbert space basis of eigenvectors $(u_n)_{n \in \mathbb{N}}$, so with the eigenvalues $\lambda_n \in \mathbb{R}$ holds:

$$Au_n = \lambda_n u_n$$

Proof

$\sigma(A)$ is countable and therefore we can write $\sigma(A) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots\} \subseteq \mathbb{R}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. $\ker(\lambda_j - A)$ is finite-dimensional. So we choose a (finite) orthonormal basis of the eigenspace. Taking these eigenvectors for all eigenvalues, we obtain a countable orthonormal system $(u_n)_{n \in \mathbb{N}}$.

$$M := \overline{\langle u_n \rangle}^{\text{closed}} \subseteq H$$

M^\perp is an invariant subspace of H under A , i.e.:

$$\tilde{A} := A|_{M^\perp} : M^\perp \rightarrow M^\perp$$

This is again symmetric and compact. We know that $\sigma(\tilde{A}) = \{0\}$.

Question: Why is $\tilde{A} = 0$?

This is not true for a general operator, e.g.:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma(A) = \{0\}$$

Answer: If A is symmetric and $\sigma(A) = \{0\}$, then one can show $A = 0$ using the following theorem 5.12:

From $\sigma(\tilde{A}) = \{0\}$ follows $r(\tilde{A}) = 0$ and since \tilde{A} is self-adjoint theorem 5.12 gives $\|\tilde{A}\| = 0$ and thus $\tilde{A} = 0$. In other words $A|_{M^\perp} = 0$.

Now choose an orthonormal Hilbert basis $(v_n)_{n \in \mathbb{N}_{\leq N}}$ of M^\perp for an $N \in \mathbb{N} \cup \{\infty\}$. Therefore $\{u_n\} \cup \{v_n\}$ is the desired orthonormal Hilbert basis of H . □_{5.10}

5.11 Definition (spectral radius)

Let $A : \mathcal{D}(A) \subset H \rightarrow H$ be a densely defined operator. Then the *spectral radius* $r(A)$ of A is defined by:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

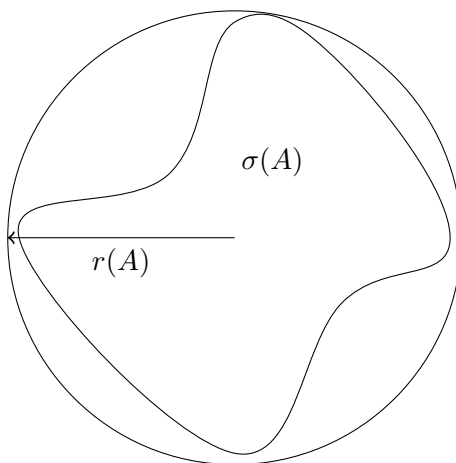


Figure 5.1: $\sigma(A) \subseteq \overline{B_{r(A)}(0)}$

5.12 Theorem

For $A \in L(H)$ holds:

$$r(A) = \limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$$

If A is symmetric, then:

$$r(A) = \|A\|$$

Proof

Recall for a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

with $a_n, z \in \mathbb{K}$ the root test (Wurzelkriterium):

– If

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} =: c < 1$$

then $|a_n z^n| < c^n$ and therefore is

$$\sum_{n=0}^{\infty} c^n$$

a convergent dominating sequence. Thus $\sum_{n=0}^{\infty} a_n z^n$ converges as well.

– If

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} =: c > 1$$

then $|a_n z^n| > c^n > 1$ for an infinite number of n . Therefore $a_n z^n$ does *not* converge to zero, which implies that $\sum_{n=0}^{\infty} a_n z^n$ does not converge as well.

– If

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} = 1$$

no conclusion is possible.

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} = |z| \cdot \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

The Radius of convergence (Konvergenzradius) is thus defined by:

$$R := \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}$$

If $|z| < R$ the sum converges absolutely and if $|z| > R$ the sum diverges.

In our setting for $A = 0$ is nothing to prove. For $\lambda \in \varrho(A) \setminus \{0\}$ we make a formal expansion:

$$\mathcal{R}_\lambda = (\lambda - A)^{-1} = \frac{1}{\lambda} \left(\mathbb{1} - \frac{A}{\lambda} \right)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} A^n \cdot \left(\frac{1}{\lambda} \right)^n$$

This is a power series in $\frac{1}{\lambda}$, but the coefficients are operators.

$$R := \frac{1}{\limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}}$$

For $\frac{1}{|\lambda|} < R$

$$\left\| \sum_{n=0}^{\infty} A^n \left(\frac{1}{\lambda} \right)^n \right\| \leq \sum_{n=0}^{\infty} \|A^n\| \frac{1}{\lambda^n}$$

converges absolutely and so

$$\sum_{n=0}^{\infty} A^n \left(\frac{1}{\lambda} \right)^n$$

converges in $L(H)$. Thus the resolvent

$$\mathcal{R}_\lambda = (\lambda - A)^{-1}$$

exists and $\sigma(A) \subseteq \overline{B_{\frac{1}{R}}(0)}$, i.e.:

$$r(A) \leq \frac{1}{R} = \limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$$

If $\frac{1}{|\lambda|} > R$

$$\left\| \sum_{n=0}^{\infty} A^n \left(\frac{1}{\lambda} \right)^n \right\|$$

diverges.

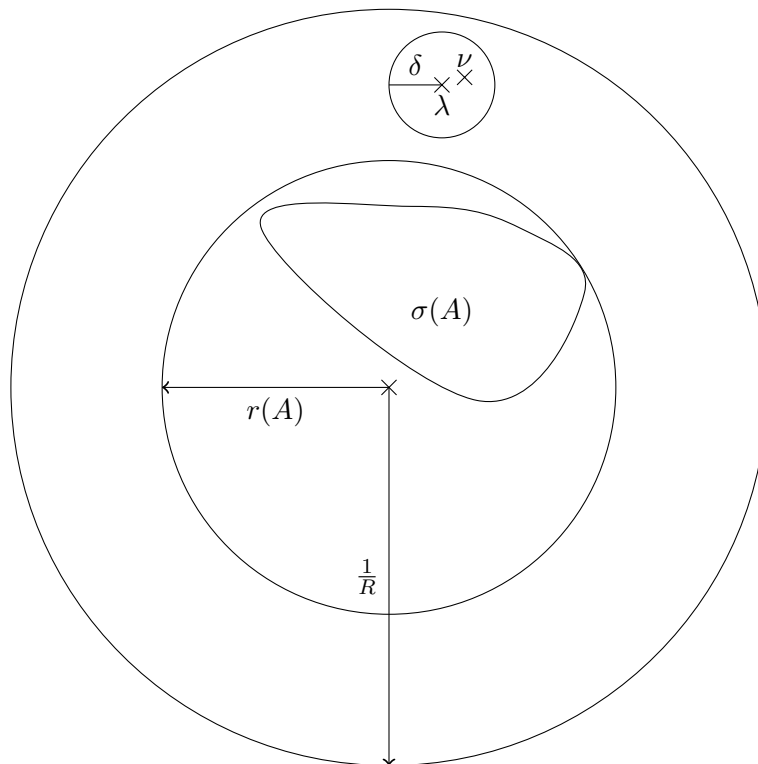


Figure 5.2: $\frac{1}{R} > r(A)$?

Why is r not smaller than $\frac{1}{R}$?

Assume that $r < \frac{1}{R}$ and choose λ with $r < |\lambda| < \frac{1}{R}$. Then \mathcal{R}_λ exists and is analytic. Consider a $\nu \in B_\delta(\lambda)$.

$$\begin{aligned} \mathcal{R}_\nu &= (\nu - A)^{-1} = ((\nu - \lambda) + (\lambda - A))^{-1} = \\ &= (((\nu - \lambda) \mathcal{R}_\lambda + \mathbb{1})(\lambda - A))^{-1} = \\ &= \mathcal{R}_\lambda (\mathbb{1} + (\nu - \lambda) \mathcal{R}_\lambda)^{-1} = \\ &= \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-(\nu - \lambda))^n \mathcal{R}_\lambda^n \end{aligned}$$

For $|\nu - \lambda| < \delta := \frac{1}{\|\mathcal{R}_\lambda\|}$ the Neumann series converges.

Thus \mathcal{R}_λ can be expanded locally in a power series, i.e. \mathcal{R}_λ is complex analytic or holomorphic.

Furthermore for $|\lambda| > \frac{1}{R}$ holds:

$$\mathcal{R}_\lambda = \sum_{n=0}^{\infty} A^n \frac{1}{\lambda^{n+1}}$$

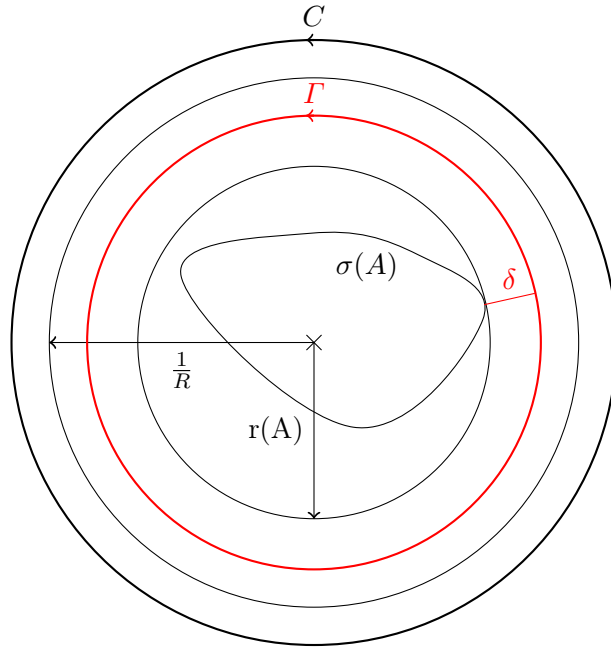


Figure 5.3: Contours Γ and C for integration

Integrate along the contour C :

$$\frac{1}{2\pi i} \oint_C \lambda^n \mathcal{R}_\lambda d\lambda = \sum_{k=0}^{\infty} A^k \underbrace{\frac{1}{2\pi i} \oint_C \frac{\lambda^n}{\lambda^{k+1}} d\lambda}_{=: I}$$

Since the geometric series converges absolutely, the summation and the integration can be interchanged. The residue theorem gives:

$$I = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Therefore we get:

$$\frac{1}{2\pi i} \oint_C \lambda^n \mathcal{R}_\lambda d\lambda = A^n$$

Choose $\Gamma = \partial B_{r+\delta}(0)$. We know, that \mathcal{R}_λ is holomorphic outside Γ . Thus we may continuously deform the contour to obtain:

$$\frac{1}{2\pi i} \oint_\Gamma \lambda^n \mathcal{R}_\lambda d\lambda = A^n$$

Thus we have:

$$\|A^n\| = \left\| \frac{1}{2\pi i} \oint_\Gamma \lambda^n \mathcal{R}_\lambda d\lambda \right\| \leq C (r + \delta)^n (r + \delta)$$

$$C := \frac{1}{2\pi} \sup_{\lambda \in I} \|\mathcal{R}_\lambda\|$$

$$\Rightarrow \quad \|A^n\|^{\frac{1}{n}} \leq (r + \delta) \left(C^{\frac{1}{n}} (r + \delta)^{\frac{1}{n}} \right) \xrightarrow{n \rightarrow \infty} r + \delta$$

Therefore:

$$\limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} \leq r + \delta$$

Since δ is arbitrary, it follows that:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} = r$$

We even conclude:

$$\|A^n\|^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} r(A)$$

Assume that A is *symmetric* (to show $\|A^n\|^{\frac{1}{n}} = \|A\|$). The Schwarz inequality gives:

$$\|A^2\| \leq \|A\| \cdot \|A\| = \|A\|^2$$

$$\|A\|^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \langle u, Au^2 \rangle \leq \sup_{\|u\|=1} \underbrace{\|u\|}_{=1} \cdot \|A^2 u\|$$

Iteratively for $n \in \mathbb{N}$:

$$\|A^{2^n}\| = \|A\|^{2^n}$$

For arbitrary $m \in \mathbb{N}$ the Schwarz inequality gives:

$$\|A^m\| \leq \|A\|^m$$

Choose n such that $2^n > m$. Then:

$$\begin{aligned} \|A\|^{2^n} &= \|A^{2^n}\| = \|A^m \cdot A^{2^n-m}\| \leq \|A^m\| \cdot \|A\|^{2^n-m} \\ \Rightarrow \quad \|A\|^m &\leq \|A\|^m \end{aligned}$$

□_{5.12}

5.13 Ritz method

Let $A \in L(H)$ be a symmetric compact operator on the separable Hilbert space H . From the Hilbert-Schmidt theorem 5.10 we know that there exists an orthonormal eigenvalue basis (u_n) of H .

$$Au_n = \lambda_n u_n$$

We now want to construct the u_n :

Consider the “expectation value” functional:

$$\begin{aligned} S : H &\rightarrow \mathbb{R} \\ u &\mapsto \langle u, Au \rangle \end{aligned}$$

This is well defined, since:

$$\overline{S(u)} = \overline{\langle u, Au \rangle} = \langle Au, u \rangle = \langle u, Au \rangle = S(u)$$

S is bounded, because:

$$|S(u)| = |\langle u, Au \rangle| \leq \|A\| \cdot \|u\|^2 \stackrel{\|u\| \leq 1}{\leq} \|A\|$$

Maximize $|S(u)|$ on $\{u \in H \mid \|u\| = 1\}$:

Choose a maximizing sequence (u_n) with $\|u_n\| = 1$ and:

$$|S(u_n)| \xrightarrow{n \rightarrow \infty} \sup_{\|u\|=1} |S(u)|$$

Since $\overline{B_1(0)}$ is weakly compact, there is a subsequence u_{k_l} , which converges weakly $u_{k_l} \rightharpoonup u$. Since A is compact, the sequence

$$v_{k_l} := Au_{k_l} \rightarrow v$$

converges and $Au = v$. As a consequence:

$$S(u_{k_l}) = \langle u_{k_l}, Au_{k_l} \rangle = \langle u_{k_l}, v_{k_l} \rangle = \underbrace{\langle u_{k_l}, v \rangle}_{\rightarrow \langle u, v \rangle} + \langle u_{k_l}, v_{k_l} - v \rangle \xrightarrow{l \rightarrow \infty} \langle u, v \rangle = \langle u, Au \rangle = S(u)$$

This follows, because:

$$|\langle u_{k_l}, v_{k_l} - v \rangle| \leq \underbrace{\|u_{k_l}\|}_{=1} \cdot \underbrace{\|v_{k_l} - v\|}_{\rightarrow 0} \xrightarrow{l \rightarrow \infty} 0$$

Thus S is weakly continuous, i.e. for any $u_k \rightharpoonup u$ converges $S(u_k) \rightarrow S(u)$.

Because (u_n) is a maximizing sequence, we get:

$$|S(u)| = \sup_{\|\tilde{u}\|=1} |S(\tilde{u})|$$

Therefore u is the desired maximizer.

– u is on the unit sphere:

The simple approach

$$\|u\|^2 \neq \lim_{l \rightarrow \infty} \|u_{k_l}\|^2$$

does not work, because u_{k_l} only converges weakly.

Example:

If (e_l) is an orthonormal Hilbert basis in a separable Hilbert space, then $e_l \rightharpoonup 0$, but:

$$\lim_{l \rightarrow \infty} \|e_l\| = 1 \neq 0 = \|0\|$$

But it holds:

$$\begin{aligned}\|u\|^2 &= \lim_{l \rightarrow \infty} |\langle u, u_{k_l} \rangle| \leq \lim_{l \rightarrow \infty} \|u_{k_l}\| \cdot \|u\| = \|u\| \\ \Rightarrow \|u\| &\leq 1\end{aligned}$$

Assume $\|u\| < 1$, then the vector $\hat{u} := \frac{u}{\|u\|}$ would satisfy the equation:

$$|S(\hat{u})| = |\langle \hat{u}, A\hat{u} \rangle| = \frac{1}{\|u\|^2} |\langle u, Au \rangle| = \frac{1}{\|u\|^2} \sup_{\|v\|=1} |S(v)| \stackrel{\|u\|<1}{>} \sup_{\|v\|=1} |S(v)|$$

This is a contradiction. Therefore u is in fact a unit vector.

- u is an eigenvector corresponding to the eigenvalue $\lambda = \langle u, Au \rangle \in \mathbb{R}$: Consider the variation for $v \in H$:

$$\tilde{u}(\tau) = u + \tau v$$

$$S\left(\frac{\tilde{u}}{\|\tilde{u}\|}\right) = \frac{\langle \tilde{u}, A\tilde{u} \rangle}{\langle \tilde{u}, \tilde{u} \rangle} = \frac{\langle u + \tau v, A(u + \tau v) \rangle}{\langle u + \tau v, u + \tau v \rangle}$$

This is called *Rayleigh quotient*. We know that $S(\tilde{u}(\tau))$ is extremal at $\tau = 0$:

$$\begin{aligned}0 &= \left. \frac{d}{d\tau} S(\tilde{u}(\tau)) \right|_{\tau=0} = \\ &= \frac{\langle u, Av \rangle + \langle v, Au \rangle + 2\tau \langle v, v \rangle}{\langle u + \tau v, u + \tau v \rangle} - \frac{\langle u + \tau v, A(u + \tau v) \rangle}{\langle u + \tau v, u + \tau v \rangle^2} \cdot (\langle v, u \rangle + \langle u, v \rangle + \tau \langle v, v \rangle) \Big|_{\tau=0} = \\ &\stackrel{A \text{ symmetric}}{=} 2 \frac{\operatorname{Re}(\langle v, Au \rangle)}{\langle u, u \rangle} - 2 \operatorname{Re}(\langle v, u \rangle) \frac{\langle u, Au \rangle}{\langle u, u \rangle^2} = \\ &\stackrel{\lambda = \frac{\langle u, Au \rangle}{\langle u, u \rangle} = 1}{=} 2 (\operatorname{Re}(\langle v, Au \rangle) - \lambda \operatorname{Re}(\langle v, u \rangle)) = 2 \operatorname{Re}(\langle v, (A - \lambda)u \rangle)\end{aligned}$$

Set $v = e^{i\varphi}w$ for any $\varphi \in \mathbb{R}$ and $w \in H$. So:

$$0 = \operatorname{Re}(\langle v, (A - \lambda)u \rangle) = \operatorname{Re}\left(e^{-i\varphi} \langle w, (A - \lambda)u \rangle\right) \quad \forall \varphi \in \mathbb{R}$$

$$\Rightarrow \langle w, (A - \lambda)u \rangle = 0 \quad \forall w \in H$$

$$\begin{aligned}(A - \lambda)u &= 0 \\ Au &= \lambda u\end{aligned}$$

- It holds $|\lambda| = \|A\|$:

There is no point ν in the spectrum of A with $|\nu| > |\lambda|$, because otherwise for all $v \in H$ with $Av = \nu v$ follows:

$$\frac{|\langle v, Av \rangle|}{\langle v, v \rangle} = |\nu| > |\lambda| = |\langle u, Au \rangle| = \sup_{w \in H} \frac{|\langle w, Aw \rangle|}{\langle w, w \rangle}$$

This is a contradiction. Thus we get:

$$|\lambda| = \sup_{\nu \in \sigma(A)} |\nu| \stackrel{\text{by definition}}{=} r(A) \stackrel{5.12}{=} \|A\|$$

Thus we have *constructed* a $u \in H$ with $\|u\| = 1$, $Au = \lambda u$ and $|\lambda| = \|A\|$. Now one can proceed inductively:

$$H_1 := \langle u \rangle^\perp$$

$$A|_{H_1} : H_1 \rightarrow H_1$$

(We saw that H_1 is invariant under A .)

Repeat the above procedure to maximize $|\langle v, Av \rangle|$ on $H_1 \cap \{v \in H \mid \|v\| = 1\}$. This gives u_1 with $\|u_1\| = 1$, $Au_1 = \lambda_1 u_1$ and:

$$|\lambda_1| = \|A|_{H_1}\| \leq \|A\| = |\lambda|$$

Now set $H_2 = \langle u, u_1 \rangle^\perp$ and proceed inductively.

This gives a sequence $u_0 := u, u_1, u_2, \dots$ of orthonormal eigenvectors, i.e. $Au_j = \lambda_j u_j$, with decreasing eigenvalues $|\lambda_j|$.

These (u_j) are an orthonormal basis. (Proof as in Theorem 5.10)

□_{5.13}

Ritz, Galerkin: Finite element method

Example: Helium molecule wave function in $H = L^2(\mathbb{R}^3, \mathbb{C})$

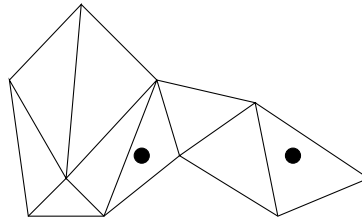


Figure 5.4: finite lattice for numerical approximation

$$A = -\frac{\hbar^2}{2m} \Delta - \frac{ze^2}{\|x - x_1\|} - \frac{ze^2}{\|x - x_2\|}$$

Now minimize

$$\frac{\langle u, Au \rangle}{\langle u, u \rangle}$$

on a finite subspace of H .

6 A few (technical) results

6.1 Dini's theorem

Let E be a metric space and $f_n : E \rightarrow \mathbb{R}$ a sequence of real valued functions.

6.1.1 Definition (point-wise/uniform convergence)

f_n converges point-wise to f if $f_n(x) \rightarrow f(x)$ converges for all $x \in E$, i.e.:

$$\forall_{x \in E} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{N(\varepsilon, x)} \quad \forall_{n \in \mathbb{N}_{\geq N}} : |f_n(x) - f(x)| < \varepsilon$$

f_n converges uniformly to f , in symbols $f_n \rightrightarrows f$, if for all $\varepsilon \in \mathbb{R}_{>0}$ exists a $N(\varepsilon)$ such that for all $n \geq N$ and all $x \in E$ holds:

$$|f_n(x) - f(x)| < \varepsilon$$

With quantifiers this is:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{N(\varepsilon)} \quad \forall_{n \in \mathbb{N}_{\geq N}} \quad \forall_{x \in E} : |f_n(x) - f(x)| < \varepsilon$$

6.1.2 Theorem

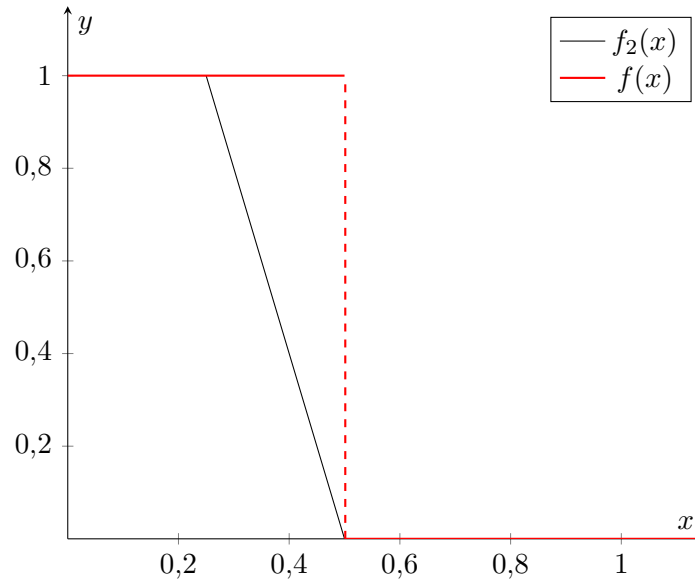
If (f_n) is a sequence of continuous functions with $f_n \rightrightarrows f$, then f is also continuous. This is not true in general for point wise convergence:

$$f_n(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2} \left(1 - \frac{1}{n}\right) \\ 0 & \text{for } x \geq \frac{1}{2} \\ n(1 - 2x) & \text{for } \frac{1}{2} \left(1 - \frac{1}{n}\right) < x < \frac{1}{2} \end{cases}$$

$f_n \rightarrow f$ converges pointwise to:

$$f(x) = \begin{cases} 1 & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{cases}$$

This f is *not* continuous.

Figure 6.1: $f_n(x)$ is continuous, but not $f(x)$ **Proof**

Show that for all $x \in E$ the ε - δ -criterion is satisfied:

Since $f_n \rightrightarrows f$ converges uniformly, there is a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ and all $x \in E$ holds:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

Because the f_n are continuous, there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $y \in B_\delta(x)$ holds:

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$$

Now follows for all $y \in B_\delta(x)$:

$$|f(y) - f(x)| \leq \underbrace{|f(y) - f_N(y)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(y) - f_N(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f(x)|}_{< \frac{\varepsilon}{3}} < \varepsilon$$

Therefore f is continuous. □_{6.1.2}

6.1.3 Definition (monotonically increasing/decreasing)

The sequence of functions (f_n) , $f_n : E \rightarrow \mathbb{R}$ is called *monotonically increasing (decreasing)* if for all $x \in E$ the real sequence $f_n(x)$ is monotonically increasing (decreasing).

6.1.4 Theorem (Dini)

Let E be a *compact* metric space, (f_n) monotone and $f_n \rightarrow f$.

If f_n and f are continuous, then the convergence $f_n \rightrightarrows f$ is uniform.

Proof

Without loss of generality we assume (f_n) is a monotonically increasing sequence (otherwise consider $-f_n$), i.e. $f_n(x) \leq f_{n+1}(x)$ for all $x \in E$ and all $n \in \mathbb{N}$.

Given $\varepsilon > 0$ we want to show:

$$\exists_{N \in \mathbb{N}} \forall_{x \in E} \forall_{n \in \mathbb{N}_{\geq N}} : |f(x) - f_n(x)| < \varepsilon$$

For any $x \in E$ there exists an $N(x)$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}_{\geq N}$ (point-wise convergence). Since both $f_{N(x)}$ and f are continuous functions, there exists a neighborhood $U(x) = B_{\delta(x)}(x)$ of x such that for all $z \in U(x)$ holds:

$$\begin{aligned} |f_{N(x)}(z) - f_{N(x)}(x)| &\leq \frac{\varepsilon}{4} \\ |f(z) - f(x)| &\leq \frac{\varepsilon}{4} \end{aligned}$$

Then follows:

$$|f_{N(x)}(z) - f(z)| \leq \underbrace{|f_{N(x)}(z) - f_{N(x)}(x)|}_{\leq \frac{\varepsilon}{4}} + \underbrace{|f_{N(x)}(x) - f(x)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f(x) - f(z)|}_{\leq \frac{\varepsilon}{4}} < \varepsilon$$

Since $f_n(z)$ is monotonically increasing, it follows that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in B_{\delta(x)}(x)$. Now use a standard compactness argument: Since E is compact, it can be covered by a finite number of these balls $B_{\delta(x_1)}(x_1), \dots, B_{\delta(x_n)}(x_n)$. Define:

$$N = \max\{N(x_1), \dots, N(x_n)\}$$

So for all $n \in \mathbb{N}_{\geq N}$ holds:

$$|f_n(x) - f(x)| < \varepsilon$$

□_{6.1.4}

6.2 Stone-Weierstraß theorem

We follow the nice (since constructive) proof by Bernstein.

6.2.1 Definition (polynomials)

Let $E = C^0([0,1])$ be the Banach space of real valued functions with norm:

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

$\mathcal{P}([0,1])$ are the *real polynomials*, i.e. for $f \in \mathcal{P}([0,1])$ there are $a_j \in \mathbb{R}$ such that:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

Clearly $\mathcal{P}([0,1]) \subseteq C^0([0,1])$ forms a subspace.

We want to show that $\mathcal{P}([0,1])$ is dense in $C^0([0,1])$.

6.2.2 Lemma

For $x \in [0,1]$ holds:

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 1$$

Proof

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = (x + 1 - x)^n = 1$$

□_{6.2.2}

6.2.3 Lemma

For $x \in [0,1]$ holds:

$$\sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$$

Obviously holds

$$(nx - k)^2 \leq 4n^2$$

and therefore:

$$\sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} \leq 4n^2 \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 4n^2$$

Proof

It holds:

$$\begin{aligned} \sum_{k=0}^n k \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^n k \frac{n!}{k! (n-k)!} x^k (1-x)^{n-k} = \\ &= 0 + \sum_{k=1}^n \frac{n \cdot (n-1)!}{(k-1)! (n-k)!} x^k (1-x)^{n-k} = \\ &= n \sum_{k=1}^n \binom{n-1}{k-1} x^k (1-x)^{n-k} = \\ &\stackrel{j:=k-1}{=} n \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1} (1-x)^{n-j-1} = \end{aligned}$$

$$= nx \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{(n-1)-j} = nx (x + 1 - x)^{n-1} = nx$$

Similarly one gets:

$$\sum_{k=0}^n k(k-1) \binom{n}{k} x^k (1-x)^{n-k} = n(n-1) \sum_{k=2}^n \binom{n-2}{k-2} x^k (1-x)^{n-k} = n(n-1) x^2$$

Together this gives:

$$\begin{aligned} \sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^n (n^2 x^2 - 2n x k + k^2) \binom{n}{k} x^k (1-x)^{n-k} = \\ &= \sum_{k=0}^n (n^2 x^2 - 2n x k + k(k-1) + k) \binom{n}{k} x^k (1-x)^{n-k} = \\ &= n^2 x^2 - 2n x \cdot nx + n(n-1) x^2 + nx = \\ &= -n^2 x^2 + n^2 x^2 - nx^2 + nx = nx(1-x) \end{aligned}$$

□_{6.2.3}

A more elegant method is to use derivatives:

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} &= (x+y)^n \\ \sum_{k=0}^n k \binom{n}{k} x^k y^{n-k} &= x \cdot \frac{d}{dx} \left(\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \right) \\ \sum_{k=0}^n k^2 \binom{n}{k} x^k y^{n-k} &= \left(x \cdot \frac{d}{dx} \right)^2 \left(\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \right) \end{aligned}$$

6.2.4 Definition

For $f \in C^0([0,1])$ define:

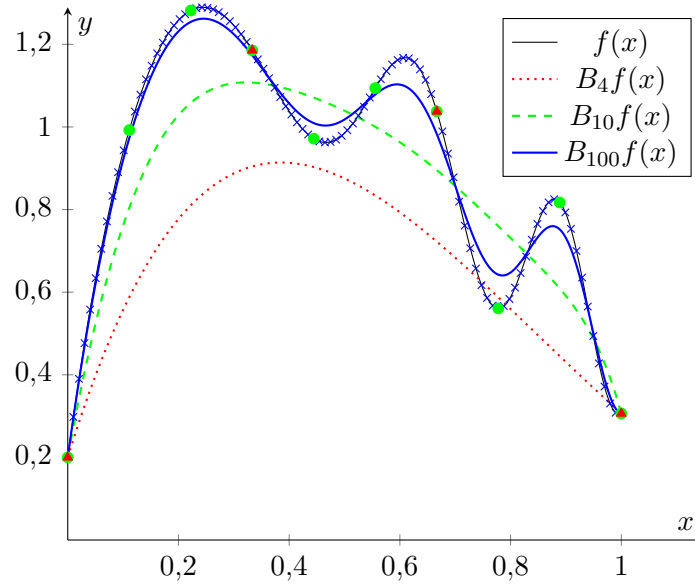
$$B_n f(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

6.2.5 Theorem (Bernstein)

For any $f \in C^0([0,1], \mathbb{R})$, $B_n f \rightrightarrows f$ converges uniformly.

Example: $f(x) = 10x \cdot e^{-3x} + \frac{1}{5} \cos((4x)^2)$

$$\begin{aligned} B_4 f(x) &\approx 0,2 \cdot (1-x)^4 + 5,2 \cdot x \cdot (1-x)^3 + 5,9 \cdot x^2 \cdot (1-x)^2 + 2,4 \cdot x^3 \cdot (1-x) + 0,3 \cdot x^4 \\ B_{10} f(x) &\approx 0,2 \cdot (1-x)^{10} + 9,4 \cdot x \cdot (1-x)^9 + 56,6 \cdot x^2 \cdot (1-x)^8 + 149,5 \cdot x^3 \cdot (1-x)^7 + \\ &\quad + 217,9 \cdot x^4 \cdot (1-x)^6 + 248,2 \cdot x^5 \cdot (1-x)^5 + 244,7 \cdot x^6 \cdot (1-x)^4 + \\ &\quad + 103,2 \cdot x^7 \cdot (1-x)^3 + 26,5 \cdot x^8 \cdot (1-x)^2 + 7,9 \cdot x^9 \cdot (1-x) + 0,3 \cdot x^{10} \end{aligned}$$

Figure 6.2: Approximation of $f(x)$ by $B_n f(x)$ **Proof**

Without loss of generality assume $f \neq 0$ (otherwise $B_n f = 0 = f$).

$$M := \|f\| > 0$$

Consider an arbitrary $\varepsilon \in \mathbb{R}_{>0}$. f is continuous on the compact set $[0,1]$ and thus uniformly continuous, i.e. there exists a $\delta \in \mathbb{R}_{>0}$ such that:

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2}$$

Choose $N \ni N \geq \frac{M}{\varepsilon \delta^2}$.

Claim: $|B_n f(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ and all $n \geq N$.

Proof: It holds:

$$f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$(B_n f - f)(x) = \sum_{k=0}^n \left(f\left(\frac{k}{n}\right) - f(x) \right) \binom{n}{k} x^k (1-x)^{n-k}$$

Define:

$$A := \left\{ k \left| \left| \frac{k}{n} - x \right| < \delta \right. \right\} \qquad B := \left\{ k \left| \left| \frac{k}{n} - x \right| \geq \delta \right. \right\}$$

We have:

$$\begin{aligned}
\sum_{k \in A} \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{< \frac{\varepsilon}{2}} \binom{n}{k} x^k (1-x)^{n-k} &< \frac{\varepsilon}{2} \sum_{k \in A} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{\varepsilon}{2} \\
\sum_{k \in B} \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{\leq 2\|f\|=2M} \binom{n}{k} x^k (1-x)^{n-k} &\leq \\
&\leq 2M \sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k} \leq \\
&\stackrel{k \in B}{\leq} \frac{2M}{n^2 \delta^2} \sum_{k=0}^n \underbrace{(k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}}_{\leq \frac{n}{4}} \leq \\
&\stackrel{n \geq N}{\leq} \frac{M}{2n\delta^2} \leq \frac{M}{2\frac{M}{\varepsilon\delta^2}\delta^2} = \frac{\varepsilon}{2}
\end{aligned}$$

Therefore holds for all $x \in [0,1]$.

$$|B_n f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□ Claim

Therefore $B_n f \Rightarrow f$ converges uniformly.

□ 6.2.5

Now generalize: Let E be a compact metric space. $C^0(E, \mathbb{R})$ with

$$\|f\| = \sup_{x \in E} |f(x)|$$

is a Banach space. Moreover, it is an algebra with the point-wise multiplication:

$$(f \cdot g)(x) := f(x) \cdot g(x)$$

The multiplication is continuous:

$$\|f \cdot g\| \leq \|f\| \cdot \|g\|$$

In summary $(C^0(E, \mathbb{R}), \|\cdot\|, +, \cdot)$ is a *Banach algebra*.

6.2.6 Theorem (Weierstraß)

The polynomials are dense in $C^0([0,1], \mathbb{R})$.

Proof

For any $f \in C^0([0,1], \mathbb{R})$, $B_n f \Rightarrow f$ converges uniformly and since the $B_n f$ are polynomials, these are dense. □ 6.2.6

6.2.7 Theorem (Stone-Weierstraß)

Let $\mathcal{A} \subseteq C^0(E, \mathbb{R})$ be a subalgebra with the following properties:

1. \mathcal{A} contains $f = 1$ and so by scalar multiplication all the constant functions.
2. \mathcal{A} separates the points of E , i.e. for all $x, y \in E$ with $x \neq y$ there exists a $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Then \mathcal{A} is dense in $C^0(E, \mathbb{R})$.

Proof

- i) There is a sequence of polynomials u_n on $[0, 1]$ such that $u_n \rightrightarrows f$ with $f(t) = \sqrt{t}$. This follows immediately from theorem 6.2.6.
- ii) If $f \in \mathcal{A}$, then $|f|$ defined by $|f|(x) := |f(x)|$ is in the closure $\overline{\mathcal{A}}$ of \mathcal{A} :
For $f \in \mathcal{A}$ define:

$$a := \|f\| = \max_{x \in E} |f(x)|$$

$$\Rightarrow \frac{f^2(x)}{a^2} \in [0, 1]$$

Then converges:

$$u_n \left(\frac{f^2(x)}{a^2} \right) \xrightarrow{n \rightarrow \infty} \sqrt{\frac{f^2(x)}{a^2}} = \frac{|f(x)|}{a}$$

The functions $u_n \left(\frac{f^2}{a^2} \right)$ lie in \mathcal{A} , since these are a polynomials of f and thus again elements of the algebra \mathcal{A} . Moreover $u_n \left(\frac{f^2}{a^2} \right)$ converges uniformly to $\frac{|f|}{a}$, because for a given $\varepsilon \in \mathbb{R}_{>0}$ exists a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq N}$ and all $t \in [0, 1]$ holds:

$$\left| u_n(t) - \sqrt{t} \right| < \varepsilon$$

Then follows with $t = \frac{f^2(x)}{a^2}$:

$$\left| u_n \left(\frac{f^2(x)}{a^2} \right) - \frac{|f|}{a} \right| < \varepsilon$$

Thus $\frac{|f|}{a} \in \overline{\mathcal{A}}$ and therefore also $|f| \in \overline{\mathcal{A}}$.

- iii) For $f, g \in \overline{\mathcal{A}}$ also $\min(f, g)$ and $\max(f, g)$ (defined point-wise) are again in $\overline{\mathcal{A}}$:

$$\min(f, g) = \frac{1}{2} (f + g - |f - g|)$$

$$\max(f, g) = \frac{1}{2} (f + g + |f - g|)$$

Choose $f_n, g_n \in \mathcal{A}$ such that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. By ii) follows $|f_n - g_n| \in \overline{\mathcal{A}}$ and $|f_n - g_n| \rightrightarrows |f - g|$. Therefore holds:

$$\overline{\mathcal{A}} \ni \min(f_n, g_n) \rightrightarrows \min(f, g) \in \overline{\mathcal{A}}$$

Similarly the claim follows for \max .

- iv) For all $x, y \in E$ with $x \neq y$ and $\alpha, \beta \in \mathbb{R}$ exists a $f \in \mathcal{A}$ such that $f(x) = \alpha$ and $f(y) = \beta$:
 For $\alpha = \beta$ we choose $f = \alpha$ as constant function.
 For $\alpha \neq \beta$ there exists, since \mathcal{A} separates points of E , a $g \in \mathcal{A}$ with $g(x) \neq g(y)$. Set $f = c_0 + c_1 g$ and choose:

$$\begin{aligned} \alpha &= c_0 + c_1 g(x) \\ \beta &= c_0 + c_1 g(y) \\ \Rightarrow c_1 &= \frac{\alpha - \beta}{g(x) - g(y)} \\ \Rightarrow c_0 &= \alpha - \frac{\alpha - \beta}{g(x) - g(y)} g(x) = \frac{\alpha g(x) - \alpha g(y) - \alpha g(x) + \beta g(x)}{g(x) - g(y)} = \\ &= \frac{\beta g(x) - \alpha g(y)}{g(x) - g(y)} \end{aligned}$$

- v) For all $f \in C^0$, $x \in E$ and $\varepsilon \in \mathbb{R}_{>0}$ there is a $g \in \overline{\mathcal{A}}$ such that

$$g(x) = f(x)$$

and for all $y \in \overline{\mathcal{A}}$ holds:

$$g(y) \leq f(y) + \varepsilon$$

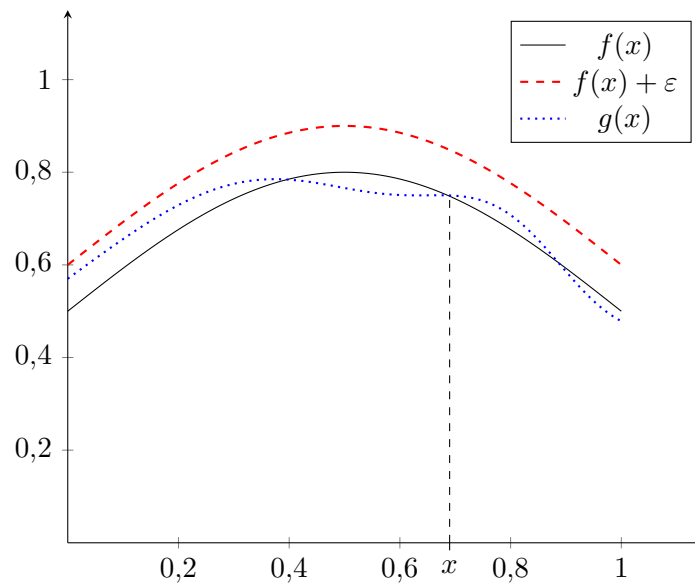
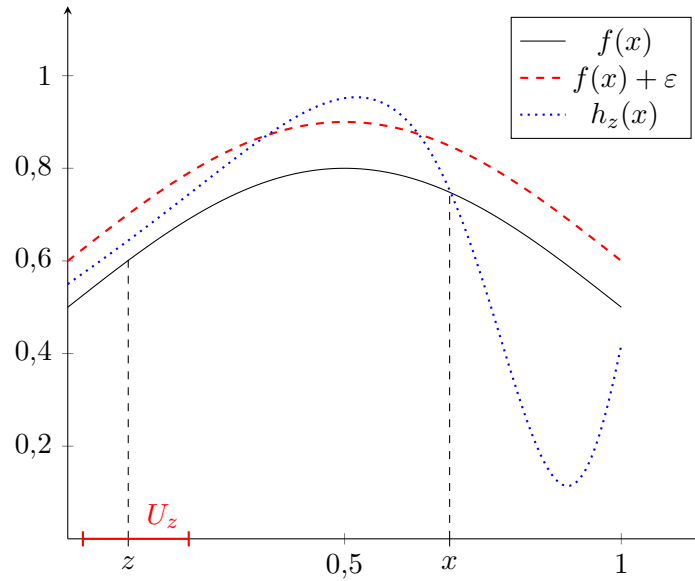


Figure 6.3: $g(x) \leq f(x) + \varepsilon$

To show this, choose for any $z \in E$ a $h_z \in \overline{\mathcal{A}}$ with $h_z(x) = f(x)$ and $h_z(z) \leq f(z) + \frac{\varepsilon}{2}$, which is possible after iv).

Since h_z is continuous, there is a neighborhood U_z of z such that $h_z \leq f + \varepsilon$ on U_z .

Figure 6.4: $h_z \leq f + \varepsilon$ on U_z

Since E is compact, we can cover it by a finite number of such neighborhoods U_{z_1}, \dots, U_{z_N} . Define:

$$g := \min \{h_{z_1}, \dots, h_{z_N}\} \in \overline{\mathcal{A}}$$

It holds $g(x) = f(x)$, because $h_{z_i}(x) = f(x)$. We also know:

$$g|_{U_j} \leq h_{z_j}|_{U_j} \leq f + \varepsilon$$

vi) $\overline{\mathcal{A}} = C^0$: Denote the function g constructed in step v) by g_x .

$$g_x(x) = f(x)$$

$$g_x \leq f + \varepsilon$$

By continuity of g_x there exists a neighborhood U_x of x such that $g_x \geq f - \varepsilon$ on U_x . By compactness we can cover E by a finite number of such neighborhoods U_{x_1}, \dots, U_{x_k} and define:

$$g := \max \{g_{x_1}, \dots, g_{x_k}\}$$

Then follows:

$$f - \varepsilon \leq g \leq f + \varepsilon$$

$$\|f - g\| < \varepsilon$$

□_{6.2.7}

Counterexample in the complex case:

$$E = [0,1] \times [0,1] \subseteq \mathbb{C}$$

Consider the set $\mathcal{A} = \mathcal{P}(z)$ of polynomials in z .

- The constant functions are in \mathcal{A} .
- \mathcal{A} separates points:
If $z_1 \neq z_2$ take $f(z) = z$ then $f(z_1) \neq f(z_2)$.

$$\overline{\mathcal{A}} = ?$$

By Morera's theorem we get:

$$\overline{\mathcal{A}} = \left\{ f \in C^0([0,1]^2) \mid |f|_{(0,1)^2} \text{ is holomorphic} \right\} \neq C^0([0,1]^2)$$

For example $f(x + iy) = x - iy$. We have $f \in C^0([0,1]^2)$, but $f \notin \overline{\mathcal{A}}$.

6.2.8 Theorem (Stone-Weierstraß, complex version)

Let $\mathcal{A} \subseteq C^0(E, \mathbb{C})$ be a subalgebra with the properties 1. and 2. from theorem 6.2.7 and additionally:

$$3. f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$$

Then \mathcal{A} is dense in $C^0(E, \mathbb{C})$.

Proof

Consider the algebras:

$$\begin{aligned} \operatorname{Re}(\mathcal{A}) &= \left\{ f + \bar{f} \mid f \in \mathcal{A} \right\} \subseteq \mathcal{A} \\ \operatorname{Im}(\mathcal{A}) &= \left\{ \frac{1}{i} (f - \bar{f}) \mid f \in \mathcal{A} \right\} \subseteq \mathcal{A} \end{aligned}$$

These are subalgebras of $C^0(E, \mathbb{R})$. By the real Stone-Weierstraß theorem we get:

$$\overline{\operatorname{Re}(\mathcal{A})} = \overline{\operatorname{Im}(\mathcal{A})} = C^0(E, \mathbb{R})$$

For given $f \in C^0(E, \mathbb{C})$ approximate $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$.

□_{6.2.8}

6.3 Arzelà-Ascoli theorem

Let K be a compact metric space and E a Banach space.

$C^0(K, E)$ is the Banach space of continuous functions $f : K \rightarrow E$ with norm:

$$\|f\| := \sup_{x \in K} \|f(x)\|_E$$

Let $\mathcal{F} \subseteq C^0(K, E)$ be a subset. Is \mathcal{F} compact?

6.3.1 Definition (relatively compact)

A subset A of a metric space is called *relatively compact*, if \overline{A} is compact.

6.3.2 Definition (equicontinuous)

A family $\mathcal{F} \subseteq C^0(K, E)$ is called *equicontinuous* (gleichgradig stetig) if for all $x \in K$ and all $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $y \in B_\delta(x)$ and for all $f \in \mathcal{F}$ holds:

$$\|f(x) - f(y)\| < \varepsilon$$

(Thus δ is independent of $f \in \mathcal{F}$.)

6.3.3 Theorem (Arzelà-Ascoli)

$\mathcal{F} \subseteq C^0(K, E)$ is relatively compact if and only if the following two conditions holds:

- i) \mathcal{F} is equicontinuous.
- ii) For every $x \in K$ the set

$$\mathcal{F}(x) := \{f(x) \mid f \in \mathcal{F}\}$$

is relatively compact in E .

Proof

„ \Rightarrow “: Assume that $\mathcal{F} \subseteq C^0(K, E)$ is relatively compact.

- i) Assume that \mathcal{F} is *not* equicontinuous. Then there exists an $\varepsilon \in \mathbb{R}_{>0}$ and sequences $x_n \in K$, $f_n \in \mathcal{F}$ and $y_n \in B_{\frac{1}{n}}(x_n)$ such that:

$$\|f_n(x_n) - f_n(y_n)\| \geq \varepsilon$$

After choosing subsequences (with the same notation), we can arrange:

$$\begin{array}{lll} x_n \rightarrow x & y_n \rightarrow x & \text{(use that } K \text{ is compact)} \\ f_n \rightarrow f & & \text{(use that } \mathcal{F} \text{ is relatively compact)} \end{array}$$

This means that there is a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{>N}$ holds for all $y \in K$:

$$\|f_n(y) - f(y)\| < \frac{\varepsilon}{3}$$

(Since convergence in $C^0(K, E)$ is the same as uniform convergence $f_n \rightrightarrows f$.)
Since f is continuous there exists a $\delta \in \mathbb{R}_{>0}$ such that for all $y \in B_\delta(x)$:

$$\|f(x) - f(y)\| < \frac{\varepsilon}{3}$$

With this we get:

$$\|f_n(x) - f_n(y)\| \leq \underbrace{\|f_n(x) - f(x)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|f(x) - f(y)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|f(y) - f_n(y)\|}_{< \frac{\varepsilon}{3}} < \varepsilon$$

This is a contradiction to $\|f_n(x_n) - f_n(y_n)\| \geq \varepsilon$.

□_i)

- ii) Consider $y_n \in \mathcal{F}(x) \subseteq E$ (to show that y_n has a convergent subsequence in E).
Then there are functions $f_n \in \mathcal{F}$ with $f_n(x) = y_n$. Since \mathcal{F} is relatively compact, a subsequence is a Cauchy sequence in $C^0(K, E)$, i.e. $\|f_{n_l} - f_{n_{l'}}\| \xrightarrow{l, l' \rightarrow \infty} 0$.

$$\|f_{n_l} - f_{n_{l'}}\| = \sup_{z \in K} \|f_{n_l}(z) - f_{n_{l'}}(z)\|_E \geq \|f_{n_l}(x) - f_{n_{l'}}(x)\|_E = \|y_{n_l} - y_{n_{l'}}\|$$

Therefore we get+:

$$\|y_{n_l} - y_{n_{l'}}\| \xrightarrow{l, l' \rightarrow \infty} 0$$

Thus (y_{n_l}) is a Cauchy sequence in E . □_{ii)}

„ \Leftarrow “: Let (f_l) be a sequence in \mathcal{F} and show that a subsequence (g_l) converges in $C^0(K, E)$:
Since K is compact, there is a countable dense subset $\{x_1, x_2, \dots\} \subseteq K$. Since $\mathcal{F}(x_1)$ is relatively compact, there is a subsequence $f_l^{(1)} \in \mathcal{F}$ of (f_l) such that $f_l^{(1)}(x_1)$ converges in E . Since $\mathcal{F}(x_2)$ is relatively compact, there is a subsequence $f_l^{(2)}$ of $f_l^{(1)}$ such that $f_l^{(2)}(x_2)$ converges. Inductively choose a subsequence $(f_l^{(n+1)})$ of $(f_l^{(n)})$ such that $f_l^{(n+1)}(x_{n+1})$ converges in E . Take the diagonal sequence $g_l := f_l^{(l)}$. This is for $l \geq n$ a subsequence of $f_l^{(n)}$, so for all $n \in \mathbb{N}$ converges $g_l(x_n) \xrightarrow{l \rightarrow \infty} y_n$.

Claim: g_n is a Cauchy sequence in $C^0(K, E)$, i.e. for all $\varepsilon \in \mathbb{R}_{>0}$ exists a $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}_{>N}$ and all $x \in K$ holds:

$$\|g_n(x) - g_m(x)\| \leq \varepsilon$$

Proof: Since \mathcal{F} is equicontinuous, for all $x \in E$ exists a $\delta \in \mathbb{R}_{>0}$ such that for all $z, z' \in B_{\delta(x)}(x)$ and all $f \in \mathcal{F}$ holds:

$$\|f(z) - f(z')\| < \frac{\varepsilon}{3}$$

We cover K by a finite number of such balls B_1, \dots, B_L . In every Ball B_l there is at least one point of $\{x_1, x_2, \dots\}$. We choose such a point $\xi_l \in B_l$. Since $(g_n(\xi_l))$ converges for every $l \in \{1, \dots, L\}$ we can choose a $N \in \mathbb{N}$ such that for all $l \in \{1, \dots, L\}$ and all $m, n \in \mathbb{N}_{>N}$ holds:

$$\|g_n(\xi_l) - g_m(\xi_l)\| < \frac{\varepsilon}{3}$$

For every $x \in K$ exists a $l \in \{1, \dots, L\}$ with $x \in B_l$.

$$\|g_n(x) - g_m(x)\| \leq \underbrace{\|g_n(x) - g_n(\xi_l)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|g_n(\xi_l) - g_m(\xi_l)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|g_m(\xi_l) - g_m(x)\|}_{< \frac{\varepsilon}{3}}$$

□_{Claim}

Therefore the subsequence (g_l) for (f_l) converges in $C^0(K, E)$, since $C^0(K, E)$ is complete, because E is a Banach space. □_{6.3.3}

Application to integral operators

Let $K \subseteq \mathbb{R}^n$ be compact. Consider an integral operator $A : C^0(K, \mathbb{R}) \rightarrow C^0(K, \mathbb{R})$, i.e.:

$$(Af)(x) = \int_K A(x, y) f(y) d^n y$$

$\mathcal{F} := A(C^0(K, \mathbb{R}))$ is equicontinuous provided that $A(., y)$ is continuous.

6.4 The Riesz representation theorem

Let K again be a compact metric space. $E = C^0(K, \mathbb{R})$ with the sup-norm is a Banach space.

Question: What is E^* ?

Consider $l \in E^*$, i.e.

$$l : E \rightarrow \mathbb{R}$$

and for all $f \in C^0(K)$ holds:

$$|l(f)| \leq C \|f\|$$

This means f is bounded or equivalently continuous.

6.4.1 Examples

Consider $K = [0, 1] \subseteq \mathbb{R}$. For any $\varphi \in L^1([0, 1])$, the functional

$$l(f) := \int_0^1 \varphi(x) f(x) dx$$

is linear and bounded:

$$|l(f)| \leq \int_0^1 |\varphi(x)| \cdot |f(x)| dx \leq \underbrace{\sup_{x \in [0, 1]} |f|}_{=\|f\|} \cdot \underbrace{\int_0^1 |\varphi(x)| dx}_{=\|\varphi\|_{L^1}}$$

It is convenient to identify $l \in E^*$ with the function $\varphi \in L^1$. We have represented l by an L^1 -function φ .

This can also be written as a *signed measure* (signiertes Maß):

$$d\mu := \varphi(x) dx$$

But not every $l \in E^*$ can be represented in this form.

Example

$$l(f) := f\left(\frac{1}{2}\right)$$

is bounded:

$$|l(f)| = \left| f\left(\frac{1}{2}\right) \right| \leq \sup_{[0, 1]} |f| = \|f\|$$

It can be represented by the Dirac measure:

$$l(f) = \int_0^1 f(x) \delta\left(x - \frac{1}{2}\right) dx = \int_0^1 f(x) d\mu$$

Here $\delta(x)$ is the δ -Distribution. $\mu = \delta_{\frac{1}{2}}$ is the Dirac measure.

$$\delta_{x_0}(\Omega) = \begin{cases} 1 & \text{if } x_0 \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

6.4.2 Definition (bounded, positive, regular measure)

Let $X \neq \emptyset$ be a set. A σ -algebra \mathcal{M} over X is a set of subsets of X such that holds:

- i) $\emptyset \in \mathcal{M}$
- ii) $A \in \mathcal{M} \Rightarrow \mathcal{C}A := X \setminus A \in \mathcal{M}$
- iii) For a countable family $(A_j)_{j \in \mathbb{N}}$ holds:

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

The elements of \mathcal{M} are called *measurable sets* (messbare Mengen).

Let K be a compact metric space. Denote by \mathfrak{M} the *Borel algebra*, i.e. the smallest σ -algebra over K , which contains all open and therefore all closed subsets of K .

A *bounded (signed) measure* is a mapping

$$\mu : \mathfrak{M} \rightarrow \mathbb{R}$$

(not $\mu : \mathfrak{M} \rightarrow \mathbb{R}^+ \cup \{0\} \cup \{\infty\}$ as before in measure theory) with the following properties:

- The empty set measures zero:

$$\mu(\emptyset) = 0$$

- σ -additivity: For $M_j \in \mathfrak{M}$ with $M_i \cap M_j = \emptyset$ for all $i \neq j$ holds:

$$\mu \left(\bigcup_{j=1}^{\infty} M_j \right) = \sum_{j=1}^{\infty} \mu(M_j)$$

μ is *positive*, if $\mu(M) \geq 0$ for all $M \in \mathfrak{M}$.

μ is *regular*, if for all $A \in \mathfrak{M}$ holds:

$$\mu(A) = \sup_{\substack{B \subseteq A \\ B \text{ compact}}} \mu(B) = \inf_{\substack{\Omega \supseteq A \\ \Omega \text{ open}}} \mu(\Omega)$$

Example

The Lebesgue measure $d^n x$ restricted to the Borel algebra on $[0,1]^n$ is a bounded, positive and regular measure.

6.4.3 Theorem (Riesz representation theorem)

Consider $l \in C^0(K, \mathbb{R})^*$. Then there is a unique bounded regular Borel measure μ (i.e. a measure on the Borel algebra \mathfrak{M}) such that for all $f \in C^0(K, \mathbb{R})$ holds:

$$l(f) = \int_K f d\mu$$

Here we only prove the case $K = [0,1]$. (We also need it for $K = [0,1]^2$.)

How can one construct positive regular Borel measures on $[0,1]$?

Lebesgue-Stieltjes integral

Let $\alpha : [0,1] \rightarrow \mathbb{R}$ be monotonically increasing (not necessarily continuous).

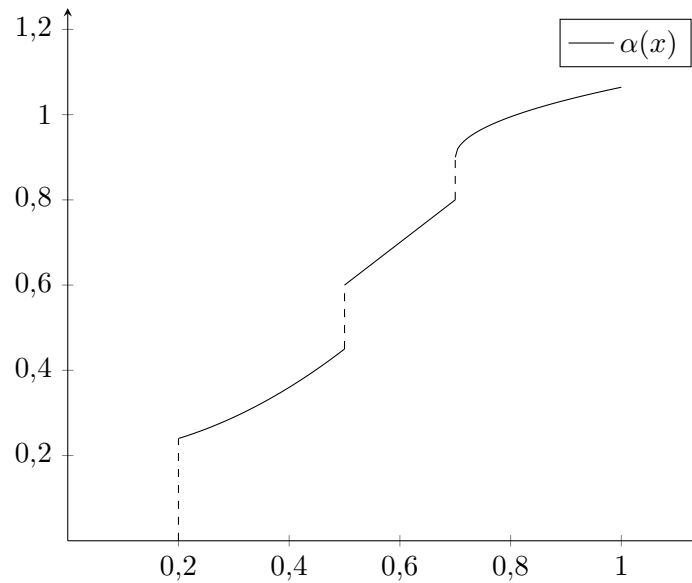


Figure 6.5: α is monotonically increasing, but not continuous

The two one-sided limits

$$\lim_{x \nearrow x_0} \alpha(x), \quad \lim_{x \searrow x_0} \alpha(x)$$

exist. In general:

$$\lim_{x \nearrow x_0} \alpha(x) \leq \alpha(x_0) \leq \lim_{x \searrow x_0} \alpha(x)$$

But equality does not need to hold. Define:

$$\mu((a,b)) := \lim_{x \nearrow b} \alpha(x) - \lim_{x \searrow a} \alpha(x)$$

By σ -additivity, this measure can be extended to a positive regular bounded Borel measure. (This can be proven exactly as for the Lebesgue integral.) The corresponding integral

$$\int_0^1 f d\mu$$

is called Lebesgue-Stieltjes integral. If $\alpha(x) = x + c$, the Lebesgue-Stieltjes integral reduces to the Lebesgue integral

6.4.4 Example

Let $\alpha \in C^1([0,1])$ be monotonically increasing. Then holds:

$$\mu((a,b)) = \alpha(b) - \alpha(a) = \int_a^b \alpha'(x) dx = \int_0^1 \chi_{(a,b)} \alpha'(x) dx$$

The corresponding Lebesgue-Stieltjes integral is:

$$\int f d\mu = \int_0^1 f(x) \cdot \alpha'(x) dx$$

The following short notation is used in general:

$$\begin{aligned} d\mu &= \alpha'(x) dx \\ d\mu &= d\alpha \end{aligned}$$

If $\alpha \in C^1([0,1])$ is not monotone, we can still set:

$$\int_0^1 f d\mu := \int_0^1 f \cdot \alpha'(x) dx$$

$d\mu$ is a signed measure.

In order to extend the Lebesgue-Stieltjes construction to functions α , which are *not* monotone (such as to obtain signed measures), we need to assume, that α has bounded variation.

6.4.5 Definition (total variation)

Let $\alpha : [0,1] \rightarrow \mathbb{R}$ be a function (not necessarily continuous).

The *total variation* (Totalvariation) is defined by:

$$(\text{TV}(\alpha))(x) := \sup_{\substack{N \in \mathbb{N} \\ 0=x_0 < \dots < x_N=x}} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

α is of *bounded variation* (beschränkte Totalvariation), $\alpha \in \mathcal{BV}([0,1])$, if $(\text{TV}(f))(1) < \infty$.

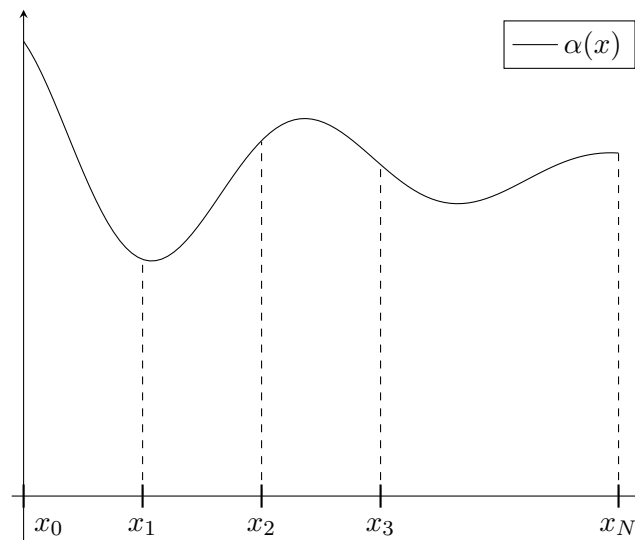


Figure 6.6: total variation of α

Note: If α is monotonically increasing, then holds:

$$(\text{TV}(\alpha))(x) = \alpha(x) - \alpha(0) < \infty$$

Thus every monotonically function has bounded variation.

But there are even continuous functions, which have unbounded variation, e.g. for large enough $p \in \mathbb{R}_{>0}$:

$$\alpha(x) = x \sin\left(\frac{1}{x^p}\right)$$

For $\alpha \in C^1([0,1])$ holds:

$$\text{TV}(\alpha)(x) = \int_0^x |\alpha'(\tau)| d\tau$$

Lemma (Properties of the total variation)

$\text{TV}(\alpha)(x)$ is monotonically increasing and:

$$\text{TV}(\alpha)(0) = 0$$

$\text{TV}(\alpha)(x) \pm \alpha(x)$ is also monotonically increasing.

Proof

Assume that $y \in \mathbb{R}_{>x}$.

$$\begin{aligned} \text{TV}(\alpha)(y) &= \sup_{\substack{N \in \mathbb{N} \\ 0=x_0 < \dots < x_N=y}} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| \geq \sup_{\substack{N \in \mathbb{N}_{\geq 2} \\ 0=x_0 < \dots < x_{N-1}=x < x_N=y}} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| \geq \\ &\geq \sup_{\substack{N \in \mathbb{N}_{\geq 2} \\ 0=x_0 < \dots < x_{N-1}=x < x_N=y}} \sum_{i=1}^{N-1} |\alpha(x_i) - \alpha(x_{i-1})| = \text{TV}(\alpha)(x) \end{aligned}$$

$$\text{TV}(\alpha)(x) \pm \alpha(x) = \pm \alpha(0) + \sup_{\substack{N \in \mathbb{N} \\ 0=x_0 < \dots < x_N=x}} \sum_{i=1}^N \underbrace{|\alpha(x_i) - \alpha(x_{i-1})| \pm (\alpha(x_i) - \alpha(x_{i-1}))}_{\geq 0}$$

Just as before this implies that

$$\text{TV}(\alpha)(x) \pm \alpha(x)$$

is monotonically increasing. □_{6.4.5}

Suppose that $f \in \mathcal{BV}([0,1])$. Then the functions

$$\begin{aligned} f_+ &= \frac{1}{2} (\text{TV}(f) + f) \\ f_- &= \frac{1}{2} (\text{TV}(f) - f) \end{aligned}$$

are monotonically increasing and:

$$f = f_+ - f_-$$

Let $d\mu_{\pm} = df_{\pm}$ be the bounded positive regular Borel measures of the corresponding Lebesgue-Stieltjes integrals. Then

$$\mu := \mu_+ - \mu_-$$

defines a bounded regular Borel measure with the property:

$$\begin{aligned} \mu((a,b)) &= \mu_+((a,b)) - \mu_-((a,b)) = \lim_{x \nearrow b} f_+(x) - \lim_{x \searrow a} f_+(x) - \lim_{x \nearrow b} f_-(x) + \lim_{x \searrow a} f_-(x) = \\ &= \lim_{x \nearrow b} f(x) - \lim_{x \searrow a} f(x) \end{aligned}$$

6.4.6 Example

Consider the Heaviside function:

$$f := \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

$d\mu := df$ has the form $\mu = \delta_{\frac{1}{2}}$.

Proof of Theorem 6.4.3 in the case $K = [0,1]$

$\mathcal{PC}([0,1])$ are the piecewise continuous functions, i.e. for all $f \in \mathcal{PC}([0,1])$ exists a $N \in \mathbb{N}$ and points $0 = x_0 < \dots < x_N = 1$ such that $f|_{(x_{i-1}, x_i)}$ is continuous and has a continuous continuation to $[x_{i-1}, x_i]$ for all $i \in \{1, \dots, N\}$.

On \mathcal{PC} we introduce the norm:

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

This makes $\mathcal{PC}([0,1])$ a Banach space.

$$C^0([0,1]) \subseteq \mathcal{PC}([0,1])$$

is a subspace, which is closed, since it is complete.

Consider $l \in C^0([0,1])^*$, i.e.

$$l : C^0([0,1]) \rightarrow \mathbb{R}$$

with:

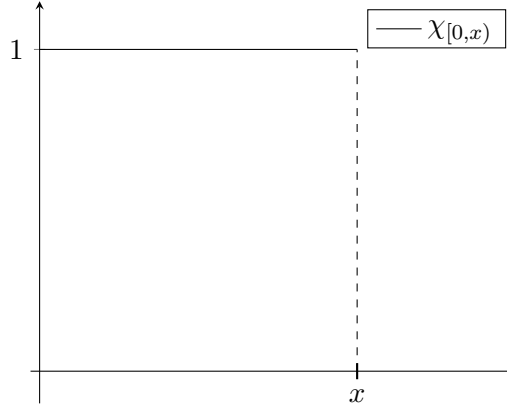
$$|l(f)| \leq C \|f\|_{C^0}$$

According to the Hahn-Banach theorem, there is an extension

$$\tilde{l} : \mathcal{PC}([0,1]) \rightarrow \mathbb{R}$$

with $\tilde{l}|_{C^0} = l$ and $|l(f)| \leq C \|f\|_{\mathcal{PC}([0,1])}$. Define $\alpha : [0,1] \rightarrow \mathbb{R}$ by:

$$\alpha(x) := \begin{cases} \tilde{l}(\chi_{[0,x)}) & \text{if } x < 1 \\ \tilde{l}(\chi_{[0,1]}) & \text{if } x = 1 \end{cases}$$

Figure 6.7: $\chi_{[0,x)}$

$l(\chi_{[0,x)})$ is ill-defined, because $\chi_{[0,x)}$ is *not* continuous.

$\tilde{l}(\chi_{[0,x)})$ is well-defined, because $\chi_{[0,x)}$ is piecewise-continuous.

– α has bounded variation: Consider:

$$0 = x_0 < \dots < x_N = 1$$

We need to show:

$$\sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| < C$$

C has to be independent of N and the (x_i) .

Define $s_i \in \{\pm 1\}$ by:

$$s_i := \begin{cases} +1 & \text{if } \alpha(x_i) - \alpha(x_{i-1}) \geq 0 \\ -1 & \text{if } \alpha(x_i) - \alpha(x_{i-1}) < 0 \end{cases}$$

Then holds:

$$\sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^N s_i (\alpha(x_i) - \alpha(x_{i-1})) = \tilde{l} \left(\sum_{i=1}^{N-1} s_i \chi_{[x_{i-1}, x_i)} + s_N \chi_{[x_{N-1}, 1]} \right)$$

Since \tilde{l} is bounded by construction, we know:

$$\begin{aligned} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| &\leq \left| \tilde{l} \left(\sum_{i=1}^{N-1} s_i \chi_{[x_{i-1}, x_i)} + s_N \chi_{[x_{N-1}, 1]} \right) \right| \leq \\ &\leq C \left\| \sum_{i=1}^{N-1} s_i \chi_{[x_{i-1}, x_i)} + s_N \chi_{[x_{N-1}, 1]} \right\| = C \end{aligned}$$

Therefore we have $\alpha \in \mathcal{BV}([0,1])$.

– Consider $d\mu := d\alpha_+ - d\alpha_-$ for the corresponding bounded regular Borel measure, where $\alpha = \alpha_+ - \alpha_-$ and α_{\pm} are monotonically increasing.

Claim: For all $f \in C^0([0,1])$ holds:

$$l(f) = \int_0^1 f d\mu$$

Proof: Consider $f \in C^0([0,1])$. Set:

$$f_n(x) := \begin{cases} \sum_{i=1}^n f\left(\frac{i}{n}\right) \cdot \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right)} & \text{if } x < 1 \\ f(1) & \text{if } x = 1 \end{cases}$$

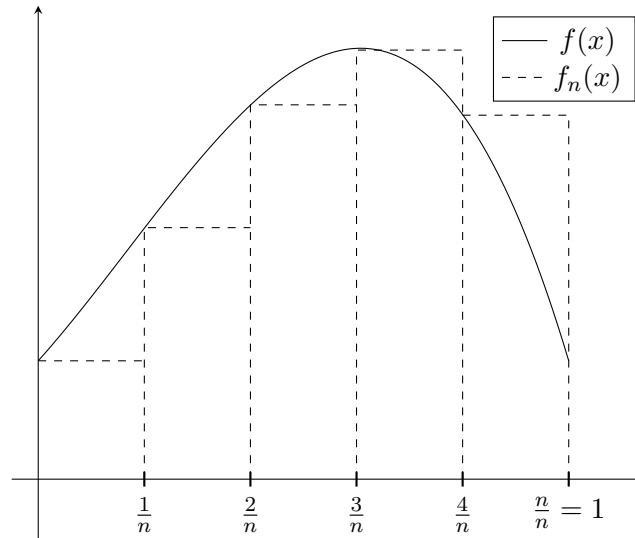


Figure 6.8: Approximation of f by $f\left(\frac{i}{n}\right)$ for $n = 5$

Since f_n is uniformly continuous, i.e. $f_n \rightrightarrows f$, we get:

$$\begin{aligned} l(f) &= \tilde{l}(f) = \tilde{l}\left(\lim_{n \rightarrow \infty} f_n\right) \stackrel{\tilde{l} \text{ continuous}}{=} \lim_{n \rightarrow \infty} \tilde{l}(f_n) = \\ &\stackrel{\text{by construction}}{=} \lim_{n \rightarrow \infty} \int_0^1 f_n d\mu \stackrel{(*)}{=} \int_0^1 \lim_{n \rightarrow \infty} f_n d\mu = \int_0^1 f d\mu \end{aligned}$$

For $(*)$ consider:

$$\left| \int_0^1 (f_n - f) d\mu \right| \leq \underbrace{\sup |f - f_n|}_{\rightarrow 0} \cdot \underbrace{\text{TV}(\alpha)(1)}_{< \infty} \xrightarrow{n \rightarrow \infty} 0$$

□ Claim

□ 6.4.3

Remarks

- Our proof only works in the case $K = [a, b] \subseteq \mathbb{R}$. (see Reed, Simon: Appendix “The Riesz-Markov Theorem”)

- In general dimension the idea would be:

$$\mu(\Omega) := \tilde{l}(\chi_\Omega)$$

But how to extend l ? So choose $f_n \rightarrow \chi_\Omega$ and define:

$$\mu(\Omega) := \lim_{n \rightarrow \infty} l(f_n)$$

(see Rudin: *Real and complex analysis*)

- Total variation of a bounded Borel measure:

$$|\mu|(\Omega) := \sup_{\substack{N \in \mathbb{N} \\ \Omega_1, \dots, \Omega_N \\ \text{with } \Omega_1 \dot{\cup} \dots \dot{\cup} \Omega_N = \Omega}} \sum_{i=1}^N |\mu(\Omega_i)|$$

$|\mu|$ is a positive bounded Borel measure. (see Rudin)

Then we can write:

$$\left| \int_K (f - f_n) d\mu \right| \leq \int_K |f - f_n| \cdot d|\mu| \leq \sup_K |f - f_n| \cdot |\mu|(K)$$

7 The Spectral Theorem for symmetric bounded operators

Let $A \in L(H)$ be symmetric and H be a separable Hilbert space. Let $p(A)$ be a polynomial in A , for example the characteristic polynomial for $A \in L(\mathbb{C}^N)$ with $p(A) = 0$. Extend this idea to functions $f(A)$ with $f \in C^0(\sigma(A))$. (Stone-Weierstraß) Then for

$$\langle u, f(A)u \rangle =: l(f)$$

holds $l \in C^0(\sigma(A))^*$. Using the Riesz representation theorem we can write:

$$\langle u, f(A)u \rangle = \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda)$$

$$d\mu_u(\lambda) = \langle u, dE_\lambda u \rangle$$

dE_λ is the so-called *spectral measure*. Then holds the spectral theorem:

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

7.1 The Spectrum of symmetric bounded operators

Let $A \in L(H)$ be symmetric, i.e. $\langle u, Av \rangle = \langle Au, v \rangle$ for all $u, v \in H$. The resolvent set is:

$$\begin{aligned} \varrho(A) &= \{ \lambda \in \mathbb{C} \mid (\lambda - A) \text{ has a continuous inverse} \} \\ \sigma(A) &= \mathbb{C} \setminus \varrho(A) \end{aligned}$$

$\varrho(A) \subseteq \mathbb{C}$ is open and so the spectrum $\sigma(A) \subseteq \mathbb{C}$ is closed. The spectral radius is:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \|A\|$$

Warning

Consider $\lambda \in \sigma(A)$, i.e. $\lambda - A$ has no continuous inverse. This does not mean $\ker(\lambda - A)$ is non-trivial. Thus λ does *not* need to be an eigenvalue!

7.1.1 Theorem

Let $A \in L(H)$ be self-adjoint. Then $\sigma(A) \subseteq \mathbb{R}$.

Proof

Consider $\lambda = \alpha + \mathbf{i}\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. We need to show that $\lambda - A$ has a continuous inverse. Introduce the following bilinear form:

$$B(x, y) = \langle x, (A - \bar{\lambda}) y \rangle = \langle (A - \lambda) x, y \rangle$$

This bilinear form satisfies the assumptions of the Lax-Milgram theorem:

- i) The sesquilinearity is clear, since the scalar product is sesquilinear.
- ii) B is bounded:

$$|\langle x, (A - \bar{\lambda}) y \rangle| \leq \|x\| \cdot \underbrace{\|A - \bar{\lambda}\|}_{\leq \|A\| + |\lambda|} \cdot \|y\| \leq C \|x\| \|y\|$$

- iii) B is bounded from below, i.e. there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that for all $x \in H$ holds:

$$|B(x, x)| \geq \varepsilon \|x\|^2$$

We know:

$$B(x, x) = \langle x, (A - \bar{\lambda}) x \rangle = \underbrace{\langle x, Ax \rangle}_{\text{real}} - \underbrace{\operatorname{Re}(\lambda \langle x, x \rangle)}_{\text{real}} - \underbrace{\mathbf{i} \operatorname{Im}(\lambda \langle x, x \rangle)}_{\text{imaginary}}$$

$$|B(x, x)| \geq |\operatorname{Im}(\lambda \langle x, x \rangle)| = |\beta| \cdot \|x\|^2$$

Set $\varepsilon := |\beta| \neq 0$.

The Lax-Milgram theorem yields that the linear functional $l(x) = \langle z, x \rangle$ can be represented as

$$l(x) = B(y, x)$$

with a unique $y = y(z) \in H$. Thus we get for all $x \in H$:

$$\begin{aligned} \langle z, x \rangle &= \langle (A - \lambda) y, x \rangle \\ \Rightarrow z &= (A - \lambda) y \end{aligned}$$

Therefore, for all $z \in H$ exists a unique $y \in H$ such that $(A - \lambda) y = z$. Thus $A - \lambda$ is invertible. The inverse $(A - \lambda)^{-1}$ is continuous due to the open mapping theorem (see Corollary 2.4.8). $\square_{7.1.1}$

7.1.2 Theorem

It holds $\sigma(A) \subseteq [a, b]$ and $a, b \in \sigma(A)$ with:

$$a := \inf_{\|u\|=1} \langle u, Au \rangle$$

$$b := \sup_{\|u\|=1} \langle u, Au \rangle$$

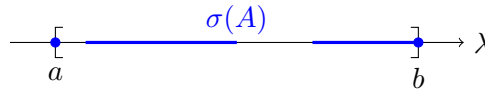


Figure 7.1: $\sigma(A) \subseteq [a, b]$ and $a, b \in \sigma(A)$

Proof

For $\lambda \in \mathbb{R}_{<a}$ holds:

$$\langle x, (A - \lambda)x \rangle = \langle x, Ax \rangle - \lambda \|x\|^2 \geq a \|x\|^2 - \lambda \|x\|^2 = \underbrace{(a - \lambda)}_{>0} \|x\|^2$$

Thus

$$\langle \cdot, \cdot \rangle_A := \langle \cdot, (A - \lambda) \cdot \rangle$$

is a scalar product on H . The corresponding norm

$$\|u\|_A := \sqrt{\langle u, u \rangle_A}$$

is equivalent to the norm $\|\cdot\|$, because it holds:

$$(a - \lambda) \|u\|^2 \leq \|u\|_A^2 = \langle u, (A - \lambda)u \rangle \leq (\|A\| - \lambda) \|u\|^2$$

For $u \in H$ and $l(w) := \langle u, w \rangle$ is $l \in H^*$. According to the Fréchet-Riesz theorem 3.1.3 (for the scalar product $\langle \cdot, \cdot \rangle_A$) there is a unique vector $v \in H$, such that for all $w \in H$ holds:

$$l(w) = \langle v, w \rangle_A$$

Thus we get for all $w \in H$:

$$\langle u, w \rangle = l(w) = \langle v, w \rangle_A = \langle v, (A - \lambda)w \rangle \stackrel{A-\lambda \text{ symmetric}}{=} \langle (A - \lambda)v, w \rangle$$

$$\Rightarrow u = (A - \lambda)v$$

Thus there exists a

$$\begin{aligned} \varphi : H &\rightarrow H \\ u &\mapsto v \end{aligned}$$

such that $u = (A - \lambda) \varphi(u)$, i.e. $A - \lambda \in L(H)$ is surjective. φ is linear and bounded according to the open mapping theorem 2.4.8. Thus we have

$$\varphi = (A - \lambda)^{-1} \in L(H)$$

and therefore $\lambda \in \varrho(A)$.

Applying the same argument to the operator $(-A)$, one sees that $(b, \infty) \subseteq \varrho(A)$.

Therefore holds $\sigma(A) \subseteq [a, b]$.

Only prove that $b \in \sigma(A)$. For $a \in \sigma(A)$ consider similarly the operator $-A$. Furthermore replace $A \rightarrow A - a$ to get $\sigma(A) \subseteq [0, b]$. We know:

$$\|A\| = r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \sup_{\lambda \in \sigma(A)} \lambda = \sup \sigma(A)$$

As a consequence we get $\|A\| \leq b$. On the other hand we have:

$$b = \sup_{\|u\|=1} \langle u, Au \rangle \leq \sup_{\|u\|=1} \|Au\| \cdot \underbrace{\|u\|}_{=1} = \|A\|$$

Thus we have $b = \|A\| = r(A)$, especially b is a limit point of the spectrum of A . Since $\sigma(A)$ is closed, it follows that $b \in \sigma(A)$. $\square_{7.1.2}$

7.2 The continuous functional calculus

7.2.1 Theorem (continuous functions of operators)

Let $A \in L(H)$ be symmetric. Then there is a unique mapping $\Phi : C^0(\sigma(A), \mathbb{C}) \rightarrow L(H)$ (remember $\sigma(A) \subseteq [a, b]$) with the following properties:

i) Φ is an involutive algebra homomorphism, i.e.:

- Φ is linear.
- $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$
- $\Phi(\overline{f}) = (\Phi(f))^*$ (involution)

ii) Φ is continuous:

$$\|\Phi(f)\|_{L(H)} \leq C \|f\|_{\infty}$$

iii) If $f(t) = t$, then $\Phi(f) = A$.

iv) If $Au = \lambda u$, i.e. $u \in H$ is an eigenvector of A , then $\Phi(f)u = f(\lambda)u$.

v) If $f \geq 0$, then $\Phi(f) \geq 0$, meaning that $\Phi(f)$ is a positive semi-definite operator, i.e. $\langle u, \Phi(f)u \rangle \geq 0$ for all $u \in H$.

vi) $\sigma(\Phi(f)) = f(\sigma(A))$ (spectral mapping theorem (spektraler Abbildungssatz))

vii) $\|\Phi(f)\|_{L(H)} = \|f\|_{\infty}$

Often we just write $\Phi(f) = f(A)$.

What if $f(t) = p(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$ is a polynomial?

$$\Phi(t) \stackrel{\text{iii)}}{=} A$$

From i) follows:

$$\Phi(1) = \Phi(1 \cdot 1) = \Phi(1) \cdot \Phi(1)$$

Therefore we get:

$$\Phi(1) = \mathbb{1}$$

Now follows:

$$\begin{aligned}\Phi(t^2) &= \Phi(t \cdot t) = \Phi(t) \cdot \Phi(t) = A \cdot A = A^2 \\ \Phi(t^l) &= A^l \\ \Phi(p) &= p(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_0 \mathbb{1}\end{aligned}$$

7.2.2 Lemma (spectral mapping theorem for polynomials)

For a complex polynomial $p \in \mathbb{P}_{\mathbb{C}}$ holds:

$$\sigma(p(A)) = p(\sigma(A))$$

Proof

- If $p = c \in \mathbb{C}$ is constant, then the lemma is trivial:

$$p(\sigma(A)) = c = \sigma(c\mathbb{1}) = \sigma(p(A))$$

So further on let p be not constant.

- $p(\sigma(A)) \subseteq \sigma(p(A))$: For $\lambda \in \sigma(A)$ and $z \in \mathbb{C}$ yields the fundamental theorem of algebra:

$$p(z) - p(\lambda) = (z - \lambda)q(z)$$

Here $q(z)$ is a new polynomial with $\deg(q) = \deg(p) - 1$. This also holds if we set $z = A$:

$$p(A) - p(\lambda) = (A - \lambda)q(A)$$

Assume $p(\lambda) \in \varrho(p(A))$, i.e. $p(A) - p(\lambda)$ has a bounded inverse. Then holds:

$$\begin{aligned}\mathbb{1} &= (p(A) - p(\lambda)) \cdot (p(A) - p(\lambda))^{-1} = (A - \lambda) \cdot q(A) \cdot (p(A) - p(\lambda))^{-1} \\ \Rightarrow (A - \lambda)^{-1} &= \underbrace{q(A)}_{\in L(H)} \cdot \underbrace{(p(A) - p(\lambda))^{-1}}_{\in L(H)} \in L(H)\end{aligned}$$

This gives $\lambda \in \varrho(A)$ in contradiction to $\lambda \in \sigma(A)$ and so $p(\lambda) \in \sigma(p(A))$.

- $\sigma(p(A)) \subseteq p(\sigma(A))$: Consider $\mu \in \sigma(p(A))$ and set $n := \deg(p)$. Using the fundamental theorem of algebra we get:

$$\begin{aligned} q(z) &:= p(z) - \mu = a(z - \lambda_1) \cdot \dots \cdot (z - \lambda_n) \\ q(A) &:= p(A) - \mu = a(A - \lambda_1) \cdot \dots \cdot (A - \lambda_n) \end{aligned}$$

If all the operators $A - \lambda_i$ had a continuous inverse, then this would hold also for their product in contradiction to the assumption $\mu \in \sigma(p(A))$. Thus one of the λ_i is in the spectrum of A . Because one of the linear factors vanishes, follows:

$$\begin{aligned} 0 &= q(\lambda_i) = p(\lambda_i) - \mu \\ \Rightarrow \mu &= p(\lambda_i) \in p(\sigma(A)) \end{aligned}$$

□_{7.2.2}

Let $p \in \mathbb{P}_{\mathbb{C}}$ be a complex polynomial.

$$(p(A))^* = \bar{p}(A)$$

Thus $p(A)$ is not symmetric.

7.2.3 Definition (normal operator)

$A \in L(H)$ is called *normal*, if $[A, A^*] = 0$.

7.2.4 Theorem

For a normal $A \in L(H)$ holds $r(A) = \|A\|$.

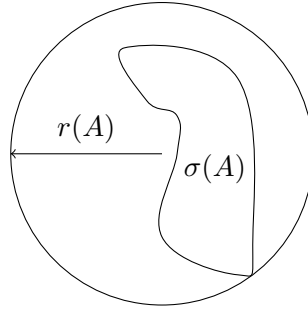


Figure 7.2: $r(A) = \|A\|$

Proof

We already proved for a general $A \in L(H)$:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} \quad (7.1)$$

For symmetric operators, we know furthermore:

$$r(A) = \|A\| = \sup_{\|u\|=1} |\langle u, Au \rangle| \quad (7.2)$$

For *normal* operators, we conclude the following: A^*A is symmetric and thus:

$$\begin{aligned} \|A\|^2 &= \sup_{\|u\|=1} \|Au\|^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \langle u, A^*Au \rangle \stackrel{(7.2)}{=} \|A^*A\| = \\ &\stackrel{(7.2)}{=} r(A^*A) \stackrel{(7.1)}{=} \lim_{n \rightarrow \infty} \|(A^*A)^n\|^{\frac{1}{n}} \end{aligned}$$

$$(A^*A)^n = \underbrace{A^*A \cdot A^*A \cdot \dots \cdot A^*A}_{n\text{-times}} \stackrel{A \text{ normal}}{=} (A^*)^n \cdot A^n$$

With

$$\|A\|^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \langle u, A^*Au \rangle \stackrel{A \text{ normal}}{=} \sup_{\|u\|=1} \langle u, AA^*u \rangle = \sup_{\|u\|=1} \langle A^*u, A^*u \rangle = \|A^*\|^2$$

we get:

$$\|(A^*A)^n\| \leq \|(A^*)^n\| \cdot \|A^n\| = \|A^n\|^2$$

It follows:

$$\|A\|^2 = \lim_{n \rightarrow \infty} \|(A^*A)^n\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \left(\|A^n\|^2 \right)^{\frac{1}{n}} \leq \|A\|^2$$

This gives:

$$\begin{aligned} \|A\|^2 &= \lim_{n \rightarrow \infty} \left(\|A^n\|^{\frac{1}{n}} \right)^2 = \left(\lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} \right)^2 = (r(A))^2 \\ &\Rightarrow r(A) = \|A\| \end{aligned}$$

□_{7.2.4}

7.2.5 Lemma

Let $A \in L(H)$ be symmetric and $p \in \mathbb{P}_{\mathbb{C}}$ a complex polynomial. Then holds:

$$\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|$$

Proof

$p(A)$ is normal and thus, according to Theorem 7.2.4 holds:

$$\|p(A)\| = \sup_{\mu \in \sigma(p(A))} |\mu| \stackrel{7.2.2}{=} \sup_{\lambda \in \sigma(A)} |p(\lambda)|$$

□_{7.2.5}

Proof of theorem 7.2.1

- For complex polynomials, we set $\Phi(p) = p(A)$. Then holds:

$$\|\Phi(p)\| = \|p(A)\| = r(p(A)) = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = \|p\|_{C^0(\sigma(A), \mathbb{C})}$$

Thus $\Phi : \mathbb{P}_{\mathbb{C}} \rightarrow L(H)$ is an isometry. ($\mathbb{P}_{\mathbb{C}} \subseteq C^0(\sigma(A), \mathbb{C})$)

Remark: If we had considered $C^0([a, b], \mathbb{C})$ with

$$a = \inf_{\|u\|=1} \langle u, Au \rangle$$

$$b = \sup_{\|u\|=1} \langle u, Au \rangle$$

then we would only have an inequality:

$$\|\Phi(p)\| \leq \|p\|_{C^0([a, b])}$$

- Moreover holds:

$$\Phi(p \cdot q) = (p \cdot q)(A) = p(A) \cdot q(A) = \Phi(p) \cdot \Phi(q)$$

$$(\Phi(p))^* = \Phi(\bar{p})$$

- Using the Stone-Weierstraß approximation theorem, Φ uniquely extends to an isometry:

$$\Phi : C^0(\sigma(A), \mathbb{C}) \rightarrow L(H)$$

This yields i), ii), iii), vii).

- More specifically, consider $f \in C^0(\sigma(A), \mathbb{C})$. Then there exist $p_n \in \mathbb{P}_{\mathbb{C}}$ such that $p_n \rightrightarrows f$ on $\sigma(A)$. ($K = \sigma(A)$ is a compact metric space.) This means:

$$\|p_n - f\|_{C^0(\sigma(A), \mathbb{C})} = \sup_{z \in \sigma(A)} |p_n(z) - f(z)| \xrightarrow{n \rightarrow \infty} 0$$

$$\|\Phi(p_n) - \Phi(p_m)\| \stackrel{\text{isometry}}{=} \|p_n - p_m\| \xrightarrow{n, m \rightarrow \infty} 0$$

Thus the operators $\Phi(p_n)$ form a Cauchy sequence in $L(H)$ and since $L(H)$ is a Banach space, this sequence converges to:

$$\Phi(f) := \lim_{n \rightarrow \infty} \Phi(p_n)$$

- iv) For $Au = \lambda u$ holds:

$$\Phi(f)u = \lim_{n \rightarrow \infty} \Phi(p_n)u = \lim_{n \rightarrow \infty} p_n(A)u = \lim_{n \rightarrow \infty} p_n(\lambda)u = f(\lambda)u$$

- vi) Now we prove the spectral mapping theorem:

„ \subseteq “: Assume $\mu \in \sigma(f(A))$, but $\mu \notin f(\sigma(A))$. Then holds $f - \mu \neq 0$ on $\sigma(A)$ and we can invert:

$$\frac{1}{f - \mu} \in C^0(\sigma(A), \mathbb{C})$$

Now follows:

$$\mathbb{1} = \Phi(1) = \Phi\left(\frac{1}{f-\mu}(f-\mu)\right) = \underbrace{\Phi\left(\frac{1}{f-\mu}\right)}_{\in L(H)} \cdot \underbrace{\Phi(f-\mu)}_{=f(A)-\mu\mathbb{1}}$$

So $f(A) - \mu\mathbb{1}$ has a bounded inverse in contradiction to the assumption $\mu \in \sigma(f(A))$.
 „ \supseteq “: Consider $\lambda \in \sigma(A)$. Choose polynomials $p_n \in \mathbb{P}_{\mathbb{C}}$ with $p_n \rightrightarrows f$. Then converges in $L(H)$:

$$p_n(A) - p_n(\lambda)\mathbb{1} \xrightarrow{n \rightarrow \infty} f(A) - f(\lambda)\mathbb{1}$$

Assume that $f(\lambda) \notin \sigma(f(A))$. Then $f(A) - f(\lambda)\mathbb{1}$ has a bounded inverse.

According to Theorem 2.5.3, the invertible operators are open in $L(H)$. Therefore there exists a $\delta \in \mathbb{R}_{>0}$ such that B has a bounded inverse for all $B \in B_\delta(f(A) - f(\lambda)\mathbb{1})$. In particular, the operators $p_n(A) - p_n(\lambda)\mathbb{1}$ have a bounded inverse for sufficiently large n . This is a contradiction to the spectral mapping theorem for polynomials 7.2.2.

v) Claim: $f \geq 0 \Rightarrow \Phi(f) \geq 0$

Let $f \in C^0(\sigma(A), \mathbb{R})$ be real-valued and $f \geq 0$. Then $g := \sqrt{f} \in C^0(\sigma(A), \mathbb{R})$ and $f = g^2$.

$$\langle u, \Phi(f)u \rangle = \langle u, \Phi(g^2)u \rangle = \langle u, \Phi(g)\Phi(g)u \rangle = \langle \Phi(\bar{g})u, \Phi(g)u \rangle = \langle \Phi(g)u, \Phi(g)u \rangle \geq 0$$

□_{7.2.1}

$\chi_\Omega(A)$ would be the projector onto the invariant subspace corresponding to the spectrum in Ω . Formally we can compute:

$$\begin{aligned} (\chi_\Omega(A))^* &= \overline{\chi_\Omega(A)} = \chi_\Omega(A) \\ \chi_\Omega(A)\chi_\Omega(A) &= \chi_\Omega^2(A) = \chi_\Omega(A) \end{aligned}$$

This motivates, why we would like to form $f(A)$ for a bounded Borel function f on $\sigma(A)$.

7.3 Spectral Measures

Let $A \in L(H)$ be symmetric. Choose a $u \in H$ (fixed).

$$\begin{aligned} \Phi_u : C^0(\sigma(A), \mathbb{R}) &\rightarrow \mathbb{R} \subseteq \mathbb{C} \\ f &\mapsto \langle u, \Phi(f)u \rangle \end{aligned}$$

$$|\Phi_u(f)| = |\langle u, \Phi(f)u \rangle| \leq \|\Phi(f)\| \cdot \|u\|^2 = \|f\|_{C^0(\sigma(A), \mathbb{R})} \cdot \|u\|^2$$

Thus ϕ_u is a bounded linear functional on $C^0(\sigma(A), \mathbb{R})$. According to the Riesz representation theorem there exists a unique regular bounded Borel measure μ_u such that:

$$\langle u, f(A)u \rangle = \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda)$$

The measure μ_u is even positive, because if $f \geq 0$, set $g = \sqrt{f}$ to get:

$$\int_{\sigma(A)} f(\lambda) d\mu_u(\lambda) = \langle u, f(A)u \rangle = \langle g(A)u, g(A)u \rangle \geq 0 \quad \forall_{f \in C^0(\sigma(A), \mathbb{R}), f \geq 0}$$

Hence by approximation follows $\mu_u(\Omega) \geq 0$ for all Borel sets $\Omega \subseteq \sigma(A)$. So μ_u is a positive measure.

The resulting integral can be defined for a more general class of functions.

A *Borel function* f is a function, which is measurable for the Borel algebra, i.e. $f^{-1}(\Omega)$ is a Borel function for all open $\Omega \subseteq \mathbb{C}$.

We use the following notation: \mathfrak{M} is the set of all Borel sets in $\sigma(A)$.

$\mathcal{B}(\sigma(A), \mathbb{R}) = L^\infty(d\mu_u)$ are the bounded Borel functions on $\sigma(A)$. We always assume:

$$\sup_{\sigma(A)} |f| < \infty$$

We define:

$$\begin{aligned} \phi_u : \mathcal{B}(\sigma(A), \mathbb{R}) &\rightarrow \mathbb{R} \\ \phi_u(f) &:= \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda) \end{aligned}$$

7.3.1 Lemma

$$|\phi_u(f)| \leq \|f\|_{L^\infty} \cdot \|u\|^2$$

Proof

For $f \in \mathcal{B}(\sigma(A), \mathbb{R})$ choose $\varphi_n \in C^0(\sigma(A), \mathbb{R})$ such that $\varphi_n \rightarrow f$ converges point-wise and $\|\varphi_n\|_\infty \leq \|f\|_\infty$. (Approximate f by step-functions and then approximate the step functions by continuous functions.)

Due to $|\varphi_n| \leq C$ and

$$\int_{\sigma(A)} C d\mu_u = C\mu_u(\sigma(A)) = C \langle u, \Phi(1)u \rangle = C \langle u, \mathbb{1}u \rangle = C \|u\|^2 < \infty$$

we can use the dominated convergence theorem:

$$\begin{aligned} \left| \int_{\sigma(A)} f d\mu_u \right| &\stackrel{\text{dominated}}{=} \lim_{\text{convergence } n \rightarrow \infty} \left| \int_{\sigma(A)} \varphi_n d\mu_n \right| = \lim_{n \rightarrow \infty} |\langle u, \Phi(\varphi_n)u \rangle| \leq \\ &\leq \lim_{n \rightarrow \infty} \|u\|^2 \cdot \|\Phi(\varphi_n)\| = \lim_{n \rightarrow \infty} \|u\|^2 \cdot \|\varphi_n\| \leq \|f\| \cdot \|u\|^2 \end{aligned}$$

□_{7.3.1}

Define using the Fréchet-Riesz theorem the unique Operator $\Phi(f)$ by:

$$\Phi_u(f) := \langle u, \Phi(f)u \rangle$$

By polarization we get:

$$B_f(u, v) = \Phi_{\frac{u+v}{2}}(f) - \Phi_{\frac{u-v}{2}}(f) - \mathbf{i}\Phi_{\frac{u+iv}{2}}(f) + \mathbf{i}\Phi_{\frac{u-iv}{2}}(f)$$

Alternatively define for $f \in C^0(\sigma(A), \mathbb{C})$:

$$\Phi_{u,v}(f) := \langle u, \Phi(f)v \rangle = \int_{\sigma(A)} f(\lambda) d\mu_{u,v}(\lambda)$$

$$B_f(u, v) := \int_{\sigma(A)} f(\lambda) d\mu_{u,v}(\lambda)$$

$d\mu_{u,v}$ is only a *complex-valued*, bounded, regular Borel measure.

7.3.2 Lemma

$B_f(u, v)$ is a *sesquilinear form*, i.e. linear in the second and anti-linear in the first argument, and it holds:

$$|B_f(u, v)| \leq \|f\| \cdot \|u\| \cdot \|v\|$$

Proof

This follows from the polarization formula and Lemma 7.3.1. □_{7.3.2}

7.3.3 Theorem

Let B be a bounded sesquilinear form, i.e.:

$$|B(u, v)| \leq C \cdot \|u\| \cdot \|v\| \quad \forall_{u, v \in H}$$

Then there is a unique operator $D \in L(H)$ with $\|D\| \leq C$ such that:

$$B(u, v) = \langle u, Dv \rangle$$

Proof

For $v \in H$ the map

$$\psi := \overline{B(\cdot, v)}$$

is a bounded linear form. According to the Fréchet-Riesz theorem 3.1.3 there exists a $w \in H$ such that for all $u \in H$ holds:

$$\psi(u) = \langle w, u \rangle$$

Then follows:

$$B(u, v) = \overline{\langle w, u \rangle} = \langle u, w \rangle$$

Thus D is uniquely determined by $Dv = w$. So $D : H \rightarrow H$ is linear and bounded by the open mapping principle 2.4.7, i.e. $D \in L(H)$ and for all $v \in H$ holds:

$$B(u, v) = \langle u, Dv \rangle$$

Choose $u = Dv$ to get:

$$\begin{aligned} B(Dv, v) &= \langle Dv, Dv \rangle = \|Dv\|^2 \\ &\leq C \cdot \|Dv\| \cdot \|v\| \end{aligned}$$

Therefore we have for all $v \in H$:

$$\begin{aligned} \|Dv\| &\leq C \cdot \|v\| \\ \|D\| &\leq C \end{aligned}$$

□_{7.3.3}

We conclude: For $f \in \mathcal{B}(\sigma(A), \mathbb{C})$ we construct $B_f(u, v)$. Then there exists a $\Phi(f) \in L(H)$ such that for all $u, v \in H$ holds:

$$\langle u, \Phi(f)v \rangle = B_f(u, v)$$

So $\Phi : \mathcal{B}(\sigma(A), \mathbb{C}) \rightarrow L(H)$ gives a functional calculus on $\mathcal{B}(\sigma(A), \mathbb{C})$, i.e. we can calculate $f(A)$ for an arbitrary Borel function.

7.3.4 Theorem (Spectral theorem in functional calculus form)

Let $A \in L(H)$ be symmetric. Then there is a unique mapping $\Phi : \mathcal{B}(\sigma(A)) \rightarrow L(H)$ with the following properties:

- i) Φ is an involutive algebra homomorphism, i.e.:

$$\begin{aligned} \Phi(f) \cdot \Phi(g) &= \Phi(f \cdot g) \\ \Phi(f)^* &= \Phi(\bar{f}) \end{aligned}$$

If $f \in C^0(\sigma(A), \mathbb{C})$, then $\Phi(f)$ agrees with the corresponding operator of the continuous functional calculus.

- ii) $\|\Phi(f)\| \leq \|f\|_\infty$
 iii) If $f_n \rightarrow f$ converges point-wise and it holds $\|f_n\|_\infty < C$, then $\Phi(f_n) \rightarrow \Phi(f)$ converges strongly, i.e. for all $u \in H$ converges in H :

$$\Phi(f_n)u \rightarrow \Phi(f)u$$

- iv) From $Au = \lambda u$ follows:

$$\Phi(f)u = f(\lambda)u$$

- v) If $f \geq 0$ holds, then $\Phi(f) \geq 0$ is positive semidefinite.
 vi) If $B \in L(H)$ commutes with A , i.e. $[A, B] = AB - BA = 0$, then $[B, \Phi(f)] = 0$. We write also $f(A) = \Phi(f)$.

Note: There is no spectral mapping theorem.

Proof

i) Prove the homomorphism property by approximation:

First step: Assume $f \in C^0(\sigma(A), \mathbb{C})$ and $g \in \mathcal{B}(\sigma(A), \mathbb{C})$. Then there exists a series $g_n \in C^0$ such that $g_n \rightarrow g$ converges point-wise and $\|g_n\|_\infty < C$. Then follows the point-wise convergence:

$$fg_n \rightarrow fg$$

We use the notation:

$$\begin{aligned} \phi_{u,v}(h) &:= \langle u, \Phi(h)v \rangle \\ \Rightarrow \phi_{u,u}(h) &= \phi_u(h) \end{aligned}$$

Since μ_u is a regular bounded Borel measure, we can apply the dominated convergence theorem:

$$\begin{aligned} \phi_{u,u}(f \cdot g) &\stackrel{\text{Definition}}{=} \int_{\sigma(A)} f \cdot g d\mu_u \stackrel{\text{dominated convergence}}{\lim_{n \rightarrow \infty}} \int_{\sigma(A)} f \cdot g_n d\mu_u = \lim_{n \rightarrow \infty} \phi_{u,u}(f, g_n) = \\ &= \lim_{n \rightarrow \infty} \langle u, \Phi(f \cdot g_n)u \rangle = \lim_{n \rightarrow \infty} \langle u, f(A) \cdot g_n(A)u \rangle = \\ &= \lim_{n \rightarrow \infty} \langle (f(A))^* u, g_n(A)u \rangle = \lim_{n \rightarrow \infty} \phi_{(f(A))^* u, u}(g_n) \end{aligned}$$

We know for all $u \in H$ using dominated convergence (see above):

$$\phi_{u,u}(g_n) \rightarrow \phi_{u,u}(g)$$

By polarization follows for all $u, v \in H$:

$$\phi_{v,u}(g_n) \rightarrow \phi_{v,u}(g)$$

This gives:

$$\begin{aligned} \phi_{u,u}(f \cdot g) &= \lim_{n \rightarrow \infty} \phi_{(f(A))^* u, u}(g_n) = \phi_{(f(A))^* u, u}(g) = \langle (f(A))^* u, \Phi(g)u \rangle \\ \Rightarrow \langle u, \Phi(f \cdot g)u \rangle &= \langle u, f(A) \cdot g(A)u \rangle \end{aligned}$$

Polarization yields:

$$\Phi(fg) = \Phi(f) \cdot \Phi(g)$$

Second Step: Consider $f, g \in \mathcal{B}$. We choose $f_n \in C^0$ with $f_n \rightarrow f$ and $\|f_n\| < C$. Then $f_n \cdot g \rightarrow f \cdot g$ converges point-wise.

$$\begin{aligned} \langle u, \Phi(f \cdot g)u \rangle &\stackrel{\text{dominated convergence}}{=} \lim_{n \rightarrow \infty} \langle u, \Phi(f_n \cdot g)u \rangle \stackrel{\text{First step}}{=} \lim_{n \rightarrow \infty} \langle u, \Phi(f_n) \cdot \Phi(g)u \rangle = \\ &= \lim_{n \rightarrow \infty} \phi_{u, g(A)u}(f_n) = \phi_{u, g(A)u}(f) = \langle u, f(A)g(A)u \rangle \\ \Rightarrow \langle u, (\Phi(fg) - \Phi(f)\Phi(g))u \rangle &= 0 \quad \forall_{u \in H} \end{aligned}$$

By polarization follows:

$$\Phi(fg) = \Phi(f)\Phi(g)$$

The involution property follows similarly. □_{i)}

iii) Claim: From point-wise convergence $f_n \rightarrow f$ and $\|f_n\| < C$ follows strong convergence $f_n(A) \rightarrow f(A)$.

a) From the dominated convergence theorem it is clear that holds:

$$\begin{aligned}\phi_u(f_n) &\rightarrow \phi_u(f) \\ \langle u, f_n(A)u \rangle &\rightarrow \langle u, f(A)u \rangle\end{aligned}$$

Polarization gives for all $u, v \in H$:

$$\langle u, f_n(A)v \rangle \rightarrow \langle u, f(A)v \rangle$$

In other words for all $v \in H$ holds:

$$f_n(A)v \rightarrow f(A)v$$

b) It holds:

$$\begin{aligned}\|f_n(A)v\|^2 &= \langle f_n(A)v, f_n(A)v \rangle = \langle v, (f_n(A))^* f_n(A)v \rangle = \\ &= \langle v, \overline{f_n}(A) f_n(A)v \rangle = \left\langle v, |f_n(A)|^2 v \right\rangle \xrightarrow[\text{convergence}]{\text{dominated}} \left\langle v, |f|^2(A)v \right\rangle = \\ &= \langle v, \overline{f}(A) f(A)v \rangle = \langle f(A)v, f(A)v \rangle = \|f(A)v\|^2\end{aligned}$$

c) Now apply the following general Lemma:

Lemma: $u_n \rightarrow u$ and $\|u_n\| \rightarrow \|u\|$ imply $u_n \rightarrow u$.

Proof:

$$\begin{aligned}\|u - u_n\| &= \langle u - u_n, u - u_n \rangle = \\ &= \|u\|^2 - 2\operatorname{Re} \underbrace{\langle u, u_n \rangle}_{\substack{\rightarrow \langle u, u \rangle \\ \text{because } u \rightarrow u_n}} + \underbrace{\|u_n\|^2}_{\substack{\rightarrow \|u\|^2 \\ \text{because } \|u_n\| \rightarrow \|u\|}} \rightarrow \|u\|^2 - 2\|u\|^2 + \|u\|^2 = 0\end{aligned}$$

□ Lemma

d) This gives:

$$f_n(A)v \rightarrow f(A)v$$

□ iii)

ii) Claim: $\|f(A)\| \leq \|f\|_\infty$ for $f \in \mathcal{B}$.

Choose $f_n \in C^0$ which converge point-wise to f and $\|f_n\|_\infty < \|f\|$.

$$\|f(A)u\| \stackrel{\text{iii)}}{=} \lim_{n \rightarrow \infty} \|f_n(A)u\| \leq \lim_{n \rightarrow \infty} \underbrace{\|f_n(A)\|}_{= \|f_n\|_\infty} \cdot \|u\| = \lim_{n \rightarrow \infty} \|f_n\|_\infty \cdot \|u\| = \|f\|_\infty \cdot \|u\|$$

$$\Rightarrow \|f(A)\| \leq \|f\|_\infty$$

□ ii)

iv) - vi) follow immediately by approximation.

□ 7.3.4

7.3.5 Remark

So far we considered Borel measures on $\sigma(A) \subseteq \mathbb{R}$. These measures can be extended to Borel measures on \mathbb{R} by defining for a Borel set $\Omega \in \mathfrak{M}(\mathbb{R})$:

$$\mu(\Omega) := \mu(\Omega \cap \sigma(A))$$

$\Omega \cap \sigma(A)$ is a Borel set of $\sigma(A)$, since $\sigma(A)$ is closed.

Now let $M \subseteq \mathfrak{M}(\mathbb{R})$ be a Borel set. $f(A)$ is well defined for any $f \in \mathcal{B}(\mathbb{R})$. With the characteristic function χ_M of M define:

$$E_M := \chi_M(A)$$

Then we get:

$$E_M^* = \overline{\chi_M}(A) = \chi_M(A) = E_M$$

$$E_M^2 = \chi_M(A) \cdot \chi_M(A) = (\chi_M \cdot \chi_M)(A) = \chi_M(A) = E_M$$

Thus E_M is symmetric and idempotent, in other words E_M is a projection operator.

The mapping $M \mapsto E_M$ is the spectral measure.

7.3.6 Definition (projection operator, spectral measure)

$P \in L(H)$ is a *projection operator* if $P^2 = P = P^*$.

An operator-valued *spectral measure* E is a mapping

$$\begin{aligned} E : \mathfrak{M}(\mathbb{R}^n) &\rightarrow L(H) \\ M &\mapsto E_M := E(M) \end{aligned}$$

with the following properties:

- i) E_M is a projection operator for all $M \in \mathfrak{M}$.
- ii) $E_\emptyset = 0$, $E_{\mathbb{R}^n} = \mathbb{1}$
- iii) For $M = \bigcup_{n=1}^{\infty} M_n$ the operator E_M is the strong limit of the partial sums $\sum_{n=1}^k E_{M_n}$:

$$E_M = \text{s-lim}_{k \rightarrow \infty} \sum_{n=1}^k E_{M_n}$$

This means that for all $u \in H$ holds:

$$E_M u = \sum_{n=1}^{\infty} (E_{M_n} u)$$

The series does not necessarily converge in the operator norm!

- iv) $E_M \cdot E_N = E_{M \cap N}$

- v) For all $u \in H$, the mapping $M \mapsto \langle u, E_M u \rangle \in \mathbb{R}$ is a (real) bounded regular Borel measure.

$\text{supp}(E)$ is the complement of the largest open set Ω with $E_\Omega = 0$, which exists due to the σ -additivity.

E is called a *compact* spectral measure if $\text{supp}(E)$ is compact.

7.3.7 Theorem

Let $A \in L(H)$ be symmetric. Then the mapping

$$E : M \mapsto \chi_M(A)$$

is a spectral measure on \mathbb{R} with $\text{supp}(E) \subseteq \sigma(A)$.

Proof

We have to show the properties from the definition 7.3.6.

i) is clear.

$$\begin{aligned}\chi_\emptyset(A) &= 0(A) = 0 \\ \chi_{\mathbb{R}}(A) &= \Phi(1) = \mathbb{1}\end{aligned}$$

So ii) is shown.

iv) follows from:

$$\chi_M(A) \cdot \chi_N(A) = (\chi_M \cdot \chi_N)(A) = \chi_{M \cap N}(A)$$

For v) consider:

$$\langle u, E_M u \rangle = \langle u, \chi_M(A) u \rangle = \phi_u(\chi_M) = \int \chi_M d\mu_u = \mu_u(M)$$

It remains to show iii) and $\text{supp}(E) \subseteq \sigma(A)$.

For the later consider $\Omega \subseteq \varrho(A)$:

$$E_\Omega = \chi_\Omega(A) = \Phi(\chi_\Omega) \stackrel{\text{extension to } \mathcal{B}(\mathbb{R})}{=} \Phi(\chi_\Omega \chi_{\sigma(A)}) = \Phi(\chi_{\Omega \cap \sigma(A)}) = \Phi(0) = 0$$

Now show iii): From

$$M = \bigcup_{j=1}^{\infty} M_j$$

follows with point-wise convergence:

$$\chi_M = \sum_{j=1}^{\infty} \chi_{M_j}$$

Theorem 7.3.4 iii) yields:

$$\text{s-lim}_{n \rightarrow \infty} \sum_{j=1}^n \underbrace{\chi_{M_j}(A)}_{=E_{M_j}} = \underbrace{\chi_M(A)}_{=E_M}$$

□_{7.3.7}

Notation

$M \mapsto E_M$ is the spectral measure, which is projection operator valued.

$M \mapsto \langle u, E_M u \rangle = \mu_u(M) = \mu_{u,u}(M)$ is the real, bounded, regular Borel measure.

$M \mapsto \langle u, E_M v \rangle = \mu_{u,v}(M)$ is the complex, bounded, regular Borel measure.

Consider the integral:

$$\int_{\mathbb{R}} f(\lambda) d\mu_u(\lambda)$$

$$d\mu_u(\lambda) = d\langle u, E_\lambda u \rangle$$

$$d\mu_{u,v}(\lambda) = d\langle u, E_\lambda v \rangle$$

7.3.8 Lemma

Let E be a spectral measure on \mathbb{R}^n and $M \in \mathfrak{M}(\mathbb{R}^n)$. Then holds for all $u, v \in H$:

$$d\langle u, E_\lambda E_M v \rangle = \chi_M(\lambda) d\langle u, E_\lambda v \rangle = d\langle E_M u, E_\lambda v \rangle$$

Proof

For all $f \in \mathcal{B}(\mathbb{R}^n)$ we have to show:

$$\int_{\mathbb{R}^n} f(\lambda) d\langle u, E_\lambda E_M v \rangle = \int_{\mathbb{R}^n} f(\lambda) \cdot \chi_M(\lambda) d\langle u, E_\lambda v \rangle$$

By approximation, it suffices to show for all $\Omega \in \mathfrak{M}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \chi_\Omega(\lambda) d\langle u, E_\lambda E_M v \rangle = \int_{\mathbb{R}^n} \chi_\Omega(\lambda) \chi_M(\lambda) d\langle u, E_\lambda v \rangle$$

Since $\int \chi_M(x) d\mu(x) = \mu(M)$, we get:

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_\Omega(\lambda) d\langle u, E_\lambda E_M v \rangle &= \langle u, E_\Omega E_M v \rangle \stackrel{\text{property iv)}}{=} \langle u, E_{\Omega \cap M} v \rangle = \\ &= \int_{\mathbb{R}^n} \chi_{\Omega \cap M} \langle u, dE_\lambda v \rangle = \int_{\mathbb{R}^n} \chi_\Omega \chi_M \langle u, dE_\lambda v \rangle \end{aligned}$$

□_{7.3.8}

We write:

$$\int_{\mathbb{R}^n} f(\lambda) d\langle u, E_\lambda v \rangle =: \left\langle u, \left(\int_{\mathbb{R}^n} f(\lambda) dE_\lambda \right) v \right\rangle$$

We will use this to define integration in $L(H)$.

7.3.9 Theorem

Let E be a spectral measure on \mathbb{R}^n and $f \in \mathcal{B}(\mathbb{R}^n)$. Then the relations

$$\int f(\lambda) d\langle u, E_\lambda v \rangle = \langle u, Av \rangle \quad \forall_{u,v \in H}$$

define a unique normal operator $A \in L(H)$, which we also denote by:

$$A = \int f(\lambda) dE_\lambda$$

Moreover:

$$A^* = \int \overline{f(\lambda)} dE_\lambda$$

Proof

We define a bilinear form $B : H \times H \rightarrow \mathbb{C}$ by:

$$B(u, v) = \int_{\mathbb{R}^n} f(\lambda) d\langle u, E_\lambda v \rangle$$

Then we have:

$$|B(u, u)| \leq \int_{\mathbb{R}^n} |f(\lambda)| \underbrace{d\langle u, E_\lambda u \rangle}_{\text{positive measure}} \leq \|f\|_\infty \cdot \left\langle u, \underbrace{E_{\mathbb{R}^n}}_{=1} u \right\rangle = \|f\|_\infty \cdot \|u\|^2$$

Polarization and estimation yields:

$$|B(u, v)| \leq \|f\|_\infty \|u\| \cdot \|v\|$$

Thus by the Fréchet-Riesz theorem, there is a unique $A \in L(H)$ with:

$$B(u, v) = \langle u, Av \rangle$$

$$\begin{aligned} \langle u, Av \rangle &= \int f(\lambda) d\langle u, E_\lambda v \rangle \\ \langle u, A^*v \rangle &= \langle v, Au \rangle = \int \overline{f(\lambda)} d\langle u, E_\lambda v \rangle \\ \Rightarrow \quad A^* &= \int \overline{f(\lambda)} dE_\lambda \end{aligned}$$

□_{7.3.9}

7.3.10 Theorem

Let E be a spectral measure on \mathbb{R}^n and $f, g \in \mathcal{B}(\mathbb{R}^n)$. Then holds:

$$\left(\int_{\mathbb{R}^n} f(\lambda) dE_\lambda \right) \left(\int_{\mathbb{R}^n} g(\lambda') dE_{\lambda'} \right) = \int_{\mathbb{R}^n} f(\lambda) g(\lambda) dE_\lambda$$

Proof

By approximation it suffices to consider the case $g = \chi_M$ for $M \in \mathfrak{M}(\mathbb{R}^n)$.

$$A := \int_{\mathbb{R}^n} f(\lambda) dE_\lambda \qquad E_M = \int_{\mathbb{R}^n} \chi_M dE_\lambda$$

For all $u, v \in H$ holds:

$$\begin{aligned} \langle u, A \cdot E_M v \rangle &= \int_{\mathbb{R}^n} f(\lambda) d \langle u, E_\lambda E_M v \rangle \stackrel{(7.3.8)}{=} \int_{\mathbb{R}^n} f(\lambda) \chi_M(\lambda) d \langle u, E_\lambda v \rangle = \\ &= \left\langle u, \int_{\mathbb{R}^n} (f \cdot \chi_M)(\lambda) dE_\lambda v \right\rangle \end{aligned}$$

$$\Rightarrow A \cdot E_M = \int_{\mathbb{R}^n} f \cdot \chi_M dE_\lambda$$

□_{7.3.10}

Physicists write:

$$E_\lambda \cdot E_\mu = \delta_{\lambda-\mu} E_\lambda$$

This follows, because E_λ is idempotent and for $\lambda \neq \mu$ holds:

$$E_\lambda E_\mu = E_{\{\lambda\}} \cdot E_{\{\mu\}} = E_{\{\lambda\} \cap \{\mu\}} = E_\emptyset = 0$$

7.3.11 Theorem (spectral decomposition of a bounded symmetric operator)

There is a one-to-one correspondence between bounded symmetric operators $A \in L(H)$ and compact spectral measures E on \mathbb{R} by:

$$A = \int_{\mathbb{R}} \lambda dE_\lambda$$

This means for a given A with corresponding spectral measure $E_M = \chi_M(A)$ holds this equation. Conversely, if E is a compact spectral measure, then this equation defines a bounded symmetric Operator and $E_M = \chi_M(A)$.

Moreover holds:

- i) $f(A) = \int_{\mathbb{R}} f(\lambda) dE_\lambda$
- ii) $\sigma(A) = \text{supp}(E)$

Proof

For a given A , let $E_M = \chi_M(A)$ be the corresponding spectral measure. Then holds for all $u, v \in H$ by construction:

$$\langle u, f(A) v \rangle = \int_{\mathbb{R}} f(\lambda) d \langle u, E_\lambda v \rangle$$

By the definition of $\int f(\lambda) dE_\lambda$ follows:

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE_\lambda$$

For the polynomial $f(\lambda) = \lambda$, i.e. $f(A) = A$, this gives:

$$A = \int_{\mathbb{R}} \lambda dE_\lambda$$

If E is a compact spectral measure, $\int_{\mathbb{R}} f(\lambda) dE_\lambda$ defines a normal operator with:

$$\left(\int_{\mathbb{R}} f(\lambda) dE_\lambda \right)^* = \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda$$

The compatibility with the spectral calculus follows from theorem 7.3.10.

Thus it remains to show $\sigma(A) \subseteq \text{supp}(E)$. Consider $\mu \notin \text{supp}(E)$. We want to show $\mu \in \varrho(A)$. Define the following bounded real function:

$$g(\lambda) := \frac{1}{\lambda - \mu} \chi_{\text{supp}(E)}$$

$$f(\lambda) := \lambda - \mu$$

$$B := \int_{\mathbb{R}} g dE_\lambda \in L(H)$$

is a well-defined integral.

$$\begin{aligned} \int_{\mathbb{R}} f(\lambda) dE_\lambda &= A - \mu \mathbb{1} \\ (A - \mu \mathbb{1}) B &= \left(\int_{\mathbb{R}} f(\lambda') dE_{\lambda'} \right) \left(\int_{\mathbb{R}} g(\lambda) dE_\lambda \right) = \int_{\mathbb{R}} f \cdot g dE_\lambda = \\ &= \int_{\mathbb{R}} \chi_{\text{supp}(E)} \underbrace{dE_\lambda}_{=0 \text{ outside of } \text{supp}(E)} = \int_{\mathbb{R}} dE_\lambda = \mathbb{1} \end{aligned}$$

Thus $B = (A - \mu \mathbb{1})^{-1}$ and therefore $\mu \in \varrho(A)$.

□_{7.3.11}

7.3.12 Corollary

For $f \in \mathcal{B}(\mathbb{R})$ holds:

$$\|f(A)\| = \sup_{\sigma(A)} \text{ess } |f|$$

Proof

„ \leq “ was already proved in theorem 7.3.4 ii).

To prove equality, we first note that $f(A)$ is a normal operator, because it holds:

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE_\lambda \quad (f(A))^* = \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda$$

$$\begin{aligned}
f(A) \cdot (f(A))^* &= \left(\int_{\mathbb{R}} f(\lambda) dE_{\lambda} \right) \left(\int_{\mathbb{R}} \overline{f(\lambda)} dE_{\lambda} \right) = \\
&= \int_{\mathbb{R}} f(\lambda) \overline{f(\lambda)} dE_{\lambda} = \int_{\mathbb{R}} \overline{f(\lambda)} f(\lambda) dE_{\lambda} = (f(A))^* f(A)
\end{aligned}$$

For a normal operator B holds:

$$\|B\| = r(B) = \sup_{x \in \sigma(B)} |x|$$

Now follows by theorem 7.3.11 ii):

$$\|f(A)\| = \sup_{x \in \sigma(f(A))} |x| = \sup(\text{supp}(f(E))) = \sup_{\lambda \in \text{supp}(E)} \text{ess } |f(\lambda)|$$

□_{7.3.12}

7.4 Simple Examples

7.4.1 Example: finite dimensions

Consider $H = \mathbb{C}^n$ and a symmetric operator $A \in L(\mathbb{C}^n)$. Choose an orthonormal eigenvector basis such that A has the matrix representation:

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The eigenvalues $\lambda_i \in \mathbb{R}$ are real, but there can be degeneracies, i.e. $\lambda_i = \lambda_j$ for some $i \neq j$.

$$A^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$$

Similarly we can compute polynomials of A .

The Stone-Weierstraß approximation yields for $f \in C^0(\sigma(A), \mathbb{C})$:

$$f(A) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix}$$

Since the spectrum

$$\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$$

is a finite set, we have $C^0(\sigma(A)) = \mathcal{B}(\sigma(A))$. The spectral measure for $\Omega \subseteq \mathbb{C}$ is:

$$E_{\Omega} := \chi_{\Omega}(A) = \begin{pmatrix} \chi_{\Omega}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \chi_{\Omega}(\lambda_n) \end{pmatrix}$$

Thus E_Ω is the projection operator on the eigenspaces, for which the eigenvalues λ lie in Ω .

$$\int f(\lambda) dE_\lambda = \sum_{j=1}^n f(\lambda_j) E_{\{\lambda_j\}}$$

More specifically, let u_j be an orthonormal eigenvector basis, $Au_j = \lambda_j u_j$ and $\langle u_i, u_j \rangle = \delta_{ij}$. Then for any $v \in \mathbb{C}^n$ let $u_1^{(\lambda)}, \dots, u_\mu^{(\lambda)}$ be all eigenvectors with the eigenvalue λ , i.e. $Au_k^{(\lambda)} = \lambda u_k^{(\lambda)}$, so

$$E_{\{\lambda\}} v = \sum_{k=1}^{\mu} u_k^{(\lambda)} \langle u_k^{(\lambda)}, v \rangle$$

is the projection on the eigenspace $\langle u^{(k)} \rangle$.

7.4.2 Example: compact operator

Let H be an infinite-dimensional Hilbert space and $A \in L(H)$ be symmetric and compact. According to the Hilbert-Schmidt theorem, there is an orthonormal eigenvector basis (u_n) , i.e.:

$$Au_n = \lambda_n u_n$$

Then $\lambda_n \rightarrow 0$, because A is compact. The λ_n have finite-dimensional eigenspaces.

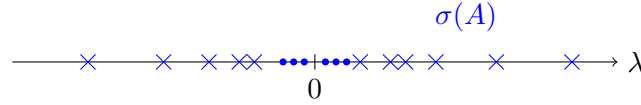


Figure 7.3: $\sigma(A)$ has only zero as limit point

$$\begin{aligned} A^2 u_n &= \lambda_n^2 u_n \\ p(A) u_n &= p(\lambda_n) u_n \end{aligned}$$

This holds for any polynomial p . The Stone-Weierstraß approximation yields for $f \in C^0(\sigma(A))$:

$$f(A) u_n = f(\lambda_n) u_n$$

The Riesz representation theorem gives

$$f(A) u_n = f(\lambda_n) u_n$$

for all $f \in \mathcal{B}(\sigma(A))$ or even $f \in \mathcal{B}(\mathbb{R})$. Then follows:

$$E_\Omega u_n := \chi_\Omega(A) u_n = \chi_\Omega(\lambda_n) u_n$$

Thus E_Ω is the projection operator to all eigenspaces whose eigenvalues λ lie in Ω . But $E_{(-\varepsilon, \varepsilon)}$ has infinite rank for all $\varepsilon > 0$.

$$A = \sum_{\lambda \in \sigma(A)} \lambda E_{\{\lambda\}}$$

$$A_N := \sum_{\substack{\lambda \in \sigma(A) \\ |\lambda| > \frac{1}{N}}} \lambda E_{\{\lambda\}}$$

is a finite-dimensional approximation of A (cf. 5.8) in the sense:

$$\|A - A_N\| \xrightarrow{N \rightarrow \infty} 0$$

More precisely we have:

$$\|A - A_N\| \leq \frac{1}{N}$$

Now consider:

$$\begin{aligned} \mathbb{1} &= \sum_{\lambda \in \sigma(A)} E_{\{\lambda\}} \\ E_N &:= \sum_{\substack{\lambda \in \sigma(A) \\ |\lambda| > \frac{1}{N}}} E_{\{\lambda\}} \end{aligned}$$

This converges strongly, but it does not converge in the operator norm:

$$\|E - E_N\| = \left\| E_{[-\frac{1}{N}, \frac{1}{N}]} \right\| = 1$$

7.4.3 Example: continuous spectrum

Consider the Hilbert space $H = L^2(\mathbb{R})$ and the function:

$$g(t) := \begin{cases} t & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

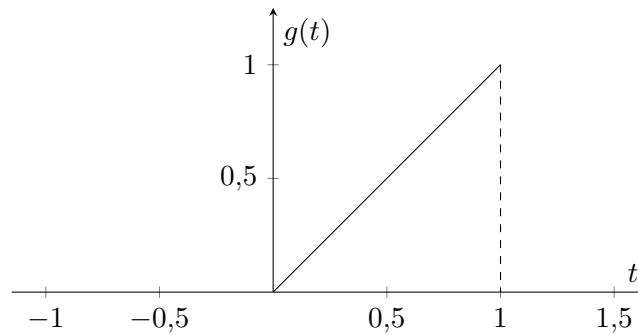


Figure 7.4: Plot of $g(t)$

$A \in L(H)$ defined by

$$(Au)(t) := g(t) \cdot u(t) = (T_g \cdot u)(t)$$

for $u \in H$ is a multiplication operator. From $|g(t)| \leq 1$ follows $\|A\| \leq 1$. As before we get:

$$A^2 = T_{g^2}$$

$$\begin{aligned} p(A) &= T_{p(g)} && \forall \text{ polynomial } p \\ f(A) &= T_{f(g)} && \forall f \in \mathcal{B}(\mathbb{R}) \end{aligned}$$

Therefore we get:

$$E_\Omega = T_{\chi_\Omega(g)}$$

$$\begin{aligned} (\chi_\Omega(g))(t) &= \begin{cases} 1 & \text{if } g(t) \in \Omega \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{g^{-1}(\Omega)} \end{aligned}$$

In general for multiplication operators holds:

$$E_\Omega = T_{\chi_\Omega(g)} = T_{\chi_{g^{-1}(\Omega)}}$$

For $\Omega = (a, b) \subseteq (0, 1)$ we get $g^{-1}(\Omega) = \Omega$ and thus $E_\Omega u = \chi_\Omega \cdot u$. If on the other hand $\Omega = \{0\}$, then holds:

$$g^{-1}(\Omega) = \mathbb{R} \setminus (0, 1) = (-\infty, 0] \cup [1, \infty)$$

Thus we get:

$$E_{\{0\}} u = \chi_{\mathbb{R} \setminus (0, 1)} u$$

The spectrum of A is $\sigma(A) = [0, 1]$. (Remember that the spectrum is always closed!)

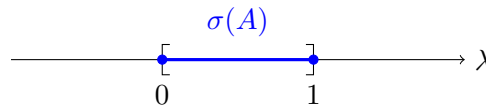


Figure 7.5: Continuous spectrum $\sigma(A)$ of A

Zero is an eigenvalue corresponding to an infinite-dimensional eigenspace, $Au = 0$ for $u|_{[0,1]} = 0$. Any $\lambda \in (0, 1]$ is *not* an eigenvalue:

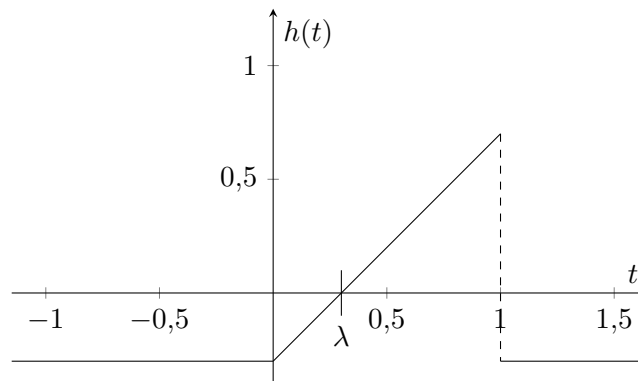


Figure 7.6: Plot of $g(t) - \lambda$

$$(A - \lambda)u = T_{g-\lambda}u$$

$$h := g - \lambda$$

$$\begin{aligned} h(x) \cdot u(x) &= 0 \\ \Leftrightarrow u &= 0 \quad \forall_{x \in \mathbb{R}, h(x) \neq 0} \\ \Leftrightarrow u &= 0 \quad \text{almost everywhere} \\ \Leftrightarrow u &= 0 \in L^2(\mathbb{R}) \end{aligned}$$

Thus the eigenvalue equation only has the trivial solution.

7.4.4 Example

Consider $H = L^2(\mathbb{R})$ and the multiplication operator $A = T_g$ for $g \in C_0^0(\mathbb{R})$. Then follows $E_\Omega = T_{g^{-1}(\Omega)}$ as before and $\sigma(A) = g(\mathbb{R})$.

That $\lambda \in \sigma(A)$ is an eigenvalue is equivalent to $g^{-1}(\{\lambda\})$ is a set of strictly positive Borel measure.

7.5 Essential and discrete spectrum

Let $A \in L(H)$ be symmetric. (The definitions are similar for normal operators or for unbounded self-adjoint operators). Let E be the corresponding spectral measure.

7.5.1 Definition (essential and discrete spectrum)

The essential spectrum $\sigma_{\text{ess}}(A)$ contains all $\lambda \in \mathbb{C}$ for which $\text{rg}(E_{B_\varepsilon(\lambda)}) = \infty$ for all $\varepsilon \in \mathbb{R}_{>0}$.

The discrete spectrum $\sigma_{\text{disc}}(A)$ contains all $\lambda \in \sigma(A)$ for which exists a $\varepsilon \in \mathbb{R}_{>0}$ such that the rank of $E_{B_\varepsilon(\lambda)}$ is finite.

Note: $\lambda \in \sigma_{\text{ess}}(A)$ implies $\lambda \in \text{supp}(E) = \sigma(A)$. Thus $\sigma(A) = \sigma_{\text{ess}}(A) \dot{\cup} \sigma_{\text{disc}}(A)$.

7.5.2 Example

Let A be a compact symmetric operator of infinite rank.

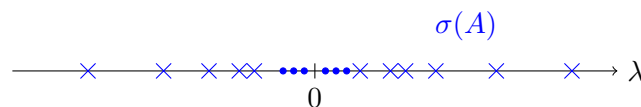


Figure 7.7: $\sigma(A)$ has only zero as limit point

Here we have:

$$\sigma_{\text{disc}} = \sigma(A) \setminus \{0\} \qquad \sigma_{\text{ess}} = \{0\}$$

7.5.3 Theorem (condition for discrete spectrum)

$\lambda \in \sigma_{\text{disc}}(A)$ holds if and only if both of the following conditions are satisfied:

- i) λ is an isolated point of $\sigma(A)$, i.e. there exists a $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(\lambda) \cap \sigma(A) = \{\lambda\}$.
- ii) λ is an eigenvalue of finite multiplicity, i.e. $\ker(A - \lambda)$ is finite-dimensional.

Proof

„ \Leftarrow “: If i) and ii) hold, then for an appropriately chosen $\varepsilon \in \mathbb{R}_{>0}$

$$E_{B_\varepsilon(\lambda)} = E_{\{\lambda\}}$$

is the projection operator on the finite-dimensional eigenspace.

„ \Rightarrow “: Consider $\lambda \in \sigma_{\text{disc}}(A)$.

- i) Choose $\varepsilon \in \mathbb{R}_{>0}$ such that $E_{B_\varepsilon(\lambda)}$ has finite rank.

$$J := E_{B_\varepsilon(\lambda)}(H)$$

is a finite-dimensional subspace of H . For $u \in J$ holds:

$$Au = AE_{B_\varepsilon(\lambda)}u = E_{B_\varepsilon(\lambda)}Au$$

Therefore follows $Au \in J$ and thus $A|_J : J \rightarrow J$ is a symmetric operator on a finite-dimensional Hilbert space. Diagonalize as in linear algebra:

$$\sigma(A|_J) = \{\lambda_1, \dots, \lambda_n\} = \sigma(A) \cap B_\varepsilon(\lambda)$$

The λ_i lie discrete and thus are isolated.

- ii) follows, because the eigenspace of A is the same as that of $A|_J$, which is finite-dimensional.

□_{7.5.3}

7.5.4 Theorem (Weyl criterion)

- i) $\lambda \in \sigma(A)$ holds if and only if there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in H such that for all $n \in \mathbb{N}$ holds $\|u_n\| = 1$ and:

$$(A - \lambda)u_n \xrightarrow{n \rightarrow \infty} 0$$

One also says, that λ is an *approximate eigenvalue*, because this can also be expressed as follows: For any $\varepsilon \in \mathbb{R}_{>0}$ there exists a $u \in H$ with $\|u\| = 1$ and $\|(A - \lambda)u\| \leq \varepsilon$.

- ii) $\lambda \in \sigma_{\text{ess}}(A)$ holds if and only if the (u_n) from above can be chosen as an orthonormal basis.

Proof

- i) For $\lambda \in \varrho(A)$ the operator $A - \lambda$ is continuously invertible, i.e. $(A - \lambda)^{-1} \in L(H)$. So for all $u \in H$ holds:

$$\|(A - \lambda)^{-1} u\| \leq C \|u\|$$

Since $A - \lambda$ is bijective, this is equivalent to:

$$\|v\| \leq C \|(A - \lambda) v\| \quad \forall_{v \in H}$$

This gives:

$$\begin{aligned} \|(A - \lambda) v\| &\geq \frac{1}{C} \|v\| \\ \|(A - \lambda) u_n\| &\geq \frac{1}{C} \|u_n\| = \frac{1}{C} \end{aligned}$$

Thus $(A - \lambda) u_n$ cannot converge to zero and thus λ is no approximate eigenvalue.

For $\lambda \in \sigma(A)$ the operator $(A - \lambda)$ has no bounded inverse. Then either $(A - \lambda)$ has a non-trivial kernel, i.e. there exists a $u \in H$ with $\|u\| = 1$ and:

$$(A - \lambda) u = 0$$

In this case one can choose $u_n := u$.

If on the other hand $(A - \lambda)$ is injective, but has no bounded inverse, then exists a sequence (u_n) with $\|(A - \lambda) u_n\| \leq \frac{1}{n} \|u_n\|$. This means that λ is an approximate eigenvalue.

- ii) This follows directly from theorem 7.5.3.

□_{7.5.4}

7.6 The Stone Formula

Let $A \in L(H)$ be symmetric, so we have $\sigma(A) \subseteq \mathbb{R}$. Thus for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the resolvent

$$R_\lambda := (A - \lambda)^{-1} \in L(H)$$

exists.

$$\begin{array}{c} \times \lambda \\ \hline \longrightarrow \mathbb{R} \end{array}$$

Figure 7.8: $\lambda \notin \mathbb{R}$

$$A = \int_{\mathbb{R}} \mu \cdot dE_\mu \qquad R_\lambda = \int_{\mathbb{R}} \frac{1}{\mu - \lambda} dE_\mu$$

$\frac{1}{\mu - \lambda} \in \mathcal{B}(\mathbb{R})$ holds, because the pole is away from the real axis.

$$(A - \lambda) R_\lambda = \left(\int_{\mathbb{R}} (\mu - \lambda) dE_\mu \right) \left(\int_{\mathbb{R}} \frac{1}{\mu - \lambda} dE_\mu \right) = \int_{\mathbb{R}} \frac{\mu - \lambda}{\mu - \lambda} dE_\mu = \int_{\mathbb{R}} dE_\mu = E_{\mathbb{R}} = \mathbb{1}$$

7.6.1 Theorem

For $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_{>0}$ holds:

$$\frac{1}{2\pi\mathbf{i}} \int_a^b (R_{\lambda+\mathbf{i}\varepsilon} - R_{\lambda-\mathbf{i}\varepsilon}) d\lambda = \frac{1}{2} (E_{(a,b)} + E_{[a,b]}) = \int_a^b \frac{1}{\mu - \lambda} dE_\mu$$

This is a convenient method for computing the spectral measure or the projection operator on eigenspaces.

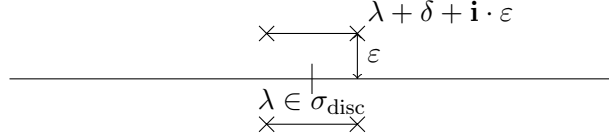


Figure 7.9: Calculating the spectral measure for a $\lambda \in \sigma_{\text{disc}}$

$$\text{s-lim}_{\delta \searrow 0} \text{s-lim}_{\varepsilon \searrow 0} \frac{1}{2\pi\mathbf{i}} \int_{\lambda-\delta}^{\lambda+\delta} (R_{\mu+\mathbf{i}\varepsilon} - R_{\mu-\mathbf{i}\varepsilon}) d\mu = E_{\{\lambda\}}$$

Proof

Let $a < b \in \mathbb{R}$ be given.

$$\phi_\varepsilon(\mu) := \frac{1}{2\pi\mathbf{i}} \int_a^b \left(\frac{1}{\mu - \lambda - \mathbf{i}\varepsilon} - \frac{1}{\mu - \lambda + \mathbf{i}\varepsilon} \right) d\lambda$$

Then holds $\phi_\varepsilon : \mathbb{R} \rightarrow \mathbb{C}$ and:

$$\begin{aligned} \phi_\varepsilon(A) &= \int_{\mathbb{R}} \phi_\varepsilon(\mu) dE_\mu = \frac{1}{2\pi\mathbf{i}} \int_a^b \int_{\mathbb{R}} \left(\underbrace{\frac{dE_\mu}{\mu - \lambda - \mathbf{i}\varepsilon}}_{=R_{\lambda+\mathbf{i}\varepsilon}} - \underbrace{\frac{dE_\mu}{\mu - \lambda + \mathbf{i}\varepsilon}}_{=R_{\lambda-\mathbf{i}\varepsilon}} \right) d\lambda = \\ &= \frac{1}{2\pi\mathbf{i}} \int_a^b (R_{\lambda+\mathbf{i}\varepsilon} - R_{\lambda-\mathbf{i}\varepsilon}) d\lambda \end{aligned}$$

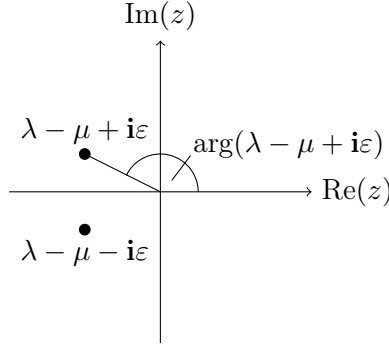
Now analyze the limit $\varepsilon \rightarrow 0$.

$$\phi_\varepsilon(\mu) = \frac{-1}{2\pi\mathbf{i}} (\ln(\lambda - \mu + \mathbf{i}\varepsilon) - \ln(\lambda - \mu - \mathbf{i}\varepsilon)) \Big|_{\lambda=a}^{\lambda=b}$$

The logarithm is cut at the negative real axis.

$$\ln(z) = \ln(|z|) + \mathbf{i} \arg(z) \qquad z = |z| e^{\mathbf{i} \arg(z)}$$

The argument of z lies in the range $(-\pi, \pi)$.

Figure 7.10: $-\pi < \arg(z) < \pi$

Thus we get:

$$\lim_{\varepsilon \searrow 0} (\ln(\lambda - \mu + i\varepsilon) - \ln(\lambda - \mu - i\varepsilon)) = \begin{cases} 0 & \text{if } \lambda - \mu > 0 \\ \pi i & \text{if } \lambda - \mu = 0 \\ 2\pi i & \text{if } \lambda - \mu < 0 \end{cases}$$

Then follows:

$$\phi(\mu) := \lim_{\varepsilon \searrow 0} \phi_\varepsilon(\mu) = \frac{-1}{2\pi i} \begin{cases} 0 & \text{if } \mu \notin [a, b] \\ -\pi i & \text{if } \mu \in \{a, b\} \\ -2\pi i & \text{if } \mu \in (a, b) \end{cases} = \begin{cases} 0 & \text{if } \mu \notin [a, b] \\ \frac{1}{2} & \text{if } \mu \in \{a, b\} \\ 1 & \text{if } \mu \in (a, b) \end{cases}$$

Thus $\phi_\varepsilon(\mu) \rightarrow \phi(\mu)$ converges point-wise.

Idea:

$$\phi_\varepsilon(A) \rightarrow \phi(A) = \frac{1}{2} (E_{[a,b]} + E_{(a,b)})$$

But how does this converge?

Consider weak convergence:

$$\langle u, \phi_\varepsilon(A) u \rangle = \int_{\mathbb{R}} \phi_\varepsilon(\mu) \underbrace{d\langle u, E_\mu u \rangle}_{=d\mu_u = d\mu_{u,u}}$$

$d\mu_u$ is a bounded regular real Borel measure. From $|\phi(\mu)| \leq 1$ follows for small enough $\varepsilon \in \mathbb{R}_{>0}$ now $|\phi_\varepsilon(\mu)| \leq 2$. Because our Borel measure is bounded, 2 is an integrable function, i.e. $2 \in L^1(\mathbb{R}, d\mu_u)$. Therefore we can use the bounded convergence theorem to get:

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \phi_\varepsilon(\mu) d\langle u, E_\mu u \rangle = \int_{\mathbb{R}} \phi(\mu) d\langle u, E_\mu u \rangle = \langle u, \phi_u(A) u \rangle$$

What about strong convergence?

We want to show for all $u \in H$ the convergence $\phi_\varepsilon(A) u \rightarrow \phi(A) u$ in H , or equivalently:

$$\begin{aligned} & (\phi_\varepsilon - \phi)(A) u \rightarrow 0 \\ \Leftrightarrow & \|(\phi_\varepsilon - \phi)(A) u\| \rightarrow 0 \end{aligned}$$

$$\|(\phi_\varepsilon - \phi)(A) u\|^2 = \langle (\phi_\varepsilon - \phi)(A) u, (\phi_\varepsilon - \phi)(A) u \rangle = \langle u, ((\phi_\varepsilon - \phi)(A))^* (\phi_\varepsilon - \phi)(A) u \rangle =$$

$$\begin{aligned}
&= \langle u, (\overline{\phi_\varepsilon} - \overline{\phi})(A) (\phi_\varepsilon - \phi)(A) u \rangle = \langle u, |\phi_\varepsilon - \phi|^2(A) u \rangle = \\
&= \int_{\mathbb{R}} \underbrace{|\phi_\varepsilon - \phi|^2(\mu)}_{\rightarrow 0 \text{ point-wise}} \underbrace{d\langle u, E_\mu u \rangle}_{\substack{\text{point-wise regular} \\ \text{Borel measure}}} \xrightarrow[\text{dominated convergence}]{\varepsilon \searrow 0} 0
\end{aligned}$$

Therefore it converges strongly.

□_{7.6.1}

8 Spectral Theorem for bounded normal operators

$A \in L(H)$ is normal if it commutes with its adjoint, i.e. $[A, A^*] = 0$. Before we considered symmetric $A \in L(H)$. Then for a complex valued function f the operator $f(A)$ is normal, but in general not symmetric, because:

$$(f(A))^* = \overline{f}(A) \stackrel{\text{in general}}{\neq} f(A)$$

$$f(A) \cdot (f(A))^* = (f \cdot \overline{f})(A) = (\overline{f} \cdot f)(A) = (f(A))^* \cdot f(A)$$

The basic idea is:

$$\frac{1}{2}(A + A^*) =: B \qquad \frac{1}{2i}(A - A^*) =: C$$

$A = B + iC$, B and C are symmetric and $[B, C] = 0$.

8.1 Theorem

Let H be a complex separable Hilbert space, $A_i \in L(H)$ for $i \in \{1, \dots, n\}$ be symmetric operators, which commute pair wise, i.e. $[A_i, A_j] = 0$ for all $i, j \in \{1, \dots, n\}$ and

$$K := \prod_{i=1}^n \underbrace{[-\|A_i\|, \|A_i\|]}_{\supseteq \sigma(A_i)} \subseteq \mathbb{R}^n$$

be compact. Then there is a mapping

$$\Phi : C^0(K, \mathbb{C}) \rightarrow L(H)$$

(notation: $\Phi(f) = f(A_1, \dots, A_n)$) with the following properties:

- i) Φ is an involutive algebra homomorphism.
- ii) $\|\Phi(f)\| \leq \|f\|_\infty = \sup_K |f|$
- iii) $\Phi(\text{pr}_i) = A_i$ for the projection maps:

$$\begin{aligned} \text{pr}_i : \mathbb{R}^n &\rightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto x_i \end{aligned}$$

Proof

Let E_i be the spectral measure of the operator A_i .

$$E_i(M) = \chi_M(A_i)$$

Let $M \subseteq K$ be a cube, i.e. $M = M_1 \times \dots \times M_n$. Define:

$$\chi_M(A_1, \dots, A_n) := \chi_{M_1}(A_1) \cdot \dots \cdot \chi_{M_n}(A_n)$$

– Now holds $[\chi_{M_i}(A_i), \chi_{M_j}(A_j)] = 0$, because from

$$[A_i, A_j] = 0$$

follows via induction for any polynomials p, q :

$$[p(A_i), q(A_j)] = 0$$

With the Stone-Weierstraß and the Riesz representation theorem follows for all Borel functions $f, g \in \mathcal{B}(\mathbb{R})$:

$$[f(A_i), g(A_j)] = 0$$

– $\chi_M(A_1, \dots, A_n)$ is a projection operator.

$$\begin{aligned} (\chi_M(A_1, \dots, A_n))^* &= \overline{\chi_{M_n}(A_n)} \cdot \dots \cdot \overline{\chi_{M_1}(A_1)} = \\ &= \chi_{M_1}(A_1) \cdot \dots \cdot \chi_{M_n}(A_n) = \chi_M(A_1, \dots, A_n) \end{aligned}$$

$$\begin{aligned} \chi_M(A_1, \dots, A_n) \cdot \chi_{M'}(A_1, \dots, A_n) &= \\ &= \chi_{M_1}(A_1) \cdot \dots \cdot \chi_{M_n}(A_n) \cdot \chi_{M'_1}(A_1) \cdot \dots \cdot \chi_{M'_n}(A_n) = \\ &= \chi_{M_1 \cap M'_1}(A_1) \cdot \dots \cdot \chi_{M_n \cap M'_n}(A_n) = \chi_{M \cap M'}(A_1, \dots, A_n) \end{aligned}$$

– Let $f = \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}$ be a step function, meaning that the M_{α} are disjoint cubes and $a_{\alpha} \in \mathbb{C}$. Define:

$$\Phi(f) = \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}(A_1, \dots, A_n)$$

Claim: This definition is well-defined, i.e. it does not depend on the decomposition of f into cells.

Proof: Suppose we have:

$$f = \sum_{\alpha=1}^N a_{\alpha} \chi_{M_{\alpha}} = \sum_{\beta=1}^{\tilde{N}} \tilde{a}_{\beta} \chi_{\tilde{M}_{\beta}}$$

Choose a joint refinement. In fact, it suffices to consider the case that \tilde{M}_{β} is already a refinement of M_{α} . Thus $M_{\alpha} = \dot{\bigcup}_{\beta \in I_{\alpha}} M_{\beta}$ and the I_{α} form a partition of $\{1, \dots, \tilde{N}\}$. Using the properties of the E_i , a direct computation gives:

$$\chi_{M_{\alpha}} = \sum_{\beta \in I_{\alpha}} \chi_{\tilde{M}_{\beta}}$$

Substitute this in the formula for f and reorder the sums, to the that the definition is well-defined. \square_{Claim}

- Verify the properties i) and ii) for step functions: By direct computation follows:

$$(\Phi(f))^* = \Phi(\bar{f})$$

$$\Phi(f) \cdot \Phi(g) = \left(\sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}} \right) \left(\sum_{\beta} a_{\beta} \chi_{M_{\beta}} \right) \stackrel{\text{as above}}{=} \sum_{\alpha, \beta} a_{\alpha} a_{\beta} \chi_{M_{\alpha} \cap M_{\beta}} = \Phi(f \cdot g)$$

$$\|\Phi(f)\| = \left\| \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}} \right\| \leq \left(\max_{\alpha} |a_{\alpha}| \right) \cdot \underbrace{\left\| \sum_{\alpha} \chi_{M_{\alpha}} \right\|}_{\leq 1} \leq \|f\|_{\infty}$$

- Now consider $f \in C^0(K, \mathbb{C})$. There is a sequence of step functions f_k such that $f_k \rightrightarrows f$ converges uniformly.

$$\|\Phi(f_k) - \Phi(f_l)\| = \Phi(f_k - f_l) \stackrel{\text{ii)}}{\leq} \sup |f_k - f_l| \xrightarrow{k, l \rightarrow \infty} 0$$

Since H is complete, $\Phi(f_k)$ converges in $L(H)$ and we define $\Phi(f) := \lim_{k \rightarrow \infty} \Phi(f_k)$. Then the properties i) and ii) remain true by continuity.

- Compute $\Phi(\text{pr}_i)$. For this let $f_k = \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}$ be a step function with $f_k(x) \rightrightarrows x$ and set $\text{pr}_i^k(x) = f_k(x_i)$, which implies $\text{pr}_i^k \rightrightarrows \text{pr}_i$.

$$\begin{aligned} \Phi(\text{pr}_i^k) &= \sum_{\alpha} a_{\alpha} \chi_{\mathbb{R} \times \dots \times \underbrace{M_{\alpha}}_{i\text{-th position}} \times \dots \times \mathbb{R}}(A_1, \dots, A_n) = \\ &= \prod_{j \neq i} \underbrace{\chi_{\mathbb{R}}(A_j)}_{=1} \cdot \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}(A_i) = \chi_{f_k}(A_i) \xrightarrow{\text{in } L(H)} A_i \end{aligned}$$

□_{8.1}

We know $\text{supp}(\chi(A_j)) = \sigma(A_j) \subseteq [-\|A_j\|, \|A_j\|]$.

$$K := [-\|A_1\|, \|A_1\|] \times [-\|A_2\|, \|A_2\|] \subseteq \mathbb{C}$$

Goal: Construct a spectral measure $\chi_M(A_1, A_2)$ on K .

- For $M = M_1 \times M_2$ (“cubes”) we set:

$$\chi_{M_1 \times M_2}(A_1, A_2) = \chi_{M_1}(A_1) \cdot \chi_{M_2}(A_2)$$

- For step functions

$$f = \sum_{\alpha=1}^N c_{\alpha} \chi_{M_1^{\alpha} \times M_2^{\alpha}}$$

we set:

$$\Phi(f) = \sum_{\alpha=1}^N c_{\alpha} \chi_{M_1^{\alpha} \times M_2^{\alpha}}(A_1, A_2)$$

- For $f \in C^0(K)$ we choose step functions f_n such that $f_n \rightrightarrows f$ converges on K .

$$\Phi(f) := \lim_{n \rightarrow \infty} \Phi(f_n)$$

This convergence is in $L(H)$.

8.2 Theorem

Now let $A \in L(H)$ be normal, i.e. $[A, A^*] = 0$, and define the symmetric bounded operators:

$$A_1 := \frac{1}{2}(A + A^*) \quad A_2 := \frac{1}{2i}(A - A^*)$$

Then follows $A = A_1 + iA_2$ and $[A_1, A_2] = 0$, which implies $[\chi_{M_1}(A_1), \chi_{M_2}(A_2)] = 0$ for all sets $M_1, M_2 \subseteq \mathbb{R}$.

$$K := [-\|A_1\|, \|A_1\|] \times [-\|A_2\|, \|A_2\|] \subseteq \mathbb{C}$$

Then there exists exactly one map

$$\Phi : C^0(K, \mathbb{C}) \rightarrow L(H)$$

with the following properties:

- i) Φ is an involutive algebra homomorphism.
- ii) $\|\Phi(f)\| \leq \|f\|_\infty$
- iii) $f(z) = z$ for $z \in K$ already implies $\Phi(f) = A$.
- iv) $Au = \lambda u$ implies $\Phi(f)u = f(\lambda)u$
- v) If f is real-valued, then $\Phi(f)$ is symmetric.
- vi) $f \geq 0$ implies $\Phi(f) \geq 0$.
- vii) For a $T \in L(H)$ with $[T, A] = [T, A^*] = 0$ follows for all $f \in C^0$:

$$[T, \Phi(f)] = 0$$

Proof

$$\begin{aligned} \text{pr}_1(x_1, x_2) &= x_1 \\ \Phi(\text{pr}_1) &= A_1 \end{aligned}$$

Choose step functions f_n of one variable, such that $f_n(x) \rightarrow x$ on $[-\|A_1\|, \|A_1\|]$. Then the functions

$$g_n(x_1, x_2) := f_n(x_1)$$

converge uniformly to pr_1 on K .

$$\begin{aligned} \Phi(g_n) &= \sum_{\alpha=1}^N c_\alpha \underbrace{\chi_{M_1^\alpha \times [-\|A_2\|, \|A_2\|]}}_{= \chi_{M_1^\alpha}(A_1) \cdot \underbrace{\chi_{[-\|A_2\|, \|A_2\|]}(A_2)}_{=1}} = \sum_{\alpha=1}^N c_\alpha \chi_{M_1^\alpha}(A_1) = f_n(A_1) \rightarrow A_1 \end{aligned}$$

This converges follows from the functional calculus for a *symmetric operator*.

Choose Φ as in Theorem 8.1 for the commuting operators A_1 and A_2 . Then i), ii) and v) follow immediately.

vi) For $f \geq 0$ there exists a $g \in C^0(K, \mathbb{R})$ with $f = g^2$.

$$\langle u, \phi(f) u \rangle = \langle u, \phi(g) \cdot \phi(g) u \rangle = \langle \phi(g) u, \phi(g) u \rangle \geq 0$$

vii) From $[T, A_1] = 0 = [T, A_2]$ follows:

$$[T, \chi_M(A_1)] = 0 = [T, \chi_M(A_2)]$$

This gives by approximation

$$[T, \chi_M(A_1, A_2)] = 0$$

for all $M \subseteq \mathbb{R}^2 \cong \mathbb{C}$.

iii) From $f(z) = z$ follows $\Phi(f) = A$.

$$\begin{aligned} z &= x_1 + \mathbf{i}x_2 \\ f(x_1, x_2) &= x_1 + \mathbf{i}x_2 \end{aligned}$$

$$\Rightarrow \quad \Phi(f) = \Phi(\text{pr}_1) + \mathbf{i}\Phi(\text{pr}_2) = A_1 + \mathbf{i}A_2 = A$$

iv) We want to show $Au = \lambda u$ implies $\Phi(f)u = f(\lambda)u$. Consider $u \in H$ with $Au = \lambda u$.

Claim: $A^*u = \bar{\lambda}u$

Proof: It holds:

$$A(A^*u) = A^*Au = A^*\lambda u = \lambda(A^*u)$$

Thus A^* maps the eigenspace $\ker(A - \lambda)$ to itself, which implies:

$$A^*u - \bar{\lambda}u \in \ker(A - \lambda)$$

For $v \in \ker(A - \lambda)$ holds:

$$\langle v, (A^* - \bar{\lambda})u \rangle = \langle (A - \lambda)v, u \rangle = 0$$

Thus we get $(A^* - \bar{\lambda})u \in \ker(A - \lambda) \cap (\ker(A - \lambda))^\perp = \{0\}$. Now we have:

$$(A^* - \bar{\lambda})u = 0$$

□_{Claim}

So we have:

$$A_1u = \lambda_1u \quad A_2u = \lambda_2u \quad \lambda = \lambda_1 + \mathbf{i}\lambda_2$$

So $\Phi(p)u = p(\lambda)u$ holds for all polynomials p . The Stone-Weierstraß theorem in two dimensions gives the result.

□_{8.2}

Now apply the Riesz representation theorem to extend the functional calculus to bounded Borel functions $\mathcal{B}(K)$.

8.3 Theorem

Let $A \in L(H)$ be normal. Then there exists a map

$$\Phi : \mathcal{B}(K, \mathbb{C}) \rightarrow L(H)$$

with the following properties:

- i) Φ is an involutive algebra homomorphism.
- ii) $\|\Phi(f)\| \leq \|f\|_{L^\infty(K)}$
- iii) For $f \in C^0$, $\Phi(f)$ coincides with the continuous functional calculus.
- iv) For point-wise converging $f_n \rightarrow f$ with $\|f\|_\infty < C$ converges $\Phi(f_n) \rightarrow \Phi(f)$ strongly.
- v) $Au = \lambda u$ implies $\Phi(f)u = f(\lambda)u$
- vi) If f is real-valued, then $\Phi(f)$ is symmetric.
 $f \geq 0$ and implies $\Phi(f) \geq 0$.
- vii) For a $T \in L(H)$ with $[T, A] = [T, A^*] = 0$ follows for all $f \in C^0$:

$$[T, \Phi(f)] = 0$$

Proof

The proof is the same as for the symmetric case. □_{8.3}

8.4 Theorem (spectral theorem for bounded normal operators)

There is a one-to-one correspondence between bounded normal operators on H and compact spectral measures via:

$$A = \int_{\mathbb{R}^2 \cong \mathbb{C}} \lambda dE_\lambda$$

Moreover holds:

- i) $f(A) = \Phi(f) = \int_{\mathbb{R}^2} f(\lambda) dE_\lambda$
- ii) $\sigma(A) = \text{supp}(E) \subseteq \mathbb{R}^2 \cong \mathbb{C}$

Proof

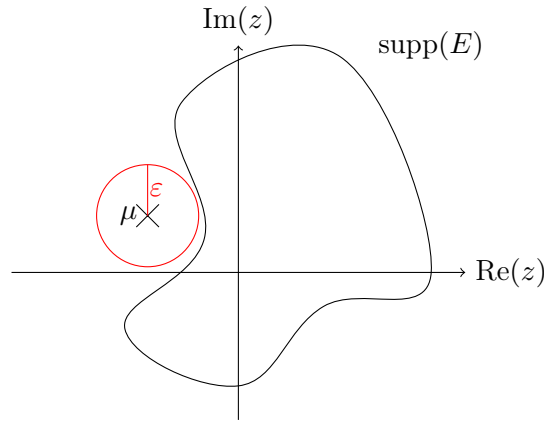
The proof is just as in the symmetric case, except for the property ii).

„ $\text{supp}(E) \supseteq \sigma(A)$ “: Consider $\mu \notin \text{supp}(E)$. Then

$$g(\lambda) := \frac{1}{\lambda - \mu} \cdot \chi_{\text{supp}(E)}$$

is a bounded Borel function, since $|g(\lambda)| \leq \frac{1}{\varepsilon}$, where $B_\varepsilon(\mu) \cap \text{supp}(E) = \emptyset$ and:

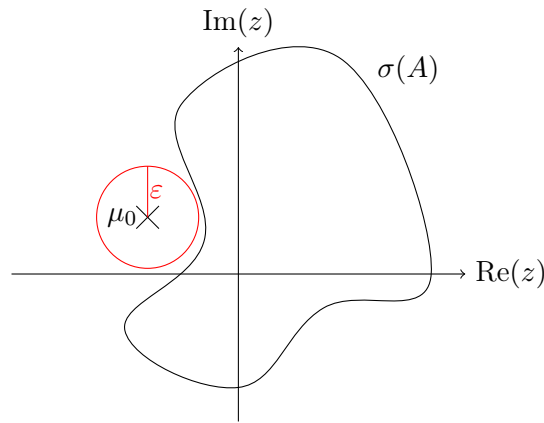
$$g(A) \cdot (A - \mu) = \int_{\mathbb{R}^2} \frac{\lambda - \mu}{\lambda - \mu} dE_\lambda = \mathbb{1}$$

Figure 8.1: $B_\varepsilon(\mu) \cap \text{supp}(E) = \emptyset$

Hence $(A - \lambda)$ has a bounded inverse and therefore $\lambda \notin \sigma(A)$.

„ $\text{supp}(E) \subseteq \sigma(A)$ “: For $\mu_0 \in \varrho(A)$ we show $\mu_0 \notin \text{supp}(E)$.

Since $\varrho(A)$ is open, there exists a $\varepsilon \in \mathbb{R}_{>0}$ with $B_\varepsilon(\mu_0) \subseteq \varrho(A)$.

Figure 8.2: $B_\varepsilon(\mu_0) \subseteq \varrho(A)$

Lemma 8.5 states: Let $B \in L(H)$ be an operator with bounded inverse and $B_n \in L(H)$ a sequence with $B_n \rightarrow B$ in $L(H)$. Then B_n^{-1} exists for large enough n and $B_n^{-1} \rightarrow B^{-1}$ converges in $L(H)$.

In particular, for $\mu_n \xrightarrow{n \rightarrow \infty} \mu_0$ converges also $(A - \mu_n)^{-1} \rightarrow (A - \mu_0)^{-1}$ in $L(H)$.

Consider now $\mu \in B_r(\mu_0)$ for any $r \in \mathbb{R}_{>0}$ and define:

$$B := (A - \mu) \cdot (A^* - \bar{\mu}) = \int |\lambda - \mu|^2 dE_\lambda$$

Now choose a $\delta \in \mathbb{R}_{>0}$ to get:

$$\begin{aligned} B + \delta &= \int (|\lambda - \mu|^2 + \delta) dE_\lambda \\ \Rightarrow (B + \delta)^{-1} &= \int \frac{1}{|\lambda - \mu|^2 + \delta} dE_\lambda \in L(H) \end{aligned}$$

Similarly follows:

$$B^p = \int |\lambda - \mu|^{2p} dE_\lambda$$

$$(B + \delta)^{-p} = \int \left(|\lambda - \mu|^2 + \delta \right)^{-p} dE_\lambda$$

For $u \in H$ with $\|u\| = 1$ holds:

$$\langle u, (B + \delta)^{-p} u \rangle = \int_{\mathbb{R}^2} \frac{1}{\left(|\lambda - \mu|^2 + \delta \right)^p} d\langle u, E_\lambda u \rangle$$

$d\langle u, E_\lambda u \rangle$ is a point-wise bounded Borel measure.

$$\begin{aligned} |\langle u, (B + \delta)^{-p} u \rangle| &\leq \underbrace{\|u\|^2}_{=1} \cdot \left\| (B + \delta)^{-1} (B + \delta)^{-(p-1)} \right\| \leq \\ &\leq \dots \leq \left\| (B + \delta)^{-1} \right\|^p \stackrel{\text{choose } r < \varepsilon}{\leq} \left\| B^{-1} \right\|^p \\ \Rightarrow \quad \liminf_{\delta} |\langle u, (B + \delta)^{-p} u \rangle| &\leq \left\| B^{-1} \right\|^p \end{aligned}$$

Remember Fatou's lemma:

$$\int \liminf_{\delta} f_{\delta} \leq \liminf_{\delta} \int f_{\delta}$$

holds if $\lim_{\delta \searrow 0} f_{\delta}$ exists point-wise. (cf. RUDIN: *Real and complex analysis*)

Applying Fatou's lemma gives:

$$\begin{aligned} \int_{\mathbb{R}^2} \liminf_{\delta} \frac{1}{\left(|\lambda - \mu|^2 + \delta \right)^p} d\langle u, E_\lambda u \rangle &= \int_{\mathbb{R}^2} \frac{1}{|\lambda - \mu|^{2p}} d\langle u, E_\lambda u \rangle \leq \\ &\leq \liminf_{\delta} \int_{\mathbb{R}^2} \frac{1}{\left(|\lambda - \mu|^2 + \delta \right)^p} d\langle u, E_\lambda u \rangle \leq \left\| B^{-1} \right\|^p \end{aligned}$$

Thus we get:

$$\left(\int \frac{1}{|\lambda - \mu|^{2p}} d\langle u, E_\lambda u \rangle \right)^{\frac{1}{p}} \leq \left\| B^{-1} \right\|$$

In other words, setting $g(\lambda) = \frac{1}{|\lambda - \mu|^2}$, we know for all $p \in \mathbb{N}_{\geq 1}$ and all $\mu \in B_{\frac{\varepsilon}{2}}(\mu_0)$:

$$\|g\|_{L^p(d\langle u, E_\lambda u \rangle)} \leq \left\| B^{-1} \right\|$$

This implies that there exists an $\varepsilon' \in \mathbb{R}_{>0}$ such that $B_{\varepsilon'}(\mu_0)$ is a set with measure zero with respect to $d\langle u, E_\lambda u \rangle$, since otherwise:

$$\left(\int \frac{1}{|\lambda - \mu|^{2p}} d\langle u, E_\lambda u \rangle \right)^{\frac{1}{p}} \geq \inf_{\lambda \in B_{\varepsilon'}(\mu_0)} \frac{1}{|\lambda - \mu|^2} \cdot \underbrace{\left(\langle u, dE_{B_{\varepsilon'}(\mu_0)} u \rangle \right)^{\frac{1}{p}}}_{>0} \xrightarrow{p \rightarrow \infty} \inf_{\lambda \in B_{\varepsilon'}(\mu_0)} \frac{1}{|\mu - \lambda|^2}$$

Since u is arbitrary (and ε' can be chosen uniformly in u) it follows that $E_{B_{\varepsilon'}(\mu_0)} = 0$ and thus $\mu_0 \notin \text{supp}(E)$. $\square_{8.4}$

8.5 Lemma

Let $B \in L(H)$ be an operator with bounded inverse and $B_n \in L(H)$ a sequence with $B_n \rightarrow B$ in $L(H)$. Then B_n^{-1} exists for large enough n and $B_n^{-1} \rightarrow B^{-1}$ converges in $L(H)$.

Proof

Use the Neumann series:

$$B_n^{-1} = (B + (B_n - B))^{-1} = (\mathbb{1} + B^{-1}(B_n - B))^{-1} B^{-1} = \sum_{k=0}^{\infty} (-B^{-1}(B_n - B))^k B^{-1}$$

This converges absolutely, if $\|B_n - B\|$ is sufficiently small. Therefore holds:

$$\|B_n^{-1} - B^{-1}\| \leq \sum_{k=1}^{\infty} \|B^{-1}\|^{k+1} \cdot \|B_n - B\|^k \xrightarrow{\|B_n - B\| \rightarrow 0} 0$$

□ Lemma

8.6 Theorem

Let $A \in L(H)$ be normal and E the corresponding spectral measure. Then holds for all $\varepsilon \in \mathbb{R}_{>0}$:

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad E_{B_\varepsilon(\lambda)} \neq 0$$

Proof

$$\lambda \in \sigma(A) \quad \Leftrightarrow \quad \lambda \in \text{supp}(E) \quad \xrightarrow{\text{definition of } \text{supp}(E)} \quad E_{B_\varepsilon(\lambda)} \neq 0$$

□_{8.6}

8.7 Theorem (spectral mapping theorem for normal operators)

Let $A \in L(H)$ be normal and $f \in C^0(\sigma(A), \mathbb{C})$. Then $\sigma(f(A)) = f(\sigma(A))$.

Note: This is not true in general for $f \in \mathcal{B}(\sigma(A), \mathbb{C})$.

Proof

- i) „ $\sigma(f(A)) \subseteq f(\sigma(A))$ “: Since $\sigma(A)$ is compact and f continuous and therefore maps compact sets to compact sets, follows:

$$f(\sigma(A)) = \overline{f(\sigma(A))}$$

We show more generally:

$$\sigma(f(A)) \subseteq \overline{f(\sigma(A))}$$

for any Borel function $f \in \mathcal{B}(\sigma(A))$. Consider $\mu \notin \overline{f(\sigma(A))}$ and set:

$$g(\lambda) = \frac{1}{f(\lambda) - \mu} \cdot \chi_{\sigma(A)}$$

This is a bounded Borel function. Thus follows:

$$g(A) \cdot (f(A) - \mu) = \int_{\mathbb{R}^2} \frac{f(\lambda) - \mu}{f(\lambda) - \mu} \chi_{\sigma(A)} dE_{\lambda} \stackrel{\sigma(A) = \text{supp}(E)}{=} \mathbb{I}$$

Hence $f(A) - \mu$ has a bounded inverse $g(A)$ and thus $\mu \in \varrho(f(A))$, i.e. $\mu \notin \sigma(f(A))$. $\square_{i)}$

ii) „ $f(\sigma(A)) \subseteq \sigma(f(A))$ “: Consider $\mu \in \sigma(A)$ and show $f(\mu) \in \sigma(f(A))$.

From $\sigma(A) = \text{supp}(E)$ follows for all $\varepsilon \in \mathbb{R}_{>0}$:

$$E_{B_{\varepsilon}(\mu)} \neq 0$$

Thus we may choose $u \neq 0$ with:

$$E_{B_{\varepsilon}(\mu)} u = u$$

Then holds:

$$\begin{aligned} \|(f(A) - f(\mu))u\|^2 &= \langle (f(A) - f(\mu))u, (f(A) - f(\mu))u \rangle = \\ &= \langle u, (\overline{f(A)} - \overline{f(\mu)}) (f(A) - f(\mu))u \rangle = \\ &= \int_{\mathbb{R}^2} |f(\lambda) - f(\mu)|^2 d\langle u, E_{\lambda} u \rangle = \\ &= \int_{\mathbb{R}^2} |f(\lambda) - f(\mu)|^2 d\langle E_{B_{\varepsilon}(\mu)} u, E_{\lambda} E_{B_{\varepsilon}(\mu)} u \rangle = \\ &= \int_{B_{\varepsilon}(\mu)} |f(\lambda) - f(\mu)|^2 d\langle u, E_{\lambda} u \rangle \leq \\ &\leq \sup_{B_{\varepsilon}(\mu)} |f(\lambda) - f(\mu)|^2 \int_{\mathbb{R}^2} d\langle u, E_{\lambda} u \rangle \leq \\ &\leq \sup_{B_{\varepsilon}(\mu)} |f(\lambda) - f(\mu)|^2 \|u\|^2 \end{aligned}$$

Since f is continuous, there exists a sequence $u_n \in H$ with $\|u_n\| = 1$ such that holds:

$$\|(f(A) - f(\mu))u_n\| \rightarrow 0$$

Hence $f(A) - f(\mu)$ has no bounded inverse and therefore follows $\mu \in \sigma(f(A))$.

$\square_{8.7}$

8.8 Corollary

For a normal $A \in L(H)$ and a $f \in C^0(\sigma(A))$ holds:

$$\|f(A)\| = \|f\|_{L^{\infty}(\sigma(A))}$$

Proof

From $(f(A))^* = \overline{f}(A)$ follows:

$$(f(A))^* f(A) = |f|^2(A) = f(A) (f(A))^*$$

Hence the operator $f(A)$ is normal.

$$\begin{aligned} \|f(A)\| &= r(f(A)) = \sup \{ |\mu| \mid \mu \in \sigma(f(A)) \} = \\ &= \sup \{ |\mu| \mid \mu \in f(\sigma(A)) \} = \sup \{ |f(\lambda)| \mid \lambda \in \sigma(A) \} = \|f\|_{L^\infty(\sigma(A))} \end{aligned}$$

□_{8.8}

Thus the mapping

$$\Phi : C^0(\sigma(A), \mathbb{C}) \rightarrow L(H)$$

is preserving the norm. Be careful to remember that

$$\Phi : C^0(\mathbb{R}^2, \mathbb{C}) \rightarrow L(H)$$

is *not* preserving the norm. Instead holds:

$$\|f(A)\| \leq \|f\|_{L^\infty(\mathbb{R})}$$

8.9 Theorem

Let $A \in L(H)$ be normal and E the corresponding spectral measure. Then μ is an eigenvalue of A if and only if $E_{\{\mu\}} \neq 0$.

Proof

„ \Leftarrow “: Assume that $E_{\{\mu\}} \neq 0$. Now choose a vector $u \neq 0$ with $E_{\{\mu\}}u = u$. Then holds:

$$\begin{aligned} \|(A - \mu)u\|^2 &= \int_{\mathbb{R}^2} |\lambda - \mu|^2 d\langle u, E_\lambda u \rangle = \int_{\mathbb{R}^2} |\lambda - \mu|^2 d\langle E_{\{\mu\}}u, E_\lambda E_{\{\mu\}}u \rangle = \\ &= \int_{\mathbb{R}^2} \underbrace{|\lambda - \mu|^2 \chi_{\{\mu\}}(\lambda)}_{=0} d\langle u, E_\lambda u \rangle = 0 \end{aligned}$$

„ \Rightarrow “: Let u be an eigenvector.

$$Au = \mu u$$

Then holds for all $f \in \mathcal{B}(\mathbb{R}^2)$ after theorem 8.3 v):

$$f(A)u = f(\mu)u$$

Choose $f = \chi_{\{\mu\}}$ to get:

$$f(A) = \chi_{\{\mu\}}(A) = E_{\{\mu\}}$$

$$\Rightarrow E_{\{\mu\}}u = u$$

Hence follows $E_{\{\mu\}} \neq 0$.

□_{8.9}

9 Cyclic vectors, the spectral theorem in its multiplicative form

Let $A \in L(H)$ be normal.

9.1 Definition (cyclic vector)

A vector $u \in H$ is called *cyclic* (with respect to A) if holds:

$$\overline{\{f(A)u \mid f \in C^0(\sigma(A), \mathbb{C})\}} = H$$

9.2 Theorem

Let $u \in H$ be a cyclic vector. Then there exists a unitary operator

$$\mathcal{U} : H \rightarrow L^2(\sigma(A), \underbrace{d\langle u, E_\lambda u \rangle}_{=d\mu_u})$$

such that for $f \in L^2(\sigma(A), d\langle u, E_\lambda u \rangle)$ and $g(\lambda) = \lambda$ holds:

$$\mathcal{U}A\mathcal{U}^{-1}f = g \cdot f$$

Proof

$$\alpha(f(A)u) + \beta(g(A)u) = (\alpha f + \beta g)(A)u$$

$$\Rightarrow I_u := \{f(A)u \mid f \in C^0(\sigma(A), \mathbb{C})\} = \langle f(A)u \mid f \in C^0(\sigma(A), \mathbb{C}) \rangle$$

By assumption, I_u is dense in H . Define

$$\mathcal{U} : I_u \rightarrow L^2(\sigma(A), d\mu_u)$$

by:

$$\mathcal{U}(f(A)u) = f$$

This is well-defined and an isometry, because:

$$\langle f(A)u, f(A)u \rangle = \int |f(\lambda)|^2 \underbrace{d\langle u, E_\lambda u \rangle}_{=d\mu_u} = \langle f, f \rangle_{L^2(\sigma(A), d\mu_u)}$$

Moreover, the image of \mathcal{U} is $C^0(\sigma(A), \mathbb{C})$ and this is dense in $L^2(\sigma(A), d\mu_u)$. Therefore \mathcal{U} can be uniquely extended by continuity to an unitary operator:

$$\mathcal{U} : H = \overline{I_u} \rightarrow \overline{C^0(\sigma(A), \mathbb{C})} = L^2(\sigma(A), d\mu_u)$$

Compute now $\mathcal{U}A\mathcal{U}^{-1}$:

$$\mathcal{U}(f(A)u) = f$$

$$\mathcal{U}A\mathcal{U}^{-1}f = \mathcal{U} \underbrace{A}_{=g(A)}(f(A)u) = \mathcal{U}((g \cdot f)(A)u) = g \cdot f$$

Using a density argument one shows that this holds for any $f \in L^2$. □_{9.2}

9.3 Examples

1. Let H be finite-dimensional and A symmetric with simple eigenvalues $\lambda_1, \dots, \lambda_n$. In an eigenvector basis holds:

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

For $v = (1, 0, \dots, 0)^T$ follows:

$$f(A)v = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} v = f(\lambda_1)v$$

Therefore this v is not cyclic. Choose $u = (1, \dots, 1)^T$ to get:

$$f(A)u = \begin{pmatrix} f(\lambda_1) \\ \vdots \\ f(\lambda_n) \end{pmatrix}$$

Since $\lambda_i \neq \lambda_j$ holds for $i \neq j$, there are $f_i \in C^0(\sigma(A))$ such that $f_i(\lambda_i) = 1$ and $f_i(\lambda_j) = 0$ for $i \neq j$. With this holds $f_i(A)u = e_i$. Therefore holds:

$$\{f(A)u \mid f \in C^0\} = H$$

2. Let A be as in 1., but with the degeneracy $\lambda_1 = \lambda_2$ and $u = (u_1, \dots, u_n)^T$. Then follows

$$f(A)u = \begin{pmatrix} f(\lambda_1)u_1 \\ \vdots \\ f(\lambda_n)u_n \end{pmatrix}$$

and the vector $v = (v_1, v_2, 0, \dots, 0)^T$ with

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \nparallel \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is not in:

$$\{f(A)u \mid f \in C^0\}$$

Hence there is no cyclic vector.

Question: What can we do if there is a cyclic vector?

9.4 Lemma

Let $A \in L(H)$ be normal and A symmetric. Then there exists an orthogonal decomposition

$$H = \bigoplus_{j \in J} H_j$$

with a finite or countable J and to every $j \in J$ there is a cyclic vector $u_j \in H_j$, i.e.:

$$H_j = \overline{\{f(A)u_j \mid f \in C^0(\sigma(A), \mathbb{C})\}}$$

Proof

Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal Hilbert basis. Choose $u_1 = e_1$ and define:

$$H_1 := \overline{\{f(A)u_1 \mid f \in C^0\}} \subseteq H$$

If $H_1 = H$, we are done. Otherwise, let $i_0 \in \mathbb{N}$ be the smallest number with $e_{i_0} \notin H_1$ and set:

$$u_2 := e_{i_0} - \text{pr}_{H_1}(e_{i_0}) = \text{pr}_{H_1^\perp}(e_{i_0})$$

$$H_2 := \overline{\{f(A)u_2 \mid f \in C^0\}} \subseteq H$$

For $H = \langle H_1, H_2 \rangle$ we stop the procedure. Otherwise choose i_1 as the smallest number such that $e_{i_1} \notin \langle H_1, H_2 \rangle$, and so on.

Proceeding inductively, we obtain that $J = \{i_k \mid k \in \mathbb{N}\}$ is finite or countable and for $j \in J$ we have:

$$H_j = \overline{\{f(A)u_j \mid f \in C^0\}}$$

– $H_i \perp H_j$ for $i \neq j$:

$$\langle f(A)u_i, g(A)u_j \rangle = \langle \underbrace{(\bar{g} \cdot f)(A)u_i}_{\in H_i}, u_j \rangle \stackrel{u_j \in H_i^\perp}{=} 0$$

The result follows by using that $\{f(A)u_i\}$ and $\{g(A)u_j\}$ are dense in H_i respectively H_j .

– The H_i generate a dense subset of H : By construction we have:

$$e_{i_k} \in \langle H_1, H_2, \dots, H_{k+2} \rangle$$

Since $i_k \geq k$ holds, every basis vector e_i is contained in $\langle H_1, H_2, \dots, H_{i+2} \rangle$. Hence the algebraic span of the (e_i) is contained in the span of the $(H_i)_{i \in J}$.

□_{9.4}

9.5 Theorem (spectral theorem in its multiplicative form)

Let $A \in L(H)$ be normal and H separable. Then there is a σ -compact measure space Ω with a finite measure μ and a unitary operator

$$\mathcal{U} : H \rightarrow L^2(\Omega, d\mu)$$

such that for $g \in L^\infty(\Omega, \mu)$ holds:

$$\mathcal{U} A \mathcal{U}^{-1} f = g \cdot f$$

Proof

Choose an orthogonal decomposition

$$H = \bigoplus_{i \in J} H_i$$

with cyclic $u_i \in H_i$. The subspaces

$$H_i = \overline{\{f(A) u_i \mid f \in C^0\}}$$

are invariant under A , i.e. $A_i := A|_{H_i} : H_i \rightarrow H_i$. Now we rescale u_i to get $\|u_i\| = 2^{-i}$.

$$\begin{aligned} \mathcal{U}_i : H_i &\rightarrow L^2(\sigma(A), \underbrace{d\langle u_i, E_\lambda u_i \rangle}_{=d\mu_{u_i}}) \\ f(A) u_i &\mapsto f \end{aligned}$$

This is just as before in theorem 9.2 unitary and for $g_i(\lambda) = \lambda$ holds:

$$\mathcal{U}_i A_i \mathcal{U}_i^{-1} f_i = g_i f_i$$

Now define:

$$\Omega := \sigma(A) \times J \qquad \Omega_i = \sigma(A) \times \{i\}$$

Thus holds:

$$\Omega = \dot{\bigcup}_{i \in J} \Omega_i$$

Define a measure:

$$\begin{aligned} \mu : \Omega_i &\rightarrow \mathbb{R}_0^+ \\ \mu(U \times \{i\}) &:= \mu_{u_i}(U) \end{aligned}$$

Extend μ by σ -additivity to a unique measure on Ω . For $U \subseteq \Omega$ we write with appropriate $U_i \subseteq \Omega_i$:

$$U = \dot{\bigcup}_{i \in I} U_i$$

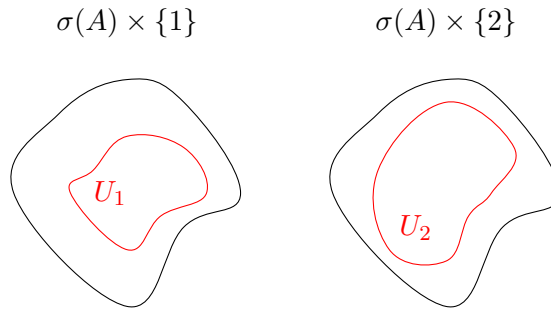


Figure 9.1: $U = \bigcup_{i \in I} U_i$

Define $\mu(U) := \sum_{i \in J} \mu(U_i)$.

$$\mu(\Omega_i) = \mu_{u_i}(\sigma(A)) = \langle u_i, \underbrace{E_{\sigma(A)}}_{=1} u_i \rangle = \|u_i\|^2 = 2^{-2i}$$

$$\mu(\Omega) = \sum_{i \in J} \mu(\Omega_i) = \sum_{i \in J} 2^{-2i} \leq 1$$

Thus μ is a bounded Borel measure.

$$\mathcal{U} := \bigoplus_{i \in J} \mathcal{U}_i : H \rightarrow L^2(\Omega, d\mu)$$

is unitary.

$$L^2(\Omega, d\mu) = \bigoplus_{i \in J} L^2(\Omega_i, d\mu_i)$$

$$\begin{array}{ccc} \mathcal{U} \uparrow & & \uparrow \mathcal{U}_i \\ H & = & \bigoplus_{i \in J} H_i \end{array}$$

$$(\mathcal{U} A \mathcal{U}^{-1}) f = \bigoplus_{i \in J} g_i \underbrace{f_i}_{\in L^2(\Omega_i, d\mu_i)}$$

Here $g_i(\{\lambda\} \times \{i\}) = \lambda$. Now

$$g := \bigoplus_{i \in J} g_i$$

is a bounded function:

$$\|g\|_{L^\infty} \leq \sup_{\lambda \in \sigma(A)} |\lambda| = r(A)$$

□_{9.5}

9.6 The pure point spectrum and the absolutely continuous spectrum

Let $A \in L(H)$ be symmetric and H separable. Then

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

gave the decomposition:

$$\sigma(A) = \sigma_{\text{disc}}(A) \dot{\cup} \sigma_{\text{ess}}(A)$$

The spectral theorem in its multiplicative form gives another decomposition of the spectrum. There exists a operator

$$\mathcal{U} : H \rightarrow L^2(\Omega, d\mu)$$

with $\mathcal{U}A\mathcal{U}^{-1}$ is the operator of multiplication by $g \in L^\infty(\Omega, d\mu)$ and $d\mu$ is a positive finite Borel measure on $\Omega = \sigma(A) \times J$. Since the spectrum is compact, it holds $\sigma(A) \subseteq [a, b] \subseteq \mathbb{R}$.

On Ω we also have the Lebesgue measure dx . According to the Raden-Nikodym theorem (that we use without proof), $d\mu$ can be decomposed as:

$$d\mu = d\mu_{\text{pp}} + d\mu_{\text{ac}} + d\mu_{\text{sing}}$$

$d\mu_{\text{pp}}$ is the *pure point*, $d\mu_{\text{ac}}$ the *absolutely continuous* and $d\mu_{\text{sing}}$ the *singular* measure. It holds

$$d\mu_{\text{ac}} = f(x) dx$$

for a $f \in L^2(\Omega, dx)$. $d\mu_{\text{pp}}$ is a weighted counting measure, i.e. there is a countable set K and $c_j \in \mathbb{R}_{>0}$ for $j \in K$ with:

$$\begin{aligned} d\mu_{\text{pp}}(\Omega) &= \sum_{j \in K} c_j \delta_{x_j} \\ \sum_{j \in K} c_j &< \infty \end{aligned}$$

This gives rise to a decomposition of the Hilbert spaces.

$$L^2(\Omega, d\mu) = L^2(\Omega, d\mu_{\text{pp}}) \oplus L^2(\Omega, d\mu_{\text{ac}}) \oplus L^2(\Omega, d\mu_{\text{sing}})$$

Applying \mathcal{U}^{-1} gives the corresponding decomposition:

$$H = H_{\text{pp}} + H_{\text{ac}} + H_{\text{sing}}$$

$$\begin{array}{ll} A|_{H_{\text{pp}}} : H_{\text{pp}} \rightarrow H_{\text{pp}} & \sigma_{\text{pp}}(A) := \sigma(A|_{H_{\text{pp}}}) \\ A|_{H_{\text{ac}}} : H_{\text{ac}} \rightarrow H_{\text{ac}} & \sigma_{\text{ac}}(A) := \sigma(A|_{H_{\text{ac}}}) \\ A|_{H_{\text{sing}}} : H_{\text{sing}} \rightarrow H_{\text{sing}} & \sigma_{\text{sing}}(A) := \sigma(A|_{H_{\text{sing}}}) \end{array}$$

10 The Spectral Theorem for Unbounded Self-Adjoint Operators

Let $A : \mathcal{D}(A) \rightarrow H$ be a densely defined linear operator with domain of definition $\mathcal{D}(A) \stackrel{\text{dense}}{\subseteq} H$.

Recall:

- A is *symmetric* if $\langle u, Av \rangle = \langle Au, v \rangle$ for all $u, v \in \mathcal{D}(A)$. (also called *formally self-adjoint*)
- A is *self-adjoint* if $A^* = A$, or equivalently:

$$\left(\forall_{u \in \mathcal{D}(A)} : \langle Au, v \rangle = \langle u, w \rangle \right) \quad \Rightarrow \quad ((w \in \mathcal{D}(A)) \wedge (Av = w))$$

10.1 Theorem (The basic criterion for self-adjointness)

Let A be a symmetric operator with dense domain of definition $\mathcal{D}(A)$. Then the following statements are equivalent.

- i) A is self-adjoint.
- ii) A is closed and $\ker(A^* \pm \mathbf{i}) = \{0\}$ (for $+$ and $-$).
- iii) $\text{im}(A \pm \mathbf{i}) = H$ (for $+$ and $-$)

Proof

„i) \Rightarrow ii)“: Let A be self-adjoint, i.e. $A = A^*$. Since A^* is always closed, it follows that A is closed. Let $\varphi \in \mathcal{D}(A^*) = \mathcal{D}(A)$ be in the kernel of $A^* \pm \mathbf{i}$, i.e. $\mp \mathbf{i}\varphi = A^*\varphi = A\varphi$. Then follows:

$$\mp \mathbf{i} \langle \varphi, \varphi \rangle = \langle \varphi, A\varphi \rangle = \langle A\varphi, \varphi \rangle = \pm \mathbf{i} \langle \varphi, \varphi \rangle$$

This shows $\|\varphi\| = 0$ and thus $\varphi = 0$.

„ii) \Rightarrow iii)“: Let A be closed and $\ker(A \pm \mathbf{i}) = \{0\}$ be trivial.

- $\text{im}(A \pm \mathbf{i})$ is dense in H . Assume conversely that there exists a $u \neq 0$ in $(\text{im}(A \pm \mathbf{i}))^\perp$. Then follows for all $v \in \mathcal{D}(A)$:

$$0 = \langle (A \pm \mathbf{i})v, u \rangle$$

So $u \in \mathcal{D}((A \pm \mathbf{i})^*) = \mathcal{D}(A^*)$ and $(A^* \mp \mathbf{i})u = 0$ in contradiction to $\ker(A^* \mp \mathbf{i}) = \{0\}$.

- $\text{im}(A \pm \mathbf{i})$ is closed in H . Let $\psi \in \overline{\text{im}(A \pm \mathbf{i})}$ lie in the closure of the image. Then there exist $\varphi_n \in \mathcal{D}(A)$ such that:

$$(A \pm \mathbf{i}) \varphi_n \rightarrow \psi$$

For any $\varphi \in \mathcal{D}(A)$ holds:

$$\|(A \pm \mathbf{i}) \varphi\|^2 = \langle (A \pm \mathbf{i}) \varphi, (A \pm \mathbf{i}) \varphi \rangle = \|A\varphi\|^2 + \|\varphi\|^2 \pm \mathbf{i} (\underbrace{\langle A\varphi, \varphi \rangle - \langle \varphi, A\varphi \rangle}_{=0, \text{ since } A \text{ is symmetric}})$$

Especially for $\varphi = \varphi_n - \varphi_m$ holds:

$$\underbrace{\|A(\varphi_n - \varphi_m)\|^2}_{\geq 0} + \underbrace{\|\varphi_n - \varphi_m\|^2}_{\geq 0} = \|(A \pm \mathbf{i})(\varphi_n - \varphi_m)\|^2 \xrightarrow[(A \pm \mathbf{i})\varphi_n \rightarrow \psi]{n, m \rightarrow \infty} 0$$

It follows:

$$\begin{aligned} \|\varphi_n - \varphi_m\| &\rightarrow 0 & \varphi_n &\rightarrow \varphi \\ \|A\varphi_n - A\varphi_m\| &\rightarrow 0 & A\varphi_n &\rightarrow \psi \mp \mathbf{i}\varphi \end{aligned}$$

Thus $(\varphi_n, A\varphi_n)$ is a Cauchy sequence in $\text{graph}(A) \subseteq H \times H$.

Since A is closed, which means by definition that $\text{graph}(A)$ is closed in $H \times H$, the limit point $(\varphi, \psi \mp \mathbf{i}\varphi)$ is in $\text{graph}(A)$. Then follows $\varphi \in \mathcal{D}(A)$ and $A\varphi = \psi \mp \mathbf{i}\varphi$, i.e. $\psi \in \text{im}(A \pm \mathbf{i})$.

„iii) \Rightarrow i)“: Assume that $\text{im}(A \pm \mathbf{i}) = H$. Consider $\varphi \in \mathcal{D}(A^*)$. Since $\text{im}(A \pm \mathbf{i}) = H$, there is a $u \in \mathcal{D}(A)$ such that $(A \pm \mathbf{i})u = (A^* \pm \mathbf{i})\varphi$. From $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ (always true for symmetric operators) follows $\varphi - u \in \mathcal{D}(A^*)$ and:

$$(A^* \pm \mathbf{i})(\varphi - u) = 0$$

Consider $w \in \ker(A^* \pm \mathbf{i}) \setminus \{0\}$. Then holds for all $\xi \in \mathcal{D}(A)$:

$$\begin{aligned} \langle (A^* \pm \mathbf{i})w, \xi \rangle &= 0 \\ \langle w, (A \mp \mathbf{i})\xi \rangle &= 0 \end{aligned}$$

Using assumption $\text{im}(A \mp \mathbf{i}) = H$ one can choose ξ such that $(A \mp \mathbf{i})\xi = w$, which means $\langle w, w \rangle = 0$, i.e. $w = 0$. Thus holds:

$$\ker(A^* \pm \mathbf{i}) = \{0\}$$

This gives $\varphi = u \in \mathcal{D}(A)$, which implies $\varphi \in \mathcal{D}(A^*)$ and thus A is self-adjoint. $\square_{10.1}$

10.2 Unbounded Multiplication Operators

Let (Ω, μ) be a measure space with a σ -finite measure μ . (For example, Ω is a σ -compact topological space and μ a positive Borel measure on Ω .)

$H = L^2(\Omega, d\mu)$ is our Hilbert space. Let $g : \Omega \rightarrow \mathbb{R}$ be measurable (and finite almost everywhere). We want to introduce T_g :

$$T_g f = g \cdot f$$

For $g \in L^\infty(\Omega, d\mu)$, T_g is a bounded symmetric operator. Suppose g is unbounded. What is $\mathcal{D}(T_g)$? How to choose $\mathcal{D}(T_g)$ such that T_g becomes self-adjoint?

Lemma

Define:

$$\mathcal{D}(T_g) = \{f \in L^2(\Omega, d\mu) \mid g \cdot f \in L^2(\Omega, d\mu)\} \subseteq L^2(\Omega, d\mu)$$

Then $T_g : \mathcal{D}(T_g) \rightarrow L^2(\Omega, d\mu)$ is self-adjoint and $\sigma_{\text{ess}}(T_g) = g(\Omega)$.

Proof

T_g is symmetric:

$$\begin{aligned} \langle T_g f, h \rangle &= \int_{\Omega} \overline{(T_g f)} h d\mu = \int_{\Omega} \overline{g(x) \cdot f(x)} h(x) d\mu(x) = \int_{\Omega} g(x) \cdot \overline{f(x)} h(x) d\mu(x) = \\ &= \int_{\Omega} \overline{f(x)} g(x) h(x) d\mu(x) = \langle f, T_g h \rangle \end{aligned}$$

T_g is self-adjoint: For $\psi \in \mathcal{D}(T_g^*)$ we show $\psi \in \mathcal{D}(T_g)$. This is equivalent to the existence of a $v \in H$ such that for all $u \in \mathcal{D}(T_g)$ holds

$$\langle T_g u, \psi \rangle = \langle u, v \rangle$$

and we have $v = T_g^* \psi$. Now we write

$$\Omega = \bigcup_N K_N$$

with $K_N \subseteq K_{N+1}$ having finite measure and set:

$$\chi_N(x) = \begin{cases} 1 & \text{if } |g(x)| \leq N \text{ and } x \in K_N \\ 0 & \text{otherwise} \end{cases}$$

So $\chi_N(x) \nearrow 1$ converges monotonously and it holds:

$$\begin{aligned} \|T_g^* \psi\|_{L^2}^2 &= \int_{\Omega} |(T_g^* \psi)(x)|^2 d\mu(x) \stackrel{\text{monotone convergence}}{=} \lim_{N \rightarrow \infty} \int_{\Omega} \chi_N(x) |(T_g^* \psi)(x)|^2 d\mu(x) = \\ &= \lim_{N \rightarrow \infty} \|\chi_N T_g^* \psi\|^2 \\ \Rightarrow \quad \|T_g^* \psi\|_{L^2} &= \lim_{N \rightarrow \infty} \|\chi_N T_g^* \psi\|_{L^2} = \lim_{N \rightarrow \infty} \sup_{\|\varphi\|=1} |\langle \varphi, \chi_N T_g^* \psi \rangle| = \\ &\stackrel{*}{=} \lim_{N \rightarrow \infty} \sup_{\|\varphi\|=1} |\langle T_g \chi_N \varphi, \psi \rangle| \end{aligned}$$

In \star we used that $\chi_N \varphi$ is in $\mathcal{D}(T_g)$. This is really the case, since for $\chi_N \varphi \in L^2(\Omega, d\mu)$ holds:

$$T_g \chi_N \varphi = \underbrace{g \cdot \chi_N}_{\text{is bounded}} \varphi = T_{g \cdot \chi_N} \varphi \in L^2(\Omega, d\mu)$$

Since the function $g \cdot \chi_N$ is bounded, the multiplication operator $T_{g \cdot \chi_N}$ is bounded and thus follows:

$$\begin{aligned} \infty > \|T_g^* \psi\| &= \lim_{N \rightarrow \infty} \sup_{\|\varphi\|=1} |\langle \varphi, \chi_N \cdot g \cdot \psi \rangle| = \lim_{N \rightarrow \infty} \|\chi_N \cdot g \cdot \psi\| = \\ &= \lim_{N \rightarrow \infty} \int_{\Omega} \chi_N(x) |g\psi|^2(x) d\mu(x) \stackrel{\text{monotone convergence}}{=} \int_{\Omega} |(g\psi)(x)|^2 d\mu(x) \end{aligned}$$

So we have $g\psi \in L^2(\Omega, d\mu)$ and thus $\psi \in \mathcal{D}(T_g)$ holds by definition of $\mathcal{D}(T_g)$.

We omit the proof that $\sigma_{\text{ess}}(T_g) = g(\Omega)$.

□_{10.2}

10.3 Theorem (The Spectral Theorem in its Multiplicative Form)

Let $A : \mathcal{D}(H) \xrightarrow{\text{dense}} H \rightarrow H$ be a self-adjoint operator and H separable. Then there is a finite measure space (M, μ) , a unitary operator $\mathcal{U} : H \rightarrow L^2(M, d\mu)$ and a measurable function $f : M \rightarrow \mathbb{R}$ such that holds:

- a) $\psi \in \mathcal{D}(A) \Leftrightarrow f \cdot \mathcal{U}\psi \in L^2(M, d\mu)$
- b) $\varphi \in \mathcal{U}(\mathcal{D}(A))$ implies $\mathcal{U}A\mathcal{U}^{-1}\varphi = f \cdot \varphi = T_f \cdot \varphi$.

Thus A is unitarily equivalent to the multiplication T_f on $L^2(M, d\mu)$ and as chosen in 10.2:

$$\mathcal{U}(\mathcal{D}(A)) = \mathcal{D}(T_f) = \{\phi \in L^2 \mid f \cdot \phi \in L^2(M, d\mu)\}$$

Proof

According to our basic criterion 10.1, the mapping

$$A \pm \mathbf{i} : \mathcal{D}(A) \rightarrow H$$

is surjective (by property iii)) and injective (by property ii)), noting:

$$\{0\} = \ker(A^* \pm \mathbf{i}) = \ker(A \pm \mathbf{i})$$

So $A \pm \mathbf{i}$ is bijective and thus the inverse $(A \pm \mathbf{i})^{-1} : H \rightarrow \mathcal{D}(A) \subseteq H$ exists.

The operators $(A \pm \mathbf{i})^{-1}$ are bounded, because for all $u \in \mathcal{D}(A)$ holds (cf. proof of 10.1):

$$\|(A + \mathbf{i})u\|^2 = \|Au\|^2 + \|u\|^2$$

Thus for $v := (A + \mathbf{i})u$ follows:

$$\begin{aligned} \|(A + \mathbf{i})^{-1}v\| &\leq \|v\| \\ \|(A + \mathbf{i})^{-1}\| &\leq 1 \end{aligned}$$

The operators $(A \pm \mathbf{i})^{-1}$ are *normal*: The resolvent identity gives:

$$\begin{aligned} (A + \mathbf{i})^{-1} - (A - \mathbf{i})^{-1} &= -2\mathbf{i} \cdot (A + \mathbf{i})^{-1} \cdot (A - \mathbf{i})^{-1} \\ (A - \mathbf{i})^{-1} - (A + \mathbf{i})^{-1} &= +2\mathbf{i} \cdot (A - \mathbf{i})^{-1} \cdot (A + \mathbf{i})^{-1} \end{aligned}$$

Together this yields:

$$\left[(A + \mathbf{i})^{-1}, (A - \mathbf{i})^{-1} \right] = 0$$

Let us compute $\left((A + \mathbf{i})^{-1} \right)^*$. For $u, v \in \mathcal{D}(A)$ holds:

$$\begin{aligned} \langle (A - \mathbf{i})u, v \rangle &\stackrel{A \text{ symmetric}}{=} \langle u, (A + \mathbf{i})v \rangle \\ \parallel &\qquad \qquad \parallel \\ \langle \underbrace{(A - \mathbf{i})u}_{=\psi}, (A + \mathbf{i})^{-1} \underbrace{(A + \mathbf{i})v}_{=\varphi} \rangle &= \langle (A - \mathbf{i})^{-1} \underbrace{(A - \mathbf{i})u}_{=\psi}, \underbrace{(A + \mathbf{i})v}_{=\varphi} \rangle \end{aligned}$$

$$\langle \psi, (A + \mathbf{i})^{-1} \phi \rangle = \langle (A - \mathbf{i})^{-1} \psi, \phi \rangle$$

Since $(A - \mathbf{i})$ and $(A + \mathbf{i})$ are surjective, this holds for all $\psi, \phi \in H$ and thus follows:

$$\begin{aligned} & \left((A + \mathbf{i})^{-1} \right)^* = (A - \mathbf{i})^{-1} \\ \Rightarrow & \left[(A + \mathbf{i})^{-1}, \left((A + \mathbf{i})^{-1} \right)^* \right] = 0 \end{aligned}$$

So $(A + \mathbf{i})^{-1}$ is normal and we can apply the spectral theorem in its multiplicative form to the operator $(A + \mathbf{i})^{-1}$. This gives:

$$\mathcal{U} : H \rightarrow L^2(M, d\mu)$$

μ is a bounded positive Borel measure on the σ -compact topological space M .

$$M = \sigma \left((A + \mathbf{i})^{-1} \right) \times J$$

And for $\varphi \in L^2(M, d\mu)$ holds

$$\left(\mathcal{U} (A + \mathbf{i})^{-1} \mathcal{U}^{-1} \right) \varphi = g \cdot \varphi$$

with a $g \in L^\infty(M, d\mu)$.

Moreover, since $(A + \mathbf{i})^{-1}$ is injective, the function g is non-zero almost everywhere: Assume conversely that there exists a $\Omega \subseteq M$ with $\mu(\Omega) \neq 0$ and $g|_\Omega = 0$. Then $\varphi := \chi_\Omega$ is a non-zero vector in $L^2(M, d\mu)$ with $g \cdot \varphi = 0$.

$$\|\varphi\|^2 = \int_M \chi_\Omega^2 d\mu = \mu(\Omega) > 0$$

Thus $\mathcal{U}^{-1}\varphi$ is a non-trivial vector in the kernel of $(A + \mathbf{i})^{-1}$, which is a contradiction to the injectivity of A .

a) Set $f = \frac{1}{g} - \mathbf{i}$. This function is measurable and finite almost everywhere.

„ \Rightarrow “: Since $(A + \mathbf{i})^{-1} : H \rightarrow \mathcal{D}(A)$ is bijective, a $\psi \in \mathcal{D}(A)$ can be written uniquely as:

$$\psi = (A + \mathbf{i})^{-1} \phi$$

$$\Rightarrow \quad \mathcal{U}\psi = \mathcal{U} (A + \mathbf{i})^{-1} \phi = \underbrace{\mathcal{U} (A + \mathbf{i})^{-1} \mathcal{U}^{-1}}_{=T_g} \mathcal{U}\phi = g\mathcal{U}\phi$$

$$f\mathcal{U}\psi = fg\mathcal{U}\phi = \underbrace{(1 - \mathbf{i}g)}_{\in L^\infty(M, d\mu)} \cdot \underbrace{\mathcal{U}\phi}_{\in L^2(M, d\mu)} \in L^2(M, d\mu)$$

„ \Leftarrow “: Assume $f\mathcal{U}\psi \in L^2(M, d\mu)$, which implies $(f + \mathbf{i})\mathcal{U}\psi \in L^2(M, d\mu)$. Now there exists a $\phi \in H$ such that holds:

$$\begin{aligned} \mathcal{U}\phi &= (f + \mathbf{i})\mathcal{U}\psi \quad / \cdot g \\ g\mathcal{U}\phi &= g(f + \mathbf{i})\mathcal{U}\psi = \mathcal{U}\psi \\ \Rightarrow \quad \psi &= \underbrace{\mathcal{U}^{-1}g\mathcal{U}}_{=(A+\mathbf{i})^{-1}} \phi = (A + \mathbf{i})^{-1} \phi \end{aligned}$$

Since $(A + \mathbf{i})^{-1} : H \rightarrow \mathcal{D}(A)$ is bijective, $\psi \in \mathcal{D}(A)$ follows.

b) We need to show for all $\varphi \in \mathcal{U}(\mathcal{D}(A))$:

$$\mathcal{U}A\mathcal{U}^{-1}\varphi = f\varphi$$

Write $\psi \in \mathcal{D}(A)$ as $\psi = (A + \mathbf{i})^{-1}\varphi$ to get just as in a) „ \Rightarrow “:

$$\begin{aligned}\mathcal{U}\psi &= g\mathcal{U}\varphi \\ \mathcal{U}\varphi &= \frac{1}{g}\mathcal{U}\psi \\ \mathcal{U}(A + \mathbf{i})\psi &= \frac{1}{g}\mathcal{U}\psi \\ \mathcal{U}A\psi &= \frac{1}{g}\mathcal{U}\psi - \mathbf{i}\mathcal{U}\psi = \left(\frac{1}{g} - \mathbf{i}\right)\mathcal{U}\psi = f\mathcal{U}\psi \\ \mathcal{U}A\mathcal{U}^{-1}\chi &\stackrel{\chi=\mathcal{U}\psi}{=} f \cdot \chi\end{aligned}$$

Finally we show that f is real-valued. For all $\psi \in \mathcal{D}(A)$ holds, because A is symmetric:

$$\begin{aligned}0 &= \operatorname{Im}(\langle \psi, A\psi \rangle) = \operatorname{Im}(\langle \psi, \mathcal{U}^{-1}f\mathcal{U}\psi \rangle) \stackrel{\mathcal{U} \text{ unitary}}{=} \operatorname{Im}(\langle \mathcal{U}\psi, f\mathcal{U}\psi \rangle) = \\ &= \int_M \operatorname{Im}(f(x)) \cdot |\mathcal{U}\psi(x)|^2 d\mu(x)\end{aligned}$$

Since $\mathcal{U}\psi$ can be any L^2 -function χ (just choose $\psi = \mathcal{U}^{-1}\chi$), it follows that $\operatorname{Im}(f) = 0$ almost everywhere. $\square_{10.3}$

Connection to the Cayley transformation

The operators

$$\begin{aligned}V &:= (A + \mathbf{i})(A - \mathbf{i})^{-1} \\ V^* &= (A + \mathbf{i})^{-1}(A - \mathbf{i})\end{aligned}$$

are unitary, because it holds:

$$\begin{aligned}V \cdot V^* &= (A + \mathbf{i})(A - \mathbf{i})^{-1}(A + \mathbf{i})^{-1}(A - \mathbf{i}) = \\ &= (A + \mathbf{i})(A + \mathbf{i})^{-1}(A - \mathbf{i})^{-1}(A - \mathbf{i}) = \mathbb{1}\end{aligned}$$

We worked here with $(A - \mathbf{i})^{-1}$.

10.4 The unbounded Functional Calculus, Projection-valued Spectral measures

Goal: Suppose E_λ is a spectral measure on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

$$f(A) = \int f(\lambda) dE_\lambda$$

So far we had $f \in \mathcal{B}(\mathbb{K})$. This gave us a bounded linear operator. We want to calculate

$$f(A) = \int f(\lambda) dE_\lambda$$

for any Borel function f , possibly unbounded. Then $f(A)$ is a possibly unbounded operator. What is $\mathcal{D}(A)$ and what is $\mathcal{D}(A^*)$?

$$\mathcal{D}(A) = \left\{ u \in H \mid \int_{\mathbb{R}} |f(\lambda)|^2 d\langle u, E_\lambda u \rangle < \infty \right\}$$

Example

$$(UAU^{-1})f = gf$$

and $g : \Omega \rightarrow \mathbb{R}$ is measurable.

$$\begin{aligned} \mathcal{D}(A) &= \{U^{-1}\varphi \mid \varphi \in L^2(\Omega, d\mu) \wedge gf \in L^2(\Omega, d\mu)\} = \\ &= U^{-1}\mathcal{D}(UAU^{-1})U \end{aligned}$$

The spectral calculus yields:

$$UA^2U^{-1} = (UAU^{-1})^2 = g^2$$

$$\Rightarrow \quad \mathcal{D}(A^2) = \{U^{-1}\varphi \mid \varphi \in L^2(\Omega, d\mu) \wedge g^2f \in L^2(\Omega, d\mu)\}$$

So the domain of definition changes.

10.4.1 Theorem (The spectral theorem in functional calculus form)

Let $A : \mathcal{D}(A) \subseteq H \rightarrow H$ be self-adjoint. Then there is a unique mapping

$$\Phi : \mathcal{B}(\mathbb{R}) \rightarrow L(H)$$

such that the following holds:

- i) Φ is an involutive algebra homomorphism.
- ii) $\|\Phi(f)\|_{L(H)} \leq \|f\|_\infty$
- iii) Let $g_n \in \mathcal{B}(\mathbb{R})$ be the elements of a sequence such that $g_n \rightarrow g$ converges point-wise and $|g_n(x)| \leq |x|$ holds. Then for every $\psi \in \mathcal{D}(A)$ converges:

$$\Phi(g_n)\psi \rightarrow \Phi(g)\psi$$

- iv) If $g_n \rightarrow g$ converges point-wise with $|g_n(x)| < C$, then holds for all $\psi \in H$ converges:

$$\Phi(g_n)\psi \rightarrow \Phi(g)\psi$$

- v) For $A\psi = \lambda\psi$ follows $\Phi(f)\psi = f(\lambda)\psi$
- vi) For $h \geq 0$ holds $\Phi(h) \geq 0$.

Proof

After a unitary transformation with the operator U from the spectral theorem in its multiplicative form, we can assume $H = L^2(M, d\mu)$ and:

$$\begin{aligned}\mathcal{D}(A) &= \{\varphi \in L^2(M, d\mu) \mid g\varphi \in L^2(M, d\mu)\} \\ A\varphi &= g\varphi \\ (\Phi(f)\varphi)(x) &= f(g(x)) \cdot \varphi(x)\end{aligned}$$

Since $f(g) \in L^\infty$ holds, define for any $\varphi \in L^2$:

$$\Phi(f)\varphi := f(g)\varphi \in L^2$$

This defines an operator in $L(H)$.

The properties i) and ii) are obvious. iii) and iv) follow from dominated convergence:

iii) It holds:

$$\begin{aligned}\Phi(f_n)\varphi &= f_n(g) \cdot \varphi \\ \Phi(f)\varphi &= f(g)\varphi \\ f_n(g) &\xrightarrow{\text{point-wise}} f(g)\end{aligned}$$

By assumption holds $|f_n(g)| \leq |g|$ and by our formula for $\mathcal{D}(A)$ follows for all $\varphi \in \mathcal{D}(A)$:

$$|f_n(g)\varphi|, |f(g)\varphi| \leq |g| \cdot |\varphi| \in L^2$$

iv) follows similarly and v) and vi) are obvious.

Uniqueness of Φ : Let $K \subseteq \mathbb{R}$ be compact and $\varphi \in L^2(K, d\mu)$. Then holds:

$$\Phi(g \cdot \chi_K)\varphi = \underbrace{\Phi(g)}_{=A} \cdot \Phi(\chi_K)\varphi = A\Phi(\chi_K)\varphi$$

On K we can approximate g using Stone-Weierstraß. Then choose a sequence $K_1 \subseteq K_2 \subseteq \dots$ of compact K_n with $\bigcup_{n \in \mathbb{N}} K_n = \mathbb{R}$. Now take the limit $n \rightarrow \infty$ and use property iii) to get $\Phi(g)\varphi = A\varphi$, which shows the uniqueness of Φ . $\square_{10.4.1}$

Now we write $\Phi(f) =: f(A)$. We can again introduce the spectral measure:

$$E_\Omega := \Phi(\chi_\Omega) = \chi_\Omega(A)$$

After a unitary transformation holds:

$$E_\Omega \varphi = \chi_\Omega(g) \cdot \varphi$$

This shows:

$$\begin{aligned}E_\Omega^* &= E_\Omega = E_\Omega^2 \\ E_U \cdot E_V &= E_{U \cap V}\end{aligned}$$

$$\begin{aligned}\langle \varphi, E_\Omega \varphi \rangle &= \int_{\mathbb{R}} |\varphi|^2 \chi_\Omega(g) d\mu \\ \langle \varphi, f(A)\varphi \rangle &= \int_{\mathbb{R}} |\varphi|^2 f(g) d\mu = \int_{\mathbb{R}} f d\langle \varphi, E_\lambda \varphi \rangle\end{aligned}$$

10.4.2 Theorem

There is a one-to-one correspondence between self-adjoint operators and projection-valued spectral measures (not necessarily with compact support) given by:

$$A = \int_{\mathbb{R}} \lambda dE_{\lambda}$$

$$\mathcal{D}(A) = \left\{ u \in H \left| \int_{\mathbb{R}} \lambda^2 d\langle u, E_{\lambda} u \rangle < \infty \right. \right\}$$

Moreover holds:

- i) $f(A) = \int_{\mathbb{R}} f(\lambda) dE_{\lambda}$ holds for all bounded Borel functions f .
- ii) If f is an unbounded Borel function, we set:

$$\mathcal{D}_f = \left\{ u \in H \left| \int_{\mathbb{R}} |f|^2 d\langle u, E_{\lambda} u \rangle < \infty \right. \right\}$$

The set $\mathcal{D}_f \subseteq H$ is dense and

$$B := \int_{\mathbb{R}} f dE_{\lambda} : \mathcal{D}_f \rightarrow H$$

is a densely defined closed operator with:

$$B^* = \int_{\mathbb{R}} \bar{f} dE_{\lambda} : \mathcal{D}_f \rightarrow H$$

(In particular, if f is real-valued, the operator B is again self-adjoint.)

Proof

- \mathcal{D}_f is dense in H : After a unitary transformation we identify H with $L^2(M, d\mu)$ and define:

$$\mathcal{D}_f = \left\{ \varphi \in L^2(M, d\mu) \left| \int |f(g)|^2 \cdot |\varphi|^2 d\mu < \infty \right. \right\}$$

(Recall $f(A) = f(g)$.) For $\psi \in L^2(M, d\mu)$, we want to show $\psi \in \overline{\mathcal{D}_f}$. To this end we set:

$$\psi_n(x) := \begin{cases} \psi(x) & \text{if } |f(g(x))| \leq n \\ 0 & \text{otherwise} \end{cases}$$

Then holds:

$$\int |f(g)|^2 \cdot |\psi_n|^2 d\mu \leq n^2 \int |\psi_n|^2 d\mu \leq n^2 \int |\psi|^2 d\mu < \infty$$

Hence follows $\psi_n \in \mathcal{D}_f$. Obviously $\psi_n \rightarrow \psi$ converges point-wise and it holds:

$$|\psi_n| \leq |\psi| \in L^2(M, d\mu)$$

Thus dominated convergence yields $\psi_n \rightarrow \psi$ in $L^2(M, d\mu)$.

- Next, $B\varphi = f(g)\varphi$ with

$$\mathcal{D}(B) = \{ \varphi \in L^2 \mid f(g)\varphi \in L^2 \}$$

is an unbounded multiplication operator. Its adjoint can be computed as in section 10.2.

□_{10.4.2}

11 Examples, Construction of Self-Adjoint extensions

The (interesting) operator $H = -\Delta_{\mathbb{R}^3} + V(x)$ requires Sobolev spaces and Fourier transform. This is discussed in the lecture partial differential equations I.

Here we only consider more simple, one-dimensional examples.

11.1 Example

Consider $A = \mathbf{i} \frac{d}{dx}$ on $H = L^2(\mathbb{R}, dx)$ with domain of definition:

$$\mathcal{D}(A) = C_0^\infty(\mathbb{R})$$

– A is symmetric: For $\psi, \phi \in C_0^\infty(\mathbb{R})$ holds:

$$\begin{aligned} \langle \psi, A\phi \rangle &= \int_{\mathbb{R}} \overline{\psi(x)} \mathbf{i} \left(\frac{d}{dx} \phi(x) \right) dx = \\ &\stackrel{\substack{\text{integration} \\ \text{by parts}}}{=} \underbrace{\overline{\psi(x)} \cdot \mathbf{i} \phi(x) \Big|_{-\infty}^{\infty}}_{=0, \text{ (compact support)}} - \int_{\mathbb{R}} (-\mathbf{i}) \left(\frac{d}{dx} \overline{\psi(x)} \right) \phi(x) dx = \\ &= \int_{\mathbb{R}} \overline{\left(\mathbf{i} \frac{d}{dx} \psi(x) \right)} \phi(x) dx = \langle A\psi, \phi \rangle \end{aligned}$$

– A is *not* self-adjoint: If A were self-adjoint, the following computation would hold:

$$\forall_{u \in \mathcal{D}(A)} : \langle Au, v \rangle = \langle u, w \rangle \quad \Rightarrow \quad (v \in \mathcal{D}(A)) \wedge (Av = w)$$

Any $v \in C_0^1(\mathbb{R}) \setminus C_0^\infty(\mathbb{R})$ is a counter example.

We could even satisfy the condition on the left by choosing $v \in C^1(\mathbb{R})$. (We need no decay assumption, since it suffices that one function has compact support). Thus follows:

$$\mathcal{D}(A^*) \subseteq C^1(\mathbb{R})$$

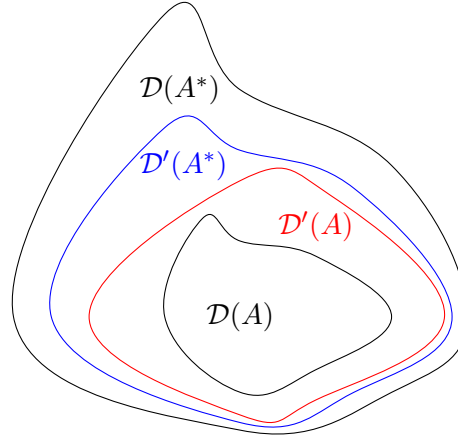


Figure 11.1: The large $\mathcal{D}'(A) \supseteq \mathcal{D}(A)$, the smaller is $\mathcal{D}'(A^*) \subseteq \mathcal{D}(A^*)$.

- $A : \mathcal{D}(A) \rightarrow H$ is essentially self-adjoint: This means that \overline{A} with $\text{graph}(\overline{A}) := \overline{\text{graph}(A)}$ is self-adjoint.

According to the basic criterion for self-adjointness (Theorem 10.1), we know:

$$A \text{ self-adjoint} \quad \Leftrightarrow \quad \text{im}(A \pm \mathbf{i}) = H$$

Therefore, for essential self-adjointness it suffices to show that $(A \pm \mathbf{i})(C_0^\infty(\mathbb{R})) \subseteq H = L^2$ is dense.

Claim: For all $v \in H$ there exists a $u \in H$ such that $(u, v) \in \overline{\text{graph}(A \pm \mathbf{i})}$.
(In other words, $\overline{A} \pm \mathbf{i}$ is surjective.)

Proof: Since $(A \pm \mathbf{i})(C_0^\infty(\mathbb{R})) \subseteq H$ is dense, there exists a sequence of $u_n \in C_0^\infty$ such that with $w_n := Au_n$ converges:

$$(A \pm \mathbf{i})u_n = w_n \pm \mathbf{i}u_n \rightarrow v$$

The estimates from the proof of the basic criterion imply:

$$w_n = Au_n \rightarrow w \qquad u_n \rightarrow u$$

This yields that $(u_n, w_n) \rightarrow (u, w)$ converges. From $(u_n, w_n) \in \text{graph}(A)$ follows $(u, w) \in \overline{\text{graph}(A)}$. □_{Claim}

Claim: $(A \pm \mathbf{i})(C_0^\infty(\mathbb{R}))$ is dense in L^2 .

Proof: The vectors in the image of $A \pm \mathbf{i}$ are of the form:

$$\mathbf{i} \frac{d}{dx} u \pm \mathbf{i} u =: v$$

From $u \in C_0^\infty$ follows $v \in C_0^\infty$. Multiply by $e^{\mp x}$ and integrate by parts to get:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\mp x} v(x) dx &= \mathbf{i} \int_{-\infty}^{\infty} e^{\mp x} \left(\left(\frac{d}{dx} \pm 1 \right) u(x) \right) dx = \\ &\stackrel{\text{integrate by parts}}{=} -\mathbf{i} \int_{-\infty}^{\infty} u(x) \left(\left(\frac{d}{dx} \pm 1 \right) e^{\mp x} \right) dx = \end{aligned}$$

$$= -\mathbf{i} \int_{-\infty}^{\infty} u(x) \underbrace{(\mp e^{\mp x} \pm e^{\mp x})}_{=0} dx = 0$$

Thus the functions in the image of $A \pm \mathbf{i}$ satisfy the condition:

$$\int_{-\infty}^{\infty} e^{\mp x} v(x) dx = 0$$

Conversely, if a function $v(x)$ satisfies this condition for $+$ and $-$, then

$$u(x) := \int_{-\infty}^x e^{\mp t} v(t) dt$$

is in $C_0^\infty(\mathbb{R})$ and $(A \pm \mathbf{i})u = v$.

Now we need to show:

$$\overline{\left\{ v \in C_0^\infty(\mathbb{R}) \mid \int e^{\pm x} v(x) dx = 0 \right\}} = H$$

Since $C_0^\infty(\mathbb{R})$ is dense in H , we only need to prove that $\psi \in C_0^\infty(\mathbb{R})$ is an element of the left set. We look for $v_n \in C_0^\infty(\mathbb{R})$ with

$$\int e^{\pm x} v_n(x) dx = 0$$

such that $v_n \rightarrow \psi$ converges in L^2 .

Choose $\eta \in C_0^\infty([0,1])$ and use the ansatz:

$$v_n = \psi + c_+ \eta(x - L) + c_- \eta(x + L)$$

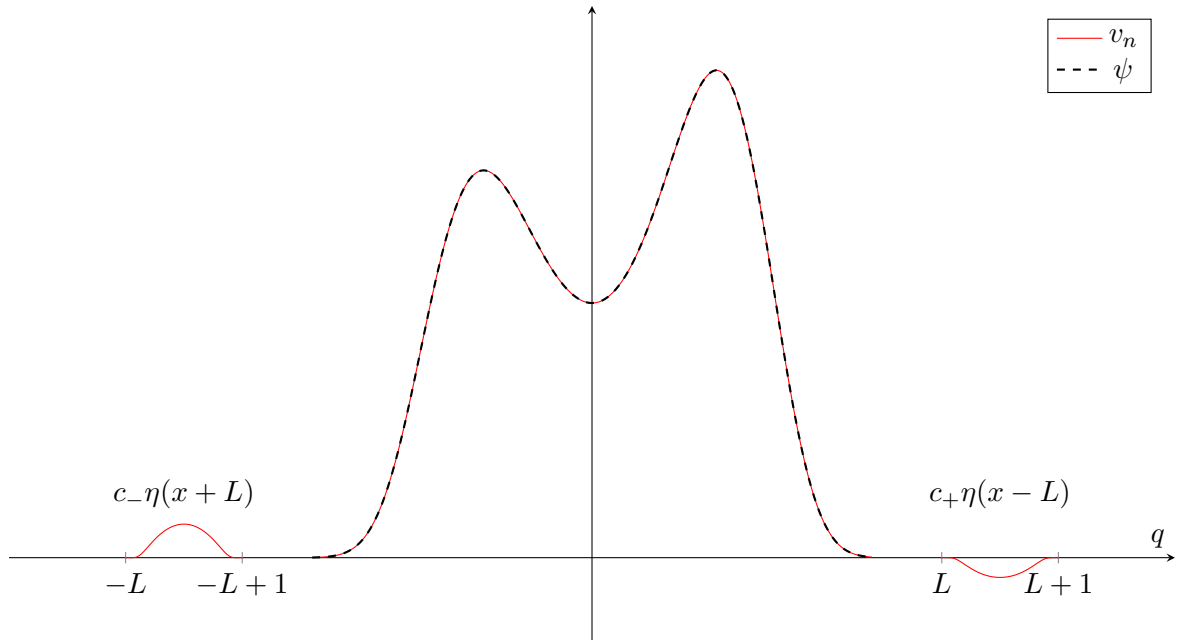


Figure 11.2: Approximation of ψ with the v_n

Then holds:

$$\begin{aligned} 0 &\stackrel{!}{=} \int_{-\infty}^{\infty} e^{\pm x} v_n(x) dx = \\ &= \int_{-\infty}^{\infty} \psi(x) dx + c_+ \underbrace{\int_{-\infty}^{\infty} e^{\pm x} \eta(x-L) dx}_{\sim e^{\pm L}} + c_- \underbrace{\int_{-\infty}^{\infty} e^{\pm x} \eta(x+L) dx}_{\sim e^{\mp L}} \end{aligned}$$

We have two conditions and two free parameters. One sees that c_+, c_- are proportional to e^{-L} . Thus $v_n \rightarrow \psi$ converges in L^2 . \square_{Claim}

Thus \overline{A} with $\mathcal{D}(\overline{A})$ (which can be described in detail) is self-adjoint.

$$\overline{A} = \int_{-\infty}^{\infty} \lambda dE_{\lambda} \quad \text{spectral theorem}$$

11.2 Example

On the Hilbert space $H = L^2([0,1], dx)$ consider the operator $A := \frac{d}{dx}$ with $\mathcal{D}(A) = C_0^\infty((0,1))$.

- a) A is *not* essentially self-adjoint. Just as in the previous example, A being essentially self-adjoint is equivalent to

$$(A \pm i)(C_0^\infty((0,1))) \subseteq H$$

being dense, or equivalently

$$M := \left\{ v \in C_0^\infty((0,1)) \mid 0 = \int_0^1 e^{\pm x} v(x) dx \right\} \subseteq H$$

being dense. For $\psi(x) = e(x) \in H$ holds for all $v \in M$:

$$\langle \psi, v \rangle = \int_0^1 \psi(x) v(x) dx = \int_0^1 e^x v(x) dx = 0$$

Therefore holds $0 \neq \psi \in M^\perp$ and M is *not* dense in H .

- b) For $f \in C_0^\infty([0,1])$ and $n \in \mathbb{Z}$ define:

$$c_n := \int_0^1 f(x) e^{2\pi i n x} dx$$

This gives rise to a unitary transformation (Plancherel theorem):

$$\begin{aligned} U : L^2([0,1]) &\rightarrow \ell_2 \\ f &\mapsto (c_n)_{n \in \mathbb{Z}} \end{aligned}$$

$$\int_0^1 |f|^2 dx = \sum_{n \in \mathbb{Z}} |c_n|^2$$

$$\hat{A}(c_n) = \left(U i \frac{d}{dx} U^{-1} \right) (c_n) = (-2\pi n c_n)_n$$

\hat{A} is a multiplication operator with:

$$\mathcal{D}(\hat{A}) = \{(c_n)_n \in \ell^2 \mid (nc_n)_n \in \ell^2\} \subseteq \ell^2$$

Then

$$\hat{A} : \mathcal{D}(\hat{A}) \rightarrow \ell^2$$

is self-adjoint. Thus

$$A : \mathcal{D}(A) := U^{-1}\mathcal{D}(\hat{A}) \rightarrow L^2$$

is self-adjoint.

11.3 Example

Consider $H = L^2(\mathbb{R}, dx)$, $A = \mathbf{i} \frac{d}{dx}$ and $T = T_g$ with a real valued g .

$$(A + T)\psi(x) = \mathbf{i} \frac{d}{dx} \psi(x) + g(x) \psi(x)$$

How to choose $\mathcal{D}(A + T)$ in order to make the operator self-adjoint?

There are two solutions:

- Friedrichs extension (by K. O. Friedrichs) for semi-bounded operators.
- Katos's method

11.4 Theorem (Kato-Rellich)

Let $A : \mathcal{D}(A) \rightarrow H$ be self-adjoint and T symmetric with $\mathcal{D}(T) \supseteq \mathcal{D}(A)$. Moreover, assume that there are constants $a, b \in \mathbb{R}_{\geq 0}$ with $b < 1$ such that for all $u \in \mathcal{D}(A)$ holds:

$$\|Tu\|^2 \leq a^2 \|u\|^2 + b^2 \|Au\|^2 \quad (11.1)$$

Then $A + T$ with

$$\mathcal{D}(A + T) = \mathcal{D}(A)$$

is self-adjoint.

T is relatively bounded with respect to A .

Proof

The inequality (11.1) implies:

$$\|Tu\| \leq a \|u\| + b \|Au\|$$

For $u \in \mathcal{D}(A)$ holds:

$$Au = (A + T)u - Tu$$

$$\begin{aligned}\|Au\| &\leq \|(A+T)u\| + \|Tu\| \leq \\ &\leq \|(A+T)u\| + a\|u\| + b\|Au\|\end{aligned}$$

This gives:

$$\|Au\| \leq \frac{1}{1-b} (\|(A+T)u\| + a\|u\|) \quad (11.2)$$

- $(A+T)$ with $\mathcal{D}(A+T) := \mathcal{D}(A)$ is closed: Choose $u_n \in \mathcal{D}(A)$ such that $u_n \rightarrow u$ and $(A+T)u_n \rightarrow w$ converge. We want to show $u \in \mathcal{D}(A)$ and $(A+T)u = w$. (11.2) implies:

$$\|A(u_n - u_m)\| \leq \frac{1}{1-b} \underbrace{\|(A+T)(u_n - u_m)\|}_{\rightarrow 0} + \frac{a}{1-b} \underbrace{\|u_n - u_m\|}_{\rightarrow 0}$$

This gives $A(u_n - u_m) \rightarrow 0$ and thus $Au_n \rightarrow v$. Since A is self-adjoint, it is closed, implying that $u \in \mathcal{D}(A)$ and $Au = v$.

- It remains to be showed that $\frac{A+T}{c} \pm \mathbf{i}$ is surjective for any $c \in \mathbb{R}_{>0}$. This is equivalent to $A+T \pm \mathbf{i}c$ being surjective. Since A is self-adjoint, we know that

$$A \pm \mathbf{i}c : \mathcal{D}(A) \rightarrow H$$

is bijective with:

$$(A \pm \mathbf{i}c)^{-1} : H \rightarrow \mathcal{D}(A)$$

This gives:

$$(A+T+\mathbf{i}c) = \underbrace{\left(T(A+\mathbf{i}c)^{-1} + \mathbb{1}\right)}_{\text{to show that this is invertible}} \underbrace{(A+\mathbf{i}c)}_{\text{invertible}}$$

We show that $\left\|T(A+\mathbf{i}c)^{-1}\right\| < 1$. Then $\mathbb{1} + T(A+\mathbf{i}c)^{-1}$ has a bounded inverse in terms of the Neumann series.

For $u \in H$ define $v := (A+\mathbf{i}c)^{-1}u \in \mathcal{D}(A)$, so it holds:

$$u = (A+\mathbf{i}c)v$$

$$\|u\|^2 = \|Av\|^2 + c^2\|v\|^2$$

$$\|v\|^2 \leq \frac{1}{c^2}\|u\|^2 \quad (11.3)$$

$$\|Av\|^2 \leq \|u\|^2 \quad (11.4)$$

We get:

$$\begin{aligned}\left\|T(A+\mathbf{i}c)^{-1}u\right\|^2 &= \|Tv\|^2 \leq a^2\|v\|^2 + b^2\|Av\|^2 \leq \\ &\stackrel{(11.4)}{\leq} a^2\|v\|^2 + b^2\|u\|^2 \leq \\ &\stackrel{(11.3)}{\leq} \frac{a^2}{c^2}\|u\|^2 + b^2\|u\|^2 = \left(\frac{a^2}{c^2} + b^2\right)\|u\|^2\end{aligned}$$

By choosing c sufficiently large, we can arrange that with $\tilde{c} < 1$ holds for all $u \in H$:

$$\left\|T(A+\mathbf{i}c)^{-1}u\right\|^2 \leq \tilde{c}\|u\|^2$$

This gives:

$$\left\|T(A+\mathbf{i}c)^{-1}\right\| < 1$$

□_{11.4}

Back to example 11.3

$A = -i \frac{d}{dx}$ is self-adjoint with $\mathcal{D}(A)$ being the domain of definition of the closure of $A : C_0^\infty(\mathbb{R}) \rightarrow \mathbb{R}$ and $T = T_g$.

If *Kato's condition* is fulfilled, i.e. for all $u \in \mathcal{D}(A)$ the inequality

$$\|T_g u\|^2 \leq a^2 \|u\|^2 + b^2 \|Au\|^2$$

with $a, b \in \mathbb{R}_{>0}$ and $b < 1$ holds, then $A + T$ is also self-adjoint.

For which g is Kato's condition satisfied?

$$\|Au\|^2 = \int_{-\infty}^{\infty} |u'(x)|^2 dx$$

(Let us assume $u \in C_0^\infty$.)

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_x^y 1 \cdot u'(t) dt \right| \stackrel{\text{Schwarz}}{\leq} \left(\int_x^y 1^2 dt \right)^{\frac{1}{2}} \cdot \left(\int_x^y |u'(t)|^2 dt \right)^{\frac{1}{2}} \leq \\ &\leq |x - y|^{\frac{1}{2}} \cdot \|Au\| \end{aligned}$$

Moreover, the mean value theorem (Mittelwertungleichung) gives for all $a \in \mathbb{R}$ the existence of a $y \in [a - \frac{1}{2}, a + \frac{1}{2}]$ such that holds:

$$|u(y)| \leq \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} |u(\tau)| d\tau \stackrel{\text{Schwarz}}{\leq} \underbrace{\left(\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} 1^2 dt \right)^{\frac{1}{2}}}_{=1} \cdot \left(\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} |u(t)|^2 dt \right)^{\frac{1}{2}} \leq \|u\|_{L^2}$$

This gives:

$$|u(x)| \leq |u(y)| + |u(x) - u(y)| \leq \|u\| + \|Au\|$$

Consider now different cases:

1. case: g is bounded, i.e. $\|g\|_\infty \leq c \in \mathbb{R}_{\geq 0}$. Then holds:

$$\begin{aligned} \|T_g u\|^2 &= \int_{\mathbb{R}} |g(x)|^2 |u(x)|^2 dx \leq c^2 \|u\|^2 \\ \Rightarrow \|T_g u\| &\leq c \|u\| \end{aligned}$$

Thus Kato's condition is satisfied with $b = 0$.

2. case: g is not bounded and $\|g\|_{L^2} < 1$. Then holds:

$$\begin{aligned} \|T_g u\|^2 &= \int_{\mathbb{R}} |g(x)|^2 |u(x)|^2 dx \leq \sup_{x \in \mathbb{R}} |u(x)|^2 \cdot \|g\|_{L^2}^2 \\ \Rightarrow \|T_g u\| &\leq (\|u\| + \|Au\|) \|g\|_{L^2} = \|g\|_{L^2} \cdot \|u\| + \|g\|_{L^2} \cdot \|Au\| \end{aligned}$$

Kato's condition is again satisfied.

3. case: $g \in L^2(\mathbb{R})$, but no bound on $\|g\|_{L^2}$. Decompose $g = g_1 + g_2$:

$$\begin{aligned} g_1^{(L)} &:= g \cdot \chi_{[-L, L]} \in L^\infty \\ g_2^{(L)} &:= g - g_1 \end{aligned}$$

From the dominated convergence theorem follows:

$$\left\| g_2^{(L)} \right\|_{L^2} \xrightarrow{L \rightarrow \infty} 0$$

Thus there exists a $L \in \mathbb{R}_{>0}$ with $\left\| g_2^{(L)} \right\| < 1$. Combining case 1 for $g_1^{(L)}$ and case 2 for $g_2^{(L)}$ shows that $A + T_g$ is again self-adjoint.

11.5 Example

Consider the operator

$$H = -\Delta_{\mathbb{R}^3} + V$$

on $L^2(\mathbb{R}^3)$ with:

$$V(x) = \begin{cases} \frac{c}{\|x\|} & \text{Coulomb potential} \\ c \cdot \frac{e^{-\|x\|}}{\|x\|} & \text{Yukawa potential} \end{cases}$$

The goal is to find $\mathcal{D}(H)$ such that H is self-adjoint.

Consider the “unperturbed operator” $-\Delta_{\mathbb{R}^3}$ on $L^2(\mathbb{R}^3)$ and use a Fourier transformation

$$\hat{A} := U(-\Delta_{\mathbb{R}^3})U^{-1}f = T_g f$$

with:

$$(T_g f)(k) = \|k\|^2 f(k)$$

Define:

$$\begin{aligned} \mathcal{D}(\hat{A}) &:= \left\{ f \in L^2(\mathbb{R}^3) \mid \|k\|^2 f(k) \in L^2(\mathbb{R}^3) \right\} \\ \mathcal{D}(-\Delta_{\mathbb{R}^3}) &:= U^{-1}(\mathcal{D}(\hat{A})) = W^{2,2}(\mathbb{R}) \end{aligned}$$

Here $W^{k,p}(\mathbb{R})$ is a Sobolov space and the special case $W^{k,2}(\mathbb{R})$ is also a Hilbert space. The norm of $W^{2,2}(\mathbb{R})$ is:

$$\|f\|_{W^{2,2}}^2 = \int \left(|f|^2 + \|\nabla f\|^2 + |\nabla^2 f|^2 \right)(x) \, d^3x$$

Functions in $W^{2,2}$ are only weakly differentiable. With elliptic estimates follows:

$$\|u\|_{W^{2,2}} \leq (1 + \varepsilon) \|\Delta u\|^2 + c \|u\|^2$$

Also the *Sobolov inequality* and the *Sobolov embedding theorem* holds:

$$\|u\|_{L^{2p}} \leq \varepsilon \|\Delta u\|^2 + c(\varepsilon) \|u\|^2$$

Kato's condition is:

$$\|Vu\|_{L^2}^2 \leq a^2 \|u\|^2 + b^2 \|\Delta u\|^2$$

With

$$\frac{1}{p} + \frac{1}{q} = 1$$

holds:

$$\begin{aligned} \|Vu\|_{L^2}^2 &\leq \int_{\mathbb{R}^3} |V(x)|^2 |u(x)|^2 \, d^3x \leq \|V\|_{2q} \cdot \underbrace{\|u\|_{2p}}_{\text{Sobolev inequality}} \leq \\ &\leq \|V\|_{2q} \left(\varepsilon \|\Delta u\|^2 + c(\varepsilon) \|u\|^2 \right) \end{aligned}$$

Now holds $b := \varepsilon \|V\|_{2q}^2 < 1$ for sufficiently small ε , provided that $\|V\|_{L^{2q}} < \infty$. This is satisfied for the Yukawa potential, but for the Coulomb potential one must work a bit harder.

Appendix

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Andreas Völklein

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