Functional Analysis

lecture by
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during the winter semester 2012/13
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Literature

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– MICHEAL REED, BARRY SIMON: *Methods of Modern Mathematical Physics I - Functional Analysis;* Acad. Press, 2010; ISBN: 978-0-12-585050-6

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(paperback, small)

– DIRK WERNER: *Funktionalanalysis*; Springer, 2011; ISBN: 978-3-642-21016-7

– JOACHIM WEIDMANN: *Lineare Operatoren in Hilberträumen, Teil I: Grundlagen*; Teubner, 2000; ISBN: 3-519-02236-2

– WALTER RUDIN: *Functional Analysis*; McGraw-Hill, 1991; ISBN: 7-111-13415-X

Zorn's Lemma:

– HANS-JOACHIM KOWALSKY, GERHARD O. MICHLER: *Lineare Algebra*; de Gruyter, 2003; ISBN: 3-11-017963-6

Contents

Motivation

In linear algebra one mainly considers finite-dimensional vector spaces with additional structures like norm $\|.\|$ or scalar product $\langle .,. \rangle$.

Let $(V, \langle .,. \rangle)$ be a finite-dimensional scalar product space and $A : V \to V$ a linear map, which is self-adjoint, that means for all $u,v \in V$:

$$\langle Au,v \rangle = \langle u,Av \rangle$$

Theorem (orthonormal eigenvector basis)

There exists an orthonormal eigenvector basis $(u_i)_{i\in\{1,\dots,n\}}$, that means with the eigenvalues $\lambda_i \in \mathbb{R}$:

$$\langle u_i,u_j \rangle = \delta_{ij} \qquad\qquad Au_i = \lambda_i u_i$$

In infinite dimensions the generalization is the *spectral theorem.*

First reformulate the result from linear algebra:
Let E_{λ_i} be the orthogonal projection operator on the eigenspace corresponding to λ_i. If this eigenspace is one dimensional, this means:

$$E_{\lambda_i} v = u_i \langle u_i,v \rangle = |u_i\rangle \langle u_i|v\rangle$$

Then one can write A as:

$$A = \sum_{i=1}^{n} \lambda_i E_{\lambda_i}$$

Theorem (spectral theorem)

Let $A \in L(H)$ be a self-adjoint (selbstadjungiert) operator, then it holds:

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E_\lambda$$

$\sigma(A) \subseteq \mathbb{R}$ is the spectrum of A and E_λ the projection-valued measure (Spektralmaß).

Applications typically are differential operators, for example:

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\Delta_{\mathbb{R}^3} : C_0^\infty\left(\mathbb{R}^3\right) \to C^\infty\left(\mathbb{R}^3\right) \quad \text{linear operator}$$

Applications in more detail are studied in the lectures on partial differential equations I + II.

0 Basic Notions

Let E be a vector space (Vektorraum), for example the finite-dimensional vector space $E \simeq \mathbb{R}^3$. In the following list the later spaces are special cases of the previous ones:

- topological vector spaces
- metric spaces with a metric $d(.,.)$ (Polish spaces if complete)
- normed spaces with norm $\|.\|$ (Banach spaces if complete)
- scalar product spaces $\langle .,. \rangle$ (Hilbert spaces if complete)

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

0.1 **Definition** (metric, ε-ball, Cauchy sequence, complete, Polish space)

A map $d : E \times E \to \mathbb{R}$ is called *metric*, if for all $x,y,z \in E$ holds:

i) $d(x,y) = d(y,x)$ (symmetry)

ii) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$ (positive definiteness)

iii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

$B_\varepsilon(x) := \left\{ z \in E \middle| d(x,z) < \varepsilon \right\}$ is called *ε-ball.*
Consider the topology generated by $B_\varepsilon(x)$: A set $\Omega \subseteq E$ is open if and only if:

$$\underset{x \in \Omega}{\forall} \underset{\varepsilon \in \mathbb{R}_{>0}}{\exists} : B_\varepsilon(x) \subseteq \Omega$$

Completeness:
$(x_n)_{n \in \mathbb{N}}$ is a *Cauchy sequence* if and only if:

$$\underset{\varepsilon \in \mathbb{R}_{>0}}{\forall} \underset{N \in \mathbb{N}}{\exists} \underset{n,m \in \mathbb{N}_{>N}}{\forall} : d(x_n,x_m) < \varepsilon$$

E is *complete* if and only if every Cauchy sequence has a limit.
A complete metric space is also called a *Polish space.*

0.2 **Definition** (norm, Banach space)

Let $(E, \|.\|)$ be a *normed space*, i.e. a $\mathbb{K}$-vector space with a map $\|.\| : E \to \mathbb{R}_{\geq 0}$ called *norm* with the following properties for $x,y \in E$ and $\lambda \in \mathbb{K}$:

i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (positive definiteness)

ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity)

iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

Define the metric $d\,(x,y) := \|x - y\|$. A complete normed spaces is called *Banach space.*

Let $A : E \to F$ be a linear map between the Banach spaces $(E, \|.\|_E)$ and $(F, \|.\|_F)$.

0.3 **Definition** (continuous, bounded)

A is *continuous* (stetig) if $A^{-1}(\Omega) \subseteq E$ is open for all open $\Omega \subseteq F$.
A is *bounded* (beschränkt) if there exists a $c \in \mathbb{R}_{>0}$ such that for all $u \in E$ holds:

$$\|Au\|_F \leq c\,\|u\|_E$$

0.4 **Lemma** (continuous $\Leftrightarrow$ bounded)

A is continuous $\Leftrightarrow$ A is bounded.

(no proof)

0.5 **Definition** (dual space, sup-norm)

The *dual space* of E is the space of continuous linear mappings from E to $\mathbb{K}$:

$$E^* = L\,(E,\mathbb{K})$$

$L\,(E,F)$ is a vector space: For $A,B \in L\,(E,F)$, $\lambda,\mu \in \mathbb{K}$ and $u \in E$ define:

$$(\lambda A + \mu B)\,(u) := \lambda A\,(u) + \mu B\,(u)$$

Define also a norm on $L\,(E,F)$, which is called *sup-norm*:

$$\|A\| := \sup_{u \in E, \|u\|_E \leq 1} \|Au\|_F$$

0.6 **Theorem**

If F is complete, so is $L\,(E,F)$.
In particular E^* is a Banach space for every E.

(no proof)

1 The Hahn-Banach Theorem and Applications

As a preparation we need Zorn's lemma.

1.1 Definition (partial ordering, chain, upper bound, maximal)

Let A be a set and $\leq$ a *partial ordering* (Halbordnung), i.e. for all $a,b,c \in A$:

i) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)

ii) $a \leq a$ (reflexivity)

iii) $a \leq b \wedge b \leq a \Rightarrow a = b$ (antisymmetry)

Note: We do *not* demand that for all $a,b \in A$ holds:

$$(a \leq b) \vee (b \leq a)$$

This is a property of a ordering relation.
$(A, \leq)$ is called *partially ordered set* (teilweise geordnete Menge).
A subset $K \subseteq A$ is called *chain* (Kette, total geordnete Teilmenge) if for all $x,y \in K$ holds:

$$(x \leq y) \vee (y \leq x)$$

An element $u \in A$ is called *upper bound* (obere Schranke) of $B \subseteq A$ if $x \leq u$ for all $x \in B$.
An element $m \in A$ is called *maximal* if $m \leq a \in A \Rightarrow m = a$.

1.2 Zorn's lemma

Let $(A, \leq)$ be a partially ordered set in which every chain has an upper bound. Then there is a maximal element.

Proof

This follows from the axiom of choice, see e.g. Kowalsky: Linear algebra.

1.3 Definition (sublinear)

Let X be a *real* vector space (without topology) and $l : X \to \mathbb{R}$ linear.
$p : X \to \mathbb{R}$ is called *sublinear* if for all $x,y \in X$ and $a \in \mathbb{R}_{>0}$:

i) $p(ax) = ap(x)$

ii) $p(x+y) \leq p(x) + p(y)$

A typical example is $p(x) = \|x\|$, but p does not need to be positive. Another example is any linear mapping.

1.4 Theorem (Hahn-Banach, real version, 1927/29)

Let X be a real vector space and $Y \subseteq X$ a subspace (Untervektorraum), $p : X \to \mathbb{R}$ sublinear and $l : Y \to \mathbb{R}$ linear with $l(y) \leq p(y)$ for all $y \in Y$.
Then there is a linear extension (Fortsetzung) $\tilde{l} : X \to \mathbb{R}$ of l to X, i.e. $\tilde{l}\big|_Y = l$, such that for all $x \in X$ holds:

$$\tilde{l}(x) \leq p(x)$$

Proof

i) Assume $Y \subsetneq X$, since otherwise there is nothing to prove. Choose a vector $z \in X \setminus Y$. We want to extend l to the span of Y and $\langle z \rangle$. $\tilde{l}(z)$ needs to be prescribed. For all $y \in Y$ and $a \in \mathbb{R}$ holds:

$$\tilde{l}(y+az) \overset{\text{linearity}}{=} l(y) + a\tilde{l}(z) \overset{\text{demand}}{\leq} p(y+az)$$

If $a = 0$, the inequality is clear. By homogeneity assumptions, it is sufficient to consider the case $a = \pm 1$. We thus demand for all $y,y' \in Y$:

$$l(y) + \tilde{l}(z) \leq p(y+z)$$
$$l(y') - \tilde{l}(z) \leq p(y'-z)$$

This is equivalent to:

$$l(y') - p(y'-z) \leq \tilde{l}(z) \leq p(y+z) - l(y)$$

We can choose $\tilde{l}(z)$ if and only if:

$$l(y') - p(y'-z) \leq p(y+z) - l(y)$$

(For example set $\tilde{l}(z) = \sup_{y' \in Y} l(y') - p(y'-z)$.)

$$\Leftrightarrow \quad l(y') + l(y) \overset{\text{neariy}}{=} l(y'+y) \leq p(y+z) + p(y'-z)$$

Now prove this inequality:
From $y' + y \in Y$ follows that $l(y+y') \leq p(y+y')$ by hypothesis. Moreover, as p is sublinear, it follows:

$$p(y+z-z+y') \leq p(y'+z) + p(y'-z)$$

So the inequality is shown. Thus l can be extended to $Y + \langle z \rangle$.

ii) Consider all extensions:

$$A := \{(Z,l)\,|Y \subseteq Z \subseteq X \text{ subspace, } l : Z \to \mathbb{R} \text{ extension of } l_Y : Y \to \mathbb{R}\}$$

This set has a partial ordering $\leq$ defined by $(Z,l) \leq (Z',l')$ if $Z \subseteq Z'$ and $l'\big|_Z = l$.
For an index set I (possibly infinite, uncountable) let $K = \{(Z_\nu,l_\nu)\,|\nu \in I\}$ be a chain, i.e. for all $(Z,l)\,,(Z',l') \in K$:

$$\left((Z,l) \leq \left(Z',l'\right)\right) \vee \left((Z,l) \leq (Z,l)\right)$$

Set $Z = \bigcup_{\nu\in I} Z_\nu$ and define $l : Z \to \mathbb{R}$ by $l\big|_{Z_\nu} = l_\nu$. (Thus suppose $u \in Z$, so there is a $\nu \in I$ with $u \in Z_\nu$. Set $l\,(u) := l_\nu\,(u)$. ν need not be unique. Suppose $u \in Z_{\nu'}$, then we know that either $Z_{\nu'} \subseteq Z_\nu$ and $l_\nu\big|_{Z_{\nu'}} = l_{\nu'}$ or $Z_\nu \subseteq Z_{\nu'}$ and $l_{\nu'}\big|_{Z_\nu} = l_\nu$. In both cases we have $l_\nu\,(u) = l_{\nu'}\,(u)$, thus $l\,(u)$ is well defined.)
This (Z,l) is an upper bound, because for all $\nu \in I$ we have $Z_\nu \subseteq Z = \bigcup_{\lambda\in I} Z_\lambda$ and l is an extension of l_ν.
With Zorn's Lemma follows, that there exists an maximal element $\left(\tilde{Y},\tilde{l}\right)$.

Claim: $\tilde{Y} = X$

Proof: Otherwise there would be a vector $u \in X \setminus \tilde{Y}$, and $\tilde{l}$ could be extended to $\tilde{Y} \oplus \langle u\rangle$, as shown in i), in contradiction to the maximality of $\tilde{l}$. Thus $\left(X = \tilde{Y},\tilde{l}\right)$ is the desired extension.

$\square_{1.4}$

1.5 Theorem (Hahn-Banach, complex version)

Let X be a complex vector space and $Y \subseteq X$ a subspace. Before, we had $l\,(x) \leq p\,(x)$ as condition, which does not make sense in the complex case, since:

$$l\left(e^{\mathbf{i}\varphi}x\right) = e^{\mathbf{i}\varphi}l\,(x) \overset{\text{in general}}{\notin} \mathbb{R}$$

Let $p : X \to \mathbb{R}$ be a *seminorm*, i.e.:

i) $p\,(ax) = |a|\,p\,(x)$ (homogeneity)

ii) $p\,(x + y) \leq p\,(x) + p\,(y)$ (triangle inequality)

Let $l : Y \to \mathbb{C}$ be a linear functional with $|l\,(y)| \leq p\,(y)$ for all $y \in Y$.
Then l can be extended to X such that $|l\,(x)| \leq p\,(x)$ holds for all $x \in X$.

Proof

We also consider X as a real vector space. (u and $\mathbf{i}u$ are then linearly independent vectors.) Decompose l into its real and imaginary parts.

$$l\,(y) = l_1\,(y) + \mathbf{i}l_2\,(y)$$
$$l_1 := \mathrm{Re}\,(l\,(y))$$
$$l_2 := \mathrm{Im}\,(l\,(y))$$

l_1 and l_2 are real-linear and:

$$l_1(\mathbf{i}y) = \mathrm{Re}(l(\mathbf{i}y)) = \mathrm{Re}(\mathbf{i}l(y)) = -\mathrm{Im}(l(y)) = -l_2(y)$$

Conversely, suppose that l_1 is real-linear. Then

$$l(x) := l_1(x) - \mathbf{i} \cdot l_1(\mathbf{i}x)$$

this is indeed a complex-linear function. We know that $|l(y)| \leq p(y)$ holds for all $y \in Y$.

$$\begin{aligned} l_1(y) = \mathrm{Re}(l(y)) &\leq |l(y)| \\ \Rightarrow \qquad l_1(y) &\leq p(y) \end{aligned}$$

Theorem 1.4 yields an real-linear extension $\tilde{l}_1 : X \to \mathbb{R}$ such that $\tilde{l}_1(x) \leq p(x)$ for all $x \in X$. Set $\tilde{l}(x) = \tilde{l}_1(x) - \mathbf{i}\,\tilde{l}_1(\mathbf{i}x)$, so that $\tilde{l} : X \to \mathbb{C}$ is complex-linear.

Claim: $\left|\tilde{l}(x)\right| \leq p(x)\ \forall_{x \in X}$

Proof: Polar decomposition:

$$\begin{aligned} \tilde{l}(x) &= re^{\mathbf{i}\varphi} \\ \left|\tilde{l}(x)\right| &= r = e^{-\mathbf{i}\varphi}\tilde{l}(x) \underset{\text{complex-linear}}{\overset{\tilde{l}\text{ is}}{=}} \tilde{l}\left(e^{-\mathbf{i}\varphi}x\right) = \mathrm{Re}\left(\tilde{l}\left(e^{-\mathbf{i}\varphi}x\right)\right) = \\ &= \tilde{l}_1\left(e^{-\mathbf{i}\varphi}x\right) \leq p\left(e^{-\mathbf{i}\varphi}x\right) \overset{\text{homogeneity}}{=} p(x) \end{aligned}$$

$\square_{\text{Claim}}$

Now to applications:

1.6 Theorem

Let $(X, \|.\|)$ be a normed $\mathbb{K}$-space (real or complex), $Y \subseteq X$ a subspace. Let φ be a continuous linear functional from Y to $\mathbb{K}$, i.e. for all $y \in Y$ holds:

$$|\varphi(y)| \leq c\,\|y\|$$

Then φ can be continued to all of X with the same supnorm, i. e.:

$$\|\tilde{\varphi}\| := \sup_{x \in X, \|x\| \leq 1} |\varphi(x)| = \|\varphi\| := \sup_{y \in Y, \|y\| \leq 1} |\varphi(y)|$$

Proof

Apply the Hahn-Banach theorem with $\varphi := c\,\|x\|$. $\square_{1.6}$

1.7 Corollary

Let X be a normed space and $u_0 \in X$ with $\|u_0\| = 1$. Then there exists a linear functional $\varphi : X \to \mathbb{K}$ such that:

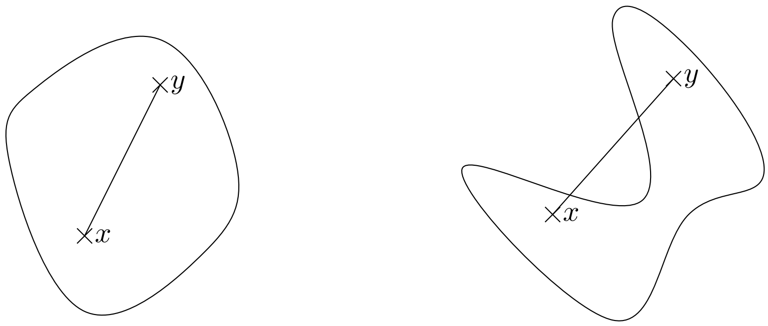
$$\varphi(u_0) = 1 \qquad\qquad \|\varphi\| = 1$$

Proof

Let $Y := \langle u_0 \rangle$ and define $\varphi_0 : \langle u_0 \rangle \to \mathbb{K}$ by $\varphi_0 (u_0) = 1$. Extend φ_0 by the Hahn-Banach theorem 1.6. $\square_{1.7}$

The Hahn-Banach theorem also has a geometric formulation. Consider only the real case:
A set $K \subseteq X$ is called *convex* if for all $x,y \in K$ and $\tau \in [0,1]$:

$$\tau x + (1 - \tau) y \in K$$



(a) konvex set (b) concave set

Figure 1.1: convexity

Geometric question:

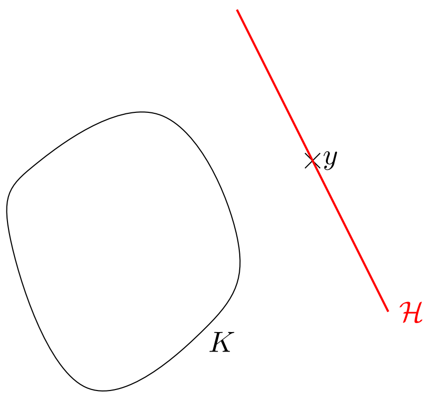


Figure 1.2: not intersecting hyperplane

Is there a hyperplane $\mathcal{H}$, which meets $y \notin K$, but does not intersect K.

1.8 **Definition** (interior point)

$x_0 \in K$ is an *interior point* (innerer Punkt) *of* K *with respect to* $u \in X$ if there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $x_0 + tu \in K$ for all $t \in (-\varepsilon,\varepsilon)$.
$x_0 \in K$ is an *interior point* if for all $u \in X$ there is a $\varepsilon = \varepsilon(u) \in \mathbb{R}_{>0}$ such that $x_0 + tu \in X$ for all $t \in (-\varepsilon,\varepsilon)$.

1.9 Theorem (geometric Hahn-Banach)

Let $K \neq \emptyset$ be convex and all points of K be interior points. Let $y \notin K$. Then there is a linear functional $l : X \to \mathbb{R}$ such that $l(x) < 1$ for all $x \in K$ and $l(y) = 1$.
$\mathcal{H} := \left\{x \in X \middle| l(x) = 1\right\}$ defines a hyperplane. Now $y \in \mathcal{H}$ and $l\big|_K < 1$ mean that K lies in one half-space.

First introduce a suitable sublinear functional. Without loss of generality, assume $0 \in K$ (otherwise shift K).

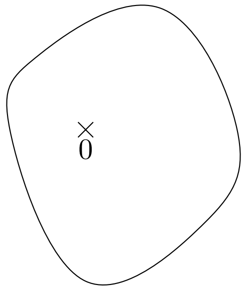


Figure 1.3: $0 \in K$

The functional $p : K \to \mathbb{R}_{\geq 0}$ with

$$p(x) := \inf \left\{ a \in \mathbb{R}_{>0} \middle| \frac{x}{a} \in K \right\}$$

is called gauge (Eichung).
Since x is an interior point, we know that $\frac{x}{a} \in K$ if $a > 1 - \varepsilon(x)$.
p is even defined on all of X, because for $x \in X$, now $\tau x \in K$ if $|\tau|$ is sufficiently small, because $0 \in K$ is an interior point.

$$p(x) < 1 \quad \Leftrightarrow \quad x \in K$$

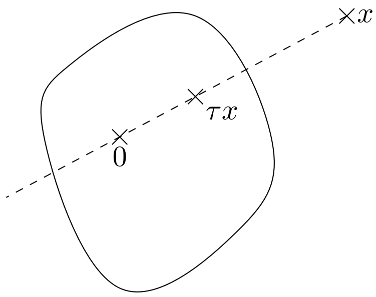


Figure 1.4: $x \notin K$, $\tau x \in K$

1.10 Lemma

p is sublinear.

Proof

The homogeneity is clear from the definition.

sub-additivity (triangle equation):
Take $x,y \in K$ and choose $a,b \in \mathbb{R}_{>0}$ such that $\frac{x}{a},\frac{y}{b} \in K$. The convexity of K implies for all $\tau \in [0,1]$:

$$\tau\frac{x}{a} + (1-\tau)\,\frac{y}{b} \in K$$

Choose $\tau = \dfrac{a}{a+b}$, then holds $1-\tau = \dfrac{b}{a+b}$, which gives:

$$\Rightarrow \quad \frac{1}{a+b}\,(x+y) \in K$$

$$p\,(x+y) \leq a+b$$

Taking the infimum over a and b gives $p\,(x+y) \leq p\,(x) + p\,(y)$:

$$p\,(x+y) = \inf \underbrace{\left\{ c \in \mathbb{R}_{>0} \middle| \frac{x+y}{c} \in K \right\}}_{\ni a+b} \leq a+b$$

$$p\,(x) = \inf \left\{ a \middle| \frac{x}{a} \in K \right\} \quad \Rightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{a\in\mathbb{R}_{>0}}{\exists} : p\,(x) \geq a - \varepsilon$$

$$p\,(y) = \inf \left\{ b \middle| \frac{x}{b} \in K \right\} \quad \Rightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{b\in\mathbb{R}_{>0}}{\exists} : p\,(y) \geq b - \varepsilon$$

$\square_{1.10}$

1.11 Lemma

$p\,(x) < 1 \Leftrightarrow x \in K$

Proof

If $x \notin K$ then $\frac{1}{a}x \notin K$ for all $0 < a < 1$ and so $p\,(x) \geq 1$.

For all $x \in K$ exists an $\varepsilon = \varepsilon\,(x) \in \mathbb{R}_{>0}$ with $(1+t)\,x \in K$ for all $t \in (-\varepsilon,\varepsilon)$.

$$\Rightarrow \quad \left(1 + \frac{\varepsilon}{2}\right) x \in K$$

$$\Rightarrow \quad p\,(x) \leq \frac{1}{1+\frac{\varepsilon}{2}} < 1$$

$\square_{1.11}$

Proof of Theorem 1.9

Introduce l on $\langle y\rangle$ by $l(y) = 1$. (Assume again that $0 \in K$ and so $y \neq 0$.)
Write $z = ay \in \langle y\rangle$ with $a \in \mathbb{R}$.

- If $a < 0$, then $l(z) = a \cdot l(y) = a < 0$ but $p(z) \geq 0$ and thus the inequality $l(z) \leq p(z)$ is trivially satisfied.
- If $a > 0$ it holds:

$$l(z) = a \underset{\Rightarrow p(y)\geq 1}{\overset{y\notin K}{\leq}} a \cdot p(y) \underset{\text{homogeneity}}{\overset{\text{positive}}{=}} p(ay) = p(z)$$

So for all $z \in \langle y\rangle$ holds $l(z) \leq p(z)$.
The Hahn-Banach Theorem yields an extension $l : X \to \mathbb{R}$ such that $l(x) \leq p(x)$ for all $x \in X$.
Therefore for all $x \in K$ we have:

$$l(x) \leq p(x) < 1$$

$\square_{1.9}$

2 Normed Spaces

Let $(E, \|.\|)$ be a normed space and let the open balls $B_\varepsilon(x) = \left\{ y \middle| \|x-y\| < \varepsilon \right\}$ generate the topology on E.

2.0.1 Definition (equivalent norms)

Two norms $\|.\|_1$ and $\|.\|_2$ are *equivalent*, if there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_2$$

2.0.2 Theorem

Equivalent norms give rise to the same topology.

(No proof)

2.0.3 Theorem

If E is finite dimensional, then any two norms on E are equivalent.

(No proof)

TODO: Rest überarbeiten

Let $F \subseteq E$ be a *closed* subspace. Define ${}^{E}\!/_{F}$ as follows:

$$x \sim y :\Leftrightarrow x - y \in F$$

defines an equivalence relation on E.

$${}^{E}\!/_{F} := {}^{E}\!/_{\sim}$$

is a vector space.

$$\|u\|_{E/F} := \inf_{\substack{\hat{u} \in E \\ \hat{u} \text{ represents } u}} \|\hat{u}\|_E$$

2.0.4 Theorem

$\left({}^{E}\!/\!_{F}, \|.\|_{{}^{E}\!/\!_{F}} \right)$ is a normed space. The closedness is essential. Suppose $F \subseteq E$ is not closed. Then there exists an $x \in {}^{E}\!/\!_{F}$ with $x \in \overline{F}$, thus there is a (x_n), $x_n \in F$ with $x_n \to x$.

Let $[x] \in {}^{E}\!/\!_{F}$ be the equivalence class. Then $[x] \neq 0$ but $\|[x]\| = \inf_{\hat{x}} \|\hat{x}\| \leq \inf \|x - x_n\| = 0$. Choose $x - x_n \stackrel{\sim}{=} x$.

Another operation. Let E and F be normed spaces.

$$E \times F = \left\{ (u,v) \middle| u \in E, v \in F \right\} \qquad \text{Cartesian product}$$

$$\|(u,v)\|_{E \times F} = \|u\|_E + \|v\|_F$$

is a norm on $E \times F$.

A complete normed space is called *Banach space*.

2.0.5 Definition

A normed space is called *separable*, if there is a countable dense subset.

2.0.6 Examples

ℓ^∞ bounded sequences $(a_n)_{n \in \mathbb{N}}$, $a_n \in \mathbb{R}$ or $\mathbb{C}$ with $\|(a_n)\|_\infty := \sup_n |a_n|$ is a Banach space.

$$A := \left\{ (a_n) \middle| a_{2n} = 0 \underset{n \in \mathbb{N}}{\forall} \right\} \subseteq \ell^\infty$$

is a closed subspace.

$${}^{\ell^\infty}\!/\!_{A} \stackrel{\sim}{=} \left\{ (a_n) \middle| a_{2n+1} = 0 \underset{n \in \mathbb{N}}{\forall} \right\}$$

$$B = \{(a_n) \text{ finite sequence}\} \subseteq \ell^\infty$$

is a subspace, but not closed in ℓ^∞. For example $a_n = \frac{1}{n} \in \ell^\infty \setminus B$. Consider $x_n \in B$ with $x_n = (a_{n_l})_{l \in \mathbb{N}}$ and:

$$a_{n_l} = \begin{cases} \frac{1}{l} & \text{if } l \leq n \\ 0 & \text{if } l > n \end{cases}$$

Then $x_n \to x$, $x = (a_n)$.

$$\overline{B} = \left\{ (a_n) \middle| a \xrightarrow{n \to \infty} 0 \right\}$$

ℓ^∞ is separable.

2.0.7 Example

Let $1 \leq p < \infty$.

$$\ell^p = \left\{ \text{sequences } (a_n) \mid \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

$$\|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \qquad \ell^p\text{-norm}$$

ℓ^p is a normed space (Hölder's inequality, Minkowski inequality). It is separable (see exercises).

2.0.8 Example

Let (Ω,μ) be a measure space.

$$L^p(\Omega) 1 \leq p < \infty \qquad \|f\|_p = \left(\int_\Omega |f(x)|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

$$L^\infty(\Omega) \qquad \|f\|_\infty = \operatorname{supess}_\Omega |f(x)| = \sup \left\{ L \middle| \mu\left(f^{-1}([L,\infty))\right) > 0 \right\}$$

2.1 Non-Compactness of the Unit Ball

Let $(E, \|.\|)$ be a normed vector space.

$$K := \overline{B_1(0)} = \left\{ x \in E \middle| \|x\| \leq 1 \right\}$$

If $\dim(E) < \infty$, K is compact by the Heine-Borel Theorem.

2.1.1 Theorem

If E is infinite-dimensional, then K is not sequentially compact (folgenkompakt). Thus we want to construct a sequence (y_n), $y_n \in K$, which has no convergent subsequence.

2.1.2 Lemma

Let $Y \subsetneq E$ be a proper (echter) closed subspace. Then there is a vector $z \in E/_{Y}$ such that:

$$\|z\| = 1 \qquad \|z - y\| > \frac{1}{2} \underset{y \in Y}{\forall}$$

TODO: Abb6

$$\overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset$$

Proof

Choose $x \in {}^{E}/_{Y}$. As ${}^{E}/_{Y}$ is open, there is a $\delta \in \mathbb{R}_{>0}$ with $B_\delta(x) \cap Y = \emptyset$. Thus:

$$\inf_{y \in Y} \|x - y\| := d > 0$$

Choose $y_0 \in Y$ such that $\|x - y_0\| < 2d$. Set $z' = x - y_0$. Then $\|z'\| < 2d$ and $\|z' - y\| \geq d$ for all $y \in Y$. Thus $z := \frac{z'}{\|z'\|}$ has the desired properties. $\square_{2.1.2}$

Proof of Theorem ??2.1.1

Choose inductively a sequence (y_n): $y_1 \in K$ arbitrary. $Y_1 := \langle y_1 \rangle$ is a one dimensional subspace, which is closed. Choose $y_2 \in K$ such that $\|y_2 - y\| > \frac{1}{2}$ for all $y \in Y_1$. (This is possible according to Lemma ??2.1.2)

Suppose $y_1, \ldots, y_n$ are given. $Y_n := \langle y_1, \ldots, y_n \rangle$ is closed. So there exists a $y_{n+1} \in K$ such that:

$$\|y_{n+1} - y\| > \frac{1}{2} \qquad \underset{y \in Y_n}{\forall}$$

This sequence has the following properties:

- $y_k \in K$
- $\forall_{k,l,k \neq l}$ then $\|y_k - y_l\| > \frac{1}{2}$, because: For example if $k < l$, then $y_k \in Y_{l-1} = \langle y_1, \ldots, y_{l-1} \rangle$ and we know by construction that $\|y_l - y\| > \frac{1}{2}$ for all $y \in Y_{l-1}$. $\Rightarrow \|y_l - y_k\| > \frac{1}{2}$

This implies that (y_k) has no convergent subspace. $\square_{2.1.2}$

2.2 Spaces of linear Mappings, Dual Spaces

Let E,F be normed spaces.

$A : E \to F$ is continuous if and only if it is bounded, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\|Au\|_F \leq C \|u\|_E \qquad \underset{u \in E}{\forall}$$

Denote by $L(E,F)$ the normed space of all bounded linear maps from E to F with:

$$\|A\| = \sup_{\|u\| \leq 1} \|Au\| = \sup_{\|u\| = 1} \|Au\|$$

2.2.1 Lemma

If $B \in L(E,F)$ and $A \in L(F,G)$ then:

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|$$
$$\|Au\| \leq \|A\| \cdot \|u\|$$

Scharz inequality or Kato inequality.

(no proof)

2.2.2 Theorem

If F is complete, so is $L(E,F)$.
Special case: $F=\mathbb{R}$, $\|x\|_{\mathbb{R}}=|x|$
$E^* := L(E,\mathbb{R})$ is the dual space.
If $\varphi \in E^*$ and $u \in E$.

$$\varphi(u) = (\varphi,u) \qquad \text{dual pairing (dt. duale Paarung)}$$

$$(.,.) : E^* \times E \to \mathbb{R}$$

is a continuous bilinear map.
For $u \in E$,

$$(.,u) : E^* \to \mathbb{R}$$

defines an element of $E^{**} = L(E^*,\mathbb{R})$. This gives rise to a linear mapping:

$$\iota : E \to E^{**}$$

2.2.3 Theorem

ι is an isometric embedding of E into E^{**}.

Proof

$$\|\iota(u)\| := \sup_{\varphi\in E^*,\|\varphi\|=1} \|(\iota(u))(\varphi)\| = \sup_{\varphi\in E^*,\|\varphi\|=1} \|\varphi(u)\| \stackrel{?}{=} \|u\|$$

$$\|\varphi\| = \sup_{v\in E,\|v\|=1} |\varphi(v)|$$

$$\|\varphi(u)\| \leq \|\varphi\| \cdot \|u\| \stackrel{\|\varphi\|=1}{=} \|u\|$$
$$\Rightarrow \quad \sup_{\varphi\in E^*,\|\varphi\|=1} \|\varphi(u)\| \leq \|u\|$$

To prove $\|\iota(u)\| \geq \|u\|$ apply the Hahn-Banach theorem.
Let $l : \langle u\rangle \to \mathbb{R}$ be the linear map with $l(u) = \|u\|$, thus:

$$\|l\| = \sup_{v\in\langle u\rangle,\|v\|=1} (l(v)) = \sup\left(l\left(\pm\frac{u}{\|u\|}\right)\right) = 1$$

By the Hahn-Banach theorem we can extend l to

$$\tilde{l} : E \to \mathbb{R}$$

with $\left\|\tilde{l}\right\| = 1$. Then:

$$\sup_{\varphi\in E^*,\|\varphi\|=1} \varphi(u) \geq \tilde{l}(u) = \|u\|$$

$\square_{2.2.3}$

$\iota : E \hookrightarrow E^*$ is an isometric embedding.

2.2.4 Definition

A Banach space is called *reflexive* (reflexiv), if ι is bijective, i.e. $E = E^{**}$.

2.2.5 Example

Let $E = \ell_1$be the space of absolutely convergent functions with $\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty$.

Let $(\lambda_n) \in \ell_\infty$ be a bounded sequence.

$$\Lambda : E \to \mathbb{R}$$
$$\Lambda((a_n)) = \sum_{n=1}^{\infty} \lambda_n a_n$$

$$|\Lambda((a_n))| = \left|\sum_{n=1}^{\infty} \lambda_n a_n\right| \le \sum_{n=1}^{\infty} |\lambda_n| \cdot |a_n| \le \|(\lambda_n)\|_\infty \sum_{n=1}^{\infty} |a_n| = \|(\lambda_n)\|_\infty \cdot \|(a_n)\|_1$$

Thus Λ is bounded and:

$$\|\Lambda\| \le \sup_{k \in \mathbb{N}} |\lambda_k|$$

It is even $\|\Lambda\| = \sup_{k\in\mathbb{N}} |\lambda_k|$.

Let E,F be Banach spaces.

$L(E,F)$ with $\|A\| = \sup_{\|u\|=1} \|Au\|$ is again a Banach space.

$E^* = L(E,\mathbb{R})$ dual space

$L : E \to E^{**}$ is norm preserving $\|Lu\| \overset{E^{**}}{=} \|u\|_E$. Therefore L is injective, because if $Lu = 0$, then $\|u\|_E = \|Lu\| = 0$ and therefore $u = 0$.

Example

$E = \ell_1$; $(\lambda_k) \in \ell_\infty$ defines a linear Functional Λ on ℓ_1.

$$\Lambda((a_k)) := \sum_{k=1}^{\infty} \lambda_k a_k$$

Λ is bounded, $\Lambda \in \ell_1^*$.

Claim: Every bounded linear functional on ℓ_1 is of this form, i. e. $\ell_1^* = \ell_\infty$.

Proof: Let $\Lambda \in \ell_1^*$. Choose $u_l \in \ell_1$ by $u_l = (0,\ldots,1,0,\ldots)$ with a one at the l-th position.

Set $\lambda_l = \Lambda(u_l)$. Then:

$$|\lambda_l| = |\Lambda(u_l)| \le \underbrace{\|\Lambda\|}_{<\infty} \cdot \underbrace{\|u_l\|}_{=1} \le \|\Lambda\| < \infty$$

So $(\lambda_l) \in \ell_\infty$.

Let (a_k) be a finite sequence, with only zeros after $k = K$. Then:

$$\Lambda\left((a_k)\right) = \Lambda\left(\sum_{k=1}^{K} a_k u_k\right) = \sum a_k \Lambda\left(u_k\right) = \sum \lambda_k a_k$$

Since the finite sequences are dense in ℓ_1, the claim follows. $\square_{2.2.5}$

So $\ell_1^* = \ell_\infty$ and one could assume $\ell_\infty^* = \ell_1$, but this is not the case (see exercises).

Thus $\ell_1^{**} \neq \ell_1$, which means, that ℓ_1 is *not* reflexive.

2.3 Weak Convergence (Schwache Konvergenz)

Let E be a Banach space and(u_n) a sequence in E.

Normal convergenz: $u_n \to u$ if and only if $\|u - u_k\| \xrightarrow{k\to\infty} 0$.

2.3.1 Definition

A sequence (u_n) in E *converges weakly* to u, $u_n \rightharpoonup u$, if for all $\varphi \in E^*$ the sequence $\varphi\left(u_n\right)$ converges to $\varphi\left(u\right)$, $\varphi\left(u_n\right) \to \varphi\left(u\right)$.

(u_n) is a *weak Cauchy sequence*, if for all $\varphi \in E^*$ the sequence $\varphi\left(u_n\right)$ is Cauchy.

2.3.2 Theorem

The weak limit is unique.

Proof

Let (u_n) be a sequence in E which converges $u_n \rightharpoonup u$ and $u_n \rightharpoonup u'$. For $\varphi \in E^*$:

$$\varphi\left(u_n\right) \to \varphi\left(u\right) \qquad\qquad \varphi\left(u_n\right) \to \varphi\left(u'\right)$$

$$0 \to \varphi\left(u - u'\right)$$

So $\varphi\left(u - u'\right) = 0$ for all $\varphi \in E^*$.

Claim: $v := u - u' = 0$

Proof: Assume convertly that $v \neq 0$.

Choose $\varphi : \langle v \rangle \to \mathbb{R}$ with $\varphi\left(v\right) = 1$. By the Hahn-Banach theorem φ can be extended continuously to E.

Therefore there exists a $\varphi \in E^*$ with $\varphi\left(v\right) = 1$, which is a contradiction to $\varphi\left(v\right) = 0$. $\square_{2.3.2}$

2.3.3 Theorem

Every convergent sequence converges weakly.

Proof

Suppose that $u_n \to u$. Let $\varphi \in E^*$, so:

$$|\varphi(u_n) - \varphi(u)| = |\varphi(u_n - u)| \leq \underbrace{\|\varphi\|}_{\in\mathbb{R}} \cdot \|u_n - u\| \to 0$$

$$\begin{aligned} \Rightarrow \quad & \varphi(u_n) \to \varphi(u) \\ \Rightarrow \quad & u_n \rightharpoonup u \end{aligned}$$

2.3.4 Example

$E = \left\{ (a_n) \,|\, a_n \xrightarrow{n\to\infty} 0 \right\} \subsetneq \ell_\infty$ with $\|(a_n)\| = \sup_n |a_n|$ is a Banach space.

Let $u_n = (0,\ldots,0,1,0,\ldots)$ be the sequence with a one at the n-th position and zeros elsewhere. For $n \neq m$ we have:

$$\|u_n - u_m\| = \sup\{0, |1|, |-1|\} = 1$$

Thus (u_n) is *not* a Cauchy sequence. Let $\varphi \in E^*$. Then φ can be represented as (see exercises):

$$\varphi((a_n)) = \sum_k \lambda_k a_k \qquad \text{with } (\lambda_k) \in \ell_1$$

$$\|\varphi\| = \sum_{k=1}^{\infty} |\lambda_k|$$

$$\varphi(u_n) = \sum_{k=1}^{\infty} \lambda_k \delta_{kn} = \lambda_n \xrightarrow{n\to\infty} 0$$

Because $(\lambda_n) \in \ell_1$ we know that $\lambda_n \to 0$. This means, that $u_k \rightharpoonup 0$. $\square_{2.3.4}$

This is used in Partial Differential Equations.

If $\mathscr{S}(u_n) \to \inf \mathscr{S}$, then not necessarily $u_n \to u$, but $u_n \rightharpoonup u$.

Consider $A_n \in L(E,F)$.

- $A_n \to A$ in $L(E,F)$, meaning that $\|A_n - A\| \to 0$. norm convergence
- $A_n u \to Au$ in F for all $u \in E$. strong convergence
- $A_n u \rightharpoonup Au$ for all $u \in E$. weak convergence

2.4 The Baire Category Theorem

Let E be a metric space (e.g. a normed space).

2.4.1 Definition

A subset $A \subseteq E$ is called *nowhere dense* (nirgends dicht), if $\overset{\circ}{\overline{A}} = \emptyset$.

A is called *of first category* (or meagre) it it can be written as a countable union of nowhere dense sets.

Otherwise it is *of second category*.

Example:

- $\mathbb{N} \subseteq \mathbb{R}$ is nowhere dense, $\overline{\mathbb{N}} = \mathbb{N}$, $\overset{\circ}{\mathbb{N}} = \emptyset$.
- $\mathbb{Q} \subseteq \mathbb{R}$: $\overline{\mathbb{Q}} = \mathbb{R}$, $\overset{\circ}{\overline{\mathbb{Q}}} = \overset{\circ}{\mathbb{R}} = \mathbb{R}$

2.4.2 Theorem (Baire)

Let $E \neq \emptyset$ be a complete metric space (Polish space). Then E is of second category.

Proof

Assume conversely that $E = \bigcup_{n \in \mathbb{N}} M_n$ and the sets M_n are nowhere dense.

Without loss of generality assume that M_n are closed, otherwise replace M_n by $\overline{M_n}$.

We shall construct inductively balls $\overline{B_n} = \overline{B_{r_n}(x_n)}$ such that $\overline{B_{n+1}} \subseteq \overline{B_n}$, $r_n < 2^{-n}$ and $B_n \cap M_n = \emptyset$ for all n.

TODO: Abb7

Then the points x_n form a Cauchy sequence, because for all $n < m$ we have $x_{n+1} \in B_n$ and so $\|x_n - x_{n+1}\| < r_n < 2^{-n}$:

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_m\| \leq \ldots \leq \\ &\leq 2^{-n} + 2^{-(n+1)} + \ldots + 2^{-(m-1)} \leq 2^{-n}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) \leq 2 \cdot 2^{-n} \end{aligned}$$

Since E is complete, $x_n \to x_0 \in E$. Then $x_0 \in \overline{B_n}$ for all n, which implies $x_0 \notin M_n$ and thus the contradiction $x_0 \notin \bigcup_n M_n = E$ follows.

Construction of the balls $\overline{B_n}$:

M_1 is nowhere dense $\Rightarrow B_1(0) \not\subseteq M_1$. So there exists a $x_1 \in B_1(0) \setminus M_1$. Since M_1 is closed, $B_1(0) \setminus M_1$ is open and therefore there exists a radius r_1 such that $B_{2r_1}(x_1)$ is contained in $B_1(0) \setminus M_1$ and thus $\overline{B_{r_1}(x_1)} \cap M_1 = \emptyset$.

Suppose $\overline{B_n}$ has been constructed. M_{n+1} is nowhere dense and closed and so there is a $x_{n+1} \in \overline{B_n} \setminus M_{n+1}$ and $r_{n+1} < 2^{-(n+1)}$ such that $B_{2r_{n+1}}(x_{n+1}) \subseteq \overline{B_n} \setminus M_{n+1}$. Then $\overline{B_{r_{n+1}}} \cap M_{n+1} = \emptyset$.

2.4.3 **Theorem** (Uniform boundedness principle, Prinzip der gleichmäßigen Beschränktheit)

Let E be a Banach space and F a normed space. Let T_i be a sequence in $L(E,F)$ which is point-wise bounded, i.e. for all $u \in E$:

$$\sup_i \|T_i u\| \leq C(u) < \infty$$

Then sup-norms of T_i are bounded:

$$\sup_i \|T_i\| = \sup_i \sup_{\|u\|=1} \|T_i u\| \leq C < \infty$$

(Thus there exists a constant $C \in \mathbb{R}_{>0}$ such that $\|T_i u\| \leq C$ for all $i \in \mathbb{N}$ and for all $u \in E$ with $\|u\| = 1$.)

Proof

Let $M_n = \left\{u \in E \middle| \sup_i \|T_i u\| \leq n\right\}$. Then M_n are closed (by continuity of $T_i \in L(E,F)$, $\|T_i u_k\| \xrightarrow{k\to\infty} \|T_i u\|$ if $u_k \to u$).

$E = \bigcup_n M_n$, because for any $u \in E$, $\sup \|T_i u\| < \infty$ and thus $u \in M_n$ if $n > \|T_1 u\|$.

If all the sets M_n had empty interior, we would get a contradiction to Baire's theorem.

So there exists an $n_0 \in \mathbb{N}$ such that $\overset{\circ}{M_n} \neq \emptyset$ and thus there are $u_0 \in E$ and $r \in \mathbb{R}_{>0}$ such that $B_r(u_0) \subseteq M_{n_0}$.

TODO: Abb8

For all $v \in B_r(u_0)$ we know that $\sup_i \|T_i v\| \leq n_0$ which is equivalent to:

$$\sup_{v \in B_r(u_0)} \|T_i v\| \leq n_0 \qquad \forall_{i \in \mathbb{N}}$$

Let $w \in B_r(0)$ be arbitrary. Then $v := u_0 + w \in B_r(u_0)$.

$$T_i w \overset{T_i \text{ linear}}{=} T_i v - T_i u_0$$

$$\|T_i w\| \leq \|T_i v\| + \|T_i u_0\| \leq n_0 + \underbrace{\sup_i \|T_i u_0\|}_{<\infty, \text{ because } T_i \text{point-wise bounded}}$$

$$\Rightarrow \quad \|T_i w\| \leq C \qquad \forall_{w \in B_r(0)}$$

$$\Rightarrow \quad \|T_i \tilde{w}\| \leq \tilde{C} = \frac{C}{r} \qquad \forall_{\tilde{w} \in \overline{B_1(0)}}$$

So $\|T_i\| \leq \tilde{C}$ for all $i \in \mathbb{N}$ and so $\|T_i\|$ is bounded. $\square_{2.4.3}$

2.4.4 **Corollary**

Let E be a normed space, not necessarily complete, and (u_n) a weak Cauchy sequence. Then $\|u_n\|$ is a bounded sequence.

Proof

$E^* = L(E,\mathbb{R})$ is a Banach space. For all $\varphi \in E^*$ we know, that $\varphi(u_n)$ is a Cauchy sequence.

$$\Rightarrow \quad |\varphi(u_n)| < C(\varphi)$$

Applying Theorem **??**2.4.3 yields:

$$|\varphi(u_n)| < C \qquad \underset{\varphi \text{ with } \|\varphi\|=1}{\forall}$$

$$\Leftrightarrow \quad \sup_{n\in\mathbb{N}} \sup_{\varphi,\|\varphi\|=1} |\varphi(u_n)| < C$$

For any $v \in E$:

$$\sup_{\varphi\in E^*,\|\varphi\|=1} |\varphi(v)| = \|v\|$$

by the Hahn-Banach theorem:

- $|\varphi(v)| \leq \|\varphi\| \cdot \|v\| = \|v\|$
- Choose $\varphi : \langle v\rangle \to \mathbb{R}$ with $\varphi(v) = \|v\|$, then $\|\varphi\|$. By Hahn-Banach theorem we can extend φ to $\tilde{\varphi} : E \to \mathbb{R}$ such that $\|\tilde{\varphi}\| = 1$. Then $\tilde{\varphi} = \|v\| \Rightarrow \sup_{\|\varphi\|=1} |\varphi(v)| \geq \|v\|$.

Thus $\sup_n \|u_n\| < C$. $\square_{2.4.4}$

TODO: Rest

Consequences of Baire's category theorem

$\to$ open mapping theorem

E,F Banach spaces:

$A \in L(E,F)$ surjective $\Rightarrow$ A is open

$A \in L(E,F)$ bijective $\Rightarrow$ A^{-} is continuous

$\Rightarrow$ Closed Graph theorem

$A : E \to F$,

$$\operatorname{graph}(A) := \left\{(u,Au) \,\middle|\, u \in E\right\} \subseteq E \times F$$

If A is linear and $\operatorname{graph}(A)$ is closed, then A is continuous.

$\operatorname{graph}(A)$ closed means: for all $u_n \in E$ with $u_n \to u$ and $Au_n \to v$, the point $(u,v) \in \operatorname{graph}(A)$, i.e. $Au = v$.

A is continuous means: for all $u_n \in E$ with $u_n \to u$, the sequence $Au_n \to v$ converges and $Au = v$.

Neumann series:

$A,B,C,\ldots \in L(E,E) = L(E)$, E Banach space.

If $\|B\| < 1$, then

$$C := \sum_{n=0}^{\infty} B^n$$

defines an element of $L(E)$.

The series converges absolutely, because:

$$\|B^n\| = \left\|B \cdot B^{n-1}\right\| \leq \|B\| \left\|B^{n-1}\right\| \leq \ldots \leq \|B\|^n$$

2.4.5 2.5.2 Theorem

$$C = (\mathbb{1} - B)^{-1}$$

Proof

$$(\mathbb{1} - B)\, C = (\mathbb{1} - B) \sum_{n=0}^{\infty} B^n = \big(\mathbb{1} + B + B^2 + \ldots\big) - \big(B + B^2 + \ldots\big) = \mathbb{1}$$

$\square_{2.4.5}$

2.4.6 Theorem

The set of all continuously invertible mappings is open in $L(E)$.

Proof

Assume that $A \in L(E)$ is continuously invertible, i.e. A^{-1} exists and $A^{-1} \in L(E)$. Set:

$$\varepsilon = \frac{1}{2\,\|A^{-1}\|}$$

Let us show, that every element of $B_\varepsilon(A) \subseteq L(E)$ is continuously invertible.
Let $C \in B_\varepsilon(A)$, i.e. $\|A - C\| < \varepsilon$.

$$C = A - (A - C) = A\big(\mathbb{1} - \underbrace{A^{-1}(A - C)}_{=:B}\big)$$

Then:

$$\|B\| \leq \big\|A^{-1}\big\| \cdot \|A - C\| = \big\|A^{-1}\big\| \cdot \frac{1}{2\,\|A^{-1}\|} = \frac{1}{2}$$

Hence $\mathbb{1} - B$ is continuously invertible by the Neumann series. Therefore:

$$C^{-1} = (\mathbb{1} - B)^{-1} \cdot A^{-1}$$

is continuous. $\square_{2.4.6}$

3 Hilbert spaces

Let H be a real ($\mathbb{K} := \mathbb{R}$) or complex ($\mathbb{K} := \mathbb{C}$) vector space with *scalar product*:

$$\langle .,. \rangle : H \times H \to \mathbb{K}$$

i) $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0 \Rightarrow u = 0$.

ii) Linear in the second and anti-linear in the first argument:

$$\langle \lambda u,v \rangle = \overline{\lambda} \langle u,v \rangle$$

iii) Symmetry: $\overline{\langle u,v \rangle} = \langle u,v \rangle$

Define corresponding norm:

$$\|u\| := \sqrt{\langle u,u \rangle}$$

3.0.1 Definition

A complete scalar product space is called *Hilbert space*.

Schwarz inequality:

$$|\langle u,v \rangle| \leq \|u\| \cdot \|v\|$$

3.0.2 **Lemma** (parallelogram equality (Parallelogramm-Gleichung))

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

TODO: Abb9

Proof

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v,u+v \rangle = \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle \\
\|u-v\|^2 &= \langle u-v,u-v \rangle = \langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle
\end{aligned}$$

$\square_{3.0.2}$

3.0.3 Definition

i) Vectors $u,v \in H$ are called *orthogonal*, $u \perp v$, if $\langle u,v \rangle = 0$.

ii) Subspaces $M_1,M_2 \subseteq H$ are orthogonal, $M_1 \perp M_2$, if $\langle u,v \rangle = 0$ for all $u \in M_1$ and $v \in M_2$.

iii) A family $(u_i)_{i \in I}$ of vectors $u_i \in H$ is called *orthonormal*, if:

$$\langle u_i,u_j \rangle = \delta_{ij}$$

3.0.4 Theorem (Bessel's inequality)

Let $(u_i)_{1 \leq i \leq N}$ be an orthonormal family. Then for all $u \in H$:

$$\begin{aligned}
\|u\|^2 &= \sum_{i=1}^{N} \langle u_i,u \rangle^2 + \left\| u - \sum_{i=1}^{N} u_i \langle u_i,u \rangle \right\|^2 \\
\|u\|^2 &\geq \sum_{i=1}^{N} \langle u_i,u \rangle^2
\end{aligned}$$

Proof

It remains to prove the equality.

$$\begin{aligned}
\left\| u - \sum_{i=1}^{N} u_i \langle u_i,u \rangle \right\|^2 &= \left\langle u - \sum_{i=1}^{N} u_i \langle u_i,u \rangle , u - \sum_{j=1}^{N} u_j \langle u_j,u \rangle \right\rangle = \\
&= \langle u,u \rangle - \sum_{j=1}^{N} \langle u_j,u \rangle \langle u,u_j \rangle - \sum_{i=1}^{N} \overline{\langle u_i,u \rangle} \langle u_i,u \rangle + \sum_{i=1}^{N} \sum_{j=1}^{N} \overline{\langle u_i,u \rangle} \langle u_j,u \rangle \underbrace{\langle u_i,u_j \rangle}_{=\delta_{ij}} = \\
&= \langle u,u \rangle - 2 \sum_{i=1}^{N} |\langle u_i,u \rangle|^2 + \sum_{i=1}^{N} |\langle u_i,u \rangle|^2 = \\
&= \langle u,u \rangle - \sum_{i=1}^{N} |\langle u_i,u \rangle|^2
\end{aligned}$$

$\square_{3.0.4}$

Isomorphism of Hilbert spaces: Let $(H_1, \langle .,. \rangle_1)$ and $(H_2, \langle .,. \rangle_2)$ be Hilbert spaces.

An isomorphism is a mapping $U : H_1 \to H_2$ which is linear, bijective and isometric (isometrisch), i.e. for all $u,v \in H_1$:

$$\langle u,v \rangle_1 = \langle Uu,Uv \rangle_2$$

Direct sum of Hilbert spaces: $H_1 \oplus H_2$

Define:

$$H := \left\{ (u,v) \,\middle|\, u \in H_1, v \in H_2 \right\}$$

$$(u_1,v_1) + (u_2,v_2) := (u_1 + u_2, v_1 + v_2)$$
$$\lambda (u,v) := (\lambda u, \lambda v)$$
$$\langle (u_1,v_1), (u_2,v_2) \rangle := \langle u_1,u_2 \rangle + \langle v_1,v_2 \rangle$$

This makes $H = H_1 \oplus H_2$ a Hilbert space, sometimes called orthogonal due to:

$$\langle (u,0), (0,v) \rangle = 0$$

3.0.5 Example

$$\ell_2 = \left\{ (a_n)_{n\in\mathbb{N}} \left| a_n \in \mathbb{K}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right. \right\}$$

Define a scalar product:

$$\langle (a_n), (b_n) \rangle := \sum_{n=1}^{\infty} \overline{a}_n \cdot b_n$$

$$\langle (a_n), (a_n) \rangle = \sum_{n=1}^{\infty} |a_n|^2 = \|a_n\|_2^2$$

$\left(\ell^2, \|.\|_2\right)$ is a Banach space. Thus $\left(\ell^2, \langle .,. \rangle\right)$ is a Hilbert space.

3.1 Projection on closed convex subsets

Let $(H, \langle .,. \rangle)$ be a Hilbert space and $K \subseteq H$ a closed convex subset.

TODO: Abb10

$u,v \in K$ $\qquad\qquad$ $w \in H \setminus K$

We want to find a vector v such that $\|v - w\| = \inf_{u\in K} \|u - w\|$.

If K were compact, then choose minimizing sequence (Minimalfolge), i.e.:

$$\|u_i - w\| \to \inf_{u\in K} \|u - w\|$$

Choose a convergent subsequence $u_{i_l} \to v$. Then by continuity:

$$\|v - w\| = \lim_{i\to\infty} \|u_i - w\| = \inf_{u\in K} \|u - w\|$$

Main application:

$K \subseteq H$ be a closed subspace.

TODO: Abb11

In this case $v - w$ will be called orthogonal to K motivating the name *orthogonal projection*.

3.1.1 Theorem (Hilbert)

There is a unique $v \in K$ with:

$$\|v - w\| = \inf_{u \in K} \|u - w\|$$

Proof

Consider a minimizing sequence u_i:

$$\|u_i - w\| \to \inf_{u \in K} \|u - w\|$$

Show that (u_i) is a Cauchy sequence.

$$\begin{aligned}
\|u_i - u_j\|^2 = \|(u_i - w) + (w - u_j)\| = \\
&\overset{3.0.2}{=} 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \|(u_i - w) - (w - u_j)\|^2 = \\
&= 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \left\|-2\left(w - \frac{u_i + u_j}{2}\right)\right\|^2 = \\
&= 2\left(\underbrace{\|u_i - w\|^2}_{\to d^2} + \underbrace{\|w - u_j\|^2}_{\to d^2} - 2\left\|\frac{u_i + u_j}{2} - w\right\|^2\right)
\end{aligned}$$

$$\|u_i - w\| \xrightarrow{i \to \infty} d := \inf_{u \in K} \|u - w\|$$

$$\|u_j - w\| \xrightarrow{j \to \infty} d := \inf_{u \in K} \|u - w\|$$

Since K is convex and $u_i, u_j \in K$, we know:

$$\frac{u_i + u_j}{2} \in K$$

$$\Rightarrow \quad \left\|\frac{u_i + u_j}{2} - w\right\| \geq d$$

Thus:

$$\|u_i - u_j\|^2 \leq 2\left(\|u_i - w\|^2 + \|w - u_j\|^2 - 2d^2\right) \xrightarrow{i,j \to \infty} 2\left(d^2 + d^2 - 2d^2\right) = 0$$

So there exists a $N \in \mathbb{N}$ such that $\|u_i - u_j\| < \varepsilon$ for all $i,j > N$. Therefore (u_i) is a Cauchy sequence. Since H is complete, we know that $u_i \to u$.

By continuity:

$$\|u - w\| = \lim_{i \to \infty} \|u_i - w\| = d$$

Uniqueness follows from the fact, that *every* minimizing sequence converges.

Namely: Let $u, \tilde{u}$ be both minimizers. The sequence $(u, \tilde{u}, u, \tilde{u}, \ldots)$ is a minimizing sequence. It converges, so $u = \tilde{u}$. $\square_{3.1.1}$

3.1.2 Corollary

Let $M \subseteq H$ be a closed subspace of H.

Then a $w \in H$ can be decomposed uniquely in the form

$$w = v + x$$

with $v \in M$ and $x \in M^{\perp}$.

We write $H = M \oplus M^{\perp}$.

TODO: Abb12

Proof

Let $v \in M$ be as in Theorem **??**3.1.1.

$$\|v - w\| = \inf_{u \in M} \|u - w\|$$

Define $x := w - v$.

Claim: $x \perp M$

Proof: Let $u \in M$. Choose $\tilde{u}(\tau) = v + \tau u \in M$ with an arbitrary $\tau \in \mathbb{K}$.

$$\begin{aligned}
\|v - w\|^2 &\overset{\text{by minimality}}{\leq} \|\tilde{u} - w\|^2 = \langle v + \tau u - w, v + \tau u - w \rangle = \\
&= \|v - w\|^2 + 2\mathrm{Re}\left(\tau \langle w - v, u \rangle\right) + |\tau|^2 \|u\|^2 \\
\Rightarrow \quad & 0 \leq 2\mathrm{Re}\left(\tau \langle x, u \rangle\right) + |\tau|^2 \|u\|^2
\end{aligned}$$

So assume conversely that $\langle x, u \rangle \neq 0$.

Then set $\tau = re^{\mathbf{i}\varphi}$, $r \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{R}$.

Choose the phase φ such that:

$$\mathrm{Re}\left(\tau \langle x, u \rangle\right) = -r \left|\langle x, u \rangle\right|$$

Then:

$$2\mathrm{Re}\left(\tau \langle x, u \rangle\right) + |\tau|^2 \|u\|^2 = -2r \left|\langle x, u \rangle\right| + r^2 \|u\|^2$$

This can be smaller than zero for sufficiently small $r \in \mathbb{R}_{>0}$. This is a contradiction.

- H real: Let $u \in M$, $\tilde{u}(\tau) = v + \tau u$

  $$\|\tilde{u} - w\|^2 = \|x\|^2 + 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 \geq \|x\|^2$$

  $f(\tau)$ has a minimum at $\tau = 0$.

  $$\begin{aligned}
  \Rightarrow \quad & f'(0) = 0 \\
  & f'(0) = 2 \langle u, x \rangle \\
  \Rightarrow \quad & 2 \langle u, x \rangle = 0 \quad \forall_{u \in M}
  \end{aligned}$$

  So $x \in M^{\perp}$.

- H complex: $\tilde{u}(\tau) = v + \tau u$, $\tau = re^{\mathbf{i}\varphi}$ with $r \geq 0$.

$$\|\tilde{u} - w\|^2 = \|x\|^2 + 2\mathrm{Re}\left(re^{-\mathbf{i}\varphi}\langle u,x\rangle\right) + r^2\|u\|^2 =: f(r,\varphi)$$

This has a minimum at $r = 0$.

$$\Rightarrow \quad 0 = \partial_r f(0,\varphi) = 2\mathrm{Re}\left(e^{-\mathbf{i}\varphi}\langle u,x\rangle\right)$$

$$\overset{\varphi \text{ arbitrary}}{\Rightarrow} \quad \langle u,x\rangle = 0$$

Uniqueness: Assume that $w = v_1 + x_1 = v_2 + x_2$ where $v_1,v_2 \in M$, $x_1,x_2 \in M^\perp$.

$$\underbrace{v_1 - v_2}_{\in M} = \underbrace{x_2 - x_1}_{\in M^\perp} \in M \cap M^\perp = \{0\}$$

Because from $u \in M \cap M^\perp$ follows $\langle u,u\rangle = 0$ and so $u = 0$. $\square_{3.1.2}$

For Banach spaces E we have E,E^*,E^{**} and a natural injection $\iota : E \hookrightarrow E^{**}$.

In Hilbert space H: Suppose $u \in H$. Define:

$$\varphi : H \to \mathbb{K}$$
$$\varphi(v) := \langle u,v\rangle$$

φ is continuous, because:

$$|\varphi(v)| \leq \|u\| \cdot \|v\| \leq C\|v\|$$

$$\iota : H \hookrightarrow H^*$$
$$\iota(u) = \varphi$$

is a linear mapping, which is injective.

3.1.3 Theorem (Fréchet-Riesz)

For any $\varphi \in H^*$ there is a unique $v \in H$ such that for all $x \in H$:

$$\varphi(x) = \langle v,x\rangle$$

In other words: $\iota : H \to H^*$ is a Banach space isomorphism.

Proof

Let $\varphi \in H^*$, without loss of generality $\varphi \neq 0$.

$$M := \ker\varphi$$

is a subspace. It is closed by continuity:

Suppose $u_n \in \ker\varphi$, $u_n \to u$. Then:

$$\varphi(u) \overset{\text{continuity}}{=} \lim_{n\to\infty}\varphi(u_n) = 0$$

So $u \in \ker\varphi$.

- $M^\perp$ is a one-dimensional subspace of H:
  $M^\perp \neq \{0\}$:
  Since $\varphi \neq 0$ there exists a $u \in H$ with $\varphi(u) \neq 0$, thus $u \notin M$.
  Now decompose $u = v + x$, $v \in M$, $u \in M^\perp$.
  TODO: Abb13
  So $x \neq 0$ thus there is a $x \neq 0$ with $x \in M^\perp$.
  $M^\perp$ is one-dimensional: Take $u,v \in M^\perp$, $u,v \neq 0$. Then $\varphi(u) \neq 0$ and $\varphi(v) \neq 0$.

  $$\varphi(\varphi(v)u - \varphi(u)v) = 0$$

  So $\varphi(v)u - \varphi(u)v \in M \cap M^\perp = \{0\}$. Thus $\varphi(v)u - \varphi(u)v = 0$, implying that u and v are linearly dependent.

- Choose $u \in M^\perp$ with $\varphi(u) = 1$ which is always possible by rescaling.

  $$\begin{aligned} v &:= \frac{u}{\|u\|^2} \\ \Rightarrow \quad \varphi(v) &= \frac{1}{\|u\|^2}\underbrace{\varphi(u)}_{=1} = \frac{1}{\|u\|^2} \\ \langle v,v\rangle &= \frac{\langle u,u\rangle}{\|u\|^4} = \frac{1}{\|u\|^2} = \varphi(v) \end{aligned}$$

- This v has the desired properties:
  Let $x \in H$, decompose:

  $$x = \underbrace{m}_{\in M} + \underbrace{\alpha v}_{\in M^\perp = \langle v\rangle}$$

  $$\begin{aligned} \Rightarrow \quad \varphi(x) &= \underbrace{\varphi(m)}_{=0} + \alpha\varphi(v) = \alpha\langle v,v\rangle = \\ &= \langle v,\alpha v\rangle = \langle v,m+\alpha v\rangle = \langle v,x\rangle \end{aligned}$$

$\square_{3.1.3}$

3.1.4 Theorem (Lax-Milgram)

Let H be a Hilbert space and $B : H \times H \to \mathbb{K}$ be a mapping with the following properties:

i) $B(x,y)$ is linear in the second an anti-linear in the first argument.

ii) $|B(x,y)| \leq C\|x\| \cdot \|y\|$ (continuity)

iii) B is symmetric ($\overline{B(x,y)} = B(y,x)$) and positive definite, i.e. $B(x,x) \geq b\|x\|^2$ with $b \in \mathbb{R}_{>0}$.

iii') $|B(x,x)| \geq b\|x\|^2$ with $b \in \mathbb{R}_{>0}$.

Then every $l \in H^*$ can be represented uniquely as:

$$l(x) = B(v,x) \qquad \underset{x\in H}{\forall}$$

Proof

First the easy case iii):

We introduce a new scalar product $\langle .,. \rangle_B$ by:

$$\langle x,y \rangle_B := B(x,y)$$

Using ii) and iii) one sees that $\|.\|_B$ is equivalent to $\|.\|$, i.e. there exists a $C \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{C}\|x\| \leq \|x\|_B \leq C\|x\|$$

According to the Fréchet-Riesz theorem, there exists a unique $v \in H$ with $\varphi(x) = \langle v,x \rangle_B = B(v,x)$ for all $x \in H$.

More difficult case iii'): Given $x \in H$

$$B(x,.) : H \to \mathbb{K}$$

is a linear bounded functional according to i) and ii), i.e. $B(x,.) \in H^*$.

According to the Fréchet-Riesz theorem there exists a unique $z \in H$ such that $B(x,y) = \langle z,y \rangle$ for all $y \in H$. This yields a mapping:

$$\begin{aligned} \varphi : H &\to H \\ x &\mapsto z \end{aligned}$$

$$B(x,y) = \langle \varphi(x), y \rangle$$

- φ is linear, because both B and $\langle .,. \rangle$ are anti-linear in their first arguments.
- $\varphi(H) \subseteq H$ is closed:

$$\begin{aligned} b\|x\|^2 &\overset{\text{iii')}}{=} |B(x,x)| = |\langle z,x \rangle| \leq \|z\| \cdot \|x\| \\ b\|x\| &\leq \|z\| \end{aligned} \tag{3.1}$$

Let $z_n \in \varphi(H)$ be a sequence with $z_n \to z \in H$. Choose x_n such that $\varphi(x_n) = z_n$, i.e. $B(x_n,y) = \langle z_n,y \rangle$ for all $y \in H$.
Due to the anti-linearity it follows that:

$$B(x_n - x_m) = \langle z_n - z_m, y \rangle$$

(3.1) yields that $\|x_n - x_m\| \leq \|z_n - z_m\|$.
Hence (x_n) is a Cauchy sequence $x_n \to x \in H$. Since B is continuous according to ii), we get:

$$\underbrace{B(x_n,y)}_{\to B(x,y)} = \underbrace{\langle z_n,y \rangle}_{\to \langle z,y \rangle}$$

So:

$$\begin{aligned} B(x,y) &= \langle z,y \rangle \\ \varphi(x) &= z \end{aligned}$$

Thus $z \in \varphi(H)$.

- $\varphi(H) = H$: Otherwise there would be a vector $y \neq 0$, $y \in \varphi(H)^{\perp}$.

  $$\Rightarrow \quad B(x,y) = \langle \varphi(x), y \rangle = 0 \quad \underset{x \in H}{\forall}$$

  In particular:

  $$\begin{aligned} & 0 = |B(y,y)| \geq b\,\|y\|^2 \\ \Rightarrow \quad & y = 0 \end{aligned}$$

  This is a contradiction.

- Let $l \in H^*$. According to Fréchet-Riesz there exists a unique $z \in H$ with $l(y) = \langle z,y \rangle$ for all $y \in H$. Now:

  $$\langle z,y \rangle = B(x,y)$$

  Claim: φ is injective.
  Proof: Suppose x,x' with $\varphi(x) = \varphi(x')$. Then follows:

  $$B\left(x - x', y\right) = \left\langle \underbrace{\varphi(x) - \varphi\left(x'\right)}_{=0}, y \right\rangle = 0$$

  Choose $y = x - x'$ so we get:

  $$\begin{aligned} B\left(x - x', x - x'\right) &= 0 \\ \Rightarrow \quad x &= x' \end{aligned}$$

  If we choose x such that $\varphi(x) = z$, $x = \varphi^{-1}(z)$.

$\square_{3.1.4}$

3.1.5 Corollary

Every Hilbert space is reflexive.

Proof

Recall $\iota : H \hookrightarrow H^{**}$. H is *reflexive*, if and only if ι is surjective, i.e. a Banach space isomorphism.

$$\begin{gathered} \tilde{\iota} : H \to H^* \\ (\iota(u))(v) = \langle u,v \rangle \end{gathered}$$

is bijective by Fréchet-Riesz. This holds also for $\bar{\iota} : H^* \to H^{**}$.

$$H \xrightarrow{\tilde{\iota}} H^* \xrightarrow{\bar{\iota}} H^{**}$$

So $\iota = \bar{\iota} \circ \tilde{\iota}$ is bijective as composition of bijective maps. $\square_{3.1.5}$

3.2 Orthonormal Bases in Separable Hilbert Spaces

3.2.1 Example

$\ell_2 = \left\{ (a_n)_{n\in\mathbb{N}} \middle| \sum_n |a_n|^2 < \infty \right\}$

$$\langle (a_n), (b_n) \rangle := \sum_n \overline{a}_n b_n$$

is a Hilbert space.

Idea: Let H be an abstract Hilbert space. Choose "orthonormal basis" (e_i).

$$H \ni u = \sum_{i=1}^{\infty} \lambda_i e_i$$
$$v = \sum_{i=1}^{\infty} \nu_i e_i$$

$$\langle u,v \rangle = \sum_{i,j=1}^{\infty} \langle \lambda_i e_i, \nu_j e_j \rangle = \sum_{i,j=1}^{\infty} \overline{\lambda_i} \nu_j \delta_{ij} = \sum_i \overline{\lambda_i} \nu_i$$

3.2.2 Definition

A system $(e_i)_{i\in J}$ is an *orthonormal system*, if $\langle e_i, e_j \rangle = \delta_{ij}$. The algebraic span is the vector space of *finite* linear combinations:

$$\langle (e_i) \rangle = \left\{ \sum_{i=1}^{N} \lambda_i e_i \middle| N \in \mathbb{N}, \lambda_i \in \mathbb{K} \right\}$$

This is a subspace of H. Now the subspace $\overline{\langle (e_i) \rangle} \subseteq H$ is called *Hilbert space span* (Hilbertraumerzeugnis).

An orthonormal system (e_i) is called a *orthonormal Hilbert space basis*, if $\overline{\langle (e_i) \rangle} = H$.

There always exists an orthonormal Hilbert space basis. (see exercises)

Any two Hilbert space bases have the same cardinality (Mächtigkeit). (see exercises)

Bernstein-Schröder Theorem

Definition: Sets A,B have the same cardinality, if there exists an bijective map $\varphi : A \to B$.

Theorem (Bernstein-Schröder):

A,B have the same cardinality, if and only if there exists an injective map from A to B and an injective map from $B \to A$.

Appendix

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Andreas Völklein

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