

# Functional Analysis

*lecture by*

PROF. DR. FELIX FINSTER

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ANDREAS VÖLKLEIN



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# Motivation

In linear algebra one mainly considers finite-dimensional vector spaces with additional structures like norm  $\|\cdot\|$  or scalar product  $\langle \cdot, \cdot \rangle$ .

Let  $(V, \langle \cdot, \cdot \rangle)$  be a finite-dimensional scalar product space and  $A : V \rightarrow V$  a linear map, which is self-adjoint, that means for all  $u, v \in V$ :

$$\langle Au, v \rangle = \langle u, Av \rangle$$

## Theorem (orthonormal eigenvector basis)

There exists an orthonormal eigenvector basis  $(u_i)_{i \in \{1, \dots, n\}}$ , that means with the eigenvalues  $\lambda_i \in \mathbb{R}$ :

$$\langle u_i, u_j \rangle = \delta_{ij} \qquad Au_i = \lambda_i u_i$$

In infinite dimensions the generalization is the *spectral theorem*.

First reformulate the result from linear algebra:

Let  $E_{\lambda_i}$  be the orthogonal projection operator on the eigenspace corresponding to  $\lambda_i$ . If this eigenspace is one dimensional, this means:

$$E_{\lambda_i} v = u_i \langle u_i, v \rangle = |u_i\rangle \langle u_i| v$$

Then one can write  $A$  as:

$$A = \sum_{i=1}^n \lambda_i E_{\lambda_i}$$

## Theorem (spectral theorem)

Let  $A \in L(H)$  be a self-adjoint (selbstadjungiert) operator, then it holds:

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

$\sigma(A) \subseteq \mathbb{R}$  is the spectrum of  $A$  and  $E_\lambda$  the projection-valued measure (Spektralmaß).

Applications typically are differential operators, for example:

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\Delta_{\mathbb{R}^3} : C_0^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3) \quad \text{linear operator}$$

Applications in more detail are studied in the lectures on partial differential equations I + II.

## 0 Basic Notions

Let  $E$  be a vector space (Vektorraum), for example the finite-dimensional vector space  $E \simeq \mathbb{R}^3$ . In the following list the later spaces are special cases of the previous ones:

- topological vector spaces
- metric spaces with a metric  $d(.,.)$  (Polish spaces if complete)
- normed spaces with norm  $\|.\|$  (Banach spaces if complete)
- scalar product spaces  $\langle ., . \rangle$  (Hilbert spaces if complete)

Let  $\mathbb{K}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ .

### 0.1 Definition (metric, $\varepsilon$ -ball, Cauchy sequence, complete, Polish space)

A map  $d : E \times E \rightarrow \mathbb{R}$  is called *metric*, if for all  $x, y, z \in E$  holds:

- i)  $d(x, y) = d(y, x)$  (symmetry)
- ii)  $d(x, y) \geq 0$  and  $d(x, y) = 0 \Leftrightarrow x = y$  (positive definiteness)
- iii)  $d(x, y) \leq d(x, z) + d(z, y)$  (triangle inequality)

$B_\varepsilon(x) := \{z \in E \mid d(x, z) < \varepsilon\}$  is called  $\varepsilon$ -ball.

Consider the topology generated by  $B_\varepsilon(x)$ : A set  $\Omega \subseteq E$  is open if and only if:

$$\forall_{x \in \Omega} \exists_{\varepsilon \in \mathbb{R}_{>0}} : B_\varepsilon(x) \subseteq \Omega$$

*Completeness:*

$(x_n)_{n \in \mathbb{N}}$  is a *Cauchy sequence* if and only if:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \exists_{N \in \mathbb{N}} \forall_{n, m \in \mathbb{N}_{>N}} : d(x_n, x_m) < \varepsilon$$

$E$  is *complete* if and only if every Cauchy sequence has a limit.

A complete metric space is also called a *Polish space*.

### 0.2 Definition (norm, Banach space)

Let  $(E, \|\cdot\|)$  be a *normed space*, i.e. a  $\mathbb{K}$ -vector space with a map  $\|\cdot\| : E \rightarrow \mathbb{R}_{\geq 0}$  called *norm* with the following properties for  $x, y \in E$  and  $\lambda \in \mathbb{K}$ :

- i)  $\|x\| \geq 0$  and  $\|x\| = 0 \Leftrightarrow x = 0$  (positive definiteness)

ii)  $\|\lambda x\| = |\lambda| \cdot \|x\|$  (homogeneity)

iii)  $\|u + v\| \leq \|u\| + \|v\|$  (triangle inequality)

Define the metric  $d(x, y) := \|x - y\|$ . A complete normed spaces is called *Banach space*.

Let  $A : E \rightarrow F$  be a linear map between the Banach spaces  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$ .

### 0.3 Definition (continuous, bounded)

$A$  is *continuous* (stetig) if  $A^{-1}(\Omega) \subseteq E$  is open for all open  $\Omega \subseteq F$ .

$A$  is *bounded* (beschränkt) if there exists a  $C \in \mathbb{R}_{>0}$  such that for all  $u \in E$  holds:

$$\|Au\|_F \leq C \|u\|_E$$

### 0.4 Lemma (continuous $\Leftrightarrow$ bounded)

$A$  is continuous  $\Leftrightarrow A$  is bounded.

(no proof)

### 0.5 Definition (dual space, sup-norm)

The *dual space* of  $E$  is the space of continuous linear mappings from  $E$  to  $\mathbb{K}$ :

$$E^* = L(E, \mathbb{K})$$

$L(E, F)$  is a vector space: For  $A, B \in L(E, F)$ ,  $\lambda, \mu \in \mathbb{K}$  and  $u \in E$  define:

$$(\lambda A + \mu B)(u) := \lambda A(u) + \mu B(u)$$

Define also a norm on  $L(E, F)$ , which is called *sup-norm*:

$$\|A\| := \sup_{u \in E, \|u\|_E \leq 1} \|Au\|_F$$

### 0.6 Theorem

If  $F$  is complete, so is  $L(E, F)$ .

In particular  $E^*$  is a Banach space for every  $E$ .

(no proof)

# 1 The Hahn-Banach Theorem and Applications

As a preparation we need Zorn's lemma.

## 1.1 Definition (partial ordering, chain, upper bound, maximal)

Let  $A$  be a set and  $\leq$  a *partial ordering* (Halbordnung), i.e. for all  $a, b, c \in A$ :

- i)  $a \leq b$  and  $b \leq c \Rightarrow a \leq c$  (transitivity)
- ii)  $a \leq a$  (reflexivity)
- iii)  $a \leq b \wedge b \leq a \Rightarrow a = b$  (antisymmetry)

*Note:* We do *not* demand that for all  $a, b \in A$  holds:

$$(a \leq b) \vee (b \leq a)$$

This is a property of a ordering relation.

$(A, \leq)$  is called *partially ordered set* (teilweise geordnete Menge).

A subset  $K \subseteq A$  is called *chain* (Kette, total geordnete Teilmenge) if for all  $x, y \in K$  holds:

$$(x \leq y) \vee (y \leq x)$$

An element  $u \in A$  is called *upper bound* (obere Schranke) of  $B \subseteq A$  if  $x \leq u$  for all  $x \in B$ .

An element  $m \in A$  is called *maximal* if  $m \leq a \in A \Rightarrow m = a$ .

## 1.2 Zorn's lemma

Let  $(A, \leq)$  be a partially ordered set in which every chain has an upper bound. Then there is a maximal element.

### Proof

This follows from the axiom of choice, see e.g. Kowalsky: Linear algebra.

### 1.3 Definition (sublinear)

Let  $X$  be a *real* vector space (without topology) and  $l : X \rightarrow \mathbb{R}$  linear.  $p : X \rightarrow \mathbb{R}$  is called *sublinear* if for all  $x, y \in X$  and  $a \in \mathbb{R}_{>0}$ :

- i)  $p(ax) = ap(x)$
- ii)  $p(x + y) \leq p(x) + p(y)$

A typical example is  $p(x) = \|x\|$ , but  $p$  does not need to be positive. Another example is any linear mapping.

### 1.4 Theorem (Hahn-Banach, real version, 1927/29)

Let  $X$  be a real vector space and  $Y \subseteq X$  a subspace (Untervektorraum),  $p : X \rightarrow \mathbb{R}$  sublinear and  $l : Y \rightarrow \mathbb{R}$  linear with  $l(y) \leq p(y)$  for all  $y \in Y$ .

Then there is a linear extension (Fortsetzung)  $\tilde{l} : X \rightarrow \mathbb{R}$  of  $l$  to  $X$ , i.e.  $\tilde{l}|_Y = l$ , such that for all  $x \in X$  holds:

$$\tilde{l}(x) \leq p(x)$$

#### Proof

- i) Assume  $Y \subsetneq X$ , since otherwise there is nothing to prove. Choose a vector  $z \in X \setminus Y$ . We want to extend  $l$  to the span of  $Y$  and  $\langle z \rangle$ .  $\tilde{l}(z)$  needs to be prescribed. For all  $y \in Y$  and  $a \in \mathbb{R}$  holds:

$$\tilde{l}(y + az) \stackrel{\text{linearity}}{=} l(y) + a\tilde{l}(z) \stackrel{\text{demand}}{\leq} p(y + az)$$

If  $a = 0$ , the inequality is clear. By homogeneity assumptions, it is sufficient to consider the case  $a = \pm 1$ . We thus demand for all  $y, y' \in Y$ :

$$\begin{aligned} l(y) + \tilde{l}(z) &\leq p(y + z) \\ l(y') - \tilde{l}(z) &\leq p(y' - z) \end{aligned}$$

This is equivalent to:

$$l(y') - p(y' - z) \leq \tilde{l}(z) \leq p(y + z) - l(y)$$

We can choose  $\tilde{l}(z)$  if and only if:

$$l(y') - p(y' - z) \leq p(y + z) - l(y)$$

(For example set  $\tilde{l}(z) = \sup_{y' \in Y} l(y') - p(y' - z)$ .)

$$\Leftrightarrow l(y') + l(y) \stackrel{\text{linearity}}{=} l(y' + y) \leq p(y + z) + p(y' - z)$$

Now prove this inequality:

From  $y' + y \in Y$  follows that  $l(y + y') \leq p(y + y')$  by hypothesis. Moreover, as  $p$  is sublinear, it follows:

$$p(y + z - z + y') \leq p(y' + z) + p(y' - z)$$

So the inequality is shown. Thus  $l$  can be extended to  $Y + \langle z \rangle$ .

ii) Consider all extensions:

$$A := \{(Z, l) \mid Y \subseteq Z \subseteq X \text{ subspace, } l : Z \rightarrow \mathbb{R} \text{ extension of } l_Y : Y \rightarrow \mathbb{R}\}$$

This set has a partial ordering  $\leq$  defined by  $(Z, l) \leq (Z', l')$  if  $Z \subseteq Z'$  and  $l'|_Z = l$ .

For an index set  $I$  (possibly infinite, uncountable) let  $K = \{(Z_\nu, l_\nu) \mid \nu \in I\}$  be a chain, i.e. for all  $(Z, l), (Z', l') \in K$ :

$$((Z, l) \leq (Z', l')) \vee ((Z', l') \leq (Z, l))$$

Set  $Z = \bigcup_{\nu \in I} Z_\nu$  and define  $l : Z \rightarrow \mathbb{R}$  by  $l|_{Z_\nu} = l_\nu$ . (Thus suppose  $u \in Z$ , so there is a  $\nu \in I$  with  $u \in Z_\nu$ . Set  $l(u) := l_\nu(u)$ .  $\nu$  need not be unique. Suppose  $u \in Z_{\nu'}$ , then we know that either  $Z_{\nu'} \subseteq Z_\nu$  and  $l_\nu|_{Z_{\nu'}} = l_{\nu'}$  or  $Z_\nu \subseteq Z_{\nu'}$  and  $l_{\nu'}|_{Z_\nu} = l_\nu$ . In both cases we have  $l_\nu(u) = l_{\nu'}(u)$ , thus  $l(u)$  is well defined.)

This  $(Z, l)$  is an upper bound, because for all  $\nu \in I$  we have  $Z_\nu \subseteq Z = \bigcup_{\lambda \in I} Z_\lambda$  and  $l$  is an extension of  $l_\nu$ .

With Zorn's Lemma follows, that there exists an maximal element  $(\tilde{Y}, \tilde{l})$ .

**Claim:**  $\tilde{Y} = X$

**Proof:** Otherwise there would be a vector  $u \in X \setminus \tilde{Y}$ , and  $\tilde{l}$  could be extended to  $\tilde{Y} \oplus \langle u \rangle$ , as shown in i), in contradiction to the maximality of  $\tilde{l}$ . Thus  $(X = \tilde{Y}, \tilde{l})$  is the desired extension.  $\square_{\text{Claim}}$

$\square_{1.4}$

## 1.5 Theorem (Hahn-Banach, complex version)

Let  $X$  be a complex vector space and  $Y \subseteq X$  a subspace. Before, we had  $l(x) \leq p(x)$  as condition, which does not make sense in the complex case, since:

$$l(e^{i\varphi}x) = e^{i\varphi}l(x) \stackrel{\text{in general}}{\notin} \mathbb{R}$$

Let  $p : X \rightarrow \mathbb{R}$  be a *seminorm*, i.e.:

- i)  $p(ax) = |a|p(x)$  (homogeneity)
- ii)  $p(x + y) \leq p(x) + p(y)$  (triangle inequality)

Let  $l : Y \rightarrow \mathbb{C}$  be a linear functional with  $|l(y)| \leq p(y)$  for all  $y \in Y$ .

Then  $l$  can be extended to  $X$  such that  $|l(x)| \leq p(x)$  holds for all  $x \in X$ .

### Proof

We also consider  $X$  as a real vector space. ( $u$  and  $\mathbf{i}u$  are then linearly independent vectors.) Decompose  $l$  into its real and imaginary parts.

$$\begin{aligned} l(y) &= l_1(y) + \mathbf{i}l_2(y) \\ l_1 &:= \operatorname{Re}(l(y)) \\ l_2 &:= \operatorname{Im}(l(y)) \end{aligned}$$

$l_1$  and  $l_2$  are real-linear and:

$$l_1(\mathbf{i}y) = \operatorname{Re}(l(\mathbf{i}y)) = \operatorname{Re}(\mathbf{i}l(y)) = -\operatorname{Im}(l(y)) = -l_2(y)$$

Conversely, suppose that  $l_1$  is real-linear. Then

$$l(x) := l_1(x) - \mathbf{i} \cdot l_1(\mathbf{i}x)$$

this is indeed a complex-linear function. We know that  $|l(y)| \leq p(y)$  holds for all  $y \in Y$ .

$$\begin{aligned} l_1(y) &= \operatorname{Re}(l(y)) \leq |l(y)| \\ \Rightarrow \quad l_1(y) &\leq p(y) \end{aligned}$$

Theorem 1.4 yields an real-linear extension  $\tilde{l}_1 : X \rightarrow \mathbb{R}$  such that  $\tilde{l}_1(x) \leq p(x)$  for all  $x \in X$ . Set  $\tilde{l}(x) = \tilde{l}_1(x) - \mathbf{i}\tilde{l}_1(\mathbf{i}x)$ , so that  $\tilde{l} : X \rightarrow \mathbb{C}$  is complex-linear.

**Claim:**  $|\tilde{l}(x)| \leq p(x) \quad \forall x \in X$

**Proof:** Polar decomposition:

$$\begin{aligned} \tilde{l}(x) &= r e^{\mathbf{i}\varphi} \\ |\tilde{l}(x)| &= r = e^{-\mathbf{i}\varphi} \tilde{l}(x) \stackrel{\tilde{l} \text{ is complex-linear}}{=} \tilde{l}(e^{-\mathbf{i}\varphi} x) = \operatorname{Re}(\tilde{l}(e^{-\mathbf{i}\varphi} x)) = \\ &= \tilde{l}_1(e^{-\mathbf{i}\varphi} x) \leq p(e^{-\mathbf{i}\varphi} x) \stackrel{\text{homogeneity}}{=} p(x) \end{aligned}$$

□<sub>Claim</sub>

□<sub>1.5</sub>

Now to applications:

## 1.6 Theorem

Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{K}$ -space (real or complex),  $Y \subseteq X$  a subspace. Let  $\varphi$  be a continuous linear functional from  $Y$  to  $\mathbb{K}$ , i.e. for all  $y \in Y$  holds:

$$|\varphi(y)| \leq \|\varphi\| \cdot \|y\|$$

Then  $\varphi$  can be continued to all of  $X$  with the same supnorm, i. e.:

$$\|\tilde{\varphi}\| := \sup_{x \in X, \|x\| \leq 1} |\varphi(x)| = \|\varphi\| := \sup_{y \in Y, \|y\| \leq 1} |\varphi(y)|$$

**Proof**

Apply the Hahn-Banach theorem with  $\tilde{\varphi} := \|\varphi\| \cdot \|x\|$ .

□<sub>1.6</sub>

## 1.7 Corollary

Let  $X$  be a normed space and  $u_0 \in X$  with  $\|u_0\| = 1$ . Then there exists a linear functional  $\varphi : X \rightarrow \mathbb{K}$  such that:

$$\varphi(u_0) = 1 \qquad \|\varphi\| = 1$$

**Proof**

Let  $Y := \langle u_0 \rangle$  and define  $\varphi_0 : \langle u_0 \rangle \rightarrow \mathbb{K}$  by  $\varphi_0(u_0) = 1$ . Extend  $\varphi_0$  by the Hahn-Banach theorem 1.6.  $\square_{1.7}$

The Hahn-Banach theorem also has a geometric formulation. Consider only the real case:  
A set  $K \subseteq X$  is called *convex* if for all  $x, y \in K$  and  $\tau \in [0, 1]$ :

$$\tau x + (1 - \tau) y \in K$$

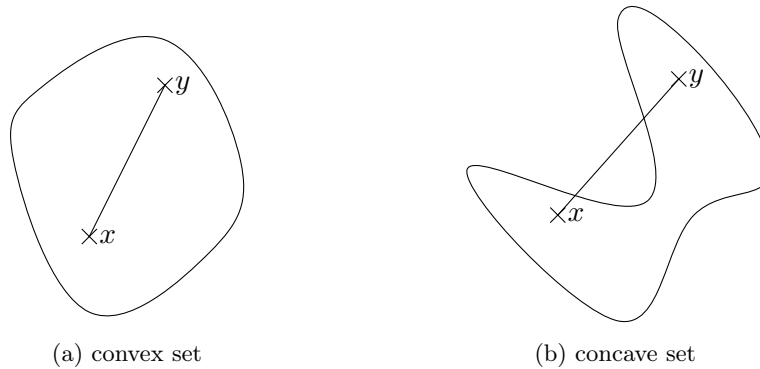


Figure 1.1: convexity

Geometric question:

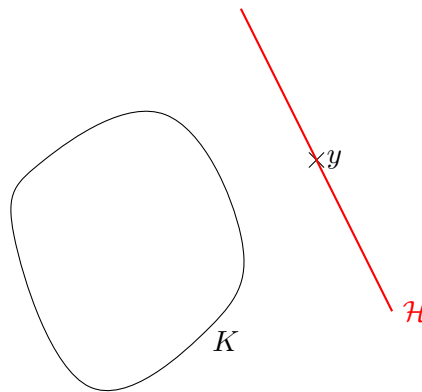


Figure 1.2: not intersecting hyperplane

Is there a hyperplane  $\mathcal{H}$ , which meets  $y \notin K$ , but does not intersect  $K$ ?

**1.8 Definition** (interior point)

$x_0 \in K$  is an *interior point* (innerer Punkt) of  $K$  with respect to  $u \in X$  if there exists an  $\varepsilon \in \mathbb{R}_{>0}$  such that  $x_0 + tu \in K$  for all  $t \in (-\varepsilon, \varepsilon)$ .

$x_0 \in K$  is an *interior point* if for all  $u \in X$  there is a  $\varepsilon = \varepsilon(u) \in \mathbb{R}_{>0}$  such that  $x_0 + tu \in K$  for all  $t \in (-\varepsilon, \varepsilon)$ .



## 1.9 Theorem (geometric Hahn-Banach)

Let  $K \neq \emptyset$  be convex and all points of  $K$  be interior points. Let  $y \notin K$ . Then there is a linear functional  $l : X \rightarrow \mathbb{R}$  such that  $l(x) < 1$  for all  $x \in K$  and  $l(y) = 1$ .

$\mathcal{H} := \{x \in X \mid l(x) = 1\}$  defines a hyperplane. Now  $y \in \mathcal{H}$  and  $l|_K < 1$  mean that  $K$  lies in one half-space.

First introduce a suitable sublinear functional. Without loss of generality, assume  $0 \in K$  (otherwise shift  $K$ ).

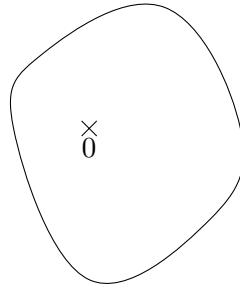


Figure 1.3:  $0 \in K$

The functional  $p : K \rightarrow \mathbb{R}_{\geq 0}$  with

$$p(x) := \inf \left\{ a \in \mathbb{R}_{>0} \mid \frac{x}{a} \in K \right\}$$

is called gauge (Eichung).

Since  $x$  is an interior point, we know that  $\frac{x}{a} \in K$  if  $a > 1 - \varepsilon(x)$ .

$p$  is even defined on all of  $X$ , because for  $x \in X$ , now  $\tau x \in K$  if  $|\tau|$  is sufficiently small, because  $0 \in K$  is an interior point.

$$p(x) < 1 \quad \Leftrightarrow \quad x \in K$$

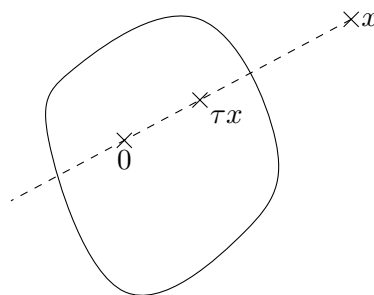


Figure 1.4:  $x \notin K$ ,  $\tau x \in K$

## 1.10 Lemma

$p$  is sublinear.

**Proof**

The homogeneity is clear from the definition.

sub-additivity (triangle equation):

Take  $x, y \in K$  and choose  $a, b \in \mathbb{R}_{>0}$  such that  $\frac{x}{a}, \frac{y}{b} \in K$ . The convexity of  $K$  implies for all  $\tau \in [0, 1]$ :

$$\tau \frac{x}{a} + (1 - \tau) \frac{y}{b} \in K$$

Choose  $\tau = \frac{a}{a+b}$ , then holds  $1 - \tau = \frac{b}{a+b}$ , which gives:

$$\Rightarrow \frac{1}{a+b} (x+y) \in K$$

$$p(x+y) \leq a+b$$

Taking the infimum over  $a$  and  $b$  gives  $p(x+y) \leq p(x) + p(y)$ :

$$p(x+y) = \inf \underbrace{\left\{ c \in \mathbb{R}_{>0} \mid \frac{x+y}{c} \in K \right\}}_{\ni a+b} \leq a+b$$

$$\begin{aligned} p(x) = \inf \left\{ a \mid \frac{x}{a} \in K \right\} &\Rightarrow \forall_{\varepsilon > 0} \exists_{a \in \mathbb{R}_{>0}} : p(x) \geq a - \varepsilon \\ p(y) = \inf \left\{ b \mid \frac{y}{b} \in K \right\} &\Rightarrow \forall_{\varepsilon > 0} \exists_{b \in \mathbb{R}_{>0}} : p(y) \geq b - \varepsilon \end{aligned}$$

□<sub>1.10</sub>

**1.11 Lemma**

$$p(x) < 1 \Leftrightarrow x \in K$$

**Proof**

If  $x \notin K$  then  $\frac{1}{a}x \notin K$  for all  $0 < a < 1$  and so  $p(x) \geq 1$ .

For all  $x \in K$  exists an  $\varepsilon = \varepsilon(x) \in \mathbb{R}_{>0}$  with  $(1+t)x \in K$  for all  $t \in (-\varepsilon, \varepsilon)$ .

$$\begin{aligned} &\Rightarrow \left(1 + \frac{\varepsilon}{2}\right)x \in K \\ &\Rightarrow p(x) \leq \frac{1}{1 + \frac{\varepsilon}{2}} < 1 \end{aligned}$$

□<sub>1.11</sub>

**Proof of Theorem 1.9**

Introduce  $l$  on  $\langle y \rangle$  by  $l(y) = 1$ . (Assume again that  $0 \in K$  and so  $y \neq 0$ .)

Write  $z = ay \in \langle y \rangle$  with  $a \in \mathbb{R}$ .

- If  $a < 0$ , then  $l(z) = a \cdot l(y) = a < 0$  but  $p(z) \geq 0$  and thus the inequality  $l(z) \leq p(z)$  is trivially satisfied.
- If  $a > 0$  it holds:

$$l(z) = a \underset{\Rightarrow p(y) \geq 1}{\overset{y \notin K}{\leq}} a \cdot p(y) \overset[\text{homogeneity}]{\text{positive}} p(ay) = p(z)$$

So for all  $z \in \langle y \rangle$  holds  $l(z) \leq p(z)$ .

The Hahn-Banach Theorem yields an extension  $l : X \rightarrow \mathbb{R}$  such that  $l(x) \leq p(x)$  for all  $x \in X$ .

Therefore for all  $x \in K$  we have:

$$l(x) \leq p(x) < 1$$

□<sub>1.9</sub>

## 2 Normed Spaces

Let  $(E, \|\cdot\|)$  be a normed space and let the open balls  $B_\varepsilon(x) = \{y \mid \|x - y\| < \varepsilon\}$  generate the topology on  $E$ .

### 2.0.1 Definition (equivalent norms)

Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are *equivalent*, if there exists a  $C \in \mathbb{R}_{>0}$  such that:

$$\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1$$

### 2.0.2 Theorem

Equivalent norms give rise to the same topology.

(No proof)

### 2.0.3 Theorem

If  $E$  is finite dimensional, then any two norms on  $E$  are equivalent.

(No proof)

### 2.0.4 Constructions (Quotient space, Cartesian product)

Let  $F \subseteq E$  be a *closed* subspace. Define the *quotient space* (Faktorraum)  $E/F$  as follows:

$$x \sim y \Leftrightarrow x - y \in F$$

defines an equivalence relation on  $E$ .

$$E/F := E/\sim$$

is a vector space.

$$\|u\|_{E/F} := \inf_{\substack{\hat{u} \in E \\ \hat{u} - u \in F}} \|\hat{u}\|_E$$

$(E/F, \|\cdot\|_{E/F})$  is a normed space. The closedness of  $F$  is essential:

Suppose  $F \subseteq E$  is not closed. Then there exists an  $x \in \overline{F} \setminus F$ , thus there is a  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \in F$

with  $x_n \rightarrow x$ .

Let  $[x] \in E/F$  be the equivalence class. Then  $[x] \neq 0$ , since  $x \notin F$ , but:

$$\|[x]\| = \inf_{\substack{\hat{x} \in E \\ \hat{x} - x \in F}} \|\hat{x}\| \stackrel{x - x_n \sim x}{\leq} \inf \|x - x_n\| = 0$$

If  $\|\cdot\|_{E/F}$  was a norm, it would imply  $[x] = 0$  and thus  $x \in F$  in contradiction to  $x \in \overline{F} \setminus F$ .

Another construction is the *Cartesian product*: Let  $E$  and  $F$  be normed spaces.

$$E \times F := \{(u, v) \mid u \in E, v \in F\}$$

$$\|(u, v)\|_{E \times F} := \|u\|_E + \|v\|_F$$

is a norm on  $E \times F$ .

### 2.0.5 Definition (separable)

A normed space is called *separable*, if there is a countable dense subset, i.e. there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  such that every nonempty open subset of the space contains at least one element of the sequence.

### 2.0.6 Examples

The space  $\ell^\infty$  of bounded sequences  $(a_n)_{n \in \mathbb{N}}$ ,  $a_n \in \mathbb{K}$  with  $\|(a_n)_{n \in \mathbb{N}}\|_\infty := \sup_n |a_n|$  is a Banach space.

$$A := \left\{ (a_n)_{n \in \mathbb{N}} \mid a_{2n} = 0 \ \forall_{n \in \mathbb{N}} \right\} \subseteq \ell^\infty$$

is a closed subspace.

$$\ell^\infty / A \cong \left\{ (a_n) \mid a_{2n+1} = 0 \ \forall_{n \in \mathbb{N}} \right\}$$

$$d := \left\{ (a_n) \mid \exists_{N \in \mathbb{N}} \forall_{n \in \mathbb{N}_{>N}} a_n = 0 \right\} \subseteq \ell^\infty$$

is a subspace, but not closed in  $\ell^\infty$ . Consider for example  $(a_n = \frac{1}{n}) =: x \in \ell^\infty \setminus d$ ,  $x_n \in d$  with  $x_n = (a_{n_l})_{l \in \mathbb{N}}$  and:

$$a_{n_l} = \begin{cases} \frac{1}{l} & \text{if } l \leq n \\ 0 & \text{if } l > n \end{cases}$$

Then converges  $x_n \rightarrow x \notin d$ , and therefore  $d$  is not closed. The closure is:

$$\overline{d} = \left\{ (a_n) \mid a \xrightarrow{n \rightarrow \infty} 0 \right\}$$

$\ell^\infty$  is not separable.

### 2.0.7 Example

For  $1 \leq p < \infty$  define

$$\ell^p = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

and the  $\ell^p$ -norm:

$$\|(a_n)\|_p := \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}$$

$\ell^p$  is a normed space (Hölder's inequality, Minkowski inequality) and also separable (see exercises).

### 2.0.8 Example

Let  $(\Omega, \mu)$  be a measure space (Maßraum).

$$\begin{aligned} L^p(\Omega) \quad (1 \leq p < \infty) \quad & \|f\|_p = \left( \int_{\Omega} |f(x)|^p d\mu \right)^{\frac{1}{p}} \\ L^{\infty}(\Omega) \quad & \|f\|_{\infty} = \sup_{\Omega} |f(x)| = \sup \{ L \in \mathbb{R} \mid \mu(f^{-1}([L, \infty))) > 0 \} \end{aligned}$$

## 2.1 Non-Compactness of the Unit Ball

Let  $(E, \|\cdot\|)$  be a normed vector space.

$$K := \overline{B_1(0)} = \{x \in E \mid \|x\| \leq 1\}$$

If  $\dim(E) < \infty$ ,  $K$  is compact by the Heine-Borel theorem.

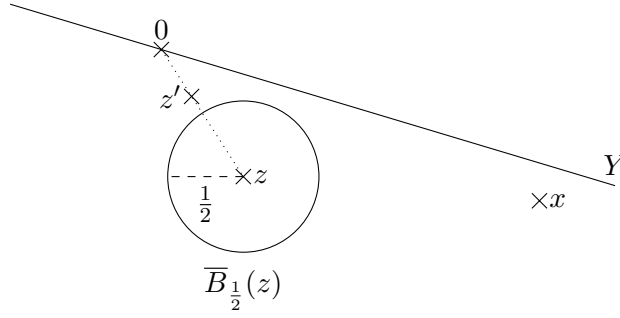
### 2.1.1 Theorem

If  $E$  is infinite-dimensional, then  $K$  is not sequentially compact (folgenkompakt), i.e. it is possible to construct a sequence  $(y_n)$ ,  $y_n \in K$ , which has no convergent subsequence.

### 2.1.2 Lemma

Let  $Y \subsetneq E$  be a proper (echter) closed subspace. Then there is a  $z \in E \setminus Y$  with  $\|z\| = 1$  such that holds:

$$\begin{aligned} & \forall_{y \in Y} : \|z - y\| > \frac{1}{2} \\ \Leftrightarrow & \overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset \end{aligned}$$


 Figure 2.1:  $\overline{B_{\frac{1}{2}}(z)} \cap Y = \emptyset$ 

### Proof

Choose  $x \in E \setminus Y \neq \emptyset$ . As  $E \setminus Y$  is open, there is a  $\delta \in \mathbb{R}_{>0}$  with  $B_\delta(x) \cap Y = \emptyset$ . Thus we can define:

$$d := \inf_{y \in Y} \|x - y\| > 0$$

Choose  $y_0 \in Y$  such that  $\|x - y_0\| < 2d$ . Set  $z' = x - y_0$ . Then  $\|z'\| < 2d$  and  $\|z' - y\| \geq d$  for all  $y \in Y$ . Thus  $z := \frac{z'}{\|z'\|}$  has the desired properties.  $\square_{2.1.2}$

### Proof of Theorem 2.1.1

Choose inductively a sequence  $(y_n)$ :  $y_1 \in K$  is arbitrary.  $Y_1 := \langle y_1 \rangle$  is a one dimensional subspace, which is closed. Choose  $y_2 \in K$  such that  $\|y_2 - y\| > \frac{1}{2}$  for all  $y \in Y_1$ , which is possible according to Lemma 2.1.2.

Suppose  $y_1, \dots, y_n$  are given.  $Y_n := \langle y_1, \dots, y_n \rangle$  is closed. So there exists a  $y_{n+1} \in K$  such that for all  $y \in Y_n$  holds:

$$\|y_{n+1} - y\| > \frac{1}{2}$$

This sequence has the following properties:

- $y_k \in K$
- For all  $k, l \in \mathbb{N}$  with  $k < l$  holds  $\|y_l - y_k\| > \frac{1}{2}$ , since  $y_k \in Y_{l-1} = \langle y_1, \dots, y_{l-1} \rangle$  and we know by construction that  $\|y_l - y\| > \frac{1}{2}$  for all  $y \in Y_{l-1}$  so especially for  $y_k \in Y_{l-1}$ .

This implies that  $(y_k)$  has no convergent subspace.  $\square_{2.1.1}$

## 2.2 Spaces of linear Mappings, Dual Spaces

Let  $E, F$  be normed spaces.

$A : E \rightarrow F$  is continuous if and only if it is bounded, i.e. there exists a  $C \in \mathbb{R}_{>0}$  such that for all  $u \in E$  holds:

$$\|Au\|_F \leq C \|u\|_E$$

Denote by  $L(E, F)$  the normed space of all bounded linear maps from  $E$  to  $F$  and define:

$$\|A\| := \sup_{\|u\| \leq 1} \|Au\| = \sup_{\|u\|=1} \|Au\|$$

### 2.2.1 Lemma

If  $B \in L(E, F)$  and  $A \in L(F, G)$  then Schwarz inequality or Kato inequality holds:

$$\begin{aligned}\|A \cdot B\| &\leq \|A\| \cdot \|B\| \\ \|Au\| &\leq \|A\| \cdot \|u\|\end{aligned}$$

(no proof)

### 2.2.2 Theorem and Definition (dual pairing)

If  $F$  is complete, so is  $L(E, F)$ .

Special case  $F = \mathbb{R}$  and  $\|x\|_{\mathbb{R}} = |x|$ :  $E^* := L(E, \mathbb{R})$  is the dual space.

For  $\varphi \in E^*$  and  $u \in E$

$$\varphi(u) = (\varphi, u)$$

is called *dual pairing* (duale Paarung).

$$(\cdot, \cdot) : E^* \times E \rightarrow \mathbb{R}$$

is a continuous bilinear map. For  $u \in E$

$$(\cdot, u) : E^* \rightarrow \mathbb{R}$$

defines an element of  $E^{**} = L(E^*, \mathbb{R})$ . This gives rise to a linear mapping:

$$\iota : E \rightarrow E^{**}$$

(no proof)

### 2.2.3 Theorem

$\iota : E \hookrightarrow E^{**}$  is an isometric embedding of  $E$  into  $E^{**}$ .

#### Proof

For  $u \in E$  holds:

$$\|\iota(u)\| := \sup_{\varphi \in E^*, \|\varphi\|=1} \|(\iota(u))(\varphi)\| = \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| \stackrel{?}{=} \|u\|$$

$$\|\varphi\| = \sup_{v \in E, \|v\|=1} |\varphi(v)|$$

$$\begin{aligned}\|\varphi(u)\| &\leq \|\varphi\| \cdot \|u\| \stackrel{\|\varphi\|=1}{=} \|u\| \\ \Rightarrow \sup_{\varphi \in E^*, \|\varphi\|=1} \|\varphi(u)\| &\leq \|u\|\end{aligned}$$



To prove  $\|\iota(u)\| \geq \|u\|$  apply the Hahn-Banach theorem:

Let  $l : \langle u \rangle \rightarrow \mathbb{R}$  be the linear map with  $l(u) = \|u\|$ , thus:

$$\|l\| = \sup_{v \in \langle u \rangle, \|v\|=1} (l(v)) = \sup \left( l \left( \pm \frac{u}{\|u\|} \right) \right) = 1$$

By the Hahn-Banach theorem we can extend  $l$  to

$$\tilde{l} : E \rightarrow \mathbb{R}$$

with  $\|\tilde{l}\| = 1$  and then holds:

$$\sup_{\varphi \in E^*, \|\varphi\|=1} \varphi(u) \stackrel{\|\tilde{l}\|=1}{\geq} \tilde{l}(u) = \|u\|$$

Therefore  $\iota$  is injective, because from  $\iota(u) = 0$  follows  $\|u\|_E = \|\iota(u)\| = 0$  and therefore  $u = 0$ .  $\square_{2.2.3}$

### 2.2.4 Definition (reflexive)

A Banach space is called *reflexive* (reflexiv) if  $\iota$  is bijective, i.e.  $E \cong E^{**}$ .

### 2.2.5 Example

Let  $\ell_1$  be the space of absolutely convergent functions with the norm:

$$\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty$$

Let  $(\lambda_n) \in \ell_{\infty}$  be a bounded sequence and define  $\Lambda \in \ell_1^*$ :

$$\begin{aligned} \Lambda : \ell_1 &\rightarrow \mathbb{R} \\ \Lambda((a_n)) &= \sum_{n=1}^{\infty} \lambda_n a_n \end{aligned}$$

$$|\Lambda((a_n))| = \left| \sum_{n=1}^{\infty} \lambda_n a_n \right| \leq \sum_{n=1}^{\infty} |\lambda_n| \cdot |a_n| \leq \|(\lambda_n)\|_{\infty} \sum_{n=1}^{\infty} |a_n| = \|(\lambda_n)\|_{\infty} \cdot \|(a_n)\|_1 < \infty$$

Thus  $\Lambda$  is bounded and:

$$\|\Lambda\| = \sup_{n \in \mathbb{N}} |\lambda_n|$$

**Claim:** Every bounded linear functional on  $\ell_1$  is of this form, i.e.  $\ell_1^* = \ell_{\infty}$ .

**Proof:** Let  $\Lambda \in \ell_1^*$ . Choose  $u_l \in \ell_1$  by  $u_l = (0, \dots, 0, 1, 0, \dots)$  with a one at the  $l$ -th position.

Setting  $\lambda_l := \Lambda(u_l)$  gives:

$$|\lambda_l| = |\Lambda(u_l)| \leq \underbrace{\|\Lambda\|}_{< \infty} \cdot \underbrace{\|u_l\|}_{=1} \leq \|\Lambda\| < \infty$$

So  $(\lambda_l) \in \ell_\infty$ .

Let  $(a_k)$  be a finite sequence, with only zeros for  $k > K \in \mathbb{N}$ . Then:

$$\Lambda((a_k)) = \Lambda\left(\sum_{k=1}^K a_k u_k\right) = \sum a_k \Lambda(u_k) = \sum \lambda_k a_k$$

Since the finite sequences are dense in  $\ell_1$ , the claim follows.  $\square_{\text{Claim}}$

So  $\ell_1^* = \ell_\infty$  and one could assume  $\ell_\infty^* = \ell_1$ , but this is not the case (see exercises).

Thus  $\ell_1^{**} \neq \ell_1$ , which means, that  $\ell_1$  is *not* reflexive.

## 2.3 Weak Convergence (Schwache Konvergenz)

Let  $E$  be a Banach space and  $(u_n)$  a sequence in  $E$ .

Normal convergence:  $u_n \rightarrow u$  if and only if  $\|u - u_n\| \xrightarrow{n \rightarrow \infty} 0$ .

### 2.3.1 Definition (weak convergence, weak Cauchy sequence)

A sequence  $(u_n)$  in  $E$  *converges weakly* to  $u$ , written as  $u_n \rightharpoonup u$ , if for all  $\varphi \in E^*$  the sequence  $\varphi(u_n)$  converges to  $\varphi(u)$ , i.e.  $\varphi(u_n) \rightarrow \varphi(u)$ .

$(u_n)$  is a *weak Cauchy sequence* if for all  $\varphi \in E^*$  the sequence  $\varphi(u_n)$  is a Cauchy sequence.

### 2.3.2 Theorem (Uniqueness of weak limit)

The weak limit is unique.

#### Proof

Let  $(u_n)$  be a sequence in  $E$ , which converges weakly to  $u$  and  $u'$ , i.e. for all  $\varphi \in E^*$  holds:

$$\begin{aligned} \varphi(u_n) &\rightarrow \varphi(u) & \varphi(u_n) &\rightarrow \varphi(u') \\ \Rightarrow \quad 0 &= \varphi(u_n - u_n) \rightarrow \varphi(u - u') \end{aligned}$$

So  $\varphi(u - u') = 0$  for all  $\varphi \in E^*$ .

**Claim:**  $v := u - u' = 0$

**Proof:** Assume to the contrary that  $v \neq 0$ .

Choose  $\varphi : \langle v \rangle \rightarrow \mathbb{R}$  with  $\varphi(v) = 1$ . By the Hahn-Banach theorem  $\varphi$  can be extended continuously to  $E$ .

Therefore exists a  $\varphi \in E^*$  with  $\varphi(v) = 1$ , which is a contradiction to  $\varphi(v) = 0$ .  $\square_{\text{Claim}}$

$\square_{2.3.2}$

### 2.3.3 Theorem (convergence implies weak convergence)

Every convergent sequence converges weakly.

**Proof**

Suppose that  $u_n \rightarrow u$ . For  $\varphi \in E^*$  follows:

$$|\varphi(u_n) - \varphi(u)| = |\varphi(u_n - u)| \leq \underbrace{\|\varphi\|}_{\in \mathbb{R}} \cdot \|u_n - u\| \rightarrow 0$$

$$\begin{aligned} \Rightarrow \quad \varphi(u_n) &\rightarrow \varphi(u) \\ \Rightarrow \quad u_n &\rightharpoonup u \end{aligned}$$

□<sub>2.3.3</sub>**2.3.4 Example**

$E = \left\{ (a_n) \left| a_n \xrightarrow{n \rightarrow \infty} 0 \right. \right\} \subsetneq \ell_\infty$  with  $\|(a_n)\| = \sup_n |a_n|$  is a Banach space.

Let  $u_n = (0, \dots, 0, 1, 0, \dots)$  be the sequence with a one at the  $n$ -th position and zeros elsewhere. For  $n \neq m$  we have:

$$\|u_n - u_m\| = \sup \{0, |1|, |-1|\} = 1$$

Thus  $(u_n)$  is *not* a Cauchy sequence. Every  $\varphi \in E^*$  can be represented with  $(\lambda_k) \in \ell_1$  as (see exercises):

$$\begin{aligned} \varphi((a_n)) &= \sum_k \lambda_k a_k \\ \|\varphi\| &= \sum_{k=1}^{\infty} |\lambda_k| < \infty \end{aligned}$$

$$\varphi(u_n) = \sum_{k=1}^{\infty} \lambda_k \delta_{kn} = \lambda_n \xrightarrow{n \rightarrow \infty} 0$$

From  $(\lambda_n) \in \ell_1$  follows  $\lambda_n \rightarrow 0$ . This means that  $u_k \rightharpoonup 0$ .

This is used in the lectures on partial differential equations.

From  $\mathcal{S}(u_n) \rightarrow \inf \mathcal{S}$  follows not necessarily  $u_n \rightarrow u$ , but  $u_n \rightharpoonup u$ .

Consider  $A_n \in L(E, F)$ .

- *norm convergence*:  $A_n \rightarrow A$  in  $L(E, F)$  means  $\|A_n - A\| \rightarrow 0$ .
- *strong convergence*:  $A_n u \rightarrow Au$  in  $F$  for all  $u \in E$ .
- *weak convergence*:  $A_n u \rightharpoonup Au$  for all  $u \in E$ , i.e. for all  $\varphi \in F^*$  holds  $\varphi(A_n u) \rightarrow \varphi(Au)$ .

**2.4 The Baire Category Theorem**

Let  $E$  be a metric space (e.g. a normed space).

### 2.4.1 Definition (nowhere dense, set of first/second category)

A subset  $A \subseteq E$  is called *nowhere dense* (nirgends dicht) if  $\overline{A}^\circ = \emptyset$ .

$A$  is called *of first category* (or *meager*) if it can be written as a countable union of nowhere dense sets. Otherwise it is *of second category*.

#### Example

- $\mathbb{N} \subseteq \mathbb{R}$  is nowhere dense:  $\overline{\mathbb{N}} = \mathbb{N}$ ,  $\mathbb{N}^\circ = \emptyset$
- $\mathbb{Q} \subseteq \mathbb{R}$  is dense:  $\overline{\mathbb{Q}} = \mathbb{R}$ ,  $\overline{\mathbb{Q}}^\circ = \mathbb{R}^\circ = \mathbb{R}$

### 2.4.2 Theorem (René Baire, 1899)

Let  $E \neq \emptyset$  be a complete metric space (Polish space). Then  $E$  is of second category.

#### Proof

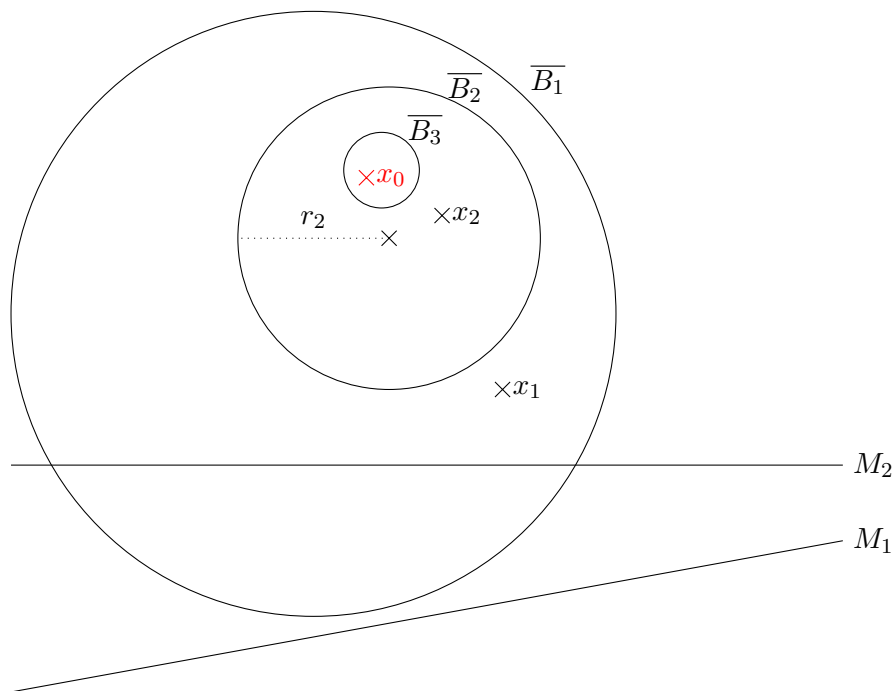


Figure 2.2:  $B_n \cap M_n = \emptyset$

Assume in contrast that  $E = \bigcup_{n \in \mathbb{N}} M_n$  and the sets  $M_n$  are nowhere dense. Without loss of generality assume that the  $M_n$  are closed, since otherwise one can replace  $M_n$  by  $\overline{M_n}$ .

We shall construct inductively balls  $\overline{B_n} = \overline{B_{r_n}(x_n)}$  such that  $\overline{B_{n+1}} \subseteq \overline{B_n}$ ,  $r_n < 2^{-n}$  and  $\overline{B_n} \cap M_n = \emptyset$  for all  $n$ .

Then the points  $x_n$  form a Cauchy sequence, because for all  $n < m \in \mathbb{N}$  we have  $x_{n+1} \in B_n$

and so  $\|x_n - x_{n+1}\| < r_n < 2^{-n}$ :

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_m\| \leq \dots \leq \\ &\leq 2^{-n} + 2^{-(n+1)} + \dots + 2^{-(m-1)} \leq 2^{-n} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \leq 2 \cdot 2^{-n} \end{aligned}$$

Since  $E$  is complete,  $x_n \rightarrow x_0 \in E$  converges. Then  $x_0 \in \overline{B_n}$  for all  $n$ , which implies  $x_0 \notin M_n$  and thus the contradiction  $x_0 \notin \bigcup_n M_n = E$  follows.

Construction of the balls  $\overline{B_n}$ :

$M_1$  is nowhere dense and therefore  $B_1(0) \not\subseteq M_1$ . So there exists a  $x_1 \in B_1(0) \setminus M_1$ . Since  $M_1$  is closed,  $B_1(0) \setminus M_1$  is open and therefore there exists a radius  $r_1$  such that  $B_{2r_1}(x_1)$  is contained in  $B_1(0) \setminus M_1$  and thus  $\overline{B_{r_1}(x_1)} \cap M_1 = \emptyset$ .

Suppose  $\overline{B_n}$  has been constructed.  $M_{n+1}$  is nowhere dense and closed and so there exists a  $x_{n+1} \in \overline{B_n} \setminus M_{n+1}$  and  $r_{n+1} < 2^{-(n+1)}$  such that  $B_{2r_{n+1}}(x_{n+1}) \subseteq \overline{B_n} \setminus M_{n+1}$ . Then follows  $\overline{B_{r_{n+1}}(x_{n+1})} \cap M_{n+1} = \emptyset$ .  $\square_{2.4.2}$

### 2.4.3 Theorem (Uniform boundedness principle, Prinzip der gleichmäßigen Beschränktheit)

Let  $E$  be a Banach space and  $F$  a normed space. Let  $T_i$  be a sequence in  $L(E, F)$  which is point-wise bounded, i.e. for all  $u \in E$ :

$$\sup_i \|T_i u\| \leq C(u) < \infty$$

Then sup-norms of  $T_i$  are bounded:

$$\sup_i \|T_i\| = \sup_i \sup_{\|u\|=1} \|T_i u\| \leq \tilde{C} < \infty$$

(Thus there exists a constant  $C \in \mathbb{R}_{>0}$  such that  $\|T_i u\| \leq C$  for all  $i \in \mathbb{N}$  and for all  $u \in E$  with  $\|u\| = 1$ .)

#### Proof

The sets  $M_n = \{u \in E \mid \sup_i \|T_i u\| \leq n\}$  are closed by continuity of the  $T_i \in L(E, F)$ , i.e. for  $u_k \rightarrow u$  converges  $\|T_i u_k\| \xrightarrow{k \rightarrow \infty} \|T_i u\|$ .

$E = \bigcup_n M_n$ , because for any  $u \in E$ ,  $\sup_i \|T_i u\| < \infty$  and thus  $u \in M_n$  for  $n > \sup_i \|T_i u\|$ .

If all the sets  $M_n$  had empty interior, we would get a contradiction to Baire's theorem.

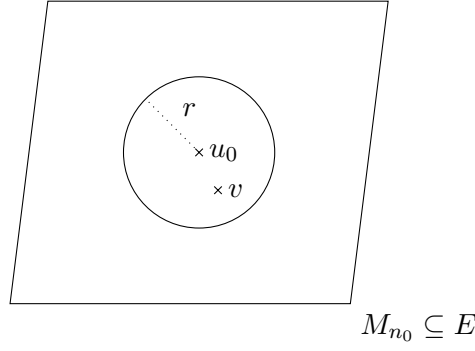
So there exists an  $n_0 \in \mathbb{N}$  such that  $M_{n_0} \neq \emptyset$  and thus there are  $u_0 \in E$  and  $r \in \mathbb{R}_{>0}$  such that  $B_r(u_0) \subseteq M_{n_0}$ .

For all  $v \in B_r(u_0)$  we know that  $\sup_i \|T_i v\| \leq n_0$  which is equivalent to:

$$\sup_{v \in B_r(u_0)} \|T_i v\| \leq n_0 \quad \forall_{i \in \mathbb{N}}$$

Let  $w \in B_r(0)$  be arbitrary. Then  $v := u_0 + w \in B_r(u_0)$ .

$$T_i w \stackrel{T_i \text{ linear}}{=} T_i v - T_i u_0$$

Figure 2.3:  $B_r(u_0) \subseteq M_{n_0}$ 

$$\|T_i w\| \leq \|T_i v\| + \|T_i u_0\| \leq n_0 + \sup_i \|T_i u_0\| < \infty$$

Here  $\sup_i \|T_i u_0\| < \infty$ , because the  $T_i$  are point-wise bounded.

$$\begin{aligned} \Rightarrow \|T_i w\| &\leq C \quad \forall w \in B_r(0) \\ \Rightarrow \|T_i \tilde{w}\| &\leq \tilde{C} = \frac{C}{r} \quad \forall \tilde{w} \in \overline{B_1(0)} \end{aligned}$$

So  $\|T_i\| \leq \tilde{C}$  for all  $i \in \mathbb{N}$  and so  $\|T_i\|$  is bounded.

□<sub>2.4.3</sub>

#### 2.4.4 Corollary

Let  $E$  be a normed space, not necessarily complete, and  $(u_n)$  a weak Cauchy sequence. Then  $\|u_n\|$  is a bounded sequence.

##### Proof

$E^* = L(E, \mathbb{R})$  is a Banach space after theorem 2.2.2, since  $\mathbb{R}$  is complete. Now we can view every  $u_n$  as operator:

$$\begin{aligned} u_n : E^* &\rightarrow \mathbb{R} \\ \varphi &\mapsto \varphi(u_n) \end{aligned}$$

So  $(u_n)$  is a sequence in  $L(E^*, \mathbb{R})$ . For all  $\varphi \in E^*$  we know that  $\varphi(u_n)$  is a Cauchy sequence and thus bounded:

$$\Rightarrow |\varphi(u_n)| < C(\varphi)$$

Applying theorem 2.4.3 yields:

$$\begin{aligned} |\varphi(u_n)| &< C \quad \forall \varphi \text{ with } \|\varphi\|=1 \\ \Leftrightarrow \sup_{n \in \mathbb{N}} \sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(u_n)| &< C \end{aligned}$$

For any  $v \in E$  we have

$$\sup_{\varphi \in E^*, \|\varphi\|=1} |\varphi(v)| = \|v\|$$

by the Hahn-Banach theorem:

- $|\varphi(v)| \leq \|\varphi\| \cdot \|v\| \stackrel{\|\varphi\|=1}{=} \|v\|$
- Choose  $\varphi : \langle v \rangle \rightarrow \mathbb{R}$  with  $\varphi(v) = \|v\|$  and so  $\|\varphi\| = 1$ . By the Hahn-Banach theorem we can extend  $\varphi$  to  $\tilde{\varphi} : E \rightarrow \mathbb{R}$  such that  $\|\tilde{\varphi}\| = 1$ . Then  $\tilde{\varphi}(v) = \|v\|$  and so  $\sup_{\|\varphi\|=1} |\varphi(v)| \geq \|v\|$ .

Thus we get  $\sup_n \|u_n\| < C$ .

□<sub>2.4.4</sub>

### 2.4.5 Corollary and Definition (Banach-Steinhaus, equicontinuous, uniformly continuous)

Let  $E, F$  be Banach spaces and  $T_i \in L(E, F)$ .

If the  $(T_i)$  are point-wise bounded, then the  $T_i$  are *equicontinuous* (gleichgradig stetig).

**Definition** (uniformly continuous, equicontinuous)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued function.

Continuity:

$$\forall_{x_0 \in \mathbb{R}} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

$f$  is called *uniformly continuous* (gleichmäßig stetig) if:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad \|x - y\| < \delta \quad \Rightarrow \quad \|f(x) - f(y)\| < \varepsilon$$

Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be a series of real-valued functions.  $(f_n)$  is called *equicontinuous* if:

$$\forall_{x_0 \in \mathbb{R}} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} \quad \forall_{n \in \mathbb{N}} : \quad \|x - x_0\| < \delta \quad \Rightarrow \quad \|f_n(x) - f_n(x_0)\| < \varepsilon$$

For a linear map  $A \in L(E, F)$  holds:

$$\begin{aligned} \|Au\| &\leq \|A\| \|u\| \\ \|Au - Au_0\| &\leq \|A\| \|u - u_0\| \end{aligned}$$

Therefore choose  $\delta = \frac{\varepsilon}{2\|A\|}$ , i.e.:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{\delta \in \mathbb{R}_{>0}} : \quad \|u\| < \delta \quad \Rightarrow \quad \|Au\| < \varepsilon$$

### Proof

Since  $(T_i)$  is point-wise bounded there is a  $C \in \mathbb{R}_{>0}$  such that for all  $i \in \mathbb{N}$  holds  $\|T_i\| \leq C$  due to the principle of uniform boundedness 2.4.3. So for all  $i \in \mathbb{N}$  holds:

$$\|T_i u\| \leq \|T_i\| \|u\| \leq C \|u\|$$

Choose  $\delta = \frac{\varepsilon}{2C}$  shows that the  $T_i$  is equicontinuous.

□<sub>2.4.5</sub>

In the following let  $E$  and  $F$  be Banach spaces.

### 2.4.6 Definition (open)

A (not necessarily linear) map  $A : E \rightarrow F$  is called *open* if the image of every open set is open. (If there exists an inverse  $A^{-1}$  then “ $A$  open” is equivalent to “ $A^{-1}$  continuous”.)

Let  $A$  be linear and open.  $B_1(0) \subseteq E$  is open, so  $A(B_1(0)) \subseteq F$  is open. Since  $0 \in A(B_1(0))$ , there is a  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_\varepsilon(0) \subseteq A(B_1(0))$ .

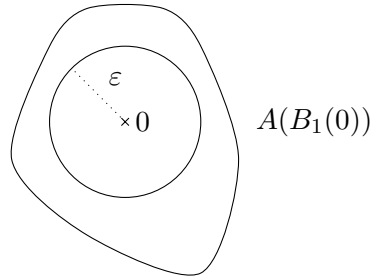


Figure 2.4:  $B_\varepsilon(0) \subseteq A(B_1(0))$

Due to the linearity holds in general:

$$B_\lambda(0) \subseteq A\left(B_{\frac{\lambda}{\varepsilon}}(0)\right)$$

In particular,  $A$  is surjective.

If  $A$  is additionally injective, then  $A$  is bijective and the openness means that  $A^{-1}$  is continuous.

### 2.4.7 Theorem (Open mapping theorem, Prinzip der offenen Abbildung)

If  $A \in L(E, F)$  is surjective, then  $A$  is open.

### 2.4.8 Corollary

If  $A \in L(E, F)$  is bijective, then  $A^{-1} \in L(F, E)$  is continuous.

#### Proof

$A$  is open following 2.4.7, since  $A$  is surjective. This means that  $A^{-1}$  is continuous.  $\square_{2.4.8}$

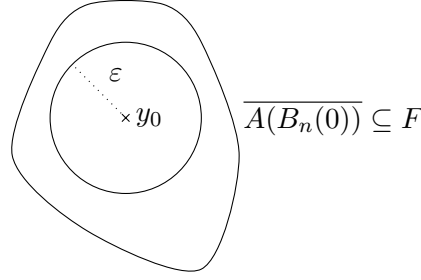
#### Proof of 2.4.7

Since  $A$  is surjective,  $F = A(E)$ . Since every element of  $E$  has a finite norm, we know:

$$\begin{aligned} E &= \bigcup_{n \in \mathbb{N}} B_n(0) \\ \Rightarrow F &= A\left(\bigcup_{n \in \mathbb{N}} B_n(0)\right) = \bigcup_{n \in \mathbb{N}} A(B_n(0)) \end{aligned}$$

According to Baire's theorem there is a  $n \in \mathbb{N}$  such that  $\overline{A(B_n(0))}^\circ \neq \emptyset$ .



Figure 2.5:  $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$ 

So there exists a  $y_0 \in A(B_n(0))$  and a  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_\varepsilon(y_0) \subseteq \overline{A(B_n(0))}$ . Since  $A$  is surjective, there is a  $x_0 \in B_n(0)$  with  $y_0 = A(x_0)$ .

$$\Rightarrow \overline{A(B_n(0) - x_0)} = \overline{A(B_n(0)) - y_0} = \overline{A(B_n(0))} - y_0 \supseteq B_\varepsilon(y_0) - y_0 = B_\varepsilon(0)$$

If  $n'$  is large enough, then  $B_n(-x_0) \subseteq B_{n'}(0)$  and so  $\overline{A(B_{n'}(0))} \supseteq B_\varepsilon(0)$ .

Since  $A$  is linear, we can rescale, i.e. there is a  $c := \frac{\varepsilon}{n'} \in \mathbb{R}_{>0}$  such that for all  $r \in \mathbb{R}_{>0}$  holds:

$$\overline{A(B_r(0))} \supseteq B_{cr}(0)$$

Now we show that every  $u \in B_c(0)$  is the image of a  $x \in B_2(0)$ , i.e.  $B_c(0) \subseteq A(B_2(0))$ :

Ansatz as a series:

$$x = \sum_{j=1}^{\infty} x_j$$

Choose  $x_1 \in B_1(0)$  with  $\|u - Ax_1\| < \frac{c}{2}$ , which is possible since  $\overline{A(B_1(0))} \supseteq B_c(0)$ .

Choose  $x_2 \in B_2(0)$  with  $\|u - Ax_1 - Ax_2\| < \frac{c}{4}$ , which is possible since  $u - Ax_1 \in B_{\frac{c}{2}}(0)$  and

$$\overline{A\left(B_{\frac{1}{2}}(0)\right)} \subseteq B_{\frac{c}{2}}(0).$$

And so on choose  $x_m \in B_{\frac{1}{2^m}}(0)$  with  $\|u - \sum_{i=1}^m Ax_i\| < \frac{c}{2^m}$ .

The series  $\sum_{i=1}^{\infty} x_i$  converges, since:

$$\left\| \sum_{j=m}^M x_j \right\| \leq \sum_{j=m}^M \|x_j\| \leq \sum_{j=m}^M 2^{-j}$$

So the sequence of partial sums is a Cauchy sequence. Because  $E$  is complete, this sequence converges.

The continuity of  $A$  yields:

$$Ax = \sum_{j=1}^{\infty} Ax_j = u$$

So there exists a  $x \in E$  with  $\|x\| < 2$  and  $Ax = u$ .

□<sub>2.4.7</sub>

$$\begin{aligned}
& \sum_{j=1}^n x_j \xrightarrow{n \rightarrow \infty} x & \|x\| < 2 \\
& \sum_{j=1}^n Ax_j \xrightarrow{n \rightarrow \infty} u \\
& \parallel \\
& A \left( \sum_{j=1}^n x_j \right) \xrightarrow[\text{continuity of } A]{n \rightarrow \infty} Ax
\end{aligned}$$

**Definition** (Graph)

For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  the *graph* is defined as:

$$\text{graph} f := \{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

For  $A : E \rightarrow F$  the *graph* is:

$$\text{graph} A := \{(u, Au) \mid u \in E\} \subseteq E \times F$$

Here  $E \times F$  is a product of normed spaces which has the norm:

$$\|(u, v)\| := \|u\|_E + \|v\|_F$$

**Lemma**

If  $A$  is continuous, then  $\text{graph} A$  is closed.

**Proof**

Let  $(u_n, Au_n) \in \text{graph} A$  be a Cauchy sequence in  $E \times F$  for Banach spaces  $E$  and  $F$ , i.e.  $u_n \rightarrow u$ . Since  $A$  is continuous, it follows:

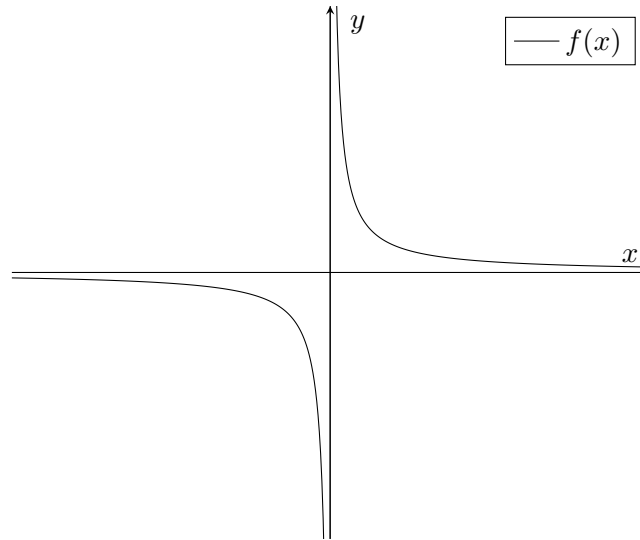
$$Au_n \rightarrow v := Au$$

Therefore  $(u, v) \in \text{graph} (A)$  and so the graph is closed. □<sub>Lemma</sub>

Consider the function:

$$\begin{aligned}
f : \mathbb{R} \setminus \{0\} &\rightarrow \mathbb{R} \\
x &\mapsto \frac{1}{x}
\end{aligned}$$

$f$  is not continuous, but  $\text{graph} (f)$  is closed in  $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ .

Figure 2.6:  $f$  is not continuous, but  $\text{graph } f$  is closed.

### 2.4.9 Theorem (Closed graph theorem, Satz vom abgeschlossenen Graphen)

Suppose a linear map  $A : E \rightarrow F$  between Banach spaces  $E$  and  $F$  has a closed graph. Then  $A$  is continuous.

$\text{graph}(A)$  closed means:

For all  $u_n \in E$  with  $u_n \rightarrow u$  and  $Au_n \rightarrow v$ , the point  $(u, v) \in \text{graph}(A)$ , i.e.  $Au = v$ .

$A$  continuous means:

For all  $u_n \in E$  with  $u_n \rightarrow u$ , the sequence  $Au_n \rightarrow v$  converges and  $Au = v$

#### Proof

On  $E \times F$  we have the norm:

$$\|(u, v)\| := \|u\|_E + \|v\|_F$$

The graph

$$G := \{(u, Au) \mid u \in E\} \subseteq E \times F$$

is a subspace of  $E \times F$ , since for  $\lambda \in \mathbb{R}$  and  $u, \tilde{u} \in E$  holds:

$$\lambda(u, Au) + (\tilde{u}, A\tilde{u}) = (\lambda u + \tilde{u}, \lambda Au + A\tilde{u}) \stackrel{A \text{ linear}}{=} (\lambda u + \tilde{u}, A(\lambda u + \tilde{u})) \in G$$

So  $G$  is complete and therefore a Banach space, since we assumed it to be closed.

Define:

$$\begin{aligned} P : G &\rightarrow E \\ (u, Au) &\mapsto u \end{aligned}$$

$$\|(u, Au)\| = \|u\| + \|Au\| \geq \|u\| = \|P(u, Au)\|$$

So for all  $w \in G$  holds  $\|Pw\| \leq \|w\|$  and therefore  $\|P\| \leq 1$ . In particular,  $P$  is continuous.  $P$  is obviously surjective and it is also injective, since:

$$P^{-1}(u) = (u, Au)$$

Following the open mapping theorem,  $P^{-1}$  is continuous, i.e. there exists a  $C \in \mathbb{R}_{>0}$  such that:

$$\|u\| + \|Au\| = \|(u, Au)\| = \|P^{-1}(u)\| \leq C \|u\|$$

Then follows:

$$\|Au\| \leq (C - 1) \|u\|$$

Therefore  $A$  is continuous. □<sub>2.4.9</sub>

## 2.5 Neumann series

Let  $E$  be a Banach space and  $A \in L(E, E) =: L(E)$ .

When is  $A$  continuously invertible?

Remember that for  $x \in \mathbb{K}$  with  $|x| < 1$  holds:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This is the geometric series.

*Idea:*  $A = \mathbb{1} - B$  with  $B \in L(E)$

*Ansatz:*  $A^{-1} := \sum_{n=0}^{\infty} B^n$

This works indeed if  $\|B\| < 1$ .

### 2.5.1 Lemma and Definition (Neumann series)

The series

$$C := \sum_{n=0}^{\infty} B^n$$

is called Neumann series (Neumannsche Reihe).

If  $\|B\| < 1$ , then  $C$  defines an element of  $L(E, E)$ , i.e. the Neumann series converges absolutely.

#### Proof

Consider the partial sums:

$$S_n := \sum_{k=0}^n B^k$$

Since  $L(E, E)$  is a Banach space, it is enough to show that  $S_n$  is a Cauchy series. Without loss of generality assume  $m > n$ :

$$\|S_n - S_m\| = \left\| \sum_{k=n}^m B^k \right\| \stackrel{\Delta \text{ inequality}}{\leq} \sum_{k=n}^m \|B^k\| \stackrel{\text{Schwarz}}{\leq} \sum_{k=n}^m \|B\|^k < c \|B\|^n \rightarrow 0$$

□<sub>2.5.1</sub>

### 2.5.2 Theorem

$$C = (\mathbb{1} - B)^{-1}$$

**Proof**

$$(\mathbb{1} - B)C = (\mathbb{1} - B) \sum_{n=0}^{\infty} B^n = (\mathbb{1} + B + B^2 + \dots) - (B + B^2 + \dots) = \mathbb{1}$$

□<sub>2.5.2</sub>

### 2.5.3 Theorem

The set of all continuously invertible mappings is open in  $L(E)$ .

**Proof**

Assume that  $A \in L(E)$  is continuously invertible, i.e.  $A^{-1}$  exists and  $A^{-1} \in L(E)$ . Set:

$$\varepsilon = \frac{1}{2\|A^{-1}\|}$$

Let us show, that every element of  $B_\varepsilon(A) \subseteq L(E)$  is continuously invertible:

Let  $C \in B_\varepsilon(A)$ , i.e.  $\|A - C\| < \varepsilon$ .

$$C = A - (A - C) = A(\mathbb{1} - \underbrace{A^{-1}(A - C)}_{=:B})$$

Then holds:

$$\|B\| \leq \|A^{-1}\| \cdot \|A - C\| < \|A^{-1}\| \cdot \frac{1}{2\|A^{-1}\|} = \frac{1}{2} < 1$$

Hence  $\mathbb{1} - B$  is continuously invertible by the Neumann series and therefore

$$C^{-1} = (\mathbb{1} - B)^{-1} \cdot A^{-1}$$

is continuous.

□<sub>2.5.3</sub>

## 3 Hilbert spaces

### Definition (scalar product)

Let  $H$  be a real ( $\mathbb{K} := \mathbb{R}$ ) or complex ( $\mathbb{K} := \mathbb{C}$ ) vector space with *scalar product*:

$$\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{K}$$

- i) Positive definiteness:  $\langle u, u \rangle \geq 0$  and  $\langle u, u \rangle = 0 \Rightarrow u = 0$ .
- ii) Linear in the second and anti-linear in the first argument:

$$\langle \lambda u, v \rangle = \bar{\lambda} \langle u, v \rangle$$

- iii) Symmetry:  $\overline{\langle u, v \rangle} = \langle u, v \rangle$

Define the corresponding norm:

$$\|u\| := \sqrt{\langle u, u \rangle}$$

### 3.0.1 Definition (Hilbert space)

A complete scalar product space is called *Hilbert space*.

The Schwarz inequality holds:

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

### 3.0.2 Lemma (parallelogram equality)

The parallelogram equality (Parallelogramm-Gleichung) is:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

### Proof

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ \|u - v\|^2 &= \langle u - v, u - v \rangle = \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\ \Rightarrow \quad \|u + v\|^2 + \|u - v\|^2 &= 2(\|u\|^2 + \|v\|^2) \end{aligned}$$

□<sub>3.0.2</sub>

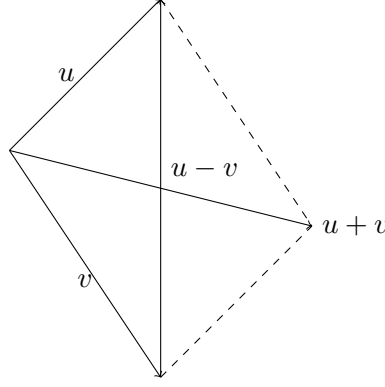


Figure 3.1: parallelogram

### 3.0.3 Definition (orthogonal, orthonormal)

- i) Vectors  $u, v \in H$  are called *orthogonal*, symbolically  $u \perp v$ , if  $\langle u, v \rangle = 0$ .
- ii) Subspaces  $M_1, M_2 \subseteq H$  are orthogonal, symbolically  $M_1 \perp M_2$ , if  $\langle u, v \rangle = 0$  for all  $u \in M_1$  and  $v \in M_2$ .
- iii) A family  $(u_i)_{i \in I}$  of vectors  $u_i \in H$  is called *orthonormal* if:

$$\langle u_i, u_j \rangle = \delta_{ij}$$

### 3.0.4 Theorem (Bessel's inequality)

Let  $(u_i)_{1 \leq i \leq N}$  be an orthonormal family. Then for all  $u \in H$  holds:

$$\begin{aligned} \|u\|^2 &= \sum_{i=1}^N \langle u_i, u \rangle^2 + \left\| u - \sum_{i=1}^N u_i \langle u_i, u \rangle \right\|^2 \\ \|u\|^2 &\geq \sum_{i=1}^N \langle u_i, u \rangle^2 \end{aligned}$$

#### Proof

$$\begin{aligned} \left\| u - \sum_{i=1}^N u_i \langle u_i, u \rangle \right\|^2 &= \left\langle u - \sum_{i=1}^N u_i \langle u_i, u \rangle, u - \sum_{j=1}^N u_j \langle u_j, u \rangle \right\rangle = \\ &= \langle u, u \rangle - \sum_{j=1}^N \langle u, u_j \rangle \langle u_j, u \rangle - \sum_{i=1}^N \overline{\langle u_i, u \rangle} \langle u_i, u \rangle + \sum_{i,j=1}^N \overline{\langle u_i, u \rangle} \langle u_j, u \rangle \underbrace{\langle u_i, u_j \rangle}_{=\delta_{ij}} = \\ &= \|u\|^2 - 2 \sum_{i=1}^N |\langle u_i, u \rangle|^2 + \sum_{i=1}^N |\langle u_i, u \rangle|^2 = \\ &= \|u\|^2 - \sum_{i=1}^N |\langle u_i, u \rangle|^2 \end{aligned}$$

□<sub>3.0.4</sub>

**Definition** (Hilbert space isomorphism)

Let  $(H_1, \langle \cdot, \cdot \rangle_1)$  and  $(H_2, \langle \cdot, \cdot \rangle_2)$  be Hilbert spaces.

A *Hilbert space isomorphism* is a mapping  $U : H_1 \rightarrow H_2$  which is linear, bijective and isometric (isometrisch), i.e. for all  $u, v \in H_1$ :

$$\langle u, v \rangle_1 = \langle Uu, Uv \rangle_2$$

**Definition** (Direct sum)

Let  $(H_1, \langle \cdot, \cdot \rangle_1)$  and  $(H_2, \langle \cdot, \cdot \rangle_2)$  be Hilbert spaces.

Define:

$$H := \{(u, v) \mid u \in H_1, v \in H_2\}$$

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$

$$\lambda(u, v) := (\lambda u, \lambda v)$$

$$\langle (u_1, v_1), (u_2, v_2) \rangle := \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$

This makes  $H =: H_1 \oplus H_2$  a Hilbert space, called *direct sum* of  $H_1$  and  $H_2$ , which is sometimes called orthogonal due to:

$$\langle (u, 0), (0, v) \rangle = 0$$

**3.0.5 Example**

$$\ell_2 = \left\{ (a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{K}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

Define a scalar product:

$$\langle (a_n), (b_n) \rangle := \sum_{n=1}^{\infty} \bar{a}_n \cdot b_n$$

$$\langle (a_n), (a_n) \rangle = \sum_{n=1}^{\infty} |a_n|^2 = \|a_n\|_2^2$$

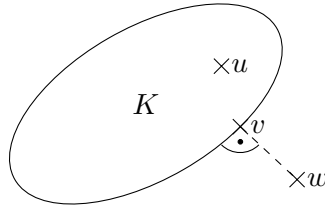
$(\ell^2, \|\cdot\|_2)$  is a Banach space. Thus  $(\ell^2, \langle \cdot, \cdot \rangle)$  is a Hilbert space.

**3.1 Projection on closed convex subsets**

Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space and  $K \subseteq H$  a closed convex subset.

$$u, v \in K \qquad w \in H \setminus K$$



Figure 3.2:  $\|v - w\| = \inf_{u \in K} \|u - w\|$ 

We want to find a vector  $v$  such that  $\|v - w\| = \inf_{u \in K} \|u - w\|$ .

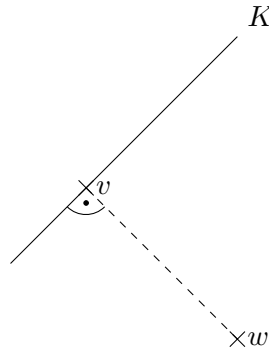
If  $K$  were compact, then choose minimizing sequence (Minimalfolge), i.e.:

$$\|u_i - w\| \rightarrow \inf_{u \in K} \|u - w\|$$

Choose a convergent subsequence  $u_{i_l} \rightarrow v$ . Then by continuity:

$$\|v - w\| = \lim_{i \rightarrow \infty} \|u_i - w\| = \inf_{u \in K} \|u - w\|$$

The main application are closed subspaces  $K \subseteq H$ .

Figure 3.3:  $v - w \perp K$ 

In this case  $v - w$  will be called orthogonal to  $K$  motivating the name *orthogonal projection*.

### 3.1.1 Theorem (Hilbert)

There is a unique  $v \in K$  with:

$$\|v - w\| = \inf_{u \in K} \|u - w\|$$

#### Proof

Consider a minimizing sequence  $u_i$ :

$$\|u_i - w\| \rightarrow \inf_{u \in K} \|u - w\| =: d$$

We show that  $(u_i)$  is a Cauchy sequence:

$$\begin{aligned}
 \|u_i - u_j\|^2 &= \|(u_i - w) + (w - u_j)\|^2 = \\
 &\stackrel{3.0.2}{=} 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \|(u_i - w) - (w - u_j)\|^2 = \\
 &= 2\|u_i - w\|^2 + 2\|w - u_j\|^2 - \left\| -2\left(w - \frac{u_i + u_j}{2}\right) \right\|^2 = \\
 &= 2\left( \underbrace{\|u_i - w\|^2}_{\rightarrow d^2} + \underbrace{\|w - u_j\|^2}_{\rightarrow d^2} - 2\left\| \frac{u_i + u_j}{2} - w \right\|^2 \right)
 \end{aligned}$$

$$\begin{aligned}
 \|u_i - w\| &\xrightarrow{i \rightarrow \infty} d = \inf_{u \in K} \|u - w\| \\
 \|u_j - w\| &\xrightarrow{j \rightarrow \infty} d = \inf_{u \in K} \|u - w\|
 \end{aligned}$$

Since  $K$  is convex and  $u_i, u_j \in K$ , we know:

$$\begin{aligned}
 \frac{u_i + u_j}{2} &\in K \\
 \Rightarrow \left\| \frac{u_i + u_j}{2} - w \right\| &\geq d
 \end{aligned}$$

Thus:

$$\|u_i - u_j\|^2 \leq 2\left(\|u_i - w\|^2 + \|w - u_j\|^2 - 2d^2\right) \xrightarrow{i,j \rightarrow \infty} 2(d^2 + d^2 - 2d^2) = 0$$

So there exists a  $N \in \mathbb{N}$  such that  $\|u_i - u_j\| < \varepsilon$  for all  $i, j > N$ . Therefore  $(u_i)$  is a Cauchy sequence. Since  $H$  is complete, we know that  $u_i \rightarrow u$  converges.

By continuity follows:

$$\|u - w\| = \lim_{i \rightarrow \infty} \|u_i - w\| = d$$

Uniqueness follows from the fact, that *every* minimizing sequence converges:

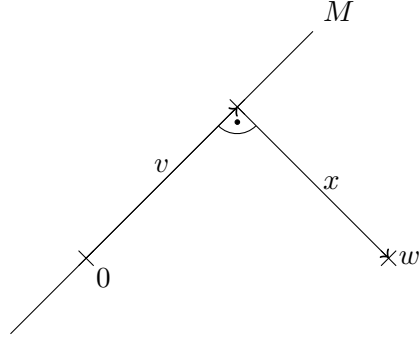
Let  $u, \tilde{u}$  be both minimizers, then the sequence  $(u, \tilde{u}, u, \tilde{u}, \dots)$  is a minimizing sequence. Since it converges,  $u = \tilde{u}$ .  $\square_{3.1.1}$

### 3.1.2 Corollary

Let  $M \subseteq H$  be a closed subspace of  $H$ . Then a  $w \in H$  can be decomposed uniquely in the form

$$w = v + x$$

with  $v \in M$  and  $x \in M^\perp$ . We write  $H = M \oplus M^\perp$ .

Figure 3.4:  $w = v + x$ **Proof**

Let  $v \in M$  be as in Theorem 3.1.1.

$$\|v - w\| = \inf_{u \in M} \|u - w\|$$

Define  $x := w - v$ .

- $H$  real: For  $u \in M$  define  $\tilde{u}(\tau) = v + \tau u$  with  $\tau \in \mathbb{R}$ .

$$\begin{aligned} \|\tilde{u} - w\|^2 &= \|x\|^2 + 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 \geq \|x\|^2 \\ 0 &\leq 2\tau \langle u, x \rangle + \tau^2 \|u\|^2 =: f(\tau) \end{aligned}$$

$f(\tau)$  has a minimum at  $\tau = 0$  and so  $f'(0) = 0$ .

$$\begin{aligned} f'(0) &= 2 \langle u, x \rangle \\ \Rightarrow 2 \langle u, x \rangle &= 0 \quad \forall_{u \in M} \end{aligned}$$

So  $x \in M^\perp$ .

- $H$  complex: Define  $\tilde{u}(\tau) = v + \tau u$ ,  $\tau = re^{i\varphi} \in \mathbb{K}$  with  $r \geq 0$ .

$$\|\tilde{u} - w\|^2 = \|x\|^2 + 2\operatorname{Re}(re^{-i\varphi} \langle u, x \rangle) + r^2 \|u\|^2 =: f(r, \varphi)$$

This has a minimum at  $r = 0$ .

$$\begin{aligned} \Rightarrow 0 &= \partial_r f(0, \varphi) = 2\operatorname{Re}(e^{-i\varphi} \langle u, x \rangle) \\ \varphi \text{ arbitrary} \Rightarrow \langle u, x \rangle &= 0 \end{aligned}$$

So  $x \in M^\perp$ .

*Uniqueness:* Assume that  $w = v_1 + x_1 = v_2 + x_2$  where  $v_1, v_2 \in M$ ,  $x_1, x_2 \in M^\perp$ .

$$\underbrace{v_1 - v_2}_{\in M} = \underbrace{x_2 - x_1}_{\in M^\perp} \in M \cap M^\perp = \{0\}$$

Because from  $u \in M \cap M^\perp$  follows  $\langle u, u \rangle = 0$  and so  $u = 0$ .

□<sub>3.1.2</sub>

For a Banach space  $E$  we have  $E, E^*, E^{**}$  and a natural injection  $\iota : E \hookrightarrow E^{**}$ .

For a Hilbert space  $H$ , suppose  $u \in H$  and define:

$$\begin{aligned}\varphi &: H \rightarrow \mathbb{K} \\ \varphi(v) &:= \langle u, v \rangle\end{aligned}$$

$\varphi$  is continuous, because:

$$|\varphi(v)| = |\langle u, v \rangle| \leq \|u\| \cdot \|v\| \leq C \|v\|$$

Now

$$\begin{aligned}\iota &: H \hookrightarrow H^* \\ \iota(u) &= \varphi\end{aligned}$$

is a linear mapping, which is injective.

### 3.1.3 Theorem (Fréchet-Riesz)

For any  $\varphi \in H^*$  there is a unique  $v \in H$  such that for all  $x \in H$ :

$$\varphi(x) = \langle v, x \rangle$$

In other words:  $\iota : H \rightarrow H^*$  is a Banach space isomorphism.

#### Proof

Let  $\varphi \in H^*$ , without loss of generality  $\varphi \neq 0$ .

$$M := \ker \varphi \subseteq H$$

is a subspace. It is closed by continuity: For  $u_n \in \ker \varphi$  with  $u_n \rightarrow u$  holds:

$$\varphi(u) \stackrel{\text{continuity}}{=} \lim_{n \rightarrow \infty} \varphi(u_n) = 0$$

So  $u \in \ker \varphi$ .

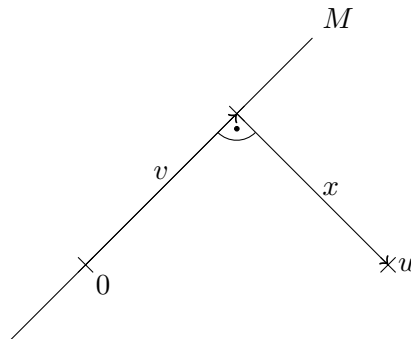


Figure 3.5:  $u = v + x$

- $M^\perp$  is a one-dimensional subspace of  $H$ :

$$M^\perp \neq \{0\}:$$

Since  $\varphi \neq 0$  there exists a  $u \in H$  with  $\varphi(u) \neq 0$ , thus  $u \notin M$ .

Now decompose  $u = v + x$ ,  $v \in M$ ,  $x \in M^\perp \setminus \{0\}$ .

$M^\perp$  is one-dimensional: Take  $u, v \in M^\perp$ ,  $u, v \neq 0$ , then  $\varphi(u) \neq 0$  and  $\varphi(v) \neq 0$ .

$$\varphi(\varphi(v)u - \varphi(u)v) = 0$$

So  $\varphi(v)u - \varphi(u)v \in M \cap M^\perp = \{0\}$ . Thus  $\varphi(v)u - \varphi(u)v = 0$ , implying that  $u$  and  $v$  are linearly dependent.

- Choose  $u \in M^\perp$  with  $\varphi(u) = 1$ , which is always possible by rescaling.

$$\begin{aligned} v &:= \frac{u}{\|u\|^2} \\ \Rightarrow \quad \varphi(v) &= \frac{1}{\|u\|^2} \underbrace{\varphi(u)}_{=1} = \frac{1}{\|u\|^2} \\ \langle v, v \rangle &= \frac{\langle u, u \rangle}{\|u\|^4} = \frac{1}{\|u\|^2} = \varphi(v) \end{aligned}$$

- This  $v$  has the desired properties:

For  $x \in H$  decompose:

$$x = \underbrace{m}_{\in M} + \underbrace{\alpha v}_{\in M^\perp = \langle v \rangle}$$

$$\begin{aligned} \Rightarrow \quad \varphi(x) &= \underbrace{\varphi(m)}_{=0} + \alpha \varphi(v) = \alpha \langle v, v \rangle = \\ &= \langle v, \alpha v \rangle = \langle v, m + \alpha v \rangle = \langle v, x \rangle \end{aligned}$$

□<sub>3.1.3</sub>

### 3.1.4 Theorem (Lax-Milgram)

Let  $H$  be a Hilbert space and  $B : H \times H \rightarrow \mathbb{K}$  be a mapping with the following properties:

- i)  $B(x, y)$  is linear in the second and anti-linear in the first argument.
- ii)  $|B(x, y)| \leq C \|x\| \cdot \|y\|$  (continuity)
- iii)  $B$  is symmetric ( $\overline{B(x, y)} = B(y, x)$ ) and positive definite, i.e.  $B(x, x) \geq b \|x\|^2$  with  $b \in \mathbb{R}_{>0}$ .
- iii')  $|B(x, x)| \geq b \|x\|^2$  with  $b \in \mathbb{R}_{>0}$ .

Then every  $l \in H^*$  can be represented uniquely as:

$$l(y) = B(x, y) \quad \forall_{y \in H}$$

**Proof**

First the easy case iii):

We introduce a new scalar product  $\langle \cdot, \cdot \rangle_B$  by:

$$\langle x, y \rangle_B := B(x, y)$$

Using ii) and iii) one sees that  $\|\cdot\|_B$  is equivalent to  $\|\cdot\|$ , i.e. there exists a  $C \in \mathbb{R}_{>0}$  such that:

$$\frac{1}{C} \|x\| \leq \|x\|_B \leq C \|x\|$$

According to the Fréchet-Riesz theorem, there exists a unique  $v \in H$  with

$$\varphi(x) = \langle v, x \rangle_B = B(v, x)$$

for all  $x \in H$ .

More difficult case iii'): Given  $x \in H$ ,

$$B(x, \cdot) : H \rightarrow \mathbb{K}$$

is a linear bounded functional according to i) and ii), i.e.  $B(x, \cdot) \in H^*$ .

According to the Fréchet-Riesz theorem there exists a unique  $z \in H$  such that  $B(x, y) = \langle z, y \rangle$  for all  $y \in H$ . This yields a mapping:

$$\begin{aligned} \varphi : H &\rightarrow H \\ x &\mapsto z \end{aligned}$$

$$B(x, y) = \langle \varphi(x), y \rangle$$

- $\varphi$  is linear, because both  $B$  and  $\langle \cdot, \cdot \rangle$  are anti-linear in their first arguments.
- $\varphi(H) \subseteq H$  is closed:

$$\begin{aligned} b \|x\|^2 &\stackrel{\text{iii}'}{\leq} |B(x, x)| = |\langle z, x \rangle| \leq \|z\| \cdot \|x\| \\ b \|x\| &\leq \|z\| \end{aligned} \tag{3.1}$$

Let  $z_n \in \varphi(H)$  be a sequence with  $z_n \rightarrow z \in H$ . Choose  $x_n$  such that  $\varphi(x_n) = z_n$ , i.e.  $B(x_n, y) = \langle z_n, y \rangle$  for all  $y \in H$ .

Due to the anti-linearity in the first argument follows that:

$$B(x_n - x_m, y) = \langle z_n - z_m, y \rangle$$

(3.1) yields that  $\|x_n - x_m\| \leq \|z_n - z_m\|$ .

Hence  $(x_n)$  is a Cauchy sequence and so  $x_n \rightarrow x \in H$  converges. Since  $B$  is continuous according to ii), we get:

$$\underbrace{B(x_n, y)}_{\rightarrow B(x, y)} = \underbrace{\langle z_n, y \rangle}_{\rightarrow \langle z, y \rangle}$$

This gives:

$$\begin{aligned} B(x, y) &= \langle z, y \rangle \\ \varphi(x) &= z \end{aligned}$$

Thus  $z$  is in  $\varphi(H)$ .

- $\varphi(H) = H$ : Otherwise there would be a vector  $y \in \varphi(H)^\perp \setminus \{0\}$  and thus for all  $x \in H$  holds.

$$B(x, y) = \langle \varphi(x), y \rangle = 0$$

In particular for  $x = y$  this gives:

$$\begin{aligned} 0 &= |B(y, y)| \geq b \|y\|^2 \\ \Rightarrow y &= 0 \end{aligned}$$

This is a contradiction and so  $\varphi(H) = H$ .

- $\varphi$  is injective: Suppose there are  $x, x' \in H$  with  $\varphi(x) = \varphi(x')$ . Then follows:

$$B(x - x', y) = \langle \underbrace{\varphi(x) - \varphi(x')}_{=0}, y \rangle = 0$$

Choose  $y = x - x'$  so we get:

$$B(x - x', x - x') = 0$$

Since  $B$  is positive definite, it follows  $x = x'$ .

- Let  $l \in H^*$ . According to Fréchet-Riesz there exists a unique  $z \in H$  with  $l(y) = \langle z, y \rangle$  for all  $y \in H$  and we have

$$\langle z, y \rangle = B(x, y)$$

for  $x = \varphi^{-1}(z)$ . So  $l(y) = B(x, y)$ .

□<sub>3.1.4</sub>

### 3.1.5 Corollary

Every Hilbert space is reflexive.

#### Proof

Recall  $\iota : H \hookrightarrow H^{**}$ .  $H$  is *reflexive* if and only if  $\iota$  is surjective, i.e. a Banach space isomorphism.

$$\begin{aligned} \tilde{\iota} : H &\rightarrow H^* \\ (\tilde{\iota}(u))(v) &= \langle u, v \rangle \end{aligned}$$

is bijective by Fréchet-Riesz. This holds also for  $\bar{\iota} : H^* \rightarrow H^{**}$ .

$$H \xrightarrow{\tilde{\iota}} H^* \xrightarrow{\bar{\iota}} H^{**}$$

So  $\iota = \bar{\iota} \circ \tilde{\iota}$  is bijective as composition of bijective maps.

□<sub>3.1.5</sub>

## 3.2 Orthonormal Bases in Separable Hilbert Spaces

### 3.2.1 Example

$$\ell_2 = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \right\}$$

with the scalar product

$$\langle (a_n), (b_n) \rangle := \sum_n \bar{a}_n b_n$$

is a Hilbert space.

Idea: Let  $H$  be an abstract Hilbert space. Choose an “orthonormal basis”  $(e_i)$ .

$$\begin{aligned} H \ni u &= \sum_{i=1}^{\infty} \lambda_i e_i \\ v &= \sum_{i=1}^{\infty} \nu_i e_i \end{aligned}$$

$$\langle u, v \rangle = \sum_{i,j=1}^{\infty} \langle \lambda_i e_i, \nu_j e_j \rangle = \sum_{i,j=1}^{\infty} \bar{\lambda}_i \nu_j \delta_{ij} = \sum_i \bar{\lambda}_i \nu_i$$

### 3.2.2 Definition (orthonormal system, Hilbert space basis, cardinality)

A system  $(e_i)_{i \in J}$  is an *orthonormal system*, if  $\langle e_i, e_j \rangle = \delta_{ij}$ . The algebraic span is the vector space of *finite* linear combinations:

$$\langle (e_i) \rangle = \left\{ \sum_{i=1}^N \lambda_i e_i \mid N \in \mathbb{N}, \lambda_i \in \mathbb{K} \right\}$$

This is a subspace of  $H$ . Now the subspace  $\overline{\langle (e_i) \rangle} \subseteq H$  is called *Hilbert space span* (Hilbertraumzeugnis).

An orthonormal system  $(e_i)$  is called a *orthonormal Hilbert space basis* if  $\overline{\langle (e_i) \rangle} = H$ .

Two sets  $A$  and  $B$  have the same cardinality if there exists a bijective map  $\varphi : A \rightarrow B$ .

### Theorem (Bernstein-Schröder)

$A$  and  $B$  have the same cardinality if and only if there exists an injective map from  $A$  to  $B$  and an injective map from  $B \rightarrow A$ .

(no proof)



A typical application of the Lax-Milgram theorem is for  $x \in \mathbb{R}^n$ , given real-valued functions  $V(x), f(x)$  and looking for  $u(x)$  that solves:

$$-\Delta u(x) + V(x)u(x) = f(x)$$

Question: Is there a solution which “decays at infinity”?

1. Weak formulation:

Suppose we have a solution  $u \in \mathcal{C}^2(\mathbb{R}^n)$

$$-\Delta u + Vu - f = 0$$

Let  $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  be a test function.

$$0 = \int_{\mathbb{R}^n} (-\Delta u + Vu - f) \eta d^n x \stackrel{\text{integration by parts}}{=} \underbrace{\int_{\mathbb{R}^n} (\langle \nabla u, \nabla \eta \rangle + Vu\eta) d^n x}_{=: B(u, \eta)} - \underbrace{\int_{\mathbb{R}^n} f \eta d^n x}_{=: l(\eta)}$$

So for all  $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  holds:

$$B(u, \eta) = l(\eta)$$

**Definition:**  $u$  is a *weak solution* of the equation  $-\Delta u + Vu = f$  if for all  $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  holds:

$$B(u, \eta) = l(\eta)$$

2. Choose the correct Hilbert space. The first idea is  $L^2(\mathbb{R}^n)$  with the scalar product:

$$\langle u, v \rangle = \int_{\mathbb{R}^n} uv d^n x$$

$$u_n(x) := e^{-|x|^2} \sin(nx_1)$$

Then for all  $n \in \mathbb{N}$  holds:

$$\|u_n\|_{L^2} \leq C$$

But  $B(u_n, u_n) \xrightarrow{n \rightarrow \infty} \infty$  diverges. Thus  $B$  is *not* continuous.  
Better choose instead:

$$\langle u, v \rangle = \int_{\mathbb{R}^n} (uv + \langle \nabla u, \nabla v \rangle) d^n x$$

The corresponding Hilbert space  $H^{1,2}(\mathbb{R}^n)$  is a Sobolev space.

$$L^2(\mathbb{R}^3) \supseteq H^{1,2}(\mathbb{R}^3) \ni u$$

Assume for simplicity that  $0 < \varepsilon \leq V \leq C < \infty$ , then we get:

$$B(u, u) = \int_{\mathbb{R}^n} (|\nabla u|^2 + Vu^2) d^n x \leq \int_{\mathbb{R}^n} (|\nabla u|^2 + Cu^2) d^n x \leq (1 + C) \|u\|_{H^{1,2}}^2$$

$$|B(u, u)| \geq \int_{\mathbb{R}^n} (|\nabla u|^2 + \varepsilon u^2) d^n x \geq \min\{1, \varepsilon\} \|u\|_{H^{1,2}}^2$$

Thus the Lax-Milgram theorem applies and yields a unique weak solution and then a regularity theorem says that  $u$  is smooth.

Consider a matrix equation

$$Au = f$$

with  $A \in \text{Symm}(\mathbb{R}^n)$  and  $f \in \mathbb{R}^n$ .

For a general existence and uniqueness result one needs that  $A$  is invertible or equivalently:

$$\forall_{u \in \mathbb{R}^n \setminus \{0\}} : Au \neq 0$$

This follows from the condition:

$$\forall_{u \in \mathbb{R}^n \setminus \{0\}} : \underbrace{\langle u, Au \rangle}_{=B(u,u)} \neq 0$$

In finite dimension this is equivalent to:

$$\forall_{u \in \mathbb{R}^n} : |B(u,u)| > b \|u\|^2$$

$(e_i)_{i \in I}$  is an orthonormal Hilbert space basis of  $H$  if

$$\langle e_i, e_j \rangle = \delta_{ij}$$

and:

$$\overline{\langle e_i \rangle} = H$$

### 3.2.3 Theorem

Let  $(e_i)_{i \in \mathbb{N}}$  be an orthonormal system. Then the mapping

$$\begin{aligned} \ell_2 &\rightarrow \overline{\langle e_i \rangle}^{\text{closed}} \subseteq H \\ (\lambda_i) &\mapsto \sum_{i \in \mathbb{N}} \lambda_i e_i \end{aligned}$$

is a Hilbert space isomorphism.

#### Proof

The mapping is well-defined and isometric:

For  $(\lambda_i) \in \ell_2$ , i.e.  $\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$  we construct:

$$u_N := \sum_{i=1}^N \lambda_i e_i \in H$$

Without loss of generality take  $M < N$ , then follows:

$$\|u_N - u_M\|^2 = \left\| \sum_{i=M}^N \lambda_i e_i \right\|^2 = \left\langle \sum_{i=M}^N \lambda_i e_i, \sum_{i=M}^N \lambda_i e_i \right\rangle = \sum_{i,j=M}^N \bar{\lambda}_i \lambda_j \underbrace{\langle e_i, e_j \rangle}_{=\delta_{ij}} = \sum_{i=M}^N |\lambda_i|^2$$

Thus  $u_N$  is a Cauchy sequence and converges since  $\overline{\langle e_i \rangle}$  is complete as a closed subset of a complete space.

$$u := \lim_{N \rightarrow \infty} u_N = \sum_{i=1}^N \lambda_i e_i$$

$$\|u\|^2 = \lim_{N \rightarrow \infty} \|u_N\|^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^N |\lambda_i|^2 = \|(\lambda_i)\|_{\ell_2}$$

The mapping is also surjective:

Let  $u \in \overline{\langle e_i \rangle}$  and  $\varepsilon > 0$ . So there exists a  $v = \sum_{i=1}^N \lambda_i e_i \in \langle e_i \rangle$  with  $\|v - u\| < \varepsilon$ .

In other words there exists a finite  $J \subseteq \mathbb{N}$  such that  $d(\langle (e_i)_{i \in J} \rangle, u) < \varepsilon$ . The vector which minimizes this distance is the orthogonal projection of  $u$  on  $\langle (e_i)_{i \in J} \rangle$  since this is a finite-dimensional subspace, which is automatically closed.

$$u_J = \sum_{i \in J} e_i \langle e_i, u \rangle$$

Choose an increasing sequence  $J_1 \subsetneq J_2 \subsetneq \dots$  of finite sets such that:

$$\|u_{J_k} - u\| \rightarrow 0 \quad \Rightarrow \quad u_{J_k} \rightarrow u$$

Thus  $u_{J_k}$  is bounded by a  $C \in \mathbb{R}_{>0}$ .

$$u_{J_k} = \sum_{i \in J_k} e_i \underbrace{\langle e_i, u \rangle}_{=\lambda_i}$$

$$C > \|u_{J_k}\| = \sum_{i \in J_k} |\lambda_i|^2$$

This gives:

$$\sum_{i \in \mathbb{N}} |\lambda_i|^2 < \infty$$

And so we get:

$$u = \sum_{i \in \mathbb{N}} \lambda_i e_i$$

□<sub>3.2.3</sub>

### 3.2.4 Theorem (Existence of Hilbert space basis)

In every Hilbert space  $H$  exists an orthonormal Hilbert space basis.

#### Proof

Consider  $(u_i)_{i \in I}$  with  $I = H$  and  $u_h = h$  for all  $h \in H$ .  $(u_i)_{i \in I}$  is obviously a generating system of  $H$ . On the set

$$X := \left\{ \tilde{I} \subseteq I \mid (u_i)_{i \in \tilde{I}} \text{ is an orthonormal system} \right\}$$

defines „ $\subseteq$ “ a partial ordering.

Let  $U \subseteq X$  be a totally ordered subset and define:

$$I_U := \bigcup_{\tilde{I} \in U} \tilde{I} \subseteq I$$

$I_U$  is an upper bound of  $U$  in  $X$  if  $I_U \in X$ . Assume  $(u_i)_{i \in I_U}$  would not be orthonormal. Then there would exist  $j, k \in I_U$  with  $\langle u_j, u_k \rangle \neq \delta_{jk}$ .

For  $j = k$  would hold  $\langle u_j, u_j \rangle \neq 1$ , but  $j$  lies in  $\tilde{I} \in U \subseteq X$  and therefor has to hold  $\langle u_j, u_j \rangle = 1$ . For  $j \neq k$  we would get  $\langle u_j, u_k \rangle \neq 0$ . But  $j$  lies in  $\tilde{I}_j \in U$  and  $k$  in  $\tilde{I}_k \subseteq U$  and  $U$  is totally ordered, i.e. either holds  $\tilde{I}_j \subseteq \tilde{I}_k$  or  $\tilde{I}_k \subseteq \tilde{I}_j$ .

Without loss of generality assume  $\tilde{I}_j \subseteq \tilde{I}_k$  (otherwise exchange  $j$  and  $k$ ). Then  $j, k \in \tilde{I}_k \in U \subseteq X$  and hence  $(u_i)_{i \in \tilde{I}_j}$  is an orthonormal system in contradiction to  $\langle u_j, u_k \rangle \neq 0$ . Therefore holds  $I_U \in X$  and thus  $I_U$  is an upper bound of  $U$ .

Using Zorn's lemma we get a maximal element  $I_{\max}$  in  $X$ . Because  $(u_i)_{i \in I_{\max}}$  is an orthonormal system and thus especially linearly independent, it suffices to show that this is an generating system of  $H$ .

Assume there exists a  $i_0 \in I$  with  $u_{i_0} \notin K := \overline{\langle (u_i)_{i \in I_{\max}} \rangle_{\text{alg.}}}$ . Since  $K \subseteq H$  is closed and convex, there is an unique projection  $v$  of  $u_{i_0}$  on  $K$  and thus  $h := u_{i_0} - v \in K^\perp$ . It holds  $h = u_h$  with  $h \in H = I$ .

Because  $I_{\max}$  is maximal, holds then  $I_{\max} \cup \{h\} \notin X$  and hence there is a  $j \in I_{\max}$  with  $\langle h, u_j \rangle \neq 0$ , because  $h = j$  cannot hold due to  $h \notin I_{\max}$ . This is a contradiction to  $h \in K^\perp$  and thus holds  $K = H$ .

Therefore  $(u_i)_{i \in I_{\max}}$  is an orthonormal Hilbert space basis of  $H$ . □<sub>3.2.4</sub>

### 3.2.5 Theorem

Let  $H$  be a Hilbert space.

- i) For any  $v \in H$  and for any orthonormal system  $\{e_j | j \in J\}$ , the set of elements  $j \in J$  for which  $\langle e_j, v \rangle = 0$  is finite or countable.
- ii) Any two Hilbert space bases of  $H$  have the same cardinality (Mächtigkeit).

#### Proof

- i) Consider  $v \in J$ . First we show that every  $n \in \mathbb{N}$ , the set  $J_n := \{j \in J | \langle e_j, v \rangle > \frac{1}{n}\}$  is finite. Indeed, by Bessel's inequality, for every finite number of elements  $e_{j_1}, \dots, e_{j_N}$  of the given orthonormal system, we have:

$$\sum_{k=1}^N |\langle e_{j_k}, v \rangle|^2 \leq \|v\|^2$$

Now suppose that for some  $n \in \mathbb{N}$ , the set  $J_n$  were not finite. Then for any  $N \in \mathbb{N}$  we could find elements  $e_{j_1}, \dots, e_{j_N}$  such that  $\langle e_{j_k}, v \rangle > \frac{1}{n}$  for all  $k \in \{1, \dots, N\}$ . Hence, for these elements holds:

$$\sum_{k=1}^N |\langle e_{j_k}, v \rangle|^2 > N \cdot \frac{1}{n^2}$$

Clearly these becomes larger than  $\|v\|$  if we make  $N$  sufficiently large. Hence all the sets  $J_n$  must be finite. But then, we see that the set

$$\{j \in J \mid \langle e_j, v \rangle \neq 0\} = \bigcup_{n \in \mathbb{N}} J_n$$

is a countable union of finite sets, and as such can be at most countable.  $\square_i)$

- ii) If  $H$  has is finite-dimensional, every Hilbert basis is a Hamel basis of  $H$  and thus the claim follows from linear algebra.

If  $H$  is infinite-dimensional, let  $(e_i)_{i \in I}$  and  $(b_j)_{j \in J}$  be two Hilbert bases of  $H$ . ( $I$  and  $J$  have infinitely many elements.)

For  $x \in H = \overline{\langle (e_i)_{i \in I} \rangle} = \overline{\langle (b_j)_{j \in J} \rangle}$  define:

$$B_x := \{j \in J \mid \langle x, b_j \rangle \neq 0\}$$

By i), the set  $B_x$  is at most countable for any  $x \in H$ . Next, let  $j \in J$  be given. Since  $\overline{\langle (e_i)_{i \in I} \rangle} = H$ , we must have  $\langle b_j, e_i \rangle \neq 0$  for some  $i \in I$ . Otherwise,  $b_j \in \overline{\langle (e_i)_{i \in I} \rangle}^\perp = \{0\}$ , which is not possible since  $b_j \neq 0$ . Therefore, we have  $j \in B_{e_i}$  for some  $i \in I$ , and since  $j \in J$  was arbitrary, it follows that  $J \subseteq \bigcup_{i \in I} B_{e_i} \subseteq I \times \mathbb{N}$ . Here the second inclusion uses that all the sets  $B_{e_j}$  are at most countable. It follows:

$$|J| \leq |I| \cdot |\mathbb{N}| = |I|$$

If we exchange the roles of  $I$  and  $J$  above, we also obtain  $|I| \leq |J|$ . By the Schröder-Bernstein theorem, we can combine both estimates to obtain that  $|I| = |J|$ .  $\square_{ii)}$

$\square_{3.2.5}$

### 3.2.6 Theorem

If  $H$  is separable, then there exists a countable orthonormal Hilbert space basis  $(e_i)_{i \in \mathbb{N}}$ . Thus  $H$  is Hilbert space isomorphic to  $\ell_2$ .

#### Proof

Since  $H$  is separable, there is a countable dense subset  $(x_i)_{i \in \mathbb{N}}$ .

1. Arrange that the  $x_i$  are linearly independent:  
Start with  $n = 1$  and  $k = 1$  set:

$$y_1 = x_1$$

If the  $y_1, \dots, y_{n-1}, x_k$  are linearly independent, we set  $y_n = x_k$  and increase  $n$  and  $k$  by one.

If the  $y_1, \dots, y_{n-1}, x_k$  are linearly dependent, we only increase  $k$  by one.

Then the  $y_i$  are linearly independent and  $\langle (y_i) \rangle = \langle (x_i) \rangle$ .

2. Gram-Schmidt procedure for orthonormalization:

$$\begin{aligned} e_1 &:= y_1 \\ e_2 &:= \frac{y_2 - e_1 \langle y_2, e_1 \rangle}{\|y_2 - e_1 \langle y_2, e_1 \rangle\|} \\ e_n &:= \frac{y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n}{\|y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n\|} \end{aligned}$$

Since the  $y_i$  are linearly independent,  $y_n - \text{Pr}_{\langle e_1, \dots, e_{n-1} \rangle} y_n$  is never zero.

Then by construction the  $e_i$  are orthonormal and  $\langle e_i \rangle = \langle x_i \rangle \subseteq H$  is dense and so  $(e_i)_{i \in \mathbb{N}}$  is a Hilbert space basis.  $\square_{3.2.6}$

### 3.3 Weak Compactness of the Closed Unit Ball

For a Banach space  $E$  *weak convergence* for  $(u_i)_{i \in \mathbb{N}}$  with  $u_i \in E$  means:

$$u_n \rightharpoonup u \quad \Leftrightarrow \quad \forall_{\varphi \in E^*} : \varphi(u_n) \rightarrow \varphi(u)$$

In Hilbert spaces, we can identify  $H^*$  with  $H$  via the Fréchet-Riesz theorem.

#### 3.3.1 Definition (weak (sequential) compactness)

$x_n \rightharpoonup x$  *converges weakly* if  $\langle y, x_n \rangle \rightarrow \langle y, x \rangle$  converges for all  $y \in H$ .

Weak compactness is for us by definition the same as *weak sequential compactness* (schwache Folgenkompaktheit):

$K \subseteq H$  is *weakly compact* if every sequence  $(x_n)$  with  $x_n \in K$  has a weakly convergent subsequence.

#### 3.3.2 Proposition

Let  $H$  be *separable* and infinite-dimensional and let  $(e_i)_{i \in \mathbb{N}}$  be an orthonormal Hilbert space basis.

Then  $e_n \rightharpoonup 0$  converges weakly.

#### Proof

Take  $y \in H$  and expand it in the basis:

$$\begin{aligned} y &= \sum_{i=1}^{\infty} y_i e_i \\ y_i &= \langle e_i, y \rangle \end{aligned}$$

We know  $(y_i)_{i \in \mathbb{N}} \in \ell_2$  and in particular  $y_i \xrightarrow{i \rightarrow \infty} 0$ , since the elements of an absolutely convergent series converge to zero. Therefore holds:

$$\langle y, e_n \rangle = \overline{y_n} \xrightarrow{n \rightarrow \infty} 0$$

Thus  $e_n \rightharpoonup 0$  converges weakly.  $\square_{3.3.2}$

### 3.3.3 Theorem (Weak Compactness of the Closed Unit Ball)

If  $H$  is *separable*, then the closed unit ball  $\overline{B_1(0)} = \{u \mid \|u\| \leq 1\}$  is weakly compact.

#### Proof

Let  $(u_l)$  be a sequence with  $u_l \in \overline{B_1(0)}$ . Choose an orthonormal Hilbert space basis  $(e_n)_{n \in \mathbb{N}}$ .

$$u_l = \sum_{n=1}^{\infty} u_{ln} e_n \quad u_{ln} = \langle e_n, u_l \rangle \quad (u_{l,n})_{n \in \mathbb{N}} \in \ell_2$$

$$|u_{ln}| = |\langle e_n, u_l \rangle| \leq \underbrace{\|e_n\|}_{=1} \cdot \|u_l\| \leq 1$$

For  $n = 1$ :  $(u_{l,1})_{l \in \mathbb{N}}$  is a bounded sequence of complex or real numbers. Therefore there exists a convergent subsequence of  $u_l$ , which we denote by  $u_l^{(1)} \in H$ . Then follows:

$$u_{l,1}^{(1)} = \langle e_1, u_l^{(1)} \rangle \xrightarrow{l \rightarrow \infty} v_1$$

For  $n = 2$ : Next we choose a subsequence  $u_l^{(2)}$  of  $u_l^{(1)}$  such that:

$$\langle e_2, u_l^{(2)} \rangle \xrightarrow{l \rightarrow \infty} v_2$$

Proceed inductively to obtain:

$$\langle e_n, u_l^{(n)} \rangle \rightarrow v_n$$

Then  $w_l = u_l^{(l)} \in \overline{B_1(0)}$  for a sequence  $(w_l)$  in  $\overline{B_1(0)}$ .

**Claim:**  $w_l \xrightarrow{l \rightarrow \infty} v := \sum_n v_n e_n$

**Proof:** We proceed as follows:

$$v_n = \lim_{l \rightarrow \infty} \langle e_n, u_l^{(n)} \rangle = \lim_{l \rightarrow \infty} \langle e_n, u_l^{(l)} \rangle = \lim_{l \rightarrow \infty} \langle e_n, w_l \rangle$$

This is because  $u_l^{(l)} = u_{l'}^{(n)}$  for  $l' \geq l$ .

1.  $(v_n) \in \ell_2$ :

$$\sum_{n=1}^N |v_n|^2 = \sum_{n=1}^N \left| \lim_{l \rightarrow \infty} \langle e_n, w_l \rangle \right|^2 \stackrel{\text{finite sum}}{=} \lim_{l \rightarrow \infty} \sum_{n=1}^N |\langle e_n, w_l \rangle|^2$$

$\underbrace{\hspace{10em}}_{\substack{\text{Bessel's} \\ \leq \\ \text{inequality}}} \|w_l\|^2 \leq 1$

So we get for all  $N \in \mathbb{N}$ :

$$\sum_{n=1}^N |v_n|^2 \leq 1$$

And thus  $(v_n) \in \ell_2$  and  $v := \sum_{n=1}^{\infty} v_n e_n$  is well-defined and has  $\|v\| \leq 1$ .

2.  $w_l \rightharpoonup v$ , i.e.  $\langle y, w_l - v \rangle \xrightarrow{l \rightarrow \infty} 0$  for all  $y \in H$ :

$$y = \sum_{n=1}^{\infty} y_n e_n$$

$$y_n = \langle e_n, y \rangle$$

$$y_{<} := \sum_{n \leq N} y_n e_n$$

$$y_{>} := \sum_{n > N} y_n e_n$$

$$\|y\|^2 = \|y_{<}\|^2 + \|y_{>}\|^2$$

$$\langle y, w_l - v \rangle = \sum_{n=1}^{\infty} y_n \langle e_n, w_l - v \rangle$$

Choose  $N \in \mathbb{N}$  so large that

$$\|y_{>}\| = \left( \sum_{n > N} |y_n|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{4}$$

to get:

$$\begin{aligned} |\langle y, w_l - v \rangle| &\leq |\langle y_{<}, w_l - v \rangle| + |\langle y_{>}, w_l - v \rangle| \leq \\ &\leq \sum_{n=1}^N |y_n| |\langle e_n, w_l - v \rangle| + \underbrace{\|y_{>}\|}_{< \frac{\varepsilon}{4}} \cdot \underbrace{\|w_l - v\|}_{\leq 2} < \sum_{n=1}^N |y_n| |\langle e_n, w_l - v \rangle| + \frac{\varepsilon}{2} \end{aligned}$$

We know  $|\langle e_n, w_l - v \rangle| \xrightarrow{l \rightarrow \infty} 0$  for each  $n$ . So we can choose  $|\langle e_n, w_l - v \rangle| \leq \frac{\varepsilon}{2}$  for  $n \leq N$  and for all  $l > L(\varepsilon)$  for a sufficiently large  $L(\varepsilon)$  and therefore:

$$|\langle y, w_l - v \rangle| \leq \varepsilon \quad \forall_{l > L(\varepsilon)}$$

Therefore  $\langle y, w_l \rangle \rightarrow \langle y, v \rangle$  converges, which means  $w_l \rightharpoonup v$ .

□<sub>Claim</sub>

□<sub>3.3.3</sub>

The corresponding statement in Banach spaces is the *Banach-Alaoglu theorem*:

Banach proved it in 1932 for separable Banach spaces using diagonal sequences.

Alaoglu proved it in 1938 for any Banach space. The proof is based on Tychonov's theorem.

We have  $E$ ,  $E^*$ ,  $E^{**}$  and an injection  $\iota : E \rightarrow E^{**}$ .

**Theorem** (Banach-Alaoglu)

The closed unit ball in  $E^*$  is *weak\*-sequentially compact*.

I.e. in simple terms:



If  $\varphi_n \in \overline{B_1(0)} \subseteq E^*$ , then there exists a subsequence  $\varphi_{n_l}$  such that  $\varphi_{n_l}(u)$  converges for all  $u \in E$ .

Application: Consider

$$E = C^0(\mathbb{R}^n)$$

with the sup-norm:

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|$$

$$E^* = \{\text{regular Borel measures}\}$$

Suppose  $\mu_n$  is a sequence of measures with  $\|\mu_n\| \leq C$  for all  $n \in \mathbb{N}$ . Then there exists a measure  $\mu$  such that  $\mu_{n_l} \rightarrow \mu$  converges as a measure.

## 4 Operators on Hilbert spaces

Let  $H$  be a Hilbert space.

$$L(H) := L(H, H)$$

is the Banach space of bounded linear operators. (An linear map on an infinite dimensional space is usually called *linear operator*.) For  $A \in L(H)$  define the norm:

$$\|A\| := \sup_{\|u\|=1} \|Au\|$$

### 4.0.1 Example

$H = L^2(\mathbb{R}, dx)$  with the Lebesgue measure  $dx$ .

$$\langle f, g \rangle = \int_{\mathbb{R}} \bar{f} g dx$$

$$A := \frac{d}{dx}$$

We would like to introduce this as an operator on  $H$ .

The inequality  $\|Au\| \leq C \|u\|$  is violated even for  $u \in C_0^\infty(\mathbb{R})$  for any constant  $C \in \mathbb{R}$ . Namely consider

$$u_n(x) = \eta(x) \sin(nx)$$

with  $\eta \in C_0^\infty(\mathbb{R})$  and  $\eta|_{[-1,1]} = 1$ . Then  $\|u_n\| < \infty$  and  $\|Au_n\| \xrightarrow{n \rightarrow \infty} \infty$ .

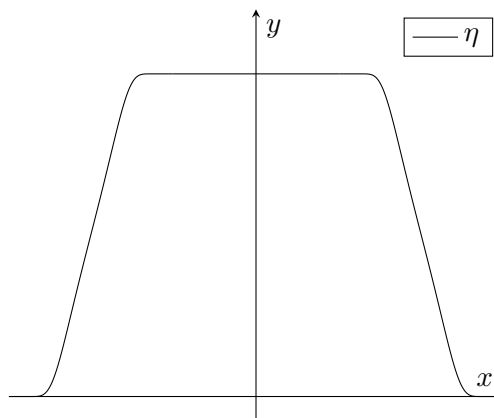


Figure 4.1:  $\eta \in C_0^\infty(\mathbb{R})$  with  $\eta|_{[-1,1]} = 1$

Moreover  $\frac{d}{dx}f$  makes no sense for every vector  $f$  in  $H$ , because  $f$  does not need to be differentiable.

Way out: Define  $A$  only on a suitable subspace  $\mathcal{D}(A)$  of  $H$ , called *domain* of definition. For example: Choose  $\mathcal{D}(A) = C_0^\infty(\mathbb{R}) \subseteq H$  and:

$$A : \mathcal{D}(A) \xrightarrow{\text{linear}} H$$

$\mathcal{D}(A)$  is dense in  $H$ , i.e.  $\overline{\mathcal{D}(A)} = H$ .

#### 4.0.2 Definition (linear operator, domain, bounded)

- i) Let  $\mathcal{D} \subseteq H$  be a dense subspace. A linear map  $A : \mathcal{D} \rightarrow H$  is called a *linear operator* on  $H$  with domain (of definition)  $\mathcal{D}$ .
- ii)  $A$  is called *bounded*, if there exists a  $C \in \mathbb{R}_{>0}$  such that for all  $u \in \mathcal{D}$  holds:

$$\|Au\| \leq C \|u\|$$

Otherwise  $A$  is called unbounded.

#### 4.0.3 Lemma

If  $A$  is a bounded operator with dense domain  $\mathcal{D} \subseteq H$ , then it can be extended by continuity to a unique operator  $A \in L(H)$ .

##### Proof

Let  $u \in H$ , not necessarily in  $\mathcal{D}$ . Since  $\overline{\mathcal{D}} = H$ , there is a sequence  $(u_l)$  in  $\mathcal{D}$  with  $u_l \rightarrow u$ .

$$\|Au_i - Au_j\| = \|A(u_i - u_j)\| \leq C \cdot \|u_i - u_j\| \xrightarrow{i,j \rightarrow \infty} 0$$

Therefore we can set:

$$Au := \lim_{l \rightarrow \infty} Au_l$$

Since  $Au_l$  converges for any sequence  $u_l \rightarrow u$ , this is well-defined.

$$\|Au\| \leftarrow \|Au_i\| \leq C \|u_i\| \rightarrow C \|u\|$$

So there exists a  $C$  such that  $\|Au\| \leq C \|u\|$  for all  $u \in H$  and therefore  $A \in L(H)$ .  $\square_{4.0.3}$

## 4.1 Isometric and unitary operators

#### 4.1.1 Definition (isometric operator)

A operator  $V : \mathcal{D}(V) \rightarrow H$  with dense domain  $\mathcal{D}(V) \subseteq H$  is called *isometric* if for all  $u \in \mathcal{D}(V)$  holds:

$$\langle Vu, Vu \rangle = \langle u, u \rangle$$

This operator is bounded, because:

$$\|Vu\| = \sqrt{\langle Vu, Vu \rangle} = \sqrt{\langle u, u \rangle} = \|u\| \stackrel{C:=1}{\leq} C \|u\|$$

Therefore we can extend it by continuity to  $H$  and

$$V : H \rightarrow H$$

is again isometric.

### The “Hilbert hotel”

Consider  $H = \ell_2$  and  $(a_i) = (a_1, a_2, \dots) \in \ell_2$ .

$$A(u_1, u_2, \dots) := (0, u_1, u_2, \dots)$$

$A$  is isometric, but it is no bijection.

Suppose you have a hotel with an infinite number of rooms and an infinite number of guest, in every room one guest.

If a new guest arrives, just move the guest from room  $n$  to room  $n + 1$  and the first room gets unoccupied, so the new guest can use it.

#### 4.1.2 Proposition

For an isometric operator  $V$  the subspace  $V(H) \subseteq H$  is closed.

#### Proof

Consider  $y \in \overline{V(H)}$  and show  $y \in V(H)$ :

There exists a  $(y_n)$  with  $y_n \in V(H)$  and  $y_n \rightarrow y$  and a  $(x_n)$  with  $V(x_n) = y_n$ . Then holds:

$$\|x_i - x_j\| \stackrel{V \text{ isometric}}{=} \|V(x_i - x_j)\| = \|y_i - y_j\| \xrightarrow{i, j \rightarrow \infty} 0$$

Thus  $x_i \rightarrow x$  converges. By continuity we get:

$$V(x) = \lim_{i \rightarrow \infty} V(x_i) = \lim_{i \rightarrow \infty} y_i = y$$

□<sub>4.1.2</sub>

#### 4.1.3 Definition (unitary operator)

If  $V : H \rightarrow H$  is an isometric operator and  $V(H) = H$ , then  $V$  is called *unitary* (unitär).

## 4.2 The Closure of an Operator

Let  $E$  and  $F$  be Banach spaces and  $A : \mathcal{D}(A) \subseteq E \rightarrow F$  be a densely defined linear operator.

$$\begin{aligned} \text{graph}(A) &:= \{(u, Au) \mid u \in \mathcal{D}(A)\} \subseteq E \times F \\ \overline{\text{graph}(A)} &\subseteq E \times F \end{aligned}$$

Try to realize this as the graph of a new operator  $\overline{A}$ .

$$\mathcal{D}(\overline{A}) := \text{pr}_1(\overline{\text{graph}A}) = \left\{ u \mid \exists_{v \in F} : (u, v) \in \overline{\text{graph}A} \right\}$$

For  $u \in \mathcal{D}(\overline{A})$  and  $(u, v) \in \overline{\text{graph}A}$  define:

$$\overline{A}u := v$$

$v$  exists by definition of  $\mathcal{D}(\overline{A})$ . Is  $v$  unique?

Suppose  $(u, v) \in \overline{\text{graph}A}$ . Then there exists a sequence  $(u_n, v_n) \in \text{graph}(A)$ , with  $(u_n, v_n) \rightarrow (u, v)$ . Equivalently:

$$\forall_{n \in \mathbb{N}} \exists_{u_n \in \mathcal{D}(A)} : (u_n \rightarrow u) \wedge (Au_n \rightarrow v)$$

Then we set  $\overline{A}u := v$ .

**Problem:** There might be two different series  $(u_n)$  and  $(\tilde{u}_n)$  with  $u_n \rightarrow u$ ,  $\tilde{u}_n \rightarrow u$ ,  $Au_n \rightarrow v$  and  $A\tilde{u}_n \rightarrow \tilde{v} \neq v$ .

### 4.2.1 Definition (closable operator)

A densely defined operator  $A$  is called closable (abschließbar) if  $\overline{\text{graph}A}$  is the graph of an operator  $B$ .

$B$  is called the *closure* of  $A$ , symbolically  $B = \overline{A}$ .

### 4.2.2 Definition (closed)

$A$  is called *closed* if  $\text{graph}A$  is a closed subset of  $E \times F$ .

### 4.2.3 Theorem (closed graph theorem)

Reformulation of 2.4.9:

If  $\mathcal{D}(A) = E$ , then  $A$  is closed if and only if  $A$  is bounded.

### 4.2.4 Example

Consider  $E = C^0([0,1])$  with the norm  $\|f\| = \sup_{x \in [0,1]} |f(x)|$ .

$$\mathcal{D}(A) = C^1([0,1]) \subseteq E$$

$$\begin{aligned} A : \mathcal{D}(A) &\rightarrow E \\ f &\mapsto f' \end{aligned}$$

$A$  is a densely defined, unbounded operator. Is  $A$  closed?

Consider  $(u, v) \in \overline{\text{graph} A}$ , i.e. there exists a sequence  $(u_n) \subseteq \mathcal{D}(A)$  with  $u_n \rightarrow u$  and  $Au_n \rightarrow v$ .  $u_n \rightarrow u$  means uniform convergence of  $u_n \rightrightarrows u$ , so  $u$  is continuous as a uniform limit of continuous functions.

$Au_n \rightarrow v$  means uniform convergence of  $Au_n \rightrightarrows v$ , so  $v$  is also continuous.

It follows that  $u \in C^1$  and  $u' = v$ .

So  $(u, v) \in \text{graph} A$  and therefore  $A$  is closed.

Consider  $F := C^1([0, 1])$  with  $\|u\| = \sup_{[0, 1]} |u| + \sup_{[0, 1]} |u'|$ . This is a Banach space.

### Remark

The closure of a closable operator is always closed.

This is obvious, because  $\text{graph} \bar{A} \stackrel{\text{def.}}{=} \overline{\text{graph} A}$ , which is closed.

### 4.2.5 Theorem (Criterion for closable)

$A$  is closable if and only if:

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \rightarrow 0) \wedge (Au_n \rightarrow v) \quad \Rightarrow \quad v = 0$$

### Proof

“ $\Rightarrow$ ”: Suppose  $A$  is closable. Thus there is an operator  $\bar{A}$  such that  $\text{graph} \bar{A} = \overline{\text{graph} A}$ .

Suppose that  $u_n \in \mathcal{D}(A)$ ,  $u_n \rightarrow 0$  and  $Au_n \rightarrow v$ . Then  $(u_n, Au_n) \rightarrow (0, v) \in \text{graph} \bar{A} = \overline{\text{graph} A}$  and thus  $v = \bar{A}(0) = 0$ .

“ $\Leftarrow$ ”: Suppose that the implication

$$(u_n \in \mathcal{D}(A)) \wedge (u_n \rightarrow 0) \wedge (Au_n \rightarrow v) \quad \Rightarrow \quad v = 0$$

holds.

Define  $\mathcal{D}(\bar{A})$  by:  $u_n \in \mathcal{D}(A)$  with  $u_n \rightarrow u$  and  $Au_n \rightarrow v$ . Then for  $u \in \mathcal{D}(\bar{A})$  set  $\bar{A}(u) = v$ .

This is well-defined: Suppose  $u_n, \tilde{u}_n \rightarrow u$ ,  $Au_n \rightarrow v$  and  $A\tilde{u}_n \rightarrow \tilde{v}$ . Then  $u_n - \tilde{u}_n \rightarrow 0$  and  $A(u_n - \tilde{u}_n) \rightarrow v - \tilde{v}$ . By assumption follows  $v - \tilde{v} = 0$ .  $\square_{4.2.5}$

## 4.3 The adjoint of a densely defined operator

Let  $A : \mathcal{D}(A) \rightarrow H$  be a linear operator with  $\overline{\mathcal{D}(A)} = H$ .

In finite-dimensional linear algebra the definition of the adjoint  $A^*$  is:

$$\langle u, Av \rangle =: \langle A^*u, v \rangle \quad \forall_{u, v \in H}$$

Here it is more complicated, since in general  $\mathcal{D}(A) \neq H$ .

$$M := \left\{ (u, w) \in H \times H \mid \forall_{v \in \mathcal{D}(A)} : \langle u, Av \rangle = \langle w, v \rangle \right\}$$

**Claim:**  $M$  is the graph of a linear map  $A^*$ .

**Proof:**  $M \neq \emptyset$  since  $(0,0) \in M$ .

- The image is unique:  $u \mapsto w$  is well-defined, as from  $(u,w), (u,w') \in M$  follows for all  $v \in \mathcal{D}(A)$ :

$$\langle w - w', v \rangle = \langle u - u, Av \rangle = 0$$

Since  $\mathcal{D}(A)$  is dense,  $w - w' = 0$  follows.

- $A^*$  is linear: For  $(u,w), (u',w') \in M$  and  $\lambda \in \mathbb{K}$  follows  $(u + \lambda u', w + \lambda w') \in M$ , which is obvious from the definition of  $M$ .  $\square_{\text{Claim}}$

### 4.3.1 Theorem

$A^*$  is closed.

**Proof**

Let  $x_n \in \mathcal{D}(A^*)$  converge to  $x \in H$  and  $A^*x_n \rightarrow y \in H$ . For  $z \in \mathcal{D}(A)$  holds:

$$\langle x, Az \rangle \stackrel{\langle \cdot, \cdot \rangle \text{ continuous}}{=} \lim_{n \rightarrow \infty} \langle x_n, Az \rangle = \lim_{n \rightarrow \infty} \langle A^*x_n, z \rangle \stackrel{\langle \cdot, \cdot \rangle \text{ continuous}}{=} \langle y, z \rangle$$

This shows  $x \in \mathcal{D}(A^*)$  and  $A^*x = y$ , so  $A^*$  is closed.  $\square_{4.3.1}$

### 4.3.2 Theorem

$A^*$  is the maximal, i.e. not extensible, operator  $S$  with the property that for all  $u \in \mathcal{D}(A)$  and  $v \in \mathcal{D}(S)$ :

$$\langle Au, v \rangle = \langle u, Sv \rangle$$

**Proof**

$$\begin{aligned} \text{graph}(S) &= \left\{ (v, w) \in \mathcal{D}(S) \times H \mid Sv = w \right\} = \\ &= \left\{ (v, w) \in \mathcal{D}(S) \times H \mid \forall_{u \in \mathcal{D}(A)} \langle Au, v \rangle = \langle u, w \rangle \right\} = \\ &= \left\{ (v, w) \in H \times H \mid \forall_{u \in \mathcal{D}(A)} \langle v, Au \rangle = \langle w, u \rangle \right\} = \text{graph}(A^*) \end{aligned}$$

$\square_{4.3.2}$

## 4.4 Symmetric and self-adjoint densely defined operators

### 4.4.1 Definition (symmetric, (essentially) self-adjoint)

- i)  $A$  is *symmetric* :  $\Leftrightarrow \forall_{u,v \in \mathcal{D}(A)} : \langle Au, v \rangle = \langle u, Av \rangle$
- ii)  $A$  is *self-adjoint* :  $\Leftrightarrow A^* = A$  (in particular,  $\mathcal{D}(A^*) = \mathcal{D}(A)$ )
- iii)  $A$  is *essentially self-adjoint* :  $\Leftrightarrow \overline{A}$  is self-adjoint

For bounded  $A$  with  $\mathcal{D}(A) = H$  all these notions coincide.

### 4.4.2 Example

Consider the operator  $A := \Delta = \sum_{i=1}^n \partial_i^2$  on  $L^2(\Omega)$  for a bounded open region  $\Omega \subseteq \mathbb{R}^n$  with  $\mathcal{D}(A) = C_0^\infty(\Omega) \stackrel{\text{dense}}{\subseteq} L^2(\Omega)$ .

- $A$  is symmetric:

$$\langle Af, g \rangle \stackrel{\text{integration by parts}}{=} \langle f, Ag \rangle$$

- Adjoint of  $\Delta$  on  $L^2$ :

$$\int \mathrm{d}^n r (\Delta f) \cdot g = \int \mathrm{d}^n r f \cdot \underbrace{h}_{\in L^2}$$

Here  $h := A^*g$ . It is sufficient to consider  $g \in H^{2,2}(\Omega)$  (Sobolev space).  $\mathcal{D}(A^*) \supsetneq \mathcal{D}(A)$

### 4.4.3 Lemma

Let  $A$  be a symmetric operator. Then  $A$  is closable and  $\overline{A}$  and  $A^*$  are extensions of  $A$  and  $\mathcal{D}(A) \stackrel{\text{i)}}{\subseteq} \mathcal{D}(\overline{A}) \stackrel{\text{ii)}}{\subseteq} \mathcal{D}(A^*)$ .

#### Proof

Let  $u_n \in \mathcal{D}(A)$  with  $u_n \rightarrow 0$  and  $Au_n \rightarrow w$ .

$$\begin{aligned} \langle Au, v \rangle &= \langle u, Av \rangle \quad \forall_{u,v \in \mathcal{D}(A)} \\ \langle w, v \rangle &\leftarrow \langle Au_n, v \rangle = \langle u_n, Av \rangle \rightarrow \langle 0, Av \rangle = 0 \end{aligned}$$

Since this holds for all  $v \in \mathcal{D}(A)$  now  $w = 0$  follows. From the criterion 4.2.5 follows that  $A$  is closable.

- i) is obvious from the definition of  $\overline{A}$ .
- ii) Take  $u \in \mathcal{D}(\overline{A})$ . Then there is a sequence  $u_n \in \mathcal{D}(A)$  with  $u_n \rightarrow u$  and  $Au_n \rightarrow \overline{A}u$ . For all  $v \in \mathcal{D}(A)$  holds:

$$\langle \overline{A}u, v \rangle \leftarrow \langle Au_n, v \rangle = \langle u_n, Av \rangle \rightarrow \langle u, Av \rangle$$

So  $u \in \mathcal{D}(A^*)$  and  $A^*u = \overline{A}u$ .



□<sub>4.4.3</sub>

„The smaller one chooses  $\mathcal{D}(A)$ , the larger becomes  $\mathcal{D}(A^*)$ .“

$$B \subseteq \mathcal{D}(A) \quad \Rightarrow \quad \mathcal{D}((A|_B)^*) \supseteq \mathcal{D}(A^*)$$

*Difficulty:* Construct  $\mathcal{D}(A)$  such that  $\mathcal{D}(A) = \mathcal{D}(A^*)$ . (More on this later in the lecture.)

## 4.5 Heisenberg's uncertainty principle

In quantum mechanics:

The Hilbert space for one dimensional problems is usually  $H = L^2(\mathbb{R})$ .

The position operator is  $x =: B$  and the momentum operator is  $\frac{\hbar}{i} \frac{d}{dx} =: A$ .

$$[A, B] := AB - BA = \frac{\hbar}{i} \mathbb{1}$$

### 4.5.1 Theorem (Winter-Wieland)

For two continuous operators  $A$  and  $B$  with  $[A, B] = c \cdot \mathbb{1}$  and  $B^n = B$  for all  $n \in \mathbb{N}_{\geq 1}$ , i.e.  $B$  is idempotent, follows  $c = 0$ .

#### Proof

Consider:

$$B^k AB^{n-k} = B^k (AB) B^{n-k-1} = B^k (BA + c\mathbb{1}) B^{n-k-1} = B^{k+1} AB^{n-k-1} + cB^{n-1}$$

$$\Rightarrow \quad cB^{n-1} = B^k AB^{n-k} - B^{k+1} AB^{n-k-1}$$

Sum this from  $k = 0$  to  $k = n - 1$ :

$$ncB^{n-1} = \sum_{k=0}^{n-1} B^k AB^{n-k} - B^{k+1} AB^{n-k-1} \stackrel{\text{telescope sum}}{=} AB^n - B^n A$$

$$n|c| \|B^{n-1}\| = \|AB^n - B^n A\| \stackrel{\Delta\text{-inequality}}{\leq} \|AB^n\| + \|B^n A\| \leq (\|AB\| + \|BA\|) \cdot \|B^{n-1}\|$$

Since this must hold for all  $n$  either  $c = 0$  or there exists a  $n \in \mathbb{N}_{>1}$  with  $\|B^{n-1}\| = 0$ , i.e.  $B^{n-1} = 0$ . Since  $B$  is idempotent follows  $B = 0$  and therefore  $[A, B] = 0$  and also  $c = 0$ . □<sub>4.5.1</sub>

Consider  $u \in \mathcal{D}(A)$  with  $\|u\| = 1$ , which represents a quantum mechanical state.

The expectation value of  $A$  in  $u$  is after the probabilistic interpretation:

$$E_u(A) := \langle u, Au \rangle$$

The “uncertainty”, i.e. the variance, is:

$$\Delta_u(A) := \|(A - E_u(A) \mathbb{1}) u\|$$

**4.5.2 Theorem** (Heisenberg's uncertainty principle)

Let  $H$  be a  $\mathbb{C}$ -Hilbert space and  $A : \mathcal{D}(A) \rightarrow H$ ,  $B : \mathcal{D}(B) \rightarrow H$  be two symmetric operators with  $\overline{\mathcal{D}(A)} = H = \overline{\mathcal{D}(B)}$ . Assume for the image domains  $\mathcal{R}$ :

$$\mathcal{R}(A) \subseteq \mathcal{D}(B) \qquad \mathcal{R}(B) \subseteq \mathcal{D}(A)$$

So  $[A, B]$  is well-defined on  $\mathcal{D}(A) \cap \mathcal{D}(B)$ .

Assume furthermore that  $[A, B] = \frac{\hbar}{i} \mathbb{1}$  with  $\hbar > 0$ .

Then for all  $u \in \mathcal{D}(A) \cap \mathcal{D}(B)$  with  $\|u\| = 1$  holds:

$$\Delta_u(A) \cdot \Delta_u(B) \geq \frac{\hbar}{2}$$

**Proof**

Replace  $A$  by  $\tilde{A} := A - E_u(A) \cdot \mathbb{1}$  and  $\tilde{B} := B - E_u(B) \cdot \mathbb{1}$ . Then holds:

$$[\tilde{A}, \tilde{B}] = \frac{\hbar}{i} \mathbb{1}$$

$$\Delta_u(A) = \|\tilde{A}u\|$$

$$\Delta_u(B) = \|\tilde{B}u\|$$

We have to show:

$$\Delta_u(A) \cdot \Delta_u(B) = \|\tilde{A}u\| \cdot \|\tilde{B}u\| \geq \frac{\hbar}{2}$$

$$\begin{aligned} \frac{\hbar}{2} &= \frac{\hbar}{2} \langle u, u \rangle = \frac{i}{2} \left\langle u, \left( \tilde{A}\tilde{B} - \tilde{B}\tilde{A} \right) u \right\rangle \stackrel{\text{symmetry}}{=} \frac{i}{2} \left( \langle \tilde{A}u, \tilde{B}u \rangle - \langle \tilde{B}u, \tilde{A}u \rangle \right) = \\ &= -\text{Im} \left( \langle \tilde{A}u, \tilde{B}u \rangle \right) \stackrel{\text{Cauchy-Schwarz}}{\leq} \|\tilde{A}u\| \cdot \|\tilde{B}u\| \end{aligned}$$

□<sub>4.5.2</sub>

**4.6 Spectrum and resolvent**

Let  $A : \mathcal{D}(A) \rightarrow H$  be a closed, densely defined operator.

**4.6.1 Definition** (continuously invertible, resolvent, spectrum)

$A$  is *continuously invertible* if and only if  $A : \mathcal{D}(A) \rightarrow H$  is bijective and  $A^{-1} : H \rightarrow \mathcal{D}(A)$  is continuous.

$$\varrho(A) := \{ \lambda \in \mathbb{K} \mid (\lambda \mathbb{1} - A) \text{ is continuously invertible} \}$$

The *resolvent* (Resolvente) is defined for  $\lambda \in \varrho(A)$  as

$$\mathcal{R}_\lambda(A) = (\lambda \mathbb{1} - A)^{-1} \in L(H)$$

and the *spectrum* of  $A$  as:

$$\sigma(A) = \mathbb{K} \setminus \varrho(A)$$

### 4.6.2 Lemma

$\varrho(A)$  is open and  $\sigma(A)$  is closed.

#### Proof

For bounded operators cf. Theorem 2.5.3.

It's method works even for unbounded operators:

Take  $\lambda, \mu \in \varrho(A)$ .

$$\begin{aligned} (A - \mu) &= (A - \lambda) + (\lambda - \mu) = \\ &= \underbrace{(A - \lambda)}_{\text{continuously invertible}} \cdot \left( \mathbb{1} + (A - \lambda)^{-1} (\lambda - \mu) \right) \end{aligned}$$

$\mathbb{1} + (A - \lambda)^{-1} (\lambda - \mu)$  is continuously invertible using the Neumann series if:

$$|\lambda - \mu| < \frac{1}{\| (A - \lambda)^{-1} \|}$$

So  $\varrho(A)$  is open and therefore the complement  $\sigma(A)$  is closed. □<sub>4.6.2</sub>

### 4.6.3 Theorem (resolvent equation)

The map  $\lambda \mapsto \mathcal{R}_\lambda(A)$  is complex analytic on  $\varrho(A)$ .

We have the *resolvent equation* (Resolventengleichung):

$$\mathcal{R}_\lambda - \mathcal{R}_\mu = -(\lambda - \mu) \mathcal{R}_\lambda \cdot \mathcal{R}_\mu$$

#### Proof

Analogy with  $\mathbb{C}$ -numbers:

$$\begin{aligned} \frac{1}{\lambda - x} - \frac{1}{\mu - x} &= \frac{\mu - \lambda}{(\lambda - x)(\mu - x)} \\ (\mu - x) - (\lambda - x) &= \mu - \lambda \end{aligned}$$

Same thing for operators:

$$\begin{aligned} (\mu - A) - (\lambda - A) &= \mu - \lambda \\ \mathcal{R}_\mu^{-1} - \mathcal{R}_\lambda^{-1} &= \mu - \lambda \quad / \mathcal{R}_\mu \cdot \quad / \cdot \mathcal{R}_\lambda \\ \mathcal{R}_\lambda - \mathcal{R}_\mu &= (\mu - \lambda) \mathcal{R}_\mu \mathcal{R}_\lambda \\ \mathcal{R}_\lambda &= \mathcal{R}_\mu + (\mu - \lambda) \mathcal{R}_\mu \mathcal{R}_\lambda \end{aligned}$$

Assume  $|\mu - \lambda| < \frac{1}{\| \mathcal{R}_\lambda \|}$ .

$$\mathcal{R}_\mu = \mathcal{R}_\lambda (1 + (\mu - \lambda) \mathcal{R}_\lambda)^{-1} = \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-1)^n (\mu - \lambda)^n \mathcal{R}_\lambda$$

This series converges absolutely and so the map is analytic in  $L(H)$ . □<sub>4.6.3</sub>

## 5 Compact Operators

Let  $E$  and  $F$  be Banach spaces and  $A \in L(E, F)$ .

**Remember:** There exists a  $C \in \mathbb{R}_{>0}$  such that for all  $u \in E$  holds:

$$\|Au\| \leq C \|u\|$$

$A$  maps bounded sets in  $E$  to bounded sets in  $F$ .

**But:** Bounded sets are not precompact in general.

### 5.1 Definition (compact operator)

$A$  is called *compact* operator if and only if  $A$  maps bounded sets to relatively compact sets, i.e. the closure is compact.

(In complete spaces relatively compact is equivalent to precompact.)

### 5.2 Example (integral operator)

Let  $E = (C^0([0,1]), \|\cdot\|_\infty)$  and consider an integral kernel  $K \in C^0([0,1] \times [0,1])$ ,  $K : E \rightarrow E$ .

$$(K\varphi)(x) := \int_0^1 K(x,y) \varphi(y) dy$$

$$\begin{aligned} |(K\varphi)(x)| &\leq \sup_y |K(x,y)| \|\varphi\| & / \sup_x \\ \|K\varphi\| &\leq C \|\varphi\| \end{aligned}$$

So  $K \in L(E)$ . Furthermore the integral kernel  $K$  is continuous and defined on a compact set. Therefore  $K$  is uniformly continuous after the Heine-Cantor theorem.

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \exists_{\delta \in \mathbb{R}_{>0}} : |K(x,y) - K(x',y)| < \varepsilon \quad \forall_{|x-x'| < \delta, y \in [0,1]}$$

$$|(K\varphi)(x) - (K\varphi)(x')| = \left| \int_0^1 (K(x,y) - K(x',y)) \varphi(y) dy \right| \leq \varepsilon \|\varphi\|_\infty$$

Let now  $B := B_M(0)$  with  $M \in \mathbb{R}_{>0}$ . Then  $K(B) \subseteq E$ .

- uniformly bounded ( $\|\varphi\| < CM$ )
- uniformly continuous

The Arzelà-Ascoli theorem yields, that  $K(B)$  is precompact and so  $K$  is a compact operator.

### 5.3 Theorem

Let  $H$  be a Hilbert space.

A compact operator  $A : H \rightarrow H$  maps weakly convergent sequences to convergent sequences.

#### Proof

Let  $x_n \rightharpoonup x$ , then  $(x_n)$  is bounded, i.e. there is a  $C \in \mathbb{R}_{>0}$  such that  $\|x_n\| < C$  for all  $n \in \mathbb{N}$ . Define  $y_n := Ax_n$ . For all  $z \in H$  holds:

$$\langle z, y_n - y \rangle = \langle z, A(x_n - x) \rangle = \langle A^* z, x_n - x \rangle \rightarrow 0$$

Therefore  $y_n \rightharpoonup y$  converges weakly. Because  $A$  is compact, every subsequence of  $y_n$  contains a convergent subsequence with limit  $\tilde{y}$ . For  $z = \tilde{y} - y$  converges:

$$0 \leftarrow \langle z, y_n - y \rangle \rightarrow \langle \tilde{y} - y, \tilde{y} - y \rangle = \|\tilde{y} - y\|$$

Therefore  $\tilde{y} = y$ .

Since this holds for every subsequence of  $y_n$  follows  $y_n \rightarrow y$ .

□<sub>5.3</sub>

### 5.4 Lemma

Consider operators  $A, B : E \rightarrow F$ .

- i) If  $A$  and  $B$  are compact, so are  $A + B$  and  $\lambda A$  for all  $\lambda \in \mathbb{K}$ .
- ii) If  $A : E \rightarrow F$  is compact (continuous) and  $B : F \rightarrow E$  continuous (compact), then  $B \circ A$  is compact.  
(In particular  $A^n$  is compact for  $A : E \rightarrow E$ .)
- iii) The compact operators form a closed subspace of  $L(E, F)$ .

#### Proof

- i) is obvious. □<sub>i</sub>
- ii) follows, since a continuous operator is bounded. □<sub>ii</sub>
- iii) Let  $(x_n)$  be bounded and  $T_k$  a convergent sequence of compact operators. By diagonal choice get a subsequence, also written  $x_n$ , such that  $T_k x_n$  converges for all  $k \in \mathbb{N}$ .

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq \underbrace{\|Tx_n - T_k x_n\|}_{\leq \|T - T_k\| \cdot \|x_n\|} + \|T_k x_n - T_k x_m\| + \underbrace{\|T_k x_m - Tx_m\|}_{\leq \|T - T_k\| \cdot \|x_m\|} \leq \\ &\leq \|T - T_k\| \cdot \|x_n\| + \|T_k x_n - T_k x_m\| + \|T - T_k\| \cdot \|x_m\| \xrightarrow{n, m, k \rightarrow \infty} 0 \end{aligned}$$

□<sub>5.4</sub>

### 5.5 Lemma (Fredholm operator)

Let  $A : E \rightarrow E$  be compact and define  $T := \mathbb{1} - A$ .  $T$  is called *Fredholm operator*.

- i)  $\ker(T)$  is finite-dimensional.
- ii) There exists a  $i \in \mathbb{N}$  such that  $\ker(T^k) = \ker(T^i)$  for all  $k \in \mathbb{N}_{>i}$ .
- iii) The image of  $T$  is closed.

#### Proof

- i)  $\ker(T) =: Z = \{u \mid u = Au\}$ . Since  $Z \cap B_1(0)$  is bounded

$$A(Z \cap B_1(0)) = Z \cap B_1(0)$$

is precompact and therefore  $Z$  is finite-dimensional.  $\square_{\text{i)}$

- ii) Define  $N_i := \ker(T^i)$ , which are closed subspaces of  $E$ , since the  $T^i$  are continuous. Suppose the claim is wrong, then  $N_j \subsetneq N_{j+1} \subsetneq \dots$ , so in particular all  $N_j$  are proper subspaces. Choose  $y_j \in N_j$  with:

$$\|y_j\| = 1 \qquad d(y_j, N_{j-1}) > \frac{1}{2}$$

This is possible after Lemma 2.1.2.

For all  $m < n$  holds:

$$Ay_n - Ay_m = y_n - \underbrace{T_{y_n} - y_m + T_{y_m}}_{\in N_{n-1}}$$

Therefore follows:

$$\|Ay_n - Ay_m\| > \frac{1}{2}$$

So  $(Ay_n)$  has no accumulation value in contradiction to the compactness of  $A$ .  $\square_{\text{ii)}$

- iii) Let  $y_k \in \text{im}(T)$  with  $y_k \rightarrow y$  and  $y_k = Tx_k$ . We want to show  $y \in \text{im}(T)$ . Define:

$$d_k := d(x_k, \ker(T)) = \inf_{z \in \ker(T)} \|x_k - z\|$$

**Claim:**  $(d_k)$  is bounded. Equivalently  $(D_k) = |\max\{1, d_k\}|$  is bounded.

**Proof:** Choose  $z_k \in \ker(T)$ ,  $w_k := x_k - z_k$  with  $\|w_k\| < 2d_k$  and  $Tw_k = y_k$ .

Assume  $D_k$  is unbounded. Since  $y_k$  is convergent and thus bounded, follows:

$$T\left(\frac{w_k}{D_k}\right) = \frac{y_k}{D_k} \xrightarrow{k \rightarrow \infty} 0$$

Now consider  $u_k := \frac{w_k}{D_k}$ . We know  $\|u_k\| < 2$  and  $T(u_k) \rightarrow 0$ .

Thus  $u_k - Au_k \rightarrow 0$ . Since  $A$  is compact, every subsequence of  $Au_k$  has a convergent subsequence, and therefore  $u_k \rightarrow 0$  converges.

The continuity of  $T$  gives:

$$T(u) = \lim_{k \rightarrow \infty} T(u_k) = 0$$

So  $u \in \ker(T)$ .

On the other hand we have for all  $z \in \ker(T)$ :

$$\begin{aligned} \|w_k - z\| &\geq D_k \\ \Rightarrow \left\| u_k - \frac{z}{D_k} \right\| &\geq 1 \end{aligned}$$

Since  $T$  is a subspace this means, that for all  $z \in \ker(T)$  holds:

$$\|u_k - z\| \geq 1$$

This is a contradiction to  $u \in \ker(T)$ .

□<sub>Claim</sub>

So  $u_k$  is bounded and  $T(w_k) = T(x_k) = y_k \rightarrow y$ . So we get:

$$w_k - Aw_k \rightarrow y$$

Since  $A$  is compact  $Aw_k$  converges and with this follows, that  $w_k \rightarrow w$  also converges. By continuity we get:

$$T(w) = \lim_{k \rightarrow \infty} T(w_k) = y$$

So  $w \in \text{im}(T)$ .

□<sub>5.5</sub>

## 5.6 Theorem (Fredholm Alternative)

Let  $A : E \rightarrow E$  be compact and define  $T := \mathbb{1} - A$ .

If the kernel  $\ker(T) = \{0\}$  is trivial, then  $T$  is continuously invertible.

### Proof

$\ker(T) = \{0\}$  means, that  $T$  is injective. We only need to show, that  $T$  is surjective, because then  $T$  is invertible and 2.4.7 yields then, that  $T$  is open and therefore  $T^{-1}$  continuous.

$\text{im}(T)$  is closed following 5.5 iii).

$\text{im}(T) = E$ , since otherwise  $T(E) \subsetneq E$ . Then the injectivity implies for all  $k \in \mathbb{N}$ :

$$T^{k+1}(E) \subsetneq \underbrace{T^k(E)}_{=E_k}$$

$E_k$  is closed for all  $k \in \mathbb{N}$ :

$$E_k = (\mathbb{1} - A)^k(E) = \underbrace{\left( \mathbb{1} + \sum_{l=1}^k (-1)^l \binom{k}{l} A^l \right)}_{A:=A_k}(E)$$

Now  $A_k$  is compact, as the compact operators form a (closed) ideal subalgebra  $\text{CP}(E)$ .

Choose  $x_k \in E_k$  with  $\|x_k\| = 1$  and  $d(x_k, E_k) > \frac{1}{2}$ , which is possible after Lemma 2.1.2. Then holds for all  $m < n$ :

$$Ax_m - Ax_n = x_m - \underbrace{Tx_m - x_n + Tx_n}_{\in H_{m+1}}$$

$$\Rightarrow \|Ax_m - Ax_n\| > \frac{1}{2}$$

This is a contradiction to the compactness of  $A$ .

Therefore  $T$  is surjective and the theorem follows.  $\square_{5.6}$

## 5.7 Theorem (Riesz-Schauder)

Let  $A \in L(H)$  be compact.

- i)  $\sigma(A)$  consists of a finite or countable set of complex numbers and 0 is the only possible accumulation point.
- ii) Every  $0 \neq \lambda \in \sigma(A)$  is an eigenvalue of finite multiplicity, i.e.  $\ker(A - \lambda)$  is finite-dimensional. That means, there exists a  $i \in \mathbb{N}$  such that for all  $k > i$  holds:

$$\ker(A - \lambda)^k = \ker(A - \lambda)^i$$

One says also that the Jordan chains are finite.

### Proof

ii) is an immediate consequence of the Lemmas 5.5 and 5.6. (Divide  $A$  by  $\lambda$ .)

i) Assume  $\lambda_n \neq 0$  are pairwise different eigenvalues. Choose eigenvectors  $x_n \in H$  such that:

$$Ax_n = \lambda_n x_n$$

$$Y_n := \langle x_1, \dots, x_n \rangle$$

Since the eigenvalues are pairwise different  $Y_n \subsetneq Y_{n+1}$  must hold, because the  $x_k$  are linearly independent.

Assume  $Y_n \subsetneq H$ , since otherwise  $H$  would be finite-dimensional and therefore  $\sigma(A)$  a finite set.

So following Lemma 2.1.2 we can choose  $y_n \in Y_n$  with  $\|y_n\| = 1$  and:

$$d(y_n, Y_{n+1}) > \frac{1}{2}$$

Since  $y_n \in Y_n$  one can find  $\alpha_j \in \mathbb{K}$  such that:

$$y_n = \sum_j \alpha_j x_j$$

Then follows:

$$(A - \lambda_n) y_n = \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) \alpha_j x_j =: \tilde{y}_n \in Y_{n-1}$$

For all  $n > m$  holds:

$$Ay_n - Ay_m = \lambda_n y_n - \underbrace{\tilde{y}_n - Ay_m}_{\in Y_{n-1}}$$



So we get:

$$\|Ay_n - Ay_m\| \geq \frac{|\lambda_n|}{2}$$

But  $(Ay_n)$  is precompact and thus for all  $\delta \in \mathbb{R}_{>0}$  exist only finitely many  $\lambda_n$  with  $|\lambda_n| > \delta$ . Therefore 0 is the only accumulation point and  $\sigma(A)$  is a countable union of finite sets and thus countable.  $\square_{5.7}$

Jordan decomposition:

$$A = \begin{pmatrix} \lambda_1 & & & & & 0 \\ & 1 & \ddots & & & \\ & & 1 & \lambda_1 & & \\ & & & \lambda_2 & & \\ & & & & 1 & \ddots \\ & & & & & 1 & \lambda_2 \\ 0 & & & & & & \ddots \end{pmatrix}$$

$$\lambda_1 - A = \begin{pmatrix} 0 & & & & & 0 \\ -1 & \ddots & & & & \\ & -1 & 0 & & & \\ & & & -\lambda_2 & & \\ & & & -1 & \ddots & \\ & & & & -1 & -\lambda_2 \\ 0 & & & & & \ddots \end{pmatrix}$$

So the first block is nilpotent. If it has  $k$  dimensions this means:

$$(\lambda_1 - A)^k = \begin{pmatrix} 0 & 0 \\ & * \\ 0 & * \end{pmatrix}$$

So  $k$  is the length of the Jordan chain.

## 5.8 Theorem

Let  $A \in L(H)$  be compact and  $H$  be a separable Hilbert space. Then  $A$  can be approximated in  $L(H)$  by operators of finite rank.

### Proof

Choose a countable orthonormal Hilbert basis  $(\varphi_j)_{j \in \mathbb{N}}$  of  $H$ , which is possible, since  $H$  is separable. Define:

$$\lambda_n := \sup_{\psi \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp, \|\psi\|=1} \|A\psi\|$$

Since  $A$  is bounded, this supremum exists. Obviously  $\lambda_1 \geq \lambda_2 \geq \dots$ . Thus  $\lambda_n \searrow \lambda \geq 0$ .

**Claim:**  $\lambda = 0$

**Proof:** Choose  $\psi_n \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp$  with  $\|\psi_n\| = 1$  and  $\|A\psi_n\| \geq \frac{\lambda}{2}$  which is possible after Lemma 2.1.2, since  $\langle \varphi_1, \dots, \varphi_n \rangle$  is a proper closed subspace of  $H$ . Write:

$$\psi_n = \sum_{j=1}^{\infty} \nu_j \varphi_j = (\nu_1, \nu_2, \dots)$$

Due to  $\psi_n \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp$  follows:

$$\psi_n = (0, \dots, 0, \nu_{n+1}, \nu_{n+2}, \dots)$$

For  $u \in H$  holds:

$$\langle u, \psi_n \rangle = \sum_{j=n+1}^{\infty} \nu_j \cdot \bar{u}_j \stackrel{\substack{\text{Schwarz} \\ \text{inequality}}}{\leq} \underbrace{\left( \sum_{j=n+1}^{\infty} |\nu_j|^2 \right)^{\frac{1}{2}}}_{=\|\psi_n\|} \cdot \left( \sum_{j=n+1}^{\infty} |u_j|^2 \right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$$

So by construction  $\psi_n \rightarrow 0$ . Therefore  $A\psi_n \rightarrow 0$  and thus  $\|A\lambda_n\| \rightarrow 0$ .

On the other hand we have  $\|A\psi_n\| \geq \frac{\lambda}{2}$  and so  $\lambda = 0$ . □<sub>Claim</sub>

Let  $P_n$  be the orthogonal projection on  $\langle \varphi_1, \dots, \varphi_n \rangle$ .

$$P_n u = \sum_{j=1}^n \varphi_j \langle \varphi_j, u \rangle$$

$AP_n$  is an operator of finite rank  $r \leq n$ , since  $\text{rank}(P_n) = n$ .

**Claim:**  $AP_n \xrightarrow{n \rightarrow \infty} A$  in  $L(H)$ .

**Proof:** Consider:

$$\|A - AP_n\| = \sup_{u \in H, \|u\|=1} \|A(\mathbb{1} - P_n)u\|$$

$(\mathbb{1} - P_n)u \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp$  and  $\|(\mathbb{1} - P_n)u\| \leq \|u\| = 1$ . ( $\mathbb{1} - P_n = P_{\langle \varphi_1, \dots, \varphi_n \rangle^\perp}$ )

Thus we get:

$$\|A - AP_n\| \leq \sup_{v \in \langle \varphi_1, \dots, \varphi_n \rangle^\perp, \|v\| \leq 1} \|Av\| = \lambda_n \xrightarrow{n \rightarrow \infty} 0$$

□<sub>Claim</sub>

□<sub>5.8</sub>

## 5.9 Lemma

Let  $A \in L(H)$  be compact and symmetric. (This implies that  $A$  is bounded and self-adjoint.) Then  $\sigma(A) \subseteq \mathbb{R}$  and if  $u$  is an eigenvector,  $Au = \lambda u$ , then its orthogonal is invariant under  $A$ .

**Proof**

For  $\lambda \in \sigma(A)$  holds  $\ker(A - \lambda) \neq \{0\}$ . Thus there exists a  $u \in \ker(A - \lambda) \setminus \{0\}$ .

$$\lambda \langle u, u \rangle = \langle u, Au \rangle = \langle Au, u \rangle = \bar{\lambda} \langle u, u \rangle$$

Since  $\|u\| \neq 0$  follows  $\lambda = \bar{\lambda}$ , which means that  $\lambda \in \mathbb{R}$ .

For  $v \in \langle u \rangle^\perp$  holds:

$$\langle Av, u \rangle = \langle v, Au \rangle = \lambda \langle v, u \rangle = 0$$

Therefore  $Av \in \langle u \rangle^\perp$ .

□<sub>5.9</sub>**5.10 Theorem (Hilbert-Schmidt)**

Let  $A \in L(H)$  be a symmetric compact operator on the separable Hilbert space  $H$ .

Then there exists an orthonormal Hilbert space basis of eigenvectors  $(u_n)_{n \in \mathbb{N}}$ , so with the eigenvalues  $\lambda_n \in \mathbb{R}$  holds:

$$Au_n = \lambda_n u_n$$

**Proof**

$\sigma(A)$  is countable and therefore we can write  $\sigma(A) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots\} \subseteq \mathbb{R}$  with  $\lambda_i \neq \lambda_j$  for  $i \neq j$ .  $\ker(\lambda_j - A)$  is finite-dimensional. So we choose a (finite) orthonormal basis of the eigenspace. Taking these eigenvectors for all eigenvalues, we obtain a countable orthonormal system  $(u_n)_{n \in \mathbb{N}}$ .

$$M := \overline{\langle u_n \rangle}^{\text{closed}} \subseteq H$$

$M^\perp$  is an invariant subspace of  $H$  under  $A$ , i.e.:

$$\tilde{A} := A|_{M^\perp} : M^\perp \rightarrow M^\perp$$

This is again symmetric and compact. We know that  $\sigma(\tilde{A}) = \{0\}$ .

**Question:** Why is  $\tilde{A} = 0$ ?

This is not true for a general operator, e.g.:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma(A) = \{0\}$$

**Answer:** If  $A$  is symmetric and  $\sigma(A) = \{0\}$ , then one can show  $A = 0$  using the following theorem 5.12:

From  $\sigma(\tilde{A}) = \{0\}$  follows  $r(\tilde{A}) = 0$  and since  $\tilde{A}$  is self-adjoint theorem 5.12 gives  $\|\tilde{A}\| = 0$  and thus  $\tilde{A} = 0$ . In other words  $A|_{M^\perp} = 0$ .

Now choose an orthonormal Hilbert basis  $(v_n)_{n \in \mathbb{N}_{\leq N}}$  of  $M^\perp$  for an  $N \in \mathbb{N} \cup \{\infty\}$ . Therefore  $\{u_n\} \cup \{v_n\}$  is the desired orthonormal Hilbert basis of  $H$ . □<sub>5.10</sub>

### 5.11 Definition (spectral radius)

Let  $A : \mathcal{D}(A) \subset H \rightarrow H$  be a densely defined operator. Then the *spectral radius*  $r(A)$  of  $A$  is defined by:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

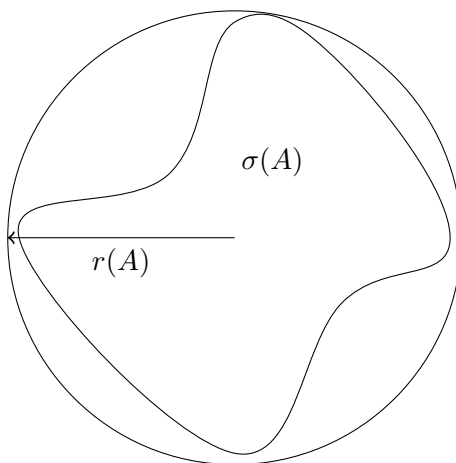


Figure 5.1:  $\sigma(A) \subseteq \overline{B_{r(A)}(0)}$

### 5.12 Theorem

For  $A \in L(H)$  holds:

$$r(A) = \limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$$

If  $A$  is symmetric, then:

$$r(A) = \|A\|$$

#### Proof

Recall for a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

with  $a_n, z \in \mathbb{K}$  the root test (Wurzelkriterium):

– If

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} =: c < 1$$

then  $|a_n z^n| < c^n$  and therefore is

$$\sum_{n=0}^{\infty} c^n$$

a convergent dominating sequence. Thus  $\sum_{n=0}^{\infty} a_n z^n$  converges as well.

– If

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} =: c > 1$$

then  $|a_n z^n| > c^n > 1$  for an infinite number of  $n$ . Therefore  $a_n z^n$  does *not* converge to zero, which implies that  $\sum_{n=0}^{\infty} a_n z^n$  does not converge as well.

– If

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} = 1$$

no conclusion is possible.

$$\limsup_{n \rightarrow \infty} |a_n z^n|^{\frac{1}{n}} = |z| \cdot \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

The Radius of convergence (Konvergenzradius) is thus defined by:

$$R := \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}}$$

If  $|z| < R$  the sum converges absolutely and if  $|z| > R$  the sum diverges.

In our setting for  $A = 0$  is nothing to prove. For  $\lambda \in \varrho(A) \setminus \{0\}$  we make a formal expansion:

$$\mathcal{R}_\lambda = (\lambda - A)^{-1} = \frac{1}{\lambda} \left( \mathbb{1} - \frac{A}{\lambda} \right)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} A^n \cdot \left( \frac{1}{\lambda} \right)^n$$

This is a power series in  $\frac{1}{\lambda}$ , but the coefficients are operators.

$$R := \frac{1}{\limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}}$$

For  $\frac{1}{|\lambda|} < R$

$$\left\| \sum_{n=0}^{\infty} A^n \left( \frac{1}{\lambda} \right)^n \right\| \leq \sum_{n=0}^{\infty} \|A^n\| \frac{1}{\lambda^n}$$

converges absolutely and so

$$\sum_{n=0}^{\infty} A^n \left( \frac{1}{\lambda} \right)^n$$

converges in  $L(H)$ . Thus the resolvent

$$\mathcal{R}_\lambda = (\lambda - A)^{-1}$$

exists and  $\sigma(A) \subseteq \overline{B_{\frac{1}{R}}(0)}$ , i.e.:

$$r(A) \leq \frac{1}{R} = \limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$$

If  $\frac{1}{|\lambda|} > R$

$$\left\| \sum_{n=0}^{\infty} A^n \left( \frac{1}{\lambda} \right)^n \right\|$$

diverges.

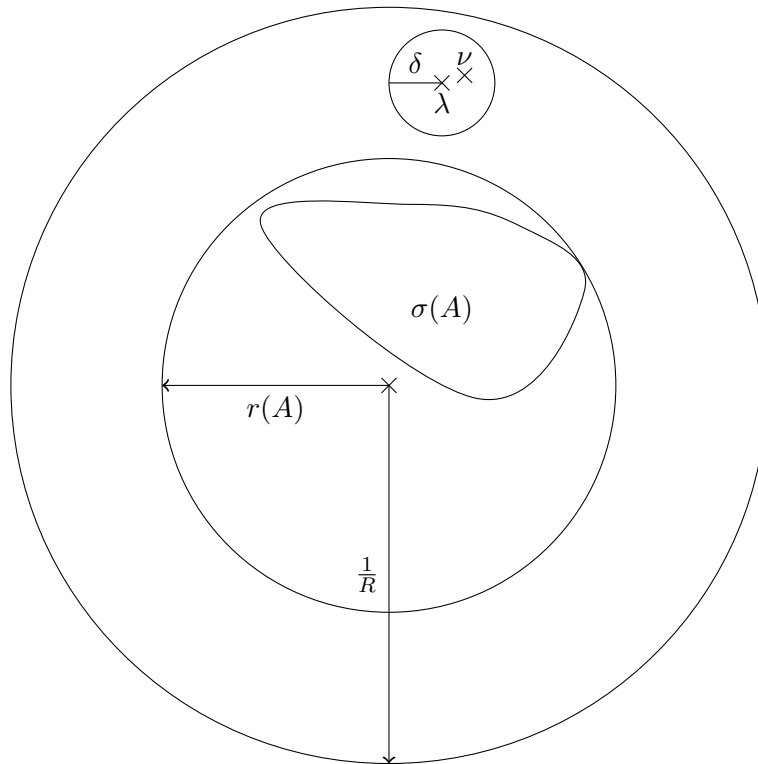


Figure 5.2:  $\frac{1}{R} > r(A)$  ?

Why is  $r$  not smaller than  $\frac{1}{R}$ ?

Assume that  $r < \frac{1}{R}$  and choose  $\lambda$  with  $r < |\lambda| < \frac{1}{R}$ . Then  $\mathcal{R}_\lambda$  exists and is analytic. Consider a  $\nu \in B_\delta(\lambda)$ .

$$\begin{aligned} \mathcal{R}_\nu &= (\nu - A)^{-1} = ((\nu - \lambda) + (\lambda - A))^{-1} = \\ &= (((\nu - \lambda) \mathcal{R}_\lambda + \mathbb{1})(\lambda - A))^{-1} = \\ &= \mathcal{R}_\lambda (\mathbb{1} + (\nu - \lambda) \mathcal{R}_\lambda)^{-1} = \\ &= \mathcal{R}_\lambda \sum_{n=0}^{\infty} (-(\nu - \lambda))^n \mathcal{R}_\lambda^n \end{aligned}$$

For  $|\nu - \lambda| < \delta := \frac{1}{\|\mathcal{R}_\lambda\|}$  the Neumann series converges.

Thus  $\mathcal{R}_\lambda$  can be expanded locally in a power series, i.e.  $\mathcal{R}_\lambda$  is complex analytic or holomorphic.

Furthermore for  $|\lambda| > \frac{1}{R}$  holds:

$$\mathcal{R}_\lambda = \sum_{n=0}^{\infty} A^n \frac{1}{\lambda^{n+1}}$$

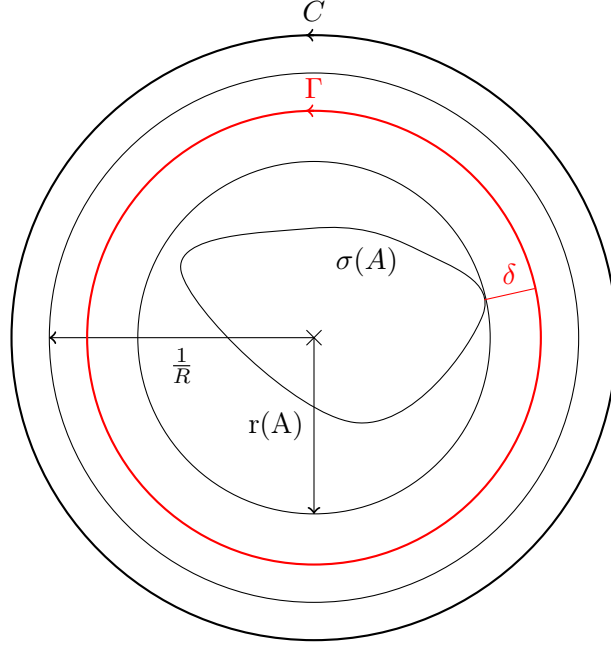


Figure 5.3: Contours  $\Gamma$  and  $C$  for integration

Integrate along the contour  $C$ :

$$\frac{1}{2\pi i} \oint_C \lambda^n \mathcal{R}_\lambda d\lambda = \sum_{k=0}^{\infty} A^k \underbrace{\frac{1}{2\pi i} \oint_C \frac{\lambda^n}{\lambda^{k+1}} d\lambda}_{=: I}$$

Since the geometric series converges absolutely, the summation and the integration can be interchanged. The residue theorem gives:

$$I = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Therefore we get:

$$\frac{1}{2\pi i} \oint_C \lambda^n \mathcal{R}_\lambda d\lambda = A^n$$

Choose  $\Gamma = \partial B_{r+\delta}(0)$ . We know, that  $\mathcal{R}_\lambda$  is holomorphic outside  $\Gamma$ . Thus we may continuously deform the contour to obtain:

$$\frac{1}{2\pi i} \oint_\Gamma \lambda^n \mathcal{R}_\lambda d\lambda = A^n$$

Thus we have:

$$\|A^n\| = \left\| \frac{1}{2\pi i} \oint_\Gamma \lambda^n \mathcal{R}_\lambda d\lambda \right\| \leq C (r + \delta)^n (r + \delta)$$

$$C := \frac{1}{2\pi} \sup_{\lambda \in I} \|\mathcal{R}_\lambda\|$$

$$\Rightarrow \quad \|A^n\|^{\frac{1}{n}} \leq (r + \delta) \left( C^{\frac{1}{n}} (r + \delta)^{\frac{1}{n}} \right) \xrightarrow{n \rightarrow \infty} r + \delta$$

Therefore:

$$\limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} \leq r + \delta$$

Since  $\delta$  is arbitrary, it follows that:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} = r$$

We even conclude:

$$\|A^n\|^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} r(A)$$

Assume that  $A$  is *symmetric* (to show  $\|A^n\|^{\frac{1}{n}} = \|A\|$ ). The Schwarz inequality gives:

$$\|A^2\| \leq \|A\| \cdot \|A\| = \|A\|^2$$

$$\|A\|^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \langle u, Au^2 \rangle \leq \sup_{\|u\|=1} \underbrace{\|u\|}_{=1} \cdot \|A^2 u\|$$

Iteratively for  $n \in \mathbb{N}$ :

$$\|A^{2^n}\| = \|A\|^{2^n}$$

For arbitrary  $m \in \mathbb{N}$  the Schwarz inequality gives:

$$\|A^m\| \leq \|A\|^m$$

Choose  $n$  such that  $2^n > m$ . Then:

$$\begin{aligned} \|A\|^{2^n} &= \|A^{2^n}\| = \|A^m \cdot A^{2^n-m}\| \leq \|A^m\| \cdot \|A\|^{2^n-m} \\ \Rightarrow \quad \|A\|^m &\leq \|A\|^m \end{aligned}$$

□<sub>5.12</sub>

### 5.13 Ritz method

Let  $A \in L(H)$  be a symmetric compact operator on the separable Hilbert space  $H$ . From the Hilbert-Schmidt theorem 5.10 we know that there exists an orthonormal eigenvalue basis  $(u_n)$  of  $H$ .

$$Au_n = \lambda_n u_n$$

We now want to construct the  $u_n$ :



Consider the “expectation value” functional:

$$\begin{aligned} S : H &\rightarrow \mathbb{R} \\ u &\mapsto \langle u, Au \rangle \end{aligned}$$

This is well defined, since:

$$\overline{S(u)} = \overline{\langle u, Au \rangle} = \langle Au, u \rangle = \langle u, Au \rangle = S(u)$$

$S$  is bounded, because:

$$|S(u)| = |\langle u, Au \rangle| \leq \|A\| \cdot \|u\|^2 \stackrel{\|u\| \leq 1}{\leq} \|A\|$$

Maximize  $|S(u)|$  on  $\{u \in H \mid \|u\| = 1\}$ :

Choose a maximizing sequence  $(u_n)$  with  $\|u_n\| = 1$  and:

$$|S(u_n)| \xrightarrow{n \rightarrow \infty} \sup_{\|u\|=1} |S(u)|$$

Since  $\overline{B_1(0)}$  is weakly compact, there is a subsequence  $u_{k_l}$ , which converges weakly  $u_{k_l} \rightharpoonup u$ . Since  $A$  is compact, the sequence

$$v_{k_l} := Au_{k_l} \rightarrow v$$

converges and  $Au = v$ . As a consequence:

$$S(u_{k_l}) = \langle u_{k_l}, Au_{k_l} \rangle = \langle u_{k_l}, v_{k_l} \rangle = \underbrace{\langle u_{k_l}, v \rangle}_{\rightarrow \langle u, v \rangle} + \langle u_{k_l}, v_{k_l} - v \rangle \xrightarrow{l \rightarrow \infty} \langle u, v \rangle = \langle u, Au \rangle = S(u)$$

This follows, because:

$$|\langle u_{k_l}, v_{k_l} - v \rangle| \leq \underbrace{\|u_{k_l}\|}_{=1} \cdot \underbrace{\|v_{k_l} - v\|}_{\rightarrow 0} \xrightarrow{l \rightarrow \infty} 0$$

Thus  $S$  is weakly continuous, i.e. for any  $u_k \rightharpoonup u$  converges  $S(u_k) \rightarrow S(u)$ .

Because  $(u_n)$  is a maximizing sequence, we get:

$$|S(u)| = \sup_{\|\tilde{u}\|=1} |S(\tilde{u})|$$

Therefore  $u$  is the desired maximizer.

–  $u$  is on the unit sphere:

The simple approach

$$\|u\|^2 \neq \lim_{l \rightarrow \infty} \|u_{k_l}\|^2$$

does not work, because  $u_{k_l}$  only converges weakly.

Example:

If  $(e_l)$  is an orthonormal Hilbert basis in a separable Hilbert space, then  $e_l \rightharpoonup 0$ , but:

$$\lim_{l \rightarrow \infty} \|e_l\| = 1 \neq 0 = \|0\|$$

But it holds:

$$\begin{aligned}\|u\|^2 &= \lim_{l \rightarrow \infty} |\langle u, u_{k_l} \rangle| \leq \lim_{l \rightarrow \infty} \|u_{k_l}\| \cdot \|u\| = \|u\| \\ \Rightarrow \|u\| &\leq 1\end{aligned}$$

Assume  $\|u\| < 1$ , then the vector  $\hat{u} := \frac{u}{\|u\|}$  would satisfy the equation:

$$|S(\hat{u})| = |\langle \hat{u}, A\hat{u} \rangle| = \frac{1}{\|u\|^2} |\langle u, Au \rangle| = \frac{1}{\|u\|^2} \sup_{\|v\|=1} |S(v)| \stackrel{\|u\|<1}{>} \sup_{\|v\|=1} |S(v)|$$

This is a contradiction. Therefore  $u$  is in fact a unit vector.

- $u$  is an eigenvector corresponding to the eigenvalue  $\lambda = \langle u, Au \rangle \in \mathbb{R}$ : Consider the variation for  $v \in H$ :

$$\tilde{u}(\tau) = u + \tau v$$

$$S\left(\frac{\tilde{u}}{\|\tilde{u}\|}\right) = \frac{\langle \tilde{u}, A\tilde{u} \rangle}{\langle \tilde{u}, \tilde{u} \rangle} = \frac{\langle u + \tau v, A(u + \tau v) \rangle}{\langle u + \tau v, u + \tau v \rangle}$$

This is called *Rayleigh quotient*. We know that  $S(\tilde{u}(\tau))$  is extremal at  $\tau = 0$ :

$$\begin{aligned}0 &= \left. \frac{d}{d\tau} S(\tilde{u}(\tau)) \right|_{\tau=0} = \\ &= \frac{\langle u, Av \rangle + \langle v, Au \rangle + 2\tau \langle v, v \rangle}{\langle u + \tau v, u + \tau v \rangle} - \frac{\langle u + \tau v, A(u + \tau v) \rangle}{\langle u + \tau v, u + \tau v \rangle^2} \cdot (\langle v, u \rangle + \langle u, v \rangle + \tau \langle v, v \rangle) \Big|_{\tau=0} = \\ &\stackrel{A \text{ symmetric}}{=} 2 \frac{\operatorname{Re}(\langle v, Au \rangle)}{\langle u, u \rangle} - 2 \operatorname{Re}(\langle v, u \rangle) \frac{\langle u, Au \rangle}{\langle u, u \rangle^2} = \\ &\stackrel{\lambda = \frac{\langle u, Au \rangle}{\langle u, u \rangle} = 1}{=} 2 (\operatorname{Re}(\langle v, Au \rangle) - \lambda \operatorname{Re}(\langle v, u \rangle)) = 2 \operatorname{Re}(\langle v, (A - \lambda)u \rangle)\end{aligned}$$

Set  $v = e^{i\varphi}w$  for any  $\varphi \in \mathbb{R}$  and  $w \in H$ . So:

$$0 = \operatorname{Re}(\langle v, (A - \lambda)u \rangle) = \operatorname{Re}\left(e^{-i\varphi} \langle w, (A - \lambda)u \rangle\right) \quad \forall \varphi \in \mathbb{R}$$

$$\Rightarrow \langle w, (A - \lambda)u \rangle = 0 \quad \forall w \in H$$

$$\begin{aligned}(A - \lambda)u &= 0 \\ Au &= \lambda u\end{aligned}$$

- It holds  $|\lambda| = \|A\|$ :

There is no point  $\nu$  in the spectrum of  $A$  with  $|\nu| > |\lambda|$ , because otherwise for all  $v \in H$  with  $Av = \nu v$  follows:

$$\frac{|\langle v, Av \rangle|}{\langle v, v \rangle} = |\nu| > |\lambda| = |\langle u, Au \rangle| = \sup_{w \in H} \frac{|\langle w, Aw \rangle|}{\langle w, w \rangle}$$

This is a contradiction. Thus we get:

$$|\lambda| = \sup_{\nu \in \sigma(A)} |\nu| \stackrel{\text{by definition}}{=} r(A) \stackrel{5.12}{=} \|A\|$$

Thus we have *constructed* a  $u \in H$  with  $\|u\| = 1$ ,  $Au = \lambda u$  and  $|\lambda| = \|A\|$ . Now one can proceed inductively:

$$H_1 := \langle u \rangle^\perp$$

$$A|_{H_1} : H_1 \rightarrow H_1$$

(We saw that  $H_1$  is invariant under  $A$ .)

Repeat the above procedure to maximize  $|\langle v, Av \rangle|$  on  $H_1 \cap \{v \in H \mid \|v\| = 1\}$ . This gives  $u_1$  with  $\|u_1\| = 1$ ,  $Au_1 = \lambda_1 u_1$  and:

$$|\lambda_1| = \|A|_{H_1}\| \leq \|A\| = |\lambda|$$

Now set  $H_2 = \langle u, u_1 \rangle^\perp$  and proceed inductively.

This gives a sequence  $u_0 := u, u_1, u_2, \dots$  of orthonormal eigenvectors, i.e.  $Au_j = \lambda_j u_j$ , with decreasing eigenvalues  $|\lambda_j|$ .

These  $(u_j)$  are an orthonormal basis. (Proof as in Theorem 5.10)

□<sub>5.13</sub>

Ritz, Galerkin: Finite element method

Example: Helium molecule wave function in  $H = L^2(\mathbb{R}^3, \mathbb{C})$

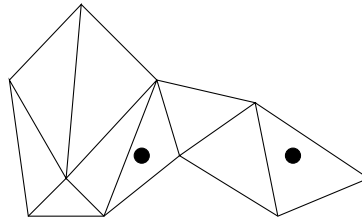


Figure 5.4: finite lattice for numerical approximation

$$A = -\frac{\hbar^2}{2m} \Delta - \frac{ze^2}{\|x - x_1\|} - \frac{ze^2}{\|x - x_2\|}$$

Now minimize

$$\frac{\langle u, Au \rangle}{\langle u, u \rangle}$$

on a finite subspace of  $H$ .

## 6 A few (technical) results

### 6.1 Dini's theorem

Let  $E$  be a metric space and  $f_n : E \rightarrow \mathbb{R}$  a sequence of real valued functions.

#### 6.1.1 Definition (point-wise/uniform convergence)

$f_n$  converges point-wise to  $f$  if  $f_n(x) \rightarrow f(x)$  converges for all  $x \in E$ , i.e.:

$$\forall_{x \in E} \quad \forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{N(\varepsilon, x)} \quad \forall_{n \in \mathbb{N}_{\geq N}} : |f_n(x) - f(x)| < \varepsilon$$

$f_n$  converges uniformly to  $f$ , in symbols  $f_n \rightrightarrows f$ , if for all  $\varepsilon \in \mathbb{R}_{>0}$  exists a  $N(\varepsilon)$  such that for all  $n \geq N$  and all  $x \in E$  holds:

$$|f_n(x) - f(x)| < \varepsilon$$

With quantifiers this is:

$$\forall_{\varepsilon \in \mathbb{R}_{>0}} \quad \exists_{N(\varepsilon)} \quad \forall_{n \in \mathbb{N}_{\geq N}} \quad \forall_{x \in E} : |f_n(x) - f(x)| < \varepsilon$$

#### 6.1.2 Theorem

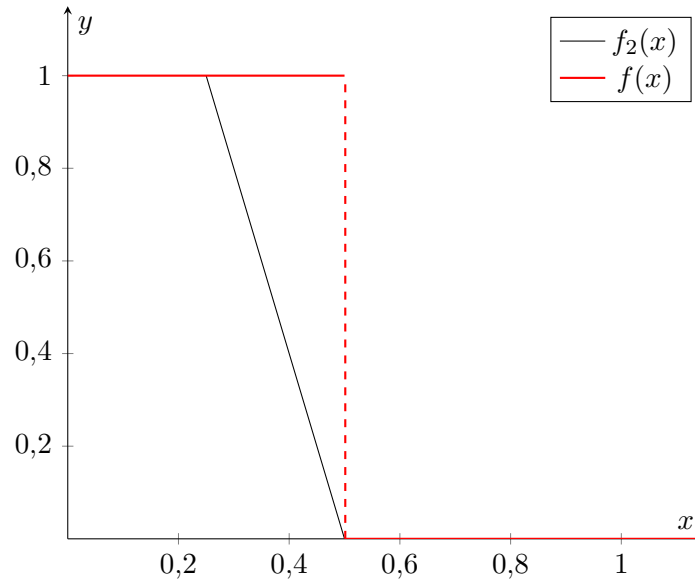
If  $(f_n)$  is a sequence of continuous functions with  $f_n \rightrightarrows f$ , then  $f$  is also continuous. This is not true in general for point wise convergence:

$$f_n(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2} \left(1 - \frac{1}{n}\right) \\ 0 & \text{for } x \geq \frac{1}{2} \\ n(1 - 2x) & \text{for } \frac{1}{2} \left(1 - \frac{1}{n}\right) < x < \frac{1}{2} \end{cases}$$

$f_n \rightarrow f$  converges pointwise to:

$$f(x) = \begin{cases} 1 & x < \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{cases}$$

This  $f$  is *not* continuous.

Figure 6.1:  $f_n(x)$  is continuous, but not  $f(x)$ **Proof**

Show that for all  $x \in E$  the  $\varepsilon$ - $\delta$ -criterion is satisfied:

Since  $f_n \rightrightarrows f$  converges uniformly, there is a  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}_{\geq N}$  and all  $x \in E$  holds:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

Because the  $f_n$  are continuous, there exists a  $\delta \in \mathbb{R}_{>0}$  such that for all  $y \in B_\delta(x)$  holds:

$$|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$$

Now follows for all  $y \in B_\delta(x)$ :

$$|f(y) - f(x)| \leq \underbrace{|f(y) - f_N(y)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(y) - f_N(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f(x)|}_{< \frac{\varepsilon}{3}} < \varepsilon$$

Therefore  $f$  is continuous. □<sub>6.1.2</sub>

**6.1.3 Definition** (monotonically increasing/decreasing)

The sequence of functions  $(f_n)$ ,  $f_n : E \rightarrow \mathbb{R}$  is called *monotonically increasing (decreasing)* if for all  $x \in E$  the real sequence  $f_n(x)$  is monotonically increasing (decreasing).

**6.1.4 Theorem** (Dini)

Let  $E$  be a *compact* metric space,  $(f_n)$  monotone and  $f_n \rightarrow f$ .

If  $f_n$  and  $f$  are continuous, then the convergence  $f_n \rightrightarrows f$  is uniform.

**Proof**

Without loss of generality we assume  $(f_n)$  is a monotonically increasing sequence (otherwise consider  $-f_n$ ), i.e.  $f_n(x) \leq f_{n+1}(x)$  for all  $x \in E$  and all  $n \in \mathbb{N}$ .

Given  $\varepsilon > 0$  we want to show:

$$\exists_{N \in \mathbb{N}} \forall_{x \in E} \forall_{n \in \mathbb{N}_{\geq N}} : |f(x) - f_n(x)| < \varepsilon$$

For any  $x \in E$  there exists an  $N(x)$  such that  $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$  for all  $n \in \mathbb{N}_{\geq N}$  (point-wise convergence). Since both  $f_{N(x)}$  and  $f$  are continuous functions, there exists a neighborhood  $U(x) = B_{\delta(x)}(x)$  of  $x$  such that for all  $z \in U(x)$  holds:

$$\begin{aligned} |f_{N(x)}(z) - f_{N(x)}(x)| &\leq \frac{\varepsilon}{4} \\ |f(z) - f(x)| &\leq \frac{\varepsilon}{4} \end{aligned}$$

Then follows:

$$|f_{N(x)}(z) - f(z)| \leq \underbrace{|f_{N(x)}(z) - f_{N(x)}(x)|}_{\leq \frac{\varepsilon}{4}} + \underbrace{|f_{N(x)}(x) - f(x)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f(x) - f(z)|}_{\leq \frac{\varepsilon}{4}} < \varepsilon$$

Since  $f_n(z)$  is monotonically increasing, it follows that  $|f_n(z) - f(z)| < \varepsilon$  for all  $z \in B_{\delta(x)}(x)$ . Now use a standard compactness argument: Since  $E$  is compact, it can be covered by a finite number of these balls  $B_{\delta(x_1)}(x_1), \dots, B_{\delta(x_n)}(x_n)$ . Define:

$$N = \max\{N(x_1), \dots, N(x_n)\}$$

So for all  $n \in \mathbb{N}_{\geq N}$  holds:

$$|f_n(x) - f(x)| < \varepsilon$$

□<sub>6.1.4</sub>

**6.2 Stone-Weierstraß theorem**

We follow the nice (since constructive) proof by Bernstein.

**6.2.1 Definition** (polynomials)

Let  $E = C^0([0,1])$  be the Banach space of real valued functions with norm:

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

$\mathcal{P}([0,1])$  are the *real polynomials*, i.e. for  $f \in \mathcal{P}([0,1])$  there are  $a_j \in \mathbb{R}$  such that:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

Clearly  $\mathcal{P}([0,1]) \subseteq C^0([0,1])$  forms a subspace.

We want to show that  $\mathcal{P}([0,1])$  is dense in  $C^0([0,1])$ .

**6.2.2 Lemma**

For  $x \in [0,1]$  holds:

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 1$$

**Proof**

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = (x + 1 - x)^n = 1$$

□<sub>6.2.2</sub>

**6.2.3 Lemma**

For  $x \in [0,1]$  holds:

$$\sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$$

Obviously holds

$$(nx - k)^2 \leq 4n^2$$

and therefore:

$$\sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} \leq 4n^2 \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 4n^2$$

**Proof**

It holds:

$$\begin{aligned} \sum_{k=0}^n k \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^n k \frac{n!}{k! (n-k)!} x^k (1-x)^{n-k} = \\ &= 0 + \sum_{k=1}^n \frac{n \cdot (n-1)!}{(k-1)! (n-k)!} x^k (1-x)^{n-k} = \\ &= n \sum_{k=1}^n \binom{n-1}{k-1} x^k (1-x)^{n-k} = \\ &\stackrel{j:=k-1}{=} n \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1} (1-x)^{n-j-1} = \\ &= nx \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{(n-1)-j} = nx (x + 1 - x)^{n-1} = nx \end{aligned}$$

Similarly one gets:

$$\sum_{k=0}^n k(k-1) \binom{n}{k} x^k (1-x)^{n-k} = n(n-1) \sum_{k=2}^n \binom{n-2}{k-2} x^k (1-x)^{n-k} = n(n-1) x^2$$

Together this gives:

$$\begin{aligned} \sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^n (n^2 x^2 - 2n x k + k^2) \binom{n}{k} x^k (1-x)^{n-k} = \\ &= \sum_{k=0}^n (n^2 x^2 - 2n x k + k(k-1) + k) \binom{n}{k} x^k (1-x)^{n-k} = \\ &= n^2 x^2 - 2n x \cdot nx + n(n-1) x^2 + nx = \\ &= -n^2 x^2 + n^2 x^2 - nx^2 + nx = nx(1-x) \end{aligned}$$

□<sub>6.2.3</sub>

A more elegant method is to use derivatives:

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} &= (x+y)^n \\ \sum_{k=0}^n k \binom{n}{k} x^k y^{n-k} &= x \cdot \frac{d}{dx} \left( \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \right) \\ \sum_{k=0}^n k^2 \binom{n}{k} x^k y^{n-k} &= \left( x \cdot \frac{d}{dx} \right)^2 \left( \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \right) \end{aligned}$$

#### 6.2.4 Definition

For  $f \in C^0([0,1])$  define:

$$B_n f(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

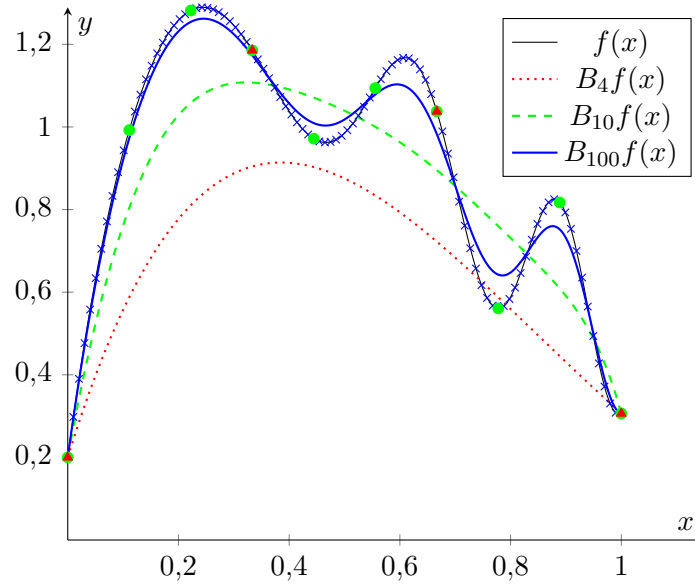
#### 6.2.5 Theorem (Bernstein)

For any  $f \in C^0([0,1], \mathbb{R})$ ,  $B_n f \rightrightarrows f$  converges uniformly.

Example:  $f(x) = 10x \cdot e^{-3x} + \frac{1}{5} \cos((4x)^2)$

$$\begin{aligned} B_4 f(x) &\approx 0,2 \cdot (1-x)^4 + 5,2 \cdot x \cdot (1-x)^3 + 5,9 \cdot x^2 \cdot (1-x)^2 + 2,4 \cdot x^3 \cdot (1-x) + 0,3 \cdot x^4 \\ B_{10} f(x) &\approx 0,2 \cdot (1-x)^{10} + 9,4 \cdot x \cdot (1-x)^9 + 56,6 \cdot x^2 \cdot (1-x)^8 + \\ &\quad + 149,5 \cdot x^3 \cdot (1-x)^7 + 217,9 \cdot x^4 \cdot (1-x)^6 + 248,2 \cdot x^5 \cdot (1-x)^5 + \\ &\quad + 244,7 \cdot x^6 \cdot (1-x)^4 + 103,2 \cdot x^7 \cdot (1-x)^3 + 26,5 \cdot x^8 \cdot (1-x)^2 + \\ &\quad + 7,9 \cdot x^9 \cdot (1-x) + 0,3 \cdot x^{10} \end{aligned}$$



Figure 6.2: Approximation of  $f(x)$  by  $B_n f(x)$ **Proof**

Without loss of generality assume  $f \neq 0$  (otherwise  $B_n f = 0 = f$ ).

$$M := \|f\| > 0$$

Consider an arbitrary  $\varepsilon \in \mathbb{R}_{>0}$ .  $f$  is continuous on the compact set  $[0,1]$  and thus uniformly continuous, i.e. there exists a  $\delta \in \mathbb{R}_{>0}$  such that:

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2}$$

Choose  $N \ni N \geq \frac{M}{\varepsilon \delta^2}$ .

**Claim:**  $|B_n f(x) - f(x)| < \varepsilon$  for all  $x \in [0,1]$  and all  $n \geq N$ .

**Proof:** It holds:

$$f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$B_n f(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$$(B_n f - f)(x) = \sum_{k=0}^n \left( f\left(\frac{k}{n}\right) - f(x) \right) \binom{n}{k} x^k (1-x)^{n-k}$$

Define:

$$A := \left\{ k \left| \left| \frac{k}{n} - x \right| < \delta \right. \right\} \qquad B := \left\{ k \left| \left| \frac{k}{n} - x \right| \geq \delta \right. \right\}$$

We have:

$$\begin{aligned}
\sum_{k \in A} \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{< \frac{\varepsilon}{2}} \binom{n}{k} x^k (1-x)^{n-k} &< \frac{\varepsilon}{2} \sum_{k \in A} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{\varepsilon}{2} \\
\sum_{k \in B} \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{\leq 2\|f\|=2M} \binom{n}{k} x^k (1-x)^{n-k} &\leq \\
&\leq 2M \sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k} \leq \\
&\stackrel{k \in B}{\leq} \frac{2M}{n^2 \delta^2} \sum_{k=0}^n \underbrace{(k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}}_{\leq \frac{n}{4}} \leq \\
&\stackrel{n \geq N}{\leq} \frac{M}{2n\delta^2} \leq \frac{M}{2\frac{M}{\varepsilon\delta^2}\delta^2} = \frac{\varepsilon}{2}
\end{aligned}$$

Therefore holds for all  $x \in [0,1]$ .

$$|B_n f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□ Claim

Therefore  $B_n f \Rightarrow f$  converges uniformly.

□ 6.2.5

Now generalize: Let  $E$  be a compact metric space.  $C^0(E, \mathbb{R})$  with

$$\|f\| = \sup_{x \in E} |f(x)|$$

is a Banach space. Moreover, it is an algebra with the point-wise multiplication:

$$(f \cdot g)(x) := f(x) \cdot g(x)$$

The multiplication is continuous:

$$\|f \cdot g\| \leq \|f\| \cdot \|g\|$$

In summary  $(C^0(E, \mathbb{R}), \|\cdot\|, +, \cdot)$  is a *Banach algebra*.

### 6.2.6 Theorem (Weierstraß)

The polynomials are dense in  $C^0([0,1], \mathbb{R})$ .

#### Proof

For any  $f \in C^0([0,1], \mathbb{R})$ ,  $B_n f \Rightarrow f$  converges uniformly and since the  $B_n f$  are polynomials, these are dense. □ 6.2.6

**6.2.7 Theorem** (Stone-Weierstraß)

Let  $\mathcal{A} \subseteq C^0(E, \mathbb{R})$  be a subalgebra with the following properties:

1.  $\mathcal{A}$  contains  $f = 1$  and so by scalar multiplication all the constant functions.
2.  $\mathcal{A}$  separates the points of  $E$ , i.e. for all  $x, y \in E$  with  $x \neq y$  there exists a  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .

Then  $\mathcal{A}$  is dense in  $C^0(E, \mathbb{R})$ .

**Proof**

- i) There is a sequence of polynomials  $u_n$  on  $[0, 1]$  such that  $u_n \rightrightarrows f$  with  $f(t) = \sqrt{t}$ . This follows immediately from theorem 6.2.6.
- ii) If  $f \in \mathcal{A}$ , then  $|f|$  defined by  $|f|(x) := |f(x)|$  is in the closure  $\overline{\mathcal{A}}$  of  $\mathcal{A}$ :  
For  $f \in \mathcal{A}$  define:

$$a := \|f\| = \max_{x \in E} |f(x)|$$

$$\Rightarrow \frac{f^2(x)}{a^2} \in [0, 1]$$

Then converges:

$$u_n \left( \frac{f^2(x)}{a^2} \right) \xrightarrow{n \rightarrow \infty} \sqrt{\frac{f^2(x)}{a^2}} = \frac{|f(x)|}{a}$$

The functions  $u_n \left( \frac{f^2}{a^2} \right)$  lie in  $\mathcal{A}$ , since these are a polynomials of  $f$  and thus again elements of the algebra  $\mathcal{A}$ . Moreover  $u_n \left( \frac{f^2}{a^2} \right)$  converges uniformly to  $\frac{|f|}{a}$ , because for a given  $\varepsilon \in \mathbb{R}_{>0}$  exists a  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}_{\geq N}$  and all  $t \in [0, 1]$  holds:

$$\left| u_n(t) - \sqrt{t} \right| < \varepsilon$$

Then follows with  $t = \frac{f^2(x)}{a^2}$ :

$$\left| u_n \left( \frac{f^2(x)}{a^2} \right) - \frac{|f|}{a} \right| < \varepsilon$$

Thus  $\frac{|f|}{a} \in \overline{\mathcal{A}}$  and therefore also  $|f| \in \overline{\mathcal{A}}$ .

- iii) For  $f, g \in \overline{\mathcal{A}}$  also  $\min(f, g)$  and  $\max(f, g)$  (defined point-wise) are again in  $\overline{\mathcal{A}}$ :

$$\min(f, g) = \frac{1}{2} (f + g - |f - g|)$$

$$\max(f, g) = \frac{1}{2} (f + g + |f - g|)$$

Choose  $f_n, g_n \in \mathcal{A}$  such that  $f_n \rightrightarrows f$  and  $g_n \rightrightarrows g$ . By ii) follows  $|f_n - g_n| \in \overline{\mathcal{A}}$  and  $|f_n - g_n| \rightrightarrows |f - g|$ . Therefore holds:

$$\overline{\mathcal{A}} \ni \min(f_n, g_n) \rightrightarrows \min(f, g) \in \overline{\mathcal{A}}$$

Similarly the claim follows for  $\max$ .

- iv) For all  $x, y \in E$  with  $x \neq y$  and  $\alpha, \beta \in \mathbb{R}$  exists a  $f \in \mathcal{A}$  such that  $f(x) = \alpha$  and  $f(y) = \beta$ :  
 For  $\alpha = \beta$  we choose  $f = \alpha$  as constant function.  
 For  $\alpha \neq \beta$  there exists, since  $\mathcal{A}$  separates points of  $E$ , a  $g \in \mathcal{A}$  with  $g(x) \neq g(y)$ . Set  $f = c_0 + c_1 g$  and choose:

$$\begin{aligned} \alpha &= c_0 + c_1 g(x) \\ \beta &= c_0 + c_1 g(y) \\ \Rightarrow c_1 &= \frac{\alpha - \beta}{g(x) - g(y)} \\ \Rightarrow c_0 &= \alpha - \frac{\alpha - \beta}{g(x) - g(y)} g(x) = \frac{\alpha g(x) - \alpha g(y) - \alpha g(x) + \beta g(x)}{g(x) - g(y)} = \\ &= \frac{\beta g(x) - \alpha g(y)}{g(x) - g(y)} \end{aligned}$$

- v) For all  $f \in C^0$ ,  $x \in E$  and  $\varepsilon \in \mathbb{R}_{>0}$  there is a  $g \in \overline{\mathcal{A}}$  such that

$$g(x) = f(x)$$

and for all  $y \in \overline{\mathcal{A}}$  holds:

$$g(y) \leq f(y) + \varepsilon$$

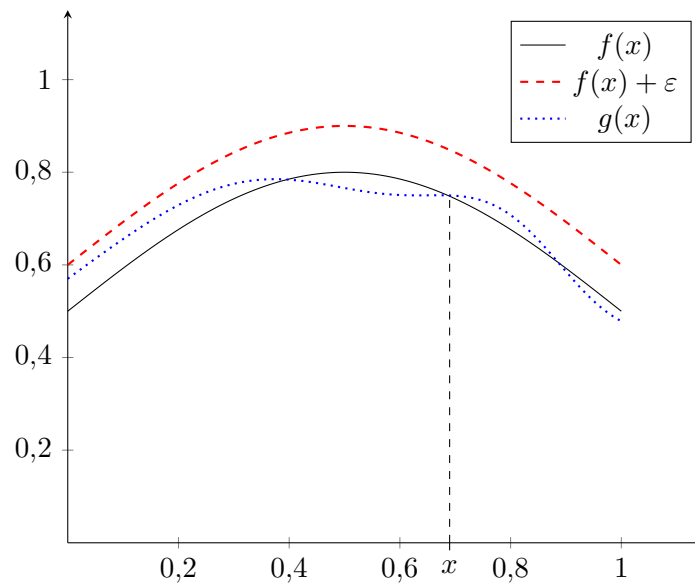
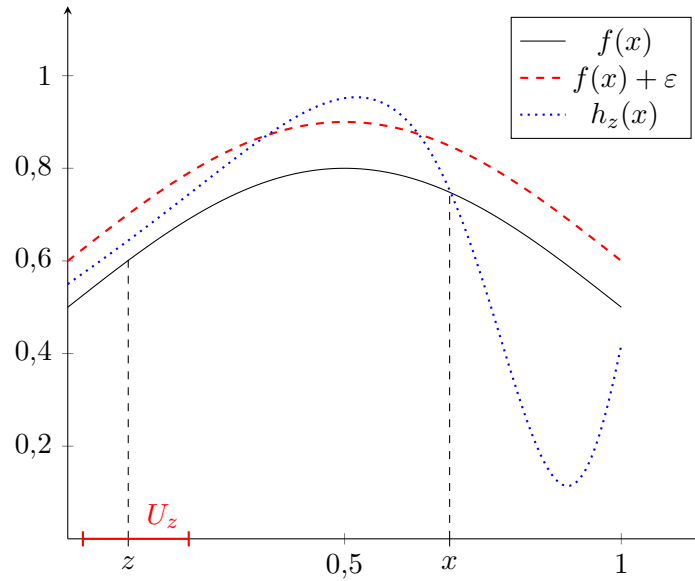


Figure 6.3:  $g(x) \leq f(x) + \varepsilon$

To show this, choose for any  $z \in E$  a  $h_z \in \overline{\mathcal{A}}$  with  $h_z(x) = f(x)$  and  $h_z(z) \leq f(z) + \frac{\varepsilon}{2}$ , which is possible after iv).

Since  $h_z$  is continuous, there is a neighborhood  $U_z$  of  $z$  such that  $h_z \leq f + \varepsilon$  on  $U_z$ .


 Figure 6.4:  $h_z \leq f + \varepsilon$  on  $U_z$ 

Since  $E$  is compact, we can cover it by a finite number of such neighborhoods  $U_{z_1}, \dots, U_{z_N}$ . Define:

$$g := \min \{h_{z_1}, \dots, h_{z_N}\} \in \overline{\mathcal{A}}$$

It holds  $g(x) = f(x)$ , because  $h_{z_i}(x) = f(x)$ . We also know:

$$g|_{U_j} \leq h_{z_j}|_{U_j} \leq f + \varepsilon$$

vi)  $\overline{\mathcal{A}} = C^0$ : Denote the function  $g$  constructed in step v) by  $g_x$ .

$$g_x(x) = f(x)$$

$$g_x \leq f + \varepsilon$$

By continuity of  $g_x$  there exists a neighborhood  $U_x$  of  $x$  such that  $g_x \geq f - \varepsilon$  on  $U_x$ . By compactness we can cover  $E$  by a finite number of such neighborhoods  $U_{x_1}, \dots, U_{x_k}$  and define:

$$g := \max \{g_{x_1}, \dots, g_{x_k}\}$$

Then follows:

$$f - \varepsilon \leq g \leq f + \varepsilon$$

$$\|f - g\| < \varepsilon$$

□<sub>6.2.7</sub>

Counterexample in the complex case:

$$E = [0,1] \times [0,1] \subseteq \mathbb{C}$$

Consider the set  $\mathcal{A} = \mathcal{P}(z)$  of polynomials in  $z$ .

- The constant functions are in  $\mathcal{A}$ .
- $\mathcal{A}$  separates points:  
If  $z_1 \neq z_2$  take  $f(z) = z$  then  $f(z_1) \neq f(z_2)$ .

$$\overline{\mathcal{A}} = ?$$

By Morera's theorem we get:

$$\overline{\mathcal{A}} = \left\{ f \in C^0([0,1]^2) \mid |f|_{(0,1)^2} \text{ is holomorphic} \right\} \neq C^0([0,1]^2)$$

For example  $f(x + iy) = x - iy$ . We have  $f \in C^0([0,1]^2)$ , but  $f \notin \overline{\mathcal{A}}$ .

### 6.2.8 Theorem (Stone-Weierstraß, complex version)

Let  $\mathcal{A} \subseteq C^0(E, \mathbb{C})$  be a subalgebra with the properties 1. and 2. from theorem 6.2.7 and additionally:

$$3. f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$$

Then  $\mathcal{A}$  is dense in  $C^0(E, \mathbb{C})$ .

#### Proof

Consider the algebras:

$$\begin{aligned} \operatorname{Re}(\mathcal{A}) &= \left\{ f + \bar{f} \mid f \in \mathcal{A} \right\} \subseteq \mathcal{A} \\ \operatorname{Im}(\mathcal{A}) &= \left\{ \frac{1}{i} (f - \bar{f}) \mid f \in \mathcal{A} \right\} \subseteq \mathcal{A} \end{aligned}$$

These are subalgebras of  $C^0(E, \mathbb{R})$ . By the real Stone-Weierstraß theorem we get:

$$\overline{\operatorname{Re}(\mathcal{A})} = \overline{\operatorname{Im}(\mathcal{A})} = C^0(E, \mathbb{R})$$

For given  $f \in C^0(E, \mathbb{C})$  approximate  $\operatorname{Re}(f)$  and  $\operatorname{Im}(f)$ .

□<sub>6.2.8</sub>

## 6.3 Arzelà-Ascoli theorem

Let  $K$  be a compact metric space and  $E$  a Banach space.

$C^0(K, E)$  is the Banach space of continuous functions  $f : K \rightarrow E$  with norm:

$$\|f\| := \sup_{x \in K} \|f(x)\|_E$$

Let  $\mathcal{F} \subseteq C^0(K, E)$  be a subset. Is  $\mathcal{F}$  compact?

### 6.3.1 Definition (relatively compact)

A subset  $A$  of a metric space is called *relatively compact*, if  $\overline{A}$  is compact.

**6.3.2 Definition** (equicontinuous)

A family  $\mathcal{F} \subseteq C^0(K, E)$  is called *equicontinuous* (gleichgradig stetig) if for all  $x \in K$  and all  $\varepsilon \in \mathbb{R}_{>0}$  there exists a  $\delta \in \mathbb{R}_{>0}$  such that for all  $y \in B_\delta(x)$  and for all  $f \in \mathcal{F}$  holds:

$$\|f(x) - f(y)\| < \varepsilon$$

(Thus  $\delta$  is independent of  $f \in \mathcal{F}$ .)

**6.3.3 Theorem** (Arzelà-Ascoli)

$\mathcal{F} \subseteq C^0(K, E)$  is relatively compact if and only if the following two conditions holds:

- i)  $\mathcal{F}$  is equicontinuous.
- ii) For every  $x \in K$  the set

$$\mathcal{F}(x) := \{f(x) \mid f \in \mathcal{F}\}$$

is relatively compact in  $E$ .

**Proof**

„ $\Rightarrow$ “: Assume that  $\mathcal{F} \subseteq C^0(K, E)$  is relatively compact.

- i) Assume that  $\mathcal{F}$  is *not* equicontinuous. Then there exists an  $\varepsilon \in \mathbb{R}_{>0}$  and sequences  $x_n \in K$ ,  $f_n \in \mathcal{F}$  and  $y_n \in B_{\frac{1}{n}}(x_n)$  such that:

$$\|f_n(x_n) - f_n(y_n)\| \geq \varepsilon$$

After choosing subsequences (with the same notation), we can arrange:

$$\begin{array}{lll} x_n \rightarrow x & y_n \rightarrow x & \text{(use that } K \text{ is compact)} \\ f_n \rightarrow f & & \text{(use that } \mathcal{F} \text{ is relatively compact)} \end{array}$$

This means that there is a  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}_{>N}$  holds for all  $y \in K$ :

$$\|f_n(y) - f(y)\| < \frac{\varepsilon}{3}$$

(Since convergence in  $C^0(K, E)$  is the same as uniform convergence  $f_n \rightrightarrows f$ .)

Since  $f$  is continuous there exists a  $\delta \in \mathbb{R}_{>0}$  such that for all  $y \in B_\delta(x)$ :

$$\|f(x) - f(y)\| < \frac{\varepsilon}{3}$$

With this we get:

$$\|f_n(x) - f_n(y)\| \leq \underbrace{\|f_n(x) - f(x)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|f(x) - f(y)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|f(y) - f_n(y)\|}_{< \frac{\varepsilon}{3}} < \varepsilon$$

This is a contradiction to  $\|f_n(x_n) - f_n(y_n)\| \geq \varepsilon$ .

□<sub>i)</sub>

- ii) Consider  $y_n \in \mathcal{F}(x) \subseteq E$  (to show that  $y_n$  has a convergent subsequence in  $E$ ).  
Then there are functions  $f_n \in \mathcal{F}$  with  $f_n(x) = y_n$ . Since  $\mathcal{F}$  is relatively compact, a subsequence is a Cauchy sequence in  $C^0(K, E)$ , i.e.  $\|f_{n_l} - f_{n_{l'}}\| \xrightarrow{l, l' \rightarrow \infty} 0$ .

$$\|f_{n_l} - f_{n_{l'}}\| = \sup_{z \in K} \|f_{n_l}(z) - f_{n_{l'}}(z)\|_E \geq \|f_{n_l}(x) - f_{n_{l'}}(x)\|_E = \|y_{n_l} - y_{n_{l'}}\|$$

Therefore we get+:

$$\|y_{n_l} - y_{n_{l'}}\| \xrightarrow{l, l' \rightarrow \infty} 0$$

Thus  $(y_{n_l})$  is a Cauchy sequence in  $E$ . □<sub>ii)</sub>

„ $\Leftarrow$ “: Let  $(f_l)$  be a sequence in  $\mathcal{F}$  and show that a subsequence  $(g_l)$  converges in  $C^0(K, E)$ :  
Since  $K$  is compact, there is a countable dense subset  $\{x_1, x_2, \dots\} \subseteq K$ . Since  $\mathcal{F}(x_1)$  is relatively compact, there is a subsequence  $f_l^{(1)} \in \mathcal{F}$  of  $(f_l)$  such that  $f_l^{(1)}(x_1)$  converges in  $E$ . Since  $\mathcal{F}(x_2)$  is relatively compact, there is a subsequence  $f_l^{(2)}$  of  $f_l^{(1)}$  such that  $f_l^{(2)}(x_2)$  converges. Inductively choose a subsequence  $(f_l^{(n+1)})$  of  $(f_l^{(n)})$  such that  $f_l^{(n+1)}(x_{n+1})$  converges in  $E$ . Take the diagonal sequence  $g_l := f_l^{(l)}$ . This is for  $l \geq n$  a subsequence of  $f_l^{(n)}$ , so for all  $n \in \mathbb{N}$  converges  $g_l(x_n) \xrightarrow{l \rightarrow \infty} y_n$ .

**Claim:**  $g_n$  is a Cauchy sequence in  $C^0(K, E)$ , i.e. for all  $\varepsilon \in \mathbb{R}_{>0}$  exists a  $N \in \mathbb{N}$  such that for all  $n, m \in \mathbb{N}_{>N}$  and all  $x \in K$  holds:

$$\|g_n(x) - g_m(x)\| \leq \varepsilon$$

**Proof:** Since  $\mathcal{F}$  is equicontinuous, for all  $x \in E$  exists a  $\delta \in \mathbb{R}_{>0}$  such that for all  $z, z' \in B_{\delta(x)}(x)$  and all  $f \in \mathcal{F}$  holds:

$$\|f(z) - f(z')\| < \frac{\varepsilon}{3}$$

We cover  $K$  by a finite number of such balls  $B_1, \dots, B_L$ . In every Ball  $B_l$  there is at least one point of  $\{x_1, x_2, \dots\}$ . We choose such a point  $\xi_l \in B_l$ . Since  $(g_n(\xi_l))$  converges for every  $l \in \{1, \dots, L\}$  we can choose a  $N \in \mathbb{N}$  such that for all  $l \in \{1, \dots, L\}$  and all  $m, n \in \mathbb{N}_{>N}$  holds:

$$\|g_n(\xi_l) - g_m(\xi_l)\| < \frac{\varepsilon}{3}$$

For every  $x \in K$  exists a  $l \in \{1, \dots, L\}$  with  $x \in B_l$ .

$$\|g_n(x) - g_m(x)\| \leq \underbrace{\|g_n(x) - g_n(\xi_l)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|g_n(\xi_l) - g_m(\xi_l)\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|g_m(\xi_l) - g_m(x)\|}_{< \frac{\varepsilon}{3}}$$

□<sub>Claim</sub>

Therefore the subsequence  $(g_l)$  for  $(f_l)$  converges in  $C^0(K, E)$ , since  $C^0(K, E)$  is complete, because  $E$  is a Banach space. □<sub>6.3.3</sub>

### Application to integral operators

Let  $K \subseteq \mathbb{R}^n$  be compact. Consider an integral operator  $A : C^0(K, \mathbb{R}) \rightarrow C^0(K, \mathbb{R})$ , i.e.:

$$(Af)(x) = \int_K A(x, y) f(y) d^n y$$

$\mathcal{F} := A(C^0(K, \mathbb{R}))$  is equicontinuous provided that  $A(., y)$  is continuous.



## 6.4 The Riesz representation theorem

Let  $K$  again be a compact metric space.  $E = C^0(K, \mathbb{R})$  with the sup-norm is a Banach space.

**Question:** What is  $E^*$ ?

Consider  $l \in E^*$ , i.e.

$$l : E \rightarrow \mathbb{R}$$

and for all  $f \in C^0(K)$  holds:

$$|l(f)| \leq C \|f\|$$

This means  $f$  is bounded or equivalently continuous.

### 6.4.1 Examples

Consider  $K = [0, 1] \subseteq \mathbb{R}$ . For any  $\varphi \in L^1([0, 1])$ , the functional

$$l(f) := \int_0^1 \varphi(x) f(x) dx$$

is linear and bounded:

$$|l(f)| \leq \int_0^1 |\varphi(x)| \cdot |f(x)| dx \leq \underbrace{\sup_{x \in [0, 1]} |f|}_{=\|f\|} \cdot \underbrace{\int_0^1 |\varphi(x)| dx}_{=\|\varphi\|_{L^1}}$$

It is convenient to identify  $l \in E^*$  with the function  $\varphi \in L^1$ . We have represented  $l$  by an  $L^1$ -function  $\varphi$ .

This can also be written as a *signed measure* (signiertes Maß):

$$d\mu := \varphi(x) dx$$

But not every  $l \in E^*$  can be represented in this form.

### Example

$$l(f) := f\left(\frac{1}{2}\right)$$

is bounded:

$$|l(f)| = \left| f\left(\frac{1}{2}\right) \right| \leq \sup_{[0, 1]} |f| = \|f\|$$

It can be represented by the Dirac measure:

$$l(f) = \int_0^1 f(x) \delta\left(x - \frac{1}{2}\right) dx = \int_0^1 f(x) d\mu$$

Here  $\delta(x)$  is the  $\delta$ -Distribution.  $\mu = \delta_{\frac{1}{2}}$  is the Dirac measure.

$$\delta_{x_0}(\Omega) = \begin{cases} 1 & \text{if } x_0 \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

**6.4.2 Definition** (bounded, positive, regular measure)

Let  $X \neq \emptyset$  be a set. A  $\sigma$ -algebra  $\mathcal{M}$  over  $X$  is a set of subsets of  $X$  such that holds:

- i)  $\emptyset \in \mathcal{M}$
- ii)  $A \in \mathcal{M} \Rightarrow \mathcal{C}A := X \setminus A \in \mathcal{M}$
- iii) For a countable family  $(A_j)_{j \in \mathbb{N}}$  holds:

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$$

The elements of  $\mathcal{M}$  are called *measurable sets* (messbare Mengen).

Let  $K$  be a compact metric space. Denote by  $\mathfrak{M}$  the *Borel algebra*, i.e. the smallest  $\sigma$ -algebra over  $K$ , which contains all open and therefore all closed subsets of  $K$ .

A *bounded (signed) measure* is a mapping

$$\mu : \mathfrak{M} \rightarrow \mathbb{R}$$

(not  $\mu : \mathfrak{M} \rightarrow \mathbb{R}^+ \cup \{0\} \cup \{\infty\}$  as before in measure theory) with the following properties:

- The empty set measures zero:

$$\mu(\emptyset) = 0$$

- $\sigma$ -additivity: For  $M_j \in \mathfrak{M}$  with  $M_i \cap M_j = \emptyset$  for all  $i \neq j$  holds:

$$\mu \left( \bigcup_{j=1}^{\infty} M_j \right) = \sum_{j=1}^{\infty} \mu(M_j)$$

$\mu$  is *positive*, if  $\mu(M) \geq 0$  for all  $M \in \mathfrak{M}$ .

$\mu$  is *regular*, if for all  $A \in \mathfrak{M}$  holds:

$$\mu(A) = \sup_{\substack{B \subseteq A \\ B \text{ compact}}} \mu(B) = \inf_{\substack{\Omega \supseteq A \\ \Omega \text{ open}}} \mu(\Omega)$$

**Example**

The Lebesgue measure  $d^n x$  restricted to the Borel algebra on  $[0,1]^n$  is a bounded, positive and regular measure.

**6.4.3 Theorem** (Riesz representation theorem)

Consider  $l \in C^0(K, \mathbb{R})^*$ . Then there is a unique bounded regular Borel measure  $\mu$  (i.e. a measure on the Borel algebra  $\mathfrak{M}$ ) such that for all  $f \in C^0(K, \mathbb{R})$  holds:

$$l(f) = \int_K f d\mu$$

Here we only prove the case  $K = [0,1]$ . (We also need it for  $K = [0,1]^2$ .)

How can one construct positive regular Borel measures on  $[0,1]$ ?

### Lebesgue-Stieltjes integral

Let  $\alpha : [0,1] \rightarrow \mathbb{R}$  be monotonically increasing (not necessarily continuous).

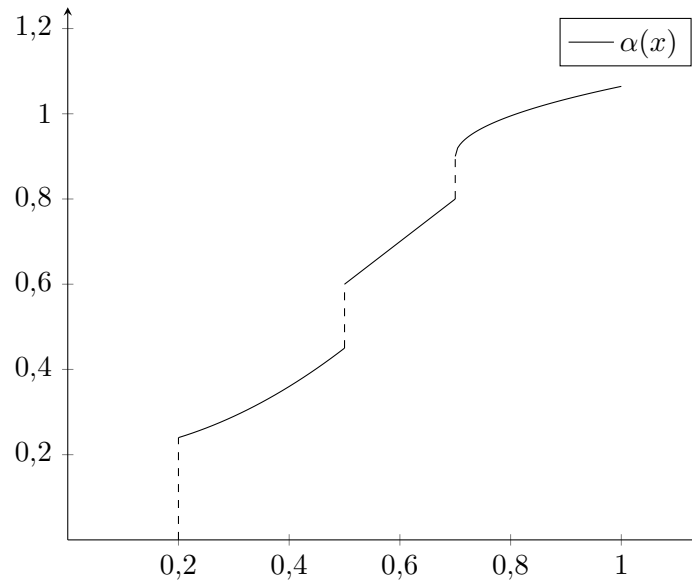


Figure 6.5:  $\alpha$  is monotonically increasing, but not continuous

The two one-sided limits

$$\lim_{x \nearrow x_0} \alpha(x), \quad \lim_{x \searrow x_0} \alpha(x)$$

exist. In general:

$$\lim_{x \nearrow x_0} \alpha(x) \leq \alpha(x_0) \leq \lim_{x \searrow x_0} \alpha(x)$$

But equality does not need to hold. Define:

$$\mu((a,b)) := \lim_{x \nearrow b} \alpha(x) - \lim_{x \searrow a} \alpha(x)$$

By  $\sigma$ -additivity, this measure can be extended to a positive regular bounded Borel measure. (This can be proven exactly as for the Lebesgue integral.) The corresponding integral

$$\int_0^1 f d\mu$$

is called Lebesgue-Stieltjes integral. If  $\alpha(x) = x + c$ , the Lebesgue-Stieltjes integral reduces to the Lebesgue integral

#### 6.4.4 Example

Let  $\alpha \in C^1([0,1])$  be monotonically increasing. Then holds:

$$\mu((a,b)) = \alpha(b) - \alpha(a) = \int_a^b \alpha'(x) dx = \int_0^1 \chi_{(a,b)} \alpha'(x) dx$$

The corresponding Lebesgue-Stieltjes integral is:

$$\int f d\mu = \int_0^1 f(x) \cdot \alpha'(x) dx$$

The following short notation is used in general:

$$\begin{aligned} d\mu &= \alpha'(x) dx \\ d\mu &= d\alpha \end{aligned}$$

If  $\alpha \in C^1([0,1])$  is not monotone, we can still set:

$$\int_0^1 f d\mu := \int_0^1 f \cdot \alpha'(x) dx$$

$d\mu$  is a signed measure.

In order to extend the Lebesgue-Stieltjes construction to functions  $\alpha$ , which are *not* monotone (such as to obtain signed measures), we need to assume, that  $\alpha$  has bounded variation.

#### 6.4.5 Definition (total variation)

Let  $\alpha : [0,1] \rightarrow \mathbb{R}$  be a function (not necessarily continuous).

The *total variation* (Totalvariation) is defined by:

$$(\text{TV}(\alpha))(x) := \sup_{\substack{N \in \mathbb{N} \\ 0=x_0 < \dots < x_N=x}} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

$\alpha$  is of *bounded variation* (beschränkte Totalvariation),  $\alpha \in \mathcal{BV}([0,1])$ , if  $(\text{TV}(f))(1) < \infty$ .

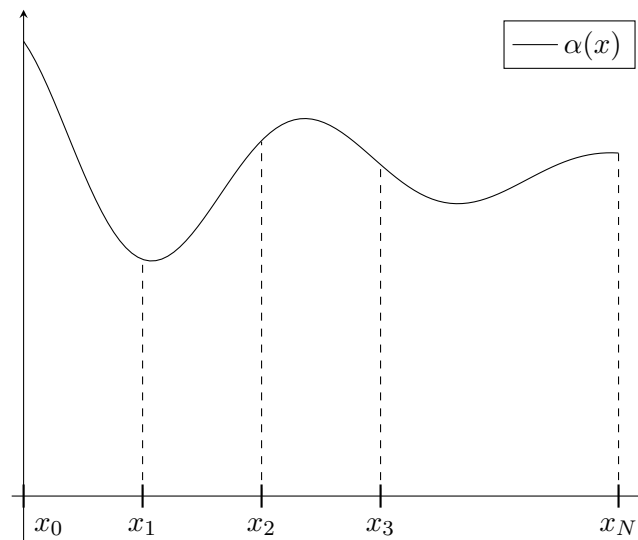


Figure 6.6: total variation of  $\alpha$

*Note:* If  $\alpha$  is monotonically increasing, then holds:

$$(\text{TV}(\alpha))(x) = \alpha(x) - \alpha(0) < \infty$$

Thus every monotonically function has bounded variation.

But there are even continuous functions, which have unbounded variation, e.g. for large enough  $p \in \mathbb{R}_{>0}$ :

$$\alpha(x) = x \sin\left(\frac{1}{x^p}\right)$$

For  $\alpha \in C^1([0,1])$  holds:

$$\text{TV}(\alpha)(x) = \int_0^x |\alpha'(\tau)| d\tau$$

**Lemma** (Properties of the total variation)

$\text{TV}(\alpha)(x)$  is monotonically increasing and:

$$\text{TV}(\alpha)(0) = 0$$

$\text{TV}(\alpha)(x) \pm \alpha(x)$  is also monotonically increasing.

**Proof**

Assume that  $y \in \mathbb{R}_{>x}$ .

$$\begin{aligned} \text{TV}(\alpha)(y) &= \sup_{\substack{N \in \mathbb{N} \\ 0=x_0 < \dots < x_N=y}} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| \geq \sup_{\substack{N \in \mathbb{N}_{\geq 2} \\ 0=x_0 < \dots < x_{N-1}=x < x_N=y}} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| \geq \\ &\geq \sup_{\substack{N \in \mathbb{N}_{\geq 2} \\ 0=x_0 < \dots < x_{N-1}=x < x_N=y}} \sum_{i=1}^{N-1} |\alpha(x_i) - \alpha(x_{i-1})| = \text{TV}(\alpha)(x) \end{aligned}$$

$$\text{TV}(\alpha)(x) \pm \alpha(x) = \pm \alpha(0) + \sup_{\substack{N \in \mathbb{N} \\ 0=x_0 < \dots < x_N=x}} \sum_{i=1}^N \underbrace{|\alpha(x_i) - \alpha(x_{i-1})| \pm (\alpha(x_i) - \alpha(x_{i-1}))}_{\geq 0}$$

Just as before this implies that

$$\text{TV}(\alpha)(x) \pm \alpha(x)$$

is monotonically increasing.

□<sub>6.4.5</sub>

Suppose that  $f \in \mathcal{BV}([0,1])$ . Then the functions

$$\begin{aligned} f_+ &= \frac{1}{2} (\text{TV}(f) + f) \\ f_- &= \frac{1}{2} (\text{TV}(f) - f) \end{aligned}$$

are monotonically increasing and:

$$f = f_+ - f_-$$

Let  $d\mu_{\pm} = df_{\pm}$  be the bounded positive regular Borel measures of the corresponding Lebesgue-Stieltjes integrals. Then

$$\mu := \mu_+ - \mu_-$$

defines a bounded regular Borel measure with the property:

$$\begin{aligned} \mu((a,b)) &= \mu_+((a,b)) - \mu_-((a,b)) = \lim_{x \nearrow b} f_+(x) - \lim_{x \searrow a} f_+(x) - \lim_{x \nearrow b} f_-(x) + \lim_{x \searrow a} f_-(x) = \\ &= \lim_{x \nearrow b} f(x) - \lim_{x \searrow a} f(x) \end{aligned}$$

### 6.4.6 Example

Consider the Heaviside function:

$$f := \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

$d\mu := df$  has the form  $\mu = \delta_{\frac{1}{2}}$ .

#### Proof of Theorem 6.4.3 in the case $K = [0,1]$

$\mathcal{PC}([0,1])$  are the piecewise continuous functions, i.e. for all  $f \in \mathcal{PC}([0,1])$  exists a  $N \in \mathbb{N}$  and points  $0 = x_0 < \dots < x_N = 1$  such that  $f|_{(x_{i-1}, x_i)}$  is continuous and has a continuous continuation to  $[x_{i-1}, x_i]$  for all  $i \in \{1, \dots, N\}$ .  
On  $\mathcal{PC}$  we introduce the norm:

$$\|f\| = \sup_{x \in [0,1]} |f(x)|$$

This makes  $\mathcal{PC}([0,1])$  a Banach space.

$$C^0([0,1]) \subseteq \mathcal{PC}([0,1])$$

is a subspace, which is closed, since it is complete.

Consider  $l \in C^0([0,1])^*$ , i.e.

$$l : C^0([0,1]) \rightarrow \mathbb{R}$$

with:

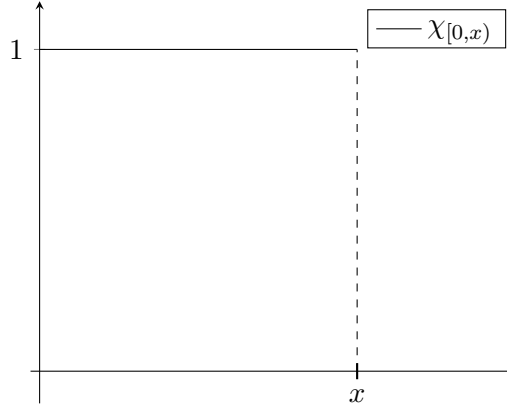
$$|l(f)| \leq C \|f\|_{C^0}$$

According to the Hahn-Banach theorem, there is an extension

$$\tilde{l} : \mathcal{PC}([0,1]) \rightarrow \mathbb{R}$$

with  $\tilde{l}|_{C^0} = l$  and  $|l(f)| \leq C \|f\|_{\mathcal{PC}([0,1])}$ . Define  $\alpha : [0,1] \rightarrow \mathbb{R}$  by:

$$\alpha(x) := \begin{cases} \tilde{l}(\chi_{[0,x)}) & \text{if } x < 1 \\ \tilde{l}(\chi_{[0,1]}) & \text{if } x = 1 \end{cases}$$

Figure 6.7:  $\chi_{[0,x)}$ 

$l(\chi_{[0,x)})$  is ill-defined, because  $\chi_{[0,x)}$  is *not* continuous.

$\tilde{l}(\chi_{[0,x)})$  is well-defined, because  $\chi_{[0,x)}$  is piecewise-continuous.

–  $\alpha$  has bounded variation: Consider:

$$0 = x_0 < \dots < x_N = 1$$

We need to show:

$$\sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| < C$$

$C$  has to be independent of  $N$  and the  $(x_i)$ .

Define  $s_i \in \{\pm 1\}$  by:

$$s_i := \begin{cases} +1 & \text{if } \alpha(x_i) - \alpha(x_{i-1}) \geq 0 \\ -1 & \text{if } \alpha(x_i) - \alpha(x_{i-1}) < 0 \end{cases}$$

Then holds:

$$\sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^N s_i (\alpha(x_i) - \alpha(x_{i-1})) = \tilde{l} \left( \sum_{i=1}^{N-1} s_i \chi_{[x_{i-1}, x_i)} + s_N \chi_{[x_{N-1}, 1]} \right)$$

Since  $\tilde{l}$  is bounded by construction, we know:

$$\begin{aligned} \sum_{i=1}^N |\alpha(x_i) - \alpha(x_{i-1})| &\leq \left| \tilde{l} \left( \sum_{i=1}^{N-1} s_i \chi_{[x_{i-1}, x_i)} + s_N \chi_{[x_{N-1}, 1]} \right) \right| \leq \\ &\leq C \left\| \sum_{i=1}^{N-1} s_i \chi_{[x_{i-1}, x_i)} + s_N \chi_{[x_{N-1}, 1]} \right\| = C \end{aligned}$$

Therefore we have  $\alpha \in \mathcal{BV}([0,1])$ .

– Consider  $d\mu := d\alpha_+ - d\alpha_-$  for the corresponding bounded regular Borel measure, where  $\alpha = \alpha_+ - \alpha_-$  and  $\alpha_{\pm}$  are monotonically increasing.

**Claim:** For all  $f \in C^0([0,1])$  holds:

$$l(f) = \int_0^1 f d\mu$$

**Proof:** Consider  $f \in C^0([0,1])$ . Set:

$$f_n(x) := \begin{cases} \sum_{i=1}^n f\left(\frac{i}{n}\right) \cdot \chi_{\left[\frac{i-1}{n}, \frac{i}{n}\right)} & \text{if } x < 1 \\ f(1) & \text{if } x = 1 \end{cases}$$

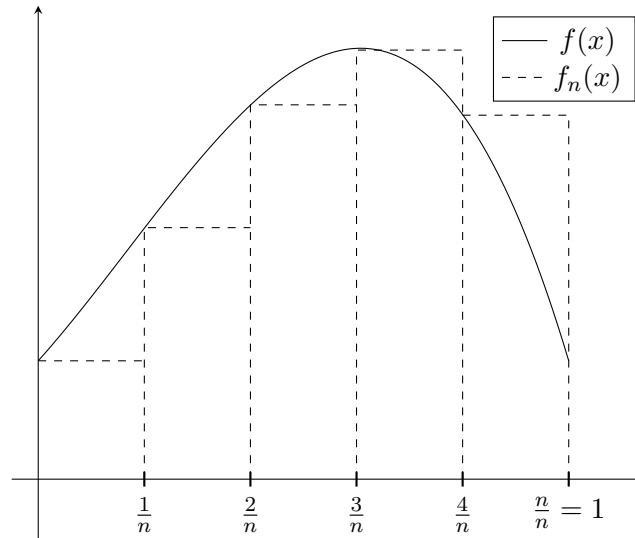


Figure 6.8: Approximation of  $f$  by  $f\left(\frac{i}{n}\right)$  for  $n = 5$

Since  $f_n$  is uniformly continuous, i.e.  $f_n \rightrightarrows f$ , we get:

$$\begin{aligned} l(f) &= \tilde{l}(f) = \tilde{l}\left(\lim_{n \rightarrow \infty} f_n\right) \stackrel{\tilde{l} \text{ continuous}}{=} \lim_{n \rightarrow \infty} \tilde{l}(f_n) = \\ &\stackrel{\text{by construction}}{=} \lim_{n \rightarrow \infty} \int_0^1 f_n d\mu \stackrel{(*)}{=} \int_0^1 \lim_{n \rightarrow \infty} f_n d\mu = \int_0^1 f d\mu \end{aligned}$$

For  $(*)$  consider:

$$\left| \int_0^1 (f_n - f) d\mu \right| \leq \underbrace{\sup |f - f_n|}_{\rightarrow 0} \cdot \underbrace{\text{TV}(\alpha)(1)}_{< \infty} \xrightarrow{n \rightarrow \infty} 0$$

□<sub>Claim</sub>

□<sub>6.4.6</sub>

## Remarks

- Our proof only works in the case  $K = [a,b] \subseteq \mathbb{R}$ . (see Reed, Simon: Appendix “The Riesz-Markov Theorem”)



- In general dimension the idea would be:

$$\mu(\Omega) := \tilde{l}(\chi_\Omega)$$

But how to extend  $l$ ? So choose  $f_n \rightarrow \chi_\Omega$  and define:

$$\mu(\Omega) := \lim_{n \rightarrow \infty} l(f_n)$$

(see Rudin: *Real and complex analysis*)

- Total variation of a bounded Borel measure:

$$|\mu|(\Omega) := \sup_{\substack{N \in \mathbb{N} \\ \Omega_1, \dots, \Omega_N \\ \text{with } \Omega_1 \dot{\cup} \dots \dot{\cup} \Omega_N = \Omega}} \sum_{i=1}^N |\mu(\Omega_i)|$$

$|\mu|$  is a positive bounded Borel measure. (see Rudin)

Then we can write:

$$\left| \int_K (f - f_n) d\mu \right| \leq \int_K |f - f_n| \cdot d|\mu| \leq \sup_K |f - f_n| \cdot |\mu|(K)$$

## 7 The Spectral Theorem for symmetric bounded operators

Let  $A \in L(H)$  be symmetric and  $H$  be a separable Hilbert space. Let  $p(A)$  be a polynomial in  $A$ , for example the characteristic polynomial for  $A \in L(\mathbb{C}^N)$  with  $p(A) = 0$ . Extend this idea to functions  $f(A)$  with  $f \in C^0(\sigma(A))$ . (Stone-Weierstraß) Then for

$$\langle u, f(A)u \rangle =: l(f)$$

holds  $l \in C^0(\sigma(A))^*$ . Using the Riesz representation theorem we can write:

$$\langle u, f(A)u \rangle = \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda)$$

$$d\mu_u(\lambda) = \langle u, dE_\lambda u \rangle$$

$dE_\lambda$  is the so-called *spectral measure*. Then holds the spectral theorem:

$$A = \int_{\sigma(A)} \lambda dE_\lambda$$

### 7.1 The Spectrum of symmetric bounded operators

Let  $A \in L(H)$  be symmetric, i.e.  $\langle u, Av \rangle = \langle Au, v \rangle$  for all  $u, v \in H$ . The resolvent set is:

$$\begin{aligned} \varrho(A) &= \{ \lambda \in \mathbb{C} \mid (\lambda - A) \text{ has a continuous inverse} \} \\ \sigma(A) &= \mathbb{C} \setminus \varrho(A) \end{aligned}$$

$\varrho(A) \subseteq \mathbb{C}$  is open and so the spectrum  $\sigma(A) \subseteq \mathbb{C}$  is closed. The spectral radius is:

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \|A\|$$

#### Warning

Consider  $\lambda \in \sigma(A)$ , i.e.  $\lambda - A$  has no continuous inverse. This does not mean  $\ker(\lambda - A)$  is non-trivial. Thus  $\lambda$  does *not* need to be an eigenvalue!

### 7.1.1 Theorem

Let  $A \in L(H)$  be self-adjoint. Then  $\sigma(A) \subseteq \mathbb{R}$ .

#### Proof

Consider  $\lambda = \alpha + \mathbf{i}\beta$  with  $\alpha, \beta \in \mathbb{R}$  and  $\beta \neq 0$ . We need to show that  $\lambda - A$  has a continuous inverse. Introduce the following bilinear form:

$$B(x, y) = \langle x, (A - \bar{\lambda}) y \rangle = \langle (A - \lambda) x, y \rangle$$

This bilinear form satisfies the assumptions of the Lax-Milgram theorem:

- i) The sesquilinearity is clear, since the scalar product is sesquilinear.
- ii)  $B$  is bounded:

$$|\langle x, (A - \bar{\lambda}) y \rangle| \leq \|x\| \cdot \underbrace{\|A - \bar{\lambda}\|}_{\leq \|A\| + |\lambda|} \cdot \|y\| \leq C \|x\| \|y\|$$

- iii)  $B$  is bounded from below, i.e. there exists an  $\varepsilon \in \mathbb{R}_{>0}$  such that for all  $x \in H$  holds:

$$|B(x, x)| \geq \varepsilon \|x\|^2$$

We know:

$$B(x, x) = \langle x, (A - \bar{\lambda}) x \rangle = \underbrace{\langle x, Ax \rangle}_{\text{real}} - \underbrace{\operatorname{Re}(\lambda \langle x, x \rangle)}_{\text{real}} - \underbrace{\mathbf{i} \operatorname{Im}(\lambda \langle x, x \rangle)}_{\text{imaginary}}$$

$$|B(x, x)| \geq |\operatorname{Im}(\lambda \langle x, x \rangle)| = |\beta| \cdot \|x\|^2$$

Set  $\varepsilon := |\beta| \neq 0$ .

The Lax-Milgram theorem yields that the linear functional  $l(x) = \langle z, x \rangle$  can be represented as

$$l(x) = B(y, x)$$

with a unique  $y = y(z) \in H$ . Thus we get for all  $x \in H$ :

$$\begin{aligned} \langle z, x \rangle &= \langle (A - \lambda) y, x \rangle \\ \Rightarrow z &= (A - \lambda) y \end{aligned}$$

Therefore, for all  $z \in H$  exists a unique  $y \in H$  such that  $(A - \lambda) y = z$ . Thus  $A - \lambda$  is invertible. The inverse  $(A - \lambda)^{-1}$  is continuous due to the open mapping theorem (see Corollary 2.4.8).  $\square_{7.1.1}$

### 7.1.2 Theorem

It holds  $\sigma(A) \subseteq [a, b]$  and  $a, b \in \sigma(A)$  with:

$$a := \inf_{\|u\|=1} \langle u, Au \rangle$$

$$b := \sup_{\|u\|=1} \langle u, Au \rangle$$

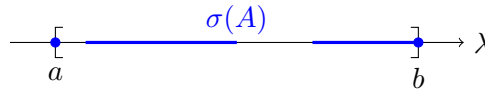


Figure 7.1:  $\sigma(A) \subseteq [a, b]$  and  $a, b \in \sigma(A)$

### Proof

For  $\lambda \in \mathbb{R}_{<a}$  holds:

$$\langle x, (A - \lambda)x \rangle = \langle x, Ax \rangle - \lambda \|x\|^2 \geq a \|x\|^2 - \lambda \|x\|^2 = \underbrace{(a - \lambda)}_{>0} \|x\|^2$$

Thus

$$\langle \cdot, \cdot \rangle_A := \langle \cdot, (A - \lambda) \cdot \rangle$$

is a scalar product on  $H$ . The corresponding norm

$$\|u\|_A := \sqrt{\langle u, u \rangle_A}$$

is equivalent to the norm  $\|\cdot\|$ , because it holds:

$$(a - \lambda) \|u\|^2 \leq \|u\|_A^2 = \langle u, (A - \lambda)u \rangle \leq (\|A\| - \lambda) \|u\|^2$$

For  $u \in H$  and  $l(w) := \langle u, w \rangle$  is  $l \in H^*$ . According to the Fréchet-Riesz theorem 3.1.3 (for the scalar product  $\langle \cdot, \cdot \rangle_A$ ) there is a unique vector  $v \in H$ , such that for all  $w \in H$  holds:

$$l(w) = \langle v, w \rangle_A$$

Thus we get for all  $w \in H$ :

$$\langle u, w \rangle = l(w) = \langle v, w \rangle_A = \langle v, (A - \lambda)w \rangle \stackrel{A-\lambda \text{ symmetric}}{=} \langle (A - \lambda)v, w \rangle$$

$$\Rightarrow u = (A - \lambda)v$$

Thus there exists a

$$\begin{aligned} \varphi : H &\rightarrow H \\ u &\mapsto v \end{aligned}$$

such that  $u = (A - \lambda) \varphi(u)$ , i.e.  $A - \lambda \in L(H)$  is surjective.  $\varphi$  is linear and bounded according to the open mapping theorem 2.4.8. Thus we have

$$\varphi = (A - \lambda)^{-1} \in L(H)$$

and therefore  $\lambda \in \rho(A)$ .

Applying the same argument to the operator  $(-A)$ , one sees that  $(b, \infty) \subseteq \rho(A)$ .

Therefore holds  $\sigma(A) \subseteq [a, b]$ .

Only prove that  $b \in \sigma(A)$ . For  $a \in \sigma(A)$  consider similarly the operator  $-A$ . Furthermore replace  $A \rightarrow A - a$  to get  $\sigma(A) \subseteq [0, b]$ . We know:

$$\|A\| = r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \sup_{\lambda \in \sigma(A)} \lambda = \sup \sigma(A)$$

As a consequence we get  $\|A\| \leq b$ . On the other hand we have:

$$b = \sup_{\|u\|=1} \langle u, Au \rangle \leq \sup_{\|u\|=1} \|Au\| \cdot \underbrace{\|u\|}_{=1} = \|A\|$$

Thus we have  $b = \|A\| = r(A)$ , especially  $b$  is a limit point of the spectrum of  $A$ . Since  $\sigma(A)$  is closed, it follows that  $b \in \sigma(A)$ .  $\square_{7.1.2}$

## 7.2 The continuous functional calculus

### 7.2.1 Theorem (continuous functions of operators)

Let  $A \in L(H)$  be symmetric. Then there is a unique mapping  $\Phi : C^0(\sigma(A), \mathbb{C}) \rightarrow L(H)$  (remember  $\sigma(A) \subseteq [a, b]$ ) with the following properties:

i)  $\Phi$  is an involutive algebra homomorphism, i.e.:

- $\Phi$  is linear.
- $\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g)$
- $\Phi(\overline{f}) = (\Phi(f))^*$  (involution)

ii)  $\Phi$  is continuous:

$$\|\Phi(f)\|_{L(H)} \leq C \|f\|_{\infty}$$

iii) If  $f(t) = t$ , then  $\Phi(f) = A$ .

iv) If  $Au = \lambda u$ , i.e.  $u \in H$  is an eigenvector of  $A$ , then  $\Phi(f)u = f(\lambda)u$ .

v) If  $f \geq 0$ , then  $\Phi(f) \geq 0$ , meaning that  $\Phi(f)$  is a positive semi-definite operator, i.e.  $\langle u, \Phi(f)u \rangle \geq 0$  for all  $u \in H$ .

vi)  $\sigma(\Phi(f)) = f(\sigma(A))$  (spectral mapping theorem (spektraler Abbildungssatz))

vii)  $\|\Phi(f)\|_{L(H)} = \|f\|_{\infty}$

Often we just write  $\Phi(f) = f(A)$ .

What if  $f(t) = p(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$  is a polynomial?

$$\Phi(t) \stackrel{\text{iii)}}{=} A$$

From i) follows:

$$\Phi(1) = \Phi(1 \cdot 1) = \Phi(1) \cdot \Phi(1)$$

Therefore we get:

$$\Phi(1) = \mathbb{1}$$

Now follows:

$$\begin{aligned}\Phi(t^2) &= \Phi(t \cdot t) = \Phi(t) \cdot \Phi(t) = A \cdot A = A^2 \\ \Phi(t^l) &= A^l \\ \Phi(p) &= p(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_0 \mathbb{1}\end{aligned}$$

### 7.2.2 Lemma (spectral mapping theorem for polynomials)

For  $p \in \mathfrak{P}(\mathbb{C})$ , i.e.  $p$  is a complex polynomial, holds:

$$\sigma(p(A)) = p(\sigma(A))$$

#### Proof

- If  $p = c \in \mathbb{C}$  is constant, then the lemma is trivial:

$$p(\sigma(A)) = c = \sigma(c\mathbb{1}) = \sigma(p(A))$$

So further on let  $p$  be not constant.

- $p(\sigma(A)) \subseteq \sigma(p(A))$ : For  $\lambda \in \sigma(A)$  and  $z \in \mathbb{C}$  yields the fundamental theorem of algebra:

$$p(z) - p(\lambda) = (z - \lambda)q(z)$$

Here  $q(z)$  is a new polynomial with  $\deg(q) = \deg(p) - 1$ . This also holds if we set  $z = A$ :

$$p(A) - p(\lambda) = (A - \lambda)q(A)$$

Assume  $p(\lambda) \in \varrho(p(A))$ , i.e.  $p(A) - p(\lambda)$  has a bounded inverse. Then holds:

$$\begin{aligned}\mathbb{1} &= (p(A) - p(\lambda)) \cdot (p(A) - p(\lambda))^{-1} = (A - \lambda) \cdot q(A) \cdot (p(A) - p(\lambda))^{-1} \\ \Rightarrow (A - \lambda)^{-1} &= \underbrace{q(A)}_{\in L(H)} \cdot \underbrace{(p(A) - p(\lambda))^{-1}}_{\in L(H)} \in L(H)\end{aligned}$$

This gives  $\lambda \in \varrho(A)$  in contradiction to  $\lambda \in \sigma(A)$  and so  $p(\lambda) \in \sigma(p(A))$ .

- $\sigma(p(A)) \subseteq p(\sigma(A))$ : Consider  $\mu \in \sigma(p(A))$  and set  $n := \deg(p)$ . Using the fundamental theorem of algebra we get:

$$\begin{aligned} q(z) &:= p(z) - \mu = a(z - \lambda_1) \cdot \dots \cdot (z - \lambda_n) \\ q(A) &:= p(A) - \mu = a(A - \lambda_1) \cdot \dots \cdot (A - \lambda_n) \end{aligned}$$

If all the operators  $A - \lambda_i$  had a continuous inverse, then this would hold also for their product in contradiction to the assumption  $\mu \in \sigma(p(A))$ . Thus one of the  $\lambda_i$  is in the spectrum of  $A$ . Because one of the linear factors vanishes, follows:

$$\begin{aligned} 0 &= q(\lambda_i) = p(\lambda_i) - \mu \\ \Rightarrow \mu &= p(\lambda_i) \in p(\sigma(A)) \end{aligned}$$

□<sub>7.2.2</sub>

Let  $p \in \mathfrak{P}(\mathbb{C})$  be a complex polynomial.

$$(p(A))^* = \bar{p}(A)$$

Thus  $p(A)$  is not symmetric.

### 7.2.3 Definition (normal operator)

$A \in L(H)$  is called *normal*, if  $[A, A^*] = 0$ .

### 7.2.4 Theorem

For a normal  $A \in L(H)$  holds  $r(A) = \|A\|$ .

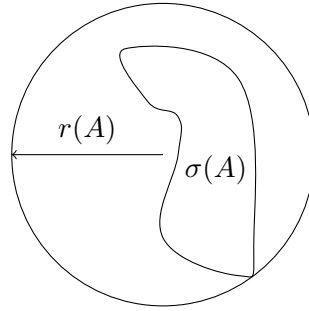


Figure 7.2:  $r(A) = \|A\|$

### Proof

We already proved for a general  $A \in L(H)$ :

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} \quad (7.1)$$

For symmetric operators, we know furthermore:

$$r(A) = \|A\| = \sup_{\|u\|=1} |\langle u, Au \rangle| \quad (7.2)$$

For *normal* operators, we conclude the following:  $A^*A$  is symmetric and thus:

$$\begin{aligned} \|A\|^2 &= \sup_{\|u\|=1} \|Au\|^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \langle u, A^*Au \rangle \stackrel{(7.2)}{=} \|A^*A\| = \\ &\stackrel{(7.2)}{=} r(A^*A) \stackrel{(7.1)}{=} \lim_{n \rightarrow \infty} \|(A^*A)^n\|^{\frac{1}{n}} \end{aligned}$$

$$(A^*A)^n = \underbrace{A^*A \cdot A^*A \cdot \dots \cdot A^*A}_{n\text{-times}} \stackrel{A \text{ normal}}{=} (A^*)^n \cdot A^n$$

With

$$\|A\|^2 = \sup_{\|u\|=1} \langle Au, Au \rangle = \sup_{\|u\|=1} \langle u, A^*Au \rangle \stackrel{A \text{ normal}}{=} \sup_{\|u\|=1} \langle u, AA^*u \rangle = \sup_{\|u\|=1} \langle A^*u, A^*u \rangle = \|A^*\|^2$$

we get:

$$\|(A^*A)^n\| \leq \|(A^*)^n\| \cdot \|A^n\| = \|A^n\|^2$$

It follows:

$$\|A\|^2 = \lim_{n \rightarrow \infty} \|(A^*A)^n\|^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \left( \|A^n\|^2 \right)^{\frac{1}{n}} \leq \|A\|^2$$

This gives:

$$\begin{aligned} \|A\|^2 &= \lim_{n \rightarrow \infty} \left( \|A^n\|^{\frac{1}{n}} \right)^2 = \left( \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} \right)^2 = (r(A))^2 \\ &\Rightarrow r(A) = \|A\| \end{aligned}$$

□<sub>7.2.4</sub>

### 7.2.5 Lemma

Let  $A \in L(H)$  be symmetric and  $p \in \mathfrak{P}(\mathbb{C})$  a complex polynomial. Then holds:

$$\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|$$

#### Proof

$p(A)$  is normal and thus, according to Theorem 7.2.4 holds:

$$\|p(A)\| = \sup_{\mu \in \sigma(p(A))} |\mu| \stackrel{7.2.2}{=} \sup_{\lambda \in \sigma(A)} |p(\lambda)|$$

□<sub>7.2.5</sub>



**Proof of theorem 7.2.1**

- For complex polynomials, we set  $\Phi(p) = p(A)$ . Then holds:

$$\|\Phi(p)\| = \|p(A)\| = r(p(A)) = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = \|p\|_{C^0(\sigma(A), \mathbb{C})}$$

Thus  $\Phi : \mathfrak{P}(\mathbb{C}) \rightarrow L(H)$  is an isometry. ( $\mathfrak{P}(\mathbb{C}) \subseteq C^0(\sigma(A), \mathbb{C})$ )

*Remark:* If we had considered  $C^0([a, b], \mathbb{C})$  with

$$a = \inf_{\|u\|=1} \langle u, Au \rangle$$

$$b = \sup_{\|u\|=1} \langle u, Au \rangle$$

then we would only have an inequality:

$$\|\Phi(p)\| \leq \|p\|_{C^0([a, b])}$$

- Moreover holds:

$$\Phi(p \cdot q) = (p \cdot q)(A) = p(A) \cdot q(A) = \Phi(p) \cdot \Phi(q)$$

$$(\Phi(p))^* = \Phi(\bar{p})$$

- Using the Stone-Weierstraß approximation theorem,  $\Phi$  uniquely extends to an isometry:

$$\Phi : C^0(\sigma(A), \mathbb{C}) \rightarrow L(H)$$

This yields i), ii), iii), vii).

- More specifically, consider  $f \in C^0(\sigma(A), \mathbb{C})$ . Then there exist  $p_n \in \mathfrak{P}(\mathbb{C})$  such that  $p_n \rightrightarrows f$  on  $\sigma(A)$ . ( $K = \sigma(A)$  is a compact metric space.) This means:

$$\|p_n - f\|_{C^0(\sigma(A), \mathbb{C})} = \sup_{z \in \sigma(A)} |p_n(z) - f(z)| \xrightarrow{n \rightarrow \infty} 0$$

$$\|\Phi(p_n) - \Phi(p_m)\| \stackrel{\text{isometry}}{=} \|p_n - p_m\| \xrightarrow{n, m \rightarrow \infty} 0$$

Thus the operators  $\Phi(p_n)$  form a Cauchy sequence in  $L(H)$  and since  $L(H)$  is a Banach space, this sequence converges to:

$$\Phi(f) := \lim_{n \rightarrow \infty} \Phi(p_n)$$

- iv) For  $Au = \lambda u$  holds:

$$\Phi(f)u = \lim_{n \rightarrow \infty} \Phi(p_n)u = \lim_{n \rightarrow \infty} p_n(A)u = \lim_{n \rightarrow \infty} p_n(\lambda)u = f(\lambda)u$$

- vi) Now we prove the spectral mapping theorem:

„ $\subseteq$ “: Assume  $\mu \in \sigma(f(A))$ , but  $\mu \notin f(\sigma(A))$ . Then holds  $f - \mu \neq 0$  on  $\sigma(A)$  and we can invert:

$$\frac{1}{f - \mu} \in C^0(\sigma(A), \mathbb{C})$$

Now follows:

$$\mathbb{1} = \Phi(1) = \Phi\left(\frac{1}{f - \mu}(f - \mu)\right) = \underbrace{\Phi\left(\frac{1}{f - \mu}\right)}_{\in L(H)} \cdot \underbrace{\Phi(f - \mu)}_{=f(A) - \mu\mathbb{1}}$$

So  $f(A) - \mu\mathbb{1}$  has a bounded inverse in contradiction to the assumption  $\mu \in \sigma(f(A))$ .  
 „ $\supseteq$ “: Consider  $\lambda \in \sigma(A)$ . Choose polynomials  $p_n \in \mathfrak{P}(\mathbb{C})$  with  $p_n \rightrightarrows f$ . Then converges in  $L(H)$ :

$$p_n(A) - p_n(\lambda)\mathbb{1} \xrightarrow{n \rightarrow \infty} f(A) - f(\lambda)\mathbb{1}$$

Assume that  $f(\lambda) \notin \sigma(f(A))$ . Then  $f(A) - f(\lambda)\mathbb{1}$  has a bounded inverse.

According to Theorem 2.5.3, the invertible operators are open in  $L(H)$ . Therefore there exists a  $\delta \in \mathbb{R}_{>0}$  such that  $B$  has a bounded inverse for all  $B \in B_\delta(f(A) - f(\lambda)\mathbb{1})$ . In particular, the operators  $p_n(A) - p_n(\lambda)\mathbb{1}$  have a bounded inverse for sufficiently large  $n$ . This is a contradiction to the spectral mapping theorem for polynomials 7.2.2.

v) Claim:  $f \geq 0 \Rightarrow \Phi(f) \geq 0$

Let  $f \in C^0(\sigma(A), \mathbb{R})$  be real-valued and  $f \geq 0$ . Then  $g := \sqrt{f} \in C^0(\sigma(A), \mathbb{R})$  and  $f = g^2$ .

$$\langle u, \Phi(f)u \rangle = \langle u, \Phi(g^2)u \rangle = \langle u, \Phi(g)\Phi(g)u \rangle = \langle \Phi(\bar{g})u, \Phi(g)u \rangle = \langle \Phi(g)u, \Phi(g)u \rangle \geq 0$$

□<sub>7.2.5</sub>

$\chi_\Omega(A)$  would be the projector onto the invariant subspace corresponding to the spectrum in  $\Omega$ . Formally we can compute:

$$\begin{aligned} (\chi_\Omega(A))^* &= \overline{\chi_\Omega(A)} = \chi_\Omega(A) \\ \chi_\Omega(A)\chi_\Omega(A) &= \chi_\Omega^2(A) = \chi_\Omega(A) \end{aligned}$$

This motivates, why we would like to form  $f(A)$  for a bounded Borel function  $f$  on  $\sigma(A)$ .

### 7.3 Spectral Measures

Let  $A \in L(H)$  be symmetric. Choose a  $u \in H$  (fixed).

$$\begin{aligned} \Phi_u : C^0(\sigma(A), \mathbb{R}) &\rightarrow \mathbb{R} \subseteq \mathbb{C} \\ f &\mapsto \langle u, \Phi(f)u \rangle \end{aligned}$$

$$|\Phi_u(f)| = |\langle u, \Phi(f)u \rangle| \leq \|\Phi(f)\| \cdot \|u\|^2 = \|f\|_{C^0(\sigma(A), \mathbb{R})} \cdot \|u\|^2$$

Thus  $\phi_u$  is a bounded linear functional on  $C^0(\sigma(A), \mathbb{R})$ . According to the Riesz representation theorem there exists a unique regular bounded Borel measure  $\mu_u$  such that:

$$\langle u, f(A)u \rangle = \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda)$$

The measure  $\mu_u$  is even positive, because if  $f \geq 0$ , set  $g = \sqrt{f}$  to get:

$$\int_{\sigma(A)} f(\lambda) d\mu_u(\lambda) = \langle u, f(A)u \rangle = \langle g(A)u, g(A)u \rangle \geq 0 \quad \forall f \in C^0(\sigma(A), \mathbb{R}), f \geq 0$$

Hence by approximation follows  $\mu_u(\Omega) \geq 0$  for all Borel sets  $\Omega \subseteq \sigma(A)$ . So  $\mu_u$  is a positive measure.

The resulting integral can be defined for a more general class of functions.

A *Borel function*  $f$  is a function, which is measurable for the Borel algebra, i.e.  $f^{-1}(\Omega)$  is a Borel function for all open  $\Omega \subseteq \mathbb{C}$ .

We use the following notation:  $\mathfrak{M}$  is the set of all Borel sets in  $\sigma(A)$ .

$\mathcal{B}(\sigma(A), \mathbb{R}) = L^\infty(d\mu_u)$  are the bounded Borel functions on  $\sigma(A)$ . We always assume:

$$\sup_{\sigma(A)} |f| < \infty$$

We define:

$$\begin{aligned} \phi_u : \mathcal{B}(\sigma(A), \mathbb{R}) &\rightarrow \mathbb{R} \\ \phi_u(f) &:= \int_{\sigma(A)} f(\lambda) d\mu_u(\lambda) \end{aligned}$$

### 7.3.1 Lemma

$$|\phi_u(f)| \leq \|f\|_{L^\infty} \cdot \|u\|^2$$

#### Proof

For  $f \in \mathcal{B}(\sigma(A), \mathbb{R})$  choose  $\varphi_n \in C^0(\sigma(A), \mathbb{R})$  such that  $\varphi_n \rightarrow f$  converges point-wise and  $\|\varphi_n\|_\infty \leq \|f\|_\infty$ . (Approximate  $f$  by step-functions and then approximate the step functions by continuous functions.)

Due to  $|\varphi_n| \leq C$  and

$$\int_{\sigma(A)} C d\mu_u = C\mu_u(\sigma(A)) = C \langle u, \Phi(1)u \rangle = C \langle u, \mathbb{1}u \rangle = C \|u\|^2 < \infty$$

we can use the dominated convergence theorem:

$$\begin{aligned} \left| \int_{\sigma(A)} f d\mu_u \right| &\stackrel{\text{dominated}}{\underset{\text{convergence}}{=}} \lim_{n \rightarrow \infty} \left| \int_{\sigma(A)} \varphi_n d\mu_n \right| = \lim_{n \rightarrow \infty} |\langle u, \Phi(\varphi_n)u \rangle| \leq \\ &\leq \lim_{n \rightarrow \infty} \|u\|^2 \cdot \|\Phi(\varphi_n)\| = \lim_{n \rightarrow \infty} \|u\|^2 \cdot \|\varphi_n\| \leq \|f\| \cdot \|u\|^2 \end{aligned}$$

□<sub>7.3.1</sub>

Define using the Fréchet-Riesz theorem the unique Operator  $\Phi(f)$  by:

$$\Phi_u(f) := \langle u, \Phi(f)u \rangle$$

By polarization we get:

$$B_f(u, v) = \Phi_{\frac{u+v}{2}}(f) - \Phi_{\frac{u-v}{2}}(f) - \mathbf{i}\Phi_{\frac{u+iv}{2}}(f) + \mathbf{i}\Phi_{\frac{u-iv}{2}}(f)$$

Alternatively define for  $f \in C^0(\sigma(A), \mathbb{C})$ :

$$\Phi_{u,v}(f) := \langle u, \Phi(f)v \rangle = \int_{\sigma(A)} f(\lambda) d\mu_{u,v}(\lambda)$$

$$B_f(u,v) := \int_{\sigma(A)} f(\lambda) d\mu_{u,v}(\lambda)$$

$d\mu_{u,v}$  is only a *complex-valued*, bounded, regular Borel measure.

### 7.3.2 Lemma

$B_f(u,v)$  is a *sesquilinear form*, i.e. linear in the second and anti-linear in the first argument, and it holds:

$$|B_f(u,v)| \leq \|f\| \cdot \|u\| \cdot \|v\|$$

#### Proof

This follows from the polarization formula and Lemma 7.3.1. □<sub>7.3.2</sub>

### 7.3.3 Theorem

Let  $B$  be a bounded sesquilinear form, i.e.:

$$|B(u,v)| \leq C \cdot \|u\| \cdot \|v\| \quad \forall_{u,v \in H}$$

Then there is a unique operator  $D \in L(H)$  with  $\|D\| \leq C$  such that:

$$B(u,v) = \langle u, Dv \rangle$$

#### Proof

For  $v \in H$  the map

$$\psi := \overline{B(\cdot, v)}$$

is a bounded linear form. According to the Fréchet-Riesz theorem 3.1.3 there exists a  $w \in H$  such that for all  $u \in H$  holds:

$$\psi(u) = \langle w, u \rangle$$

Then follows:

$$B(u,v) = \overline{\langle w, u \rangle} = \langle u, w \rangle$$

Thus  $D$  is uniquely determined by  $Dv = w$ . So  $D : H \rightarrow H$  is linear and bounded by the open mapping principle 2.4.7, i.e.  $D \in L(H)$  and for all  $v \in H$  holds:

$$B(u,v) = \langle u, Dv \rangle$$

Choose  $u = Dv$  to get:

$$\begin{aligned} B(Dv, v) &= \langle Dv, Dv \rangle = \|Dv\|^2 \\ &\leq C \cdot \|Dv\| \cdot \|v\| \end{aligned}$$

Therefore we have for all  $v \in H$ :

$$\begin{aligned} \|Dv\| &\leq C \cdot \|v\| \\ \|D\| &\leq C \end{aligned}$$

□<sub>7.3.3</sub>

We conclude: For  $f \in \mathcal{B}(\sigma(A), \mathbb{C})$  we construct  $B_f(u, v)$ . Then there exists a  $\Phi(f) \in L(H)$  such that for all  $u, v \in H$  holds:

$$\langle u, \Phi(f)v \rangle = B_f(u, v)$$

So  $\Phi : \mathcal{B}(\sigma(A), \mathbb{C}) \rightarrow L(H)$  gives a functional calculus on  $\mathcal{B}(\sigma(A), \mathbb{C})$ , i.e. we can calculate  $f(A)$  for an arbitrary Borel function.

### 7.3.4 Theorem (Spectral theorem in functional calculus form)

Let  $A \in L(H)$  be symmetric. Then there is a unique mapping  $\Phi : \mathcal{B}(\sigma(A)) \rightarrow L(H)$  with the following properties:

i)  $\Phi$  is an involutive algebra homomorphism, i.e.:

$$\begin{aligned} \Phi(f) \cdot \Phi(g) &= \Phi(f \cdot g) \\ \Phi(f)^* &= \Phi(\overline{f}) \end{aligned}$$

If  $f \in C^0(\sigma(A), \mathbb{C})$ , then  $\Phi(f)$  agrees with the corresponding operator of the continuous functional calculus.

ii)  $\|\Phi(f)\| \leq \|f\|_\infty$

iii) If  $f_n \rightarrow f$  converges point-wise and it holds  $\|f_n\|_\infty < C$ , then  $\Phi(f_n) \rightarrow \Phi(f)$  converges strongly, i.e. for all  $u \in H$  converges in  $H$ :

$$\Phi(f_n)u \rightarrow \Phi(f)u$$

iv) From  $Au = \lambda u$  follows:

$$\Phi(f)u = f(\lambda)u$$

v) If  $f \geq 0$  holds, then  $\Phi(f) \geq 0$  is positive semidefinite.

vi) If  $B \in L(H)$  commutes with  $A$ , i.e.  $[A, B] = AB - BA = 0$ , then  $[B, \Phi(f)] = 0$ . We write also  $f(A) = \Phi(f)$ .

*Note:* There is no spectral mapping theorem.

**Proof**

i) Prove the homomorphism property by approximation:

*First step:* Assume  $f \in C^0(\sigma(A), \mathbb{C})$  and  $g \in \mathcal{B}(\sigma(A), \mathbb{C})$ . Then there exists a series  $g_n \in C^0$  such that  $g_n \rightarrow g$  converges point-wise and  $\|g_n\|_\infty < C$ . Then follows the point-wise convergence:

$$fg_n \rightarrow fg$$

We use the notation:

$$\begin{aligned} \phi_{u,v}(h) &:= \langle u, \Phi(h)v \rangle \\ \Rightarrow \quad \phi_{u,u}(h) &= \phi_u(h) \end{aligned}$$

Since  $\mu_u$  is a regular bounded Borel measure, we can apply the dominated convergence theorem:

$$\begin{aligned} \phi_{u,u}(f \cdot g) &\stackrel{\text{Definition}}{=} \int_{\sigma(A)} f \cdot g d\mu_u \stackrel{\text{dominated convergence}}{\lim_{n \rightarrow \infty}} \int_{\sigma(A)} f \cdot g_n d\mu_u = \lim_{n \rightarrow \infty} \phi_{u,u}(f, g_n) = \\ &= \lim_{n \rightarrow \infty} \langle u, \Phi(f \cdot g_n)u \rangle = \lim_{n \rightarrow \infty} \langle u, f(A) \cdot g_n(A)u \rangle = \\ &= \lim_{n \rightarrow \infty} \langle (f(A))^* u, g_n(A)u \rangle = \lim_{n \rightarrow \infty} \phi_{(f(A))^* u, u}(g_n) \end{aligned}$$

We know for all  $u \in H$  using dominated convergence (see above):

$$\phi_{u,u}(g_n) \rightarrow \phi_{u,u}(g)$$

By polarization follows for all  $u, v \in H$ :

$$\phi_{v,u}(g_n) \rightarrow \phi_{v,u}(g)$$

This gives:

$$\begin{aligned} \phi_{u,u}(f \cdot g) &= \lim_{n \rightarrow \infty} \phi_{(f(A))^* u, u}(g_n) = \phi_{(f(A))^* u, u}(g) = \langle (f(A))^* u, \Phi(g)u \rangle \\ \Rightarrow \quad \langle u, \Phi(f \cdot g)u \rangle &= \langle u, f(A) \cdot g(A)u \rangle \end{aligned}$$

Polarization yields:

$$\Phi(fg) = \Phi(f) \cdot \Phi(g)$$

*Second Step:* Consider  $f, g \in \mathcal{B}$ . We choose  $f_n \in C^0$  with  $f_n \rightarrow f$  and  $\|f_n\| < C$ . Then  $f_n \cdot g \rightarrow f \cdot g$  converges point-wise.

$$\begin{aligned} \langle u, \Phi(f \cdot g)u \rangle &\stackrel{\text{dominated convergence}}{=} \lim_{n \rightarrow \infty} \langle u, \Phi(f_n \cdot g)u \rangle \stackrel{\text{First step}}{=} \lim_{n \rightarrow \infty} \langle u, \Phi(f_n) \cdot \Phi(g)u \rangle = \\ &= \lim_{n \rightarrow \infty} \phi_{u, g(A)u}(f_n) = \phi_{u, g(A)u}(f) = \langle u, f(A)g(A)u \rangle \\ \Rightarrow \quad \langle u, (\Phi(fg) - \Phi(f)\Phi(g))u \rangle &= 0 \quad \forall_{u \in H} \end{aligned}$$

By polarization follows:

$$\Phi(fg) = \Phi(f)\Phi(g)$$

The involution property follows similarly. □<sub>i)</sub>

iii) Claim: From point-wise convergence  $f_n \rightarrow f$  and  $\|f_n\| < C$  follows strong convergence  $f_n(A) \rightarrow f(A)$ .

a) From the dominated convergence theorem it is clear that holds:

$$\begin{aligned}\phi_u(f_n) &\rightarrow \phi_u(f) \\ \langle u, f_n(A)u \rangle &\rightarrow \langle u, f(A)u \rangle\end{aligned}$$

Polarization gives for all  $u, v \in H$ :

$$\langle u, f_n(A)v \rangle \rightarrow \langle u, f(A)v \rangle$$

In other words for all  $v \in H$  holds:

$$f_n(A)v \rightarrow f(A)v$$

b) It holds:

$$\begin{aligned}\|f_n(A)v\|^2 &= \langle f_n(A)v, f_n(A)v \rangle = \langle v, (f_n(A))^* f_n(A)v \rangle = \\ &= \langle v, \overline{f_n}(A) f_n(A)v \rangle = \left\langle v, |f_n(A)|^2 v \right\rangle \xrightarrow[\text{convergence}]{\text{dominated}} \left\langle v, |f|^2(A)v \right\rangle = \\ &= \langle v, \overline{f}(A) f(A)v \rangle = \langle f(A)v, f(A)v \rangle = \|f(A)v\|^2\end{aligned}$$

c) Now apply the following general Lemma:

**Lemma:**  $u_n \rightarrow u$  and  $\|u_n\| \rightarrow \|u\|$  imply  $u_n \rightarrow u$ .

**Proof:**

$$\begin{aligned}\|u - u_n\| &= \langle u - u_n, u - u_n \rangle = \\ &= \|u\|^2 - 2\operatorname{Re} \underbrace{\langle u, u_n \rangle}_{\substack{\rightarrow \langle u, u \rangle \\ \text{because } u \rightarrow u_n}} + \underbrace{\|u_n\|^2}_{\substack{\rightarrow \|u\|^2 \\ \text{because } \|u_n\| \rightarrow \|u\|}} \rightarrow \|u\|^2 - 2\|u\|^2 + \|u\|^2 = 0\end{aligned}$$

□ Lemma

d) This gives:

$$f_n(A)v \rightarrow f(A)v$$

□<sub>i)</sub>

ii) Claim:  $\|f(A)\| \leq \|f\|_\infty$  for  $f \in \mathcal{B}$ .

Choose  $f_n \in C^0$  which converge point-wise to  $f$  and  $\|f_n\|_\infty < \|f\|$ .

$$\|f(A)u\| \stackrel{\text{iii)}}{=} \lim_{n \rightarrow \infty} \|f_n(A)u\| \leq \lim_{n \rightarrow \infty} \underbrace{\|f_n(A)\|}_{= \|f_n\|_\infty} \cdot \|u\| = \lim_{n \rightarrow \infty} \|f_n\|_\infty \cdot \|u\| = \|f\|_\infty \cdot \|u\|$$

$$\Rightarrow \|f(A)\| \leq \|f\|_\infty$$

□<sub>i)</sub>

iv) - vi) follow immediately by approximation.

□<sub>7.3.4</sub>

### 7.3.5 Remark

So far we considered Borel measures on  $\sigma(A) \subseteq \mathbb{R}$ . These measures can be extended to Borel measures on  $\mathbb{R}$  by defining for a Borel set  $\Omega \in \mathfrak{M}(\mathbb{R})$ :

$$\mu(\Omega) := \mu(\Omega \cap \sigma(A))$$

$\Omega \cap \sigma(A)$  is a Borel set of  $\sigma(A)$ , since  $\sigma(A)$  is closed.

Now let  $M \subseteq \mathfrak{M}(\mathbb{R})$  be a Borel set.  $f(A)$  is well defined for any  $f \in \mathcal{B}(\mathbb{R})$ . With the characteristic function  $\chi_M$  of  $M$  define:

$$E_M := \chi_M(A)$$

Then we get:

$$E_M^* = \overline{\chi_M}(A) = \chi_M(A) = E_M$$

$$E_M^2 = \chi_M(A) \cdot \chi_M(A) = (\chi_M \cdot \chi_M)(A) = \chi_M(A) = E_M$$

Thus  $E_M$  is symmetric and idempotent, in other words  $E_M$  is a projection operator.

The mapping  $M \mapsto E_M$  is the spectral measure.

### 7.3.6 Definition (projection operator, spectral measure)

$P \in L(H)$  is a *projection operator* if  $P^2 = P = P^*$ .

An operator-valued *spectral measure*  $E$  is a mapping

$$\begin{aligned} E : \mathfrak{M}(\mathbb{R}^n) &\rightarrow L(H) \\ M &\mapsto E_M := E(M) \end{aligned}$$

with the following properties:

- i)  $E_M$  is a projection operator for all  $M \in \mathfrak{M}$ .
- ii)  $E_\emptyset = 0$ ,  $E_{\mathbb{R}^n} = \mathbb{1}$
- iii) For  $M = \bigcup_{n=1}^{\infty} M_n$  the operator  $E_M$  is the strong limit of the partial sums  $\sum_{n=1}^k E_{M_n}$ :

$$E_M = \text{s-lim}_{k \rightarrow \infty} \sum_{n=1}^k E_{M_n}$$

This means that for all  $u \in H$  holds:

$$E_M u = \sum_{n=1}^{\infty} (E_{M_n} u)$$

The series does not necessarily converge in the operator norm!

- iv)  $E_M \cdot E_N = E_{M \cap N}$

- v) For all  $u \in H$ , the mapping  $M \mapsto \langle u, E_M u \rangle \in \mathbb{R}$  is a (real) bounded regular Borel measure.

$\text{supp}(E)$  is the complement of the largest open set  $\Omega$  with  $E_\Omega = 0$ , which exists due to the  $\sigma$ -additivity.

$E$  is called a *compact* spectral measure if  $\text{supp}(E)$  is compact.



### 7.3.7 Theorem

Let  $A \in L(H)$  be symmetric. Then the mapping

$$E : M \mapsto \chi_M(A)$$

is a spectral measure on  $\mathbb{R}$  with  $\text{supp}(E) \subseteq \sigma(A)$ .

#### Proof

We have to show the properties from the definition 7.3.6.

i) is clear.

$$\begin{aligned}\chi_\emptyset(A) &= 0(A) = 0 \\ \chi_{\mathbb{R}}(A) &= \Phi(1) = \mathbb{1}\end{aligned}$$

So ii) is shown.

iv) follows from:

$$\chi_M(A) \cdot \chi_N(A) = (\chi_M \cdot \chi_N)(A) = \chi_{M \cap N}(A)$$

For v) consider:

$$\langle u, E_M u \rangle = \langle u, \chi_M(A) u \rangle = \phi_u(\chi_M) = \int \chi_M d\mu_u = \mu_u(M)$$

It remains to show iii) and  $\text{supp}(E) \subseteq \sigma(A)$ .

For the later consider  $\Omega \subseteq \varrho(A)$ :

$$E_\Omega = \chi_\Omega(A) = \Phi(\chi_\Omega) \stackrel{\text{extension to } \mathcal{B}(\mathbb{R})}{=} \Phi(\chi_\Omega \chi_{\sigma(A)}) = \Phi(\chi_{\Omega \cap \sigma(A)}) = \Phi(0) = 0$$

Now show iii): From

$$M = \bigcup_{j=1}^{\infty} M_j$$

follows with point-wise convergence:

$$\chi_M = \sum_{j=1}^{\infty} \chi_{M_j}$$

Theorem 7.3.4 iii) yields:

$$\text{s-lim}_{n \rightarrow \infty} \sum_{j=1}^n \underbrace{\chi_{M_j}(A)}_{=E_{M_j}} = \underbrace{\chi_M(A)}_{=E_M}$$

□<sub>7.3.7</sub>

**Notation**

$M \mapsto E_M$  is the spectral measure, which is projection operator valued.

$M \mapsto \langle u, E_M u \rangle = \mu_u(M) = \mu_{u,u}(M)$  is the real, bounded, regular Borel measure.

$M \mapsto \langle u, E_M v \rangle = \mu_{u,v}(M)$  is the complex, bounded, regular Borel measure.

Consider the integral:

$$\int_{\mathbb{R}} f(\lambda) d\mu_u(\lambda)$$

$$\begin{aligned} d\mu_u(\lambda) &= d\langle u, E_\lambda u \rangle \\ d\mu_{u,v}(\lambda) &= d\langle u, E_\lambda v \rangle \end{aligned}$$

**7.3.8 Lemma**

Let  $E$  be a spectral measure on  $\mathbb{R}^n$  and  $M \in \mathfrak{M}(\mathbb{R}^n)$ . Then holds for all  $u, v \in H$ :

$$d\langle u, E_\lambda E_M v \rangle = \chi_M(\lambda) d\langle u, E_\lambda v \rangle = d\langle E_M u, E_\lambda v \rangle$$

**Proof**

For all  $f \in \mathcal{B}(\mathbb{R}^n)$  we have to show:

$$\int_{\mathbb{R}^n} f(\lambda) d\langle u, E_\lambda E_M v \rangle = \int_{\mathbb{R}^n} f(\lambda) \cdot \chi_M(\lambda) d\langle u, E_\lambda v \rangle$$

By approximation, it suffices to show for all  $\Omega \in \mathfrak{M}(\mathbb{R}^n)$ :

$$\int_{\mathbb{R}^n} \chi_\Omega(\lambda) d\langle u, E_\lambda E_M v \rangle = \int_{\mathbb{R}^n} \chi_\Omega(\lambda) \chi_M(\lambda) d\langle u, E_\lambda v \rangle$$

Since  $\int \chi_M(x) d\mu(x) = \mu(M)$ , we get:

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_\Omega(\lambda) d\langle u, E_\lambda E_M v \rangle &= \langle u, E_\Omega E_M v \rangle \stackrel{\text{property iv)}}{=} \langle u, E_{\Omega \cap M} v \rangle = \\ &= \int_{\mathbb{R}^n} \chi_{\Omega \cap M} \langle u, dE_\lambda v \rangle = \int_{\mathbb{R}^n} \chi_\Omega \chi_M \langle u, dE_\lambda v \rangle \end{aligned}$$

□<sub>7.3.8</sub>

We write:

$$\int_{\mathbb{R}^n} f(\lambda) d\langle u, E_\lambda v \rangle =: \left\langle u, \left( \int_{\mathbb{R}^n} f(\lambda) dE_\lambda \right) v \right\rangle$$

We will use this to define integration in  $L(H)$ .

### 7.3.9 Theorem

Let  $E$  be a spectral measure on  $\mathbb{R}^n$  and  $f \in \mathcal{B}(\mathbb{R}^n)$ . Then the relations

$$\int f(\lambda) d\langle u, E_\lambda v \rangle = \langle u, Av \rangle \quad \forall_{u,v \in H}$$

define a unique normal operator  $A \in L(H)$ , which we also denote by:

$$A = \int f(\lambda) dE_\lambda$$

Moreover:

$$A^* = \int \overline{f(\lambda)} dE_\lambda$$

#### Proof

We define a bilinear form  $B : H \times H \rightarrow \mathbb{C}$  by:

$$B(u, v) = \int_{\mathbb{R}^n} f(\lambda) d\langle u, E_\lambda v \rangle$$

Then we have:

$$|B(u, u)| \leq \int_{\mathbb{R}^n} |f(\lambda)| \underbrace{d\langle u, E_\lambda u \rangle}_{\text{positive measure}} \leq \|f\|_\infty \cdot \left\langle u, \underbrace{E_{\mathbb{R}^n}}_{=1} u \right\rangle = \|f\|_\infty \cdot \|u\|^2$$

Polarization and estimation yields:

$$|B(u, v)| \leq \|f\|_\infty \|u\| \cdot \|v\|$$

Thus by the Fréchet-Riesz theorem, there is a unique  $A \in L(H)$  with:

$$B(u, v) = \langle u, Av \rangle$$

$$\begin{aligned} \langle u, Av \rangle &= \int f(\lambda) d\langle u, E_\lambda v \rangle \\ \langle u, A^*v \rangle &= \langle v, Au \rangle = \int \overline{f(\lambda)} d\langle u, E_\lambda v \rangle \\ \Rightarrow A^* &= \int \overline{f(\lambda)} dE_\lambda \end{aligned}$$

□<sub>7.3.9</sub>

### 7.3.10 Theorem

Let  $E$  be a spectral measure on  $\mathbb{R}^n$  and  $f, g \in \mathcal{B}(\mathbb{R}^n)$ . Then holds:

$$\left( \int_{\mathbb{R}^n} f(\lambda) dE_\lambda \right) \left( \int_{\mathbb{R}^n} g(\lambda') dE_{\lambda'} \right) = \int_{\mathbb{R}^n} f(\lambda) g(\lambda) dE_\lambda$$

**Proof**

By approximation it suffices to consider the case  $g = \chi_M$  for  $M \in \mathfrak{M}(\mathbb{R}^n)$ .

$$A := \int_{\mathbb{R}^n} f(\lambda) dE_\lambda \quad E_M = \int_{\mathbb{R}^n} \chi_M dE_\lambda$$

For all  $u, v \in H$  holds:

$$\begin{aligned} \langle u, A \cdot E_M v \rangle &= \int_{\mathbb{R}^n} f(\lambda) d \langle u, E_\lambda E_M v \rangle \stackrel{(7.3.8)}{=} \int_{\mathbb{R}^n} f(\lambda) \chi_M(\lambda) d \langle u, E_\lambda v \rangle = \\ &= \left\langle u, \int_{\mathbb{R}^n} (f \cdot \chi_M)(\lambda) dE_\lambda v \right\rangle \end{aligned}$$

$$\Rightarrow A \cdot E_M = \int_{\mathbb{R}^n} f \cdot \chi_M dE_\lambda$$

□<sub>7.3.10</sub>

Physicists write:

$$E_\lambda \cdot E_\mu = \delta_{\lambda-\mu} E_\lambda$$

This follows, because  $E_\lambda$  is idempotent and for  $\lambda \neq \mu$  holds:

$$E_\lambda E_\mu = E_{\{\lambda\}} \cdot E_{\{\mu\}} = E_{\{\lambda\} \cap \{\mu\}} = E_\emptyset = 0$$

**7.3.11 Theorem** (spectral decomposition of a bounded symmetric operator)

There is a one-to-one correspondence between bounded symmetric operators  $A \in L(H)$  and compact spectral measures  $E$  on  $\mathbb{R}$  by:

$$A = \int_{\mathbb{R}} \lambda dE_\lambda$$

This means for a given  $A$  with corresponding spectral measure  $E_M = \chi_M(A)$  holds this equation. Conversely, if  $E$  is a compact spectral measure, then this equation defines a bounded symmetric Operator and  $E_M = \chi_M(A)$ .

Moreover holds:

- i)  $f(A) = \int_{\mathbb{R}} f(\lambda) dE_\lambda$
- ii)  $\sigma(A) = \text{supp}(E)$

**Proof**

For a given  $A$ , let  $E_M = \chi_M(A)$  be the corresponding spectral measure. Then holds for all  $u, v \in H$  by construction:

$$\langle u, f(A) v \rangle = \int_{\mathbb{R}} f(\lambda) d \langle u, E_\lambda v \rangle$$

By the definition of  $\int f(\lambda) dE_\lambda$  follows:

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE_\lambda$$

For the polynomial  $f(\lambda) = \lambda$ , i.e.  $f(A) = A$ , this gives:

$$A = \int_{\mathbb{R}} \lambda dE_\lambda$$

If  $E$  is a compact spectral measure,  $\int_{\mathbb{R}} f(\lambda) dE_\lambda$  defines a normal operator with:

$$\left( \int_{\mathbb{R}} f(\lambda) dE_\lambda \right)^* = \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda$$

The compatibility with the spectral calculus follows from theorem 7.3.10.

Thus it remains to show  $\sigma(A) \subseteq \text{supp}(E)$ . Consider  $\mu \notin \text{supp}(E)$ . We want to show  $\mu \in \varrho(A)$ . Define the following bounded real function:

$$g(\lambda) := \frac{1}{\lambda - \mu} \chi_{\text{supp}(E)}$$

$$f(\lambda) := \lambda - \mu$$

$$B := \int_{\mathbb{R}} g dE_\lambda \in L(H)$$

is a well-defined integral.

$$\begin{aligned} \int_{\mathbb{R}} f(\lambda) dE_\lambda &= A - \mu \mathbb{1} \\ (A - \mu \mathbb{1}) B &= \left( \int_{\mathbb{R}} f(\lambda') dE_{\lambda'} \right) \left( \int_{\mathbb{R}} g(\lambda) dE_\lambda \right) = \int_{\mathbb{R}} f \cdot g dE_\lambda = \\ &= \int_{\mathbb{R}} \underbrace{\chi_{\text{supp}(E)} \lambda}_{=0 \text{ outside of } \text{supp}(E)} dE_\lambda = \int_{\mathbb{R}} dE_\lambda = \mathbb{1} \end{aligned}$$

Thus  $B = (A - \mu \mathbb{1})^{-1}$  and therefore  $\mu \in \varrho(A)$ .

□<sub>7.3.11</sub>

### 7.3.12 Corollary

For  $f \in \mathcal{B}(\mathbb{R})$  holds:

$$\|f(A)\| = \sup_{\sigma(A)} \text{ess } |f|$$

#### Proof

„ $\leq$ “ was already proved in theorem 7.3.4 ii).

To prove equality, we first note that  $f(A)$  is a normal operator, because it holds:

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE_\lambda \quad (f(A))^* = \int_{\mathbb{R}} \overline{f(\lambda)} dE_\lambda$$

$$\begin{aligned}
f(A) \cdot (f(A))^* &= \left( \int_{\mathbb{R}} f(\lambda) dE_{\lambda} \right) \left( \int_{\mathbb{R}} \overline{f(\lambda)} dE_{\lambda} \right) = \\
&= \int_{\mathbb{R}} f(\lambda) \overline{f(\lambda)} dE_{\lambda} = \int_{\mathbb{R}} \overline{f(\lambda)} f(\lambda) dE_{\lambda} = (f(A))^* f(A)
\end{aligned}$$

For a normal operator  $B$  holds:

$$\|B\| = r(B) = \sup_{x \in \sigma(B)} |x|$$

Now follows by theorem 7.3.11 ii):

$$\|f(A)\| = \sup_{x \in \sigma(f(A))} |x| = \sup(\text{supp}(f(E))) = \sup_{\lambda \in \text{supp}(E)} \text{ess } |f(\lambda)|$$

□<sub>7.3.12</sub>

## 7.4 Simple Examples

### 7.4.1 Example: finite dimensions

Consider  $H = \mathbb{C}^n$  and a symmetric operator  $A \in L(\mathbb{C}^n)$ . Choose an orthonormal eigenvector basis such that  $A$  has the matrix representation:

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The eigenvalues  $\lambda_i \in \mathbb{R}$  are real, but there can be degeneracies, i.e.  $\lambda_i = \lambda_j$  for some  $i \neq j$ .

$$A^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$$

Similarly we can compute polynomials of  $A$ .

The Stone-Weierstraß approximation yields for  $f \in C^0(\sigma(A), \mathbb{C})$ :

$$f(A) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix}$$

Since the spectrum

$$\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$$

is a finite set, we have  $C^0(\sigma(A)) = \mathcal{B}(\sigma(A))$ . The spectral measure for  $\Omega \subseteq \mathbb{C}$  is:

$$E_{\Omega} := \chi_{\Omega}(A) = \begin{pmatrix} \chi_{\Omega}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \chi_{\Omega}(\lambda_n) \end{pmatrix}$$

Thus  $E_\Omega$  is the projection operator on the eigenspaces, for which the eigenvalues  $\lambda$  lie in  $\Omega$ .

$$\int f(\lambda) dE_\lambda = \sum_{j=1}^n f(\lambda_j) E_{\{\lambda_j\}}$$

More specifically, let  $u_j$  be an orthonormal eigenvector basis,  $Au_j = \lambda_j u_j$  and  $\langle u_i, u_j \rangle = \delta_{ij}$ . Then for any  $v \in \mathbb{C}^n$  let  $u^{(k)}$  be all eigenvectors with the eigenvalue  $\lambda$ , so

$$E_{\{\lambda\}} v = \sum_k u^{(k)} \langle u^{(k)}, v \rangle$$

is the projection on the eigenspace  $\langle u^{(k)} \rangle$ .

#### 7.4.2 Example: compact operator

Let  $H$  be an infinite-dimensional Hilbert space and  $A \in L(H)$  be symmetric and compact. According to the Hilbert-Schmidt theorem, there is an orthonormal eigenvector basis  $(u_n)$ , i.e.:

$$Au_n = \lambda_n u_n$$

Then  $\lambda_n \rightarrow 0$ , because  $A$  is compact. The  $\lambda_n$  have finite-dimensional eigenspaces.

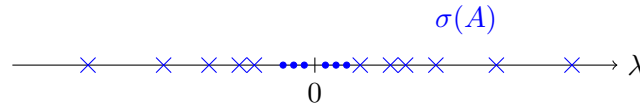


Figure 7.3:  $\sigma(A)$  has only zero as limit point

$$\begin{aligned} A^2 u_n &= \lambda_n^2 u_n \\ p(A) u_n &= p(\lambda_n) u_n \end{aligned}$$

This holds for any polynomial  $p$ . The Stone-Weierstraß approximation yields for  $f \in C^0(\sigma(A))$ :

$$f(A) u_n = f(\lambda_n)$$

The Riesz representation theorem gives

$$f(A) u_n = f(\lambda_n)$$

for all  $f \in \mathcal{B}(\sigma(A))$  or even  $f \in \mathcal{B}(\mathbb{R})$ . Then follows:

$$E_\Omega u_n := \chi_\Omega(A) u_n = \chi_\Omega(\lambda_n) u_n$$

Thus  $E_\Omega$  is the projection operator to all eigenspaces whose eigenvalues  $\lambda$  lie in  $\Omega$ . But  $E_{(-\varepsilon, \varepsilon)}$  has infinite rank for all  $\varepsilon > 0$ .

$$A = \sum_{\lambda \in \sigma(A)} \lambda E_{\{\lambda\}}$$

$$A_N := \sum_{\substack{\lambda \in \sigma(A) \\ |\lambda| > \frac{1}{N}}} \lambda E_{\{\lambda\}}$$

is a finite-dimensional approximation of  $A$  (cf. 5.8) in the sense:

$$\|A - A_N\| \xrightarrow{N \rightarrow \infty} 0$$

More precisely we have:

$$\|A - A_N\| \leq \frac{1}{N}$$

Now consider:

$$\begin{aligned} \mathbb{1} &= \sum_{\lambda \in \sigma(A)} E_{\{\lambda\}} \\ E_N &:= \sum_{\substack{\lambda \in \sigma(A) \\ |\lambda| > \frac{1}{N}}} E_{\{\lambda\}} \end{aligned}$$

This converges strongly, but it does not converge in the operator norm:

$$\|E - E_N\| = \left\| E_{[-\frac{1}{N}, \frac{1}{N}]} \right\| = 1$$

### 7.4.3 Example: continuous spectrum

Consider the Hilbert space  $H = L^2(\mathbb{R})$  and the function:

$$g(t) := \begin{cases} t & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

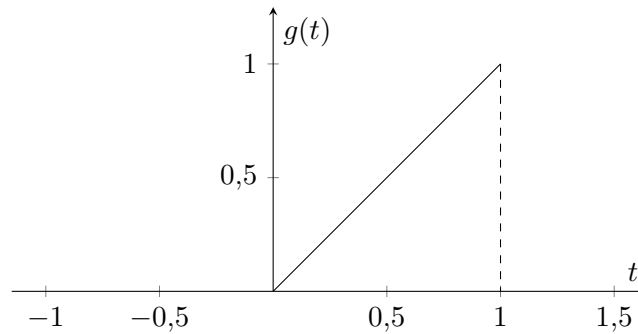


Figure 7.4: Plot of  $g(t)$

$A \in L(H)$  defined by

$$(Au)(t) := g(t) \cdot u(t) = (T_g \cdot u)(t)$$



for  $u \in H$  is a multiplication operator. From  $|g(t)| \leq 1$  follows  $\|A\| \leq 1$ . As before we get:

$$\begin{aligned} A^2 &= T_{g^2} \\ p(A) &= T_{p(g)} \quad \forall \text{ polynomial } p \\ f(A) &= T_{f(g)} \quad \forall f \in \mathcal{B}(\mathbb{R}) \end{aligned}$$

Therefore we get:

$$E_\Omega = T_{\chi_\Omega(g)}$$

$$\begin{aligned} (\chi_\Omega(g))(t) &= \begin{cases} 1 & \text{if } g(t) \in \Omega \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{g^{-1}(\Omega)} \end{aligned}$$

In general for multiplication operators holds:

$$E_\Omega = T_{\chi_\Omega(g)} = T_{\chi_{g^{-1}(\Omega)}}$$

For  $\Omega = (a, b) \subseteq (0, 1)$  we get  $g^{-1}(\Omega) = \Omega$  and thus  $E_\Omega u = \chi_\Omega \cdot u$ . If on the other hand  $\Omega = \{0\}$ , then holds:

$$g^{-1}(\Omega) = \mathbb{R} \setminus (0, 1) = (-\infty, 0] \cup [1, \infty)$$

Thus we get:

$$E_{\{0\}} u = \chi_{\mathbb{R} \setminus (0, 1)} u$$

The spectrum of  $A$  is  $\sigma(A) = [0, 1]$ . (Remember that the spectrum is always closed!)

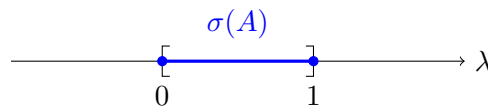


Figure 7.5: Continuous spectrum  $\sigma(A)$  of  $A$

Zero is an eigenvalue corresponding to an infinite-dimensional eigenspace,  $Au = 0$  for  $u|_{[0,1]} = 0$ . Any  $\lambda \in (0, 1]$  is *not* an eigenvalue:

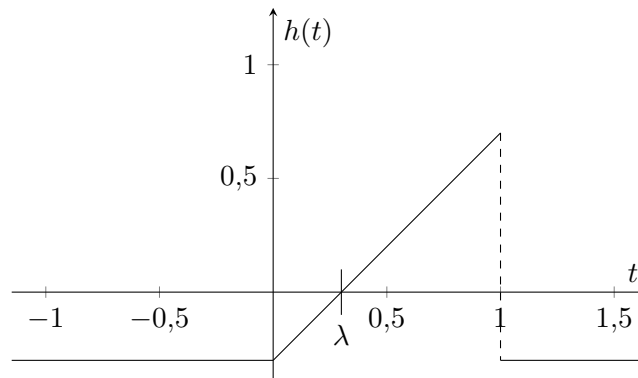


Figure 7.6: Plot of  $g(t) - \lambda$

$$(A - \lambda)u = T_{g-\lambda}u$$

$$h := g - \lambda$$

$$\begin{aligned} h(x) \cdot u(x) &= 0 \\ \Leftrightarrow u &= 0 \quad \forall_{x \in \mathbb{R}, h(x) \neq 0} \\ \Leftrightarrow u &= 0 \quad \text{almost everywhere} \\ \Leftrightarrow u &= 0 \in L^2(\mathbb{R}) \end{aligned}$$

Thus the eigenvalue equation only has the trivial solution.

#### 7.4.4 Example

Consider  $H = L^2(\mathbb{R})$  and the multiplication operator  $A = T_g$  for  $g \in C_0^0(\mathbb{R})$ . Then follows  $E_\Omega = T_{g^{-1}(\Omega)}$  as before and  $\sigma(A) = g(\mathbb{R})$ .

That  $\lambda \in \sigma(A)$  is an eigenvalue is equivalent to  $g^{-1}(\{\lambda\})$  is a set of strictly positive Borel measure.

### 7.5 Essential and discrete spectrum

Let  $A \in L(H)$  be symmetric. (The definitions are similar for normal operators or for unbounded self-adjoint operators). Let  $E$  be the corresponding spectral measure.

#### 7.5.1 Definition (essential and discrete spectrum)

The essential spectrum  $\sigma_{\text{ess}}(A)$  contains all  $\lambda \in \mathbb{C}$  for which  $\text{rg}(E_{B_\varepsilon(\lambda)}) = \infty$  for all  $\varepsilon \in \mathbb{R}_{>0}$ .

The discrete spectrum  $\sigma_{\text{disc}}(A)$  contains all  $\lambda \in \sigma(A)$  for which exists a  $\varepsilon \in \mathbb{R}_{>0}$  such that the rank of  $E_{B_\varepsilon(\lambda)}$  is finite.

*Note:*  $\lambda \in \sigma_{\text{ess}}(A)$  implies  $\lambda \in \text{supp}(E) = \sigma(A)$ . Thus  $\sigma(A) = \sigma_{\text{ess}}(A) \dot{\cup} \sigma_{\text{disc}}(A)$ .

#### 7.5.2 Example

Let  $A$  be a compact symmetric operator of infinite rank.

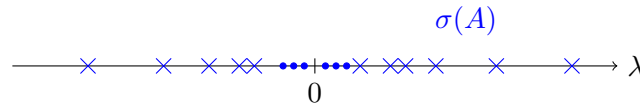


Figure 7.7:  $\sigma(A)$  has only zero as limit point

Here we have:

$$\sigma_{\text{disc}} = \sigma(A) \setminus \{0\} \qquad \sigma_{\text{ess}} = \{0\}$$

**7.5.3 Theorem** (condition for discrete spectrum)

$\lambda \in \sigma_{\text{disc}}(A)$  holds if and only if both of the following conditions are satisfied:

- i)  $\lambda$  is an isolated point of  $\sigma(A)$ , i.e. there exists a  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_\varepsilon(\lambda) \cap \sigma(A) = \{\lambda\}$ .
- ii)  $\lambda$  is an eigenvalue of finite multiplicity, i.e.  $\ker(A - \lambda)$  is finite-dimensional.

**Proof**

„ $\Leftarrow$ “: If i) and ii) hold, then for an appropriately chosen  $\varepsilon \in \mathbb{R}_{>0}$

$$E_{B_\varepsilon(\lambda)} = E_{\{\lambda\}}$$

is the projection operator on the finite-dimensional eigenspace.

„ $\Rightarrow$ “: Consider  $\lambda \in \sigma_{\text{disc}}(A)$ .

- i) Choose  $\varepsilon \in \mathbb{R}_{>0}$  such that  $E_{B_\varepsilon(\lambda)}$  has finite rank.

$$J := E_{B_\varepsilon(\lambda)}(H)$$

is a finite-dimensional subspace of  $H$ . For  $u \in J$  holds:

$$Au = AE_{B_\varepsilon(\lambda)}u = E_{B_\varepsilon(\lambda)}Au$$

Therefore follows  $Au \in J$  and thus  $A|_J : J \rightarrow J$  is a symmetric operator on a finite-dimensional Hilbert space. Diagonalize as in linear algebra:

$$\sigma(A|_J) = \{\lambda_1, \dots, \lambda_n\} = \sigma(A) \cap B_\varepsilon(\lambda)$$

The  $\lambda_i$  lie discrete and thus are isolated.

- ii) follows, because the eigenspace of  $A$  is the same as that of  $A|_J$ , which is finite-dimensional.

□<sub>7.5.3</sub>

**7.5.4 Theorem** (Weyl criterion)

- i)  $\lambda \in \sigma(A)$  holds if and only if there exists a sequence  $(u_n)_{n \in \mathbb{N}}$  in  $H$  such that for all  $n \in \mathbb{N}$  holds  $\|u_n\| = 1$  and:

$$(A - \lambda)u_n \xrightarrow{n \rightarrow \infty} 0$$

One also says, that  $\lambda$  is an *approximate eigenvalue*, because this can also be expressed as follows: For any  $\varepsilon \in \mathbb{R}_{>0}$  there exists a  $u \in H$  with  $\|u\| = 1$  and  $\|(A - \lambda)u\| \leq \varepsilon$ .

- ii)  $\lambda \in \sigma_{\text{ess}}(A)$  holds if and only if the  $(u_n)$  from above can be chosen as an orthonormal basis.

**Proof**

- i) For  $\lambda \in \varrho(A)$  the operator  $A - \lambda$  is continuously invertible, i.e.  $(A - \lambda)^{-1} \in L(H)$ . So for all  $u \in H$  holds:

$$\|(A - \lambda)^{-1} u\| \leq C \|u\|$$

Since  $A - \lambda$  is bijective, this is equivalent to:

$$\|v\| \leq C \|(A - \lambda) v\| \quad \forall_{v \in H}$$

This gives:

$$\begin{aligned} \|(A - \lambda) v\| &\geq \frac{1}{C} \|v\| \\ \|(A - \lambda) u_n\| &\geq \frac{1}{C} \|u_n\| = \frac{1}{C} \end{aligned}$$

Thus  $(A - \lambda) u_n$  cannot converge to zero and thus  $\lambda$  is no approximate eigenvalue.

For  $\lambda \in \sigma(A)$  the operator  $(A - \lambda)$  has no bounded inverse. Then either  $(A - \lambda)$  has a non-trivial kernel, i.e. there exists a  $u \in H$  with  $\|u\| = 1$  and:

$$(A - \lambda) u = 0$$

In this case one can choose  $u_n := u$ .

If on the other hand  $(A - \lambda)$  is injective, but has no bounded inverse, then exists a sequence  $(u_n)$  with  $\|(A - \lambda) u_n\| \leq \frac{1}{n} \|u_n\|$ . This means that  $\lambda$  is an approximate eigenvalue.

- ii) This follows directly from theorem 7.5.3.

□<sub>7.5.4</sub>

**7.6 The Stone Formula**

Let  $A \in L(H)$  be symmetric, so we have  $\sigma(A) \subseteq \mathbb{R}$ . Thus for any  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  the resolvent

$$R_\lambda := (A - \lambda)^{-1} \in L(H)$$

exists.

$\times \lambda$

—————→  $\mathbb{R}$

Figure 7.8:  $\lambda \notin \mathbb{R}$

$$A = \int_{\mathbb{R}} \mu \cdot dE_\mu \qquad R_\lambda = \int_{\mathbb{R}} \frac{1}{\mu - \lambda} dE_\mu$$

$\frac{1}{\mu - \lambda} \in \mathcal{B}(\mathbb{R})$  holds, because the pole is away from the real axis.

$$(A - \lambda) R_\lambda = \left( \int_{\mathbb{R}} (\mu - \lambda) dE_\mu \right) \left( \int_{\mathbb{R}} \frac{1}{\mu - \lambda} dE_\mu \right) = \int_{\mathbb{R}} \frac{\mu - \lambda}{\mu - \lambda} dE_\mu = \int_{\mathbb{R}} dE_\mu = E_{\mathbb{R}} = \mathbb{1}$$

### 7.6.1 Theorem

For  $\lambda \in \mathbb{R}$  and  $\varepsilon \in \mathbb{R}_{>0}$  holds:

$$\frac{1}{2\pi\mathbf{i}} \int_a^b (R_{\lambda+\mathbf{i}\varepsilon} - R_{\lambda-\mathbf{i}\varepsilon}) d\lambda = \frac{1}{2} (E_{(a,b)} + E_{[a,b]}) = \int_a^b \frac{1}{\mu - \lambda} dE_\mu$$

This is a convenient method for computing the spectral measure or the projection operator on eigenspaces.

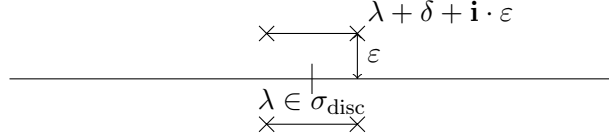


Figure 7.9: Calculating the spectral measure for a  $\lambda \in \sigma_{\text{disc}}$

$$\text{s-lim}_{\delta \searrow 0} \text{s-lim}_{\varepsilon \searrow 0} \frac{1}{2\pi\mathbf{i}} \int_{\lambda-\delta}^{\lambda+\delta} (R_{\mu+\mathbf{i}\varepsilon} - R_{\mu-\mathbf{i}\varepsilon}) d\mu = E_{\{\lambda\}}$$

#### Proof

Let  $a < b \in \mathbb{R}$  be given.

$$\phi_\varepsilon(\mu) := \frac{1}{2\pi\mathbf{i}} \int_a^b \left( \frac{1}{\mu - \lambda - \mathbf{i}\varepsilon} - \frac{1}{\mu - \lambda + \mathbf{i}\varepsilon} \right) d\lambda$$

Then holds  $\phi_\varepsilon : \mathbb{R} \rightarrow \mathbb{C}$  and:

$$\begin{aligned} \phi_\varepsilon(A) &= \int_{\mathbb{R}} \phi_\varepsilon(\mu) dE_\mu = \frac{1}{2\pi\mathbf{i}} \int_a^b \int_{\mathbb{R}} \left( \underbrace{\frac{dE_\mu}{\mu - \lambda - \mathbf{i}\varepsilon}}_{=R_{\lambda+\mathbf{i}\varepsilon}} - \underbrace{\frac{dE_\mu}{\mu - \lambda + \mathbf{i}\varepsilon}}_{=R_{\lambda-\mathbf{i}\varepsilon}} \right) d\lambda = \\ &= \frac{1}{2\pi\mathbf{i}} \int_a^b (R_{\lambda+\mathbf{i}\varepsilon} - R_{\lambda-\mathbf{i}\varepsilon}) d\lambda \end{aligned}$$

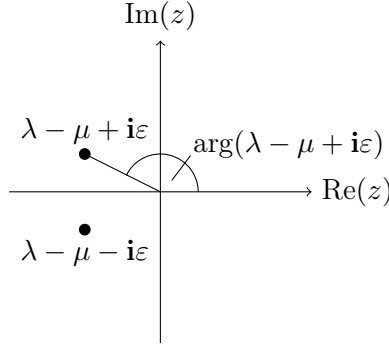
Now analyze the limit  $\varepsilon \rightarrow 0$ .

$$\phi_\varepsilon(\mu) = \frac{-1}{2\pi\mathbf{i}} (\ln(\lambda - \mu + \mathbf{i}\varepsilon) - \ln(\lambda - \mu - \mathbf{i}\varepsilon)) \Big|_{\lambda=a}^{\lambda=b}$$

The logarithm is cut at the negative real axis.

$$\ln(z) = \ln(|z|) + \mathbf{i} \arg(z) \qquad z = |z| e^{\mathbf{i} \arg(z)}$$

The argument of  $z$  lies in the range  $(-\pi, \pi)$ .

Figure 7.10:  $-\pi < \arg(z) < \pi$ 

Thus we get:

$$\lim_{\varepsilon \searrow 0} (\ln(\lambda - \mu + i\varepsilon) - \ln(\lambda - \mu - i\varepsilon)) = \begin{cases} 0 & \text{if } \lambda - \mu > 0 \\ \pi i & \text{if } \lambda - \mu = 0 \\ 2\pi i & \text{if } \lambda - \mu < 0 \end{cases}$$

Then follows:

$$\phi(\mu) := \lim_{\varepsilon \searrow 0} \phi_\varepsilon(\mu) = \frac{-1}{2\pi i} \begin{cases} 0 & \text{if } \mu \notin [a, b] \\ -\pi i & \text{if } \mu \in \{a, b\} \\ -2\pi i & \text{if } \mu \in (a, b) \end{cases} = \begin{cases} 0 & \text{if } \mu \notin [a, b] \\ \frac{1}{2} & \text{if } \mu \in \{a, b\} \\ 1 & \text{if } \mu \in (a, b) \end{cases}$$

Thus  $\phi_\varepsilon(\mu) \rightarrow \phi(\mu)$  converges point-wise.

*Idea:*

$$\phi_\varepsilon(A) \rightarrow \phi(A) = \frac{1}{2} (E_{[a,b]} + E_{(a,b)})$$

But how does this converge?

Consider weak convergence:

$$\langle u, \phi_\varepsilon(A) u \rangle = \int_{\mathbb{R}} \phi_\varepsilon(\mu) \underbrace{d\langle u, E_\mu u \rangle}_{=d\mu_u = d\mu_{u,u}}$$

$d\mu_u$  is a bounded regular real Borel measure. From  $|\phi(\mu)| \leq 1$  follows for small enough  $\varepsilon \in \mathbb{R}_{>0}$  now  $|\phi_\varepsilon(\mu)| \leq 2$ . Because our Borel measure is bounded, 2 is an integrable function, i.e.  $2 \in L^1(\mathbb{R}, d\mu_u)$ . Therefore we can use the bounded convergence theorem to get:

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \phi_\varepsilon(\mu) d\langle u, E_\mu u \rangle = \int_{\mathbb{R}} \phi(\mu) d\langle u, E_\mu u \rangle = \langle u, \phi_u(A) u \rangle$$

What about strong convergence?

We want to show for all  $u \in H$  the convergence  $\phi_\varepsilon(A) u \rightarrow \phi(A) u$  in  $H$ , or equivalently:

$$\begin{aligned} & (\phi_\varepsilon - \phi)(A) u \rightarrow 0 \\ \Leftrightarrow & \|(\phi_\varepsilon - \phi)(A) u\| \rightarrow 0 \end{aligned}$$

$$\begin{aligned}
 \|(\phi_\varepsilon - \phi)(A)u\|^2 &= \langle (\phi_\varepsilon - \phi)(A)u, (\phi_\varepsilon - \phi)(A)u \rangle = \langle u, ((\phi_\varepsilon - \phi)(A))^* (\phi_\varepsilon - \phi)(A)u \rangle = \\
 &= \langle u, (\overline{\phi_\varepsilon - \phi})(A)(\phi_\varepsilon - \phi)(A)u \rangle = \langle u, |\phi_\varepsilon - \phi|^2(A)u \rangle = \\
 &= \int_{\mathbb{R}} \underbrace{|\phi_\varepsilon - \phi|^2(\mu)}_{\substack{\rightarrow 0 \text{ point-wise} \\ \text{Borel measure}}} \underbrace{d\langle u, E_\mu u \rangle}_{\substack{\text{point-wise regular} \\ \text{Borel measure}}} \xrightarrow[\substack{\text{dominated} \\ \text{convergence}}]{\varepsilon \searrow 0} 0
 \end{aligned}$$

Therefore it converges strongly.

□<sub>7.6.1</sub>

## 8 Spectral Theorem for bounded normal operators

$A \in L(H)$  is normal if it commutes with its adjoint, i.e.  $[A, A^*] = 0$ . Before we considered symmetric  $A \in L(H)$ . Then for a complex valued function  $f$  the operator  $f(A)$  is normal, but in general not symmetric, because:

$$(f(A))^* = \overline{f}(A) \stackrel{\text{in general}}{\neq} f(A)$$

$$f(A) \cdot (f(A))^* = (f \cdot \overline{f})(A) = (\overline{f} \cdot f)(A) = (f(A))^* \cdot f(A)$$

The basic idea is:

$$\frac{1}{2}(A + A^*) =: B \qquad \frac{1}{2i}(A - A^*) =: C$$

$A = B + iC$ ,  $B$  and  $C$  are symmetric and  $[B, C] = 0$ .

### 8.1 Theorem

Let  $H$  be a complex separable Hilbert space,  $A_i \in L(H)$  for  $i \in \{1, \dots, n\}$  be symmetric operators, which commute pair wise, i.e.  $[A_i, A_j] = 0$  for all  $i, j \in \{1, \dots, n\}$  and

$$K := \prod_{i=1}^n \underbrace{[-\|A_i\|, \|A_i\|]}_{\supseteq \sigma(A_i)} \subseteq \mathbb{R}^n$$

be compact. Then there is a mapping

$$\phi : C^0(K, \mathbb{C}) \rightarrow L(H)$$

(notation:  $\phi(f) = f(A_1, \dots, A_n)$ ) with the following properties:

- i)  $\phi$  is an involutive algebra homomorphism.
- ii)  $\|\phi(f)\| \leq \|f\|_\infty = \sup_K |f|$
- iii)  $\phi(\text{pr}_i) = A_i$  for the projection maps:

$$\begin{aligned} \text{pr}_i : \mathbb{R}^n &\rightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto x_i \end{aligned}$$



**Proof**

Let  $E_i$  be the spectral measure of the operator  $A_i$ .

$$E_i(M) = \chi_M(A_i)$$

Let  $M \subseteq K$  be a cube, i.e.  $M = M_1 \times \dots \times M_n$ . Define:

$$\chi_M(A_1, \dots, A_n) := \chi_{M_1}(A_1) \cdot \dots \cdot \chi_{M_n}(A_n)$$

– Now holds  $[\chi_{M_i}(A_i), \chi_{M_j}(A_j)] = 0$ , because from

$$[A_i, A_j] = 0$$

follows via induction for any polynomials  $p, q$ :

$$[p(A_i), q(A_j)] = 0$$

With the Stone-Weierstraß and the Riesz representation theorem follows for all Borel functions  $f, g \in \mathcal{B}(\mathbb{R})$ :

$$[f(A_i), g(A_j)] = 0$$

–  $\chi_M(A_1, \dots, A_n)$  is a projection operator.

$$\begin{aligned} (\chi_M(A_1, \dots, A_n))^* &= \overline{\chi_{M_n}(A_n)} \cdot \dots \cdot \overline{\chi_{M_1}(A_1)} = \\ &= \chi_{M_1}(A_1) \cdot \dots \cdot \chi_{M_n}(A_n) = \chi_M(A_1, \dots, A_n) \end{aligned}$$

$$\begin{aligned} \chi_M(A_1, \dots, A_n) \cdot \chi_{M'}(A_1, \dots, A_n) &= \\ &= \chi_{M_1}(A_1) \cdot \dots \cdot \chi_{M_n}(A_n) \cdot \chi_{M'_1}(A_1) \cdot \dots \cdot \chi_{M'_n}(A_n) = \\ &= \chi_{M_1 \cap M'_1}(A_1) \cdot \dots \cdot \chi_{M_n \cap M'_n}(A_n) = \chi_{M \cap M'}(A_1, \dots, A_n) \end{aligned}$$

– Let  $f = \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}$  be a step function, meaning that the  $M_{\alpha}$  are disjoint cubes and  $a_{\alpha} \in \mathbb{C}$ . Define:

$$\phi(f) = \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}(A_1, \dots, A_n)$$

**Claim:** This definition is well-defined, i.e. it does not depend on the decomposition of  $f$  into cells.

**Proof:** Suppose we have:

$$f = \sum_{\alpha=1}^N a_{\alpha} \chi_{M_{\alpha}} = \sum_{\beta=1}^{\tilde{N}} \tilde{a}_{\beta} \chi_{\tilde{M}_{\beta}}$$

Choose a joint refinement. In fact, it suffices to consider the case that  $\tilde{M}_{\beta}$  is already a refinement of  $M_{\alpha}$ . Thus  $M_{\alpha} = \bigcup_{\beta \in I_{\alpha}} \tilde{M}_{\beta}$  and the  $I_{\alpha}$  form a partition of  $\{1, \dots, \tilde{N}\}$ . Using the properties of the  $E_i$ , a direct computation gives:

$$\chi_{M_{\alpha}} = \sum_{\beta \in I_{\alpha}} \chi_{\tilde{M}_{\beta}}$$

Substitute this in the formula for  $f$  and reorder the sums, to see that the definition is well-defined. □<sub>Claim</sub>

- Verify the properties i) and ii) for step functions: By direct computation follows:

$$(\phi(f))^* = \phi(\bar{f})$$

$$\phi(f) \cdot \phi(g) = \left( \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}} \right) \left( \sum_{\beta} a_{\beta} \chi_{M_{\beta}} \right) \stackrel{\text{as above}}{=} \sum_{\alpha, \beta} a_{\alpha} a_{\beta} \chi_{M_{\alpha} \cap M_{\beta}} = \phi(f \cdot g)$$

$$\|\phi(f)\| = \left\| \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}} \right\| \leq \left( \max_{\alpha} |a_{\alpha}| \right) \cdot \underbrace{\left\| \sum_{\alpha} \chi_{M_{\alpha}} \right\|}_{\leq 1} \leq \|f\|_{\infty}$$

- Now consider  $f \in C^0(K, \mathbb{C})$ . There is a sequence of step functions  $f_k$  such that  $f_k \rightrightarrows f$  converges uniformly.

$$\|\phi(f_k) - \phi(f_l)\| = \phi(f_k - f_l) \stackrel{\text{ii)}}{\leq} \sup |f_k - f_l| \xrightarrow{k, l \rightarrow \infty} 0$$

Since  $H$  is complete,  $\phi(f_k)$  converges in  $L(H)$  and we define  $\phi(f) := \lim_{k \rightarrow \infty} \phi(f_k)$ . Then the properties i) and ii) remain true by continuity.

- Compute  $\phi(\text{pr}_i)$ . For this let  $f_k = \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}$  be a step function with  $f_k(x) \rightrightarrows x$  and set  $\text{pr}_i^k(x) = f_k(x_i)$ , which implies  $\text{pr}_i^k \rightrightarrows \text{pr}_i$ .

$$\begin{aligned} \phi(\text{pr}_i^k) &= \sum_{\alpha} a_{\alpha} \chi_{\mathbb{R} \times \dots \times \underbrace{M_{\alpha}}_{i\text{-th position}} \times \dots \times \mathbb{R}}(A_1, \dots, A_n) = \\ &= \prod_{j \neq i} \underbrace{\chi_{\mathbb{R}}(A_j)}_{=1} \cdot \sum_{\alpha} a_{\alpha} \chi_{M_{\alpha}}(A_i) = \chi_{f_k}(A_i) \xrightarrow{\text{in } L(H)} A_i \end{aligned}$$

□<sub>8.1</sub>

## 8.2 Theorem

Now let  $A \in L(H)$  be normal, i.e.  $[A, A^*] = 0$ , and define the symmetric bounded operators:

$$A_1 := \frac{1}{2}(A + A^*) \quad A_2 := \frac{1}{2i}(A - A^*)$$

Then follows  $A = A_1 + iA_2$  and  $[A_1, A_2] = 0$ .

$$K := [-\|A_1\|, \|A_1\|] \times [-\|A_2\|, \|A_2\|] \subseteq \mathbb{C}$$

Then there exists exactly one map

$$\phi : C^0(K, \mathbb{C}) \rightarrow L(H)$$

with the following properties:

1.  $\phi$  is an involutive algebra homomorphism.

2.  $\|\phi(f)\| \leq \|f\|_\infty$
3.  $f(z) = z$  for  $z \in K$  already implies  $\phi(f) = A$ .
4.  $Au = \lambda u$  implies  $\phi(f)u = f(\lambda)u$
5. If  $f$  is real-valued and  $A$  symmetric, then  $\phi(f)$  is symmetric.
6.  $f \geq 0$  and  $A$  symmetric implies  $\phi(f) \geq 0$ .
7. For a  $T \in L(H)$  with  $[T, A] = [T, A^*] = 0$  follows for all  $f \in C^0$ :

$$[T, \phi(f)] = 0$$

### Proof

Choose  $\phi$  as in Theorem 8.1 for the commuting operators  $A_1$  and  $A_2$ . Then i), ii), iii) and v) follow immediately.

- vi) For  $f \geq 0$  there exists a  $g \in C^0(K, \mathbb{R})$  with  $f = g^2$ .

$$\langle u, \phi(f)u \rangle = \langle u, \phi(g) \cdot \phi(g)u \rangle = \langle \phi(g)u, \phi(g)u \rangle \geq 0$$

- vii) From  $[T, A_1] = 0 = [T, A_2]$  follows:

$$[T, \chi_M(A_1)] = 0 = [T, \chi_M(A_2)]$$

This gives by approximation

$$[T, \chi_M(A_1, A_2)] = 0$$

for all  $M \subseteq \mathbb{R}^2 \cong \mathbb{C}$ .

# Appendix

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Andreas Völklein

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