

# Integrated Course Ib

## Theoretical quantum mechanics

*lecture by*

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## ATTENTION

This script does *not* replace the lecture.

Therefore it is recommended *strongly* to attend the lecture.

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# 1 The Formalism and its interpretation

As any theory about physical phenomenons, quantum mechanics requires

- i) kinematical aspects: The (mathematical) space where physical states are represented.  
Example from classical mechanics: points in phase space
- ii) the definition of observables: Which quantities can be measured and how to represent them in the space of states?  
Example from classical mechanics: any function  $f(\vec{r}, \vec{p})$   
(This is more complicated in quantum mechanics.)
- iii) an dynamical law: How do states evolve in “time”?  
Example from classical mechanics: Hamilton’s equations determine  $(\vec{q}(t), \vec{p}(t))$  depending on  $(\vec{q}(t_0), \vec{p}(t_0))$ .

## 1.1 The kinematical aspects of quantum mechanics

The FIRST POSTULATE says:

“The state of a quantum mechanical system is represented as a normalized vector in a (complex) Hilbert space.

Vectors, which differ only by a phase, represent the same state.”

To understand what this means, we introduce some concepts:

- a) *Hilbert space*: A (finite or infinite dimensional) complete vector space with a positive definite scalar product. Following Dirac, elements in the Hilbert space  $\mathcal{H}$  are called *kets* and we represent them by  $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ .

For complex numbers  $a, b \in \mathbb{C}$  also  $a|\psi\rangle + b|\phi\rangle \in \mathcal{H}$  is a element of the Hilbert space.

The sum is associative and the product by scalars is distributive.

- b) *Scalar product*: The operation associating a complex number to each pair of states

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$$

$$(|\psi\rangle, |\phi\rangle) \mapsto \langle \psi, \phi \rangle$$

with the properties

$$\begin{aligned} \langle \eta, a\phi + b\psi \rangle &= a \langle \eta, \phi \rangle + b \langle \eta, \psi \rangle & \langle \psi, \phi \rangle &= \langle \phi, \psi \rangle^* \\ \langle \psi, \psi \rangle &\geq 0 & \langle \psi, \psi \rangle = 0 &\Leftrightarrow |\psi\rangle = 0 \end{aligned}$$

for  $a, b \in \mathbb{C}$  and  $|\phi\rangle, |\psi\rangle, |\eta\rangle \in \mathcal{H}$  is called the scalar product.

The *norm* of a state  $|\psi\rangle$  is then defined as:

$$\|\psi\| := \| |\psi\rangle \| := \sqrt{\langle \psi, \psi \rangle}$$

If  $\| |\psi\rangle \| = 1$ , then  $|\psi\rangle$  is said to be *normalized*.

- c) *Phase*: A Complex number  $z \in \mathbb{C}$  with unit norm, that means  $zz^* = 1$ . It can be represented via a real number  $\alpha \in \mathbb{R}$  as  $z = e^{i\alpha}$ .

The physical state associated with  $|\psi\rangle$  and  $e^{i\alpha} |\psi\rangle$  is THE SAME.

- d) *Complete basis*: A family of kets  $(|\phi_i\rangle)_{i \in I \subseteq \mathbb{N}}$  such that for ALL states  $|\psi\rangle \in \mathcal{H}$ , there is a family of complex numbers  $(c_i)_{i \in I \subseteq \mathbb{N}}$  (depending on  $|\psi\rangle$ ) with:

$$|\psi\rangle = \sum_{i \in I} c_i |\phi_i\rangle$$

If  $\langle \phi_i, \phi_j \rangle = \delta_{ij}$ , then the basis is *complete* and *orthonormal*.

In the case of an uncountably infinite vector space the basis  $(|q\rangle)_{q \in \mathbb{R}}$  can be written as a function of a real variable. The representation of  $|\psi\rangle$  then is an infinite sum, that is an integral

$$|\psi\rangle = \int \psi(q) |q\rangle dq$$

and a complete and orthonormal basis is characterized by  $\langle q, q' \rangle = \delta(q - q')$ .

- e) *Adjoint*: For each Hilbert space  $\mathcal{H}$  there is another Hilbert space  $\mathcal{H}^*$  called *dual*, with elements  $\langle f| \in \mathcal{H}^*$ , which are LINEAR FUNCTIONALS acting on  $\mathcal{H}$ :

$$\begin{aligned} \langle f| : \mathcal{H} &\rightarrow \mathbb{C} \\ |\psi\rangle &\mapsto \langle f | \psi \rangle \end{aligned}$$

$\langle \psi | \phi \rangle$  is called *bracket*. Riesz theorem says, that there is a ONE TO ONE correspondence between  $\mathcal{H}$  and  $\mathcal{H}^*$ :

$$\forall_{|\psi\rangle \in \mathcal{H}} \quad \exists_{\langle \psi| \in \mathcal{H}^*} \quad \forall_{|\phi\rangle \in \mathcal{H}} : \langle \psi | \phi \rangle = \langle \psi, \phi \rangle, \quad \|\langle \psi| \| = \| |\psi\rangle \|$$

The so associated  $\langle \psi|$  is the *adjoint* of  $|\psi\rangle$ , called *bra*, and we write:

$$\begin{aligned} \dagger : \mathcal{H} &\rightarrow \mathcal{H}^* \\ |\psi\rangle &\mapsto (|\psi\rangle)^\dagger = \langle \psi| \end{aligned}$$

The function  $\dagger$  is semilinear, that means for  $a, b \in \mathbb{C}$  and  $|\psi\rangle, |\phi\rangle \in \mathcal{H}$  is:

$$(a |\psi\rangle + b |\phi\rangle)^\dagger = a^* \langle \psi| + b^* \langle \phi|$$

- f) *Representation*: Assume that  $(|\phi_i\rangle)_{i \in I \subseteq \mathbb{N}}$  (respectively  $(|q\rangle)_{q \in \mathbb{R}}$ ) is a complete orthonormal basis. The ket  $|\psi\rangle$  is said to “be represented” in that basis by associating:

discrete case	continous case
$ \psi\rangle \mapsto \begin{pmatrix} \langle \phi_1   \psi \rangle \\ \langle \phi_2   \psi \rangle \\ \vdots \end{pmatrix} \in \mathbb{C}^{ I }$	$ \psi\rangle \mapsto \langle q   \psi \rangle =: \psi(q)$
$\langle \psi   \mapsto \begin{pmatrix} \langle \phi_1   \psi \rangle^* \\ \langle \phi_2   \psi \rangle^* \\ \vdots \end{pmatrix} \in \mathbb{C}^{ I }$	$\langle \psi   \mapsto \langle q   \psi \rangle^* = \psi(q)^*$

This  $\psi(q)$  is the wave function.

- g) *Exterior product*: The object  $|\phi\rangle \langle \psi| : \mathcal{H} \rightarrow \mathcal{H}$  is a linear operator acting on  $|\eta\rangle \in \mathcal{H}$  defined by

$$(|\phi\rangle \langle \psi|) |\eta\rangle := \underbrace{\langle \psi | \eta \rangle}_{\in \mathbb{C}} |\phi\rangle$$

with the adjoint:

$$(|\phi\rangle \langle \psi|)^\dagger := |\psi\rangle \langle \phi|$$

Example from linear algebra:

$$\vec{v} = (v_1, \dots, v_n)^T \qquad \vec{u} = (u_1, \dots, u_n)^T$$

$$\vec{v}^T \vec{u} = \sum_{i=1}^n v_i u_i$$

$$\vec{u} \vec{v}^T = \begin{pmatrix} u_1 v_1 & \dots & u_1 v_n \\ \vdots & \ddots & \vdots \\ u_n v_1 & \dots & u_n v_n \end{pmatrix}$$

## 1.2 Observables and measurements in quantum mechanics

What can be observed and how measurements affect quantum states is encoded into the following two POSTULATES:

“Observables in quantum mechanics are represented by LINEAR HERMITIAN OPERATORS on  $\mathcal{H}$ .”

“The results of a measurement of the physical quantity represented by an observable can only take values belonging to the SPECTRUM of the observable. Just after measurement, that gives one of the eigenvalues of the observable, the state belongs to corresponding eigenspace.”

- a)  $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$  is a *linear* operator if and only if:

$$\hat{A}(a|\psi\rangle + b|\phi\rangle) = a\hat{A}|\psi\rangle + b\hat{A}|\phi\rangle$$

For every linear operator  $\hat{A}$  there is the identity:

$$\langle\psi|\hat{A}\phi\rangle = \langle\phi|\hat{A}^\dagger\psi\rangle^*$$

$\hat{A}^\dagger$  is called the *adjoint* of  $\hat{A}$ . If and only if  $\hat{A} = \hat{A}^\dagger$  then  $\hat{A}$  is called HERMITIAN.

- b) The *spectrum* of an operator  $\hat{Q}$  is defined by a set of numbers  $Q_i$ , called *eigenvalues*, that fulfill the equation:

$$\hat{Q}|Q_i\rangle = Q_i|Q_i\rangle$$

The  $|Q_i\rangle$  are the corresponding eigenvectors. If there is more than one eigenvector for the same eigenvalue then the spectrum is called *degenerated* and the different eigenvectors are denoted by  $|Q_i^{(d)}\rangle$  with  $d \in D \subseteq \mathbb{N}$ .

Hermitian operators have REAL eigenvalues and ORTHOGONAL eigenvectors.

That means, if  $\hat{A} = \hat{A}^\dagger$ ,  $\hat{A}|a_i\rangle = a_i|a_i\rangle$ , then  $a_i = a_i^*$  and  $\langle a_i|a_j\rangle = 0$  if  $i \neq j$ .

If the eigenvectors are normalized, then we can write  $\langle a_i|a_j\rangle = \delta_{ij}$ .

- c) The *spectrum decomposition* of a Hermitian operator ( $\hat{A} = \hat{A}^\dagger$ ) with

discrete case	continous case
$\hat{A} a_i\rangle = a_i a_i\rangle$	$\hat{A} a\rangle = a a\rangle$

is given by:

$$\hat{A} = \sum_{i \in I} a_i |a_i\rangle \langle a_i| \qquad \hat{A} = \int a |a\rangle \langle a| da$$

In linear algebra a symmetric matrix  $M = M^T$  with eigenvalues  $m_i$  for  $i \in \{1, \dots, n\}$  can be written as:

$$M = \begin{pmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & m_n \end{pmatrix} = m_1 \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} + \dots + m_n \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

- d) The *unit operator*  $\hat{I}$  defined by

$$\forall_{|\phi\rangle \in \mathcal{H}} : \hat{I}|\phi\rangle = |\phi\rangle$$

can be written as

discrete case	continous case
$\hat{I} = \sum_{i \in I}  a_i\rangle \langle a_i $	$\hat{I} = \int  a\rangle \langle a  da$

where  $(|a_i\rangle)_{i \in I \subseteq \mathbb{N}}$  (respectively  $(|a\rangle)_{a \in \mathbb{R}}$ ) is a complete orthonormal basis.



e) The *projection* of  $|\psi\rangle$  along a basis vector  $|a_i\rangle$  (respectively  $|a\rangle$ ) is:

discrete case	continous case
$( a_i\rangle \langle a_i )  \psi\rangle = \langle a_i   \psi \rangle  a_i\rangle$	$( a\rangle \langle a )  \psi\rangle = \langle a   \psi \rangle  a\rangle$

f) The *algebra* of the operators  $\hat{Q}_1$  and  $\hat{Q}_2$  is:

$$\begin{aligned} (\hat{Q}_1 \hat{Q}_2) |\psi\rangle &:= \hat{Q}_1 (\hat{Q}_2 |\psi\rangle) \neq (\hat{Q}_2 \hat{Q}_1) |\psi\rangle \\ (\hat{Q}_1 \hat{Q}_2)^\dagger &:= \hat{Q}_2^\dagger \hat{Q}_1^\dagger \end{aligned}$$

The *commutator* is defined by:

$$[\hat{Q}_1, \hat{Q}_2] := \hat{Q}_1 \hat{Q}_2 - \hat{Q}_2 \hat{Q}_1$$

It is bilinear and has the properties:

i) It is antisymmetric:

$$[\hat{Q}_1, \hat{Q}_2] = - [\hat{Q}_2, \hat{Q}_1]$$

ii) Jacobi identity:

$$[\hat{Q}_1, [\hat{Q}_2, \hat{Q}_3]] + [\hat{Q}_2, [\hat{Q}_3, \hat{Q}_1]] + [\hat{Q}_3, [\hat{Q}_1, \hat{Q}_2]] = 0$$

iii) Leibniz identity:

$$[\hat{Q}_1, \hat{Q}_2 \hat{Q}_3] = \hat{Q}_2 [\hat{Q}_1, \hat{Q}_3] + [\hat{Q}_1, \hat{Q}_2] \hat{Q}_3$$

Due to these identities the commutator is analogous to the classical Poisson bracket.

Two observables  $(\hat{A} = \hat{A}^\dagger, \hat{B} = \hat{B}^\dagger)$  are called *compatible* if and only if  $[\hat{A}, \hat{B}] = 0$ .

Theorem: If  $\hat{A}$  and  $\hat{B}$  are compatible then there exists a basis  $(|k_i\rangle)_{i \in I \subseteq \mathbb{N}}$  such that:

$$\begin{aligned} \hat{A} |k_i\rangle &= a_i |k_i\rangle \\ \hat{B} |k_i\rangle &= b_i |k_i\rangle \end{aligned}$$

Therefore the states  $|k_i\rangle$  have well defined properties  $(a_i, b_i)$ .

g) i) The *matrix representation* of an operator WITH RESPECT TO THE BASIS  $(|a_i\rangle)_{i \in \{1, \dots, n\}}$  (respectively  $(|q\rangle)_{q \in \mathbb{R}}$ ) is:

discrete case	continuous case
$\hat{Q} \rightarrow \begin{pmatrix} \langle a_1   \hat{Q}   a_1 \rangle & \dots & \langle a_1   \hat{Q}   a_n \rangle \\ \vdots & \ddots & \vdots \\ \langle a_n   \hat{Q}   a_1 \rangle & \dots & \langle a_n   \hat{Q}   a_n \rangle \end{pmatrix} = \left( \langle a_i   \hat{Q}   a_j \rangle \right)_{ij} =: Q_{ij}$	$\hat{Q} \rightarrow \langle q   \hat{Q}   q' \rangle = Q(q, q')$

ii) The *trace* of  $\hat{Q}$  is defined by

$$\begin{array}{cc} \text{discrete case} & \text{continuous case} \\ \text{Tr}(\hat{Q}) = \sum_{i=1}^n \langle a_i | \hat{Q} | a_i \rangle & \text{Tr}(\hat{Q}) = \int Q(q, q) dq \end{array}$$

ans is independent of the basis.

Therefore the equation

$$\hat{Q} |\psi\rangle = |\phi\rangle$$

can be written in this basis by:

i) Multiplying  $\langle a_i |$  from the left:  $\langle a_i | \hat{Q} | \phi \rangle = \langle a_i | \phi \rangle$

ii) Insert an unit operator after  $\hat{Q}$ :

$$\langle a_i | \hat{Q} \hat{I} | \phi \rangle = \left\langle a_i \left| \hat{Q} \left( \sum_{j=1}^n |a_j\rangle \langle a_j| \right) \right| \phi \right\rangle = \sum_{j=1}^n \langle a_i | \hat{Q} | a_j \rangle \langle a_j | \phi \rangle = \langle a_i | \phi \rangle$$

This is the component form of the vector equation:

$$\begin{pmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

The continuous version is

$$\int Q(q, q') \psi(q') dq' = \phi(q)$$

with  $Q(q, q') := \langle q | \hat{Q} | q' \rangle$ ,  $\psi(q') = \langle q' | \psi \rangle$  and  $\phi(q) = \langle q | \phi \rangle$ .

h) Change of basis and unitary transformations.

i) An operator  $\hat{U}$  is called *unitary* if and only if  $\hat{U}^T = \hat{U}^{-1}$ .

ii) Unitary transformations (“transformation” is the same as “operator”) preserve scalar products. For  $|\psi\rangle, |\phi\rangle \in \mathcal{H}$  we define:

$$|\psi'\rangle := \hat{U} |\psi\rangle \qquad |\phi'\rangle := \hat{U} |\phi\rangle$$

Then the scalar product is the same:

$$\langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle$$

iii) Unitary transformations represent changes of the basis.

$$\begin{aligned} |\psi\rangle &= \hat{I} |\psi\rangle = \left( \sum_i |a_i\rangle \langle a_i| \right) |\psi\rangle = \sum_{i \in I} \langle a_i | \psi \rangle |a_i\rangle \\ \langle b_i | \psi \rangle &= \sum_{j \in I} \langle b_i | a_j \rangle \langle a_j | \psi \rangle \end{aligned}$$

$$\begin{pmatrix} \psi_1^{(b)} \\ \vdots \\ \psi_N^{(b)} \end{pmatrix} = \mathbb{U} \cdot \begin{pmatrix} \psi_1^{(a)} \\ \vdots \\ \psi_N^{(a)} \end{pmatrix}$$

$$[\mathbb{U}]_{ij} = U_{ij} = \langle b_i | a_j \rangle$$

iv) For operators the transformations are:

$$\begin{aligned} Q_{ij}^{(b)} &= \langle b_i | \hat{Q} | b_j \rangle = \langle b_i | \hat{I} \hat{Q} \hat{I} | b_j \rangle = \sum_{l,m \in I} \langle b_i | a_l \rangle \langle a_l | \hat{Q} | a_m \rangle \langle a_m | b_j \rangle = \\ &= \sum_{l,m \in I} U_{il} Q_{lm}^{(a)} U_{mj} = \sum_{l,m} U_{il} Q_{lm}^{(a)} (U_{jm})^* \\ Q^{(b)} &= \mathbb{U} Q^{(a)} \mathbb{U}^\dagger \end{aligned}$$

i) One can describe functions mapping one operator to another by their Taylor series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n \\ c_n &= \frac{1}{n!} \partial_x^n f(x) \big|_{x=0} \\ f(\hat{Q}) &:= \sum_{n=0}^{\infty} c_n \hat{Q}^n \end{aligned}$$

This is very nice. But this does not work for more than one operator, if they do not commute.

$$\begin{aligned} f(x, y) &= \sum_{n,m} c_{n,m} x^n y^m \\ f(\hat{Q}_1, \hat{Q}_2) &\neq \sum_{n,m} c_{n,m} \hat{Q}_1^n \hat{Q}_2^m \\ xy &\rightarrow \begin{cases} \hat{Q}_1 \hat{Q}_2 \\ \hat{Q}_2 \hat{Q}_1 \\ \frac{1}{2} (\hat{Q}_1 \hat{Q}_2 + \hat{Q}_2 \hat{Q}_1) \end{cases} \end{aligned}$$

In a basis of eigenvectors  $|q_i\rangle$ , that means  $\hat{Q} |q_i\rangle = q_i |q_i\rangle$ , one can write:

$$\begin{array}{cc} \text{discrete case} & \text{continuous case} \\ f(\hat{Q}) = \sum_i f(q_i) |q_i\rangle \langle q_i| & f(\hat{Q}) = \int f(q) |q\rangle \langle q| dq \end{array}$$

j) An operator can also depend on parameters:

$$\mathbb{R} \ni t \mapsto \hat{A}(t) \in \text{End}_{\mathbb{C}}(\mathcal{H})$$

It has all the nice properties, but one has to be careful not to change the ordering of the operators, if they do not commute:

$$\frac{d}{dt} (\hat{A}(t) \hat{B}(t)) = \left( \frac{d\hat{A}(t)}{dt} \right) \hat{B}(t) + \hat{A}(t) \left( \frac{d\hat{B}(t)}{dt} \right)$$

$$\frac{d\hat{A}(t)}{dt} = \hat{B}(t) \quad \Rightarrow \quad \hat{A}(t) = \int_0^t \hat{B}(t') dt' + \hat{A}_0$$

Extra property:

$$\frac{d}{dt} (\langle \psi(t) | \phi(t) \rangle) = \left( \frac{d\langle \psi(t) |}{dt} \right) | \phi(t) \rangle + \langle \psi(t) | \left( \frac{d| \phi(t) \rangle}{dt} \right)$$

As norm one usually uses:

$$\|\hat{A}\|^2 = \text{Tr}(\hat{A}\hat{A}^\dagger)$$

### 1.3 What does quantum actually predict?

Think about classical mechanics:

$$(x(0), p(0)) \xrightarrow{\text{evolution}} (x(t), p(t)) \rightarrow f(x(t), p(t)) \text{ is known}$$

So classical mechanics is DETERMINISTIC (the state of a system is determined by the initial conditions) and REALISTIC (the value you are going to measure already exists before the actual measurement).

Back to quantum mechanics:

The full dynamics is encoded in the postulate:

“In each quantum mechanical system there exists an observable  $\hat{H}(t)$ , called *Hamiltonian* which is assumed to be BOUNDED from below (and it is a nice operator). The solution of the equation

$$i\hbar \partial_t \hat{U}(t, t_0) = \hat{H}(t) \hat{U}(t, t_0) \quad \hat{U}(t_0, t_0) = \hat{I} \quad \text{for } t > t_0$$

is an unitary operator  $\hat{U}$ , called TIME EVOLUTION OPERATOR. The time evolution of an initial state  $|\psi(t_0)\rangle$  is given by  $|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle$ .”

Comments:

- a) If  $\partial_t \hat{H}(t) = 0$ , the one can directly solve the Schrödinger equation:

$$\hat{U}(t, t_0) = e^{\frac{-i}{\hbar} \hat{H}(t-t_0)} \quad \text{for } t > t_0 \quad (1.1)$$

- b)  $\hat{U}$  has a SEMIGROUP property:

$$\hat{U}(t, t') \hat{U}(t', t_0) = \hat{U}(t, t_0) \quad t > t' > t_0$$

Semigroup means:

$$\hat{U}(t, t_0)^{-1} = \hat{U}(t, t_0)^\dagger \neq \hat{U}(t_0, t)$$

- c) Constructing  $\hat{H}(t)$  is the main step to define a quantum mechanical system, at the end it relies on experimental verification.

d) The formal solution of (1.1) is found by iteration:

$$\begin{aligned}
 \hat{U}(t, t_0) &= \hat{I} - \frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') \hat{U}(t', t_0) dt' & t > t' \\
 &= \hat{I} - \frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') \left( \hat{I} - \frac{i}{\hbar} \int_{t_0}^{t'} \hat{H}(t'') \hat{U}(t'', t_0) dt'' \right) dt' = \\
 &= \hat{I} - \frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') dt' + \left( \frac{-i}{\hbar} \right)^2 \int_{t_0}^t \int_{t_0}^{t'} \hat{H}(t') \hat{H}(t'') \hat{U}(t'', t_0) dt'' dt' = \\
 &= \tau e^{\frac{-i}{\hbar} \int_{t_0}^t \hat{H}(t') dt'}
 \end{aligned}$$

$\tau$  is the TIME ORDERING OPERATOR.

$$\tau \left( \hat{H}(t) \hat{H}(t') \right) = \begin{cases} \hat{H}(t) \hat{H}(t') & t > t' \\ \hat{H}(t') \hat{H}(t) & t < t' \end{cases}$$

e) Stationary states: Assume  $\partial_t \hat{H}(t) = 0$ , then the eigenstates of  $\hat{H}$  are defined by the time-independent Schrödinger equation:

$$\hat{H} |e_n\rangle = e_n |e_n\rangle$$

Their time evolution is:

$$|e_n(t)\rangle = e^{-\frac{i}{\hbar} \hat{H} t} |e_n\rangle = \underbrace{e^{-\frac{i e_n}{\hbar} t}}_{\text{phase}} |e_n\rangle = e^{-i \omega_n t} |e_n\rangle$$

For an arbitrary initial state  $|\psi(t_0)\rangle$  we can calculate the final state:

$$\begin{aligned}
 |\psi(t)\rangle &= e^{-\frac{i}{\hbar} \hat{H} \cdot (t-t_0)} |\psi(t_0)\rangle = e^{-\frac{i}{\hbar} \hat{H} \cdot (t-t_0)} \hat{I} |\psi(t_0)\rangle = \\
 &= \sum_{n=0}^{\infty} \langle e_n | \psi(t_0) \rangle e^{-i \omega_n \cdot (t-t_0)} |e_n\rangle
 \end{aligned}$$

f) Instead of finding  $\hat{U}$  we can solve the SCHRÖDINGER EQUATION directly for  $|\psi(t)\rangle$ :

$$\begin{aligned}
 i\hbar \partial_t |\psi(t)\rangle &= \hat{H}(t) |\psi(t)\rangle \\
 |\psi(t=t_0)\rangle &= |\psi(t_0)\rangle
 \end{aligned}$$

## 1.4 The probabilistic interpretation of quantum mechanics (Born's rule)

“If the states of a system at time  $t$  is given by the normalized state  $|\psi(t)\rangle$ , then the PROBABILITY to get the outcome  $a_i$  when the observable  $\hat{A}$  is measured is given by:”

discrete case	continuous case
$P(a_i, t) =  \langle a_i   \psi(t) \rangle ^2$	$P(q, t) =  \langle q   \psi(t) \rangle ^2 dq$

a)

discrete case

$$\sum_i P(a_i, t) = 1$$

continuous case

$$\int P(q, t) dq = 1$$

$\langle a_i | \psi(t) \rangle$  is called probability Amplitude,  $P(q, t)$  is the density of probability and the probability to obtain an outcome between  $q$  and  $q + dq$  is  $|\langle q | \psi(t) \rangle|^2$ .

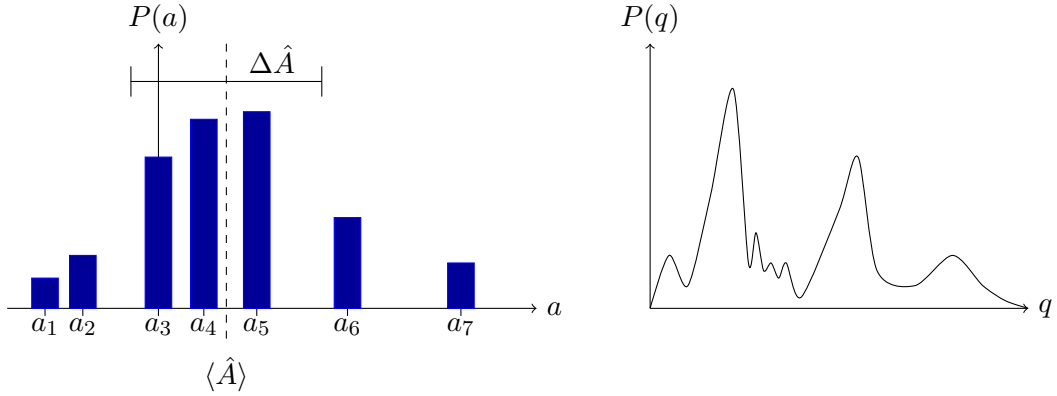
b) The expectation value is given by:

$$\begin{aligned} \bar{A} &:= \sum_i a_i P(a_i, t) = \sum_i a_i |\langle a_i | \psi(t) \rangle|^2 = \sum_i a_i \langle a_i | \psi(t) \rangle \cdot \langle a_i | \psi(t) \rangle^* = \\ &= \sum_i a_i \langle \psi(t) | a_i \rangle \cdot \langle a_i | \psi(t) \rangle = \langle \psi(t) | \left( \sum_i a_i |a_i\rangle \cdot \langle a_i| \right) | \psi(t) \rangle = \\ &= \langle \psi(t) | \hat{A} | \psi(t) \rangle =: \langle \hat{A} \rangle \end{aligned}$$

$$\begin{aligned} \text{Var} \hat{A} &= \langle \hat{A}^2 \rangle - (\langle \hat{A} \rangle)^2 = \langle \psi(t) | \hat{A}^2 | \psi(t) \rangle - \langle \psi(t) | \hat{A} | \psi(t) \rangle^2 = \\ &= \langle (\hat{A} - \hat{I} \langle \hat{A} \rangle)^2 \rangle = \sum (a_i - \bar{a}_i)^2 P(a_i, t) = (\Delta \hat{A})^2 \end{aligned}$$

The only case, one gets

$$\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = 0$$

is, if  $|\psi(t)\rangle = |a_i\rangle$ .

c) Consider the eigenstates of the Hamiltonian:

$$\hat{H} |e_n\rangle = e_n |e_n\rangle$$

If at  $t = 0$  the state is  $|e_n\rangle$ , then the time evolution is very simple:

$$\begin{aligned} |e_n(t)\rangle &= e^{-i\omega_n t} |e_n\rangle \\ \Rightarrow \langle e_n(t) | \hat{A} | e_n(t) \rangle &= \langle e_n | \hat{A} | e_n \rangle \\ \partial_t \langle \hat{A} \rangle &= 0 \end{aligned}$$

d) Define  $|\eta\rangle = N(|\psi\rangle + |\phi\rangle)$ . What is the probability  $P_\eta(a_i)$ ?

$$\begin{aligned} |\langle a_i | \eta \rangle|^2 &= N^2 |\langle a_i | \psi \rangle + \langle a_i | \phi \rangle|^2 = \\ &= N^2 \left( |\langle a_i | \psi \rangle|^2 + |\langle a_i | \phi \rangle|^2 + 2\text{Re}(\langle a_i | \psi \rangle \langle a_i | \phi \rangle^*) \right) = \\ &= N^2 (P_\psi(a_i) + P_\phi(a_i) + \text{something}) \end{aligned}$$

Therefore it is not enough to know the probabilities, one also has to know the phase of the states.

## 1.5 “Quantization”:

### Introducing the “fundamental” or basic observables

#### 1.5.1 General approach

Historically, it all started with the Hydrogen atom. (In this system the Hilbert space  $\mathcal{H}$  is infinite-dimensional.) It is called “canonical” quantization.

- i)  $\dim \mathcal{H} = \infty$  means, there can be discrete as well as continuous eigenvalues.
- ii) Every quantum mechanical observable is a function of two basic observables, the position operator  $\hat{q}_\alpha$  and the momentum operator  $\hat{p}_\alpha$ , satisfying:

$$[\hat{q}_\alpha, \hat{p}_\beta] = i\hbar \delta_{\alpha\beta} \hat{I} \qquad [\hat{q}_\alpha, \hat{q}_\beta] = 0 = [\hat{p}_\alpha, \hat{p}_\beta]$$

Therefore you can define an eigenbasis  $|\vec{q}\rangle$  with:

$$\hat{q}_\alpha |\vec{q}\rangle = q_\alpha |\vec{q}\rangle \qquad \hat{p}_\alpha |\vec{p}\rangle = p_\alpha |\vec{p}\rangle \qquad q_\alpha, p_\alpha \in \mathbb{R}$$

$$\langle \vec{q} | \vec{q}' \rangle = \delta(\vec{q} - \vec{q}') \qquad \langle \vec{p} | \vec{p}' \rangle = \delta(\vec{p} - \vec{p}')$$

- iii) The Hamiltonian operator is given in terms of the classical Hamiltonian

$$H(\vec{q}, \vec{p}) = \left( \frac{1}{2m} \sum_\alpha p_\alpha^2 \right) + V(\vec{q})$$

as:

$$\hat{H} = H(\hat{q}_\alpha, \hat{p}_\alpha) = \left( \frac{1}{2m} \sum_\alpha \hat{p}_\alpha^2 \right) + V(\hat{q}_\alpha)$$

This works ONLY in Cartesian coordinates! In other coordinates the product  $pq$  occurs in  $H$ , but how to quantize  $qp$ ? The operators do not commute:  $\hat{q}\hat{p} \neq \hat{p}\hat{q}$

For a magnetic field with vector potential  $\vec{A}(\vec{q})$  the Hamiltonian is

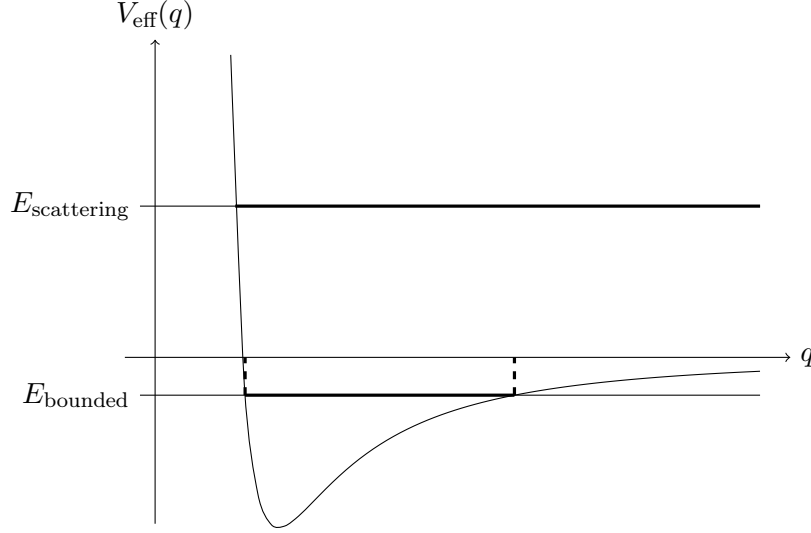
$$H = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A}(\vec{q}) \right)^2$$

and also depends on products of  $\vec{p}$  and  $\vec{q}$ .

$$\hat{H} = \sum_n e_n |e_n\rangle \langle e_n| + \int_\Omega e(\kappa) |e(\kappa)\rangle \langle e(\kappa)| d\kappa$$

$$\begin{aligned} \langle e_n | e_{n'} \rangle &= \delta_{n,n'} && \text{bounded} \\ \langle e(\kappa) | e(\kappa') \rangle &= \delta(\kappa - \kappa') && \text{scattering} \end{aligned}$$

A typical effective potential is:



- iv) The representations of a state  $|\psi\rangle$  in the position or momentum eigenstates are called WAVEFUNCTIONS. This means:

$$|\psi\rangle = \int \underbrace{\langle \vec{q} | \psi \rangle}_{=\psi(\vec{q})} |\vec{q}\rangle d\vec{q} \qquad |\psi\rangle = \int \underbrace{\langle \vec{p} | \psi \rangle}_{=\tilde{\psi}(\vec{p})} |\vec{p}\rangle d\vec{p}$$

Born's rule says, that  $|\psi(\vec{q})|^2 d\vec{q}$  is the probability to find the particle between  $\vec{q}$  and  $\vec{q} + (dq_1, dq_2, \dots)$  and  $|\tilde{\psi}(\vec{p})|^2 d\vec{p}$  is the probability to find the particle between  $\vec{p}$  and  $\vec{p} + (dp_1, dp_2, \dots)$ .

### 1.5.2 Algebraic properties of $\hat{q}$ and $\hat{p}$

Consider the following commutators:

$$\begin{aligned} [\hat{q}_\alpha, \hat{p}_\beta^2] &= \hat{p}_\beta [\hat{q}_\alpha, \hat{p}_\beta] + [\hat{q}_\alpha, \hat{p}_\beta] \hat{p}_\beta = 2i\hbar\delta_{\alpha\beta}\hat{p}_\beta \\ [\hat{q}_\alpha, \hat{p}_\beta^3] &= \hat{p}_\beta^2 [\hat{q}_\alpha, \hat{p}_\beta] + \hat{p}_\beta [\hat{q}_\alpha, \hat{p}_\beta] \hat{p}_\beta + [\hat{q}_\alpha, \hat{p}_\beta] \hat{p}_\beta^2 = 3i\hbar\delta_{\alpha\beta}\hat{p}_\beta^2 \\ &\vdots \\ [\hat{q}_\alpha, \hat{p}_\beta^n] &= \sum_{k=1}^n \hat{p}_\beta^{n-k} [\hat{q}_\alpha, \hat{p}_\beta] \hat{p}_\beta^{k-1} = ni\hbar\delta_{\alpha\beta}\hat{p}_\beta^{n-1} \end{aligned}$$

Since every analytic function can be represented by its Taylor series, one gets:

$$\begin{aligned} [\hat{q}_\alpha, f(\hat{p}_\beta)] &= i\hbar\delta_{\alpha\beta}\partial_x f(x) \big|_{x=\hat{p}_\beta} =: i\hbar\delta_{\alpha\beta}\partial_{\hat{p}_\beta} f(\hat{p}_\beta) \\ [\hat{p}_\alpha, f(\hat{q}_\beta)] &= -i\hbar\delta_{\alpha\beta}\partial_{\hat{q}_\beta} f(\hat{q}_\beta) \end{aligned}$$



One can show:

$$\hat{q} e^{-\frac{i}{\hbar} \hat{p} q_0} |q\rangle = (q + q_0) |q + q_0\rangle$$

This follows, because  $e^{-\frac{i}{\hbar} \hat{p} q_0} |q\rangle$  is an eigenstate of  $\hat{q}$  with eigenvalue  $q + q_0$  and therefore:

$$e^{-\frac{i}{\hbar} \hat{p} q_0} |q\rangle = |q + q_0\rangle$$

One can also show:

$$e^{\frac{i}{\hbar} \hat{q} p_0} |p\rangle = |p + p_0\rangle$$

This is similar to:

$$e^{-\frac{i}{\hbar} \hat{H} t} |\psi(t_0)\rangle = |\psi(t_0 + t)\rangle$$

### 1.5.3 The Schrödinger equation in position and momentum representation

Consider the Hamiltonian:

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{q})$$

I want to “represent”:

$$\begin{aligned} \hat{H} |e_n\rangle &= e_n |e_n\rangle \\ \hat{H} |e\rangle &= e |e\rangle \end{aligned}$$

For position representation, we apply  $\langle q|$  to the equation, use the definitions  $\psi_n(q) := \langle q | e_n \rangle$  and  $\tilde{\psi}_n(p) := \langle p | e_n \rangle$ , and multiply with the unit operator  $\hat{I} = \int |p\rangle \langle p| dp$  to get:

$$\begin{aligned} \frac{1}{2m} \langle q | \hat{p}^2 \cdot \hat{I} | e_n \rangle + V(q) \langle q | e_n \rangle &= e_n \langle q | e_n \rangle \\ \frac{1}{2m} \int p^2 \langle q | p \rangle \tilde{\psi}_n(p) dp + V(q) \psi_n(q) &= e_n \psi_n(q) \end{aligned}$$

In general one gets:

$$\langle q | f(\hat{p}) | \psi \rangle = f(-\hbar \partial_q) \psi(q)$$

The recipe is:

$$\begin{array}{ll} \hat{p} \mapsto -i\hbar \partial_q & \text{one dimension} \\ \hat{p} \mapsto -i\hbar \vec{\nabla} & \text{three dimensions} \end{array}$$

In three dimensions this is:

$$\begin{array}{ll} \text{discrete case} & \text{continuous case} \\ \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{q}) \right) \psi_n(\vec{q}) = e_n \psi_n(\vec{q}) & \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{q}) \right) \psi_k(\vec{q}) = e(k) \psi_k(\vec{q}) \end{array}$$

The state is only in the discrete case normalizable:

$$\langle e_n, e_n \rangle = \int |\psi_n(\vec{q})|^2 d\vec{q} < \infty \quad \langle e(k), e(k) \rangle = \int |\psi_k(\vec{q})|^2 d\vec{q} = \infty$$

Time evolution:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{q})\right) \hat{U}(\vec{q}, \vec{q}', t) = i\hbar \partial_t \hat{U}(\vec{q}, \vec{q}', t) \quad \hat{U}(\vec{q}, \vec{q}', t=0) = \delta(\vec{q} - \vec{q}')$$

It is much simpler to solve the Schrödinger equation for one state:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{q})\right) \psi(\vec{q}, t) = i\hbar \partial_t \psi(\vec{q}, t) \quad \psi(\vec{q}, t=0) = \psi_0(\vec{q})$$

### 1.5.4 The Heisenberg and the Schrödinger “pictures”

Consider:

$$\begin{aligned} \langle \hat{A} \rangle_t &:= \langle \psi(t) | \hat{A} | \psi(t) \rangle \\ |\psi(t)\rangle &= \hat{U}(t, t_0) |\psi(t_0)\rangle \end{aligned}$$

This quantity admits a two-folded interpretation:

$$\begin{aligned} \langle \hat{A} \rangle_t &= \left( \langle \psi(t_0) | \hat{U}(t, t_0)^\dagger \right) \hat{A} \left( \hat{U}(t, t_0) | \psi(t_0) \rangle \right) \quad \text{the state evolves in time (Schrödinger)} \\ &= \langle \psi(t_0) | \underbrace{\left( \hat{U}(t, t_0)^\dagger \hat{A} \hat{U}(t, t_0) \right)}_{=: \hat{A}_H(t, t_0)} | \psi(t_0) \rangle \quad \text{the operator evolves in time (Heisenberg)} \end{aligned}$$

$\hat{A}_H(t, t_0)$  is the HEISENBERG OPERATOR, that is the operator  $\hat{A}$  in the Heisenberg representation. If  $\hat{A} = \hat{A}^\dagger$ , the also  $\hat{A}_H$  is Hermitian.

$$\hat{A}_H(t = t_0, t_0) = \hat{A}$$

An equation of motion for  $\hat{A}$  is easy to derive:

$$\begin{aligned} \partial_t \hat{A}_H(t, t_0) &= \partial_t \left( \hat{U}(t, t_0)^\dagger \hat{A}(t) \hat{U}(t, t_0) \right) = \partial_t \left( \hat{U}^\dagger \right) \hat{A} \hat{U} + \hat{U}^\dagger \partial_t \left( \hat{A} \right) \hat{U} + \hat{U}^\dagger \hat{A} \partial_t \left( \hat{U} \right) = \\ &\stackrel{\hat{H}\hat{U} = i\hbar \partial_t \hat{U}}{=} \left( -\frac{i}{\hbar} \hat{H} \hat{U} \right)^\dagger \hat{A} \hat{U} + \hat{U}^\dagger \left( \partial_t \hat{A} \right) \hat{U} + \hat{U}^\dagger \hat{A} \left( -\frac{i}{\hbar} \hat{H} \hat{U} \right) = \\ &\stackrel{\hat{U}\hat{U}^\dagger = 1}{=} -\frac{i}{\hbar} \left[ \hat{A}_H(t, t_0), \hat{H}_H(t, t_0) \right] + \left( \partial_t \hat{A}(t) \right)_H \end{aligned}$$

Consider for example  $\partial_t \hat{H} = 0$ ,  $\hat{A}_1 = \hat{q}$  and  $\hat{A}_2 = \hat{p}$ :

$$\frac{d}{dt} \hat{q}_H = \frac{1}{m} \hat{p}_H \quad \frac{d}{dt} \hat{p}_H = -\partial_q V(q) \big|_{q=\hat{q}_H}$$

This leads to the Ehrenfest theorem:

$$\begin{aligned} m \frac{d^2}{dt^2} \hat{q}_H(t) &= -\partial_{\hat{q}_H} V(\hat{q}_H) \\ \hat{q}_H(t = t_0) &= \hat{q} \\ \hat{p}_H(t = t_0) &= \hat{p} \end{aligned}$$

For  $\hat{q}_H$  and  $\hat{p}_H$  and the usual Hamiltonian  $\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{q}, t)$  they satisfy “Newton’s” equations. Take expectation values  $\langle \psi(t_0) | \dots | \psi(t_0) \rangle$ .

$$m \frac{d^2}{dt^2} \langle \hat{q} \rangle_t = -\langle \psi(t_0) | \partial_{\hat{q}_H} V(\hat{q}_H) | \psi(t_0) \rangle \stackrel{\text{in general}}{\neq} -\partial_{\langle \hat{q} \rangle_t} V(\langle \hat{q} \rangle_t)$$

But for  $V(\hat{q}, t) = f_1(t) \hat{q}^2 + f_2(t) \hat{q} + f_3(t)$  the partial derivative  $\partial_{\hat{q}_H} V(\hat{q}_H)$  is linear in  $\hat{q}_H$  and therefore  $\langle \hat{q} \rangle_t$  describes the classical trajectory with the initial conditions  $\langle \psi(t_0) | \hat{q} | \psi(t_0) \rangle$  and  $\langle \psi(t_0) | \hat{p} | \psi(t_0) \rangle$ .

Comment: The best compromise to do quantum mechanics in a way, that looks similar (at least for short times) to classical mechanics is GAUSSIAN wavepacket.

A wavepacket is a superposition:

$$|\psi\rangle = \int \tilde{\psi}(p) |p\rangle dp$$

A Gaussian wavepacket is:

$$\begin{aligned} |\psi_{(q_0, p_0)}\rangle &= \int \psi_{(q_0, p_0)}(q) |q\rangle dq \\ &= \int \tilde{\psi}_{(q_0, p_0)}(p) |p\rangle dp \end{aligned}$$

$$\psi_{(q_0, p_0)}(q) = \frac{1}{\sqrt[4]{2\pi}\sigma} \exp\left(-\frac{1}{4} \frac{(q - q_0)^2}{\sigma^2}\right) \exp\left(-\frac{i}{\hbar} q p_0\right) \quad (1.2)$$

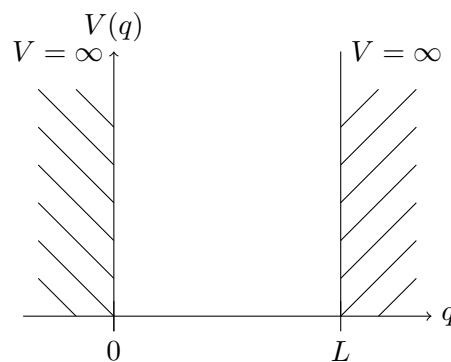
For this one gets:

$$(\Delta \hat{q})^2 (\Delta \hat{p})^2 = \frac{\hbar^2}{4}$$

$$\begin{aligned} \langle \psi_{(q_0, p_0)} | \hat{q} | \psi_{(q_0, p_0)} \rangle &= q_0 \\ \langle \psi_{(q_0, p_0)} | \hat{p} | \psi_{(q_0, p_0)} \rangle &= p_0 \end{aligned}$$

## 2 Basic applications

### 2.1 Boxes

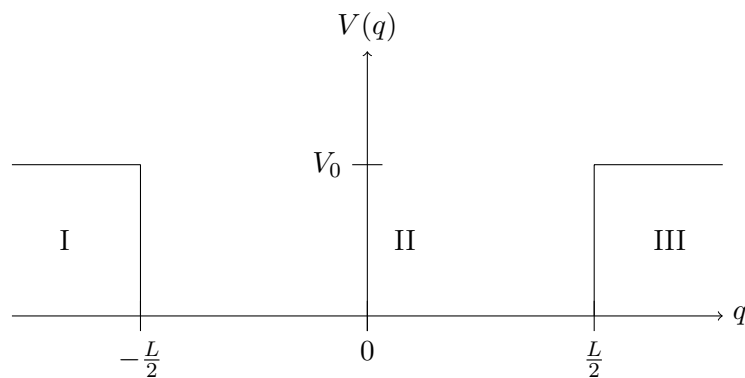


The normalized eigenstates are:

$$\psi_n(q) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}q\right)$$

$$e_n = n^2 e_0$$

### 2.2 Square well



- i) Energy  $E < V_0$  smaller than potential wall, leads to a bound state.

Write the equation:

$$\begin{aligned} \left( -\frac{\hbar^2}{2m} \partial_q^2 + V(q) \right) \psi(q) &= E \psi(q) \\ \partial_q^2 \psi(q) &= -k(E)^2 \psi(q) \\ k(E) &= +\sqrt{\frac{2m}{\hbar^2} (E - V(q))} \end{aligned}$$

In the three regions I, II and III one gets:

$$\begin{aligned} k_I(E) &= i |k_I(E)| \\ k_{II}(E) &= |k_{II}(E)| \\ k_{III}(E) &= i |k_{III}(E)| \end{aligned}$$

ii) Write the general solution:

$$\begin{aligned} \psi_{I(q)} &= A_I e^{|k_I(E)|q} + B_I e^{-|k_I(E)|q} \\ \psi_{II(q)} &= A_{II} e^{i|k_{II}(E)|q} + B_{II} e^{-i|k_{II}(E)|q} \\ \psi_{III(q)} &= A_{III} e^{|k_{III}(E)|q} + B_{III} e^{-|k_{III}(E)|q} \end{aligned}$$

iii) Normalization:  $B_I = A_{III} = 0$

Continuity: For a finite potential  $\psi$  and  $\partial_q \psi'$  are continuous.

$$\begin{aligned} \psi_I \left( -\frac{L}{2} \right) &= \psi_{II} \left( -\frac{L}{2} \right) & \partial_q \psi_I \left( -\frac{L}{2} \right) &= \partial_q \psi_{II} \left( -\frac{L}{2} \right) \\ \psi_{II} \left( \frac{L}{2} \right) &= \psi_{III} \left( \frac{L}{2} \right) & \partial_q \psi_{II} \left( \frac{L}{2} \right) &= \partial_q \psi_{III} \left( \frac{L}{2} \right) \end{aligned}$$

Now we have four equations and four variables, but one expects infinitely many solutions.

$$\mathbb{M} \begin{pmatrix} A_I \\ A_{II} \\ B_{II} \\ B_{III} \end{pmatrix} = 0 \quad \det \mathbb{M}(E) \stackrel{!}{=} 0$$

## 2.3 The harmonic oscillator (in one dimension)

The potential is:

$$V(\hat{q}) = \frac{m\omega_0^2}{2} \hat{q}^2$$

For all energies  $e_n \in \mathbb{R}_{\geq 0}$  we have:

$$\begin{aligned} \hat{H} |e_n\rangle &= e_n |e_n\rangle \\ \langle e_n | e_n \rangle &= 1 \end{aligned}$$

i) The Schrödinger equation is:

$$\left( -\frac{\hbar^2}{2m} \partial_q^2 + \frac{m\omega_0^2}{2} q^2 \right) \psi_n(q) = e_n \psi_n(q) \quad \psi(q \rightarrow \pm\infty) = 0$$

ii) Use dimensionless variables:

$$q = \alpha x \qquad \phi(x) := \psi_n(\alpha x)$$

$$\begin{aligned} \left( -\frac{\hbar^2}{2m} \frac{1}{\alpha^2} \partial_q^2 + \frac{m\omega_0^2}{2} \alpha^2 q^2 \right) \phi(q) &= e_n \phi(x) \\ \left( \partial_x^2 - \frac{m^2 \omega_0^2}{\hbar^2} \alpha^4 x^2 \right) \phi(q) &= -\frac{2m}{\hbar^2} \alpha^2 e_n \phi(x) \end{aligned}$$

Choose:

$$\alpha := \sqrt{\frac{\hbar}{m\omega_0}} \qquad \lambda := \frac{2e_n}{\hbar\omega_0}$$

Then the problem reduces to:

$$\boxed{(\partial_x^2 + (\lambda - x^2)) \phi(x) = 0 \qquad \psi(x \rightarrow \pm\infty) = 0}$$

iii) Check the asymptotic behavior, that is, when  $x$  goes to  $\pm\infty$ :

$$\phi(x) \approx \exp\left(-\frac{1}{2}x^2\right)$$

Assume a solution:

$$\phi(x) = u(x) \exp\left(-\frac{1}{2}x^2\right)$$

iv) Then we get for  $u(x)$  the differential equation:

$$(\partial_x^2 - 2x\partial_x + (\lambda - 1)) u(x) = 0$$

Solve this with a power series, by assuming, that  $u(x)$  is analytic:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

This leads to the „recursion relation“:

$$a_{n+2} = \frac{(2n+1-\lambda)}{(n+1)(n+2)} a_n$$

Or equivalently:

$$a_{n-2} = \frac{(n-1)n}{(2n-3-\lambda)} a_n$$

v) Check again the asymptotic behavior, to get an idea, how  $u(x)$  looks like, when  $n \rightarrow \infty$ :

$$\frac{a_{n+2}}{a_n} \xrightarrow{n \gg 1} \frac{2}{n}$$

But we have

$$\exp\left(-\frac{x^2}{2}\right) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!} \left(-\frac{1}{2}\right)^n}_{=: c_n} x^{2n} = \sum_{n=0}^{\infty} c_n x^{2n}$$

with:

$$\frac{c_{n+1}}{c_n} = \frac{n!}{(n+1)!} \cdot \left(-\frac{1}{2}\right)^{n+1} \cdot (-2)^n = \frac{-1}{2(n+1)} \xrightarrow{n \gg 1} -\frac{1}{2n}$$

Therefore  $u(x \rightarrow \infty)$  grows FASTER THAN  $\exp\left(\frac{1}{2}x^2\right)$  and therefore:

$$\exp\left(-\frac{1}{2}x^2\right) u(x) \rightarrow \infty$$

If the series does not stop,  $\phi(x)$  is *not* normalizable.

So there has to exist a „ $n^*$ “ such that:

$$\begin{aligned} 2n^* + 1 - \lambda &= 0 \\ \lambda &= 2n^* + 1 \\ e_n &= \hbar\omega_0 \left(n^* + \frac{1}{2}\right) \end{aligned}$$

This gives:

$$\psi_n(q) = N_n \exp\left(-\frac{1}{2} \left(\frac{q}{\alpha}\right)^2\right) H_n\left(\frac{q}{\alpha}\right)$$

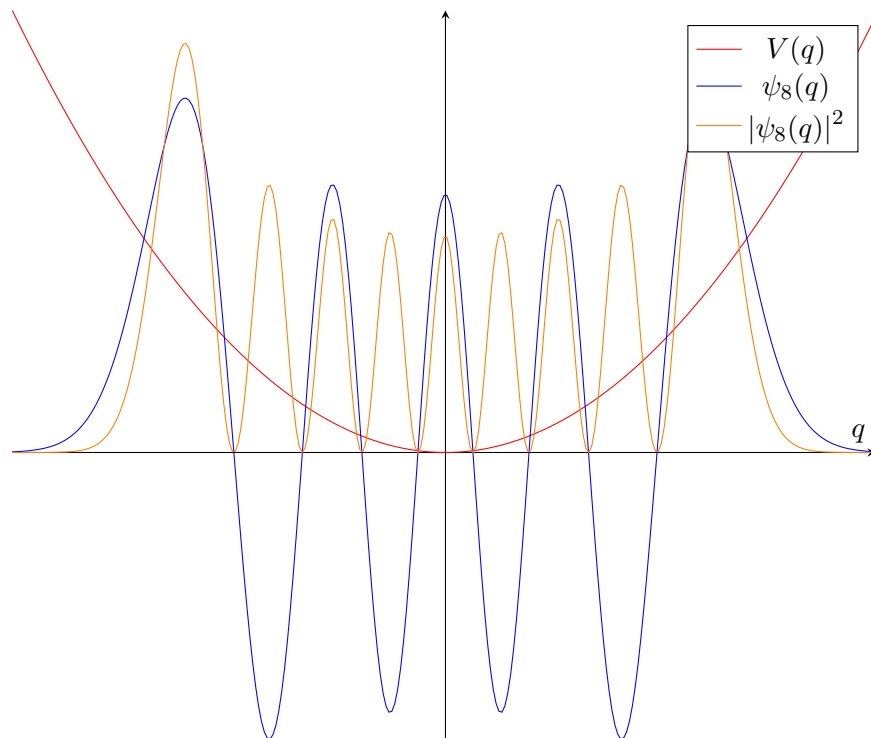
The polynomials  $H_n(x)$  are called the HERMITE POLYNOMIALS and are explicitly:

$$H_n(x) = (-1)^n \sum_{k_1+2k_2=n} \frac{n!}{k_1! \cdot k_2!} (-1)^{k_1+k_2} (2x)^{k_1}$$

$N_n$  is fixed by:

$$\int_{-\infty}^{\infty} |\psi_n(q)|^2 dq = 1$$

This looks like:



Then came Dirac:

### 2.3.1 The number representation

Introduce two operators:

$$\begin{aligned} \text{„destruction“} \quad \hat{a} &= \sqrt{\frac{m\omega_0}{2\hbar}} \left( \hat{q} + \frac{\mathbf{i}}{m\omega_0} \hat{p} \right) & \text{„creation“} \quad \hat{a}^\dagger &= \sqrt{\frac{m\omega_0}{2\hbar}} \left( \hat{q} - \frac{\mathbf{i}}{m\omega_0} \hat{p} \right) \end{aligned}$$

It is easy to calculate:

$$[\hat{a}, \hat{a}^\dagger] = 1$$

Introduce the „number“ operator:

$$\hat{n} := \hat{a}^\dagger \hat{a}$$

Then we can find easily:

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{m\omega_0^2}{2} \hat{q}^2 = \hbar\omega_0 \left( \hat{n} + \frac{1}{2} \right)$$

Therefore the eigenstates of  $\hat{n}$  are automatically the eigenstates of  $\hat{H}$ .

It also satisfies:

$$[\hat{n}, \hat{a}] = -\hat{a} \quad [\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger$$

We want to find  $|\sigma\rangle$  such that:

$$\hat{n} |\sigma\rangle = \sigma |\sigma\rangle$$



We note:

$$\hat{n}(\hat{a}|\sigma\rangle) = (\hat{n}\hat{a})|\sigma\rangle = (\hat{a}\hat{n} - [\hat{a}, \hat{n}])|\sigma\rangle = (\hat{a}\hat{n} - \hat{a})|\sigma\rangle = (\sigma - 1)|\sigma\rangle$$

$\hat{a}|\sigma\rangle$  is an eigenstate of  $\hat{n}$  with eigenvalue  $\sigma - 1$ , so we get:

$$\hat{a}|\sigma\rangle = N_\sigma |\sigma - 1\rangle$$

From  $\langle\sigma|\sigma\rangle = \langle\sigma - 1|\sigma - 1\rangle = 1$  follows:

$$\boxed{\hat{a}|\sigma\rangle = \sqrt{\sigma} |\sigma - 1\rangle}$$

Analogously we have:

$$\boxed{\hat{a}^\dagger|\sigma\rangle = \sqrt{\sigma + 1} |\sigma + 1\rangle}$$

Take a  $\sigma > 0$  and apply  $\hat{a}$  to  $|\sigma\rangle$  many times to get, that  $\sigma$  must be an integer, because otherwise the normalization constant  $\sqrt{\sigma}$  would be imaginary.

There is a special state  $|0\rangle$ , the „vacuum“.

Let us call the states  $|n\rangle$  for  $n \in \mathbb{N}_0$ .

$$\hat{n}|n\rangle = n|n\rangle \quad \hat{a}|n\rangle = \sqrt{n}|n - 1\rangle \quad \hat{a}^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle$$

$$\hat{H}|n\rangle = e_n|n\rangle \quad e_n = \hbar\omega_0 \left(n + \frac{1}{2}\right)$$

If we know  $|0\rangle$ , we can construct an arbitrary  $|n\rangle$ :

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

To calculate  $\langle q|0\rangle$  we use  $\hat{a}|0\rangle = 0$  and represent.

$$\begin{aligned} \langle q|\hat{a}|0\rangle &= 0 \\ \left\langle q \left| \hat{q} + \frac{\mathbf{i}}{m\omega_0} \hat{p} \right| 0 \right\rangle &= 0 \\ \left( q + \frac{\mathbf{i}}{m\omega_0} (-i\hbar\partial_q) \right) \psi_0(q) &= 0 \\ \left( \frac{\hbar}{m\omega_0} \partial_q + q \right) \psi_0(q) &= 0 \\ \psi_0(q) &= N_0 \exp\left(-\frac{1}{2} \left(\frac{q}{\alpha}\right)^2\right) \end{aligned}$$

$N_0$  is fixed by normalization:

$$\int_{-\infty}^{\infty} |\psi_0(q)|^2 dq = 1$$

For example  $\psi_1(q)$  is given by:

$$\begin{aligned} \psi_1(q) &= \langle q|1\rangle = \langle q|\hat{a}^\dagger|0\rangle = \sqrt{\frac{m\omega_0}{2\hbar}} \left( q - \frac{\mathbf{i}}{m\omega_0} (-i\hbar\partial_q) \right) \psi_0(q) = \\ &= N_0 \sqrt{\frac{m\omega_0}{2\hbar}} \left( q - \frac{\hbar}{m\omega_0} \partial_q \right) \exp\left(-\frac{1}{2} \left(\frac{q}{\alpha}\right)^2\right) = \\ &= N_1 \exp\left(-\frac{1}{2} \left(\frac{q}{\alpha}\right)^2\right) \cdot H_1\left(\frac{q}{\alpha}\right) \end{aligned}$$

### 2.3.2 The dynamics of the harmonic oscillator

The Hamiltonian is:

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{m\omega_0^2}{2} \hat{q}^2$$

In principal we need:

$$\hat{U}(t) := \exp\left(-\frac{\mathbf{i}}{\hbar} \hat{H} t\right) = \exp\left(-\frac{\mathbf{i}}{\hbar} \left(\frac{1}{2m} \hat{p}^2 + \frac{m\omega_0^2}{2} \hat{q}^2\right) t\right)$$

Let's calculate in the position representation:

$$\langle q | \hat{U}(t) | q' \rangle =: U(q, q', t)$$

Remember, what  $\hat{U}$  does:

$$|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$$

In position representation this is:

$$\psi(q, t) = \int U(q, q', t) \psi(q', 0) dq'$$

In some books the notation for  $U(q, q', t)$  is  $K(q, q', t)$ .

$$\begin{aligned} U(q, q', t) &= \sum_n \exp\left(-\frac{\mathbf{i}}{\hbar} E_n t\right) \langle q | n \rangle \langle n | q' \rangle = \\ &= \sum_n \exp\left(-\mathbf{i}\omega_0 \left(n + \frac{1}{2}\right) t\right) N_n \exp\left(-\frac{1}{2} \left(\frac{q}{\alpha}\right)^2\right) H_n\left(\frac{q}{\alpha}\right) \cdot \\ &\quad \cdot \left(N_n \exp\left(-\frac{1}{2} \left(\frac{q'}{\alpha}\right)^2\right) H_n\left(\frac{q'}{\alpha}\right)\right)^* = \\ &= \sum_n \exp\left(-\mathbf{i}\omega_0 \left(n + \frac{1}{2}\right) t\right) N_n^2 \exp\left(-\frac{q^2 + q'^2}{2\alpha^2}\right) H_n\left(\frac{q}{\alpha}\right) H_n\left(\frac{q'}{\alpha}\right) = \\ &= \sqrt{\frac{m\omega_0}{2\pi\mathbf{i}\hbar \sin(\omega_0 t)}} \cdot \exp\left(\frac{\mathbf{i}}{\hbar} \left(\frac{m\omega_0}{2 \sin(\omega_0 t)}\right) (q^2 + q'^2) \cos(\omega_0 t) - 2qq'\right) \end{aligned}$$

To check this, consider the limes  $\omega_0 \rightarrow 0$ , for which we get the free particle:

$$U(q, q', t) \xrightarrow{\omega_0 \rightarrow 0} \sqrt{\frac{m}{2\pi\mathbf{i}\hbar t}} \cdot \exp\left(\frac{\mathbf{i}}{\hbar} \frac{m}{2t} (q - q')^2\right)$$

Let's try the Heisenberg picture:

$$\begin{aligned} \frac{d}{dt} \hat{q}_H(t) &= \frac{\mathbf{i}}{\hbar} [H(\hat{q}_H, \hat{p}_H), \hat{q}_H] = \frac{1}{m} \hat{p}_H \\ \frac{d}{dt} \hat{p}_H(t) &= \frac{\mathbf{i}}{\hbar} [H(\hat{q}_H, \hat{p}_H), \hat{p}_H] = -m\omega_0 \hat{q}_H \end{aligned}$$

The initial conditions are:

$$\begin{aligned}\hat{q}_H(t=0) &= \hat{q} \\ \hat{p}_H(t=0) &= \hat{p}\end{aligned}$$

Remember the definition of an operator in the Heisenberg picture:

$$\hat{q}_H = \exp\left(\frac{\mathbf{i}}{\hbar}\hat{H}t\right) \hat{q} \exp\left(-\frac{\mathbf{i}}{\hbar}\hat{H}t\right)$$

Now we define the NESTED COMMUTATOR as

$$\begin{aligned}[\hat{B}, \hat{A}]^{(0)} &:= \hat{A} \\ [\hat{B}, \hat{A}]^{(n+1)} &:= [\hat{B}, [\hat{B}, \hat{A}]^{(n)}]\end{aligned}$$

and use the operator identity for  $\lambda \in \mathbb{C}$  and arbitrary operators  $\hat{A}, \hat{B}$ :

$$\begin{aligned}\exp(\mathbf{i}\lambda\hat{B}) \hat{A} \exp(-\mathbf{i}\lambda\hat{B}) &= \sum_{n=0}^{\infty} \frac{(\mathbf{i}\lambda)^n}{n!} [\hat{B}, \hat{A}]^{(n)} \\ &\approx \hat{A} + \mathbf{i}\lambda [\hat{B}, \hat{A}] + \frac{(\mathbf{i}\lambda)^2}{2!} [\hat{B}, [\hat{B}, \hat{A}]]\end{aligned}$$

If  $[\hat{A}, \hat{B}] = 0$ , we get just  $\hat{A}$ .

If  $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ , we get  $\hat{A} + \mathbf{i}\lambda [\hat{B}, \hat{A}]$ .

Now we come back to the problem at hand:

$$\lambda = \frac{t}{\hbar} \qquad \hat{B} = \hat{H} \qquad \hat{A} = \hat{q}$$

We now:

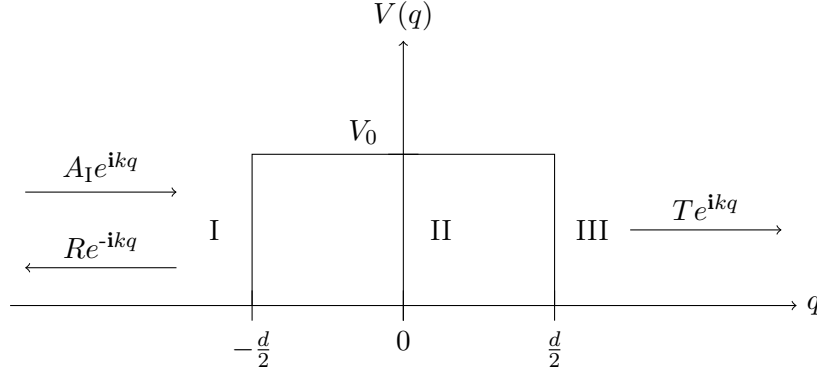
$$\begin{aligned}[\hat{H}, \hat{q}] &= \frac{1}{2m} [\hat{p}^2, \hat{q}] + \underbrace{[V(\hat{q}), \hat{q}]}_{=0} = \frac{1}{2m} (\hat{p} [\hat{p}, \hat{q}] + [\hat{p}, \hat{q}] \hat{p}) = -\frac{\mathbf{i}\hbar}{m} \hat{p} \\ [\hat{H}, \hat{p}] &= \frac{1}{2m} \underbrace{[\hat{p}^2, \hat{p}]}_{=0} + \frac{m\omega_0^2}{2} [\hat{q}^2, \hat{p}] = \mathbf{i}\hbar m\omega_0^2 \hat{q}\end{aligned}$$

Together this inductively gives:

$$\begin{aligned}\hat{q}_H(t) &= \hat{q} \cos(\omega_0 t) + \frac{1}{m\omega_0} \hat{p} \sin(\omega_0 t) \\ \hat{p}_H(t) &= m\omega_0 \hat{q} \sin(\omega_0 t) + \hat{p} \cos(\omega_0 t)\end{aligned}$$

## 2.4 The tunneling effect

Look at the potential:



The problem is unbounded and therefore the spectrum is continuous and every energy  $E \in \mathbb{R}_{>0}$  is allowed.

Imagine that we inject the particle from the left with a certain energy  $E$ .

$$k := \frac{\sqrt{2mE}}{\hbar} \qquad k_V := \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

The goal is to find  $R$  and  $T$  as functions of the Energy  $E$ , the height  $V_0$  and width  $d$  of the potential.

Business as usual:

The boundary condition is, that  $A_I$  is know and the coefficient of  $e^{-ikq}$  in the region III is zero. We consider only  $E < V_0$ .

$$\begin{aligned} \psi_I(q) &= A_I e^{ikq} + R e^{-ikq} \\ \psi_{II}(q) &= A_{II} e^{k_V q} + B_{II} e^{-k_V q} \\ \psi_{III}(q) &= T e^{ikq} \end{aligned}$$

We have the four variables  $R, T, A_{II}$  and  $B_{II}$ . Now we impose the matching conditions:

$$\begin{aligned} \psi_I\left(-\frac{d}{2}\right) &= \psi_{II}\left(-\frac{d}{2}\right) & \psi'_I\left(-\frac{d}{2}\right) &= \psi'_{II}\left(-\frac{d}{2}\right) \\ \psi_I\left(\frac{d}{2}\right) &= \psi_{II}\left(\frac{d}{2}\right) & \psi'_I\left(\frac{d}{2}\right) &= \psi'_{II}\left(\frac{d}{2}\right) \end{aligned}$$

This system looks like:

$$\mathbb{M} \cdot \begin{pmatrix} B \\ A_{II} \\ B_{II} \\ T \end{pmatrix} = A_I \begin{pmatrix} -e^{-ik\frac{d}{2}} \\ -ik e^{ik\frac{d}{2}} \\ 0 \\ 0 \end{pmatrix}$$

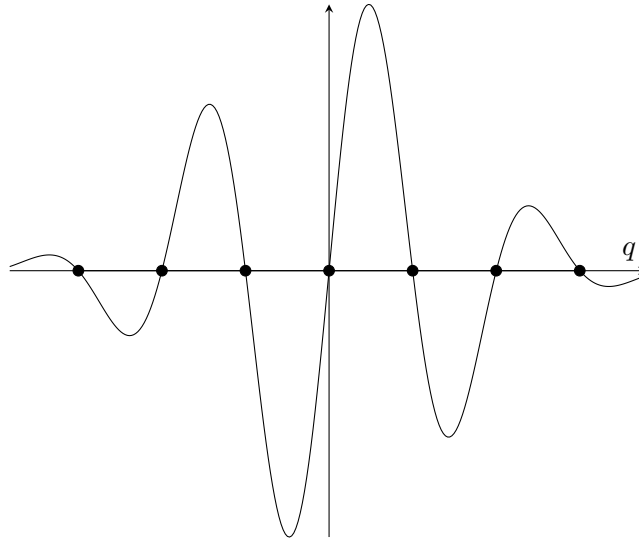
For  $k_V d \gg 1$  this roughly gives:

$$\begin{aligned} T(E) &\sim A_I e^{-\frac{k_V(E)d}{2}} \\ \frac{T(E, d)}{T(E, d')} &\sim e^{-\frac{k_V(E)}{2}(d-d')} \end{aligned}$$

## 2.5 Two formal consequences of the position representation of the Schrödinger equation

### 2.5.1 Oscillation theorem (1D, bounded)

If  $\langle q | e_n \rangle$  has  $k$ -nodes, that means  $\psi_n(q) = 0$  is  $k$  times, then  $\langle q | e_{n+1} \rangle$  has  $k + 1$  nodes.



### 2.5.2 The continuity equation

## 3 Symmetries in quantum mechanics (Angular momentum)

## 4 Perturbation theory (time-independent)

## 5 Perturbation theory (time-dependent)



## 6 Many-particle systems

## 7 Scattering theory

# Appendix

# Acknowledgements

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Andreas Völklein

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