Kommutative Algebra

Vorlesung von
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im Sommersemester 2012
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Stand: 17. April 2012

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Literatur

- Michael Francis Atiyah, Ian G. Macdonald: *Introduction to commutative algebra*, Westview Press, 1994; ISBN: 0-201-40751-5
- David Eisenbud: *Commutative algebra with a view toward algebraic geometry*, Springer, 2004; ISBN: 3-540-94269-6
- Hideyuki Matsumura: *Commutative ring theory*, Cambridge University Press, 2005 ISBN: 0-521-36764-6
- Nicolas Bourbaki: *Commutative algebra*, Springer, 1991 ISBN: 3-540-64239-0

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1 Ringe und Ideale

1.1 Definition (Ring, Ringhomomorphismus)

i) Ein (*unitärer*, *kommutativer*) *Ring* ist ein Tupel $(A, +, \cdot, 0, 1)$ mit den Eigenschaften:

a) $(A, +, 0)$ ist eine abelsche Gruppe.

b) Für alle $x, y, z \in A$ gelten:

$$\begin{aligned} (xy)\,z &= x\,(yz) && \text{(Assoziativität)} \\ x\,(y+z) &= xy + xz && \text{(Distributivität)} \\ xy &= yx && \text{(Kommutativität)} \\ x \cdot 1 &= x && \text{(neutrales Element)} \end{aligned}$$

ii) Sind A und B Ringe, so ist ein *Ringhomomorphismus* (*von* A *nach* B) (Abkürzung: Ringhom.) eine Abbildung $f : A \to B$, sodass für alle $x, y \in A$ gilt:

a) $f(x+y) = f(x) + f(y)$

b) $f(xy) = f(x)\,f(y)$

c) $f(1) = 1$

Aus a) folgt direkt $f(0) = 0$, da Ringe additive Gruppen sind.

Aus b) folgt aber nicht c), da Ringe im Allgemeinen keine multiplikativen Gruppen sind.

1.2 Beispiel

i) Bekannte Ringe sind $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}[\mathbf{i}] = \left\{a + b\mathbf{i} \middle| a, b \in \mathbb{Z}\right\} \subseteq \mathbb{C}$, der Polynomring $A[X]$ eines Rings A und der Produktring $A \times B$ der Ringe A und B.

ii) Für jeden Ring A existiert genau ein Ringhomomorphismus $\mathbb{Z} \to A$.

Für jeden Ring A ist die Abbildung von Mengen

$$\begin{aligned} \left\{f : \mathbb{Z}[X] \to A \middle| f \text{ ist Ringhom.}\right\} &\xrightarrow{\sim} A \\ f &\mapsto f(X) \end{aligned}$$

bijektiv. (Die Ringhomomorphismen f sind die Einsetzungshomomorphismen.)

1.3 Definition (Unterring)

Sei A ein Ring.
Eine Teilmengen $B \subseteq A$ heißt *Unterring* (*von* A), wenn für alle $x,y \in B$ gilt:

i) $x - y \in B$, $x \cdot y \in B$

ii) $1 \in B$

In diesem Fall ist $\left(B, + \big|_{B \times B}, \cdot \big|_{B \times B}, 0, 1\right)$ wieder ein Ring.

1.4 Beispiel

$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ sind Unterringe, nicht aber $\mathbb{N} \subseteq \mathbb{Z}$.

1.5 Definition und Bemerkung (Ideal, Quotientenring)

Sei A ein Ring.

i) Eine Teilmenge $I \subseteq A$ heißt *Ideal* (*von* A), falls folgende Eigenschaften gelten:

   a) $I \subseteq (A, +, 0)$ ist eine Untergruppe.
   b) Für alle $x \in A$ und $y \in I$ gilt $xy \in I$.

ii) Ist $I \subseteq A$ ein Ideal, so ist die für alle $x,y \in A$ durch

$$x \equiv y \mod I \qquad :\Leftrightarrow \qquad x - y \in I$$

(lies: „x *ist kongruent zu* y *modulo* I“) auf A definierte Relation eine Äquivalenzrelation und es existiert genau eine Ringstruktur A/I, sodass die kanonische Abbildung

$$\begin{aligned} \pi : A &\to A/I \\ x &\mapsto \pi(x) := [x] := (x \mod I) \end{aligned}$$

ein Ringhomomorphismus ist.
Es gilt $\ker(\pi) = I$. Der Ring A/I heißt *Quotientenring* (*von* A *bezüglich* I).

iii) Sind $I \subseteq A$ ein Ideal und B ein Ring, so ist die Abbildung

$$\begin{aligned} \left\{\overline{f} : A/I \to B \,\middle|\, \overline{f} \text{ ist Ringhom.}\right\} &\xrightarrow{\sim} \left\{f : A \to B \,\middle|\, f \text{ ist Ringhom.}, f(I) = 0\right\} \\ \overline{f} &\mapsto \overline{f} \circ \pi \end{aligned}$$

bijektiv.
Skizze:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi"'] & B \\ A/I \arrow[ur, "\exists!_{\overline{f}} : f = \overline{f} \circ \pi \Leftrightarrow f(I) = 0 (\Leftrightarrow I \subseteq \ker(f))"'] & \end{tikzcd}$$

(universelle Eigenschaft des Quotientenrings)

1.6 Beispiel (Kern, Bild)

i) Ist $f : A \to B$ ein Ringhomomorphismus, so ist

$$\ker(f) := \left\{x \in A \middle| f(x) = 0\right\} \subseteq A$$

ein Ideal und das Bild

$$\operatorname{im}(f) = \left\{f(x) \middle| x \in A\right\} \subseteq B$$

ein Unterring.

(Ist $B \subseteq A$ ein Unterring, so ist $\iota : B \hookrightarrow A$ ein Ringhomomorphismus mit $\operatorname{im}(\iota) = B$.)

ii) Sind A ein Ring und $n \in \mathbb{N}_{>0}$, so gilt:

$$\left|\left\{f : \mathbb{Z}/_{n\mathbb{Z}} \to A \text{ ist Ringhom.}\right\}\right| = \begin{cases} 1 & \text{falls } \underbrace{1 + \ldots + 1}_{n\text{-mal}} = 0 \text{ in } A \\ 0 & \text{sonst} \end{cases}$$

(Verwende 1.2 ii) und 1.5 iii) mit $I = n\mathbb{Z} \subseteq A = \mathbb{Z}$.)

1.7 Proposition (Ideale des Quotientenrings)

Seien A ein Ring, $I \subseteq A$ ein Ideal und $\pi : A \to A/_I$ der kanonische Ringhomomorphismus. Dann ist die Abbildung

$$\begin{aligned} \Phi : M := \left\{J \middle| J \subseteq A \text{ Ideal mit } I \subseteq J\right\} &\xrightarrow{\sim} N := \left\{\overline{J} \middle| \overline{J} \subseteq A/_I \text{ Ideal}\right\} \\ J &\mapsto \Phi(J) := \pi(J) \end{aligned}$$

wohldefiniert und bijektiv und erfüllt:
Für alle Ideale $J_1, J_2 \subseteq A$ mit $I \subseteq J_1, J_2$ gilt:

$$J_1 \subseteq J_2 \Leftrightarrow \Phi(J_1) \subseteq \Phi(J_2)$$

Beweis

Ist $J \in M$, so ist $\Phi(J) = \pi(J) \subseteq A/_I$ als Bild des Ringhomomorphismus π ein Unterring von $A/_I$ und somit ist insbesondere $(\pi(J), +, 0)$ eine Untergruppe.
Seien $x \in A/_I$ und $y \in \Phi(J)$ gegeben. Wegen $\Phi(J) = \pi(J)$ und $A/_I = \pi(A)$ gibt es ein $x_0 \in A$ und ein $y_0 \in J$ mit $x = \pi(x_0)$ und $y = \pi(y_0)$.

$$xy = \pi(x_0)\,\pi(y_0) = \pi(x_0 y_0)$$

Da J ein Ideal ist, folgt aus $y_0 \in J$ und $x_0 \in A$ schon $x_0 y_0 \in J$. Daher ist $xy = \pi(x_0 y_0) \in \pi(J)$, weswegen $\pi(J)$ ein Ideal ist. Also ist Φ wohldefiniert.
Betrachte die Abbildung:

$$\begin{aligned} \Psi : N &\to M \\ \overline{J} &\mapsto \pi^{-1}\left(\overline{J}\right) \end{aligned}$$

Seien $\overline{J} \subseteq A/_I$ ein Ideal und $a,b \in \Psi\left(\overline{J}\right) = \pi^{-1}\left(\overline{J}\right)$, das heißt $\pi(a),\pi(b) \in \overline{J}$. Dann gilt:

$$\pi(a-b) = \underbrace{\pi(a)}_{\in \overline{J}} - \underbrace{\pi(b)}_{\in \overline{J}} \overset{\overline{J}\text{ Ideal}}{\in} \overline{J}$$

Also ist $a-b \in \pi^{-1}\left(\overline{J}\right)$ und somit $\left(\Psi\left(\overline{J}\right),+,0\right)$ eine Untergruppe von A.
Seien $x \in A$ und $y \in \Psi\left(\overline{J}\right)$, das heißt $\pi(y) \in \overline{J}$, so gilt:

$$\pi(x \cdot y) = \pi(x)\underbrace{\pi(y)}_{\in \overline{J}} \overset{\overline{J}\text{ Ideal}}{\in} \overline{J}$$

Also ist $\Psi\left(\overline{J}\right) \subseteq A$ ein Ideal.
Da für alle Ideale $\overline{J} \in A/_I$ schon $0 \in \overline{J}$ gilt, folgt $\pi^{-1}\left(\overline{J}\right) \supseteq \pi^{-1}(0) = \ker(\pi) = I$. Daher ist Ψ wohldefiniert.
Nun gilt:

$$\begin{aligned}
(\Phi \circ \Psi)\left(\overline{J}\right) &= \left(\pi \circ \pi^{-1}\right)\left(\overline{J}\right) \overset{\pi\text{ surjektiv}}{=} \overline{J} \\
(\Psi \circ \Phi)(J) &= \left(\pi^{-1} \circ \pi\right)(J) = \left\{a+b \middle| a \in J, b \in I\right\} \overset{I \subseteq J}{=} J
\end{aligned}$$

Also ist Ψ die Umkehrabbildung zu Φ, weswegen Φ bijektiv ist. $\square_{1.7}$

1.8 Beispiel

Die Ideale des Ringes $\mathbb{Z}/_{6\mathbb{Z}} = \{\overline{0},\overline{1},\ldots,\overline{5}\}$ sind genau die Hauptideale:

$$\left(\overline{0}\right), \left(\overline{3}\right), \left(\overline{2}\right), \left(\overline{1}\right) \subseteq \mathbb{Z}/_{6\mathbb{Z}}$$

1.9 Definition (Integritätsring)

i) Sei A ein Ring. Ein Element $x \in A$ heißt genau dann *Nullteiler*, wenn es ein $x \in A \setminus \{0\}$ mit $xy = 0$ gibt.

ii) Ein *Integritätsring* (Abkürzung: IR) ist ein Ring $A \neq \{0\}$, in dem $0 \in A$ der einzige Nullteiler ist.

1.10 Beispiel

i) $\mathbb{Z}$ ist ein Integritätsring.

ii) Ist A ein Integritätsring, so ist auch $A[X]$ ein Integritätsring. (Gradformel!)

iii) Der Ring $A := \mathbb{Z}[X]/_{\left(X^2\right)}$ ist kein Integritätsring, denn für $\overline{X} := \left(X \mod \left(X^2\right)\right)$ gilt $\overline{X} \neq 0$, aber auch $\overline{X} \cdot \overline{X} = 0$, weswegen $\overline{X} \in A$ ein Nullteiler ist.
Auch $\mathbb{Z}/_{6\mathbb{Z}}$ ist kein Integritätsring, denn $\underbrace{\overline{2}}_{\neq \overline{0}} \cdot \underbrace{\overline{3}}_{\neq \overline{0}} = \overline{0}$.

1.11 Definition (nilpotent)

Sei A ein Ring. Ein $x \in A$ heißt genau dann *nilpotent*, wenn es ein $n \in \mathbb{N}_{>0}$ gibt mit $x^n = 0$.

1.12 Beispiel

i) Ist $0 \neq x \in A \neq \{0\}$ nilpotent, so ist x ein Nullteiler, denn für $N := \min\left\{n \in \mathbb{N}_{>0} \middle| x^n = 0\right\}$ gilt $N \geq 1$ und wegen der Minimalität von N ist $x^{N-1} \neq 0$. Also folgt aus

$$0 = x^N = \underbrace{x}_{\neq 0} \cdot \underbrace{x^{N-1}}_{\neq 0}$$

schon, dass x ein Nullteiler ist.

ii) In dem Produktring $A := \mathbb{Z} \times \mathbb{Z}$ ist $x := (1,0)$ ein Nullteiler, da $x \cdot (0,1) = (0,0) = 0$ gilt, aber nicht nilpotent, denn für alle $n \in \mathbb{N}_{>0}$ gilt:

$$x^n = (1,0)^n = (1^n,0^n) = (1,0) = x \neq 0$$

1.13 Definition und Bemerkung (Einheit, Einheitengruppe)

Sei A ein Ring. Ein $x \in A$ heißt genau dann *Einheit* (*in* A), wenn es ein $y \in A$ mit $xy = 1$ gibt. Die Menge $A^* = \left\{x \in A \middle| x \text{ ist Einheit}\right\}$ der Einheiten ist eine kommutative Gruppe bezüglich der Multiplikation und heißt *die Einheitengruppe von A*.
(Beachte: $\{0\}^* = \{0 = 1\}$)

1.14 Beispiel

i) $\mathbb{Z}^* = \{\pm 1\}$, $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$.

ii) $\mathbb{Z}[\mathbf{i}]^* = \{\pm 1, \pm \mathbf{i}\}$

iii) Ist A ein Integritätsring, dann ist $(A[X])^* = A^*$. (Folgt aus der Gradformel.)

1.15 Beispiel und Definition

Sei A ein Ring.

i) Für jedes $x \in A$ ist $(x) := \left\{xy \middle| y \in A\right\} \subseteq A$ ein Ideal. Es heißt *das von x erzeugte Hauptideal* (Abkürzung: HI).

ii) Ist A ein Integritätsring, so gilt für alle $x,y \in A$:

$$(x) = (y) \quad \Leftrightarrow \quad \underset{u \in A^*}{\exists} : x = uy$$

Insbesondere gilt:

$$(x) = 1 = A \quad \Leftrightarrow \quad x \in A^* \tag{1.1}$$

(Beachte: (1.1) gilt für jeden Ring A.)

iii) A heißt genau dann *Hauptidealring* (Abkürzung: HIR), wenn A ein Integritätsring ist, in dem jedes Ideal $I \subseteq A$ ein Hauptideal ist.

iv) A heißt genau dann *Körper*, wenn $A^* = A \setminus \{0\}$ und $A \neq \{0\}$ ist.

1.16 Beispiel

i) Die Ringe $\mathbb{Z}, \mathbb{Z}[\mathbf{i}]$ und $k[X]$ für einen Körper k, nicht aber $\mathbb{Z}[X]$ oder $k[X,Y]$ sind Hauptidealringe.

   Die Ringe $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ und $\mathbb{F}_p := \mathbb{Z}/_{p\mathbb{Z}}$ für eine Primzahl p sind Körper, nicht aber $\mathbb{Z}$.

ii)

Anhang

Danksagungen

Mein besonderer Dank geht an Professor Naumann, der diese Vorlesung hielt und es mir gestattete, diese Vorlesungsmitschrift zu veröffentlichen.
Außerdem möchte ich mich ganz herzlich bei allen bedanken, die durch aufmerksames Lesen Fehler gefunden und mir diese mitgeteilt haben.

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