Partial Differential Equations I

lecture by

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during the summer semester 2013

revision and layout in LyX by

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This script does *not* replace the lecture.

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A transparent copy of the recent version of this document can be downloaded from:

`https://github.com/andiv/PDE1`

Literature

Elliptic and parabolic partial differential equations:

- Jürgen Jost: *Partial Differential Equations*; Springer, 2007
  ISBN: 978-0-387-49318-3; doi: 10.1007/978-0-387-49319-0
  (good book, but not all details, small errors)
- Lawrence C. Evans: *Partial Differential Equations*; American Mathematical Society, 2010
  ISBN: 978-0-8218-4974-3
  (part of the lecture follows this book, lots of details)
- David Gilborg, Neil S. Trudinger: *Elliptic Partial Differential Equations of second order*; Springer, 2001
  ISBN: 3-540-41160-7
  (classic textbook, complete treatment)

Hyperbolic partial differential equations (for the lecture “Partial Differential Equations II”):

- Fritz John: *Partial Differential Equations*; Springer, 1999
  ISBN: 0-387-90609-6
- Michael E. Taylor: *Partial Differential Equations I - III*; Springer, 1997
  ISBN: 0-387-94653-5, 0-387-94651-9, 0-387-94652-7
  (nice detailed text books)
- Joel Smoller: *Shock waves and reaction-diffusion equations*; Springer, 1994
  ISBN: 3-540-94259-9
  (nicely presented, good motivations, covers most of the material)
- Friedrich Sauvigny: Partial Differential Equations I-II; Springer, 2012
  ISBN: 978-1-4471-2981-3, 978-1-4471-2984-4;
  doi: 10.1007/978-1-4471-2981-3, 10.1007/3-540-27540-1
- and many more . . .

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1 A brief introduction

An ordinary differential equation (ODE) can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}u\left(t\right)=\dot{u}\left(t\right)=v\left(t,u\right)$$
$$u:I\subseteq\mathbb{R}\to\mathbb{R}^N$$

This equation involves only derivatives with respect to *one* variable t.

$$\frac{\partial}{\partial x}f\left(x,y\right)+\frac{\partial}{\partial y}f\left(x,y\right)=0$$

This is an example for a partial differential equation.

1.1 Definition (Partial Differential Equation)

A *partial differential equation* (PDE) is a (scalar) equation, which involves partial derivatives of an unknown function $u:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}$. We always assume that $\Omega\subseteq\mathbb{R}^n$ is open.
More generally, a *system of partial differential equations* is a system of equations involving partial derivatives of a function $u:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^N$.
Similarly one can define partial differential equations on manifolds.
For ordinary differential equations we considered the initial-value problem:

$$\dot{u}\left(t\right)=v\left(t,u\right)\qquad\qquad u\left(t_0\right)=u_0$$

For partial differential equations one considers
- the initial-value problem and
- the boundary-value problem.

1.2 Examples

1. Cauchy-Riemann equations: Let

$$f:\Omega\overset{\text{open}}{\subseteq}\mathbb{C}\to\mathbb{C}$$

be holomorphic.

$$f=a+\mathbf{i}b\qquad\qquad a:=\mathrm{Re}\left(f\right)\qquad\qquad b:=\mathrm{Im}\left(f\right)$$

$$\frac{\partial a}{\partial x}=\frac{\partial b}{\partial y}\qquad\qquad\frac{\partial b}{\partial x}=-\frac{\partial a}{\partial y}$$

This is a system of two partial differential equations.

$$u := \begin{pmatrix} a \\ b \end{pmatrix} \qquad\qquad u : \Omega \subseteq \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2$$

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = \frac{\partial b}{\partial x \partial y} - \frac{\partial b}{\partial y \partial x} = 0$$

$$\Rightarrow \qquad \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{=:\Delta} a = 0 \qquad\qquad \Delta b = 0$$

This is the *Laplace equation* with the *Laplace operator* Δ. Solutions of the Laplace equation are called *harmonic functions*.

2. Let (M, g) be a Riemannian manifold. Here exists the Laplace-Beltrami operator $\Delta\backslash$.
   - In the special case $M = \mathbb{R}^n$ we have:

   $$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$
   $$\Delta\varphi = 0$$

   - With the Riemannian metric g_{ij} we can define:

   $$\Delta\backslash\varphi = g^{ij}\nabla_i\nabla_j\varphi = \operatorname{div}(\operatorname{grad}(\varphi)) = \frac{1}{\sqrt{\det(g)}}\partial_j\left(\sqrt{\det(g)}g^{jk}\partial_k\varphi\right)$$

   This gives an elliptic equation.
3. Newton's gravitational law: Let $\varrho(x)$ be the mass density and $\varphi(x)$ the Newtonian potential.

   $$\Delta\varphi = \underbrace{-4\pi\varrho}_{\text{inhomogeneity}}$$

   Such an inhomogeneous Laplace equation is usually referred to as Poisson equation and it is elliptic.
4. Heat flow equation (Wärmeleitungsgleichung): Let $\varphi(t, x)$ be the temperature at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^n$.

   $$\partial_t\varphi(t, x) = \Delta\varphi(t, x)$$

   This is a parabolic equation.
5. The Schrödinger equation is a parabolic equation:

   $$\mathbf{i}\hbar\partial_t\psi(t, x) = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi(t, x)$$

   Additionally to the the heat flow equation there is the potential V, but more important there is a factor of $\mathbf{i}$ in front of the partial derivative. The time-independent Schrödinger equation is:

   $$E\psi(x) = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi(x)$$

   This is similar to the Poisson equation and also elliptic.

6. The wave equation
$$\left(\partial_t^2 - \Delta_x\right) \psi\left(t, x\right) = 0$$
is hyperbolic. We will consider it in the lecture "Partial Differential Equations II".
7. Maxwell's equations: $E\left(t, x\right)$ is the electric field and $B\left(t, x\right)$ the magnetic field.
$$\begin{aligned} \operatorname{div}\left(E\right) &= 4\pi\varrho && \text{Gauss law} \\ \operatorname{rot}\left(E\right) &= -\partial_t B && \text{Maxwell} \\ \operatorname{div}\left(B\right) &= 0 && \\ \operatorname{rot}\left(B\right) &= 4\pi j - \partial_t E && \text{Faraday} \end{aligned}$$
This is a system of 8 partial differential equation.
8. Einstein's field equation:
$$R_{ij} - \frac{1}{2} R g_{ij} = 4\pi\kappa T_{ij}$$
This is a geometric partial differential equation. R_{ij} is the Ricci curvature, R the scalar curvature and T_{ij} the energy-momentum tensor. It is a system of 10 partial differential equations.
9. Equations of relativistic quantum mechanics:
$$\left(-\partial_t^2 + \Delta\right) \psi = m^2 \psi$$
This is the Klein-Gordon equation with the mass m.
$$\mathbf{i}\gamma^j \partial_j \psi = m\psi$$
This is the Dirac equation, a system of 4 complex-valued or 8 real-valued partial differential equations for a particle with spin $\frac{1}{2}$.
10. Water waves can be described by the Korteweg-de Vries equation:
$$\partial_t u + u \partial_x u + \partial_x^3 u = 0$$

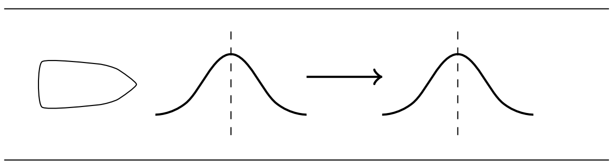

Figure 1.1: Solitons (discovered by John Russel in 1834): When the ship suddenly stops, the water flows on along the channel. This wave moves with a constant speed and its shape stays the same.

11. Shock waves: Burger's equation
$$\partial_t u + u \partial_x u = 0$$
is hyperbolic.
12. Turbulence can be described by the incompressible Navier-Stokes equations for the velocity $v : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.
$$\begin{aligned} \operatorname{div}\left(v\right) &= 0 \\ \rho \partial_t v^j + \rho v^i \partial_i v^j - \eta \Delta v^j &= -\partial_j P \end{aligned}$$
Here ϱ is the gas density, P the pressure and η the viscosity.

1.3 Classification

I) The *order* of a partial differential equation is the highest order of the derivatives in it.

$$
\begin{aligned}
\Delta u &= f && \text{second order} \\
\partial_t \varphi &= \Delta \varphi && \text{second order} \\
\partial_t u + u \partial_x u + \partial_x^3 u &= 0 && \text{third order}
\end{aligned}
$$

II) Algebraic classification:

a) *Linear* equations: The unknown function u and its derivatives appear only linearly.

$$
\begin{aligned}
\partial_t u &= u && \text{linear} \\
\partial_t u + u \partial_x u &= 0 && \text{non-linear}
\end{aligned}
$$

b) Linear *homogeneous* equations: If u is a solution, then λu for $\lambda \in \mathbb{R}$ is also a solution.

$$
\begin{aligned}
\Delta u &= 0 && \text{linear homogeneous} \\
\Delta u &= \varrho && \text{linear inhomogeneous} \\
\Delta\backslash u &= 0 && \text{linear homogeneous}
\end{aligned}
$$

c) Linear with *constant coefficients*:

$$
\begin{aligned}
\Delta u &= \rho && \text{linear with constant coefficients} \\
\Delta\backslash u &= \varrho && \text{in general non-constant coefficients}
\end{aligned}
$$

III) Classification by type: elliptic, parabolic, hyperbolic
Here we only consider scalar second order equations with $x \in \Omega \subseteq \mathbb{R}^n$.

$$
F\left(x, u, \mathrm{D}u, \mathrm{D}^2 u\right) = 0
$$
$$
F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}
$$

$$
A_{ij} := \frac{\partial F\left(x, u, p_i, p_{ij}\right)}{\partial p_{ij}}
$$

is a symmetric $n \times n$ matrix.

- If A is positive definite, the equation is called *elliptic*.
- If A has $n-1$ positive and one negative eigenvalue, the equation is called *hyperbolic*.
- If A has $n-1$ positive eigenvalues and a non-trivial kernel, the equation is called *parabolic*.
- If all eigenvalues are negative or $n-1$ are negative, then we replace F by $-F$.
  All other case of *mixed type* are difficult and we do not consider them in this lecture.

1.4 Examples

Consider the Poisson equation:

$$
\Delta u = \varrho
$$

$$\begin{aligned} F\left(x, u, \mathrm{D}u, \mathrm{D}^2 u\right) &= -\varrho\left(x\right) + \delta^{ij}\partial_{ij}u \\ F\left(x, u, p_i, p_{ij}\right) &= -\varrho\left(x\right) + \delta^{ij}p_{ij} \\ A_{ij} = \frac{\partial F\left(x, u, p_i, p_{ij}\right)}{\partial p_{ij}} &= \delta_{ij} \end{aligned}$$

So we have $A = \mathbb{1}$ and thus the equation is elliptic.
Now consider the inhomogeneous wave equation:

$$\left(\partial_t^2 - \Delta\right)\phi\left(t, x_1, x_2, x_3\right) = \varrho$$

$$F\left(x, u, p_i, p_{ij}\right) = \varrho\left(x\right) + \eta^{ij}p_{ij} \qquad \eta = \mathrm{diag}\left(-1, 1, 1, 1\right)$$

So $A = \eta$ has one negative and three positive eigenvalues which means that the equation is hyperbolic.

$$\partial_t\phi = \Delta\phi$$

$$\begin{aligned} F\left(x, u, \mathrm{D}\phi, \mathrm{D}^2\phi\right) &= -\partial_0\phi + \sum_{i,j=1}^{3}\delta^{ij}\partial_{ij}\phi \\ A_{ij} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore the equation is parabolic.

2 Distributions and Fourier transform

Motivation

We want to solve partial differential equations with constant coefficients in $\Omega = \mathbb{R}^n$, for example:

$$\left(-\partial_t^2 + \Delta\right)\phi = 0$$

Now we make a "plane wave ansatz" with $t, \omega \in \mathbb{R}$ and $k, x \in \mathbb{R}^{n-1}$:

$$\begin{aligned} \phi(t,x) &= e^{-\mathbf{i}\omega t + \mathbf{i}\langle k,x\rangle} \\ \partial_t \phi(t,x) &= -\mathbf{i}\omega\phi(t,x) \\ \partial_j \phi(t,x) &= \mathbf{i}k_j\phi(t,x) \end{aligned}$$

This gives an algebraic equation:

$$\begin{aligned} \left(-(-\mathbf{i}\omega)^2 + (\mathbf{i}k)^2\right)\phi &= 0 \\ \Leftrightarrow \qquad \omega^2 &= k^2 \end{aligned}$$

We also want to differentiate non-smooth functions, e.g.:

$$\Delta_{\mathbb{R}^3}\frac{1}{|x|} = -4\pi\delta(x)$$

$\delta(x)$ is called Dirac δ-distribution.

2.1 The Schwartz space and distributions

Laurent Schwartz was the first to investigate distributions systematically. He was awarded the fields medal for this.

2.1.1 Definition (Multi-index)

For $\mathbb{R}^n$ we denote indices by $i, j, k \in \{1, \ldots, n\}$. We call $\alpha = (i_1, \ldots, i_k)$ with $i_l \in \{1, \ldots, n\}$ a *multi-index*. $|\alpha| := k$ is called the *order* or *absolute value* of the multi-index.
With this we can write differentials of order k as

$$\mathrm{D}^\alpha := \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} \tag{2.1}$$

and homogeneous polynomials of degree k in the components of a vector $x = \left(x^1, \ldots, x^n\right)$ as:

$$x^\alpha := x^{i_1} \cdots x^{i_k} \tag{2.2}$$

For $f \in C^\infty(\mathbb{R}^n)$ and $r, s \in \mathbb{N}$ we define the Schwartz norm:

$$\|f\|_{r,s} := \sum_{\alpha, |\alpha| \le r} \sum_{\beta, |\beta| \le s} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \mathrm{D}^\beta f(x) \right| \tag{2.3}$$
$$\|f\|_{0,0} = \sup_{x \in \mathbb{R}^n} |f(x)| = \|f\|_{C^0}$$

2.1.2 Definition (Schwartz space)

The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ is the vector space of all $f \in C^\infty(\mathbb{R}^n)$ for which all Schwartz norms are finite, i.e. for all $r, s \in \mathbb{N}$ holds:

$$\|f\|_{r,s} < \infty$$

This space is an infinite-dimensional vector space.
On a normed space $(E, \|.\|)$, the topology is given by the open sets.
$\Omega \subseteq E$ is defined as *open* if holds:

$$\underset{x \in \Omega}{\forall} \underset{\varepsilon > 0}{\exists} : B_\varepsilon(x) \subseteq \Omega$$

A subset $\Omega \subseteq \mathcal{S}(\mathbb{R}^n)$ is called *open* if for every $f \in \Omega$ there is a $\varepsilon > 0$ and $r, s \in \mathbb{N}$ such that holds:

$$\left\{ g \in \mathcal{S} \middle| \|g - f\|_{r,s} < \varepsilon \right\} \subseteq \Omega \tag{2.4}$$

Note: This topology is fine, because it involves many open sets, since the condition for open only involves the statement "there are $r, s \in \mathbb{N}$".
Convergence $f_n \to f$ in $\mathcal{S}$ means that *every* open neighborhood U of f contains almost all f_n. For a finer topology, the condition for a sequence to converge is stronger.

2.1.3 Theorem

Convergence $f_n \to f$ in $\mathcal{S}$ is equivalent to the convergence $\|f_n - f\|_{r,s} \to 0$ for all $r, s \in \mathbb{N}$.

Proof

"$\Rightarrow$": Suppose that $f_n \to f$ converges. By definition of the convergence, every open neighborhood of f contains almost all f_n. For all $r, s \in \mathbb{N}$ the sets $U_\varepsilon^{r,s} := \left\{ g \middle| \|g - f\|_{r,s} < \varepsilon \right\}$ are open by definition. So the inequality

$$\|f_n - f\|_{r,s} < \varepsilon$$

holds for almost all f_n and thus converges $\|f_n - f\|_{r,s} \to 0$.
"$\Leftarrow$": Assume that $\|f_n - f\|_{r,s} \to 0$ converges for all $r, s \in \mathbb{N}$. Let A be an open neighborhood of f. This means by definition that there exist $r, s \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$ with $U_\varepsilon^{r,s} \subseteq A$. For this (r, s) we know that $\|f_n - f\|_{r,s} \to 0$ converges. Hence there exists a $N \in \mathbb{N}$ such that $\|f_n - f\|_{r,s} < \varepsilon$ holds for all $n \in \mathbb{N}_{>N}$, in other words $f_n \in U_\varepsilon^{r,s} \subseteq A$. So $f_n \to f$ converges in $\mathcal{S}$. $\square_{2.1.3}$

A vector space with a topology generated by a family of norms or semi-norms is a *uniform space* and is called *topological vector space.*

2.1.4 Definition (tempered distribution)

Let $\mathcal{S}^*(\mathbb{R}^n)$ be the dual space of $\mathcal{S}(\mathbb{R}^n)$. It is called the space of *tempered distributions* (temperierte Distributionen).

In linear algebra for a finite-dimensional vector space V, the dual space $V^* = L(V, \mathbb{R})$ is the space of linear functionals. V^* is again a vector space with $\dim(V^*) = \dim(V)$.
Here $\mathcal{S}(\mathbb{R}^n)$ is an infinite-dimensional vector space with a topology. $\mathcal{S}^*(\mathbb{R}^n)$ is the space of all *continuous* linear functionals.

In a Banach space $(E, \|.\|)$ holds: A linear functional $A : E \to \mathbb{R}$ is continuous if and only if A is bounded, i.e. $|Au| \leq c\|U\|$ for all $u \in E$.

2.1.5 Lemma

A linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ is continuous if and only if there are $r, s \in \mathbb{N}$ and a $c \in \mathbb{R}_{>0}$ such that holds:

$$|Tf| \leq c\|f\|_{r,s} \tag{2.5}$$

Proof

"$\Leftarrow$": Assume that (2.5) holds for some $r, s \in \mathbb{N}$. We want to show that T is continuous. To this end, let $f_n \to f$ be a convergent series in $\mathcal{S}$. Our task is to show that $Tf_n \to Tf$ converges.
The convergence $f_n \to f$ implies $\|f_n - f\|_{r',s'} \to 0$ for all $r', s' \in \mathbb{N}$ and thus in particular for r, s satisfying the inequality (2.5). By linearity follows:

$$|Tf_n - Tf| = |T(f_n - f)| \leq c\|f_n - f\|_{r,s} \xrightarrow{n\to\infty} 0$$

So T maps convergent sequences to convergent sequences and is thus continuous.

"$\Rightarrow$": Assume that T is continuous. Then the preimage of open sets is open, in particular $T^{-1}(B_1(0)) \subseteq \mathcal{S}(\mathbb{R}^n)$ is open. So there exist $r, s \in \mathbb{N}$ and a $\varepsilon \in \mathbb{R}_{>0}$ such that holds:

$$T^{-1}(B_1(0)) \supseteq U_\varepsilon^{r,s} := \left\{g \middle| \|g\|_{r,s} < \varepsilon\right\}$$

This implies:

$$\|g\|_{r,s} < \varepsilon \quad \Rightarrow \quad g \in T^{-1}(B_1(0))$$

Now $g \in T^{-1}(B_1(0))$ means $|Tg| < 1$. For any $f \in \mathcal{S}(\mathbb{R}^n)$ apply this to $g = \frac{f}{\lambda}$ with $\lambda \in \mathbb{R}_{>0}$.

$$\begin{aligned} \frac{1}{\lambda}\|f\|_{r,s} < \varepsilon \quad &\Rightarrow \quad \frac{1}{\lambda}|Tf| < 1 \qquad / \cdot \lambda \\ \|f\|_{r,s} < \lambda\varepsilon \quad &\Rightarrow \quad |Tf| < \lambda \end{aligned}$$

Now choose $\lambda = \frac{2}{\varepsilon}\|f\|_{r,s}$, so the left side holds, which implies:

$$|Tf| < \frac{2}{\varepsilon}\|f\|_{r,s} \qquad \underset{f\in\mathcal{S}(\mathbb{R}^n)}{\forall}$$

$\square_{2.1.5}$

2.1.6 Example (δ-distribution)

a) Consider the following functional:

$$\begin{aligned}\delta : \mathcal{S}(\mathbb{R}) &\to \mathbb{R}\\ f &\mapsto f(0)\end{aligned}$$

This is obviously linear.

$$|\delta(f)| = |f(0)| \leq \sup_{\mathbb{R}} |f| = \|f\|_{0,0}$$

Hence δ is continuous, which means that $\delta \in \mathcal{S}^*(\mathbb{R})$ is a tempered distribution.
A convenient *notation* with $f \in \mathcal{S}(\mathbb{R})$ is:

$$\delta(f) = \int_{\mathbb{R}} f(x)\,\delta(x)\,\mathrm{d}x$$

b) In higher dimension $n \in \mathbb{N}$ we define:

$$\begin{aligned}\delta : \mathcal{S}(\mathbb{R}^n) &\to \mathbb{R}\\ f &\mapsto f(0)\end{aligned}$$

Again holds $|\delta(f)| \leq \|f\|_{0,0}$. The physicists' notation for this is:

$$\begin{aligned}\delta(f) &= \int_{\mathbb{R}^n} f(x)\,\delta^{(n)}(x)\,\mathrm{d}^n x\\ \delta^{(n)}(x) &= \delta\left(x^1\right)\cdots\delta\left(x^n\right)\end{aligned}$$

Remark

δ can also be introduced as a *measure* on $\mathbb{R}^n$, the *Dirac measure.* For $A \subseteq \mathbb{R}^n$ define:

$$\delta(x) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{otherwise} \end{cases}$$

Then for $f \in C^0(\mathbb{R}^n)$ the expression

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\delta(x) = f(0)$$

makes mathematical sense as an integral.
This is useful because convergence theorems and so on from measure theory are available. The problem is, that this does not work for every distribution and thus is not general enough for most purposes, e.g. the derivative $\delta'(x)$ is a distribution, but cannot be written as a measure.

2.1.7 Example (integral operator)

a) Consider $g \in C^\infty(\mathbb{R}^n)$ with at most polynomial growth, i.e. there are $c \in \mathbb{R}_{>0}$ and $r \in \mathbb{N}$ such that holds:

$$|g(x)| \leq c\left(1 + |x|^r\right)$$

Now define:

$$T_g : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$$
$$f \mapsto \int_{\mathbb{R}^n} g(x) f(x) \, \mathrm{d}^n x$$

The integral here is just the Lebesgue integral and it exists:
For $f \in \mathcal{S}(\mathbb{R}^n)$ holds $\|f\|_{r,s} < \infty$ for all $r, s \in \mathbb{N}$.

$$\sup_{\mathbb{R}} |f| + \sup_{x\in\mathbb{R}} \left(|x|^s \cdot |f(x)|\right) \le \|f\|_{0,s} < \infty$$
$$\Rightarrow \quad \sup_{x\in\mathbb{R}} \left((1+|x|^s) |f(x)|\right) \le \|f\|_{0,s}$$
$$\Rightarrow \quad |f(x)| \le \frac{\|f\|_{0,s}}{1+|x|^s} \qquad \forall_{s\in\mathbb{N}}$$

So we get:

$$T_g f = \int g(x) f(x) \, \mathrm{d}^n x \le \int c\left(1+|x|^r\right) \frac{\|f\|_{0,s}}{(1+|x|^s)} \mathrm{d}^n x =$$
$$\overset{\text{polar coordinates}}{\underset{\rho:=|x|}{=}} c\,\|f\|_{0,s} \quad \underbrace{\mu\left(S^{n-1}\right)}_{\text{volume of unit sphere}} \quad \int_0^\infty \rho^{n-1} \frac{1+\rho^r}{1+\rho^s} \mathrm{d}\rho \overset{s>r+n}{<} \infty$$

This is finite if and only if the integrand decays faster than ρ^{-1}, i.e. $n-1+r-s<-1$ and thus $s > n+r$. Since $s \in \mathbb{N}$ is arbitrary, the integral exists.
Continuity: The previous estimate implies with $s = n+r+1$:

$$|T_g f| \le C(g,n) \|f\|_{0,s}$$

Thus $T_g \in \mathcal{S}^*(\mathbb{R}^n)$ is a tempered distribution.

b) Chose $g(x) = e^x$ and define:

$$T_g f := \int_{-\infty}^{\infty} f(x) e^x \mathrm{d}x$$

This is *not* a well-defined tempered distribution. Namely, choose:

$$f(x) = \frac{1}{\cosh\left(\frac{x}{2}\right)}$$

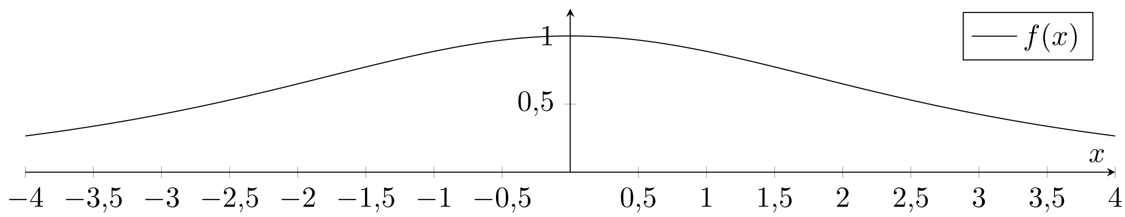

Figure 2.1: $f(x)$ decays rapidly.

$f(x)$ and all its derivatives decay rapidly (exponentially fast $\sim e^{-\frac{x}{2}}$) at $\pm\infty$, so $f \in \mathcal{S}$ is a Schwartz function. But $T_g f$ diverges:

$$T_g f = \int_{-\infty}^{\infty} \frac{e^x}{\cosh\left(\frac{x}{2}\right)} \mathrm{d}x = +\infty$$

Appendix

Acknowledgements

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Andreas Völklein