Partial Differential Equations I

lecture by

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during the summer semester 2013

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A transparent copy of the recent version of this document can be downloaded from:

`https://github.com/andiv/PDE1`

Literature

Elliptic and parabolic partial differential equations:

- Jürgen Jost: *Partial Differential Equations*; Springer, 2007;
  ISBN: 978-0-387-49318-3; doi: 10.1007/978-0-387-49319-0
  (good book, but not all details, small errors)
- Lawrence C. Evans: *Partial Differential Equations*; American Mathematical Society, 2010; ISBN: 978-0-8218-4974-3
  (part of the lecture follows this book, lots of details)
- David Gilbarg, Neil S. Trudinger: *Elliptic Partial Differential Equations of second order*; Springer, 2001; ISBN: 3-540-41160-7
  (classic textbook, complete treatment)
- Robert A. Adams, John J. F. Fournier: *Sobolev Spaces*; Acadademic Press, 2009;
  ISBN: 978-0-12-044143-3
- Richard Courant, David Hilbert: *Methods of Mathematical Physics: Partial Differential Equations, Volume II*; Wiley, 2008;
  ISBN: 978-0471504399; doi: 10.1002/9783527617234
  (Lebesgue spine)
- Walter Rudin: *Real and Complex Analysis*; McGraw-Hill, 2009;
  ISBN: 978-0-07-054234-1
  (Urysohn's Lemma)

Hyperbolic partial differential equations (for the lecture "Partial Differential Equations II"):

- Fritz John: *Partial Differential Equations*; Springer, 1999
  ISBN: 0-387-90609-6
- Michael E. Taylor: *Partial Differential Equations I - III*; Springer, 1997
  ISBN: 0-387-94653-5, 0-387-94651-9, 0-387-94652-7
  (nice detailed text books)
- Joel Smoller: *Shock waves and reaction-diffusion equations*; Springer, 1994
  ISBN: 3-540-94259-9
  (nicely presented, good motivations, covers most of the material)
- Friedrich Sauvigny: Partial Differential Equations I-II; Springer, 2012
  ISBN: 978-1-4471-2981-3, 978-1-4471-2984-4;
  doi: 10.1007/978-1-4471-2981-3, 10.1007/3-540-27540-1
- and many more . . .

Contents

1 A Brief Introduction

An ordinary differential equation (ODE) can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \dot{u}(t) = v(t, u)$$
$$u : I \subseteq \mathbb{R} \to \mathbb{R}^N$$

This equation involves only derivatives with respect to *one* variable t.

$$\frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y) = 0$$

This is an example for a partial differential equation.

1.1 Definition (Partial Differential Equation)

A *partial differential equation* (PDE) is a (scalar) equation, which involves partial derivatives of an unknown function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. We always assume that $\Omega \subseteq \mathbb{R}^n$ is open.
More generally, a *system of partial differential equations* is a system of equations involving partial derivatives of a function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$.
Similarly one can define partial differential equations on manifolds.
For ordinary differential equations we considered the initial-value problem:

$$\dot{u}(t) = v(t, u) \qquad\qquad u(t_0) = u_0$$

For partial differential equations one considers
- the initial-value problem and
- the boundary-value problem.

1.2 Examples

1. Cauchy-Riemann equations: Let

   $$f : \Omega \overset{\text{open}}{\subseteq} \mathbb{C} \to \mathbb{C}$$

   be holomorphic.

   $$f = a + \mathbf{i}b \qquad a := \mathrm{Re}(f) \qquad b := \mathrm{Im}(f)$$

   $$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad\qquad \frac{\partial b}{\partial x} = -\frac{\partial a}{\partial y}$$

This is a system of two partial differential equations.

$$u := \begin{pmatrix} a \\ b \end{pmatrix} \qquad\qquad u : \Omega \subseteq \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2$$

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = \frac{\partial b}{\partial x \partial y} - \frac{\partial b}{\partial y \partial x} = 0$$

$$\Rightarrow \qquad \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{=:\Delta} a = 0 \qquad\qquad \Delta b = 0$$

This is the *Laplace equation* with the *Laplace operator* (or *Laplacian*) Δ. Solutions of the Laplace equation are called *harmonic functions.*

2. Let (M, g) be a Riemannian manifold. Here exists the Laplace-Beltrami operator $\Delta\backslash$.
   - In the special case $M = \mathbb{R}^n$ we have:

   $$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$
   $$\Delta \varphi = 0$$

   - With the Riemannian metric g_{ij} we can define:

   $$\Delta\backslash \varphi = g^{ij} \nabla_i \nabla_j \varphi = \operatorname{div}(\operatorname{grad}(\varphi)) = \frac{1}{\sqrt{\det(g)}} \partial_j \left(\sqrt{\det(g)} g^{jk} \partial_k \varphi\right)$$

   This gives an elliptic equation.
3. Newton's gravitational law: Let $\varrho(x)$ be the mass density and $\varphi(x)$ the Newtonian potential.

   $$\Delta \varphi = \underbrace{-4\pi\varrho}_{\text{inhomogeneity}}$$

   Such an inhomogeneous Laplace equation is usually referred to as Poisson equation and it is elliptic.
4. Heat flow equation (Wärmeleitungsgleichung): Let $\varphi(t, x)$ be the temperature at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^n$.

   $$\partial_t \varphi(t, x) = \Delta \varphi(t, x)$$

   This is a parabolic equation.
5. The Schrödinger equation is a parabolic equation:

   $$\mathbf{i}\hbar \partial_t \psi(t, x) = \left(-\frac{\hbar^2}{2m}\Delta + V\right) \psi(t, x)$$

   Additionally to the the heat flow equation there is the potential V, but more important there is a factor of $\mathbf{i}$ in front of the partial derivative. The time-independent Schrödinger equation is:

   $$E\psi(x) = \left(-\frac{\hbar^2}{2m}\Delta + V\right) \psi(x)$$

   This is similar to the Poisson equation and also elliptic.

6. The wave equation
$$\left(\partial_t^2 - \Delta_x\right) \psi(t,x) = 0$$
is hyperbolic. We will consider it in the lecture "Partial Differential Equations II".
7. Maxwell's equations: $E(t,x)$ is the electric field and $B(t,x)$ the magnetic field.
$$\begin{aligned} \operatorname{div}(E) &= 4\pi\varrho && \text{Gauss law} \\ \operatorname{rot}(E) &= -\partial_t B && \text{Maxwell} \\ \operatorname{div}(B) &= 0 \\ \operatorname{rot}(B) &= 4\pi j - \partial_t E && \text{Faraday} \end{aligned}$$
This is a system of 8 partial differential equation.
8. Einstein's field equation:
$$R_{ij} - \frac{1}{2} R g_{ij} = 4\pi\kappa T_{ij}$$
This is a geometric partial differential equation. R_{ij} is the Ricci curvature, R the scalar curvature and T_{ij} the energy-momentum tensor. It is a system of 10 partial differential equations.
9. Equations of relativistic quantum mechanics:
$$\left(-\partial_t^2 + \Delta\right)\psi = m^2\psi$$
This is the Klein-Gordon equation with the mass m.
$$\mathbf{i}\gamma^j \partial_j \psi = m\psi$$
This is the Dirac equation, a system of 4 complex-valued or 8 real-valued partial differential equations for a particle with spin $\frac{1}{2}$.
10. Water waves can be described by the Korteweg-de Vries equation:
$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

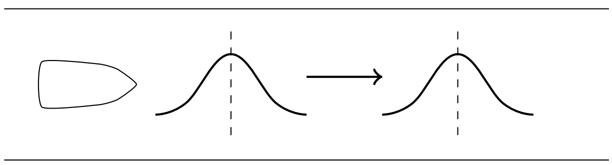


Figure 1.1: Solitons (discovered by John Russel in 1834): When the ship suddenly stops, the water flows on along the channel. This wave moves with a constant speed and its shape stays the same.

11. Shock waves: Burger's equation
$$\partial_t u + u\partial_x u = 0$$
is hyperbolic.
12. Turbulence can be described by the incompressible Navier-Stokes equations for the velocity $v : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.
$$\begin{aligned} \operatorname{div}(v) &= 0 \\ \rho\partial_t v^j + \rho v^i \partial_i v^j - \eta\Delta v^j &= -\partial_j P \end{aligned}$$
Here ϱ is the gas density, P the pressure and η the viscosity.

1.3 Classification

I) The *order* of a partial differential equation is the highest order of the derivatives in it.

$\Delta u = f$	second order
$\partial_t \varphi = \Delta\varphi$	second order
$\partial_t u + u\partial_x u + \partial_x^3 u = 0$	third order

II) Algebraic classification:

a) *Linear* equations: The unknown function u and its derivatives appear only linearly.

$\partial_t u = u$	linear
$\partial_t u + u\partial_x u = 0$	non-linear

b) Linear *homogeneous* equations: If u is a solution, then λu for $\lambda \in \mathbb{R}$ is also a solution.

$\Delta u = 0$	linear homogeneous
$\Delta u = \varrho$	linear inhomogeneous
$\Delta\backslash u = 0$	linear homogeneous

c) Linear with *constant coefficients*:

$\Delta u = \rho$	linear with constant coefficients
$\Delta\backslash u = \varrho$	in general non-constant coefficients

III) Classification by type: elliptic, parabolic, hyperbolic
Here we only consider scalar second order equations with $x \in \Omega \subseteq \mathbb{R}^n$.

$$F\left(x, u, \mathrm{D}u, \mathrm{D}^2 u\right) = 0$$
$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$$

$$A_{ij} := \frac{\partial F\left(x, u, p_i, p_{ij}\right)}{\partial p_{ij}}$$

is a symmetric $n \times n$ matrix.

- If A is positive definite, the equation is called *elliptic*.
- If A has $n-1$ positive and one negative eigenvalue, the equation is called *hyperbolic*.
- If A has $n-1$ positive eigenvalues and a non-trivial kernel, the equation is called *parabolic*.
- If all eigenvalues are negative or $n-1$ are negative, then we replace F by $-F$.
  All other case of *mixed type* are difficult and we do not consider them in this lecture.

1.4 Examples

Consider the Poisson equation:

$$\Delta u = \varrho$$

$$\begin{aligned} F\left(x, u, \mathrm{D}u, \mathrm{D}^2 u\right) &= -\varrho\left(x\right) + \delta^{ij}\partial_{ij}u \\ F\left(x, u, p_i, p_{ij}\right) &= -\varrho\left(x\right) + \delta^{ij}p_{ij} \\ A_{ij} = \frac{\partial F\left(x, u, p_i, p_{ij}\right)}{\partial p_{ij}} &= \delta_{ij} \end{aligned}$$

So we have $A = \mathbb{1}$ and thus the equation is elliptic.
Now consider the inhomogeneous wave equation:

$$\left(\partial_t^2 - \Delta\right)\phi\left(t, x_1, x_2, x_3\right) = \varrho$$

$$F\left(x, u, p_i, p_{ij}\right) = \varrho\left(x\right) + \eta^{ij}p_{ij} \qquad \eta = \operatorname{diag}\left(-1, 1, 1, 1\right)$$

So $A = \eta$ has one negative and three positive eigenvalues which means that the equation is hyperbolic.

$$\partial_t\phi = \Delta\phi$$

$$\begin{aligned} F\left(x, u, \mathrm{D}\phi, \mathrm{D}^2\phi\right) &= -\partial_0\phi + \sum_{i,j=1}^{3}\delta^{ij}\partial_{ij}\phi \\ A_{ij} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore the equation is parabolic.

2 Distributions and Fourier Transform

Motivation

We want to solve partial differential equations with constant coefficients in $\Omega = \mathbb{R}^n$, for example:

$$\left(-\partial_t^2 + \Delta\right)\phi = 0$$

Now we make a "plane wave ansatz" with $t, \omega \in \mathbb{R}$ and $k, x \in \mathbb{R}^{n-1}$:

$$\begin{aligned} \phi(t,x) &= e^{-\mathbf{i}\omega t + \mathbf{i}\langle k,x\rangle} \\ \partial_t \phi(t,x) &= -\mathbf{i}\omega\phi(t,x) \\ \partial_j \phi(t,x) &= \mathbf{i}k_j\phi(t,x) \end{aligned}$$

This gives an algebraic equation:

$$\begin{aligned} \left(-\left(-\mathbf{i}\omega\right)^2 + \left(\mathbf{i}k\right)^2\right)\phi &= 0 \\ \Leftrightarrow \qquad \omega^2 &= k^2 \end{aligned}$$

We also want to differentiate non-smooth functions, e.g.:

$$\Delta_{\mathbb{R}^3}\frac{1}{|x|} = -4\pi\delta(x)$$

$\delta(x)$ is called Dirac δ-distribution.

2.1 The Schwartz Space and Distributions

Laurent Schwartz was the first to investigate distributions systematically. He was awarded the fields medal for his research.

2.1.1 Definition (Multi-Index)

For $\mathbb{R}^n$ we denote indices by $i, j, k \in \{1, \ldots, n\}$. We call $\alpha = (i_1, \ldots, i_k)$ with $i_l \in \{1, \ldots, n\}$ a *multi-index*. $|\alpha| := k$ is called the *order* or *absolute value* of the multi-index.
With this we can write differentials of order k as

$$\mathrm{D}^\alpha := \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} \tag{2.1}$$

and homogeneous polynomials of degree k in the components of a vector $x = \left(x^1, \ldots, x^n\right)$ as:

$$x^\alpha := x^{i_1} \cdots x^{i_k} \tag{2.2}$$

For $f \in C^\infty(\mathbb{R}^n)$ and $r, s \in \mathbb{N}$ we define the *Schwartz norm*:

$$\|f\|_{r,s} := \sum_{\alpha, |\alpha| \le r} \sum_{\beta, |\beta| \le s} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \mathrm{D}^\beta f(x) \right| \tag{2.3}$$

For example for $r = 0 = s$ we have:

$$\|f\|_{0,0} = \sup_{x \in \mathbb{R}^n} |f(x)| = \|f\|_{C^0}$$

2.1.2 Definition (Schwartz Space)

The *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ is the vector space of all $f \in C^\infty(\mathbb{R}^n)$ for which all Schwartz norms are finite, i.e. for all $r, s \in \mathbb{N}$ holds:

$$\|f\|_{r,s} < \infty$$

This space is an infinite-dimensional vector space.
On a normed space $(E, \|.\|)$, the topology is given by the open sets.
$\Omega \subseteq E$ is defined as *open* if holds:

$$\underset{x \in \Omega}{\forall} \ \underset{\varepsilon > 0}{\exists} \ : \ B_\varepsilon(x) \subseteq \Omega$$

A subset $\Omega \subseteq \mathcal{S}(\mathbb{R}^n)$ is called *open* if for every $f \in \Omega$ there is a $\varepsilon > 0$ and $r, s \in \mathbb{N}$ such that holds:

$$\left\{ g \in \mathcal{S} \,\middle|\, \|g - f\|_{r,s} < \varepsilon \right\} \subseteq \Omega \tag{2.4}$$

Note: This topology is fine, because it involves many open sets, since the condition for open only involves the statement “there are $r, s \in \mathbb{N}$”.
Convergence $f_n \to f$ in $\mathcal{S}$ means that *every* open neighborhood U of f contains almost all f_n. For a finer topology, the condition for a sequence to converge is stronger.

2.1.3 Theorem (Criterion for Convergence)

Convergence $f_n \to f$ in $\mathcal{S}$ is equivalent to the convergence $\|f_n - f\|_{r,s} \to 0$ for all $r, s \in \mathbb{N}$.

Proof

“$\Rightarrow$”: Suppose that $f_n \to f$ converges. By definition of the convergence, every open neighborhood of f contains almost all f_n. For all $r, s \in \mathbb{N}$ the sets $U_\varepsilon^{r,s} := \left\{ g \,\middle|\, \|g - f\|_{r,s} < \varepsilon \right\}$ are open by definition. So the inequality

$$\|f_n - f\|_{r,s} < \varepsilon$$

holds for almost all f_n and thus converges $\|f_n - f\|_{r,s} \to 0$.
“$\Leftarrow$”: Assume that $\|f_n - f\|_{r,s} \to 0$ converges for all $r, s \in \mathbb{N}$. Let A be an open neighborhood of f. This means by definition that there exist $r, s \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$ with $U_\varepsilon^{r,s} \subseteq A$. For this (r, s) we know that $\|f_n - f\|_{r,s} \to 0$ converges. Hence there exists a $N \in \mathbb{N}$ such that $\|f_n - f\|_{r,s} < \varepsilon$ holds for all $n \in \mathbb{N}_{>N}$, in other words $f_n \in U_\varepsilon^{r,s} \subseteq A$. So $f_n \to f$ converges in $\mathcal{S}$. $\square_{2.1.3}$

A vector space with a topology generated by a family of norms or semi-norms is a *uniform space* and is called *topological vector space.*

2.1.4 Definition (Tempered Distribution)

Let $\mathcal{S}^*(\mathbb{R}^n)$ be the dual space of $\mathcal{S}(\mathbb{R}^n)$. It is called the space of *tempered distributions* (temperierte Distributionen).

In linear algebra for a finite-dimensional vector space V, the dual space $V^* = L(V, \mathbb{R})$ is the space of linear functionals. V^* is again a vector space with $\dim(V^*) = \dim(V)$.
Here $\mathcal{S}(\mathbb{R}^n)$ is an infinite-dimensional vector space with a topology. $\mathcal{S}^*(\mathbb{R}^n)$ is the space of all *continuous* linear functionals.

In a Banach space $(E, \|.\|)$ holds: A linear functional $A : E \to \mathbb{R}$ is continuous if and only if A is bounded, i.e. $|Au| \le c\,\|U\|$ for all $u \in E$.

2.1.5 Lemma (Criterion for Continuity)

A linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ is continuous if and only if there are $r, s \in \mathbb{N}$ and a $c \in \mathbb{R}_{>0}$ such that holds:

$$|Tf| \le c\,\|f\|_{r,s} \tag{2.5}$$

Proof

"$\Leftarrow$": Assume that (2.5) holds for some $r, s \in \mathbb{N}$. We want to show that T is continuous. To this end, let $f_n \to f$ be a convergent series in $\mathcal{S}$. Our task is to show that $Tf_n \to Tf$ converges.
The convergence $f_n \to f$ implies $\|f_n - f\|_{r',s'} \to 0$ for all $r', s' \in \mathbb{N}$ and thus in particular for r, s satisfying the inequality (2.5). By linearity follows:

$$|Tf_n - Tf| = |T(f_n - f)| \le c\,\|f_n - f\|_{r,s} \xrightarrow{n\to\infty} 0$$

So T maps convergent sequences to convergent sequences and is thus continuous.

"$\Rightarrow$": Assume that T is continuous. Then the preimage of open sets is open, in particular $T^{-1}(B_1(0)) \subseteq \mathcal{S}(\mathbb{R}^n)$ is open. So there exist $r, s \in \mathbb{N}$ and a $\varepsilon \in \mathbb{R}_{>0}$ such that holds:

$$T^{-1}(B_1(0)) \supseteq U_\varepsilon^{r,s} := \left\{ g \middle| \|g\|_{r,s} < \varepsilon \right\}$$

This implies:

$$\|g\|_{r,s} < \varepsilon \quad \Rightarrow \quad g \in T^{-1}(B_1(0))$$

Now $g \in T^{-1}(B_1(0))$ means $|Tg| < 1$. For any $f \in \mathcal{S}(\mathbb{R}^n)$ apply this to $g = \frac{f}{\lambda}$ with $\lambda \in \mathbb{R}_{>0}$.

$$\begin{aligned} \frac{1}{\lambda}\|f\|_{r,s} < \varepsilon \quad &\Rightarrow \quad \frac{1}{\lambda}|Tf| < 1 \qquad / \cdot \lambda \\ \|f\|_{r,s} < \lambda\varepsilon \quad &\Rightarrow \quad |Tf| < \lambda \end{aligned}$$

Now choose $\lambda = \frac{2}{\varepsilon}\|f\|_{r,s}$, so the left side holds, which implies:

$$|Tf| < \frac{2}{\varepsilon}\|f\|_{r,s} \qquad \underset{f \in \mathcal{S}(\mathbb{R}^n)}{\forall}$$

$\square_{2.1.5}$

2.1.6 Example (δ-Distribution)

a) Consider the following functional:

$$\begin{aligned} \delta : \mathcal{S}(\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto f(0) \end{aligned} \tag{2.6}$$

This is obviously linear.

$$|\delta(f)| = |f(0)| \leq \sup_{\mathbb{R}} |f| = \|f\|_{0,0}$$

Hence δ is continuous, which means that $\delta \in \mathcal{S}^*(\mathbb{R})$ is a tempered distribution.
A convenient *notation* with $f \in \mathcal{S}(\mathbb{R})$ is:

$$\delta(f) = \int_{\mathbb{R}} f(x)\, \delta(x)\, \mathrm{d}x$$

b) In higher dimension $n \in \mathbb{N}$ we define:

$$\begin{aligned} \delta : \mathcal{S}(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto f(0) \end{aligned}$$

Again holds $|\delta(f)| \leq \|f\|_{0,0}$. The physicists' notation for this is:

$$\begin{aligned} \delta(f) &= \int_{\mathbb{R}^n} f(x)\, \delta^{(n)}(x)\, \mathrm{d}^n x \\ \delta^{(n)}(x) &= \delta\left(x^1\right) \cdots \delta(x^n) \end{aligned}$$

Remark

δ can also be introduced as a *measure* on $\mathbb{R}^n$, the *Dirac measure*. For $A \subseteq \mathbb{R}^n$ define:

$$\delta(x) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

Then for $f \in C^0(\mathbb{R}^n)$ the expression

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}\delta(x) = f(0)$$

makes mathematical sense as an integral.

This is useful because convergence theorems and so on from measure theory are available. The problem is, that this does not work for every distribution and thus is not general enough for most purposes, e.g. the derivative $\delta'(x)$ is a distribution, but cannot be written as a measure.

2.1.7 Example (Integral Operator)

a) Consider $g \in C^\infty(\mathbb{R}^n)$ with at most polynomial growth, i.e. there are $c \in \mathbb{R}_{>0}$ and $r \in \mathbb{N}$ such that holds:

$$|g(x)| \leq c\left(1 + |x|^r\right)$$

Now define:

$$\begin{aligned} T_g : \mathcal{S}(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \int_{\mathbb{R}^n} g(x) f(x) \, \mathrm{d}^n x \end{aligned}$$

The integral here is just the Lebesgue integral and it exists:
For $f \in \mathcal{S}(\mathbb{R}^n)$ holds $\|f\|_{r,s} < \infty$ for all $r, s \in \mathbb{N}$.

$$\begin{aligned} & \sup_{\mathbb{R}} |f| + \sup_{x \in \mathbb{R}} \left(|x|^{\tilde{r}} \cdot |f(x)| \right) \leq \|f\|_{\tilde{r},0} < \infty \\ \Rightarrow \quad & \sup_{x \in \mathbb{R}} \left(\left(1 + |x|^{\tilde{r}} \right) |f(x)| \right) \leq \|f\|_{\tilde{r},0} \\ \Rightarrow \quad & |f(x)| \leq \frac{\|f\|_{\tilde{r},0}}{1 + |x|^{\tilde{r}}} \qquad \forall_{\tilde{r} \in \mathbb{N}} \end{aligned}$$

So we get:

$$T_g f = \int g(x) f(x) \, \mathrm{d}^n x \leq \int c \left(1 + |x|^r \right) \frac{\|f\|_{\tilde{r},0}}{\left(1 + |x|^{\tilde{r}} \right)} \mathrm{d}^n x =$$

$$\overset{\text{polar coordinates}}{\underset{\rho := |x|}{=}} c \, \|f\|_{\tilde{r},0} \underbrace{\mu\left(S^{n-1} \right)}_{\text{volume of unit sphere}} \int_0^\infty \rho^{n-1} \frac{1 + \rho^r}{1 + \rho^{\tilde{r}}} \mathrm{d}\rho \overset{\tilde{r} > r+n}{<} \infty$$

This is finite if and only if the integrand decays faster than ρ^{-1}, i.e. $n - 1 + r - \tilde{r} < -1$ and thus $\tilde{r} > n + r$. Since $\tilde{r} \in \mathbb{N}$ is arbitrary, the integral exists.
Continuity: The previous estimate implies with $\tilde{r} = n + r + 1$:

$$|T_g f| \leq C(g, n) \, \|f\|_{\tilde{r},0}$$

Thus $T_g \in \mathcal{S}^*(\mathbb{R}^n)$ is a tempered distribution.

b) Chose $g(x) = e^x$ and define:

$$T_g f := \int_{-\infty}^{\infty} f(x) \, e^x \mathrm{d}x$$

This is *not* a well-defined tempered distribution. Namely, choose:

$$f(x) = \frac{1}{\cosh\left(\frac{x}{2}\right)}$$

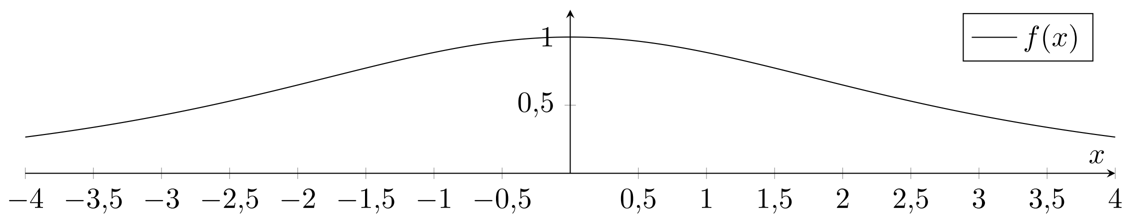


Figure 2.1: $f(x)$ decays rapidly.

$f(x)$ and all its derivatives decay rapidly (exponentially fast $\sim e^{-\frac{x}{2}}$) at $\pm\infty$, so $f \in \mathcal{S}$ is a Schwartz function. But $T_g f$ diverges:

$$T_g f = \int_{-\infty}^{\infty} \frac{e^x}{\cosh\left(\frac{x}{2}\right)} \mathrm{d}x = +\infty$$

2.1.8 Remark (Schwartz Functions as Distributions)

The mapping

$$\begin{aligned} T : \mathcal{S}(\mathbb{R}^n) &\to \mathcal{S}^*(\mathbb{R}^n) \\ f &\mapsto T_f \end{aligned}$$

is injective.

Proof

If T was injective, there were $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ with $T_{f_1} = T_{f_2}$ and $f_1 \neq f_2$. By linearity this would imply $T_g = 0$ with $g = f_1 - f_2 \neq 0$ and we could choose a $y \in \mathbb{R}^n$ with $g(y) \neq 0$. By continuity follows $g > 0$ or $g < 0$ in a neighborhood U of y. Now choose a test function h with $\operatorname{supp}(h) \subseteq U$ and $h \geq 0$. Then follows the contradiction:

$$0 = T_g(h) = \int_{\mathbb{R}^n} g(x)\, h(x)\, \mathrm{d}^n x \neq 0$$

$\square_{2.1.8}$

Thus we can regard distributions as "generalized functions". Namely we identify a function g with T_g. (Later on many people often do not distinguish between g and T_g.)

Operations on Schwartz Functions and Distributions

- $\mathcal{S}^*(\mathbb{R}^n)$ is a vector space with addition $T+S$ and scalar multiplication $\alpha \cdot f$ for distributions $T, S \in \mathcal{S}^*(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$.
- Multiplication of a distribution by a Schwartz function is defined for $T \in \mathcal{S}^*(\mathbb{R}^n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$ as:

  $$(fT)(g) := T(f \cdot g) \tag{2.8}$$

  This is well defined, because $f \cdot g$ is again a Schwartz function and for $h \in \mathcal{S}(\mathbb{R}^n)$ holds:

  $$\begin{aligned} (fT_h)(g) &\overset{\text{definition of } fT_h}{=} T_h(f \cdot g) \overset{\text{definition of } T_h}{=} \int_{\mathbb{R}^n} h(x)\,(f \cdot g)(x)\, \mathrm{d}^n x = \\ &= \int_{\mathbb{R}^n} (f \cdot h)(x)\, g(x)\, \mathrm{d}^n x = T_{fh}(g) \end{aligned}$$

  So this definition is extends the multiplication of Schwartz functions to distributions. But we still have to show, that this operation gives a continuous functional.

2.1.9 Definition (regular/singular distribution)

A tempered distribution T is called *regular* distribution if there is a $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ (locally integrable, i.e. integrable on every compact interval) with $T = T_g$. Otherwise, T is called *singular*.
For example $\delta(x)$ is singular.

2.1.10 Lemma (Multiplication of a Distribution by a Schwartz Function)

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function and $T \in \mathcal{S}^*(\mathbb{R}^n)$ a distribution. Then fT is a *continuous* linear functional on $\mathcal{S}(\mathbb{R}^n)$, in other words $fT \in \mathcal{S}^*(\mathbb{R}^n)$ is also a distribution.

Proof

According to Lemma 2.1.5, our task is to show that there are $r, s \in \mathbb{N}$ and a $C \in \mathbb{R}_{>0}$ with:

$$|(fT)(g)| \leq C \|g\|_{r,s}$$

Since T is continuous, there exist $r, s \in \mathbb{N}$ and a $\tilde{C} \in \mathbb{R}_{>0}$ with:

$$|T(fg)| \leq \tilde{C} \|fg\|_{r,s}$$

Thus it remains to show that there is a $c(r,s) \in \mathbb{R}_{>0}$ such that for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ holds:

$$\|fg\|_{r,s} \leq c(r,s) \|f\|_{r,s} \cdot \|g\|_{r,s}$$

This inequality can be proven by induction in s.
Induction basis $s = 0$:

$$\begin{aligned}
\|fg\|_{r,0} &= \sum_{|\alpha| \leq r} \sup_{x \in \mathbb{R}^n} (|x^\alpha f(x) g(x)|) \leq \\
&\leq \sup_{y \in \mathbb{R}^n} |g(y)| \sum_{|\alpha| \leq r} \sup_{x \in \mathbb{R}^n} (|x^\alpha f(x)|) = \\
&= \|g\|_{0,0} \cdot \|f\|_{r,0} \leq \|g\|_{r,0} \cdot \|f\|_{r,0}
\end{aligned}$$

Induction step $s \rightsquigarrow s+1$: Assume that the statement holds for a $s \in \mathbb{N}$ for all $r \in \mathbb{N}$. Let β be a multi-index with $|\beta| = s+1$, i.e. $\beta = (i_1, \ldots, i_{s+1})$. Now set:

$$\hat{\beta} := (i_1, \ldots, i_s) \qquad j := i_{s+1} \qquad \mathrm{D}^\beta = \mathrm{D}^{\hat{\beta}} \frac{\partial}{\partial x^j}$$

It holds:

$$\mathrm{D}^\beta (fg) = \mathrm{D}^{\hat{\beta}} \frac{\partial}{\partial x^j} (fg) = \mathrm{D}^{\hat{\beta}} \left(\left(\frac{\partial}{\partial x^j} f \right) g + f \left(\frac{\partial}{\partial x^j} g \right) \right)$$

$$\begin{aligned}
\|fg\|_{r,s+1} &= \|f \cdot g\|_{r,s} + \sum_{\substack{|\alpha| \leq r \\ |\beta| = s+1}} \sup_{x \in \mathbb{R}} \left| x^\alpha \mathrm{D}^\beta (f \cdot g)(x) \right| = \\
&= \|fg\|_{r,s} + \sum_{\substack{|\alpha| \leq r \\ |\hat{\beta}| = s}} \sum_{j=1}^n \sup_{x \in \mathbb{R}} \left| x^\alpha \mathrm{D}^{\hat{\beta}} \left(\left(\frac{\partial}{\partial x^j} f(x) \right) g(x) + f(x) \left(\frac{\partial}{\partial x^j} g(x) \right) \right) \right| \leq \\
&\underset{\text{hypothesis}}{\overset{\text{induction}}{\leq}} c(r,s) \|f\|_{r,s} \|g\|_{r,s} + \sum_{j=1}^n c(r,s) \left(\left\| \frac{\partial}{\partial x^j} f \right\|_{r,s} \|g\|_{r,s} + \|f\|_{r,s} \left\| \frac{\partial}{\partial x^j} g \right\|_{r,s} \right) \leq \\
&\leq c(r,s) \|f\|_{r,s} \|g\|_{r,s} + n \cdot c(r,s) \left(\|f\|_{r,s+1} \|g\|_{r,s} + \|f\|_{r,s} \|g\|_{r,s+1} \right) \leq \\
&\leq \underbrace{(2n+1)\, c(r,s)}_{=c(r,s+1)} \|f\|_{r,s+1} \|g\|_{r,s+1}
\end{aligned}$$

$\Box_{2.1.10}$

2.1.11 Example (Derivative of the δ-Distribution)

We make a formal computation:

$$\int_{\mathbb{R}} \delta'(x) f(x)\,\mathrm{d}x = \int_{\mathbb{R}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\delta(x)\right) f(x)\,\mathrm{d}x \overset{\text{integration}}{\underset{\text{by parts}}{=}} -\int_{\mathbb{R}} \delta(x) f'(x)\,\mathrm{d}x = -f'(0)$$

This motivates us to *define*:

$$\begin{aligned} \delta' : \mathcal{S}(\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto -f'(0) \end{aligned} \tag{2.9}$$

This is obviously linear and it is continuous, because for all $f \in \mathcal{S}(\mathbb{R})$ holds:

$$\left|\delta'(f)\right| = \left|f'(0)\right| \leq \|f\|_{0,1}$$

Hence we have $\delta' \in \mathcal{S}^*(\mathbb{R})$.

Remark

In contrast to δ, the distribution δ' cannot be introduced as a measure:

$$\begin{aligned} \delta(\Omega) &= \begin{cases} 1 & \text{if } 0 \in \Omega \\ 0 & \text{otherwise} \end{cases} \\ \delta'(\Omega) &= ? \end{aligned}$$

2.1.12 Definition (Distributional Derivative, Convolution)

- For a tempered distribution $T \in \mathcal{S}^*(\mathbb{R}^n)$ we define the *distributional derivative* $\mathrm{D}^\alpha T$ by:

  $$(\mathrm{D}^\alpha T)(f) := (-1)^{|\alpha|}\, T(\mathrm{D}^\alpha f) \tag{2.10}$$

  $\mathrm{D}^\alpha T$ is a distribution, since it is a continuous functional:

  $$|(\mathrm{D}^\alpha T)(f)| = |T(\mathrm{D}^\alpha f)| \overset{T \text{ continuous}}{\leq} C\,\|\mathrm{D}^\alpha f\|_{r,s} \leq C\,\|f\|_{r,s+|\alpha|}$$

  So we have a mapping $\mathrm{D}^\alpha : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.
- The convolution (Faltung) for $f, g \in \mathcal{S}(\mathbb{R}^n)$ is defined as:

  $$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\,\mathrm{d}^n y \tag{2.11}$$

2.1.13 Lemma (Commutativity and Associativity of the Convolution)

The convolution is commutative and associative, i.e. for $f, g, h \in \mathcal{S}(\mathbb{R}^n)$ holds:

$$f * g = g * f \qquad\qquad f * (g * h) = (f * g) * h \tag{2.12}$$

Proof

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, \mathrm{d}^n y \underset{\mathrm{d}^n z = \mathrm{d}^n y}{\overset{z := x-y}{=}} \int_{\mathbb{R}^n} f(z)\, g(x-z)\, \mathrm{d}^n z = (g * f)(x)$$

Associativity follows analogously using Fubini's theorem, which can be applied, since the function $(y,z) \mapsto f(x-y)\, g(y-z)\, h(z)$ is an element of $\mathcal{S}\left(\mathbb{R}^{2n}\right) \subseteq L^1\left(\mathbb{R}^{2n}\right)$:

$$\begin{aligned}
f * (g * h)(x) &= \int_{\mathbb{R}^n} f(x-y)\,(g * h)(y)\, \mathrm{d}^n y = \\
&= \int_{\mathbb{R}^n} f(x-y) \left(\int_{\mathbb{R}^n} g(y-z)\, h(z)\, \mathrm{d}^n z \right) \mathrm{d}^n y = \\
&\underset{\mathrm{d}^n \tilde{y} = \mathrm{d} y}{\overset{\tilde{y} := y - z}{=}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - \tilde{y} - z)\, g(\tilde{y})\, h(z)\, \mathrm{d}^n \tilde{y} \mathrm{d}^n z = \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x - z - \tilde{y})\, g(\tilde{y})\, \mathrm{d}^n \tilde{y} \right) h(z)\, \mathrm{d}^n z = \\
&= \int_{\mathbb{R}^n} (f * g)(x-z)\, h(z)\, \mathrm{d}^n z = \\
&= (f * g) * h(x)
\end{aligned}$$

$\square_{2.1.13}$

2.1.14 Proposition (Convolution is Continuous)

The convolution $* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is continuous.

Proof

$$\|f * g\|_{r,s} = \sum_{\substack{|\alpha| \leq r \\ |\beta| \leq s}} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \mathrm{D}_x^\beta \int_{\mathbb{R}^n} f(x-y)\, g(y)\, \mathrm{d}^n y \right|$$

The derivative may be commuted with the integral:

$$\begin{aligned}
\frac{\partial}{\partial x_j} (f * g)(x) &= \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left((f * g)(x + \varepsilon e_j) - (f * g)(x) \right) = \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} \frac{f(x + \varepsilon e_j - y) - f(x-y)}{\varepsilon} g(y)\, \mathrm{d}^n y
\end{aligned}$$

Using the estimate

$$\begin{aligned}
\frac{f(x + \varepsilon e_j - y) - f(x-y)}{\varepsilon} &= \frac{1}{\varepsilon} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} \left(f(x + \varepsilon \tau e_j - y) \right) \mathrm{d}\tau = \\
&= \frac{1}{\varepsilon} \int_0^1 (\partial_j f)(x + \varepsilon \tau e_j - y)\, \varepsilon \mathrm{d}\tau \\
\Rightarrow \qquad \left| \frac{f(x + \varepsilon e_j - y) - f(x-y)}{\varepsilon} \right| &\leq \sup_{\mathbb{R}^n} |\partial_j f| \leq \|f\|_{0,1}
\end{aligned}$$

we get:

$$\left|\frac{f(x+\varepsilon e_j-y)-f(x-y)}{\varepsilon}g(y)\right| \le \|f\|_{0,1}\cdot|g(y)|\in L^1(\mathbb{R}^n)$$

Thus the dominated convergence theorem implies:

$$\frac{\partial}{\partial x_j}(f*g)(x)=\int_{\mathbb{R}^n}\frac{\partial}{\partial x^j}f(x-y)\,g(y)\,\mathrm{d}^n y$$

By induction follows:

$$x^\alpha \mathrm{D}^\beta \int_{\mathbb{R}} f(x-y)\,g(y)\,\mathrm{d}^n y = x^\alpha\int_{\mathbb{R}^n}\left(\mathrm{D}^\beta f\right)(x-y)\,g(y)\,\mathrm{d}^n y$$

Now we treat the x^α:

$$x^\alpha=((x-y)+y)^\alpha=\sum_{\gamma,\delta\text{ with }\alpha=\gamma+\delta} c_{\gamma\delta}(x-y)^\gamma\cdot y^\delta$$

Here holds $|\gamma|,|\delta|\le|\alpha|$. Now we can estimate:

$$\left|(x-y)^\gamma\,\mathrm{D}^\beta f(x)\right|\le\|f\|_{r,s}$$

Hence we get:

$$\begin{aligned}\|f*g\|_{r,s}&\le c(r,s)\,\|f\|_{r,s}\sum_\delta\int_{\mathbb{R}^n}\left|y^\delta g(y)\right|\cdot\frac{(1+|y|)^{n+1}}{(1+|y|)^{n+1}}\mathrm{d}^n y\le\\&\le c(r,s)\,\|f\|_{r,s}\,\|g\|_{n+1+r,0}\underbrace{\int_{\mathbb{R}^n}\frac{\mathrm{d}^n y}{(1+|y|)^{n+1}}}_{<\infty}\end{aligned}$$

$\square_{2.1.14}$

2.1.15 Definition (Convolution with Distribution)

How can we define the convolution of $T\in S^*(\mathbb{R}^n)$ with $f\in S(\mathbb{R}^n)$? $f*T_g$ should be equal to T_{f*g}. For $h\in\mathcal{S}(\mathbb{R}^n)$ holds:

$$\begin{aligned}T_{f*g}(h)&=\int_{\mathbb{R}^n}(f*g)(x)\cdot h(x)\,\mathrm{d}^n x=\\&=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}f(x-y)\,g(y)\,\mathrm{d}^n y\right)\cdot h(x)\,\mathrm{d}^n x\end{aligned}$$

By Fubini's theorem we may interchange the order of integration to get:

$$\begin{aligned}T_{f*g}(h)&=\int_{\mathbb{R}^n}g(y)\left(\int_{\mathbb{R}^n}f(x-y)\cdot h(x)\,\mathrm{d}^n x\right)\mathrm{d}^n y=\\&\overset{\tilde f(z):=f(-z)}{=}\int_{\mathbb{R}^n}g(y)\left(\int_{\mathbb{R}^n}\tilde f(y-x)\cdot h(x)\,\mathrm{d}^n x\right)\mathrm{d}^n y=\\&=\int_{\mathbb{R}^n}g(y)\left(\tilde f*h\right)(y)\,\mathrm{d}^n y=T_g\left(\tilde f*h\right)\end{aligned}$$

So for a distribution $T\in S^*(\mathbb{R}^n)$ we define the *convolution* as:

$$\begin{gathered}*:\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}^*(\mathbb{R}^n)\to\mathcal{S}^*(\mathbb{R}^n)\\(f*T)(h):=T\left(\tilde f*h\right)\end{gathered}\tag{2.13}$$

For $S,T\in\mathcal{S}^*(\mathbb{R}^n)$, $S*T$ and $S\cdot T$ are ill-defined in general. For example $\delta(x)\cdot\delta(x)$ makes no sense, as well as T_f*T_f for $f=1$.

2.2 The Fourier Transform

First consider the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ and later on $\mathcal{S}^*(\mathbb{R}^n)$.

2.2.1 Definition (Fourier Transform)

Define linear functionals $\mathcal{F}$ and $\overline{\mathcal{F}}$ on $\mathcal{S}$:

$$(\mathcal{F}f)(p) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\mathbf{i}px} f(x)\,\mathrm{d}^n x \tag{2.14}$$

$$\left(\overline{\mathcal{F}}f\right)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\mathbf{i}px} f(p)\,\mathrm{d}^n p \tag{2.15}$$

The integrals are well-defined and finite, because f has suitable decay properties at infinity. An alternative convention, which is not convenient here, because it has less symmetry, is:

$$(\mathcal{F}f)(p) := \int_{\mathbb{R}^n} e^{-\mathbf{i}px} f(x)\,\mathrm{d}^n x$$

$$\left(\overline{\mathcal{F}}f\right)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}px} f(p)\,\mathrm{d}^n p$$

2.2.2 Proposition (Fourier Transform)

$\mathcal{F}$ and $\overline{\mathcal{F}}$ are well-defined linear operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Proof

The linearity is clear. We still have to show, that all norms $\|\mathcal{F}f\|_{r,s}$ are finite. First consider the norm $\|.\|_{0,0}$:

$$\begin{aligned}|(\mathcal{F}f)(p)| &\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)| \cdot \frac{(1+|x|)^{n+1}}{(1+|x|)^{n+1}} \mathrm{d}^n x \le \\ &\le \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_{n+1,0} \underbrace{\int_{\mathbb{R}^n} \frac{\mathrm{d}^n x}{(1+|x|)^{n+1}}}_{<\infty} \le c\,\|f\|_{n+1,0}\end{aligned}$$

Now we consider $\left|p^\alpha \mathrm{D}^\beta (\mathcal{F}f)(p)\right|$.

$$\begin{aligned}\frac{\partial}{\partial p^j}(\mathcal{F}f)(p) &= \frac{\partial}{\partial p^j} \int e^{-\mathbf{i}px} f(x)\,\mathrm{d}^n x = \ldots = \int \left(\frac{\partial}{\partial p^j} e^{-\mathbf{i}px}\right) f(x)\,\mathrm{d}^n x = \\ &= \int \left(-\mathbf{i}x^j\right) e^{-\mathbf{i}px} f(x)\,\mathrm{d}^n x\end{aligned}$$

That the derivative and the integral can be interchanged is shown as follows:

$$\frac{\partial}{\partial p^j} \int e^{-\mathbf{i}px} f(x)\,\mathrm{d}^n x = \lim_{\varepsilon \searrow 0} \int \frac{e^{-\mathbf{i}(p+\varepsilon e_j)x} - e^{-\mathbf{i}px}}{\varepsilon} f(x)\,\mathrm{d}^n x$$

$$\frac{e^{-\mathbf{i}(p+\varepsilon e_j)x} - e^{-\mathbf{i}px}}{\varepsilon} = \frac{1}{\varepsilon} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} e^{-\mathbf{i}(p+\varepsilon\tau e_j)x} \mathrm{d}\tau = -\mathbf{i}e_j x \int_0^1 e^{-\mathbf{i}(p+\varepsilon\tau e_j)x} \mathrm{d}\tau$$

$$\Rightarrow \qquad \left|\frac{e^{-\mathbf{i}(p+\varepsilon e_j)x}-e^{-\mathbf{i}px}}{\varepsilon}\right| \leq \|x\| \cdot \int_0^1 \underbrace{\left|e^{-\mathbf{i}(p+\varepsilon\tau e_j)x}\right|}_{=1} \mathrm{d}\tau = \|x\|$$

$$\left|\frac{e^{-\mathbf{i}(p+\varepsilon e_j)x}-e^{-\mathbf{i}px}}{\varepsilon} f(x)\right| \leq \|x\| \cdot |f(x)| \leq \frac{\|x\|}{1+\|x\|^{n+2}} \|f\|_{n+2,0} =: h(x) \in L^1(\mathbb{R}^n)$$

This allows us to apply the dominated convergence theorem to take the limit $\varepsilon \to 0$ inside the integral. Iteration of this process gives:

$$\begin{aligned}
\mathrm{D}^\beta(\mathcal{F}f)(p) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}} (-\mathbf{i})^{|\beta|} x^\beta f(x) e^{-\mathbf{i}px} \mathrm{d}^n x \\
p^\alpha \mathrm{D}^\beta(\mathcal{F}f)(p) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}} (-\mathbf{i})^{|\beta|} x^\beta f(x) p^\alpha e^{-\mathbf{i}px} \mathrm{d}^n x \\
p^j e^{-\mathbf{i}px} &= \mathbf{i} \frac{\partial}{\partial x^j} e^{-\mathbf{i}px} \\
\Rightarrow \qquad p^\alpha e^{-\mathbf{i}px} &= \mathbf{i}^{|\alpha|} \mathrm{D}_x^\alpha e^{-\mathbf{i}px} \\
p^\alpha \mathrm{D}^\beta(\mathcal{F}f)(p) &= \frac{(-\mathbf{i})^{|\beta|}\,\mathbf{i}^{|\alpha|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \underbrace{\left(x^\beta f(x)\right)}_{\text{rapid decay}} \left(\mathrm{D}_x^\alpha e^{-\mathbf{i}px}\right) \mathrm{d}^n x = \\
&\underset{\text{by parts}}{\overset{\text{integration}}{=}} \frac{(-\mathbf{i})^{|\beta|}\,\mathbf{i}^{|\alpha|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (-1)^{|\alpha|} \mathrm{D}_x^\alpha \left(x^\beta f(x)\right) e^{-\mathbf{i}px} \mathrm{d}^n x = \\
&= \frac{(-\mathbf{i})^{|\alpha|+|\beta|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathrm{D}_x^\alpha \left(x^\beta f(x)\right) e^{-\mathbf{i}px} \mathrm{d}^n x
\end{aligned}$$

From the computation we did earlier in the proof we know:

$$\left|\mathrm{D}_x^\alpha(\mathcal{F}f)(p)\right| \leq C \left\|\mathrm{D}_x^\alpha\left(x^\beta f\right)\right\|_{n+1,0} \leq \tilde{C}(\alpha,\beta) \|f\|_{|\beta|+n+1,|\alpha|}$$

$$\|\mathcal{F}f\|_{r,s} \leq \tilde{c}(s,r,n) \|f\|_{s+n+1,r}$$

Therefore $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is well-defined. The same follows analogously for $\overline{\mathcal{F}}$. $\square_{2.2.2}$

So we have the following correspondence:

$$-\mathbf{i}x^j \leftrightarrow \frac{\partial}{\partial p^j} \tag{2.16}$$

$$-\mathbf{i}\frac{\partial}{\partial x^j} \leftrightarrow p^j \tag{2.17}$$

Here the derivatives always act on f or $\mathcal{F}f$ and not on $e^{-\mathbf{i}px}$.

$$x^\alpha \mathrm{D}^\beta f \leftrightarrow \mathbf{i}^{|\alpha|+|\beta|} \mathrm{D}_p^\alpha p^\beta (\mathcal{F}f)(p) = \mathbf{i}^{|\alpha|+|\beta|} p^\beta \mathrm{D}^\alpha (\mathcal{F}f)(p) + \text{lower order terms}$$

Suppose we had worked with $\|f\|_{0,k} = |f|_{C^k}$ as family of norms. Then the norms of the Fourier transform of a function with finite norms would not necessarily be finite.

2.2.3 Theorem (Plancherel, Convergence Generating Factor)

$\mathcal{F}$ and $\overline{\mathcal{F}}$ are inverse to each other:

$$\overline{\mathcal{F}}\mathcal{F} = \mathcal{F}\overline{\mathcal{F}} = \mathbb{1}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \tag{2.18}$$

Proof

$$\left(\mathcal{F}\overline{\mathcal{F}}f\right)(p) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\mathbf{i}px}\underbrace{\left(\int_{\mathbb{R}^n} e^{\mathbf{i}qx} f(q)\,\mathrm{d}^n q\right)}_{(\overline{\mathcal{F}}f)(x)}\mathrm{d}^n x \overset{?}{=} f(p)$$

$$\overset{?}{=} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(q)\left(\int_{\mathbb{R}^n} e^{-\mathbf{i}(p-q)x}\mathrm{d}^n x\right)\mathrm{d}^n q$$

The problem here is, that $e^{-\mathbf{i}(p-q)x}$ does not decay at infinity, so the integral is not well-defined. Instead we have to introduce a *convergence generating factor* $e^{-\varepsilon x^2}$ and, after integrating, calculate the limes $\varepsilon \to 0$.

$$\left(\mathcal{F}\overline{\mathcal{F}}f\right)(p) = \frac{1}{(2\pi)^n}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n} e^{-\mathbf{i}px}e^{-\varepsilon x^2}\left(\int_{\mathbb{R}^n} e^{\mathbf{i}qx} f(q)\,\mathrm{d}^n q\right)\mathrm{d}^n x =$$

$$\overset{\text{Fubini}}{=} \frac{1}{(2\pi)^n}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n} f(q)\left(\int_{\mathbb{R}^n} e^{-\mathbf{i}(p-q)x}e^{-\varepsilon x^2}\mathrm{d}^n x\right)\mathrm{d}^n q$$

The resulting Gaussian integral can be computed in closed form. In one dimension it is:

$$\int_{\mathbb{R}} e^{-\mathbf{i}\lambda x}e^{-\varepsilon x^2}\mathrm{d}x = \int_{\mathbb{R}} e^{-\varepsilon\left(x+\frac{\mathbf{i}\lambda}{2\varepsilon}\right)^2-\frac{\lambda^2}{4\varepsilon}}\mathrm{d}x = e^{-\frac{\lambda^2}{4\varepsilon}}\int_{\mathbb{R}} e^{-\varepsilon\left(x+\frac{\mathbf{i}\lambda}{2\varepsilon}\right)^2}\mathrm{d}x =$$

$$\underset{\mathrm{d}z=\sqrt{\varepsilon}\mathrm{d}x}{\overset{z=\sqrt{\varepsilon}\left(x+\frac{\mathbf{i}\lambda}{2\varepsilon}\right)}{=}} e^{-\frac{\lambda^2}{4\varepsilon}}\int_{\mathbb{R}+\frac{\mathbf{i}\lambda}{2\sqrt{\varepsilon}}} e^{-z^2}\frac{\mathrm{d}z}{\sqrt{\varepsilon}} =$$

$$\underset{\text{deformation}}{\overset{\text{contour}}{=}} \frac{e^{-\frac{\lambda^2}{4\varepsilon}}}{\sqrt{\varepsilon}}\underbrace{\int_{\mathbb{R}} e^{-z^2}\mathrm{d}z}_{=\sqrt{\pi}} = \sqrt{\frac{\pi}{\varepsilon}}e^{-\frac{\lambda^2}{4\varepsilon}}$$

So we get:

$$\left(\mathcal{F}\overline{\mathcal{F}}f\right)(p) = \frac{1}{(2\pi)^n}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n} f(q)\left(\frac{\pi}{\varepsilon}\right)^{\frac{n}{2}} e^{-\frac{(p-q)^2}{4\varepsilon}}\mathrm{d}^n q$$

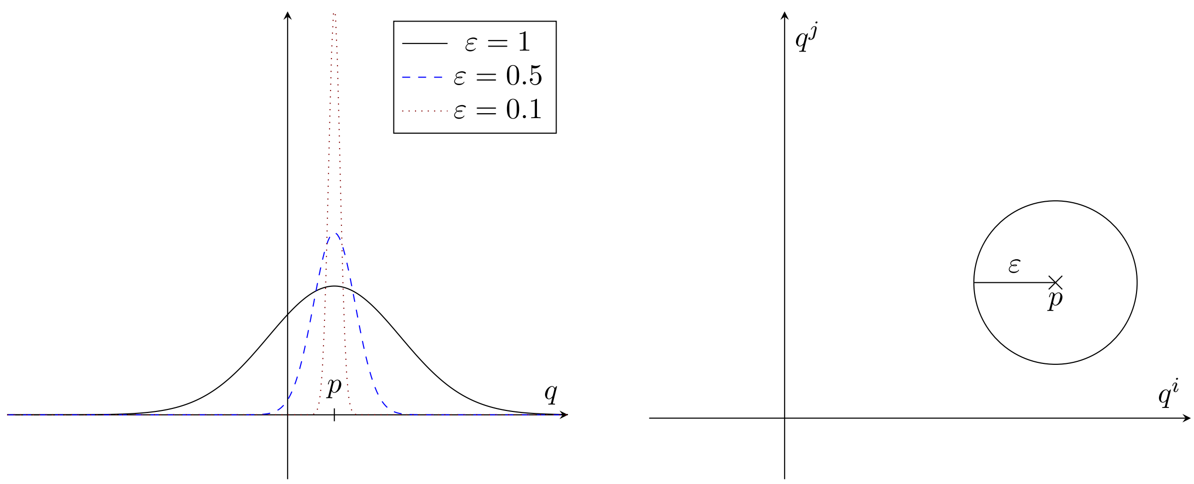


Figure 2.2: The Gaussian gets very narrow and very high as ε decreases.

Estimate the integral as follows:

$$\left(\mathcal{F}\overline{\mathcal{F}}f\right)(p) = \lim_{\varepsilon \searrow 0} \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} f(p)\, e^{-\frac{(p-q)^2}{4\varepsilon}} \mathrm{d}^n q + \int_{\mathbb{R}^n} \left(f(q) - f(p)\right) e^{-\frac{(p-q)^2}{4\varepsilon}} \mathrm{d}^n q \right)$$

The first integral gives:

$$\int_{\mathbb{R}^n} e^{-\frac{(p-q)^2}{4\varepsilon}} \mathrm{d}^n q \underset{\mathrm{d}^n z = \frac{\mathrm{d}^n q}{(4\varepsilon)^{\frac{n}{2}}}}{\overset{z=\frac{p-q}{2\sqrt{\varepsilon}}}{=}} (4\varepsilon)^{\frac{n}{2}} \underbrace{\int_{\mathbb{R}^n} e^{-z^2} \mathrm{d}^n q}_{=\pi^{\frac{n}{2}}} = (4\pi\varepsilon)^{\frac{n}{2}}$$

So we get:

$$\left(\mathcal{F}\overline{\mathcal{F}}f\right)(p) = f(p) + \lim_{\varepsilon \searrow 0} \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(f(q) - f(p)\right) e^{-\frac{(p-q)^2}{4\varepsilon}} \mathrm{d}^n q$$

It remains to show that the second summand goes to zero for $\varepsilon \to 0$. We use the following scaling argument:

$$\begin{aligned} \frac{1}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(f(q) - f(p)\right) e^{-\frac{(p-q)^2}{4\varepsilon}} \mathrm{d}^4 q &\underset{\mathrm{d}^n q = \varepsilon^{\frac{n}{2}} \mathrm{d}^n u}{\overset{u=\frac{p-q}{\sqrt{\varepsilon}}}{=}} \frac{1}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(f\left(p - \sqrt{\varepsilon} u\right) - f(p)\right) e^{-\frac{u^2}{4}} \varepsilon^{\frac{n}{2}} \mathrm{d}^n u = \\ &= \int_{\mathbb{R}^n} \underbrace{\left(f\left(p - \sqrt{\varepsilon} u\right) - f(p)\right)}_{\xrightarrow{\varepsilon \searrow 0} 0 \text{ pointwise}} e^{-\frac{u^2}{4}} \mathrm{d}^n u \end{aligned}$$

For the integrand holds:

$$\underbrace{\left(f\left(p - \sqrt{\varepsilon} u\right) - f(p)\right)}_{\xrightarrow{\varepsilon \searrow 0} 0 \text{ pointwise}} e^{-\frac{u^2}{4}} \leq \|f\|_{0,0}\, e^{-\frac{u^2}{4}} \in L^1(\mathbb{R}^n)$$

So the dominated convergence theorem can be applied to get:

$$\left(\mathcal{F}\overline{\mathcal{F}}f\right)(p) = f(p) + \lim_{\varepsilon \searrow 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \underbrace{\lim_{\varepsilon \searrow 0} \left(f\left(p - \sqrt{\varepsilon} u\right) - f(p)\right)}_{=0} e^{-\frac{u^2}{4}} \mathrm{d}^n u = f(p)$$

$\overline{\mathcal{F}}\mathcal{F} = \mathbb{1}$ follows analogously. $\square_{2.2.3}$

We want to generalize the Fourier transform to $\mathcal{S}^*(\mathbb{R}^n)$. We begin with the case T_g with $g \in \mathcal{S}(\mathbb{R}^n)$. We want:

$$\mathcal{F}(T_g) = T_{\mathcal{F}g}$$

$$T_{\mathcal{F}g}(f) = \int_{\mathbb{R}^n} (\mathcal{F}g)(p)\, f(p)\, \mathrm{d}^n p = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-\mathrm{i}px} g(x)\, \mathrm{d}^n x \right) f(p)\, \mathrm{d}^n p$$

Fubini's theorem allows us to interchange the order of integration:

$$T_{\mathcal{F}g}(f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(x) \underbrace{\left(\int_{\mathbb{R}^n} e^{-\mathrm{i}px} f(p)\, \mathrm{d}^n p \right)}_{=\mathcal{F}f(x)} \mathrm{d}^n x = \int_{\mathbb{R}^n} g(x)\, (\mathcal{F}f)(x)\, \mathrm{d}^n x = \mathrm{T}_g(\mathcal{F}f)$$

Since we want $(\mathcal{F}T_g)(f) = T_{\mathcal{F}g} = T_g(\mathcal{F}f)$, this motivates the following general definition:

2.2.4 Definition (Fourier Transform of Distributions)

$\mathcal{F}, \overline{\mathcal{F}} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$ are defined by their action on a test function $f \in \mathcal{S}(\mathbb{R}^n)$:

$$(\mathcal{F}T)(f) := T(\mathcal{F}f) \tag{2.19}$$
$$\left(\overline{\mathcal{F}}T\right)(f) := T\left(\overline{\mathcal{F}}f\right) \tag{2.20}$$

It holds:

$$|(\mathcal{F}T)(f)| = |T(\mathcal{F}f)| \underset{\Rightarrow \exists\, r,s \in \mathbb{N}, c \in \mathbb{R}_{>0}}{\overset{T \in \mathcal{S}^*(\mathbb{R}^n)}{\leq}} c\,\|\mathcal{F}f\|_{r,s} \leq \tilde{c}\,\|f\|_{s+n+1,r}$$

Thus $\mathcal{F}T$ is indeed a tempered distribution.

2.2.5 Theorem (Plancherel for Distributions)

Plancherel's theorem holds on $\mathcal{S}^*(\mathbb{R}^n)$ as well:

$$\mathcal{F}\overline{\mathcal{F}} = \overline{\mathcal{F}}\mathcal{F} = \mathbb{1}_{\mathcal{S}^*(\mathbb{R}^n)} \tag{2.21}$$

Proof

$$\left(\mathcal{F}\overline{\mathcal{F}}T\right)(f) \overset{\text{Definition 2.2.4}}{=} \left(\overline{\mathcal{F}}T\right)(\mathcal{F}f) = T\left(\overline{\mathcal{F}}\mathcal{F}f\right) \overset{\text{Plancherel 2.2.3}}{=} T(f)$$

Since this holds for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $T \in \mathcal{S}^*(\mathbb{R}^n)$ it follows:

$$\mathcal{F}\overline{\mathcal{F}} = \mathbb{1}_{\mathcal{S}^*(\mathbb{R}^n)}$$

The same follows for $\overline{\mathcal{F}}\mathcal{F}$. $\square_{2.2.5}$

2.2.6 Examples

1. The Fourier transform of the δ-Distribution can be calculated as follows:

$$\begin{aligned}(\mathcal{F}\delta)(f) &= \delta(\mathcal{F}f) = (\mathcal{F}f)(0) = \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\mathrm{i}px} f(x)\,\mathrm{d}^n x \bigg|_{p=0} = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} f(x)\,\mathrm{d}^n x = T_{(2\pi)^{-\frac{n}{2}}}(f)\end{aligned}$$

   This means:

$$\mathcal{F}\delta = T_{(2\pi)^{-\frac{n}{2}}}$$

   Or, in a more computational manner, one can write this as:

$$\mathcal{F}\delta = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\mathrm{i}px} \delta(x)\,\mathrm{d}^n x = \frac{1}{(2\pi)^{\frac{n}{2}}}$$

   This is not satisfying from a mathematical point of view, because one does not know, that the usual formula for the Fourier transform also works for distributions.

2. Consider T_f with $f(p) = e^{\mathbf{i}py}$ for a given $y \in \mathbb{R}^n$.

$$\begin{aligned}(\mathcal{F}T_f)(h) = T_f(\mathcal{F}h) = T_f\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{-\mathbf{i}px}h(x)\,\mathrm{d}^n x\right) &= \\ = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{\mathbf{i}py}\left(\int e^{-\mathbf{i}px}h(x)\,\mathrm{d}^n x\right)\mathrm{d}^n p &= \\ = (2\pi)^{\frac{n}{2}}\left(\overline{\mathcal{F}}\mathcal{F}\right)h(y) = (2\pi)^{\frac{n}{2}}\,h(y)\end{aligned}$$

So we get:

$$\mathcal{F}T_f h = (2\pi)^{\frac{n}{2}}\,h(y)$$
$$(\mathcal{F}T_f)(x) = (2\pi)^{\frac{n}{2}}\,\delta^{(n)}(x-y)$$

Formally one can write:

$$(\mathcal{F}f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\mathbf{i}px}e^{\mathbf{i}py}\mathrm{d}^n p = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\mathbf{i}p(x-y)}\mathrm{d}^n p$$

This is ill-defined, but physicists use the formal relation:

$$\int_{\mathbb{R}^n} e^{-\mathbf{i}p(x-y)}\mathrm{d}^n p = (2\pi)^n\,\delta^{(n)}(x-y)$$

3. Consider $T = T_g$ with $g(p) = p^2 e^{\mathbf{i}py}$ for a given $y \in \mathbb{R}$.

$$\begin{aligned}(\mathcal{F}T_g)(f) = T_g(\mathcal{F}f) = \int_{-\infty}^{\infty} g(p)(\mathcal{F}f)(p)\,\mathrm{d}p &= \\ = \int g(p)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-\mathbf{i}px}\mathrm{d}x\right)\mathrm{d}p\end{aligned}$$

One cannot interchange the integrals, because the integral

$$\int_{-\infty}^{\infty}\underbrace{g(p)\,e^{-\mathbf{i}p\alpha}}_{\notin L^1}\mathrm{d}p$$

does not exist. Therefore we work again with a convergence generating factor, which we can due to the dominated convergence theorem:

$$\begin{aligned}(\mathcal{F}T_g)(f) &= \lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} g(p)\,e^{-\varepsilon|p|}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-\mathbf{i}px}\mathrm{d}x\right)\mathrm{d}p = \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} g(p)\,e^{-\varepsilon|p|}e^{-\mathbf{i}px}\mathrm{d}p\right)\mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} p^2 e^{\mathbf{i}py}e^{-\varepsilon|p|}e^{-\mathbf{i}px}\mathrm{d}p\right)\mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} (\mathbf{i}\partial_x)^2\,e^{-\mathbf{i}p(x-y)-\varepsilon|p|}\mathrm{d}p\right)\mathrm{d}x = \\ &\overset{\text{Lebesgue theorem}}{=} \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} f(x)\,(\mathbf{i}\partial_x)^2\left(\int_{-\infty}^{\infty} e^{-\mathbf{i}p(x-y)-\varepsilon|p|}\mathrm{d}p\right)\mathrm{d}x\end{aligned}$$

Now one can decompose the integral into integrals from $-\infty$ to 0 and from 0 to ∞ and calculate the result.

$$\begin{aligned}
(\mathcal{F}T_g)(f) &= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f(x)\,(\mathbf{i}\partial_x)^2 \left(\int_{-\infty}^{0} e^{-\mathbf{i}p(x-y)+\varepsilon p}\mathrm{d}p + \int_{0}^{\infty} e^{-\mathbf{i}p(x-y)-\varepsilon p}\mathrm{d}p\right)\mathrm{d}x = \\
&= \frac{-1}{\sqrt{2\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f(x)\,\partial_x^2 \left(\frac{1}{-\mathbf{i}(x-y)+\varepsilon} - \frac{1}{-\mathbf{i}(x-y)-\varepsilon}\right)\mathrm{d}x = \\
&= \frac{-1}{\sqrt{2\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f(x)\,\partial_x^2 \left(\frac{-\mathbf{i}(x-y)-\varepsilon-(-\mathbf{i}(x-y)+\varepsilon)}{-(x-y)^2-\varepsilon^2}\right)\mathrm{d}x = \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f(x)\,\partial_x^2 \left(\frac{2\varepsilon}{(x-y)^2+\varepsilon^2}\right)\mathrm{d}x = \\
&\overset{\substack{\text{integration}\\ \text{by parts}}}{=} \sqrt{\frac{2}{\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f''(x)\cdot\frac{\varepsilon}{(x-y)^2+\varepsilon^2}\mathrm{d}x = \\
&\overset{\substack{z:=\frac{x-y}{\varepsilon}\\ \mathrm{d}z=\frac{\mathrm{d}x}{\varepsilon}}}{=} \sqrt{\frac{2}{\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f''(\varepsilon z+y)\cdot\frac{\varepsilon}{\varepsilon^2 z^2+\varepsilon^2}\varepsilon\mathrm{d}z = \\
&= \sqrt{\frac{2}{\pi}} \lim_{\varepsilon\searrow 0} \int_{-\infty}^{\infty} f''(\varepsilon z+y)\cdot\frac{1}{z^2+1}\mathrm{d}z = \\
&= \sqrt{\frac{2}{\pi}} f''(y) \underbrace{\int_{-\infty}^{\infty} \frac{1}{z^2+1}\mathrm{d}z}_{=\pi} = \sqrt{2\pi} f''(y)
\end{aligned}$$

So we get:

$$(\mathcal{F}T_g)(x) = \sqrt{2\pi}\,\delta''(x-y)$$

4. Consider the following Hilbert space:

$$L^2(\mathbb{R}^n) = \left\{ f:\mathbb{R}^n\to\mathbb{R} \,\middle|\, f \text{ measurable with } \int |f|^2\,\mathrm{d}^n x < \infty \right\}$$

For $f \in L^2(\mathbb{R}^n)$ define:

$$\begin{aligned}
T_f(g) &:= \int_{\mathbb{R}^n} f(x)\,g(x)\,\mathrm{d}^n x \\
|T_f(g)| &\overset{\text{Schwarz}}{\leq} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \cdot c(n)\, \|g\|_{n+1,0}
\end{aligned}$$

Here we used:

$$|g(x)| \leq \tilde{c}(n) \frac{\|g\|_{n+1,0}}{(1+|x|)^{n+1}}$$

So we have $T_f \in \mathcal{S}^*(\mathbb{R}^n)$.

$$\begin{aligned}
L^2(\mathbb{R}^n) &\hookrightarrow \mathcal{S}^*(\mathbb{R}^n) \xrightarrow{\mathcal{F}} \mathcal{S}^*(\mathbb{R}^n) \\
f &\mapsto \quad T_f \quad \mapsto \mathcal{F}T_f
\end{aligned}$$

2.2.7 Theorem (Fourier Transform is isometry)

The mappings $\mathcal{F}, \overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isometries, i.e.:

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} \tag{2.22}$$

Due to $\mathcal{F}\overline{\mathcal{F}} = 1$ they are even unitary transformations.

Proof

Consider first $f, g \in \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$.

$$\begin{aligned}\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \overline{(\mathcal{F}f)(x)}\,(\mathcal{F}g)(x)\,\mathrm{d}^n x \stackrel{?}{=} \int \overline{f}(x)\,g(x)\,\mathrm{d}^n x \\ &= T_{\overline{\mathcal{F}}\overline{f}}(\mathcal{F}g) = \left(\overline{\mathcal{F}}T_{\overline{f}}\right)(\mathcal{F}g) = T_{\overline{f}}\left(\overline{\mathcal{F}}\mathcal{F}g\right) = \\ &\stackrel{\overline{\mathcal{F}}\mathcal{F}=1}{=} T_{\overline{f}}(g) = \int \overline{f}(x)\,g(x)\,\mathrm{d}^n x\end{aligned}$$

$\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. Because $\mathcal{F}$ is continuous, $\mathcal{F}$ is also isometric on $L^2(\mathbb{R}^n)$.
For $f \in L^2(\mathbb{R}^n)$ choose $f_n \in \mathcal{S}(\mathbb{R}^n)$ with $f_n \to f$ converging in $L^2(\mathbb{R}^n)$. Then holds for all $n \in \mathbb{N}$:

$$\|f_n\|_{L^2} = \|\mathcal{F}f_n\|$$

In the limes $n \to \infty$ we get, since $\mathcal{F}$ is continuous:

$$\|f\| = \|\mathcal{F}f\|$$

$\square_{2.2.7}$

2.3 Applications to Partial Differential Equations with Constant Coefficients

Consider as example the Poisson equation

$$\Delta u = f$$

in $\mathbb{R}^n$ with a given f and assume for simplicity $f \in \mathcal{S}(\mathbb{R}^n)$. After a Fourier transform and defining $\hat{u} := \mathcal{F}u$ and $\hat{f} = \mathcal{F}f$ we get:

$$\left(-\|p\|^2\right)\hat{u}(p) = \hat{f}(p)$$

$$\Rightarrow \qquad \hat{u}(p) = -\frac{\hat{f}(p)}{\|p\|^2}$$

Then $\overline{\mathcal{F}}\hat{u}$ is a solution of the Poisson equation.

- For $\|p\|^{-2}\hat{f}(p) \in \mathcal{S}(\mathbb{R}^n)$ the method works directly and one gets a $u \in \mathcal{S}(\mathbb{R}^n)$.
- In the case of $n \geq 3$, $\hat{u} = -\|p\|^{-2}\hat{f}(p)$ is a regular distribution. (If $n < 3$ the integral does not necessarily converge.)

  $$(T_{\hat{u}})(g) := \int \left(-\frac{\hat{f}(p)}{\|p\|^2}\right) g(p)\,\mathrm{d}^n p$$

  Therefore $u := \overline{\mathcal{F}}T_{\hat{u}}$ is a distributional solution of the Poisson equation, so we get $\Delta u = T_f$.

  Problem: The distributional solution is not unique, because e.g.

  $$\hat{u}(p) = -\frac{\hat{f}(p)}{\|p\|^2} + c\delta(p)$$

  is also a solution. Therefore we have to specify the behavior of $u(x)$ in the limit $\|x\| \to \infty$.

3 The Laplace Equation in $\Omega \subseteq \mathbb{R}^n$

In this chapter we always consider an open subset $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$. With the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \tag{3.1}$$

the Laplace equation can be written as:

$$\Delta u = 0$$

The inhomogeneous Laplace equation is called Poisson equation:

$$\Delta u = f$$

For both one needs to specify boundary conditions on $\partial\Omega$ to get a unique solution.

3.0 Reminder

3.0.1 Theorem (Gauss's Theorem)

If Y is a smooth vector field on the closure $\overline{\Omega}$ and $\partial\Omega$ is smooth with outer normal ν, then holds:

$$\int_\Omega \operatorname{div}(Y) \underbrace{\mathrm{d}^n x}_{=\mathrm{d}\mu} = \int_{\partial\Omega} \langle Y, \nu \rangle \,\mathrm{d}\mu_{\partial\Omega} \tag{3.2}$$

(Proof omitted)

3.0.2 Theorem (Green's Identities)

For $u, w \in C^\infty\left(\overline{\Omega}\right)$ holds:

$$\int_\Omega w \Delta u \mathrm{d}\mu_\Omega = \int_{\partial\Omega} w\left(\nabla_\nu u\right) \mathrm{d}\mu_{\partial\Omega} - \int_\Omega \langle \nabla w, \nabla u \rangle \,\mathrm{d}\mu_\Omega \tag{3.3}$$

$$\int_\Omega \left(w\left(\Delta u\right) - \left(\Delta w\right) u\right) \mathrm{d}\mu_\Omega = \int_{\partial\Omega} \left(w\left(\nabla_\nu u\right) - \left(\nabla_\nu w\right) u\right) \mathrm{d}\mu_{\partial\Omega} \tag{3.4}$$

Proof

We use $\Delta = \operatorname{div}\operatorname{grad}$ and integrate by parts:

$$\int_\Omega w\Delta u \mathrm{d}\mu_\Omega = \int_\Omega w \cdot \operatorname{div}(\nabla u)\,\mathrm{d}\mu_\Omega = \int_\Omega \operatorname{div}(w\nabla u) - \langle \nabla w, \nabla u\rangle\,\mathrm{d}\mu_\Omega$$

Then one can use Gauss's theorem to get the first identity. Now one subtracts the identity with w and u commuted:

$$\begin{aligned} w\Delta u - u\Delta w &= \operatorname{div}(w\nabla u) - \langle \nabla w, \nabla u\rangle - (\operatorname{div}(u\nabla w) - \langle \nabla u, \nabla w\rangle) = \\ &\overset{\langle .,. \rangle \text{ symmetric}}{=} \operatorname{div}(w\nabla u) - \operatorname{div}(u\nabla w) \end{aligned}$$

Using Gauss's theorem the second identity follows. $\square_{3.0.2}$

3.1 Representation Formulas for Harmonic Functions

3.1.1 Definition (Harmonic Functions)

A function $u \in C^2\left(\overline{\Omega}\right)$ is called *harmonic*, if the Laplacian vanishes:

$$\Delta u = 0$$

The harmonic functions form a vector space.

Examples

- Constant or linear functions
- $u(x_1, x_2) = x_1^2 - x_2^2$ is harmonic on $\mathbb{R}^2$.
- Holomorphic functions on $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$

Consider now *spherically symmetric harmonic functions*. For this we choose polar coordinates (r, ϑ, φ) in $\mathbb{R}^3$:

$$\begin{aligned} x &= r\cos(\vartheta) \\ y &= r\sin(\vartheta)\cos(\varphi) \\ z &= r\sin(\vartheta)\sin(\varphi) \end{aligned}$$

More general in $\mathbb{R}^n$ with $n \in \mathbb{N}_{\geq 2}$ we choose $r = \|x\|$ and $\omega \in S^{n-1}$. Regard $\mathbb{R}^n$ with the Euclidian metric as a Riemannian manifold (M, g). Polar coordinates give a special chart on $\Omega \subseteq M$ with $0 \notin \Omega$. The we can calculate $\Delta =$ Δ\ as Laplace-Beltrami operator in polar coordinates. The metric is:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g_{S^{n-1}} \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} g_{S^{n-1}}^{-1} \end{pmatrix}$$

$$\begin{aligned} \det(g) &= r^{2(n-1)} g_{S^{n-1}} \\ \sqrt{\det(g)} &= r^{n-1}\sqrt{\det(g_{S^{n-1}})} \end{aligned}$$

Now the Laplace-Beltrami operator can be calculated with the *Koszul formula*:

$$\begin{aligned}\Delta\backslash u = \nabla_j \nabla^j u &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^j}\left(\sqrt{\det(g)} g^{jk} \frac{\partial}{\partial x^k} u\right) = \\ &= \frac{1}{r^{n-1}\sqrt{\det(g_{S^{n-1}})}} \frac{\partial}{\partial x^j}\left(r^{n-1}\sqrt{\det(g_{S^{n-1}})} g^{jk} \frac{\partial}{\partial x^k} u\right) = \\ &= \frac{1}{r^{n-1}\sqrt{\det(g_{S^{n-1}})}} \frac{\partial}{\partial r}\left(r^{n-1}\sqrt{\det(g_{S^{n-1}})} \frac{\partial}{\partial r} u\right) + \\ &\quad + \sum_{j,k=2}^{n} \frac{1}{r^{n-1}\sqrt{\det(g_{S^{n-1}})}} \frac{\partial}{\partial x^j}\left(r^{n-1}\sqrt{\det(g_{S^{n-1}})} \frac{1}{r^2} (g_{S^{n-1}})^{jk} \frac{\partial}{\partial x^k} u\right) = \\ &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r} u\right) + \frac{1}{r^2}\ \Delta\backslash_{S^{n-1}} u\end{aligned} \tag{3.5}$$

The important formula is:

$$\Delta\backslash = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\ \cdot\right) + \frac{1}{r^2}\ \Delta\backslash_{S^{n-1}} \tag{3.6}$$

For spherically symmetric solutions $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$, the Laplace equation reads:

$$\Delta\backslash \Gamma = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r} \Gamma(r)\right) = 0$$

This gives:

$$\begin{aligned} r^{n-1} \frac{\partial}{\partial r} \Gamma(r) &= c \\ \frac{\partial}{\partial r} \Gamma(r) &= c r^{1-n} \end{aligned}$$

$$\Gamma(r) = a + C \int^r \tau^{1-n} \mathrm{d}\tau = \begin{cases} a + \tilde{c} \ln(r) & \text{if } n = 2 \\ a + \frac{c}{r^{n-2}} & \text{if } n > 2 \end{cases}$$

Now we choose specific values for a and c or $\tilde{c}$.

3.1.2 Definition (Fundamental Solution of Laplace Equation)

The *fundamental solution* Γ of the Laplace equation in $\mathbb{R}^n$ is defined by:

$$\Gamma(x, y) = \Gamma(\|x - y\|) := \begin{cases} \frac{1}{2\pi} \ln(\|x - y\|) & \text{if } n = 2 \\ \frac{1}{n(2-n)\omega_n} \|x - y\|^{2-n} & \text{if } n > 2 \end{cases} \tag{3.7}$$

Here $\omega_n := \mu(B_1(0))$ is the Lebesgue measure of the unit ball.
For example for $n = 3$ we have $\omega_3 = \frac{4\pi}{3}$ and thus:

$$\Gamma(x, y) = \frac{1}{3(-1)\frac{4\pi}{3}} \cdot \frac{1}{\|x - y\|} = -\frac{1}{4\pi} \frac{1}{\|x - y\|} \tag{3.8}$$

3.1.3 Theorem (Green's representation)

Let $\Omega \subseteq \mathbb{R}^n$ be open with smooth boundary and $u \in C^2\left(\overline{\Omega}\right)$. Then for any $y \in \Omega$ holds:

$$u(y) = \int_{\partial\Omega} \left(u(x) \nabla_\nu \Gamma(x,y) - \Gamma(x,y) \nabla_\nu u(x)\right) \mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega \Gamma(x,y) (\Delta u)(x) \,\mathrm{d}\mu(x) \tag{3.9}$$

For harmonic functions with $\Delta u = 0$, this simplifies to:

$$u(y) = \int_{\partial\Omega} \left(u(x) \nabla_\nu \Gamma(x,y) - \Gamma(x,y) \nabla_\nu u(x)\right) \mathrm{d}\mu_{\partial\Omega}(x) \tag{3.10}$$

Thus u has an explicit representation in terms of its boundary values on $\partial\Omega$.

Proof

Choose $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(y) \subseteq \Omega$.

$$\int_{\Omega \setminus B_\varepsilon(y)} \underbrace{\Gamma(x,y)}_{\text{smooth}} \underbrace{(\Delta u)(x)}_{\text{continuous}} \mathrm{d}\mu(x)$$

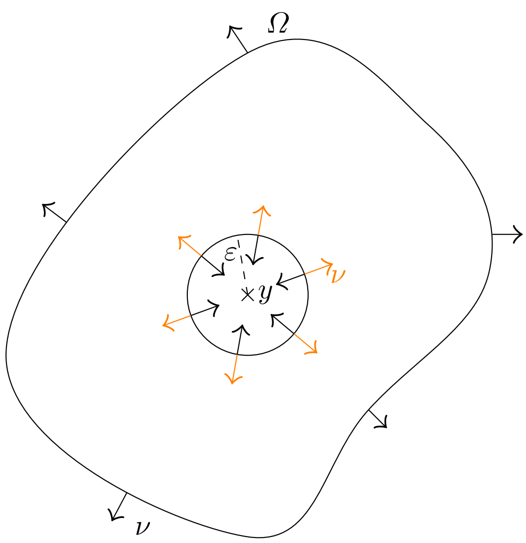


Figure 3.1: Outer normal ν of $\Omega \setminus B_\varepsilon(y)$ (black) and of $B_\varepsilon(y)$ (orange)

We apply the second Green's identity to obtain with $v(x) := \Gamma(x,y)$:

$$\begin{aligned}\int_{\Omega \setminus B_\varepsilon(y)} \big(v \cdot \Delta u - \underbrace{(\Delta v)}_{=0} u\big) \mathrm{d}\mu &= \int_{\partial(\Omega \setminus B_\varepsilon(y))} \left(v (\nabla_\nu u) - (\nabla_\nu v) u\right) \mathrm{d}\mu_{\partial\Omega} = \\ &= \int_{\partial\Omega} \left(\Gamma(x,y) \nabla_\nu u(x) - (\nabla_\nu \Gamma)(x,y) u(x)\right) \mathrm{d}\mu_{\partial\Omega} + \\ &\quad - \int_{\partial B_\varepsilon(y)} \left(\Gamma(x,y) \nabla_\nu u(x) - \nabla_\nu \Gamma(x,y) u(x)\right) \mathrm{d}\mu_{\partial B_\varepsilon(y)}\end{aligned}$$

The minus in front of $\int_{\partial B_\varepsilon(y)}$ comes from the fact, that the outer normal of $\partial(\Omega \setminus B_\varepsilon(y))$ shows in the opposite direction of the outer normal of $\partial B_\varepsilon(y)$. The left side of the integral gives in the limit $\varepsilon \searrow 0$:

$$\lim_{\varepsilon \searrow 0} \int_{\Omega \setminus B_\varepsilon(y)} v \cdot \Delta u \mathrm{d}\mu = \int_\Omega v \cdot \Delta u \mathrm{d}\mu = \int_\Omega \Gamma(x,y) \cdot (\Delta u)(x) \,\mathrm{d}\mu(x)$$

We estimate the integral over $\partial B_\varepsilon(y)$ in the limit $\varepsilon \searrow 0$:

$$\left| \int_{\partial B_\varepsilon(y)} \Gamma(x,y) \nabla_\nu u(x) \,\mathrm{d}\mu_{\partial B_\varepsilon(y)} \right| \leq \Gamma(\varepsilon) \sup_{B_\varepsilon(y)} \|\nabla u\| \cdot \underbrace{n\omega_n}_{\substack{\text{surface area of} \\ \text{the unit sphere}}} \varepsilon^{n-1} \sim$$

$$\sim \begin{cases} \varepsilon & \text{if } n > 2 \\ \varepsilon \ln(\varepsilon) & \text{if } n = 2 \end{cases} \xrightarrow{\varepsilon \searrow 0} 0$$

Now we expand the second part around $\varepsilon = 0$:

$$\begin{aligned} \int_{\partial B_\varepsilon(y)} (\nabla_\nu \Gamma)(x,y) \, u(x) \,\mathrm{d}\mu_{\partial B_\varepsilon(y)} &= \underbrace{\frac{\partial}{\partial \varepsilon} \Gamma(\varepsilon)}_{= \frac{1}{n\omega_n} \varepsilon^{1-n}} \underbrace{\int_{\partial B_\varepsilon(y)} u(x) \,\mathrm{d}\mu_{B_\varepsilon(y)}}_{= u(y) n\omega_n \varepsilon^{n-1} + o_0(\varepsilon^{n-1})} = \\ &= u(y) + o_0\left(\varepsilon^0\right) \xrightarrow{\varepsilon \searrow 0} u(y) \end{aligned}$$

This gives:

$$\int_\Omega \Gamma(x,y) (\Delta u)(x) \,\mathrm{d}\mu(x) = \int_{\partial\Omega} \left(\Gamma(x,y) \nabla_\nu u(x) - (\nabla_\nu \Gamma)(x,y) \, u(x)\right) \mathrm{d}\mu_{\partial\Omega} + u(y)$$

$$\Rightarrow \qquad u(y) = \int_{\partial\Omega} \left(u(x) \nabla_\nu \Gamma(x,y) - \Gamma(x,y) \nabla_\nu u(x)\right) \mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega \Gamma(x,y) (\Delta u)(x) \,\mathrm{d}\mu(x)$$

$\square_{3.1.3}$

3.1.4 Corollary (Laplacian of Fundamental Solution is Delta Distribution)

For any $\varphi \in C_0^\infty(\Omega)$ holds:

$$\varphi(y) = \int_\Omega \Gamma(x,y) (\Delta\varphi)(x) \,\mathrm{d}\mu(x) \tag{3.11}$$

This can also be expressed in terms of distributions as:

$$\Delta_x \Gamma(x,y) = \delta^{(n)}(x-y) \tag{3.12}$$

More correctly, for fixed $y \in \mathbb{R}^n$, $T(x) := T_{\Gamma(x,y)}$ defines a regular distribution. Equation (3.11) means, that for all $\varphi \in C_0^\infty(\Omega) \subseteq \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned} & (\Delta T)(\varphi) = \varphi(y) \\ \Leftrightarrow \qquad & \Delta T = \delta_y \end{aligned}$$

Proof

Since the support of φ lies inside of Ω, the first term in Green's representation vanishes:

$$\begin{aligned}\varphi(y) &= \int_{\partial\Omega} \left(\varphi(x)\,\nabla_\nu \Gamma(x,y) - \Gamma(x,y)\,\nabla_\nu \varphi(x)\right) \mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega \Gamma(x,y)\,(\Delta\varphi)(x)\,\mathrm{d}\mu(x) = \\ &= \int_\Omega \Gamma(x,y)\,(\Delta\varphi)(x)\,\mathrm{d}\mu(x)\end{aligned}$$

$\square_{3.1.4}$

Next we investigate the existence of solutions.

$$\Delta u = 0 \qquad u\big|_{\partial\Omega} = u_0 \qquad \text{Dirichlet problem} \tag{3.13}$$

$$\Delta u = 0 \qquad \nabla_\nu u\big|_{\partial\Omega} = u_1 \qquad \text{Neumann problem} \tag{3.14}$$

The problem is, that the representation

$$u(y) = \int_{\partial\Omega} \left(u(x)\,\nabla_\nu \Gamma(x,y) - \Gamma(x,y)\,\nabla_\nu u(x)\right) \mathrm{d}\mu_{\partial\Omega}(x)$$

needs booth u and $\nabla_\nu u$ on the boundary. Suppose we want to solve the Dirichlet problem. Then $u_0 = u\big|_{\partial\Omega}$ is given, but $\nabla_\nu u\big|_{\partial\Omega}$ is unknown.

3.2 The Green's Function

Consider the Dirichlet problem for the Poisson equation:

$$\Delta u(x) = f(x) \qquad \underset{x\in\Omega}{\forall} \qquad u\big|_{\partial\Omega} = \varphi \tag{3.15}$$

Assume $u \in C^2\left(\overline{\Omega}\right)$, $f \in C^0\left(\overline{\Omega}\right)$ and $\varphi \in C^2(\partial\Omega)$.

3.2.1 Definition (Green's function)

A function $G(x,y)$ defined for $x, y \in \overline{\Omega}$ with $x \neq y$ is called *Green's function* for the domain Ω if the following conditions are satisfied:

i) For all $x \in \partial\Omega$ and $y \neq x$ holds $G(x,y) = 0$.

ii) $h(x,y) := G(x,y) - \Gamma(x,y)$ is in $C^2(\Omega)$, even for $x = y$, and harmonic in $x \in \Omega$.

3.2.2 Proposition (Solution of Dirichlet Problem)

For a solution u of the Dirichlet problem for the Poisson equation holds:

$$u(y) = \int_{\partial\Omega} u(x)\,\nabla_\nu G(x,y)\,\mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega f(x)\,G(x,y)\,\mathrm{d}\mu(x) \tag{3.16}$$

Proof

$$\int_{\Omega} h(x,y)(\Delta u)(x)\,\mathrm{d}\mu(x) = \int_{\Omega} h(x,y)\, f(x)\,\mathrm{d}\mu(x) = \underset{\text{identity}}{\overset{\text{2. Green's}}{=}} \int_{\partial\Omega} \left(h(x,y)(\nabla_\nu u)(x) - (\nabla_\nu h(x,y))\cdot u(x)\right)\mathrm{d}\mu_{\partial\Omega}(x)$$

Now we add the Green's representation

$$\int_{\Omega} \Gamma(x,y)\, f(x)\,\mathrm{d}\mu(x) = \int_{\partial\Omega} \left(\Gamma(x,y)(\nabla_\nu u)(x) - (\nabla_\nu \Gamma(x,y))\, u(x)\right)\mathrm{d}\mu_{\partial\Omega}(x) + u(y)$$

to get:

$$\int_{\Omega} G(x,y)\, f(x)\,\mathrm{d}\mu(x) = \int_{\partial\Omega} \left(G(x,y)(\nabla_\nu u)(x) - (\nabla_\nu G(x,y))\, u(x)\right)\mathrm{d}\mu_{\partial\Omega}(x) + u(y)$$

Since $G(x,y) = 0$ on $\partial\Omega$, the proposition follows. $\square_{3.2.2}$

3.2.3 Theorem (Symmetry of the Green's Function)

For all $x, y \in \Omega$ with $x \neq y$ holds:

$$G(x,y) = G(y,x) \tag{3.17}$$

Proof

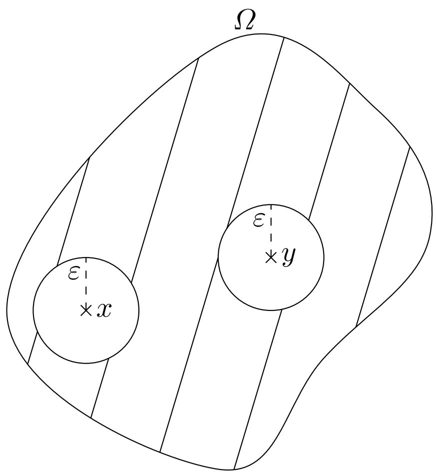


Figure 3.2: $B_\varepsilon(x) \subseteq \Omega$, $B_\varepsilon(y) \subseteq \Omega$, $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$

Choose $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint subsets of Ω.

$$u(z) := G(z,x) \qquad\qquad v(z) := G(z,y)$$

It holds $u, v \in C^2(\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y)))$ and u, v are harmonic. Moreover for $z \in \partial\Omega$ holds:

$$u(z) = G(z,x) = 0 \qquad\qquad v(z) = G(z,y) = 0$$

Apply again the second Green's identity:

$$0 = \int_{\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))} \big(v \underbrace{(\Delta u)}_{=0} - \underbrace{(\Delta v)}_{=0} u\big) \mathrm{d}\mu =$$

$$= \int_{\partial\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))} (v (\nabla_\nu u) - (\nabla_\nu v) u) \, \mathrm{d}\mu_{\partial\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))}$$

Moreover, the boundary values on $\partial\Omega$ vanish. We conclude:

$$0 = \int_{\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)} (v (\nabla_\nu u) - (\nabla_\nu v) u) \, \mathrm{d}\mu_{\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)}$$

Consider the integral over $\partial B_\varepsilon(x)$ first. Since $G = \Gamma + h$ and $h \in C^2$ is bounded, we know for all $z \in \partial B_\varepsilon(x)$:

$$u(z) = G(z,x) \sim \Gamma(z,x) \sim \begin{cases} \ln(\varepsilon) & \text{if } n = 2 \\ \varepsilon^{2-n} & \text{if } n > 2 \end{cases}$$

Since $\nabla_\nu v$ is also bounded due to $v \in C^2(B_\varepsilon(x))$, we get:

$$\int_{\partial B_\varepsilon(x)} (\nabla_\nu v) \, u \mathrm{d}_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon \searrow 0} 0$$

The other term gives:

$$\int_{\partial B_\varepsilon(x)} v(\nabla_\nu u) \, \mathrm{d}_{\partial B_\varepsilon(x)} \underset{=\tilde{c}\varepsilon^{1-n} + o_0(\varepsilon^0)}{\overset{\nabla_\nu u = \frac{\partial}{\partial\varepsilon}\Gamma(\varepsilon) + o_0(\varepsilon^0) =}{=}} cv(x) \, \varepsilon^{1-n} \underbrace{\int_{\partial B_\varepsilon(x)} 1 \mathrm{d}\mu_{\partial B_\varepsilon(x)}}_{=n\omega_n \varepsilon^{n-1}} + o_0(\varepsilon) \xrightarrow{\varepsilon \searrow 0} cv(x)$$

Here the constant $c \neq 0$ does not vanish. Now follows:

$$\int_{\partial B_\varepsilon(x)} ((\nabla_\nu v) \, u - v (\nabla_\nu u)) \, \mathrm{d}\mu_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon \searrow 0} cv(x)$$

$$\int_{\partial B_\varepsilon(y)} ((\nabla_\nu v) \, u - v (\nabla_\nu u)) \, \mathrm{d}\mu_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon \searrow 0} cu(y)$$

Adding these two integrals gives:

$$0 = c(v(x) - u(y))$$

$$\Rightarrow \qquad G(x,y) = v(x) = u(y) = G(y,x)$$

$\square_{3.2.3}$

Any solution u of the Dirichlet problem

$$\Delta u = f \qquad\qquad u\big|_{\partial\Omega} = \varphi$$

has the representation:

$$u(y) = \int_{\partial\Omega} u(x) \, \nabla_\nu G(x,y) \, \mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega G(x,y) \, f(x) \, \mathrm{d}\mu(x)$$

We define:

$$u(y) := \int_{\partial\Omega} \varphi(x) \nabla_\nu G(x,y) \,\mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega G(x,y) f(x) \,\mathrm{d}\mu(x)$$

Since G is symmetric, $G(x,y) = G(y,x)$, the equation $\Delta_x G(x,y) = \delta^{(n)}(x-y)$ implies $\Delta_y G(x,y) = \delta^{(n)}(x-y)$ as well. As a consequence, a formal computation gives:

$$\Delta_y u(y) = \int_{\partial\Omega} \varphi(x) \nabla_\nu \delta^{(n)}(x-y) \,\mathrm{d}\mu_{\partial\Omega}(x) + \int_\Omega \delta^{(n)}(x-y) f(x) \,\mathrm{d}\mu(x)$$

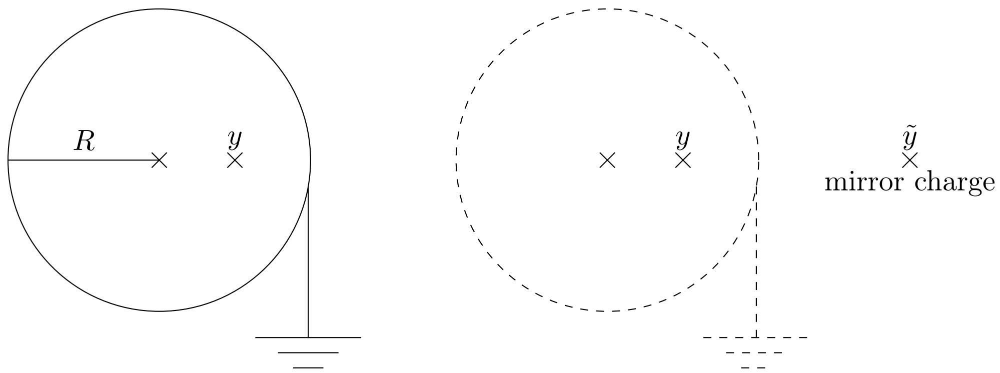
For $y \in \Omega$ the first term vanishes, so we get:

$$\Delta_y u(y) = f(y)$$

This formal calculation can be made rigorous, once an explicit Green's function is given. Then one can also check, whether the boundary conditions are satisfied.

3.2.4 Example (Green's function for $B_R(0)$)

Now we want to construction the Green's function for $\Omega := B_R(0) \subseteq \mathbb{R}^n$. Consider an electric charge inside an earthed, electrically conducting sphere, that screens the electric field, so that it vanishes outside the sphere. The electric field inside the sphere can be calculated using the concept of a mirror charge, which ensures, that the electric field is perpendicular to the sphere.



(a) charge in earthed sphere (b) equivalent mirror charge outside sphere

Figure 3.3: The mirror charge replicates the boundary conditions of the sphere.

The position of the mirror charge must be:

$$\tilde{y} = \frac{R^2}{y^2} \cdot y$$

$$\Rightarrow \qquad \|y\| \cdot \|\tilde{y}\| = R^2$$

Compare this with the inversion with respect to the unit circle (Spiegelung am Einheitskreis) in the complex plane:

$$z \mapsto \frac{1}{\bar{z}} = \frac{z}{|z|^2}$$

This motivates the ansatz:

$$G(x,y) = \begin{cases} \Gamma(\|x-y\|) - \Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) & \text{if } y \neq 0 \\ \Gamma(\|x\|) - \Gamma(R) & \text{if } y = 0 \end{cases}$$

Let us verify that G has all the required properties:

$$G(x,y) - \Gamma(x,y) = \begin{cases} -\Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) & \text{if } y \neq 0 \\ -\Gamma(R) & \text{if } y = 0 \end{cases}$$

For $x, y \in B_R(0)$ follows $\tilde{y} \notin B_R(0)$ and thus $x \neq \tilde{y}$. So $\Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)$ is smooth if $y \neq 0$. To see smoothness in the case $y = 0$, we rewrite:

$$\begin{aligned} \Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) &= \Gamma\left(\sqrt{\frac{y^2}{R^2}\left(x^2+\tilde{y}^2-2\langle x,\tilde{y}\rangle\right)}\right) = \\ &= \Gamma\left(\sqrt{\frac{y^2}{R^2}\left(x^2+\frac{R^4}{y^2}-2\frac{R^2}{y^2}\langle x,y\rangle\right)}\right) = \\ &= \Gamma\left(\sqrt{\frac{x^2y^2}{R^2}+R^2-2\langle x,y\rangle}\right) \end{aligned} \tag{3.18}$$

The argument is smooth at $y = 0$ and it holds:

$$\lim_{y\to 0} \Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) = \Gamma(R)$$

So $G(x,y) - \Gamma(x,y)$ is in C^2 for $x, y \in \Omega$.
For $x \in \partial\Omega$, i.e. $\|x\| = R$, holds:

$$\begin{aligned} G(x,y) &= \Gamma(\|x-y\|) - \Gamma\left(\sqrt{\frac{\|x\|^2\|y\|^2}{R^2}+R^2-2\langle x,y\rangle}\right) = \\ &\overset{\|x\|=R}{=} \Gamma(\|x-y\|) - \Gamma\left(\sqrt{\|y\|^2+\|x\|^2-2\langle x,y\rangle}\right) = \\ &= \Gamma(\|x-y\|) - \Gamma\left(\sqrt{\|x-y\|^2}\right) = 0 \end{aligned}$$

Thus $G(x,y)$ satisfies the boundary condition $G(x,y) = 0$ for $x \in \partial\Omega$.
Now we show that $G(x,y) - \Gamma(x,y)$ is harmonic:

$$\begin{aligned} G(x,y) - \Gamma(x,y) &= -\Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) \\ \nabla\Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) &= \Gamma'\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\frac{\|y\|}{R}\nabla\|x-\tilde{y}\| \\ \Delta\Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) &= \Gamma''\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\left(\frac{\|y\|}{R}\right)^2(\nabla\|x-\tilde{y}\|)^2 + \\ &\quad + \Gamma'\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\frac{\|y\|}{R}\Delta\|x-\tilde{y}\| \end{aligned}$$

We know:

$$\begin{aligned} 0 = \Delta\Gamma &= \frac{1}{r^{n-1}}\partial_r \left(r^{n-1}\partial_r \Gamma\right) \\ \Rightarrow \qquad 0 &= \Gamma''(r) + \frac{n-1}{r}\Gamma'(r) \end{aligned}$$

With this follows:

$$\begin{aligned} \Delta\Gamma\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right) &= -(n-1)\frac{R}{\|y\|\,\|x-\tilde{y}\|}\Gamma'\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\left(\frac{\|y\|}{R}\right)^2\left(\nabla\|x-\tilde{y}\|\right)^2 + \\ &\quad + \Gamma'\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\frac{\|y\|}{R}\Delta\|x-\tilde{y}\| = \\ &= \frac{\|y\|}{R}\Gamma'\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\left(\frac{-(n-1)}{\|x-\tilde{y}\|}\left(\nabla\|x-\tilde{y}\|\right)^2 + \Delta\|x-\tilde{y}\|\right) = \\ &= \frac{\|y\|}{R}\Gamma'\left(\frac{\|y\|}{R}\|x-\tilde{y}\|\right)\left(\frac{-(n-1)}{\|x-\tilde{y}\|}\left(\frac{x-\tilde{y}}{\|x-\tilde{y}\|}\right)^2 + \frac{n-1}{\|x-\tilde{y}\|}\right) = 0 \end{aligned}$$

Thus $G(x,y)$ is the desired Green's function. From (3.18) one sees explicitly:

$$G(x,y) = G(y,x)$$

We hope that the solution of the Dirichlet problem

$$\Delta u = f \qquad\qquad u\big|_{\partial B_R(0)} = \varphi$$

is given by the Green's representation:

$$u(y) = \int_{B_R(0)} G(x,y)\,f(x)\,\mathrm{d}\mu(x) + \int_{B_R(0)} \nabla_\nu G(x,y)\,\varphi(x)\,\mathrm{d}\mu_{\partial B_R(0)}(x)$$

Computing $\nabla_\nu G(x,y)$ for $x \in \partial B_R(0)$ gives:

$$\nabla_\nu G(x,y) = \frac{R^2 - y^2}{n\omega_n R}\cdot\frac{1}{\|x-y\|^n}$$

3.2.5 Theorem (Poisson representation)

The function

$$u(y) := \begin{cases} \frac{R^2-y^2}{n\omega_n R}\int_{\partial B_R(0)} \frac{\varphi(x)}{\|x-y\|^n}\mathrm{d}\mu_{\partial B_R(0)} & \text{if } y \in B_R(0) \\ \varphi(y) & \text{if } y \in \partial B_R(0) \end{cases} \tag{3.19}$$

with $\varphi \in C^0(\partial B_R(0))$ has the following properties:

- $u \in C^0\left(\overline{B_R(0)}\right)$
- $u \in C^2(B_R(0))$
- u is harmonic in $B_R(0)$.

Proof

This can be shown using Green's representation, computing the boundary values and justifying that the y-derivative may be taken inside the integral. $\square_{3.2.5}$

3.3 The Mean Value Theorem and the Maximum Principle for Harmonic Functions

3.3.1 Theorem (Mean Value Formulas)

A continuous function $u : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^n$ is harmonic, i.e. $u \in C^2(\Omega)$ and $\Delta u = 0$, if and only if for all $x_0 \in \Omega$ and all balls $B_r(x_0) \subseteq \Omega$ one of the following mean value formulas holds:

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(x)\, \mathrm{d}\mu_{\partial B_R}(x) \qquad \text{(spherical mean)} \tag{3.20}$$

$$u(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u(x)\, \mathrm{d}\mu(x) \qquad \text{(mean over ball)} \tag{3.21}$$

In this case, the other formula holds as well.

Proof

"$\Rightarrow$": Let $u \in C^2(\Omega)$ be harmonic and $B_r(x_0) \subseteq \Omega$. Then $B_r(y) \subseteq \Omega$ holds also for all y in a neighborhood of x_0. The function

$$H(x,y) = \Gamma(x,y) - \Gamma(r)$$

coincides up to a constant with the fundamental solution and it holds $H(x,y) = 0$ for $x \in \partial B_r(y)$. Using Green's representation gives:

$$\begin{aligned} u(y) = &\int_{B_r(y)} \underbrace{(\Delta u)(x)}_{=0} \cdot H(x,y)\, \mathrm{d}\mu(x) + \\ &+ \int_{\partial B_r(y)} \Big(u(x) \nabla_\nu H(x,y) - \underbrace{H(x,y)}_{=0} \nabla_\nu u(x) \Big) \mathrm{d}\mu_{\partial B_r(y)}(x) = \\ = &\int_{\partial B_r(y)} u(x) \nabla_\nu H(x,y)\, \mathrm{d}\mu_{\partial B_r(y)}(x) \end{aligned}$$

$$\nabla_\nu H(x,y) = \nabla_{\nu,x} \Gamma(\|x-y\|) \overset{r:=\|x-y\|}{=} \frac{\partial}{\partial r} \Gamma(r) = \frac{1}{n\omega_n r^{n-1}}$$

For $y := x_0$ follows the spherical mean formula:

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(x)\, \mathrm{d}\mu_{\partial B_R}(x)$$

Now follows:

$$\begin{aligned} \int_{B_r(y)} u(x)\, \mathrm{d}\mu(x) &\overset{\text{Fubini}}{=} \int_0^r \mathrm{d}\rho \int_{\partial B_\rho(y)} u(x)\, \mathrm{d}\mu_{\partial B_\rho(y)}(x) = \int_0^r u(y)\, n\omega_n \rho^{n-1} \mathrm{d}\rho = \\ &= u(y)\, n\omega_n \cdot \frac{1}{n} r^n = \omega_n r^n u(y) \end{aligned} \tag{3.22}$$

"$\Leftarrow$": Let $u \in C^0(\Omega)$ be continuous. If the formula for the spherical mean holds, the computation (3.22) gives the formula for means over balls. Let us show that the formula for means over balls implies the formula for spherical means. So assume for fixed y and all $r < r_0$:

$$u(y) = \frac{1}{\omega_n r^n} \int_{B_r(y)} u(x)\, \mathrm{d}\mu(x)$$

Thus using Fubini's theorem follows:

$$r^n u(y) = \frac{1}{\omega_n} \overbrace{\int_0^r \mathrm{d}\rho \underbrace{\int_{\partial B_\rho(y)} u(x)\,\mathrm{d}\mu_{\partial B_\rho(y)}(x)}_{\text{continuous in } \rho}}^{C^1 \text{ in } r}$$

Differentiation on both sides with respect to r gives the formula for spherical means:

$$n r^{n-1} u(y) = \frac{1}{\omega_n} \int_{\partial B_r(y)} u(x)\,\mathrm{d}\mu_{\partial B_r(y)}(x)$$

Next we show, that u is smooth in Ω. The idea is to mollify by convolution. Choose:

$$\varrho(t) := \begin{cases} c_n e^{\frac{1}{t^2-1}} & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases} \in C_0^\infty(\mathbb{R})$$

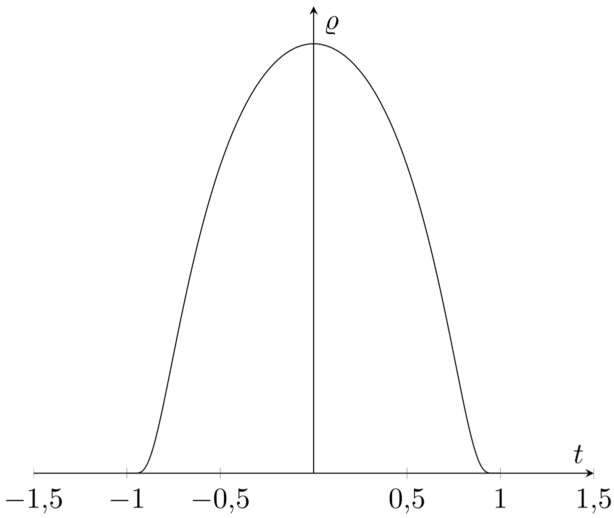


Figure 3.4: $\varrho(t)$ is smooth and has compact support $[-1, 1]$.

Then holds $\varrho(\|x\|) \in C_0^\infty(\mathbb{R}^n)$. Choose c_n such that holds:

$$1 = \int_{\mathbb{R}^n} \varrho(\|x\|)\,\mathrm{d}^n x$$

Now for $\varepsilon \in \mathbb{R}_{>0}$ set:

$$\varrho_\varepsilon(x, y) := \frac{1}{\varepsilon^n} \varrho\left(\frac{\|x - y\|}{\varepsilon}\right)$$

Then still holds:

$$1 = \int_{\mathbb{R}^n} \varrho_\varepsilon(x, y)\,\mathrm{d}^n x$$

Also we know $\varrho_\varepsilon(x, .) \in C_0^\infty(B_{2\varepsilon}(x))$. Choose ε so small that $B_{2\varepsilon}(y) \subseteq \Omega$ and use ϱ_ε as our convolution kernel to define:

$$u_\varepsilon(y) := \int_\Omega \varrho_\varepsilon(x, y)\, u(x)\,\mathrm{d}\mu(x)$$

Now u_ε is a smooth function, because $\varrho_\varepsilon(x,.)$ is smooth and any derivative with respect to y can be exchanged with the integral, since the integration volume is compact.

$$u_\varepsilon(y) = \frac{1}{\varepsilon^n} \int_\Omega \varrho\left(\frac{\|x-y\|}{\varepsilon}\right) u(x)\, \mathrm{d}\mu(x)$$

Choose polar coordinates around y and use Fubini to get:

$$\begin{aligned} u_\varepsilon(y) &= \frac{1}{\varepsilon^n} \int_0^{2\varepsilon} \mathrm{d}r \int_{\partial B_r(y)} \varrho\left(\frac{r}{\varepsilon}\right) u(x)\, \mathrm{d}\mu_{\partial B_r(y)}(x) = \\ &\overset{\text{spherical mean}}{=} \frac{1}{\varepsilon^n} \int_0^{2\varepsilon} \mathrm{d}r \varrho\left(\frac{r}{\varepsilon}\right) u(y)\, n\omega_n r^{n-1} = \\ &= u(y) \frac{1}{\varepsilon^n} \int_0^{2\varepsilon} \mathrm{d}r \int_{\partial B_r(y)} 1 \cdot \varrho\left(\frac{r}{\varepsilon}\right) \mathrm{d}\mu_{\partial B_r(y)}(x) = \\ &= u(y) \int_{\mathbb{R}^n} \varrho_\varepsilon(x,y)\, \mathrm{d}^n x = u(y) \end{aligned}$$

Thus u is smooth.
Compute Δu using the theorem of Gauss:

$$\int_{B_r(y)} (\Delta u(x))\, \mathrm{d}\mu(x) = \int_{\partial B_r(y)} \nabla_\nu u \mathrm{d}\mu_{\partial B_r(y)}$$

With

$$\omega := \frac{x-y}{\|x-y\|} \in S^{n-1}$$

we get:

$$\begin{aligned} \int_{B_r(y)} (\Delta u(x))\, \mathrm{d}\mu(x) &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial}{\partial r} u(y+r\omega)\, \mathrm{d}\mu_{\partial B_1(0)}(\omega) = \\ &= r^{n-1} \frac{\partial}{\partial r} \left(\frac{1}{r^{n-1}} \int_{\partial B_r(y)} u(x)\, \mathrm{d}\mu_{\partial B_r(y)}(x) \right) = \\ &\overset{\substack{\text{mean value} \\ \text{property}}}{=} r^{n-1} \frac{\partial}{\partial r} \left(\frac{1}{r^{n-1}} u(y)\, n\omega_n r^{n-1} \right) = 0 \end{aligned}$$

In the limes $r \to 0$ follows $\Delta u(y) = 0$ since u is smooth. Thus u is harmonic. $\square_{3.3.1}$

3.3.2 Theorem (Strong Maximum Principle)

Let $u \in C^2(\Omega)$ be a harmonic function on an open and connected subset $\Omega \subseteq \mathbb{R}^n$. If a point $x_0 \in \Omega$ with

$$u(x_0) = \sup_{x\in\Omega}(u)$$

or

$$u(x_0) = \inf_{x\in\Omega}(u)$$

exists, then u is constant.

Proof

Since $-u$ is also harmonic, it suffices to consider the case $u(x_0) = \sup_\Omega(u)$, so assume

$$u(x_0) = \sup_\Omega(u) =: M$$

and define:

$$\Omega_M := \left\{x \in \Omega \middle| u(x) = M\right\}$$

Clearly, Ω_M is not empty, because $x_0 \in \Omega_M$. By continuity, Ω_M is closed in Ω with respect to the relative topology. Thus it suffices to show that Ω_M is open. Consider $y \in \Omega_M$, i.e. $u(y) = M$, and choose $r \in \mathbb{R}_{>0}$ with $B_r(y) \subseteq \Omega$.

$$0 = u(y) - M \underset{\text{property}}{\overset{\text{mean value}}{=}} \frac{1}{\omega_n r^n} \int_{B_r(y)} \underbrace{(u(x) - M)}_{\leq 0} \mathrm{d}\mu(x)$$

Since $u(x) \leq M$, the integrand is non-positive and continuous. So $u(x) = M$ for all $x \in B_r(y)$ and thus follows $B_r(y) \subseteq \Omega_M$. This means that Ω_M is open. Since it is also non-empty, closed and connected, the only possibility is $\Omega_M = \Omega$. $\square_{3.3.2}$

3.3.3 Theorem (Weak Maximum Principle)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and $u \in C^0\left(\overline{\Omega}\right)$ be harmonic. Then holds for all $x \in \Omega$:

$$\min_{y \in \partial\Omega}(u(y)) \leq u(x) \leq \max_{y \in \partial\Omega}(u(y))$$

Proof

If the statement were false, there would be an interior maximum or minimum at point $x_0 \in \Omega$, since $\overline{\Omega}$ is compact, which ensures that the maximum and minimum of u are attained in $\overline{\Omega}$. Let $\tilde{\Omega} \subseteq \Omega$ be the connected component of Ω which contains x_0.

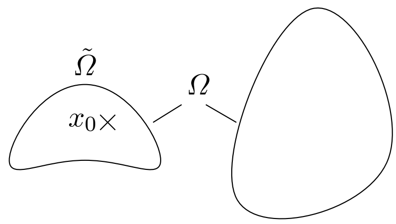


Figure 3.5: $\tilde{\Omega}$ is the connected component of Ω that contains x_0.

The strong maximum principle implies $u = u(x_0)$ on $\tilde{\Omega}$. Thus follows:

$$\min_{\partial\tilde{\Omega}}(u) = u(x_0) = \max_{\partial\tilde{\Omega}}(u)$$

$$\Rightarrow \quad \min_{\partial\Omega}(u) \leq \min_{\partial\tilde{\Omega}}(u) = u(x_0) = \max_{\partial\tilde{\Omega}}(u) \leq \max_{\partial\Omega}(u)$$

Therefore x_0 cannot be an interior maximum or minimum in contradiction to our assumption. $\square_{3.3.3}$

3.3.4 Corollary (Uniqueness of Solutions of the Poisson Equation)

Let $u, v \in C^0\left(\overline{\Omega}\right) \cap C^2(\Omega)$ be solutions of the Poisson equation:

$$\Delta u = f = \Delta v$$

If moreover holds $u \leq v$ on $\partial\Omega$, then follows $u \leq v$ in Ω.
In particular, $u = v$ on $\partial\Omega$ implies $u = v$ in Ω.

Proof

The function $h := u - v$ is harmonic

$$\Delta h = \Delta u - \Delta v = f - f = 0$$

and on $\partial\Omega$ holds $h \leq 0$. The weak maximum principle gives $h \leq 0$ in Ω. $\square_{3.3.4}$

3.3.5 Corollary (Liouville's Theorem)

If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded, then u is constant.

Proof

Choose $y_1, y_2 \in \mathbb{R}^n$. Then for any $r \in \mathbb{R}_{>0}$ holds for $i \in \{1, 2\}$.

$$u(y_i) = \frac{1}{\omega_n r^n} \int_{B_r(y_i)} u(x)\,\mathrm{d}\mu(x)$$

The difference is:

$$u(y_1) - u(y_2) = \frac{1}{\omega_n r^n} \left(\int_{B_r(y_1)} u(x)\,\mathrm{d}\mu(x) - \int_{B_r(y_2)} u(x)\,\mathrm{d}\mu(x) \right)$$

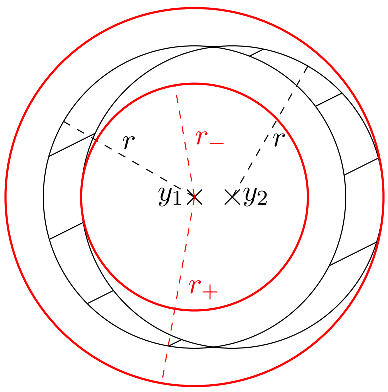


Figure 3.6: $r_- := r - \|y_1 - y_2\|$, $r_+ := r + \|y_1 - y_2\|$

So we get:

$$u(y_1) - u(y_2) \leq \frac{1}{\omega_n r^n} \underbrace{\sup_{\mathbb{R}^n}(|u|)}_{<\infty} \mu\left(B_r(y_1) \setminus B_r(y_2) \cup B_r(y_2) \setminus B_r(y_1)\right) \leq$$

$$
\begin{aligned}
&\leq \frac{1}{\omega_n r^n} \sup_{\mathbb{R}^n} (|u|) \left(\mu\left(B_{r+\|y_2-y_1\|}(y_1)\right) - \mu\left(B_{r-\|y_2-y_1\|}(y_1)\right)\right) \leq \\
&\leq \frac{1}{\omega_n r^n} \sup_{\mathbb{R}^n} (|u|) \left(\omega_n (r + \|y_2 - y_1\|)^n - \omega_n (r - \|y_2 - y_1\|)^n\right) = \\
&= \sup_{\mathbb{R}^n} (|u|) \frac{1}{r^n} \left(2n \|y_2 - y_1\| r^{n-1} + \dots\right) \xrightarrow{r\to\infty} 0
\end{aligned}
$$

This gives $u(y_1) = u(y_2)$ and thus u is constant. $\square_{3.3.5}$

3.4 The Harnack Inequality for Harmonic Functions

3.4.1 Theorem (Harnack inequality)

Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ harmonic and non-negative. Then for every compact, (for simplicity) connected subset $\Omega' \subseteq \Omega$ there is a constant $c = c(\Omega, \Omega') \in \mathbb{R}$, which is independent of u, such that holds:

$$\sup_{\Omega'} (u) \leq c \cdot \inf_{\Omega'} (u)$$

Proof

We begin with the case $\Omega' = B_r(x_0)$ and $B_{4r}(x_0) \subseteq \overline{\Omega}$. Choose $y_1, y_2 \in \Omega'$.

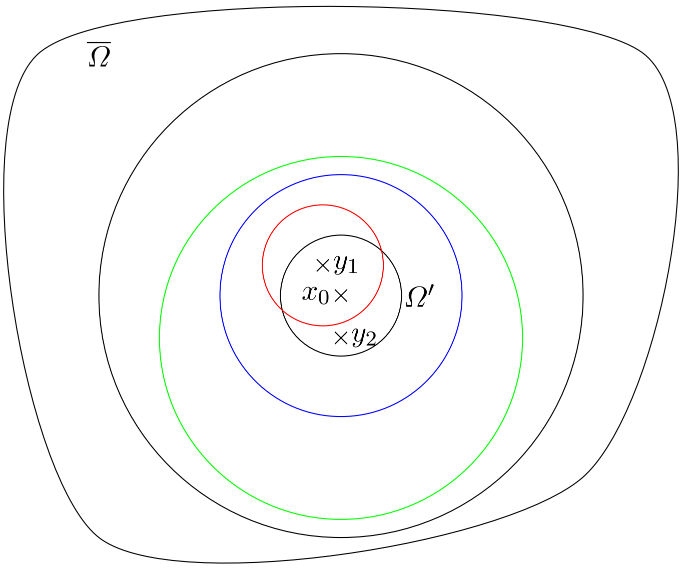


Figure 3.7: $B_r(y_1) \subseteq B_{2r}(x_0) \subseteq B_{3r}(y_2) \subseteq B_{4r}(x_0) \subseteq \overline{\Omega}$

It holds:

$$
\begin{aligned}
u(y_1) &= \frac{1}{\omega_n r^n} \int_{B_r(y_1)} u(x)\, \mathrm{d}\mu(x) \overset{u\geq 0}{\leq} \frac{1}{\omega_n r^n} \int_{B_{2r}(x_0)} u(x)\, \mathrm{d}\mu(x) \leq \\
&\leq \frac{1}{\omega_n r^n} \int_{B_{3r}(y_2)} u(x)\, \mathrm{d}\mu(x) \leq \frac{3^n}{\omega_n (3r)^n} \int_{B_{3r}(y_2)} u(x)\, \mathrm{d}\mu(x) = 3^n u(y_2)
\end{aligned}
$$

Taking the supremum over y_1 and the infimum over y_2 we get:

$$\sup_{B_r(x_0)} (u) \le 3^n \inf_{B_r(x_0)} (u)$$

Now consider a general compact subset $\Omega' \subseteq \Omega$. If Ω' is empty, we have nothing to show, so consider a non-empty Ω'. We want to find a $r \in \mathbb{R}_{>0}$ such that for every $x \in \Omega'$ holds $B_{4r}(x) \subseteq \Omega$. For $\Omega = \mathbb{R}^n$ we can choose any r. Otherwise $\partial\Omega \subseteq \mathbb{R}^n$ is non-empty and closed. Since Ω' is also non-empty and even compact, the distance $\operatorname{dist}(\Omega', \partial\Omega)$ is attained. This must be larger than zero, because from $\partial\Omega \cap \Omega = \emptyset$ and $\Omega' \subseteq \Omega$ follows $\partial\Omega \cap \Omega' = \emptyset$. So we can choose any $r \in \mathbb{R}_{>0}$ with:

$$r < \frac{1}{4} \operatorname{dist}\left(\Omega', \partial\Omega\right)$$

We cover Ω' by a finite number $\Omega_1, \ldots, \Omega_N$ of balls of radius r. On each of the sets Ω_l holds:

$$\sup(u) \le 3^n \inf(u)$$

One can proceed inductively as follows: Without loss of generality holds $\sup_{\Omega'}(u) \le \sup_{\Omega_1}(u)$, otherwise change the numbering of the Ω_i. Since Ω' is connected, one can choose a finite number $\Omega_2, \ldots, \Omega_k$ from the balls of the covering such that for $i \in \{1, \ldots, k-1\}$ holds

$$\Omega_i \cap \Omega_{i+1} \neq \emptyset$$

and $\inf_{\Omega_k}(u) \le \inf_{\Omega'}(u)$. If $\inf_{\Omega_1}(u) \le \inf_{\Omega'}(u)$, we can choose $k = 1$. It holds:

$$\sup_{\Omega_i}(u) \le 3^n \inf_{\Omega_i}(u) \overset{z_i \in \Omega_i \cap \Omega_{i+1}}{\le} 3^n u(z_i) \le 3^n \sup_{\Omega_{i+1}}(u)$$

Inductively one gets after $k-1$ steps:

$$\sup_{\Omega'}(u) \le \sup_{\Omega_1}(u) \le 3^{(k-1)n} \sup_{\Omega_k}(u) \le 3^{kn} \inf_{\Omega_k}(u) \le 3^{kn} \inf_{\Omega'}(u)$$

In the worst case of u, we have $k = N$ and therefore for any u holds:

$$\sup_{\Omega'}(u) \le 3^{Nn} \inf_{\Omega'}(u)$$

Thus the theorem is proven. $\Box_{3.4.1}$

3.4.2 Corollary (Harnack's Convergence Theorem)

Let $u_n : \Omega \to \mathbb{R}^n$ be a monotonically increasing sequence of harmonic functions. Assume that there is a $y \in \Omega$, where $u_n(y)$ is a bounded sequence.
Then the functions u_n converge locally uniformly, i.e. uniformly in a small neighborhood of y, to a harmonic function u.

Proof

Since the sequence $u_n(y)$ is bounded and monotonically increasing, it is convergent, i.e. for all $\varepsilon \in \mathbb{R}_{>0}$ there exists a $N \in \mathbb{N}$ such that for all $k > m > N$ holds:

$$u_k(y) - u_m(y) < \varepsilon$$

The function $u_k - u_m$ is non-negative and harmonic. Now choose a bounded and connected neighborhood Ω' of y with $\overline{\Omega'} \subseteq \Omega$. According to the Harnack inequality holds:

$$\sup_{\overline{\Omega'}} (u_k - u_m) \leq c\left(\Omega, \overline{\Omega'}\right) \cdot \inf_{\overline{\Omega'}} (u_k - u_m) \leq c\left(u_k(y) - u_m(y)\right) \xrightarrow{k,m\to\infty} 0$$

Hence u_k converges *uniformly* in Ω' to a function $u \in C^0(\Omega')$. The function u is harmonic, because for all $x_0 \in \Omega'$ and $r \in \mathbb{R}_{>0}$ such that $B_r(x_0) \subseteq \Omega'$ holds:

$$u_k(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u_k(x)\, \mathrm{d}\mu(x)$$

From $u_k(x_0) \to u(x_0)$ and $u_k(x) \rightrightarrows u(x)$ follows:

$$u(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u(x)\, \mathrm{d}\mu(x)$$

So u has the mean-value property and is thus harmonic. $\square_{3.4.2}$

4 Perron Method of Sub- and Supersolutions

Consider the Dirichlet problem on a bounded $\Omega \subseteq \mathbb{R}^n$ with $\varphi \in C^2(\partial\Omega)$:

$$\Delta u = 0 \qquad\qquad u\big|_{\partial\Omega} = \varphi$$

The Perron method, named after Oskar Perron (1880-1972), yields the existence of a solution if $\partial\Omega$ is sufficiently smooth. This is a classical method, in the sense that it works always with continuous functions and no weak solutions, but only classical solutions. The idea is to find the solution as a supremum of subsolutions:

$$u(x) := \sup_{v \text{ subsolution}} (v(x))$$

For any domain this gives a harmonic function, but $\partial\Omega$ has to be sufficiently smooth, for the boundary condition to be satisfied.

4.1 **Definition** (Sub- and Superharmonic Functions, Compactly Contained, Subsolution)

A function $u \in C^0(\Omega)$ is called *subharmonic* (*superharmonic*) in Ω, if for every ball $B \Subset \Omega$ (*compactly contained* in Ω, i.e. $\overline{B} \subseteq \Omega$) and every harmonic function $h \in C^0(\overline{B})$, $h\big|_B \in C^2(B)$ with $u \leq h$ (respectively $u \geq h$) in ∂B follows $u \leq h$ (respectively $u \geq h$) in all of B.

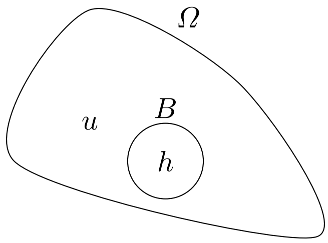


Figure 4.1: $h \in C^0(\overline{B})$

Note that every harmonic function is subharmonic and superharmonic by the weak maximum principle.
For $\varphi \in C^2(\partial\Omega)$ we consider the family S_φ of functions defined as:

$$S_\varphi := \left\{ v \in C^0(\overline{\Omega}) \,\middle|\, v \text{ subharmonic}, v\big|_{\partial\Omega} \leq \varphi\big|_{\partial\Omega} \right\}$$

Because the constant $v := \min_{\partial\Omega} \varphi$ is subharmonic and satisfies $v\big|_{\partial\Omega} \leq \varphi\big|_{\partial\Omega}$, we know $S_\varphi \neq \emptyset$. The constant is bigger than minus infinity, because Ω is bounded and thus $\partial\Omega$ is compact. A function $v \in S_\varphi$ is called *subsolution*. Now define the function u by:

$$u(x) := \sup_{v \in S_\varphi} (v(x)) \in \mathbb{R} \cup \{\infty\}$$

We hope, that u is the solution of the Dirichlet problem.

4.2 Lemma

If u_1 and u_2 are subharmonic then $\max(u_1, u_2)$ is also subharmonic.

Proof

Let $B \Subset \Omega$ and $h \in C^2(\Omega)$ be harmonic.

$$\max(u_1, u_2)\big|_{\partial B} \leq h\big|_{\partial B}$$

Then follows $u_1\big|_{\partial B} \leq h\big|_{\partial B}$ and $u_2\big|_{\partial B} \leq h\big|_{\partial B}$ in B, since u_1 and u_2 are subharmonic. Therefore holds also for the maximum in B:

$$\max(u_1, u_2)\big|_{B} \leq h\big|_{B}$$

$\square_{4.2}$

4.3 Theorem (Supremum of Subsolutions is Harmonic)

The function

$$u(x) := \sup_{v \in S_\varphi} (v(x)) \in \mathbb{R} \cup \{\infty\}$$

is a harmonic function in Ω.
After proving this theorem, we still need to verify that u satisfies the boundary condition.

4.4 Definition (Harmonic Lift)

Let $u \in C^0(\Omega)$ be subharmonic in Ω and $B \Subset \Omega$ a ball. The function $\overline{v} \in C^0(\Omega)$ defined by

$$\overline{v}\big|_{\Omega \setminus B} = v\big|_{\Omega \setminus B} \qquad \Delta \overline{v}\big|_{B} = 0$$

is referred to as the *harmonic lift* of v in B.

4.5 Lemma (Harmonic Lift Exists)

The harmonic lift exists.

Proof

Consider the Dirichlet problem on the ball B:

$$\Delta u\big|_B = 0 \qquad\qquad u\big|_{\partial B} = v\big|_{\partial B}$$

This Dirichlet problem has an explicit solution u given by the Poisson representation. Set:

$$\overline{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \setminus B \\ u(x) & \text{if } x \in B \end{cases}$$

$\square_{4.5}$

Note that $\overline{v} \geq v$ follows by the definition of subharmonic, because $u \geq v$ holds in B and thus $\overline{v} \geq v$ holds in Ω.

4.6 Lemma (Harmonic Lift is Subharmonic)

The harmonic lift is again subharmonic.

Proof

Let $\overline{v}$ be the harmonic lift of v in B.
We want to prove, that for all balls $B' \Subset \Omega$ and for all harmonic functions u with $u \geq \overline{v}$ on $\partial B'$ holds $u \geq \overline{v}$ in B'.
If $B' \cap B = \emptyset$ this follows, because $\overline{v}\big|_{\Omega \setminus B} = v$ and v is subharmonic.

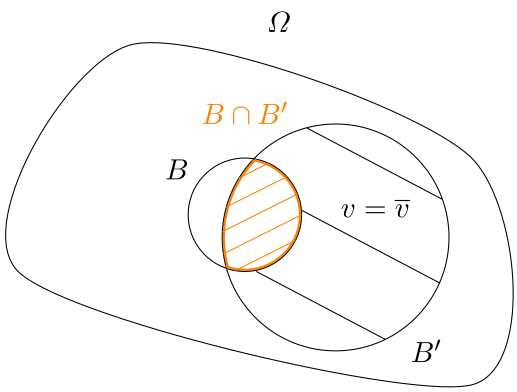


Figure 4.2: $B' \cap B \neq \emptyset$

Otherwise let $u \in C^2(B')$ be harmonic and $u \geq \overline{v}$ on $\partial B'$. We know $v \leq \overline{v}$ in Ω thus follows:

$$v\big|_{\partial B'} \leq \overline{v}\big|_{\partial B'} \leq u\big|_{\partial B'}$$
$$\overset{\text{subharmonic}}{\Rightarrow} v\big|_{B'} \leq u\big|_{B'}$$

Therefore we have $\overline{v} = v \leq u$ in $B' \setminus B$ and thus $\overline{v} \leq u$ on $\partial(B \cap B')$. Since the functions $\overline{v}$ and u are both harmonic in $B \cap B'$, the weak maximum principle implies $\overline{v} \leq u$ in $B \cap B'$. $\square_{4.6}$

4.7 Proposition (Maximum Principle for Subharmonic Functions)

Let $\Omega \subseteq \mathbb{R}^n$ be open and connected and $u \in C^0\left(\overline{\Omega}\right)$ subharmonic.

a) Strong maximum principle: If there is a $x_0 \in \Omega$ with $u(x_0) = \sup_\Omega (u)$, then u is constant.

b) Weak maximum principle: $u(x) \leq \sup\limits_{\partial\Omega}(u)$

Proof

b) follows from a) just as in the proof of Theorem 3.3.3.

To prove a), suppose that $u(x_0) = \sup_\Omega (u)$ holds, but u is *not* constant. Then there is a $y \in \Omega$ and a $r \in \mathbb{R}_{>0}$ such that $u(y) = \sup_{x\in\Omega} u(x)$, but $u\big|_{\partial B_r(y)}$ is not constant.

Let $c(\tau)$ be a curve joining $c(0) = x_0$ with $c(1) = z$.

$$\tilde{\tau} := \sup\left\{\tau \middle| u(c(\tau)) = u(x_0)\right\} < 1$$

Choose $y := c(\tilde{\tau})$ and r small enough, so that $B_r(y) \subseteq \Omega$.

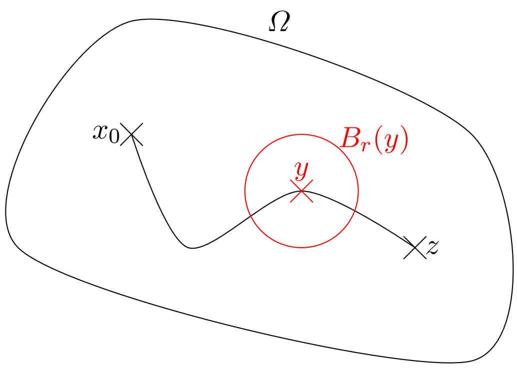


Figure 4.3: $u(z) < u(x_0) = u(y)$; $u\big|_{\partial B_r(y)}$ is not constant.

Let $\overline{u}$ be the harmonic lift of u in B_r. Then holds:

$$\overline{u}(y) \geq u(y) \geq \sup_{\partial B_r(y)}(u) = \sup_{\partial B_r(y)}(\overline{u})$$

By the weak maximum principle, which states $\overline{u}(y) \leq \sup_{\partial B_r(y)}(\overline{u})$, follows that $\overline{u}$ is constant in $B_r(y)$. This implies that $\overline{u}\big|_{\partial B_r} = u\big|_{\partial B_r}$ is constant, which is a contradiction to u not being constant on ∂B_r. $\square_{4.7}$

4.8 Proposition

If v is subharmonic and u superharmonic in Ω and $v \leq u$ holds on $\partial\Omega$, then in Ω holds either $v < u$ or $v = u$, in which case u and v are harmonic.

Proof

This is proven in the exercises. $\square_{4.8}$

Proof of Theorem 4.3

Define $u(x) := \sup_{v \in S_\varphi}(v(x))$. According to Proposition 4.8 we know, since the constant function $\sup_{\partial\Omega}(\varphi)$ is superharmonic:

$$\begin{aligned} & v(x) \le \sup_{\partial\Omega}(\varphi) \qquad \forall_{v \in S_\varphi} \\ \Rightarrow \qquad & u(x) \le \sup_{\partial\Omega}(\varphi) \end{aligned}$$

From $\inf_{\partial\Omega}(\varphi) \in S_\varphi$ and because u is the supremum of the functions in S_φ, we also know:

$$\inf_{\partial\Omega}(\varphi) \le u(x)$$

Hence u is a bounded function on Ω. Consider $y \in \Omega$. By definition of the supremum, there is a sequence $v_n \in S_\varphi$ such that $v_n(y) \to u(y)$ converges. We can assume that the functions v_n are bounded, because $v_n \le u$ bounds from above and if v_n is not bounded from below, we can replace it by $\max(v_n, \inf_{\partial\Omega}(\varphi))$. This is again a subsolution.
Choose $r \in \mathbb{R}_{>0}$ such that $B_r(y) \Subset \Omega$ and let $\overline{v}_n$ be the harmonic lift of v_n in $B_r(y)$. Then $v_n \le \overline{v}_n$ holds in Ω and $\overline{v}_n(y) \le u(y)$, since $\overline{v}_n$ is again a subsolution. Hence follows:

$$\underbrace{v_n(y)}_{\to u(y)} \le \overline{v}_n(y) \le u(y)$$

Thus we have $\overline{v}_n(y) \to u(y)$. Possibly after choosing a subsequence, the sequence $\overline{v}_n(y)$ is monotonically increasing. Moreover, we can assume that the sequence v_n is monotonically increasing in Ω. Namely, we can replace v_n by $\tilde{v}_n$ defined by:

$$\begin{aligned} &\tilde{v}_1 = v_1 \\ &\tilde{v}_2 := \max(v_1, v_2) \\ &\vdots \\ &\tilde{v}_n := \max(v_1, \ldots, v_n) \end{aligned}$$

These are again subharmonic and they are monotonically increasing. Then the $\overline{v}_n$ are also monotonically increasing in $B_r(y)$ using the maximum principle:

$$(\overline{v}_{n+1} - \overline{v}_n)\big|_{\partial B_r(y)} = (v_{n+1} - v_n)\big|_{\partial B_r(y)} \ge 0$$

The weak maximum principle implies $\overline{v}_{n+1} - \overline{v}_n \ge 0$ in $B_r(y)$.
Applying Harnack's convergence theorem, we conclude

$$\overline{v}_n \rightrightarrows \tilde{v} \text{ in } B_r(y)$$

and $\overline{v}$ is harmonic.
Now we show $u = \tilde{v}$ in $B_r(y)$. Obviously holds $\tilde{v} \le u$, because $\overline{v}_n \le u$ holds for all $n \in \mathbb{N}$.
Thus assume that there is a point $x \in B_r(y)$ with $\tilde{v}(x) < u(x)$. Then there exists a subsolution $s \in S_\varphi$ with $\tilde{v}(x) < s(x) < u(x)$, because $u(x) = \sup_{v \in S_\varphi}(v(x))$. Thus we also have:

$$v_n(x) \le \tilde{v}(x) < s(x)$$

Now form the sequence:

$$w_n := \max(s, v_n)$$

These are again subharmonic, from $v_n(x) < s(x)$ follows $w_n(x) = s(x)$, and for the harmonic lifts holds $\overline{w}_n \geq \overline{v}_n$. Then repeating the above construction for v_n replaced by w_n, we find $\overline{w}_n \rightrightarrows \tilde{w}$ converges uniformly to a harmonic $\tilde{w}$. Then follows $\tilde{v} \leq \tilde{w} \leq u$ just as before and thus $\tilde{v} - \tilde{w} \leq 0$ in B. Additionally $\tilde{v}$ and $\tilde{w}$ are harmonic and we know:

$$\tilde{v}(y) = \tilde{w}(y) = u(y)$$

So $\tilde{v} - \tilde{w}$ attains its maximum in $y \in B$ and the strong maximum principle gives $\tilde{v} = \tilde{w}$ in B. From $w_n(x) > s(x)$ follows $\tilde{w}(x) \geq s(x) > \tilde{v}(x)$. This is a contradiction. $\square_{4.3}$

Question: Is u also a solution of the Dirichlet problem?

4.9 Proposition (Existing Solution is Perron Solution)

If the Dirichlet problem has a solution, then

$$u(x) = \sup_{v \in S_\varphi} (v(x))$$

equals this solution.

Proof

Let $\tilde{u}$ be a solution of the Dirichlet problem. Then $\tilde{u}$ is also a subsolution, i.e. $\tilde{u} \in S_\varphi$. Therefore holds $u(x) \geq \tilde{u}(x)$ for all $x \in \Omega$. Hence $u - \tilde{u}$ is harmonic and $(u - \tilde{u})\big|_{\partial\Omega} = 0$. By the weak maximum principle follows $u = \tilde{u}$ in Ω. $\square_{4.9}$

4.10 Definition (Barrier, Local Barrier)

A function $w \in C^0\left(\overline{\Omega}\right)$ is called *barrier* at $\xi \in \partial\Omega$, if holds:

i) w is superharmonic.

ii) In $\overline{\Omega} \setminus \{\xi\}$ holds $w > 0$ and $w(\xi) = 0$.

We show first that the barrier is a local concept.

A *local barrier* in $\xi \in \partial\Omega$ is a function $w \in C^0\left(\overline{\Omega} \cap B_\varepsilon(\xi)\right)$ for $\varepsilon \in \mathbb{R}_{>0}$ with:

i) w is superharmonic in $\Omega \cap B_\varepsilon(\xi)$.

ii) In $\overline{\Omega \cap B_\varepsilon(\xi)} \setminus \{\xi\}$ holds $w > 0$ and $w(\xi) = 0$.

4.11 Theorem (Barrier is Local Concept)

If there is a local barrier, there is also a barrier.

Proof

Let $w \in C^0\left(\overline{\Omega \cap B_\varepsilon(\xi)}\right)$ be a local barrier. With

$$m := \inf_{B_\varepsilon(\xi) \setminus B_{\frac{\varepsilon}{2}}(\xi)} (w) > 0$$

we choose:

$$\overline{w}(x) := \begin{cases} \min(w(x), m) & x \in \overline{\Omega \cap B_{\frac{\varepsilon}{2}}(\xi)} \\ m & x \notin \overline{\Omega \cap B_{\frac{\varepsilon}{2}}(\xi)} \end{cases}$$

This is a barrier. $\square_{4.11}$

4.12 Definition (regular point)

A border-point $\xi \in \partial\Omega$ is called *regular* if there is a barrier at ξ.

4.13 Theorem (Solution of Dirichlet Problem with Regular Border)

If ξ is a regular border-point, then holds for the function $u(x) := \sup_{v \in S_\varphi}(v(x))$:

$$\lim_{\Omega \ni x \to \xi} u(x) = \varphi(\xi)$$

So if every border point is regular, the Perron solution is the solution of the Dirichlet problem.

Proof

We construct a supersolution. Because ξ is a regular border-point, there is a barrier w at ξ. Consider $\varepsilon \in \mathbb{R}_{>0}$ and define:

$$M := \sup_{\partial\Omega}(|\varphi|)$$

Now choose $\delta, k \in \mathbb{R}_{>0}$ such that holds:

$$|\varphi(x) - \varphi(\xi)| < \varepsilon \qquad \underset{x \in B_\delta(\xi) \cap \partial\Omega}{\forall}$$

$$kw(x) \geq 2M \qquad \underset{x \in \partial\Omega \setminus B_\delta(\xi)}{\forall}$$

Then

$$u_+(x) := \varphi(\xi) + \varepsilon + kw(x)$$

is a supersolution and

$$u_-(x) := \varphi(\xi) - \varepsilon - kw(x)$$

a subsolution. (These are sub-/superharmonic, because $w(x)$ is superharmonic and thus $-w(x)$ subharmonic and constants are harmonic.)

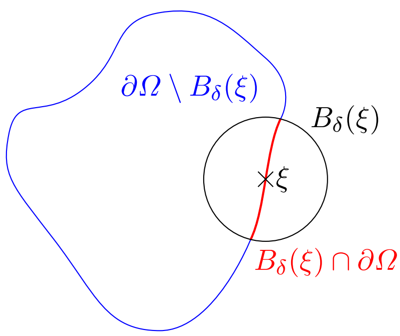


Figure 4.4: Construction of u_+ and u_-

The ε ensures $u_-(x) \leq \varphi(x) \leq u_+(x)$ for $x \in B_\delta(\xi) \cap \partial\Omega$ and for $x \in \partial\Omega \setminus B_\delta(\xi)$ this follows from $kw \geq 2M$. Now the maximum principle implies:

$$\varphi(\xi) - \varepsilon - kw(x) \leq u(x) \leq \varphi(\xi) + \varepsilon + kw(x)$$

Due to $\lim_{x\to\xi}(w(x)) = 0$ and because ε can be chosen arbitrarily, follows:

$$\lim_{x\to\xi} u(x) = \varphi(\xi)$$

$\square_{4.13}$

4.14 Theorem (Existence of Solution for Dirichlet Problem)

The following statements are equivalent:

1. The Dirichlet problem has for every $\varphi \in C^0(\partial\Omega)$ a solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$.
2. Every border-point of Ω is regular.

Proof

"2. $\Rightarrow$ 1.": If every border-point is regular, then the Perron solution $u(x)$ is the solution of the Dirichlet problem after Theorem 4.13.

"1. $\Rightarrow$ 2.": For $\xi \in \partial\Omega$ let u be the solution of the following Dirichlet problem:

$$\Delta u\big|_{\Omega} = 0$$
$$u(x)\big|_{\partial\Omega} = \|x - \xi\|\big|_{\partial\Omega}$$

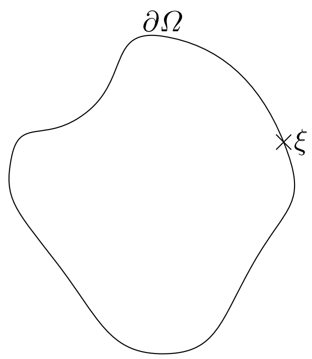


Figure 4.5: $u(\xi) = 0$, $u\big|_{\partial\Omega\setminus\{\xi\}} > 0$

Were $u(y) = 0$ for a $y \in \Omega$, we had an interior minimum, and thus the maximum principle would give the contradiction $u = 0$. So this u is a barrier. $\square_{4.14}$

How can one construct barriers? Under what geometrical conditions do barriers exist?

Consider the dimension $n = 2$. Without loss of generality we assume $\xi = 0$. We consider this as a problem in $\mathbb{C} \cong \mathbb{R}^2$ and define:

$$w(z) = -\mathrm{Re}\left(\frac{1}{\ln(z)}\right)$$

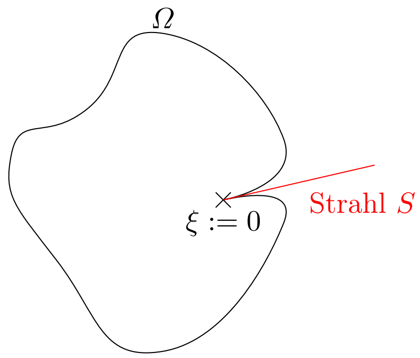


Figure 4.6: Ω under consideration

Choose the branch cut of the logarithm on S. Such an S with $\Omega \cap S = \emptyset$ and $\xi \in S$ can always be chosen locally. Thus $\ln(z)$ is holomorphic in $\mathbb{C} \setminus S$ and therefore $(\ln(z))^{-1}$ is holomorphic on $\mathbb{C} \setminus S$, which means that w is harmonic on $\mathbb{C} \setminus S$ and especially continuous on $\mathbb{C} \setminus S$.

To study the behaviour at the origin, we choose the polar representation:

$$z = re^{\mathbf{i}\varphi}$$
$$\ln(z) = \ln(r) + \mathbf{i}\varphi + 2\pi\mathbf{i}n$$

Here $n \in \mathbb{Z}$ depends on the branch of the logarithm.

$$-\frac{1}{\ln(z)} = -\frac{1}{\ln(r) + \mathbf{i}\varphi + 2\pi\mathbf{i}n}$$
$$-\mathrm{Re}\left(\frac{1}{\ln(z)}\right) = -\frac{\ln(r)}{(\ln(r))^2 + (\varphi + 2\pi n)^2} \xrightarrow{r \to 0} 0$$

This converges locally uniformly in φ.

Alternatively, this follows from the Riemann mapping theorem.

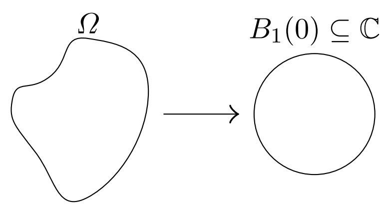


Figure 4.7: Mapping from Ω to $B_1(0)$

Just transform from any simply connected Ω to $B_1(0) \subseteq \mathbb{C}$, solve the Dirichlet problem in the ball and transform back.

Consider now the case $n \geq 3$: One gets barriers if additional conditions are satisfied.

4.15 Definition (Exterior Sphere/Cone Condition)

An open subset $\Omega \subseteq \mathbb{R}^n$ satisfies in $\xi \in \partial\Omega$ the *exterior sphere condition* if there is a closed ball $K \subseteq \mathbb{R}^n$ with radius R such that $K \cap \overline{\Omega} = \{\xi\}$.

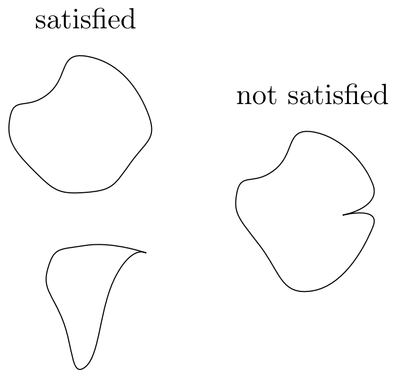


Figure 4.8: Exterior Sphere Condition

Now choose:

$$w(x) = -\mathit{\Gamma}(\|x - y\|) + \mathit{\Gamma}(\|\xi - y\|)$$

This function is harmonic and non-negative. Furthermore $w(x) = 0$ implies $x = \xi$. So w is a barrier and we can solve the Dirichlet problem.

An improvement is the so called *exterior cone condition*.

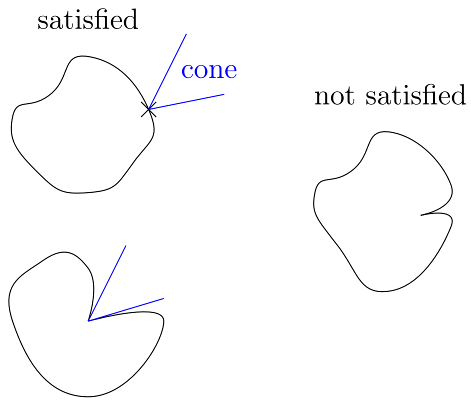


Figure 4.9: Exterior Cone Condition

Here one can also construct a barrier.

4.16 Example (Dirichlet Problem without Solution, Lebesgue Spine)

A counter example for the solvability of the Dirichlet problem with continuous boundary values was found in 1912 by Lebesgue. The details can be found in the book of Courant and Hilbert.

We use the Ansatz:

$$u(x) = \int \mathit{\Gamma}(\|x - y\|)\, \varrho(y)\, \mathrm{d}^n y$$

Here $\varrho(y)$ can be viewed as "charge density". Then follows:

$$\Delta u(x) = \varrho(x)$$

$$\Delta u(x) = 0 \qquad \text{if } \varrho(x) = 0$$

Consider the charge distribution on the x-axis:

$$\varrho(x,y,z) = \begin{cases} \delta(y)\,\delta(z)\,(-4\pi x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

This is the so-called Lebesgue spine in $\mathbb{R}^3$.

$$\begin{aligned} u(x,y,z) &= \int_0^1 \Gamma\left(\|(\xi,0,0)-(x,y,z)\|\right)(-4\pi\xi)\,\mathrm{d}\xi = \\ &= \int_0^1 \frac{\xi \mathrm{d}\xi}{\sqrt{(\xi-x)^2+y^2+z^2}}\mathrm{d}\xi \end{aligned}$$

Define $\zeta := \sqrt{y^2+z^2}$. The integral gives:

$$u(x,\zeta) = A(x,\zeta) - 2x\ln(\zeta)$$

$$A(x,\zeta) = \sqrt{(1-x)^2+\zeta^2} - \sqrt{x^2+\zeta^2} + x\ln\left(\left|\left(1+x+\sqrt{(1-x)^2+\zeta^2}\right)\left(x+\sqrt{x^2+\zeta^2}\right)\right|\right)$$

A is continuous at the origin:

$$\begin{aligned} 1+x+\sqrt{(1-x)^2+\zeta^2} &\to 2 \\ \lim_{(x,\zeta)\to 0}\left(x+\sqrt{x^2+\zeta^2}\right) &= \lim_{x\to 0}(2x) \\ \lim_{x\to 0} x\ln(x) &= 0 \\ \Rightarrow \qquad \lim_{(x,\zeta)\to 0} A(x,\zeta) &= 1 \end{aligned}$$

But $x\ln(\zeta)$ is not continuous at the origin. Consider for example:

$$\begin{aligned} \zeta &= |x|^\alpha & \qquad x\ln(\zeta) &= x\alpha\ln(|x|) \xrightarrow{x\to 0} 0 \\ \zeta &= e^{-\frac{c}{x}} & \qquad x\ln(\zeta) &= x\frac{-c}{x} \xrightarrow{x\to 0} -c \end{aligned}$$

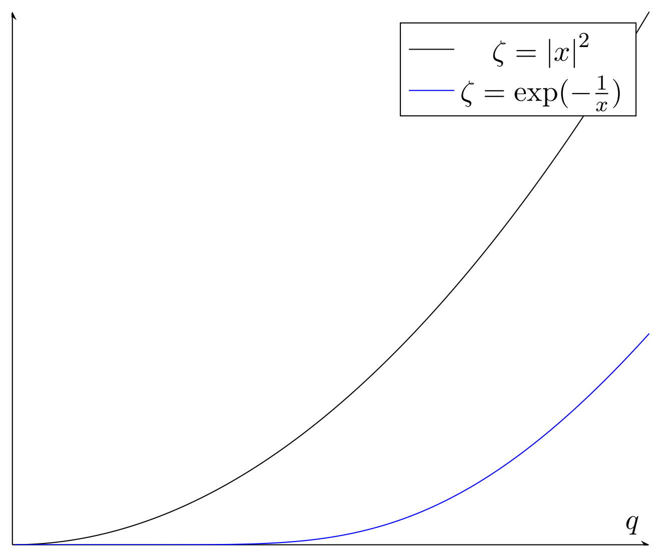


Figure 4.10: $\lim_{(x,\zeta)\to 0} A(x,\zeta)$ has different values for different path.

So the function $-2x\ln(\zeta)$ is constant along the curve $\left\{\zeta = e^{-\frac{c}{2x}} \middle| x > 0\right\}$, but is not continuous at the origin. Consider the level set $u(x,\zeta) = 1 + C$ for $C \in \mathbb{R}_{>0}$.

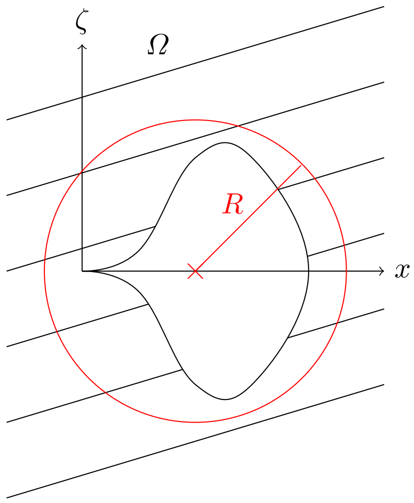


Figure 4.11: The level set $u(x,\zeta) = 1 + C$ defines the region Ω.

We have:

$$\Delta u\big|_{\Omega} = 0$$
$$u\big|_{\partial\Omega} = 1 + C$$

But u is *not* continuous at $(x,y,z) = 0$. Let B be a ball such that $\partial B \cap \partial\Omega = \emptyset$. Now perform an inversion:

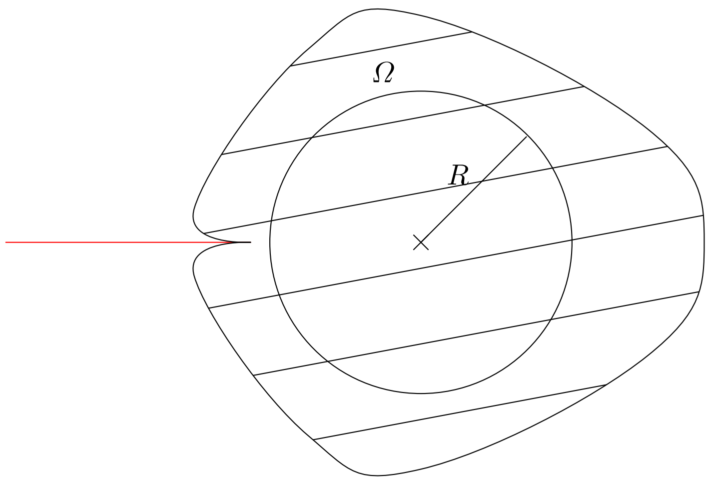


Figure 4.12: Ω inverted

For formulas see the book of Courant and Hilbert.

$$\Delta u\big|_{\Omega} = 0$$
$$u\big|_{\partial\Omega} = 1 + C$$

u is not continous in $\overline{\Omega}$.

5 The Maximum Principle for Linear Elliptic Equations

Again $\Omega \subseteq \mathbb{R}^n$ is an open domain and $u \in C^2(\Omega, \mathbb{R})$ a scalar function. Let L be a differential operator of second order, linear and elliptic.

$$Lu(x) = \sum_{i,j=1}^{n} a^{ij}(x)\,\partial_{ij}u(x) + \sum_{i=1}^{n} b^i(x)\,\partial_i u(x) + c(x)\,u(x)$$

Usually we use the summation convention to omit the sums. We make the following assumpions:

i) *Uniform ellipticity*: There exists a $\lambda \in \mathbb{R}_{>0}$ such that for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$ holds:

$$a^{ij}(x)\,\xi_i\xi_j \geq \lambda \cdot |\xi|^2$$

ii) The coefficients are *uniformly bounded*: There exists a $K \in \mathbb{R}_{>0}$ such that for all $x \in \Omega$ and $i, j \in \{1, \ldots, n\}$ holds:

$$\left|a^{ij}(x)\right|, \left|b^i(x)\right|, |c(x)| \leq K$$

Consider solutions of the homogeneous equation $Lu = 0$. Our goal is to derive a maximum principle.

5.1 Example (One-Dimensional)

We begin with a one-dimensional example.

$$u''(x) + cu(x) = 0$$

a) $c < 0$: There are two fundamental solutions, i.e. every solution is a linear combination:

$$u_1(x) = e^{\sqrt{|c|}x}$$
$$u_2(x) = e^{-\sqrt{|c|}x}$$

Consider $\Omega = (0,1)$ and impose Dirichlet boundary conditions $u(0) = 0 = u(1)$.

$$u = \alpha u_1 + \beta u_2$$

$$u(0) = 0 \quad \Rightarrow \quad \alpha = -\beta \quad \Rightarrow \quad u = \alpha\left(e^{\sqrt{|c|}x} - e^{-\sqrt{|c|}x}\right)$$
$$u(1) = 0 \quad \Rightarrow \quad \alpha\left(e^{\sqrt{|c|}} - e^{-\sqrt{|c|}}\right) = 0 \quad \Rightarrow \quad \alpha = 0$$

Thus the Dirichlet problem has a unique solution $u = 0$, in accordance with the maximum principle.

b) $c > 0$: The functions

$$u_1(x) = \sin\left(\sqrt{c}x\right)$$
$$u_2(x) = \cos\left(\sqrt{c}x\right)$$

are fundamental solutions.

$$u = \alpha u_1 + \beta u_2$$
$$u(0) = 0 \qquad \Rightarrow \qquad \beta = 0$$
$$u(1) = 0 = \alpha \sin\left(\sqrt{c}\right)$$

In the case of $\sin\left(\sqrt{c}\right) = 0$, the parameter α is arbitrary, otherwise follows $\alpha = 0$ and thus $u = 0$.
For example for $c = \pi^2$ with $\alpha = 1$ we have:

$$u(x) = \sin(\pi x)$$

This is a non-trivial solution of the Dirichlet problem and it has an interior maximum. So the maximum principle is violated.

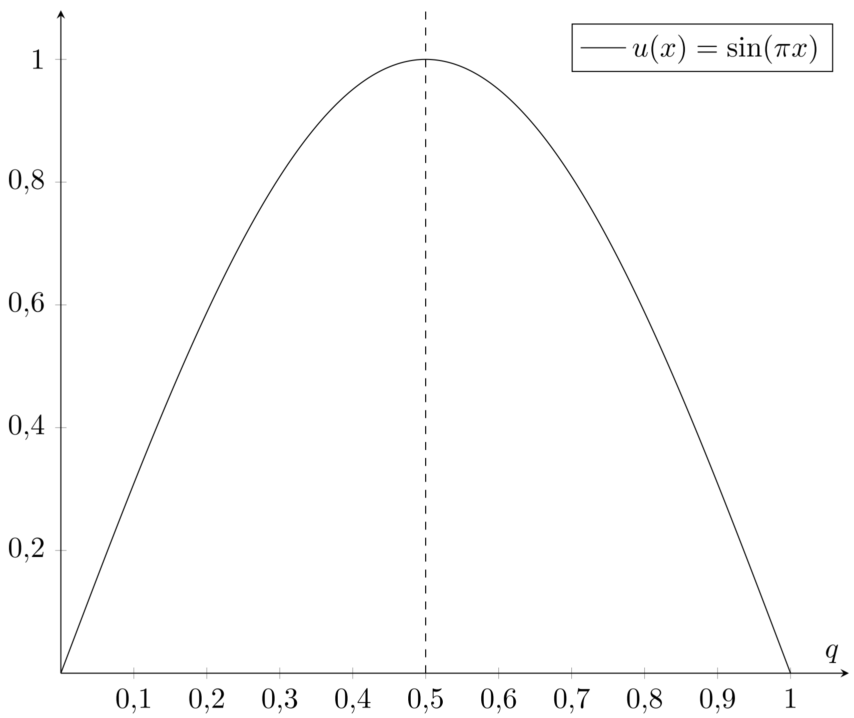


Figure 5.1: $u(x)$ has an interior maximum.

5.2 Theorem (Weak Maximum Principle)

Let $\Omega \subseteq \mathbb{R}^n$ be bounded and $c(x) = 0$. Assume $Lu \geq 0$ (or $Lu \leq 0$) in all of Ω and $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$. Then follows:

$$\sup_{\Omega}(u) = \sup_{\partial\Omega}(u) \qquad \left(\inf_{\Omega}(u) = \inf_{\partial\Omega}(u)\right)$$

Proof

Only consider the case $Lu \geq 0$, since for $Lu \leq 0$ holds $L(-u) \geq 0$.

a) Assume $Lu > 0$ and $\sup_{\Omega}(u) > \sup_{\partial\Omega}(u)$. Then the function u has a maximum at $x_0 \in \Omega$. (Here we use that Ω is bounded and thus $\overline{\Omega}$ is compact.) This implies

$$\partial_i u(x_0) = 0$$

and that $\partial_{ij} u(x_0)$ is negative semi-definite.

$$Lu(x_0) = \underbrace{a^{ij}(x_0)}_{\text{pos. def.}} \underbrace{\partial_{ij} u(x_0)}_{\text{neg. sem.-def.}} \leq 0$$

This is a contradiction to $Lu > 0$.

b) Assume only $Lu \geq 0$. The idea is to modify u such that a) applies.

$$Le^{\gamma x_1} = a^{11}\partial_{x_1}\partial_{x_1}e^{\gamma x_1} + b^1\partial_{x_1}e^{\gamma x_1} = \left(\gamma^2 a^{11} + \gamma b^1\right)e^{\gamma x_1}$$

We know $a^{11}(x) > \lambda$ and $\left|b^1\right| < K$. For $\gamma > \frac{K}{\lambda}$ we have:

$$Le^{\gamma x_1} > 0$$

Now consider for $\varepsilon \in \mathbb{R}_{>0}$:

$$L\left(u + \varepsilon e^{\gamma x_1}\right) = Lu + \varepsilon Le^{\gamma x_1} > 0$$

Thus one can apply a) to conclude:

$$\sup_{\Omega}\left(u + \varepsilon e^{\gamma x_1}\right) = \sup_{\partial\Omega}\left(u + \varepsilon e^{\gamma x_1}\right)$$

Since this holds for all $\varepsilon \in \mathbb{R}_{>0}$, we get also:

$$\sup_{\Omega}(u) = \sup_{\partial\Omega}(u)$$

$\square_{5.2}$

5.3 Corollary

For $c(x) \leq 0$ and $Lu \geq 0$ (or $Lu \leq 0$) holds with $u^+ := \max(u, 0)$ (and $u^- := \min(u, 0)$):

$$\sup_{\Omega}(u) \leq \sup_{\partial\Omega}\left(u^+\right) \qquad \left(\inf_{\Omega}(u) \geq \inf_{\partial\Omega}\left(u^-\right)\right)$$

Proof

We show this only for $Lu \geq 0$, since for $Lu \leq 0$ one can consider $-u$. L without the zero order term is:

$$L_0 := a^{ij}(x)\,\partial_{ij} + b^i(x)\,\partial_i$$

Now consider the domain:

$$\Omega^+ := \left\{x \in \Omega \middle| u(x) > 0\right\}$$

Ω^+ is again an open, bounded domain and in Ω^+ holds:

$$L_0 u \geq L_0 u + \underbrace{cu}_{\leq 0} = Lu \geq 0$$

Then 5.2 yields:

$$\sup_{\Omega^+}(u) = \sup_{\partial\Omega^+}(u) \leq \sup_{\partial\Omega}\left(u^+\right)$$

If in Ω holds $u \leq 0$, there is nothing to prove. Otherwise holds:

$$\sup_{\Omega}(u) = \sup_{\Omega^+}(u) = \sup_{\partial\Omega^+}(u) \leq \sup_{\partial\Omega}\left(u^+\right)$$

$\square_{5.3}$

5.4 Theorem

Let $c \in \mathbb{R}_{\leq 0}$ and $u, v \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ satisfy the inequalities $Lu \geq Lv$ in Ω and $u \leq v$ on $\partial\Omega$. Then follows $u \leq v$ in Ω.

Proof

$$\begin{aligned} L(u-v)\big|_{\Omega} &\geq 0 \\ (u-v)\big|_{\partial\Omega} &\leq 0 \end{aligned}$$

Apply Corollary 5.3 to obtain:

$$\sup_{\Omega}(u-v) \leq 0 \qquad \Rightarrow \qquad u\big|_{\Omega} \leq v\big|_{\Omega}$$

$\square_{5.4}$

Now we want to show the strong maximum principle, i.e. $u(x) = \max_{\Omega}(u)$ for $x \in \Omega$ implies that u is constant.

5.5 Definition (Interior Sphere Condition)

A boundary point $x_0 \in \partial\Omega$ satisfies the *interior sphere condition* if there is a ball $B \subseteq \Omega$ such that $x_0 = \partial B$.

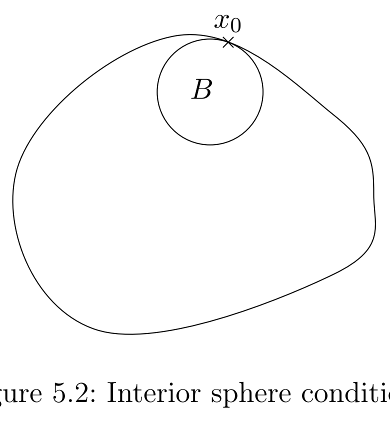


Figure 5.2: Interior sphere condition

5.6 Lemma

Assume $c = 0$ and $Lu \geq 0$ in Ω. Moreover, at the boundary point $x_0 \in \partial\Omega$ the following conditions should hold:

i) u is continuous at x_0.

ii) $u(x_0) > u(x)$ for all $x \in \Omega$.

iii) The interior sphere condition is satisfied.

Then the normal derivative of u at x_0, if it exists, satisfies the inequality:

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

We take the *outer* normal.

Note: The statement $\frac{\partial u}{\partial \nu}(x_0) \geq 0$ is obvious. The point is to prove the *strict* inequality.

Proof

Let $B = B_R(y)$ be the sphere of the interior sphere condition.

$$v(x) := e^{-\alpha r^2} - e^{-\alpha R^2} \qquad r := \|x - y\| \qquad r^2 = (x-y)_i (x-y)^i$$

$$\partial_i r^2 = 2(x-y)_i$$
$$\partial_{ij} r^2 = 2\delta_{ij}$$

$$\partial_i e^{-\alpha r^2} = e^{-\alpha r^2}(-2\alpha)(x-y)_i$$
$$\partial_{ij} e^{-\alpha r^2} = e^{-\alpha r^2} 4\alpha^2 (x-y)_i (x-y)_j + e^{-\alpha r^2}(-2\alpha)\delta_{ij}$$

$$Lv = e^{-\alpha r^2}\Big(4\alpha^2 \underbrace{a^{ij}(x)(x-y)_i(x-y)_j}_{\leq \lambda \|x-y\|^2} - 2\alpha \underbrace{\left(a^{ij}\delta_{ij} + b^i (x-y)_i\right)}_{|\ldots| \leq n^2K + nKR}\Big)$$

For $\varrho < R$ consider $B_R(y) \setminus B_\varrho(y)$.

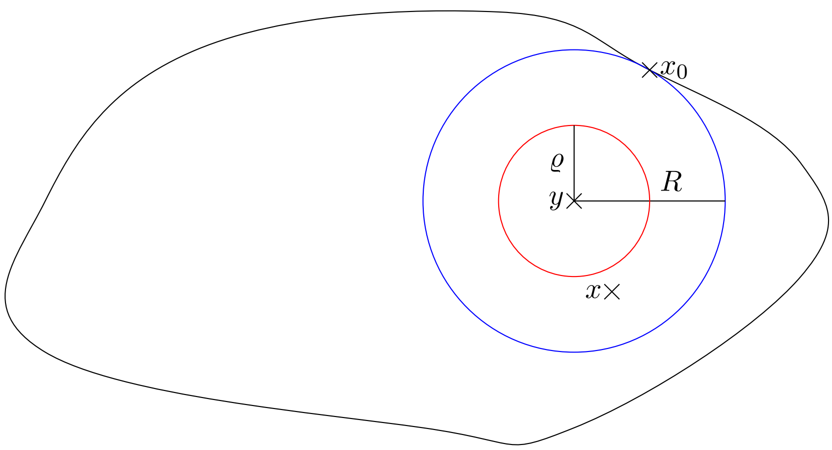


Figure 5.3: $Lv \geq 0$ in $B_R(y) \setminus B_\varrho(y)$

By choosing α sufficiently large, we can arrange that $Lv \geq 0$ in $B_R(y) \setminus B_\varrho(y)$. It holds $u(x) - u(x_0) < 0$ on $\partial B_\varrho(y)$.

On $\partial B_R(y)$ holds $v = 0$ and $u(x) - u(x_0) \leq 0$. Thus follows

$$u - u(x_0) + \varepsilon v \leq 0$$

on $\partial(B_R(y) \setminus B_\varrho(y))$ for small enough $\varepsilon \in \mathbb{R}_{>0}$ and:

$$L(u - u(x_0) + \varepsilon v) \geq 0$$

The weak maximum principle implies

$$u - u(x_0) + \varepsilon v \leq 0$$

in $B_R(y) \setminus B_\varrho(y)$. Now we compute the normal derivative.

$$0 \leq \frac{\partial}{\partial \nu}(u - u(x_0) + \varepsilon v)\big|_{x_0} = \frac{\partial u}{\partial \nu}\bigg|_{x_0} + \varepsilon \frac{\partial v}{\partial \nu}$$

$$\frac{\partial v}{\partial \nu} = v'(r) = -2\alpha r e^{-\alpha r^2}\big|_{r=R} < 0$$

Therefore follows:

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

$\square_{5.6}$

5.7 Theorem (Hopf's Maximum Principle)

Assume that L is uniformly elliptic with uniformly bounded coefficients and $c = 0$. Furthermore assume $Lu \geq 0$ (or $Lu \leq 0$) in Ω, but not necessarily bounded. If u attains a maximum (respectively minimum) in Ω, then u is constant.

For $c \leq 0$, the function u cannot attain a non-negative maximum (respectively a non-positively minimum) in Ω, unless u is constant.

This theorem can be viewed as a generalization of Liouville's theorem to $\mathbb{R}^n$.

There were two important mathematicians Hopf:

- Heinz Hopf (1894-1971): Hopf fibration, Hopf algebra, ...
- Ebehard Hopf (1902-1983): Hopf's maximum principle is named after him.
  1936 he was Professor in Leipzig, in 1944 he went to Munich as a successor of Caratheodory. After the war, he went to Bloomington in the USA.

Proof

Assume that $Lu \geq 0$ and that u attains a maximum M in Ω, but u is not constant. Define:

$$\Omega^- := \left\{x \in \Omega \middle| u(x) < M\right\} \neq \emptyset$$

Since $\Omega^- \subsetneq \Omega$ is a proper subset, $\partial\Omega^- \cap \Omega$ is not empty. For $y \in \Omega^-$ let $B_R(y)$ be the largest ball, which does not intersect $\Omega \setminus \Omega^-$. By choosing a $x_0 \in \Omega^-$ sufficiently close to $\partial\Omega^- \cap \Omega$, one can arrange $B := B_R(x_0) \subseteq \Omega$.

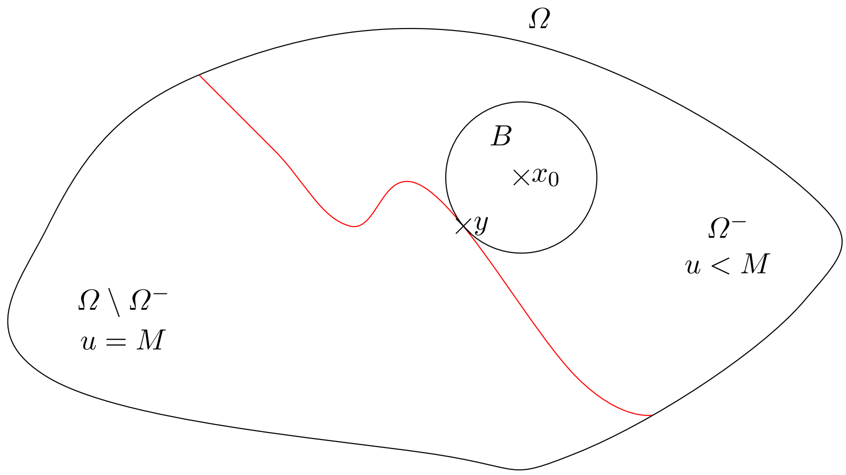


Figure 5.4: $B_R(x_0) \subseteq \Omega$

It holds $\overline{B} \cap \partial\Omega^- \neq \emptyset$, since otherwise the radius would not be maximal. Choose $y \in \overline{B} \cap \partial\Omega^-$, i.e. $u(y) = M$. We want to apply the previous Lemma to Ω^- and the boundary point y. We check the conditions:

1. u is continuous in y, because we have $u \in C^2(\Omega)$.
2. We have $u(y) = M$, but $u < M$ in Ω^-.
3. The interior sphere condition holds by construction.

So we get:

$$\left.\frac{\partial u}{\partial \nu}\right|_y > 0$$

On the other hand, $\left.\mathrm{D}u\right|_y$ vanishes, because u has a maximum at y. This is a contradiction. $\square_{5.7}$

6 Weak Solutions, Sobolev Spaces

We can write the elliptic operator

$$L = a^{ij}\partial_{ij} + b^i\partial_i + c$$

in divergence form:

$$\begin{aligned}0 = Lu &= a^{ij}\partial_{ij}u + b^i\partial_i u + cu = \\ &= \partial_i\left(a^{ij}\partial_j u\right) + \underbrace{\left(-\partial_i a^{ij} + b^j\right)}_{=:\tilde{b}^j}\partial_j u + cu = \\ &= \partial_i\left(a^{ij}\partial_j u\right) + \tilde{b}^j\partial_j u + cu\end{aligned}$$

We consider $u \in C^2(\Omega)$. Let $\eta \in C_0^\infty(\Omega)$ be a "test function". We evaluate weakly:

$$\int_\Omega (Lu)\,\eta \mathrm{d}^n x = \int_\Omega \left(\partial_i\left(a^{ij}\partial_j u\right) + \tilde{b}^i\partial_i u + cu\right)\eta \mathrm{d}^n x$$

Now we apply Gauß' divergence theorem. The boundary terms vanish due to $\eta\big|_{\partial\Omega} = 0$.

$$\int_\Omega (Lu)\,\eta \mathrm{d}^n x = \int_\Omega \left(-a^{ij}\left(\partial_j u\right)\left(\partial_i \eta\right) + \left(\tilde{b}^i\partial_i u\right)\eta + cu\eta\right)\mathrm{d}^n x$$

Now $Lu = 0$ is equivalent to $\int_\Omega (Lu)\,\eta \mathrm{d}^n x = 0$, since $(Lu)\,\eta$ is continuous for any $\eta \in C_0^\infty(\Omega)$.

$$0 = \int_\Omega \left(-a^{ij}\left(\partial_j u\right)\left(\partial_i \eta\right) + \left(\tilde{b}^i\partial_i u\right)\eta + cu\eta\right)\mathrm{d}^n x$$

This works just as well for $u \in C^1(\Omega)$. One can do even better, by only demanding $\partial_i u \in L^1(\Omega)$. The partial derivative is defined weakly. For $u \in C^1(\Omega)$ assume:

$$\partial_i u = v$$

Equivalently holds:

$$\int_\Omega v\eta \mathrm{d}^n x = \int_\Omega \left(\partial_i u\right)\eta \mathrm{d}^n x = -\int_\Omega u\left(\partial_i \eta\right)\mathrm{d}^n x \qquad \forall_{\eta \in C_0^\infty(\Omega)}$$

The last condition can be stated for $u \in L^2(\Omega)$.

What are the resulting function spaces?

These are Sobolov space spaces $H^{n,2}(\Omega) = W^{n,2}(\Omega)$ (Hilbert spaces, in this case $H^{1,2}(\Omega)$) or more general the Sobolov spaces of n-times weakly differentiable functions $W^{n,p}(\Omega)$ with $\mathrm{D}^\beta f \in L^p$ for all multi-indices β with $|\beta| \leq n$ (Banach spaces).

Writing the equation in divergence form is also of advantage for using *variational methods*. For simplicity for $\Delta u = 0$ in Ω holds equivalently:

$$\int_\Omega (\partial_i u)\left(\partial^i \eta\right) \mathrm{d}^n x = 0 \qquad \underset{\eta \in C_0^\infty(\Omega)}{\forall}$$

The Dirichlet energy is defined as:

$$E(u) := \frac{1}{2} \int_\Omega \|\nabla u\|^2 \mathrm{d}^n x$$

Minimize E in a suitable Sobolev space ($H^{1,2}(\Omega) \cap \ldots$). Suppose a minimizer u exists.

$$E(u + \tau\eta) \geq E(u) \qquad \underset{\eta \in C_0^\infty}{\forall} \ \underset{\tau \in \mathbb{R}}{\forall}$$

If $E(u + \tau\eta)$ is differentiable, now follows:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} E(u + \tau\eta) \bigg|_{\tau=0}$$

For the Dirichlet energy this gives for all $\eta \in C_0^\infty$:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{1}{2} \int_\Omega \partial_i (u + \tau\eta)\, \partial^i (u + \tau\eta)\, \mathrm{d}^n x \bigg|_{\tau=0} = \\
&= \int_\Omega \partial_i (u + \tau\eta)\, \partial^i \eta \mathrm{d}^n x \bigg|_{\tau=0} = \\
&= \int_\Omega (\partial_i u)\left(\partial^i \eta\right) \mathrm{d}^n x
\end{aligned}$$

Thus holds $\Delta u = 0$ in the weak sense.
With regularity theory one can proof, that u is smooth.

6.1 Hölder Spaces

Let $\Omega \subseteq \mathbb{R}^n$ be open.

6.1.1 Definition (Hölder Continuous, Hölder Norm)

A function $u : \Omega \to \mathbb{R}^m$ is called *Hölder continuous* with exponent $\alpha \in \mathbb{R}_{\geq 0}$, if the *Hölder norm* is finite:

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{C^0(\Omega)} + \sup_{x \neq y \in \Omega} \left(\frac{\|u(x) - u(y)\|}{\|x - y\|^\alpha} \right) < \infty$$

This implies:

$$\|u(x) - u(y)\| \leq \sup_{x \neq y \in \Omega} \left(\frac{\|u(x) - u(y)\|}{\|x - y\|^\alpha} \right) \cdot \|x - y\|^\alpha$$

We often use the short notation:

$$\|\cdot\|_{C^{0,\alpha}(\Omega)} =: \|\cdot\|_{0,\alpha}$$

Obviously holds $C^{0,\alpha}(\Omega) \subseteq C^0(\Omega)$.

$C^{0,0}(\Omega)$ are bounded continuous functions.

$C^{0,1}(\Omega)$ are bounded, Lipschitz continuous functions.

This can also be defined locally: $f \in C^{0,\alpha}_{\text{loc}}(\Omega)$ means that for all $x \in \Omega$ exists a neighborhood $U \subseteq \Omega$ such that $f\big|_U \in C^{0,\alpha}(U)$.

A typical example is $u(x) = \sqrt{|x|}$ on $[-1,1]$ with $f \in C^{0,\frac{1}{2}}(\Omega)$.

$$C^{k,\alpha}(\Omega) := \left\{ f \in C^k \middle| \mathrm{D}^\beta f \in C^{0,\alpha} \underset{\text{multi-index } \beta, |\beta| \le k}{\forall} \right\}$$

$$\|f\|_{k,\alpha} = \sum_{\beta, |\beta| \le k} \left\| \mathrm{D}^\beta f \right\|_{0,\alpha}$$

6.1.2 Proposition (Hölder Continuous Functions are Complete)

$\left(C^{k,\alpha}(\Omega), \|.\|_{k,\alpha}\right)$ is a Banach space.

Proof

Let $u_n \in C^{k,\alpha}(\Omega)$ be a Cauchy sequence. Then u_n is also a Cauchy sequence in $C^k(\Omega)$. Since $C^k(\Omega)$ is complete, $\mathrm{D}^\beta u_n \rightrightarrows \mathrm{D}^\beta u$ converges uniformly for all multi-indices β with $|\beta| \le k$.

Next we have by uniform convergence:

$$\frac{\mathrm{D}^\beta u_n(x) - \mathrm{D}^\beta u_n(y)}{\|x-y\|^\alpha} \xrightarrow{n\to\infty} \frac{\mathrm{D}^\beta u(x) - \mathrm{D}^\beta u(y)}{\|x-y\|^\alpha}$$

Therefore the Hölder norms also converge. $\square_{6.1.2}$

6.2 L^p-Spaces, Approximate Theorems

$$L^p(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \middle| \int_\Omega |f(x)|^p \,\mathrm{d}^n x < \infty \right\}$$

$$\|f\|_p := \left(\int |f(x)|^p \,\mathrm{d}^n x \right)^{\frac{1}{p}}$$

6.2.1 Definition and Lemma (Mollifier)

A *mollifier* is a function $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(x) \in \mathbb{R}_{\ge 0}$ and $\int_{\mathbb{R}^n} \eta(x) \,\mathrm{d}^n x = 1$. Example:

$$\eta(x) = \begin{cases} ce^{-\frac{1}{\|x\|^2 - 1}} & \text{if } \|x\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

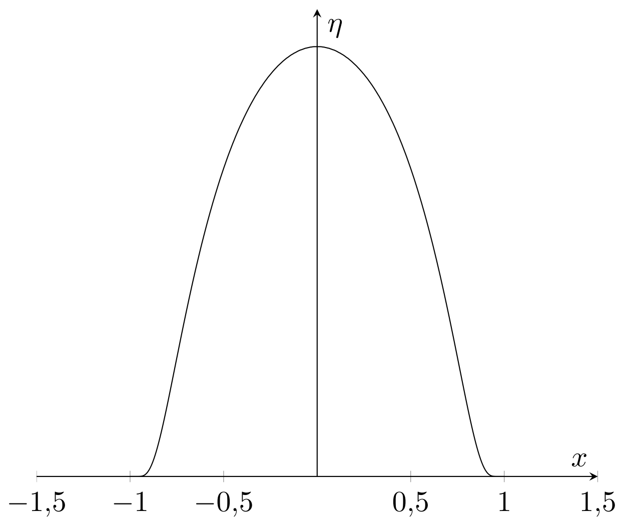


Figure 6.1: $\eta(x)$ is smooth and has compact support $[-1, 1]$.

For $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\varepsilon \in \mathbb{R}_{>0}$ define the mollified function as:

$$u_\varepsilon(x) = \int_\Omega \underbrace{\frac{1}{\varepsilon^n}\eta\left(\frac{x-y}{\varepsilon}\right)}_{=:\eta_\varepsilon(x-y)} u(y)\,\mathrm{d}^n y = (\eta_\varepsilon * u)(x)$$

Now holds $u_\varepsilon \in C^\infty(\Omega')$ for every $\Omega' \Subset \Omega$.

Proof

The support satisfies $\operatorname{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(0)$. Define $\varepsilon_0 := \operatorname{dist}(\Omega', \partial\Omega) > 0$. For $\varepsilon < \varepsilon_0$, we have for $x \in \Omega'$:

$$u_\varepsilon(x) = \int_{\Omega'} \eta_\varepsilon(x-y)\,u(y)\,\mathrm{d}^n y$$

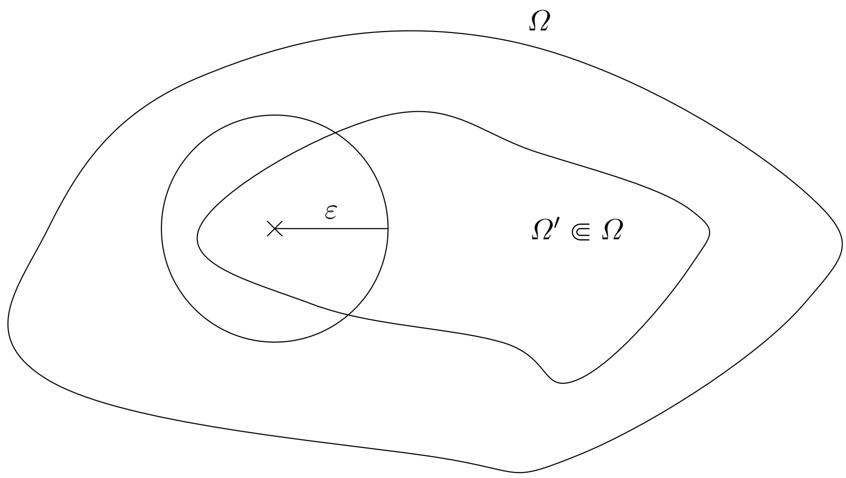


Figure 6.2: $\operatorname{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(0) \subseteq \Omega$

$$\frac{u_\varepsilon(x+he_i)-u_\varepsilon(x)}{h}=\int_{\Omega'}\underbrace{\frac{1}{h}\left(\eta_\varepsilon(x+he_i-y)-\eta_\varepsilon(x-y)\right)}_{\xrightarrow{h\to 0}\partial_i\eta_\varepsilon(x-y)}u(y)\,\mathrm{d}^n y$$

The limit $h\searrow 0$ can be taken inside the integral by Lebesgue's dominated convergence theorem, since both $|u|$ and η_ε are integrable. $\square_{6.2.1}$

6.2.2 Lemma (Mollified Function Converges Locally Uniformly)

For $u\in C^0(\Omega)$ the functions $u_\varepsilon \overset{\text{loc}}{\rightrightarrows} u$ converge locally uniformly.

Proof

$$u_\delta(x)=\frac{1}{\delta^n}\int_\Omega \eta\left(\frac{x-y}{\delta}\right)u(y)\,\mathrm{d}^n y$$

For $x_0\in\Omega$ we choose $\varrho\in\mathbb{R}_{>0}$ such that $B_{2\varrho}(x_0)\subseteq\Omega$. Let us show the uniform convergence $u_\delta\rightrightarrows u$ in $B_\varrho(x_0)$. We always assume $\delta<\varrho$. Then for all $x\in B_\varrho(x_0)$ holds:

$$\begin{aligned}u_\delta(x)&=\frac{1}{\delta^n}\int_\Omega \eta\left(\frac{x-y}{\delta}\right)u(y)\,\mathrm{d}^n y=\frac{1}{\delta^n}\int_{\mathbb{R}^n} \eta\left(\frac{x-y}{\delta}\right)u(y)\,\mathrm{d}^n y=\\ &\underset{\mathrm{d}^n z=\frac{\mathrm{d}^n y}{\delta^n}}{\overset{z:=\frac{x-y}{\delta}}{=}}\frac{1}{\delta^n}\int_{\mathbb{R}^n}\eta(z)\,u(x-\delta z)\,\delta^n\mathrm{d}^n z=\\ &=\int_{\mathbb{R}^n}\eta(z)\,u(x-\delta z)\,\mathrm{d}^n z\end{aligned}$$

u is uniformly continuous on the compact set $\overline{B_\delta(x)}$. For all $\varepsilon\in\mathbb{R}_{>0}$, there exists a $\delta\in\mathbb{R}_{>0}$ such that for all $y,y'\in\overline{B_\delta(x)}$ holds:

$$\left\|y-y'\right\|<\delta \qquad\Rightarrow\qquad \left|u(y)-u\left(y'\right)\right|<\varepsilon$$

$$\begin{aligned}\Rightarrow\qquad (u_\delta-u)(x)&=\int_{\mathbb{R}^n}\eta(z)\left(u(x-\delta z)-u(x)\right)\mathrm{d}^n z\\ |(u_\delta-u)(x)|&\le\int_{\mathbb{R}^n}\eta(z)\left|u(x-\delta z)-u(x)\right|\mathrm{d}^n z\le\varepsilon\int_{\mathbb{R}^n}\eta=\varepsilon\end{aligned}$$

$\square_{6.2.2}$

6.2.3 Lemma (Mollified Function Converges)

For $u\in L^p_{\text{loc}}(\Omega)$ (or $u\in L^p(\Omega)$) converges $u_\varepsilon\xrightarrow{\varepsilon\to 0}u$ in $L^p_{\text{loc}}(\Omega)$ (respectively in $L^p(\Omega)$).

Proof

For $\Omega' \Subset \Omega$ holds for small enough $\varepsilon \in \mathbb{R}_{>0}$:

$$\operatorname{dist}\left(\Omega', \partial\Omega\right) > 2\varepsilon$$

The definition of $u_\varepsilon(x)$ for $x \in \Omega'$ is:

$$u_\varepsilon(x) = \int_\Omega \frac{1}{\varepsilon^n} \eta\left(\frac{x-y}{\varepsilon}\right) u(y) \, \mathrm{d}^n y$$

By a variable transformation $z := \frac{x-y}{\varepsilon}$ follows due to $\operatorname{supp}(\eta) = B_1(0)$:

$$u_\varepsilon(x) = \int_{\|z\| \le 1} \eta(z) \, u(y + \varepsilon z) \, \mathrm{d}^n z$$

Now define the measure $\mathrm{d}\varrho := \eta(z) \, \mathrm{d}z$ and apply Hölder's inequality:

$$|u_\varepsilon(x)| = \left| \int_{\|z\| \le 1} u(x + \varepsilon z) \, \mathrm{d}\varrho \right| \overset{\text{Hölder}}{\le} \underbrace{\left(\int_{\|z\| \le 1} 1 \mathrm{d}\varrho \right)^{\frac{1}{q}}}_{=1} \left(\int_{\|z\| \le 1} |u|^p (x + \varepsilon z) \, \mathrm{d}\varrho \right)^{\frac{1}{p}}$$

This implies:

$$|u_\varepsilon(x)|^p \le \int_{\|z\| \le 1} |u|^p (x + \varepsilon z) \, \eta(z) \, \mathrm{d}z$$

Integrating over x yields:

$$\begin{aligned} \int_{\Omega'} |u_\varepsilon(x)|^p \, \mathrm{d}x &\le \int_{\Omega'} \mathrm{d}x \int_{\|z\| \le 1} |u|^p (x + \varepsilon z) \, \eta(z) \, \mathrm{d}z = \\ &\overset{\text{Fubini}}{=} \int_{\|z\| \le 1} \mathrm{d}z \eta(z) \int_{\Omega'} \mathrm{d}x \, |u|^p (x + \varepsilon z) \le \\ &\le \underbrace{\int_{\|z\| \le 1} \mathrm{d}z \eta(z)}_{=1} \int_{\Omega''} \mathrm{d}x \, |u|^p (x) = \int_{\Omega''} \mathrm{d}x \, |u|^p (x) \end{aligned}$$

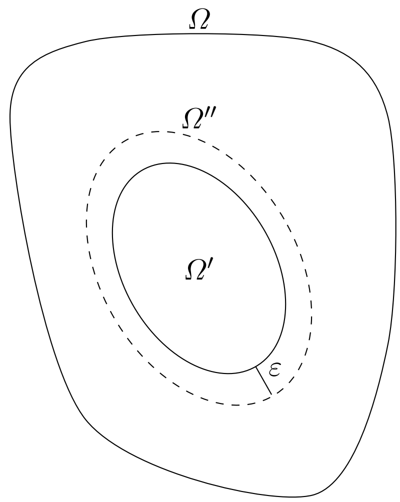


Figure 6.3: $\Omega'' := \left\{ x + \varepsilon z \middle| x \in \Omega', \|z\| \le 1 \right\}$

This means:

$$\|u_\varepsilon\|_{L^p(\Omega')} \leq \|u\|_{L^p(\Omega'')}$$

We want to apply the Lemma 6.2.2. For $u \in L^p_{\mathrm{loc}}(\Omega)$ and $\varepsilon \in \mathbb{R}_{>0}$ choose $\omega \in C^0(\Omega)$ with:

$$\|u - \omega\|_{L^p(\Omega'')} \leq \varepsilon$$

This is possible after Urysohn's lemma (cf. the book of Rudin). With

$$\|\omega_\varepsilon - u_\varepsilon\|_{L^p(\Omega')} \leq \|\omega - u\|_{L^p(\Omega'')} \leq \varepsilon$$

this gives:

$$\begin{aligned}
\|u - u_\varepsilon\|_{L^p(\Omega')} &\leq \|u - \omega\|_{L^p(\Omega')} + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} + \|\omega_\varepsilon - u_\varepsilon\|_{L^p(\Omega')} \leq \\
&\leq \|u - \omega\|_{L^p(\Omega'')} + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} + \|\omega - u\|_{L^p(\Omega'')} \leq \\
&\leq 2\,\|u - \omega\|_{L^p(\Omega'')} + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} \leq \\
&\leq 2\varepsilon + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')}
\end{aligned}$$

By Lemma 6.2.2 holds, since ω is continuous:

$$\|\omega - \omega_\varepsilon\|_{L^p(\Omega')} \xrightarrow{\varepsilon \to 0} 0$$

Therefore converges $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(\Omega)$.
To get the statement without "loc", extend u by 0 to all of $\mathbb{R}^n$. If Ω is bounded, $u_\varepsilon \to u$ converges in $L^p(\Omega)$ as above, as $\overline{\Omega}$ is compact, so $\overline{\Omega} \Subset \mathbb{R}^n$. If Ω is not bounded, we choose a ball $B_R(0)$ such that $\|u\|_{L^p(\Omega \setminus B_R(0))} \leq \varepsilon$. This is possible, since u is integrable on Ω. $\square_{6.2.3}$

6.2.4 **Corollary** (Dense Subsets of Integrable Functions)

$C^\infty(\Omega)$ and $C^{k,\alpha}(\Omega)$ are dense in $L^p_{\mathrm{loc}}(\Omega)$ and $C^\infty_0(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

6.3 Sobolev Spaces

6.3.1 **Definition** (Weak Derivative)

Consider $u, v \in L^1_{\mathrm{loc}}(\Omega)$ and a multi-index α. The function u is the α-th *weak derivative* of v if and only if for all test functions $\eta \in C^\infty_0(\Omega)$ holds:

$$\int_\Omega v(x)\,(-1)^{|\alpha|}\,\mathrm{D}^\alpha \eta(x)\,\mathrm{d}x = \int_\Omega u(x)\,\eta(x)\,\mathrm{d}x$$

This reminds of the definition of the distributional derivative $T_u = \mathrm{D}^\alpha T_v$ by

$$T_u(\eta) = (\mathrm{D}^\alpha T_v)(\eta) := (-1)^{|\alpha|}\,T_v(\mathrm{D}^\alpha \eta)$$

for all Schwartz functions η.
Here it is more general, because we have less test functions $\eta \in C^\infty_0$, but on the other hand less general due to $v \in L_1^{\mathrm{loc}}$, so T_v is a kind of "regular distribution".

6.3.2 Definition (Sobolev Space)

The *Sobolev space* $H^{k,p}(\Omega)$ for $k \in \mathbb{N}_{\geq 0}$ and $1 \leq p \leq \infty$ consists of L^1_{loc} functions $u : \Omega \to \mathbb{C}$ such that for all multi-indices α with $|\alpha| \leq k$, the weak derivative $\mathrm{D}^\alpha u$ exists and is in $L^p(\Omega)$.

$$\|u\|_{H^{k,p}(\Omega)} := \left(\sum_{|\alpha|\leq k} \int_\Omega |\mathrm{D}^\alpha u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

A special case of interest is $p = 2$, where we have an associated scalar product.

$$\langle u, v\rangle_{H^{k,2}(\Omega)} := \sum_{|\alpha|\leq k} \int_\Omega \overline{\mathrm{D}^\alpha u} \cdot \mathrm{D}^\alpha v \mathrm{d}x$$

6.3.3 Theorem (Sobolev Spaces are Complete)

The spaces $H^{k,p}(\Omega)$ are Banach spaces and Hilbert spaces for $p = 2$.

Proof

We have to show completeness. Let u_n be a Cauchy sequence in $H^{k,p}(\Omega)$. Completeness of $L^p(\Omega)$ implies that $u_n^\alpha := \mathrm{D}^\alpha u_n \xrightarrow{n\to\infty} v^\alpha$ converges in $L^p(\Omega)$. Then holds

$$v := \lim_{n\to\infty} u_n \in H^{k,p}(\Omega)$$

and $\partial^\alpha v = v^\alpha$, because for all $\eta \in C_0^\infty(\Omega)$ holds:

$$\int_{\mathrm{supp}(\eta)} v\,(-1)^{|\alpha|}\,\mathrm{D}^\alpha \eta \mathrm{d}x \leftarrow \int u_n\,(-1)^{|\alpha|}\,\mathrm{D}^\alpha \eta \mathrm{d}x = \int u_n^\alpha \eta \mathrm{d}x \to \int_{\mathrm{supp}(\eta)} v^\alpha \eta \mathrm{d}x$$

This is true, because convergence in L^p implies convergence in L^1_{loc}. $\square_{6.3.3}$

What is the relation between the weak derivative and mollifying?

6.3.4 Lemma (Weak Derivative and Mollifying Commute)

Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be α-times *weakly differentiable*, i.e. $\mathrm{D}^\alpha u$ exists for the multi-index α. Let $\Omega' \Subset \Omega$ be a compact subset with $\mathrm{dist}(\Omega', \partial\Omega) > \varepsilon \in \mathbb{R}_{>0}$. Then holds:

$$(\mathrm{D}^\alpha u)_\varepsilon = \mathrm{D}^\alpha u_\varepsilon$$

Proof

$$\begin{aligned}\mathrm{D}^\alpha u_\varepsilon(x) &= \int_{\mathbb{R}^n} \mathrm{D}_x^\alpha \eta_\varepsilon(x-y)\,u(y) = \int_{\mathbb{R}^n} (-1)^{|\alpha|}\,\mathrm{D}_y^\alpha \eta_\varepsilon(x-y)\,u(y)\,\mathrm{d}y = \\ &\overset{\text{integration}}{\underset{\text{by parts}}{=}} \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\,\mathrm{D}^\alpha u(y)\,\mathrm{d}y = (\mathrm{D}^\alpha u)_\varepsilon\end{aligned}$$

$\square_{6.3.4}$

6.3.5 Theorem (Dense Subset of Sobolev Space)

$C^\infty(\Omega) \cap H^{k,p}(\Omega)$ is dense in $H^{k,p}(\Omega)$.

Proof

Let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of Ω by relatively compact sets Ω_n, i.e. $\Omega_n \Subset \Omega_{n+1} \Subset \Omega$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$, with $\Omega_0 \neq \emptyset$. Define $A_n := \Omega_{n+1} \setminus \overline{\Omega}_n$. The A_n form a locally finite open covering.

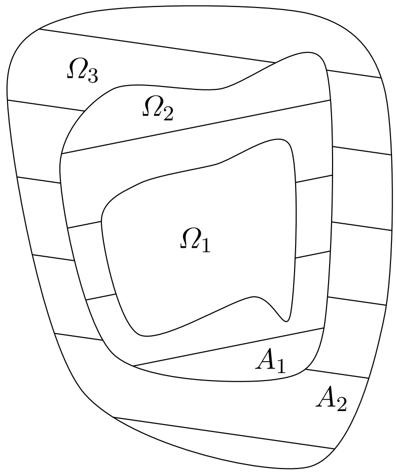


Figure 6.4: A_n form locally finite open covering

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a subordinate partition of unity, i.e. $\varphi_n \in C_0^\infty(A_n)$ and $\sum_n \varphi_n = 1$. For $u \in H^{k,p}(\Omega)$ holds $\varphi_n u \in H^{k,p}(A_n)$.

Now $(\varphi_n u)_{\varepsilon_n} \xrightarrow{\varepsilon_n \to 0} \varphi_n u$ converges in $L^p(A_n)$ after Lemma 6.2.3 due to $\varphi_n u \in L^p(A_n)$. Since weak derivative and mollifying commute (Lemma 6.3.4), follows in $L^p(A_n)$:

$$\mathrm{D}^\alpha\left((\varphi_n u)_{\varepsilon_n}\right) = (\mathrm{D}^\alpha \varphi_n u)_{\varepsilon_n} \xrightarrow{\varepsilon_n \to 0} \mathrm{D}^\alpha \varphi_n u$$

So $(\varphi_n u)_{\varepsilon_n} \xrightarrow{\varepsilon_n \to 0} \varphi_n u$ converges in $H^{k,p}(\Omega)$. Therefore one can choose $\varepsilon_n \in \mathbb{R}_{>0}$ such that for every $\delta \in \mathbb{R}_{>0}$ holds:

$$\left\|(\varphi_n u)_{\varepsilon_n} - \varphi_n u\right\|_{H^{k,p}(\Omega)} \leq \frac{\delta}{2^n}$$

Define:

$$\tilde{u} = \sum_{n=1}^{\infty} (\varphi_n u)_{\varepsilon_n} \in C^\infty(\Omega)$$

This $\tilde{u}$ is smooth, because the sum is locally finite and the $(\varphi_n u)_{\varepsilon_n}$ are smooth. So we get:

$$\|u - \tilde{u}\|_{H^{k,p}(\Omega)} \leq \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta$$

$\square_{6.3.5}$

Warning: $C_0^\infty(\Omega)$ is in general *not* dense in $H^{k,p}(\Omega)$. This motivates the following definition:

6.3.6 Definition (Sobolev Spaces for Dirichlet Boundary)

The *Sobolev spaces for Dirichlet boundary* condition are defined as:

$$H_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{H^{k,p}} \subseteq H^{k,p}$$

We have:

$$\begin{array}{ccccc} C^\infty(\Omega) \cap H^{k,p}(\Omega) & \overset{\text{dense}}{\subseteq} & H^{k,p}(\Omega) \subseteq L^p(\Omega) & & \\ \cup\!\! \mid & & \cup\!\! \mid & \text{(in general not dense)} \\ C_0^\infty(\Omega) & \overset{\text{dense}}{\subseteq} & H_0^{k,p}(\Omega) & & \end{array}$$

Is $C^\infty\left(\overline{\Omega}\right) \cap H^{k,p}(\Omega)$ dense in $H_0^{k,p}(\Omega)$? In general it is not, but for smooth boundary of Ω, it is.

6.3.7 Theorem (Elementary Properties of Sobolev Functions)

i) For $u \in H^{k,p}(\Omega)$ holds $\mathrm{D}^\alpha u \in H^{k-|\alpha|,p}(\Omega)$.

ii) For $\Omega' \subseteq \Omega$ and $u \in H^{k,p}(\Omega)$ holds $u\big|_{\Omega'} \in H^{k,p}(\Omega')$.

iii) Leibniz rule: For $u \in H^{k,p}(\Omega)$, $v \in C_0^\infty$ and $|\alpha| \leq k$ holds:

$$vu \in H^{k,p}$$

$$\mathrm{D}^\alpha(vu) = \sum_{\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{D}^\beta v \mathrm{D}^{\alpha-\beta} u$$

For multi-indices $\alpha = (i_1, \ldots, i_r)$ and $\beta = (j_1, \ldots, j_s)$ with $i_l, j_l \in \{1, \ldots, n\}$ let α_m be the number indices $i_l = m$ and analog β_m. The Binomial coefficient is defined as follows:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}$$

iv) Chain rule: For $f \in C^1$, $f' \in L^\infty$ and u weakly differentiable holds:

$$\mathrm{D}(f \circ u) = f'(u)\,\mathrm{D}u$$

Proof

i) and ii) are obvious.

iii) For brevity consider the first derivatives only.

$$\begin{aligned} \int (vu)(-1)(\mathrm{D}\eta)\,\mathrm{d}x &\underset{\text{for } C^\infty}{\overset{\text{Leibniz}}{=}} \int u(-1)(\mathrm{D}(\eta v) - \eta(\mathrm{D}v))\,\mathrm{d}x = \\ &\underset{\text{by parts}}{\overset{\text{integtation}}{=}} \int ((\mathrm{D}u)\eta v + u\eta \mathrm{D}v)\,\mathrm{d}x = \int (v\mathrm{D}u + u\mathrm{D}v)\eta \mathrm{d}x \\ \Rightarrow \qquad \mathrm{D}(uv) &= (\mathrm{D}u)v + u(\mathrm{D}v) \end{aligned}$$

Be careful: A product of two $H^{k,p}$ functions need not be in $H^{k,p}$ in general!

iv) We use an approximation argument: Consider $\Omega' \Subset \Omega$, $u_n \in C^\infty(\Omega')$ with $u_n \to u$ and $\mathrm{D}u_n \to \mathrm{D}u$ converging in $L^1(\Omega')$. The chain rule in C^1 gives $\mathrm{D}(f \circ u_n) = f'(u_n)\,\mathrm{D}u_n$. Now evaluate in the weak sense with a test function $\eta \in C_0^\infty(\Omega)$:

$$\int (f \circ u_n)(-\mathrm{D}\eta)\,\mathrm{d}x = \int f'(u_n)(\mathrm{D}u_n)\,\eta \mathrm{d}x$$

We have to show, that we can take the limit on both sides.

$$\int_{\Omega'} |f(u_n) - f(u)|\,|\mathrm{D}\eta|\,\mathrm{d}x \leq \sup|\mathrm{D}\eta| \sup|f'| \int_{\Omega'} (u_n - u)\,\mathrm{d}x \xrightarrow{n\to\infty} 0$$

The general technique

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n - a|\,|b_n| + |a|\,|b_n - b|$$

gives for the other side:

$$\left|\int_{\Omega'} f'(u_n)\,\mathrm{D}u_n - f'(u)(\mathrm{D}u)\,\eta \mathrm{d}x\right| \leq$$
$$\leq \int_{\Omega'} |f'(u_n) - f'(u)|\,|\mathrm{D}u|\,|\eta|\,\mathrm{d}x + \sup|f'| \cdot \int_{\Omega'} |\mathrm{D}u_n - \mathrm{D}u|\,|\eta|\,\mathrm{d}x$$

The second term converges to zero, because $\mathrm{D}u_n \to \mathrm{D}u$ converges in $L^1(\Omega')$.
The first term converges to zero, because as $u_n \to u$ converges almost everywhere, $f'(u_n) \to f'(u_n)$ converges almost everywhere and due to$f' \in C^0$ we can apply the dominated convergence theorem to take the limit under the integral. $\square_{6.3.7}$

6.4 Traces of Sobolev Functions at the Boundary

Note: These traces have nothing to do with traces in linear algebra.
Consider $u \in H^{k,p}(\Omega)$.
Questions: What is $u\big|_{\partial\Omega}$? In which space is it?
Assume for simplicity that $\partial\Omega$ is C^1, i.e. $\overline{\Omega}$ is a manifold with C^1-boundary $\partial\Omega$.

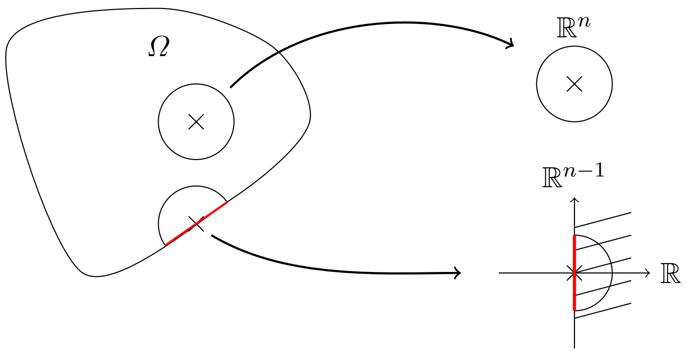


Figure 6.5: Ω is a manifold with C^1-boundary.

Thus for every $p \in \partial\Omega$ exists a neighborhood $p \in U \subseteq \overline{\Omega}$ open in the relative topology and a mapping

$$\varphi : U \to \mathbb{R}^n$$

with $\varphi(U) \subseteq \mathbb{R}^n_+ \left\{x \in \mathbb{R}^n \middle| x^n \geq 0\right\}$ and $\varphi(\partial\Omega \cap U) \subseteq \left\{x \in \mathbb{R}^n \middle| x^n = 0\right\}$ such that the mapping $\varphi : U \to \varphi(U)$ is a C^1-Diffeomorphism.

Moreover we assume that Ω is bounded. Thus $\partial\Omega$ is compact and therefore we can work with a finite number of charts (φ_i, U_i) with $i \in \{1, \dots, N\}$ and $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$.

6.4.1 Theorem and Definition (Trace)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^1-boundary. Then there is a unique bounded operator $T : H^{1,p}(\Omega) \to L^p(\partial\Omega)$, called *trace*, such that for all $u \in H^{1,p}(\Omega) \cap C^0\left(\overline{\Omega}\right)$ holds for all $x \in \partial\Omega$:

$$(Tu)(x) = u(x)$$

Proof

First consider $u \in C^1\left(\overline{\Omega}\right)$. ($C^1\left(\overline{\Omega}\right) \subseteq H^{1,p}(\Omega)$ holds, because Ω is bounded.) We show that T is bounded, i.e.

$$\|Tu\|_{L^p(\partial\Omega)} \leq K \|u\|_{H^{1,p}(\Omega)}$$

with a $C \in \mathbb{R}_{>0}$ independent of u. Then T has a unique continuation to $\overline{C^1\left(\overline{\Omega}\right)}^{H^{1,p}} = H^{1,p}(\Omega)$, because C^1 is dense in $H^{1,p}$.

1. Let $u \in C^1\left(\overline{\Omega}\right)$, $y \in \partial\Omega$ and assume that $\partial\Omega$ is flat in a neighborhood U of y, i.e. a hyperplane. By a rotation in $\mathbb{R}^n$, we can arrange that $U \cap \Omega = U \cap \mathbb{R}^n_+$. Set $\tilde{x} = \left(x^1, \dots, x^{n-1}\right)$.

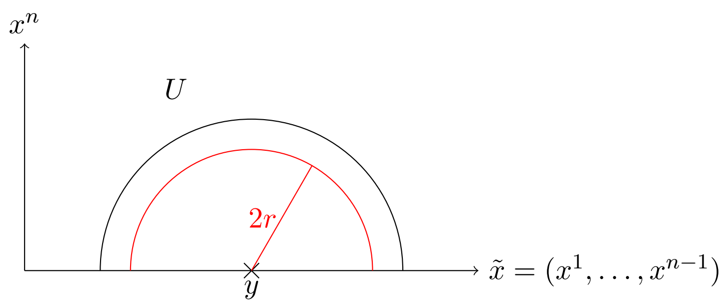


Figure 6.6: $U \cap \Omega = U \cap \mathbb{R}^n_+$

Choose $r \in \mathbb{R}_{>0}$ with $B_{2r}(y) \cap \mathbb{R}^n_+ \subseteq U$ and $\eta \in C_0^\infty(B_{2r}(y))$ with $\eta\big|_{B_r(y)} = 1$ and $\eta \geq 0$. Define $\Gamma := B_r(y) \cap \partial\Omega$.

$$\begin{aligned}
\int_\Gamma |u(\tilde{x}, 0)|^p \, \mathrm{d}^{n-1}\tilde{x} \leq \int_{\mathbb{R}^{n-1}} \underbrace{\eta(\tilde{x}, 0)\, |u(\tilde{x}, 0)|^p}_{\geq 0} \mathrm{d}^{n-1}\tilde{x} &= \\
= \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty -\frac{\partial}{\partial x^n} \left(\eta(x)\, |u(x)|^p\right) \mathrm{d}x^n \right) \mathrm{d}^{n-1}\tilde{x} &= \\
\overset{\text{Fubini}}{=} -\int_{\mathbb{R}^n_+} \frac{\partial}{\partial x^n} \left(\eta(x)\, |u(x)|^p\right) \mathrm{d}^n x &= \\
= \int_{\mathbb{R}^n_+} -\frac{\partial \eta}{\partial x^n} |u(x)|^p - \operatorname{sgn}(u(x))\, \eta p u^{p-1} \left(\frac{\partial}{\partial x^n} u\right) \mathrm{d}^n x &
\end{aligned}$$

The derivative of $|u(x)|^p$ does not exist for $p = 1$ and $u = 0$, but any point with $u = 0$ does not contribute to the original integral. We estimate:

$$\int_\Gamma |u(\tilde{x}, 0)|^p \, \mathrm{d}^{n-1}\tilde{x} \le \left|\frac{\partial \eta}{\partial x^n}\right|_{C^0} \int_{\mathbb{R}^n_+ \cap \Omega} |u|^p \, \mathrm{d}^n x + |\eta|_{C^0} \, p \cdot \int_{\mathbb{R}^n_+ \cap \Omega} |u|^{p-1} |\partial_n u| \, \mathrm{d}^n x$$

Now we use Young's inequality for $1 \le p \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

This yields for $q = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$:

$$|\partial_n u| \, |u|^{p-1} \le \frac{|\partial_n u|^p}{p} + \frac{|u|^{q(p-1)}}{q} = \frac{|\partial_n u|^p}{p} + \frac{|u|^p}{q}$$

So we get:

$$\begin{aligned}
\int_\Gamma |u(\tilde{x}, 0)|^p \, \mathrm{d}^{n-1}\tilde{x} &\le \left|\frac{\partial \eta}{\partial x^n}\right|_{C^0} \int_{\mathbb{R}^n_+ \cap \Omega} |u|^p \, \mathrm{d}^n x + |\eta|_{C^0} \, p \cdot \int_{\mathbb{R}^n_+ \cap \Omega} \left(\frac{|\partial_n u|^p}{p} + \frac{|u|^p}{q}\right) \mathrm{d}^n x \le \\
&\le \left|\frac{\partial \eta}{\partial x^n}\right|_{C^0} \int_{\mathbb{R}^n_+ \cap \Omega} \left(|u|^p + |\partial_n u|^p\right) \mathrm{d}^n x + \\
&\qquad + |\eta|_{C^0} \left(1 + \frac{p}{q}\right) \int_{\mathbb{R}^n_+ \cap \Omega} \left(|\partial_n u|^p + |u|^p\right) \mathrm{d}^n x = \\
&= \left(\left|\frac{\partial \eta}{\partial x^n}\right|_{C^0} + |\eta|_{C^0} \left(1 + \frac{p}{q}\right)\right) \int_{\mathbb{R}^n_+ \cap \Omega} \left(|u|^p + |\partial_n u|^p\right) \mathrm{d}^n x \le \\
&\le C \, \|u\|^p_{H^{1,p}(\Omega)}
\end{aligned}$$

All in all follows:

$$\|Tu\|_{L^p(\Gamma)} \le C^{\frac{1}{p}} \, \|u\|_{H^{1,p}(\Omega)}$$

2. Now consider a general Ω. We cover the boundary by a finite number $N \in \mathbb{N}$ of balls with charts (φ_i, U_i). Let $\eta_i(\tilde{x})$ be a subordinate partition of unity to get:

$$\begin{aligned}
\int_{\partial\Omega} |u|^p \, \mathrm{d}\mu_{\partial\Omega} &= \sum_{i=1}^N \int_{\varphi_i(U_i)} \underbrace{\eta_i(\tilde{x})}_{\le C(\Omega)} \underbrace{\left|\left(u \circ \varphi_i^{-1}\right)(\tilde{x})\right|^p}_{=|u(\tilde{y})|^p} \underbrace{\left|\det\left(\frac{\partial \varphi}{\partial \tilde{x}}\right)\right| \mathrm{d}^{n-1}\tilde{x}}_{=\mathrm{d}^{n-1}\tilde{y}} \le \\
&\le C(\Omega) \sum_{i=1}^N \int_{\Gamma_i} |u(\tilde{y})|^p \, \mathrm{d}^{n-1}\tilde{y}
\end{aligned}$$

With the first step follows:

$$\begin{aligned}
\int_{\partial\Omega} |u|^p \, \mathrm{d}\mu_{\partial\Omega} &\le C(\Omega) \sum_{i=1}^N \int_{\Gamma_i} \left(|u|^p + |\partial^n u|^p\right) \mathrm{d}^{n-1}\tilde{x} \le \\
&\le \tilde{C}(\Omega) \int_\Omega \left(|u|^p + |\partial^n u|^p\right) \mathrm{d}^n x
\end{aligned}$$

This gives for all $u \in C^1\left(\overline{\Omega}\right)$:

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \leq \underbrace{\left(\tilde{C}\left(\Omega\right)\right)^{\frac{1}{p}}}_{=:K} \|u\|_{H^{1,p}}$$

3. Now we use an approximation argument for $u \in H^{1,p}(\Omega)$. Since $C^1\left(\overline{\Omega}\right)$ is dense in $H^{1,p}(\Omega)$, we can take a sequence $u_n \in C^1\left(\overline{\Omega}\right)$ with $u_n \to u$ converging in $H^{1,p}(\Omega)$. Thus holds:

$$\|T(u_n - u_m)\|_{L^p(\partial\Omega)} = \left\|u_n\big|_{\partial\Omega} - u_m\big|_{\partial\Omega}\right\|_{L^p(\partial\Omega)} \leq K\,\|u_n - u_m\|_{H^{1,p}(\Omega)} \xrightarrow{n,m\to\infty} 0$$

Thus Tu_n converges in $L^p(\partial\Omega)$. So we define:

$$Tu := \lim_{n\to\infty} Tu_n$$

This is determined uniquely by the demand that T be continuous. Then the inequality also holds in the limit:

$$\|Tu\|_{L^p(\partial\Omega)} \leftarrow \|Tu_n\|_{L^p(\partial\Omega)} \leq K\,\|u_n\|_{H^{1,p}(\Omega)} \to K\,\|u\|_{H^{1,p}(\Omega)}$$

$\square_{6.4.1}$

6.4.2 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $1 \leq p < \infty$. Then for every $u \in H^{1,p}(\Omega)$ holds $u \in H_0^{1,p}(\Omega)$ if and only if $T(u) = 0$.

Proof

"$\Rightarrow$": For $u \in H_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{H^{1,p}(\Omega)}$ exists a sequence $u_n \in C_0^\infty(\Omega)$ with $u_n \to u$ in $H^{1,p}$ and $Tu_n = 0$. So we get:

$$0 = Tu_n \to Tu$$

"$\Leftarrow$": The proof is quite technical and can be found in the book of Evans. $\square_{6.4.2}$

6.5 The Gagliardo-Nirenberg Inequalities

Question: Consider $u \in C_0^1(\mathbb{R}^n)$. For which p and q does the inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq c\,\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \qquad \underset{u\in C_0^1(\mathbb{R}^n)}{\forall}$$

hold? This inequality does not holds for $u \in C^1(\mathbb{R}^n)$, because u can be an arbitrarily large constant with vanishing derivative.

We consider the scaling behavior, i.e. the behavior under the substitution $x \rightsquigarrow \lambda x$ with $\lambda \in \mathbb{R}_{>0}$.

$$u_\lambda(x) := u(\lambda x) \qquad\qquad \mathrm{D}u_\lambda(x) = \lambda \mathrm{D}u\big|_{\lambda x}$$

$$\|\mathrm{D}u_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \|\mathrm{D}u_\lambda\|^p \,\mathrm{d}^n x\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} \left\|\lambda \mathrm{D}u\big|_{\lambda x}\right\|^p \mathrm{d}^n x\right)^{\frac{1}{p}} = \lambda \left(\int_{\mathbb{R}^n} \left\|\mathrm{D}u\big|_{\lambda x}\right\|^p \mathrm{d}^n x\right)^{\frac{1}{p}} =$$

$$\underset{\mathrm{d}^n y = \lambda^n \mathrm{d}^n x}{\overset{y := \lambda x}{=}} \lambda \left(\int_{\mathbb{R}^n} \|\mathrm{D}u\,(y)\|^p \frac{\mathrm{d}^n y}{\lambda^n}\right)^{\frac{1}{p}} = \lambda^{1-\frac{n}{p}} \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int |u\,(\lambda x)|^q \,\mathrm{d}^n x\right)^{\frac{1}{q}} \overset{y=\lambda x}{=} \left(\int |u\,(x)|^q \frac{\mathrm{d}^n y}{\lambda^n}\right)^{\frac{1}{q}} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$$

The inequality can only be true, if both scalings are the same:

$$\begin{aligned} 1 - \frac{n}{p} &\overset{!}{=} -\frac{n}{q} \\ \frac{1}{q} &= \frac{1}{p} - \frac{1}{n} \\ q &= \frac{np}{n-p} \end{aligned}$$

q is called the *Sobolev conjugate*, usually denoted by p^*.

6.5.1 Theorem (Gagliardo-Nirenberg-Sobolev Inequality)

For $1 \leq p < n$ there is a constant $c \in \mathbb{R}_{>0}$ such that holds:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \,\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \qquad \underset{u \in C_0^1(\mathbb{R}^n)}{\forall}$$

One can choose:

$$c = \frac{p\,(n-1)}{n-p}$$

6.5.2 Lemma (Generalized Hölder Inequalities)

Consider $1 \leq p_1, \ldots, p_m < \infty$ with:

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n} = 1$$

Then for any $u_i \in L^{p_i}\,(\varOmega)$ holds:

$$\int_\varOmega |u_1 \cdots u_m| \,\mathrm{d}^n x \leq \|u_1\|_{L^{p_1}(\varOmega)} \cdots \|u_m\|_{L^{p_m}(\varOmega)}$$

Proof

We proceed inductively: $\frac{1}{p_1} + \frac{1}{q_1} = 1$

$$\begin{aligned} \int |u_1| \cdot |u_2 \cdots u_m| \,\mathrm{d}^n x &\leq \|u\|_{p_1} \cdot \left(\int |u_2 \cdots u_m|^{q_1} \mathrm{d}^n x\right)^{\frac{1}{q_1}} \leq \\ &\leq \|u\|_{p_1} \cdot \left(\int |u_2|^{q_1} |u_3 \cdots u_m|^{q_1} \mathrm{d}^n x\right)^{\frac{1}{q_1}} \end{aligned}$$

Applying Hölder's inequality with exponents r, s with $\frac{1}{r}+\frac{1}{s}=1$ such that $rq_1 = p_2$ gives:

$$\int |u_2|^{q_1} |u_3 \cdots u_m|^{q_1} \, \mathrm{d}^n x \leq \underbrace{\|u_2^{q_1}\|_r}_{=\|u_2\|_{rq_1}^{q_1}} \cdot \left(\int |u_3|^{q_1 s} |u_4 \cdots u_m|^{q_1 s} \, \mathrm{d}^n x \right)^{\frac{1}{s}}$$

$$\int |u_1| \cdot |u_2 \cdots u_m| \, \mathrm{d}^n x \leq \|u\|_{p_1} \cdot \|u_2\|_{p_2} \left(\int |u_3|^{q_1 s} |u_4 \cdots u_m|^{q_1 s} \, \mathrm{d}^n x \right)^{\frac{1}{q_1 s}}$$

Iteratively this gives the the generalized Hölder Inequality. $\Box_{6.5.2}$

Proof of Theorem 6.5.1

We need to find a constant $c \in \mathbb{R}_{>0}$ such that for $u \in C_0^1(\mathbb{R}^n)$ holds:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \, \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

1. Consider the case $p = 1$, i.e. $p^* = \frac{n \cdot 1}{n-1}$. We have to show:

$$\|u\|_{L^{\frac{n}{n-1}}} \leq \|\mathrm{D}u\|_{L^1}$$

For any $i \in \{1, \ldots, n\}$ the fundamental theorem of calculus yields:

$$\begin{aligned}
u(x) &= \int_{-\infty}^{x_i} \frac{\partial}{\partial x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1} \ldots, x_n) \, \mathrm{d}y_i \\
|u(x)| &\leq \int_{-\infty}^{\infty} \|\mathrm{D}u(x_1, \ldots, y_i, \ldots, x_n)\| \, \mathrm{d}y_i \\
|u(x)|^{\frac{n}{n-1}} &\leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \|\mathrm{D}u(x_1, \ldots, y_i, \ldots, x_n)\| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}}
\end{aligned}$$

We proceed inductively. First we integrate over x_1:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} \, \mathrm{d}x_1 &\leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \|\mathrm{D}u(x_1, \ldots, y_i, \ldots, x_n)\| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 = \\
&= \underbrace{\left(\int_{-\infty}^{\infty} \|\mathrm{D}u(y_1, x_2, \ldots, x_n)\| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}}}_{\text{independent of } x_1} \cdot \\
&\quad \cdot \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \|\mathrm{D}u(x_1, \ldots, y_i, \ldots, x_n)\| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1
\end{aligned}$$

The generalized Hölder's inequality for the $n-1$ exponents

$$\begin{aligned}
& p_2 = \ldots = p_n = n-1 \\
\Rightarrow \qquad & \frac{1}{p_2} + \ldots + \frac{1}{p_n} = 1
\end{aligned}$$

gives:

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} \, \mathrm{d}x_1 \leq \left(\int_{-\infty}^{\infty} \|\mathrm{D}u(y_1, x_2, \ldots, x_n)\| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \cdot$$

$$\cdot \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}y_i \left\| \mathrm{D}u\left(x_1, \dots, y_i, \dots, x_n\right) \right\| \right)^{\frac{1}{n-1}}$$

Next, we integrate over x_2 (and don't write the boundary of the integral any more):

$$\begin{aligned} \int \left| u\left(x\right) \right|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \leq & \underbrace{\left(\int \mathrm{d}x_1 \int \mathrm{d}y_i \left\| \mathrm{D}u\left(x_1, y_2, x_3, \dots, x_n\right) \right\| \right)^{\frac{1}{n-1}}}_{\text{independent of } x_2} \cdot \\ & \cdot \int \mathrm{d}x_2 \left(\int \mathrm{d}y_i \left\| \mathrm{D}u\left(y_1, x_2, \dots, x_n\right) \right\| \right)^{\frac{1}{n-1}} \cdot \\ & \cdot \prod_{i=3}^{n} \left(\int \mathrm{d}x_1 \int \mathrm{d}y_i \left\| \mathrm{D}u\left(x_1, \dots, y_i, \dots, x_n\right) \right\| \right) \end{aligned}$$

We again have $n-1$ factors, so we can apply the generalized Hölder inequality with exponents $n-1$ to get:

$$\begin{aligned} \int \left| u\left(x\right) \right|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \leq & \left(\int \mathrm{d}x_1 \int \mathrm{d}y_i \left\| \mathrm{D}u\left(x_1, y_2, x_3, \dots, x_n\right) \right\| \right)^{\frac{1}{n-1}} \cdot \\ & \cdot \left(\int \mathrm{d}x_2 \int \mathrm{d}y_i \left\| \mathrm{D}u\left(y_1, x_2, \dots, x_n\right) \right\| \right)^{\frac{1}{n-1}} \cdot \\ & \cdot \prod_{i=3}^{n} \left(\int \mathrm{d}x_2 \int \mathrm{d}x_1 \int \mathrm{d}y_i \left\| \mathrm{D}u\left(x_1, \dots, y_i, \dots, x_n\right) \right\| \right)^{\frac{1}{n-1}} \end{aligned}$$

After also integrating over $x_3, \dots, x_n$ and then renaming $y_i \to x_i$ we get:

$$\int_{\mathbb{R}^n} \left| u\left(x\right) \right|^{\frac{n}{n-1}} \mathrm{d}^n x \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} \mathrm{d}^n x \left\| \mathrm{D}u\left(x\right) \right\| \right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} \mathrm{d}^n x \left\| \mathrm{D}u\left(x\right) \right\| \right)^{\frac{n}{n-1}}$$

$$\left\| u \right\|_{L^{\frac{n}{n-1}}} = \left(\int_{\mathbb{R}^n} \left| u\left(x\right) \right|^{\frac{n}{n-1}} \mathrm{d}^n x \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} \mathrm{d}^n x \left\| \mathrm{D}u\left(x\right) \right\| = \left\| \mathrm{D}u \right\|_{L^1}$$

2. The general case $1 < p < n$ follows from the first case by using Hölder's inequality: We have to prove for $p^* = \frac{np}{n-p}$:

$$\left\| u \right\|_{L^{p^*}\left(\mathbb{R}^n\right)} = \left(\int_{\mathbb{R}^n} \left| u\left(x\right) \right|^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \leq c \left(\int_{\mathbb{R}^n} \left| \mathrm{D}u\left(x\right) \right|^{p} \mathrm{d}x \right)^{\frac{1}{p}} = c \left\| \mathrm{D}u \right\|_{L^p\left(\mathbb{R}^n\right)}$$

Now introduce $v := |u|^c$ such that $|u|^{p^*} = v^{\frac{n}{n-1}}$:

$$\begin{aligned} & c \frac{n}{n-1} = p^* = \frac{np}{n-p} \\ \Rightarrow \qquad & c = \frac{\left(n-1\right)p}{n-p} \end{aligned}$$

Now follows:

$$\left(\int \left| u \right|^{p^*} \mathrm{d}^n x \right)^{\frac{n-1}{n}} = \left(\int v^{\frac{n}{n-1}} \mathrm{d}^n x \right)^{\frac{n-1}{n}} = \left\| v \right\|_{L^{\frac{n}{n-1}}} \leq$$

$$\overset{1.}{\leq} \int |\mathrm{D}v| \,\mathrm{d}^n x \leq c \int |u|^{c-1} |\mathrm{D}u| \,\mathrm{d}^n x$$

Applying Hölder's inequality with $\tilde{q}$ and $\tilde{p}$ gives:

$$\left(\int |u|^{p^*} \,\mathrm{d}^n x \right)^{\frac{n-1}{n}} \leq c \left(\int |u|^{\tilde{q}(c-1)} \,\mathrm{d}^n x \right)^{\frac{1}{\tilde{q}}} \left(\int |\mathrm{D}u|^{\tilde{p}} \,\mathrm{d}^n x \right)^{\frac{1}{\tilde{p}}}$$

We choose $\tilde{q}$ such that $\tilde{q}\,(c-1) = p^*$ holds.

$$\tilde{q} = \frac{p^*}{c-1} = \frac{\frac{np}{n-p}}{\frac{(n-1)p}{n-p} - 1} = \frac{np}{(n-1)\,p - (n-p)} = \frac{np}{np-n} = \frac{p}{p-1}$$

$$\tilde{p} = \frac{1}{1-\frac{1}{\tilde{q}}} = \frac{1}{1-\frac{p-1}{p}} = \frac{p}{p-(p-1)} = p$$

We get:

$$\left(\int |u|^{p^*} \,\mathrm{d}^n x \right)^{\frac{n-1}{n}} \leq c \left(\int |u|^{p^*} \,\mathrm{d}^n x \right)^{\frac{p-1}{p}} \left(\int |\mathrm{D}u|^{p} \,\mathrm{d}^n x \right)^{\frac{1}{p}}$$

$$\left(\int |u|^{p^*} \,\mathrm{d}^n x \right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq c \,\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{p\,(n-1) - n\,(p-1)}{np} = \frac{np-p-np+n}{np} = \frac{n-p}{np} = \frac{1}{p^*}$$

Finally follows:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} = \left(\int |u|^{p^*} \,\mathrm{d}^n x \right)^{\frac{1}{p^*}} \leq c \,\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

$\square_{6.5.1}$

6.5.3 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded domain with C^1-boundary. Then there is a constant $C \in \mathbb{R}_{>0}$ such that for all $u \in H^{1,p}(\Omega)$ holds:

$$\|u\|_{L^{p^*}(\Omega)} \leq C \,\|u\|_{H^{1,p}(\Omega)}$$

Recall:

$$\|u\|_{H^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\mathrm{D}u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

Proof

Consider $u \in H^{1,p}(\Omega)$ and choose Ω' with $\Omega \Subset \Omega'$. Then u can be extended to a function $\overline{u} \in H^{1,p}_0(\Omega')$ with:

$$\|\overline{u}\|_{H^{1,p}(\Omega')} \leq \tilde{c} \,\|u\|_{H^{1,p}(\Omega)}$$

Namely choose $u_n \in C^1(\Omega)$ with $u_n \to u$ in $H^{1,p}(\Omega)$ and extend u_n to $C_0^1(\Omega')$ to obtain functions $\overline{u}_n$ with:

$$\|\overline{u}_n\|_{H^{1,p}(\Omega')} \leq \tilde{c}\,\|u_n\|_{H^{1,p}(\Omega)}$$

A subsequence of the $\overline{u}_n$ converges to $\overline{u}$ in $H_0^{1,p}(\Omega')$. The constant $\tilde{c}$ can be chosen independently of u, because one can construct a linear mapping

$$A : C^1(\Omega) \to C_0^1\left(\Omega'\right)$$

with:

$$\|Au\|_{H^{1,p}(\Omega')} \leq c\,\|u\|_{H^{1,p}(\Omega)} \qquad \underset{u\in C^1(\Omega)}{\forall}$$

Extending $\overline{u}$ by zero to $\mathbb{R}^n$ we get:

$$\|\overline{u}\|_{H^{1,p}(\mathbb{R}^n)} \leq \tilde{c}\,\|u\|_{H^{1,p}(\Omega)}$$

Now follows:

$$\begin{aligned}\|u\|_{L^{p^*}(\Omega)} \leq \|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} &\overset{6.5.1}{\leq} c\,\|\mathrm{D}\overline{u}\|_{L^p(\mathbb{R}^n)} \leq \\ &\leq c\,\|\overline{u}\|_{H^{1,p}(\mathbb{R}^n)} \leq \underbrace{c\tilde{c}}_{=:C}\,\|u\|_{H^{1,p}(\Omega)}\end{aligned}$$

$\square_{6.5.3}$

6.5.4 Theorem (Poincaré's inequality)

Let Ω be a bounded open domain, $1 \leq p < \infty$ and $1 \leq q \leq p^*$. Then there is a $C(\Omega) \in \mathbb{R}$ such that for all $u \in H_0^{1,p}(\Omega)$ holds:

$$\|u\|_{L^q(\Omega)} \leq C(\Omega)\,\|\mathrm{D}u\|_{L^p(\Omega)}$$

Proof

For $u \in H_0^{1,p}(\Omega)$ there is by definition a sequence $u_k \in C_0^1(\Omega)$ with $u_k \to u$ converging in $H_0^{1,p}(\Omega)$. We extend u_n by zero to $\mathbb{R}^n$ and apply the Gagliardo-Nirenberg inequality:

$$\|u_k\|_{L^{p^*}(\Omega)} = \|u_k\|_{L^{p^*}(\mathbb{R}^n)} \leq c\,\|\mathrm{D}u_k\|_{L^p(\mathbb{R}^n)} = c\,\|\mathrm{D}u_k\|_{L^p(\Omega)} \xrightarrow{k\to\infty} c\,\|\mathrm{D}u\|_{L^p(\Omega)}$$

For the left sides holds:

$$\|u_k - u_l\|_{L^{p^*}(\Omega)} \leq c\,\|\mathrm{D}(u_k - u_l)\|_{L^p(\mathbb{R}^n)} \leq c\,\|u_k - u_l\|_{H^{1,p}(\Omega)} \xrightarrow{k,l\to\infty} 0$$

Therefore $u_k \to u$ converges in L^{p^*}. In the limit we get:

$$\|u\|_{L^{p^*}(\Omega)} \leq c\,\|\mathrm{D}u\|_{L^p(\Omega)}$$

Now we have:

$$\|u\|_{L^q(\Omega)} = \left(\int 1 \cdot |u|^q\,\mathrm{d}^n x\right)^{\frac{1}{q}}$$

Finally, we choose Hölder exponents r, s with

$$sq = p^*$$
$$s = \frac{p^*}{q} \geq 1$$

to get:

$$\|u\|_{L^q(\Omega)} \leq (\mu(\Omega))^{\frac{1}{r}} \|u\|_{L^{p^*}(\Omega)} \leq \underbrace{c\,(\mu(\Omega))^{\frac{1}{r}}}_{=:C(\Omega)} \|\mathrm{D}u\|_{L^p(\Omega)}$$

$\Box_{6.5.4}$

6.5.5 Corollary

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $1 \leq p < n$. Then $\|\mathrm{D}u\|_{L^p(\Omega)}$ is a norm on $H_0^{1,p}(\Omega)$ which is equivalent to the $H^{1,p}$-norm.

Proof

The $H^{1,p}$-norm

$$\|u\|_{H^{1,p}} = \left(\|u\|_{L^p}^p + \|\mathrm{D}u\|_{L^p}^p\right)^{\frac{1}{p}}$$

is equivalent to:

$$\|u\|_{L^p(\Omega)} + \|\mathrm{D}u\|_{L^p(\Omega)}$$

It holds:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c\,\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

$$\|\mathrm{D}u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|\mathrm{D}u\|_{L^p(\Omega)} \overset{p<p^*}{\leq} \tilde{c}\ \underbrace{\|u\|_{L^{p^*}(\Omega)}}_{\leq c\|\mathrm{D}u\|_{L^p(\Omega)}} + \|\mathrm{D}u\|_{L^p(\Omega)} \leq (1 + \tilde{c}c)\,\|\mathrm{D}u\|_{L^p(\Omega)}$$

$\Box_{6.5.5}$

6.6 The Morrey Inequalities

This traces back to Charles Morrey (1907-1984).

6.6.1 Theorem (Morrey Inequalities)

For $n < p \leq \infty$ exists a constant $C \in \mathbb{R}_{>0}$ (depending only on n and p), such that for $u \in C^1(\mathbb{R}^n) \cap H^{1,p}(\mathbb{R}^n)$ and $\gamma := 1 - \frac{n}{p}$ holds:

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\,\|u\|_{H^{1,p}(\mathbb{R}^n)}$$

The exponent γ can be found with a scaling argument.

Proof

We define:

$$\fint_U f(y)\,\mathrm{d}^n y := \frac{1}{\mu(U)} \int_U f(y)\,\mathrm{d}^n y$$

1. **Claim:**

$$\fint_{B_r(x)} |u(x) - u(y)|\,\mathrm{d}^n y \leq C_1(n) \cdot \int_{B_r(x)} \frac{\|\mathrm{D}u(y)\|}{\|x-y\|^{n-1}} \mathrm{d}^n y$$

Proof: Choose $B_r(x)$, $s \in (0,r)$ and a unit vector $w \in \partial B_1(0)$.

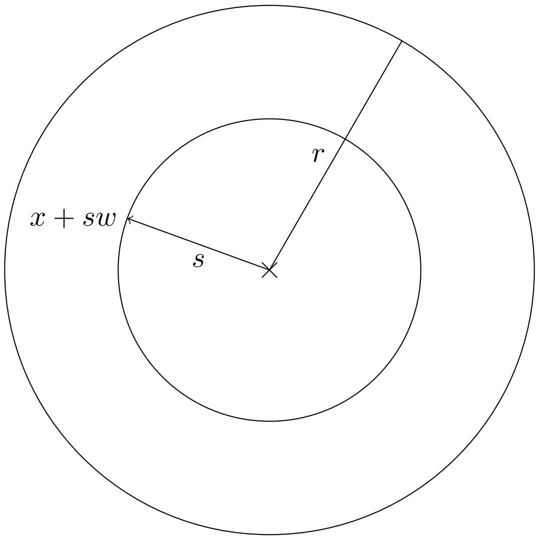


Figure 6.7: $x + sw \in B_r(x)$

$$|u(x+sw) - u(x)| = \left| \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} u(x+tw)\,\mathrm{d}t \right| \leq \int_0^s \|\mathrm{D}u(x+tw)\| \cdot \underbrace{\|w\|}_{=1} \mathrm{d}t$$

Integration gives:

$$\begin{aligned}
&\int_{\partial B_1(0)} |u(x+sw) - u(x)|\,\mathrm{d}\mu_{\partial B_1(0)}(w) \leq \\
&\qquad \leq \int_0^s \mathrm{d}t \int_{\partial B_1(0)} \|\mathrm{D}u(x+tw)\|\,\mathrm{d}\mu_{\partial B_1(0)}(w) = \\
&\qquad \overset{z:=x+tw}{=} \int_0^s \frac{\mathrm{d}t}{t^{n-1}} \int_{\partial B_t(x)} \|\mathrm{D}u(z)\|\,\mathrm{d}\mu_{\partial B_t(x)}(z) = \\
&\qquad \overset{t=\|x-z\|}{=} \int_0^s \mathrm{d}t \int_{\partial B_t(x)} \frac{\|\mathrm{D}u(z)\|}{\|x-z\|^{n-1}} \mathrm{d}\mu_{\partial B_t(x)}(z) = \\
&\qquad \overset{\text{Fubini}}{=} \int_{B_s(x)} \frac{\|\mathrm{D}u(y)\|}{\|x-y\|^{n-1}} \mathrm{d}\mu_{B_s(x)}(y) \leq \int_{B_r(x)} \frac{\|\mathrm{D}u(y)\|}{\|x-y\|^{n-1}} \mathrm{d}^n y
\end{aligned}$$

Multiplying by s^{n-1} and integrating over s gives:

$$\int_0^r \mathrm{d}s \cdot s^{n-1} \int_{\partial B_1(0)} |u(x+sw) - u(x)| \, \mathrm{d}\mu_{\partial B_1(0)}(w) \leq \int_0^r \mathrm{d}s \cdot s^{n-1} \int_{B_r(x)} \frac{\|\mathrm{D}u(y)\|}{\|x-y\|^{n-1}} \mathrm{d}^n y$$

$$\overset{\text{Fubini}}{\Rightarrow} \qquad \int_{\partial B_r(x)} |u(y) - u(x)| \, \mathrm{d}\mu_{B_r(x)}(y) \leq \frac{r^n}{n} \int_{B_r(x)} \frac{\|\mathrm{D}u(y)\|}{\|x-y\|^{n-1}} \mathrm{d}^n y$$

Dividing by r^n gives the claim. $\square_{1.}$

2. Now estimate u point-wise:

$$|u(x)| = |u(x)| \underbrace{⨍_{B_1(x)} \mathrm{d}^n y}_{=1} \leq ⨍_{B_1(x)} |u(x) - u(y)| \, \mathrm{d}^n y + ⨍_{B_1(x)} |u(y)| \, \mathrm{d}^n y \leq$$

$$\overset{1.}{\leq} C_1 \cdot \int_{B_1(x)} \frac{\|\mathrm{D}u(y)\|}{\|x-y\|^{n-1}} \mathrm{d}^n y + \frac{1}{\omega_n} \int_{B_1(x)} |u(y)| \, \mathrm{d}^n y \leq$$

$$\underset{p^{-1}+q^{-1}=1}{\overset{\text{Hölder}}{\leq}} C_1 \|\mathrm{D}u\|_{L^q(B_1(x))} \left(\int_{B_1(x)} \left(\frac{1}{\|x-y\|^{n-1}} \right)^q \mathrm{d}^n y \right)^{\frac{1}{q}} +$$

$$+ \|u\|_{L^p(B_1(x))} \underbrace{\frac{1}{\omega_n} \left(\int_{B_1(x)} 1^q \mathrm{d}^n y \right)^{\frac{1}{q}}}_{=:C_2} \leq$$

$$\leq C_1 \|\mathrm{D}u\|_{L^q(\mathbb{R}^n)} \left(\int_{B_1(x)} \left(\frac{1}{\|x-y\|^{n-1}} \right)^q \mathrm{d}^n y \right)^{\frac{1}{q}} + C_2 \|u\|_{L^p(\mathbb{R}^n)}$$

Is the pole integrable? The order of the pole is:

$$(n-1)\, q = (n-1) \frac{p}{p-1} \overset{?}{<} n$$

$$(n-1)\, p \overset{?}{<} n\,(p-1)$$

$$-p \overset{?}{<} -n$$

$$n < p$$

This holds by assumption, thus the pole is integrable and we get:

$$\left| \int_{B_1(x)} \frac{1}{\|x-y\|^{(n-1)q}} \right| \leq C_3 < \infty$$

Thus holds:

$$|u(x)| \leq C_1 C_3 \|\mathrm{D}u\|_{L^q(B_1(x))} + C_2 \|u\|_{L^p(\mathbb{R}^n)} \leq C_4 \|u\|_{H^{1,p}(\mathbb{R}^n)}$$

3. Now estimate $|u(x) - u(y)|$. For $x, y \in \mathbb{R}^n$ define $r := \|x-y\|$ and $W := B_r(x) \cap B_r(y)$.

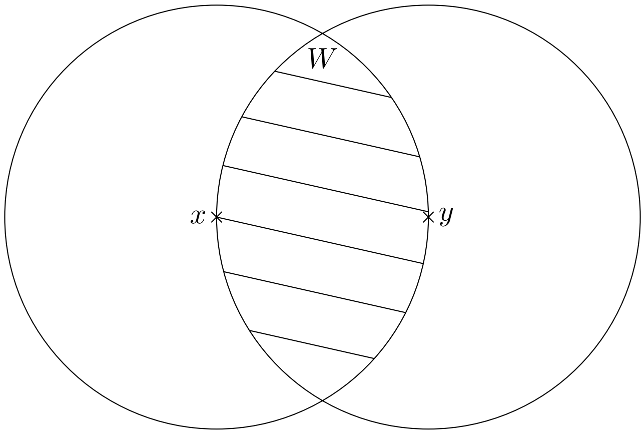


Figure 6.8: Intersection of spheres

Thus we get:

$$|u(x) - u(y)| = \fint_W |u(x) - u(y)| \, \mathrm{d}^n z \leq \fint_W |u(x) - u(z)| \, \mathrm{d}^n z + \fint_W |u(z) - u(y)| \, \mathrm{d}^n z$$

$$\fint_W |u(x) - u(z)| \, \mathrm{d}^n z \leq \frac{1}{\mu(W)} \int_{B_r(x)} |u(x) - u(z)| \, \mathrm{d}^n z = C_5(n) \fint_{B_r(x)} |u(x) - u(z)| \, \mathrm{d}^n z \tag{6.1}$$

The constant

$$C_5(n) = \frac{\mu(B_r)}{\mu(W)}$$

is independent of r, due to the proportionalities $\mu(B_r) \sim r^n \sim \mu(W)$. With 1. follows:

$$\begin{aligned}
\fint_W |u(x) - u(z)| \, \mathrm{d}^n z &\leq C_5(n) \fint_{B_r(x)} |u(x) - u(z)| \, \mathrm{d}^n z \leq \\
&\leq C_5(n) \, C_1(n) \cdot \int_{B_r(x)} \frac{\|\mathrm{D}u(z)\|}{\|x - z\|^{n-1}} \mathrm{d}^n z \leq \\
&\overset{\text{Hölder}}{\leq} C_6(n) \, \|\mathrm{D}u\|_{L^p(B_r(x))} \left(\int_{B_r(x)} \frac{1}{\|x - z\|^{(n-1)\frac{p}{p-1}}} \mathrm{d}^n z \right)^{\frac{p-1}{p}}
\end{aligned}$$

$$\begin{aligned}
\int_{B_r(x)} \frac{1}{\|x - z\|^{(n-1)\frac{p}{p-1}}} \mathrm{d}^n z &= \int_0^r \mathrm{d}s \, (n\omega_n) \, s^{n-1} \frac{1}{s^{(n-1)\frac{p}{p-1}}} = C_7(n) \, r \cdot r^{n-1} \frac{1}{r^{(n-1)\frac{p}{p-1}}} = \\
&= C_7(n) \, r^{n - \frac{(n-1)p}{p-1}} = C_7(n) \, r^{\frac{np - n - np + p}{p-1}} = C_7(n) \, r^{\frac{p-n}{p-1}}
\end{aligned}$$

Thus follows:

$$\fint_W |u(x) - u(z)| \, \mathrm{d}^n z \leq \underbrace{C_6(n) \, (C_7(n))^{\frac{p-1}{p}}}_{=:C_8(n)} \|\mathrm{D}u\|_{L^p(B_r(x))} \, r^{\frac{p-n}{p-1} \cdot \frac{p-1}{p}}$$

Estimating the other integral in (6.1) in the same way, we obtain:

$$|u(x) - u(y)| \leq \underbrace{2C_8(n)}_{=:C_9(n)} r^{1 - \frac{n}{p}} \, \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

With $\gamma := 1 - \frac{n}{p}$ follows for all $x, y \in \mathbb{R}^n$ due to $r = \|x - y\|$:

$$\frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \leq C_9(n) \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

4. Combining 2. and 3., we get:

$$\begin{aligned}\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} &= \|u\|_{C^0(\mathbb{R}^n)} + \sup_{x \neq y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \leq \\ &\leq C_4(n) \|u\|_{H^{1,p}(\mathbb{R}^n)} + C_9(n) \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{H^{1,p}(\mathbb{R}^n)}\end{aligned}$$

$\square_{6.6.1}$

We use a denseness argument to extend the statement to $H^{1,p}$-functions and $C^{0,\alpha}$-functions. For $u \in H^{1,p}(\mathbb{R}^n)$, weak derivatives $\mathrm{D}^{(i)}u$ exist and are in L^p. Note that $u \in L^p$ stands for an equivalence class of function which differ on a set of measure zero. The inequality

$$|u(x)| \leq c \|u\|_{H^{1,p}(\mathbb{R}^n)}$$

makes no sense on these classes.

6.6.2 Theorem (Hölder Continuous Representative)

Let $\Omega \subseteq \mathbb{R}^n$ be bounded with C^1-boundary. Assume $n < p \leq \infty$ and $u \in H^{1,p}(\Omega)$. Then there is a representative u^* of u in $H^{1,p}(\Omega)$, which is Hölder continuous $u \in C^{0,\gamma}\left(\overline{\Omega}\right)$ with $\gamma = 1 - \frac{n}{p}$ and:

$$\|u^*\|_{C^{0,\gamma}} \leq c \|u\|_{H^{1,p}(\Omega)}$$

In what follows, we always choose this representative and omit the star.

Proof

For $u \in H^{1,p}(\Omega)$ there are $u_m \in C_0^1(\mathbb{R}^n)$ with $u_m\big|_\Omega \to u$ converging in $H^{1,p}(\Omega)$ and

$$\|u_m\|_{H^{1,p}(\mathbb{R}^n)} \leq c \|u_m\|_{H^{1,p}(\Omega)}$$

just as in the proof of Theorem 6.5.3.
According to Theorem 6.6.1, the Morrey inequalities hold:

$$\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u_m - u_l\|_{H^{1,p}(\mathbb{R}^n)} \leq cC \|u_m - u_l\|_{H^{1,p}(\Omega)} \xrightarrow{m,l\to\infty} 0$$

Thus u_l is a Cauchy sequence in $C^{0,\gamma}\left(\overline{\Omega}\right)$. Hence converges $u_l \to u^* \in C^{0,\gamma}\left(\overline{\Omega}\right)$.

$$\|u^*\|_{C^{0,\gamma}\left(\overline{\Omega}\right)} \xleftarrow{l\to\infty} \|u_l\|_{C^{0,\gamma}\left(\overline{\Omega}\right)} \leq c \|u_l\|_{H^{1,p}(\Omega)} \xrightarrow{l\to\infty} c \|u\|_{H^{1,p}(\Omega)}$$

$\square_{6.6.2}$

7 Construction of Weak Solutions of Linear Elliptic PDEs

Let $\Omega \subseteq \mathbb{R}^n$ be a open domain with smooth boundary and $n \in \mathbb{N}_{>2}$. Consider the linear partial differential operator in divergence form:

$$Lu = -\frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^j}u(x)\right) + c(x)\,u(x)$$

Assume that the a^{ij} are uniformly elliptic, i.e. for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$ holds:

$$a^{ij}(x)\,\xi_i\xi_j \geq c\,\|\xi\|^2$$

Also assume that the coefficients are smooth and uniformly bounded $\left|a^{ij}\right|, |c| \leq K$. The first order term is missing!
We consider the Dirichlet problem:

$$\begin{aligned} Lu &= f \in C^\infty\left(\overline{\Omega}\right) \quad \text{in } \Omega \\ u\big|_{\partial\Omega} &= u_0 \in C^\infty(\partial\Omega) \end{aligned}$$

Examples:
- Poisson equation in $\Omega \subseteq \mathbb{R}^n$
- Let (M, g) be a Riemannian manifold and $\Omega \subseteq M$ be contained in a coordinate chart (x, U).

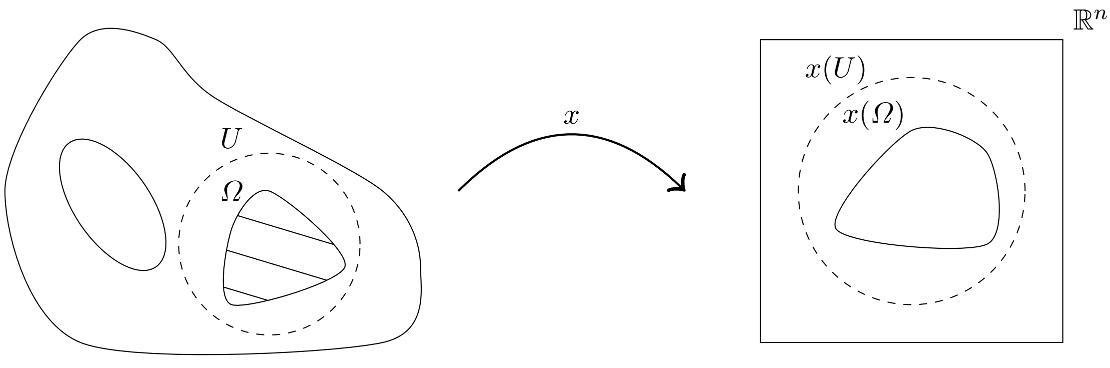


Figure 7.1: The chart maps Ω to $\mathbb{R}^n$.

The Laplace-Beltrami-Operator is:

$$\Delta\backslash u = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}g^{ij}\frac{\partial}{\partial x^j}u\right)$$

$$\Delta\backslash u = f \qquad \Leftrightarrow \qquad \frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}g^{ij}\frac{\partial}{\partial x^j}u\right) = \sqrt{\det(g)}f = \tilde{f}$$

This is in the desired divergence form.

Reduction to the case $u_0 = 0$: Let $v \in C^\infty\left(\overline{\Omega}\right)$ with $v\big|_{\partial\Omega} = u_0$. Set $\tilde{u} := u - v$.

$$\begin{aligned} L\tilde{u} &= f - Lv =: \tilde{f} \\ \tilde{u}\big|_{\partial\Omega} &= 0 \end{aligned}$$

A solution $\tilde{u}$ of this Dirichlet problem yields a solution $u = \tilde{u} + v$ of the original problem. Therefore, from now on we consider the problem:

$$\begin{aligned} Lu &= f \\ u\big|_{\partial\Omega} &= 0 \end{aligned}$$

7.1 The Dirichlet Principle

Define the *action*:

$$S := \int\left(a^{ij}(x)\left(\frac{\partial}{\partial x^i}u(x)\right)\left(\frac{\partial}{\partial x^j}u(x)\right) + c(x)\,u^2(x) - 2f(x)\,u(x)\right)\mathrm{d}^n x$$

Suppose u is a critical point (for example a minimizer), then holds for $\eta \in C_0^\infty(\Omega)$ formally:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}S(u + \tau\eta)\big|_{\tau=0} = \\ &= \int_\Omega\left(2a^{ij}(x)\left(\frac{\partial}{\partial x^i}u(x)\right)\left(\frac{\partial}{\partial x^j}\eta(x)\right) + 2c(x)\,u(x)\,\eta(x) - 2f(x)\,\eta(x)\right)\mathrm{d}^n x = \\ &\overset{\substack{\text{integration}\\ \text{by parts}}}{=} 2\int_\Omega \eta(x)\underbrace{\left(-\frac{\partial}{\partial x^j}\left(a^{ij}(x)\left(\frac{\partial}{\partial x^i}u(x)\right)\right) + c(x)\,u(x) - f(x)\right)}_{=Lu-f}\mathrm{d}^n x \end{aligned}$$

This has to hold for any η and thus follows $Lu - f = 0$.

Assume $u \in H^{1,2}(\Omega)$ for the integral over the two partial derivatives in the action being well defined. To ensure the boundary value $u\big|_{\partial\Omega} = 0$ we assume even $u \in H_0^{1,2}(\Omega)$. Then holds:

$$\begin{aligned} &\int_\Omega \underbrace{a^{ij}(\partial_i u)(\partial_j u)}_{\geq 0}\mathrm{d}^n x \leq \sup_\Omega\left(\left|a^{ij}\right|\right)\int_\Omega \|\mathrm{D}u\|^2\,\mathrm{d}^n x \leq \sup_\Omega\left(\left|a^{ij}\right|\right)\|u\|^2_{H_0^{1,2}(\Omega)} < \infty \\ &\int_\Omega \left|c(x)\,u^2(x)\right|\mathrm{d}^n x \leq \sup_\Omega(|c|)\int_\Omega u^2(x)\,\mathrm{d}^n x \leq \sup_\Omega(|c|)\,\|u\|^2_{H_0^{1,2}(\Omega)} < \infty \\ &\int_\Omega 2\,|f(x)\,u(x)|\,\mathrm{d}^n x \leq 2\,\|f\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)} \leq 2\,\|f\|_{L^2(\Omega)}\,\|u\|_{H_0^{1,2}(\Omega)} < \infty \end{aligned}$$

We always assume that Ω is bounded. Then $f \in C^\infty\left(\overline{\Omega}\right)$ implies $f \in L^2(\Omega)$.

Furthermore, we know from the Gagliardo-Nirenberg estimate

$$H_0^{1,2}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

with $p = 2$ and:

$$\frac{1}{p^*} = \frac{1}{2} - \frac{1}{n} < \frac{1}{2}$$

$$p^* = \frac{2n}{n-2} > 2$$

This is an *imbedding* (*embedding*), i.e. an injective mapping.
If Ω has finite volume, which it has in our case, since we assume that Ω is bounded, it follows:

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega u^2(x)\,\mathrm{d}^n x\right)^{\frac{1}{2}} \leq \left(\int_\Omega u^{2\tilde{p}}(x)\,\mathrm{d}^n x\right)^{\frac{1}{2\tilde{p}}} (\mu(\Omega))^{\frac{1}{2\tilde{q}}} \leq \tilde{c}\,\|u\|_{L^{2\tilde{p}}(\Omega)}$$

We chose $\tilde{p}$ such that $2\tilde{p} = p^*$ and use Poincaré's inequality:

$$\|u\|_{L^2(\Omega)} \leq \tilde{c}\,\|u\|_{L^{p^*}(\Omega)} \leq \underbrace{\tilde{c}C}_{=:c}\,\|\mathrm{D}u\|_{L^2(\Omega)}$$

Thus the action can be viewed as a mapping:

$$S : H_0^{1,2}(\Omega) \to \mathbb{R}$$

7.1.1 Theorem (Action Bounded from Below)

S is bounded from below, i.e. $S(u) \geq -K \in \mathbb{R}_{\leq 0}$ for all $u \in H_0^{1,2}(\Omega)$.

Proof

The uniform ellipticity means:

$$a^{ij}(\partial_i u)(\partial_j u) \geq \underbrace{\lambda}_{>0}\,\|\mathrm{D}u\|^2$$

Moreover holds $C(x) \geq 0$. Therefore follows:

$$\begin{aligned}
S(u) &\geq \int_\Omega \lambda\,\|\mathrm{D}u\|^2\,\mathrm{d}^n x - \|f\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} \geq \\
&\overset{\text{Poincaré}}{\geq} \frac{\lambda}{c^2}\,\|u\|^2_{L^2(\Omega)} - \|f\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)} = \\
&= \frac{\lambda}{c^2}\left(\|u\|_{L^2(\Omega)} - \frac{c^2}{2\lambda}\,\|f\|_{L^2(\Omega)}\right)^2 - \frac{c^2}{4\lambda}\,\|f\|^2_{L^2(\Omega)} \geq \\
&\geq -\frac{c^2}{4\lambda}\,\|f\|^2_{L^2(\Omega)} =: -K
\end{aligned}$$

$\square_{7.1.1}$

We now choose a *minimizing sequence* (Minimalfolge), i.e. a sequence $u_n \in H_0^{1,2}(\Omega)$ with:

$$S(u_n) \to \inf_{H_0^{1,2}(\Omega)}(S)$$

Hope:

- A subsequence of the u_n converges (weakly?) in $H_0^{1,2}(\Omega)$ to $u \in H_0^{1,2}(\Omega)$.
- $S(u) = \lim_{n\to\infty} S(u_n) = \inf_{H_0^{1,2}(\Omega)} S$

Such a minimizer satisfies the following Euler-Lagrange equation:

7.1.2 Theorem (Euler-Lagrange equation)

If $u \in H^{1,2}(\Omega)$ satisfies $S(u) = \inf_{H_0^{1,2}(\Omega)}(S)$, then holds for all $\eta \in C_0^\infty(\Omega)$:

$$\int_\Omega \left(u \left(-\frac{\partial}{\partial x^i} a^{ij} \frac{\partial}{\partial x^j} \eta + c\eta \right) - f\eta \right) \mathrm{d}^n x = 0$$

Thus u is a weak solution of the equation $Lu = f$.
For $u \in C^2(\Omega)$, we can integrate by parts twice, which would give:

$$\int (Lu - f)\, \eta \mathrm{d}^n x = 0$$

Proof

$$\begin{aligned} S(u+\tau\eta) &= \int_\Omega a^{ij} \left(\partial_i (u+\tau\eta)\right) \left(\partial_j (u+\tau\eta)\right) + c\,(u+\tau\eta)^2 - 2f\,(u+\tau\eta)\, \mathrm{d}^n x = \\ &= \tau^2 \int_\Omega \left(a^{ij} (\partial_i \eta)(\partial_j \eta) + c\eta^2 \right) \mathrm{d}^n x + 2\tau \int_\Omega \left(a^{ij} (\partial_i u)(\partial_j \eta) + cu\eta - f\eta \right) \mathrm{d}^n x + \\ &\quad + \int_\Omega \left(a^{ij} (\partial_i u)(\partial_j u) + cu^2 - 2fu \right) \mathrm{d}^n x \end{aligned}$$

$S(\tau)$ has a minimum at $\tau = 0$. Therefore holds:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau} S(u+\tau\eta)\Big|_{\tau=0} = 2 \int_\Omega \left(a^{ij} (\partial_i u)(\partial_j \eta) + cu\eta - f\eta \right) \mathrm{d}^n x \\ 0 &= \int_\Omega \left(u\partial_i \left(-a^{ij} (\partial_j \eta) \right) + cu\eta - f\eta \right) \mathrm{d}^n x \end{aligned}$$

$\square_{7.1.2}$

7.2 Existence of Minimizers

Let (u_n) be a minimizing sequence. To show is $u_n \to u$ and:

$$S(u) = \kappa := \inf_{H_0^{1,2}(\Omega)} (S)$$

More generally, let $\mathcal{H}$ be a Hilbert space. (In our case we have $\mathcal{H} = H_0^{1,2}(\Omega)$.)

$$\begin{aligned} A(u,v) &:= \int_\Omega \left(a^{ij} (\partial_i u)(\partial_j v) + cuv \right) \mathrm{d}^n x \\ l(u) &:= -2 \int_\Omega f u \mathrm{d}^n x \\ S(u) &= A(u,u) + l(u) \end{aligned}$$

The bilinear map $A : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and the linear map $l : \mathcal{H} \to \mathbb{R}$ are continuous:

$$\begin{aligned} |A(u,v)| &\le |||A|||\, \|u\| \cdot \|v\| \\ |l(u)| &\le |||l|||\, \|u\| \end{aligned}$$

Moreover A is symmetric:

$$A(u,v) = A(v,u)$$

7.2.1 Definition (Coercive)

A is called coercive (koerziv, manchmal auch elliptisch) if there is a $\lambda \in \mathbb{R}_{>0}$ such that for all $u \in \mathcal{H}$ holds:

$$A(u,u) \geq \lambda \|u\|_{\mathcal{H}}^2$$

In our problem, this condition is fulfilled due to the uniform ellipticity:

$$A(u,u) = \int_\Omega a^{ij} (\partial_i u)(\partial_j u) \, \mathrm{d}^n x \underset{\text{ellipticity}}{\overset{\text{uniform}}{\geq}} \lambda \int \|\mathrm{D}u\|^2 \, \mathrm{d}^n x \geq$$
$$\overset{\text{Poincaré}}{\geq} \frac{\lambda}{2} \int \|\mathrm{D}u\|^2 \, \mathrm{d}^n x + \frac{\lambda}{2c^2} \int \|u\|^2 \, \mathrm{d}^n x \geq \tilde{c} \|u\|_{H_0^{1,2}(\Omega)}^2$$

7.2.2 Theorem (Unique Minimizer)

Let $(\mathcal{H}, \langle ., . \rangle)$ be a Hilbert space with norm $\|.\|_{\mathcal{H}}$, $V \subseteq \mathcal{H}$ closed and convex,

$$A : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

bilinear, symmetric, continuous and coercive and

$$l : \mathcal{H} \to \mathbb{R}$$

linear and continuous.
Then the action $S(u) := A(u,u) + l(u)$ has a unique minimizer in V.

Proof

The action is bounded from below:

$$S(v) \geq \lambda \|v\|_{\mathcal{H}}^2 - |||l||| \, \|v\|_{\mathcal{H}} \underset{\text{the square}}{\overset{\text{completing}}{\geq}} -\frac{|||l|||^2}{2\lambda}$$

Let $\kappa := \inf_V (S)$. We choose a minimizing sequence $u_n \in V$ with $S(u_n) \to \kappa$. We want to show $u_n \to u$. Then by continuity follows:

$$S(u) = \lim_{n\to\infty} (S(u_n)) = \kappa$$

Since $\mathcal{H}$ is a Hilbert space and thus complete and V is closed, we only have to verify the Cauchy condition, to show the existence. Because V is convex holds $\frac{1}{2}(u_n + u_m) \in V$. So we get:

$$\kappa = \inf_V (S) \leq S\left(\frac{u_n + u_m}{2}\right) = A\left(\frac{u_n + u_m}{2}, \frac{u_n + u_m}{2}\right) + l\left(\frac{u_n + u_m}{2}\right) =$$
$$= \frac{1}{4} A(u_n, u_n) + \frac{1}{4} A(u_m, u_m) + \frac{1}{2} A(u_n, u_m) + \frac{1}{2} l(u_n) + \frac{1}{2} l(u_m) =$$
$$= \frac{1}{2} S(u_n) + \frac{1}{2} S(u_m) - \frac{1}{4} A(u_n, u_n) - \frac{1}{4} A(u_m, u_m) + \frac{1}{2} A(u_n, u_m) =$$
$$= \underbrace{\frac{1}{2} S(u_n)}_{\to \frac{\kappa}{2}} + \underbrace{\frac{1}{2} S(u_m)}_{\to \frac{\kappa}{2}} \underbrace{- \frac{1}{4} A(u_n - u_m, u_n - u_m)}_{\leq 0}$$

Thus converges $A\left(u_n - u_m, u_n - u_m\right) \xrightarrow{n,m\to\infty} 0$.

$$\|u_n - u_m\|^2 \leq \frac{1}{\lambda} A\left(u_n - u_m, u_n - u_m\right) \xrightarrow{n,m\to\infty} 0$$

Hence (u_n) is a Cauchy sequence and $u_n \to u$.
Uniqueness: Let u_1, u_2 be minimizers, i.e. $S\left(u_1\right) = \kappa = S\left(u_2\right)$. Then holds:

$$\kappa \leq S\left(\frac{u_1 + u_2}{2}\right) = \underbrace{\frac{1}{2} S\left(u_1\right) + \frac{1}{2} S\left(u_2\right)}_{=\kappa} \underbrace{- \frac{1}{4} A\left(u_1 - u_2, u_1 - u_2\right)}_{\leq 0}$$

$$\begin{aligned} \Rightarrow \quad & A\left(u_1 - u_2, u_1 - u_2\right) = 0 \\ & \Rightarrow \quad u_1 = u_2 \end{aligned}$$

$\square_{7.2.2}$

7.2.3 Corollary

Under the conditions of Theorem 7.2.2 and if in addition V is a closed linear subspace of $\mathcal{H}$, there is a unique $u \in V$ such that for all $\varphi \in V$ holds:

$$2A\left(u, \varphi\right) + l\left(\varphi\right) = 0$$

Proof

For a minimizer u, $S\left(u + \tau\varphi\right)$ is minimal at $\tau = 0$. This is a polynomial in τ and thus holds:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} S\left(u + \tau\varphi\right)\big|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(A\left(u + \tau\varphi, u + \tau\varphi\right) + l\left(u + \tau\varphi\right)\right)\big|_{\tau=0} = 2A\left(u, \varphi\right) + l\left(\varphi\right)$$

$\square_{7.2.3}$

8 Convex Variational Problems

(cf. J. Jost: *Partial Differential Equations,* Section 8.6)
We consider the action

$$S(u) := \int_{\Omega} f(x, \mathrm{D}u)\, \mathrm{d}^n x$$

for $u \in H_0^{1,2}(\Omega)$. $\Omega \subseteq \mathbb{R}^n$ is assumed to be bounded with smooth boundary.

8.1 Theorem (Existence of Minimizer for S)

Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a function with the following properties:

i) $f(., v) : \Omega \to \mathbb{R}$ is measurable for any $v \in \mathbb{R}^n$.

ii) $f(x, .) : \mathbb{R}^n \to \mathbb{R}$ is convex for all $x \in \Omega$, i.e. for all $v, w \in \mathbb{R}^n$ and $\tau \in [0, 1]$ holds:

$$f(x, \tau v + (1-\tau) w) \leq \tau f(x, v) + (1-\tau) f(x, w)$$

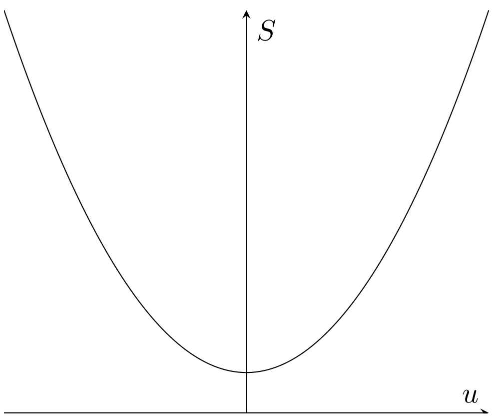
iii) Coercivity: There is a constant $\kappa \in \mathbb{R}_{>0}$ and an integrable function $\gamma \in L^1(\Omega)$ such that for all $x \in \Omega$ and $v \in \mathbb{R}^n$ holds:

$$f(x, v) \geq -\gamma(x) + \kappa \|v\|^2$$

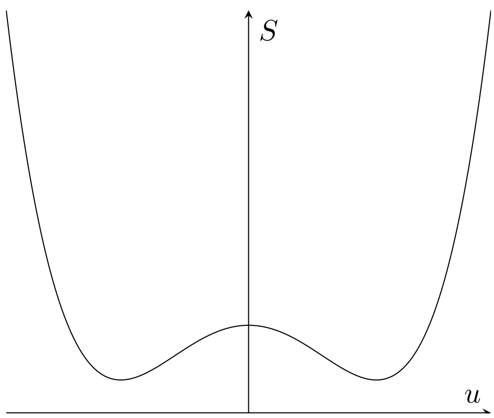
Then S attains its minimum, i.e. there is a $u_0 \in H_0^{1,2}(\Omega)$ with:

$$S(u_0) = \inf_{u \in H_0^{1,2}(\Omega)} (S(u))$$

Convexity is crucial for uniqueness.



(a) convex: unique minimum



(b) not convex: two minima

Figure 8.1: one-dimensional analogon

Convexity and coercivity are also essential for proving existence.

$$\begin{aligned} S : H_0^{1,2}(\Omega) &\to \mathbb{R} \cup \{\infty\} \\ u &\mapsto \int_\Omega f(x, \mathrm{D}u)\, \mathrm{d}^n x = \underbrace{\int_\Omega \underbrace{(f(x, \mathrm{D}u) + \gamma)}_{\geq 0} \mathrm{d}^n x}_{\text{could be } \infty} - \int_\Omega \underbrace{\gamma}_{\in L^1} \mathrm{d}^n x \end{aligned}$$

Therefore S is bounded from below.

8.2 Lemma (Lower Semi-Continuous)

The functional

$$J(v) = \int_\Omega f(x, v(x))\, \mathrm{d}^n x$$

for $v \in L^2(\Omega, \mathbb{R}^n)$ is lower semi-continuous, i.e. if $v_k \to v$ converges in $L^2(\Omega, \mathbb{R}^n)$, then holds:

$$J(v) \leq \liminf_{k\to\infty} J(v_k)$$

Proof

The function $v \in L^2(\Omega)$ is clearly measurable, and since $f(x, .)$ is continuous (every convex function is continuous) and $f(., v)$ is measurable, it follows that $f(x, v(x))$ is measurable.

In more detail: Introduce a new function:

$$\begin{aligned} g : \Omega \times \Omega &\to \mathbb{R} \\ g(x, y) &= f(x, v(y)) \end{aligned}$$

$g(x, .)$ is measurable, because $f(x, .)$ is measurable, and $g(., y)$ is measurable as a composition of a measurable function with a continuous function. Let

$$\begin{aligned} \iota : \Omega &\to \Omega \times \Omega \\ x &\mapsto (x, x) \end{aligned}$$

be the natural injection, which is continuous. So the composition $f(x, v(x)) = g(\iota(x))$ is measurable, because g is measurable and ι continuous.

Let $v_n \to v$ converge in $L^2(\Omega, \mathbb{R}^n)$, then $v_k(x) \to v(x)$ converges for almost all $x \in \Omega$. Since $f(x, .)$ and $\|.\|$ are continuous, it holds:

$$f(x, v_k(x)) - \kappa \|v_k(x)\|^2 \to \underbrace{f(x, v(x)) - \kappa \|v(x)\|}_{\geq -\gamma(x) \in L^1(\Omega)}$$

Question:

$$\lim_{k\to\infty} \int f(x, v_k(x)) - \kappa \|v(x)\|^2 \mathrm{d}^n x \stackrel{?}{=} \int f(x, v(x)) - \kappa \|v(x)\|^2 \mathrm{d}^n x$$

In general this is wrong, because the integrand is not necessarily bounded from above. But Fatou's lemma yields:

$$\liminf_{k\to\infty} \int f(x, v_k(x)) - \kappa \|v_k(x)\|^2 \mathrm{d}^n x \geq \int f(x, v(x)) - \kappa \|v(x)\|^2 \mathrm{d}^n x$$

Example to illuminate Fatou's Lemma:

$$g_k(x) = \sqrt{k} e^{-kx^2}$$

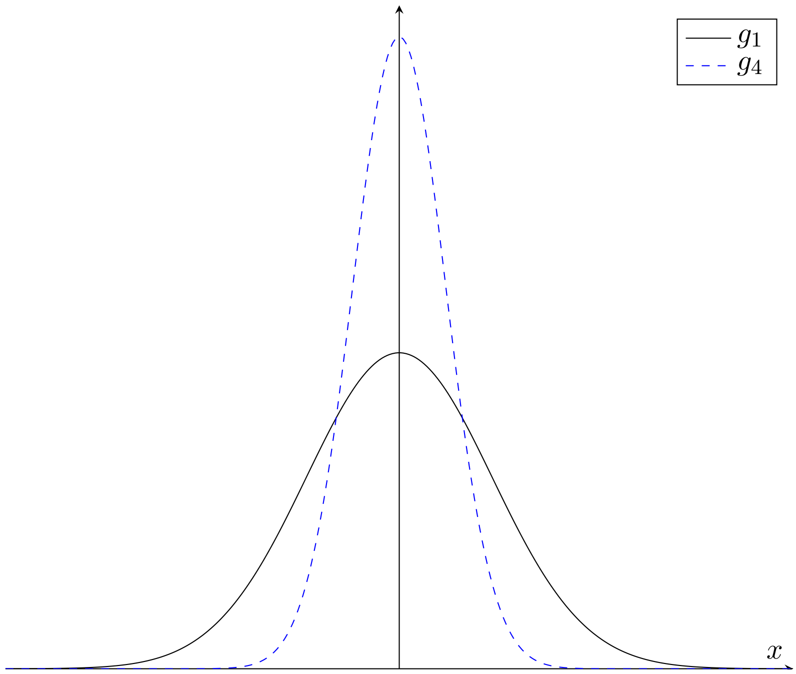


Figure 8.2: Example for Fatou's Lemma

$g_k(x) \to 0$ converges almost everywhere, but the integral does not converge to zero:

$$\int g_k(x)\,\mathrm{d}^n x = c > 0 = \int 0 \mathrm{d}^n x$$

Fatou's Lemma yields:

$$\liminf_{k\to\infty} \left(J(v_k) - \kappa \int_\Omega \|v_k\|^2 \,\mathrm{d}^n x \right) \geq J(v) - \kappa \int_\Omega \|v(x)\|^2 \,\mathrm{d}^n x$$

Since $v_k \to v$ converges in $L^2(\Omega)$, also $\int_\Omega \|v_k\|^2 \,\mathrm{d}^n x \to \int_\Omega \|v\|^2 \,\mathrm{d}^n x$ converges and thus follows:

$$\liminf_{k\to\infty} J(v_k) \geq J(v)$$

$\square_{8.2}$

8.3 Lemma (Convex)

J is convex.

Proof

$$J(\tau v + (1-\tau) w) = \int_\Omega f(x, \tau v(x) + (1-\tau) w(x))\,\mathrm{d}^n x \leq$$

$$\begin{aligned}&\le \int_\Omega \tau f\left(x, v\left(x\right)\right) + \left(1-\tau\right) f\left(x, w\left(x\right)\right) \mathrm{d}^n x = \\ &= \tau J\left(v\right) + \left(1-\tau\right) J\left(w\right)\end{aligned}$$

$\square_{8.3}$

To summarize:

8.4 Lemma

The functional

$$S\left(u\right) = \int_\Omega f\left(x, \mathrm{D}u\right) \mathrm{d}^n x = J\left(\mathrm{D}u\right)$$

is convex and lower semi-continuous on $H_0^{1,2}\left(\Omega\right)$.

Proof

For $u \in H_0^{1,2}\left(\Omega\right)$ follows $\mathrm{D}u \in L^2\left(\Omega\right)$ and the claim follows from the Lemmata 8.2 and 8.3. $\square_{8.4}$

8.5 Theorem (Existence of Minimizer for S_λ)

Let $\left(\mathcal{H}, \langle ., . \rangle\right)$ be a Hilbert space and

$$S : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$$

be a functional which is bounded from below and $S \neq \infty$. For every $\lambda \in \mathbb{R}_{>0}$ and $u \in \mathcal{H}$, the functional

$$S_\lambda\left(y\right) := S\left(y\right) + \lambda \left\|u - y\right\|^2$$

has a unique minimizer $u_\lambda \in \mathcal{H}$, i.e.:

$$S\left(u_\lambda\right) + \lambda \left\|u - u_\lambda\right\|^2 = \inf_{y \in \mathcal{H}} S_\lambda\left(y\right)$$

If $\left\|u_\lambda\right\|$ is bounded in the limit $\lambda \searrow 0$, then

$$\lim_{\lambda \searrow 0} u_\lambda := u_0$$

exists and u_0 is a minimizer of S, i.e.:

$$S\left(u_0\right) = \inf_{y \in \mathcal{H}} S\left(y\right)$$

Note: u_0 may still depend on u, so the uniqueness is unclear at the moment. (Later we prove uniqueness.)
Before coming to the proof of the theorem, let us apply it to our variational problem.

Proof of Theorem 8.1

$$S(y) = \int_\Omega f(x, \mathrm{D}y)\, \mathrm{d}^n x$$

For $S \neq \infty$ we have nothing to prove, so we assume $S \neq \infty$.

1. S is bounded from below: By coercivity holds:

$$S(y) \geq \int_\Omega \left(-\gamma(x) + \kappa \|\mathrm{D}y\|^2\right) \mathrm{d}^n x$$

Now we use Poincaré's inequality, which holds, since Ω is bounded.

$$\|y\|^2_{H_0^{1,2}(\Omega)} \leq C(\Omega) \int_\Omega \|\mathrm{D}y\|^2\, \mathrm{d}^n x \overset{\kappa>0}{\leq} \frac{C(\Omega)}{\kappa} \left(S(y) + \underbrace{\int_\Omega \gamma(x)\, \mathrm{d}^n x}_{=:\tilde{C}<\infty}\right)$$

Thus follows:

$$S(y) \geq \frac{\kappa}{C(\Omega)} \|y\|_{H_0^{1,2}(\Omega)} - \tilde{C} \geq -\tilde{C}$$

2. $\|u_\lambda\|$ stays bounded for $\lambda \searrow 0$:

$$\begin{aligned}\|u_\lambda\|^2_{H_0^{1,2}(\Omega)} &\leq \frac{C(\Omega)}{\kappa} \left(S(u_\lambda) + \tilde{C}\right) \leq \frac{C(\Omega)}{\kappa} \Big(\underbrace{S(u_\lambda) + \lambda \|u - u_\lambda\|^2}_{=S_\lambda(u_\lambda)} + \tilde{C}\Big) \leq \\ &\overset{S_\lambda(u_\lambda) \leq S_\lambda(u)}{\leq} \frac{C(\Omega)}{\kappa} \left(S(u) + \lambda \|u - u\|^2 + \tilde{C}\right) = \frac{C(\Omega)}{\kappa} \left(S(u) + \tilde{C}\right)\end{aligned}$$

This upper bound is uniform in λ.

This completes the proof, assuming that Theorem 8.5 holds. $\square_{8.1}$

Proof of Theorem 8.5

Let y_k be a minimizing sequence of the functional $S(y) + \lambda \|u - y\|^2$. Set:

$$\begin{aligned} y_{k,l} &:= \frac{1}{2}(y_k + y_l) \\ s_\lambda &:= \inf_{y \in \mathcal{H}} \left(S(y) + \lambda \|u - y\|^2\right)\end{aligned}$$

Then holds:

$$\begin{aligned}\|u - y_{k,l}\|^2 &= \left\|u - \frac{1}{2}(y_k + y_l)\right\|^2 = \left\|\frac{1}{2}(u - y_k) + \frac{1}{2}(u - y_l)\right\|^2 = \\ &= \frac{1}{4} \|(u - y_k) + (u - y_l)\|^2 = \\ &\overset{\text{parallelogram}}{\underset{\text{law}}{=}} -\frac{1}{4} \|(u - y_k) - (u - y_l)\|^2 + \frac{1}{2} \|u - y_k\|^2 + \frac{1}{2} \|u - y_l\|^2 = \\ &= -\frac{1}{4} \|y_l - y_k\|^2 + \frac{1}{2} \|u - y_k\|^2 + \frac{1}{2} \|u - y_l\|\end{aligned}$$

So we get:

$$\begin{aligned}
s_\lambda &\leq S\left(y_{k,l}\right) + \lambda \left\|u - y_{k,l}\right\|^2 \leq \\
&\overset{S \text{ convex}}{\leq} \frac{1}{2} S\left(y_k\right) + \frac{1}{2} S\left(y_l\right) + \frac{\lambda}{2}\left\|u - y_k\right\|^2 + \frac{\lambda}{2}\left\|u - y_l\right\|^2 - \frac{\lambda}{4}\left\|y_k - y_l\right\|^2 = \\
&= \underbrace{\frac{1}{2}\left(S\left(y_k\right) + \lambda\left\|u - y_k\right\|^2\right)}_{\to \frac{1}{2} s_\lambda} + \underbrace{\frac{1}{2}\left(S\left(y_l\right) + \lambda\left\|u - y_l\right\|^2\right)}_{\to \frac{1}{2} s_\lambda} \underbrace{- \frac{\lambda}{4}\left\|y_k - y_l\right\|^2}_{\leq 0}
\end{aligned}$$

Therefore converges $y_k - y_l \xrightarrow{k,l \to \infty} 0$ and thus $y_k \to u_\lambda$. Now u_λ is a minimizer, because S is lower semi-continuous.

The limit u_λ is unique, because every minimizing sequence converges, so for another minimizer v_λ we consider the sequence $\left(u_\lambda, v_\lambda, u_\lambda, v_\lambda, \ldots\right)$ and see $u_\lambda = v_\lambda$.

For $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ with $\lambda_1 < \lambda_2$ holds, because u_{λ_1} is the minimizer of $S\left(.\right) + \lambda_1\left\|u - .\right\|^2$.

$$\begin{aligned}
S\left(u_{\lambda_2}\right) + \lambda_1\left\|u_{\lambda_2} - u\right\|^2 &\geq S\left(u_{\lambda_1}\right) + \lambda_1\left\|u_{\lambda_1} - u\right\| \qquad / + \left(\lambda_2 - \lambda_1\right)\left\|u - u_{\lambda_2}\right\|^2 \\
S\left(u_{\lambda_2}\right) + \lambda_2\left\|u - u_{\lambda_2}\right\|^2 &\geq S\left(u_{\lambda_1}\right) + \lambda_2\left\|u - u_{\lambda_1}\right\|^2 + \left(\lambda_1 - \lambda_2\right)\left(\left\|u - u_{\lambda_1}\right\|^2 - \left\|u - u_{\lambda_2}\right\|^2\right)
\end{aligned}$$

Now holds, because u_{λ_2} is the minimizer of $S\left(.\right) + \lambda_2\left\|u - .\right\|^2$.

$$S\left(u_{\lambda_2}\right) + \lambda_2\left\|u - u_{\lambda_2}\right\|^2 \leq S\left(u_{\lambda_1}\right) + \lambda_2\left\|u - u_{\lambda_1}\right\|^2$$

In total we get:

$$\begin{aligned}
S\left(u_{\lambda_1}\right) + \lambda_2\left\|u - u_{\lambda_1}\right\|^2 &\geq S\left(u_{\lambda_1}\right) + \lambda_2\left\|u - u_{\lambda_1}\right\|^2 + \left(\lambda_1 - \lambda_2\right)\left(\left\|u - u_{\lambda_1}\right\|^2 - \left\|u - u_{\lambda_2}\right\|^2\right) \\
0 &\geq \underbrace{\left(\lambda_1 - \lambda_2\right)}_{<0}\left(\left\|u - u_{\lambda_1}\right\|^2 - \left\|u - u_{\lambda_2}\right\|^2\right) \\
0 &\leq \left\|u - u_{\lambda_1}\right\|^2 - \left\|u - u_{\lambda_2}\right\|^2 \\
\left\|u - u_{\lambda_2}\right\|^2 &\leq \left\|u - u_{\lambda_1}\right\|^2
\end{aligned}$$

Thus $\left\|u - u_\lambda\right\|$ is increasing as $\lambda \searrow 0$.

By assumption holds uniformly in λ:

$$\left\|u - u_\lambda\right\| \leq \left\|u\right\| + \left\|u_\lambda\right\| < C$$

Thus $\left\|u - u_\lambda\right\|$ converges as $\lambda \searrow 0$, i.e. for all $\varepsilon \in \mathbb{R}_{>0}$ exists a λ_0 such that for all $\lambda_1, \lambda_2 \in \left(0, \lambda_0\right)$ holds:

$$\left|\left\|u - u_{\lambda_1}\right\|^2 - \left\|u - u_{\lambda_2}\right\|^2\right| < \frac{\varepsilon}{2}$$

Now again we use the convexity. Define:

$$u_{1,2} := \frac{1}{2}\left(u_{\lambda_1} + u_{\lambda_2}\right)$$

Without loss of generality we assume $S\left(u_{\lambda_1}\right) \geq S\left(u_{\lambda_2}\right)$ so we get:

$$\begin{aligned}
S\left(u_{1,2}\right) &\leq S\left(u_{\lambda_1}\right) \\
0 &\leq S\left(u_{\lambda_1}\right) - S\left(u_{1,2}\right)
\end{aligned}$$

$$
\begin{aligned}
0 &\le S\left(u_{\lambda_1}\right) + \lambda_1 \left\|u - u_{1,2}\right\|^2 - \left(S\left(u_{1,2}\right) + \lambda_1 \left\|u - u_{1,2}\right\|^2\right) = \\
&\overset{\text{parallelogram}}{\underset{\text{law}}{=}} S\left(u_{\lambda_1}\right) + \lambda_1 \left(\frac{1}{2}\left\|u - u_{\lambda_1}\right\|^2 + \frac{1}{2}\left\|u - u_{\lambda_2}\right\|^2 - \frac{1}{4}\left\|u_{\lambda_1} - u_{\lambda_2}\right\|^2\right) - S_{\lambda_1}\left(u_{1,2}\right) = \\
&= S\left(u_{\lambda_1}\right) + \lambda_1 \left\|u - u_{\lambda_1}\right\|^2 + \frac{\lambda_1}{2}\left(\left\|u - u_{\lambda_2}\right\|^2 - \left\|u - u_{\lambda_1}\right\|^2\right) - \frac{\lambda_1}{4}\left\|u_{\lambda_1} - u_{\lambda_2}\right\|^2 - S_{\lambda_1}\left(u_{1,2}\right) \le \\
&\le S_{\lambda_1}\left(u_{1,2}\right) + \frac{\lambda_1}{2}\left(\left\|u - u_{\lambda_2}\right\|^2 - \left\|u - u_{\lambda_1}\right\|^2\right) - \frac{\lambda_1}{4}\left\|u_{\lambda_1} - u_{\lambda_2}\right\|^2 - S_{\lambda_1}\left(u_{1,2}\right) = \\
&= \frac{\lambda_1}{2}\left(\left\|u - u_{\lambda_2}\right\|^2 - \left\|u - u_{\lambda_1}\right\|^2\right) - \frac{\lambda_1}{4}\left\|u_{\lambda_1} - u_{\lambda_2}\right\|^2 \le \\
&\le \frac{\lambda_1}{4}\varepsilon - \frac{\lambda_1}{4}\left\|u_{\lambda_1} - u_{\lambda_2}\right\|^2
\end{aligned}
$$

This gives:

$$\left\|u_{\lambda_1} - u_{\lambda_2}\right\|^2 \le \varepsilon$$

Since ε is arbitrary, we conclude that the limit $\lim\limits_{\lambda \searrow 0} u_\lambda =: u_0$ exists.

By lower semi-continuity of S follows:

$$S\left(u_0\right) \le \lim_{\lambda \searrow 0} S\left(u_\lambda\right) = \inf_{u \in \mathcal{H}} S(u)$$

This implies:

$$S\left(u_0\right) = \inf_{u \in \mathcal{H}} S(u)$$

$\square_{8.5}$

9 The Finite Element Method

We return to the action of Chapter 7.

$$S(u) = A(u,u) + l(u)$$

- $|A(u,v)| \leq |||A||| \cdot \|u\| \cdot \|v\|$, $A(u,u) \geq \lambda \|u\|^2$, $\lambda \in \mathbb{R}_{>0}$
- $|l(u)| \leq |||l||| \cdot \|u\|$

We showed that there is a unique minimizer $u \in \mathcal{H}$.

Question: How can we approximate u?

To this end, we choose $V \subseteq \mathcal{H}$ finite-dimensional (e.g. triangulations, lattice approximations, wavelets). Then we can minimize $S\big|_V : V \to \mathbb{R}$. The minimizer can be found numerically.

Typically, one considers $V_1 \subseteq V_2 \subseteq \ldots \subseteq \mathcal{H}$, a *filtration* (Ausschöpfung) of $\mathcal{H}$.

9.1 Lemma

Let u be the minimizer of S on $\mathcal{H}$ (unknown!) and u_V the minimizer of S on V (known!). Then holds:

$$\|u - u_V\| \leq \frac{|||A|||}{\lambda} \inf_{v \in V} \|u - v\|$$

Proof

The Euler-Lagrange equations for $\varphi \in \mathcal{H}$ are:

$$2A(u,\varphi) + l(\varphi) = 0$$

If we minimize only on the subspace V, we get for $\varphi \in V$.

$$2A(u_V,\varphi) + l(\varphi) = 0$$

(We consider only variations $u_V + \tau\varphi$ with $\varphi \in V$.) Taking the difference gives for $\varphi \in V$:

$$A(u - u_V, \varphi) = 0$$

For $v \in V$ holds:

$$\begin{aligned}\|u - u_V\|^2 &\overset{\text{coercive}}{\leq} \frac{1}{\lambda} A(u - u_V, u - u_V) = \frac{1}{\lambda} A(u - u_V, u - u_V - v + v) = \\ &\overset{\text{linearity}}{=} \frac{1}{\lambda} A(u - u_V, u - v) + \frac{1}{\lambda} \underbrace{A(u - u_V, \underbrace{v - u_V}_{=:\varphi \in V})}_{=0} = \frac{1}{\lambda} A(u - u_V, u - v) \leq\end{aligned}$$

$$\overset{\text{boundedness}}{\leq} \frac{|||A|||}{\lambda} \|u - u_V\| \cdot \|u - v\|$$

So for all $v \in V$ we get:

$$\|u - u_V\| \leq \frac{|||A|||}{\lambda} \|u - v\|$$

Taking the infimum over $v \in V$ gives the result. $\square_{9.1}$

For a filtration converges $\inf_{v \in V_n} \|u - v\| \xrightarrow{n \to \infty} 0$ and thus $u_{V_n} \to u$.

10 Imbedding Theorems

Recapitulation: Let $\Omega \subseteq \mathbb{R}^n$ be bounded with C^1-boundary.

- Gagliardo-Nirenberg inequality for functions $u \in H_0^{1,p}(\Omega)$ vanishing on the boundary:

$$\|u\|_{L^{p^*}(\Omega)} \leq C(n)\,\|\mathrm{D}u\|_{L^p(\Omega)}$$
$$p^* = \frac{np}{n-p}$$

- On $H^{1,p}(\Omega)$ without zero boundary conditions for $1 \leq p < n$:

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega)\,\|u\|_{H^{1,p}(\Omega)}$$

  Using Hölder one finds that the inequalities also hold, if the p^* is replaced by $q < p^*$.

- Morrey inequalities for $n < p < \infty$ and $u \in H^{1,p}(\Omega)$:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq c\,\|u\|_{H^{1,p}(\Omega)}$$
$$\gamma = 1 - \frac{n}{p}$$

We have:

$$H^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{p^*}(\Omega) & 1 \leq p < n \\ C^{0,\gamma}\left(\overline{\Omega}\right) & n < p \quad \text{(imbedding of equivalence classes)} \end{cases}$$

Question: Are these imbeddings compact?

Definition: For Banach spaces E, F, a linear map $A : E \to F$ is called *compact*, if A maps bounded sets to relatively compact sets, i.e. sets, whose closure is compact.

One can compose the imbeddings:

$$H^{2,p}(\Omega) \hookrightarrow H^{1,p^*}(\Omega) \hookrightarrow \begin{cases} L^{p^{**}}(\Omega) \\ C^{0,\tilde{\gamma}}(\Omega) \end{cases}$$

10.1 Theorem (Generalized Gagliardo-Nirenberg/Morrey)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with C^k-boundary and $u \in H^{k,p}(\Omega)$.

i) For $k < \frac{n}{p}$ holds $u \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and:

$$\|u\|_{L^q(\Omega)} \leq C(\Omega)\,\|u\|_{H^{k,p}(\Omega)}$$

ii) For $k > \dfrac{n}{p}$ holds $u \in C^{k-\left\lfloor \frac{n}{p} \right\rfloor+1,\gamma}(\Omega)$ with

$$\gamma \begin{cases} = \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \in (0,1) & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases}$$

and:

$$\|u\|_{H^{k-\left\lfloor \frac{n}{p} \right\rfloor-1,\gamma}(\Omega)} \leq C(\Omega)\, \|u\|_{H^{k,p}(\Omega)}$$

Proof

i) Consider $u \in H^{k,p}(\Omega)$ and $k \in \mathbb{N}_{\geq 1}$. Then for all multi-indices α with $|\alpha| \leq k$ holds $\mathrm{D}^\alpha u \in L^p(\Omega)$. The Gagliardo-Nirenberg inequality gives for all multi-indices β with $|\beta| \leq k-1$:

$$\left\|\mathrm{D}^\beta u\right\|_{L^{p^*}(\Omega)} \leq c\, \|u\|_{H^{1,p}(\Omega)}$$

Therefore holds $u \in H^{k-1,p^*}(\Omega)$ and:

$$\|u\|_{H^{k-1,p^*}} \leq C\, \|u\|_{H^{k,p}(\Omega)}$$
$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

Iterating gives $u \in H^{k-2,p^{**}}(\Omega)$ and:

$$\|u\|_{H^{k-2,p^{**}}(\Omega)} \leq C\, \|u\|_{H^{k,p}(\Omega)}$$
$$\frac{1}{p^{**}} = \frac{1}{p*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$$

After k steps we get $u \in H^{0,q}(\Omega) = L^q(\Omega)$ and:

$$\|u\|_{L^q(\Omega)} \leq C\, \|u\|_{H^{k,p}(\Omega)}$$
$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

We always need $p < n$. Due to $p^* > p$ this means $\frac{1}{p} - \frac{k-1}{n} > \frac{1}{n}$, i.e. $p^{-1} > \frac{k}{n}$, $k < \frac{n}{p}$.

ii) Case $\frac{n}{p} \notin \mathbb{N}$: Proceed as before

$$H^{k,p}(\Omega) \hookrightarrow H^{k-1,p^*}(\Omega) \hookrightarrow \ldots \hookrightarrow H^{k-l,r}(\Omega)$$

$$\frac{1}{r} = \frac{1}{p} - \frac{l}{n}$$
$$r = \frac{pn}{n-pl}$$

until $r > n$. (If this does not happen, we are in case i).)

$$r = \frac{pn}{n-pl} > n$$

$$\begin{aligned} p &> n - pl \\ l + 1 &> \frac{n}{p} \end{aligned}$$

We have to choose the smallest l, since as soon as $r > n$ occurs, we cannot continue our iteration. Thus we have:

$$\begin{aligned} & l \leq \frac{n}{p} < l + 1 \\ \Rightarrow \qquad & l = \left\lfloor \frac{n}{p} \right\rfloor \end{aligned}$$

Morrey's inequality yields for all multi-indices α with $|\alpha| \leq k - l - 1$:

$$\mathrm{D}^\alpha u \in C^{0,1-\frac{n}{r}}\left(\overline{\Omega}\right)$$

(For $k - l = 1$ we know $u \in H^{1,r} \hookrightarrow C^{0,1-\frac{n}{r}}$ and for $k - l > 1$ holds $\mathrm{D}^\alpha u \in H^{1,r} \hookrightarrow C^{0,1-\frac{n}{r}}$ for all multi-indices α with $|\alpha| \leq k - l - 1$.)
We get $u \in C^{q,1-\frac{n}{r}}\left(\overline{\Omega}\right)$ with:

$$\begin{aligned} q = k - l - 1 &= k - \left\lfloor \frac{n}{p} \right\rfloor - 1 \\ 1 - \frac{n}{r} = 1 - \left(\frac{n}{p} - l\right) &= 1 - \frac{n}{p} + \left\lfloor \frac{n}{p} \right\rfloor \end{aligned}$$

iii) Case $\frac{n}{p} \in \mathbb{N}$: Proceed as in case ii).

$$H^{k,p}(\Omega) \hookrightarrow \ldots \hookrightarrow H^{k-l,r}(\Omega)$$

$$\frac{1}{r} = \frac{1}{p} - \frac{l}{n} = \frac{1}{n}\left(\frac{n}{p} - l\right)$$

We proceed until $\frac{n}{p} - l = 1$, i.e. $l = \frac{n}{p} - 1$, $\frac{1}{r} = \frac{1}{n}$. Thus holds $u \in H^{k-l,n}(\Omega)$.
Applying the Gagliardo-Nirenberg inequality, we conclude $u \in H^{k-l-1,q}$ for all $n \leq q < \infty$. Thus for all multi-indices α with $|\alpha| \leq k - l - 1 = k - \frac{n}{p}$ holds $\mathrm{D}^\alpha u \in L^q(\Omega)$. Applying Morrey's inequality yields for all $n < q < \infty$ and $|\alpha| \leq k - \frac{n}{p} - 1$:

$$\mathrm{D}^\alpha u \in C^{0,1-\frac{n}{q}}\left(\overline{\Omega}\right)$$

This implies $u \in C^{k-\frac{n}{p}-1,\gamma}\left(\overline{\Omega}\right)$ with $\gamma \in (0,1)$.

$\square_{10.1}$

10.2 Definition (compact imbedding)

Let X, Y be Banach spaces with an imbedding $X \hookrightarrow Y$ (also written as $X \subseteq Y$). Then the imbedding is called *compact* (notation $X \Subset Y$) if holds:

i) For all $u \in X$ holds:

$$\|u\|_Y \leq c\,\|u\|_X$$

ii) X-bounded sequences (u_n) (i.e. for all n holds $\|u_n\|_X \leq C$) have a convergent subsequence in Y.

10.3 Theorem (Arzelá-Ascoli)

Consider a sequence (f_k) of real-valued functions on a compact $K \subseteq \mathbb{R}^n$ with the properties.

1. The (f_k) are *uniformly bounded*, i.e. for all $k \in \mathbb{N}$ and $x \in K$ holds $|f_k(x)| \leq M$.
2. The (f_k) are *equicontinuous* (gleichgradig stetig), i.e. for all $\varepsilon \in \mathbb{R}_{>0}$ exists a $\delta \in \mathbb{R}_{>0}$ such that holds:

$$\|x - y\| < \delta \quad \Rightarrow \quad \underset{k \in \mathbb{N}}{\forall} |f_k(x) - f_k(y)| < \varepsilon$$

   Note that δ is independent of k.

Then (f_k) has a subsequence which converges uniformly, i.e. $f_k \rightrightarrows f$ converges uniformly in K or equivalently $f_k \to f$ converges in $C^0(K, \mathbb{R})$.

Proof

For $\varepsilon \in \mathbb{R}_{>0}$ choose a $\delta \in \mathbb{R}_{>0}$ satisfying the equicontinuity condition. Then for every ball $B_\delta(x)$ the following holds:

$$y \in B_{\delta(x)} \quad \Rightarrow \quad \underset{k \in \mathbb{N}}{\forall} |f_k(x) - f_k(y)| < \varepsilon$$

Since K is compact, we can cover it by a finite number of such balls $B_\delta(x_1), B_\delta(x_2), \ldots, B_\delta(x_L)$. The sequence $f_k(x_1)$ is bounded by the uniform boundedness condition and thus has a convergent subsequence. Iteratively choosing subsequences (For simplicity we denote the subsequences again by f_k.) we can arrange that $f_k(x_l) \xrightarrow{k \to \infty} f(x_l)$ converges for all $l \in \{1, \ldots, L\}$. In particular there is a $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}_{>N}$ and all $l \in \{1, \ldots, L\}$ holds:

$$|f_k(x_l) - f(x_l)| < \varepsilon$$

For $y \in K$ holds $y \in B_\delta(x_l)$ for some l, since the balls cover K. Therefore holds for $j, k \in \mathbb{N}_{>N}$:

$$|f_j(y) - f_k(y)| \leq \underbrace{|f_j(y) - f_j(x_l)|}_{<\varepsilon \text{ (equicontinuity)}} + \underbrace{|f_j(x_l) - f(x_l)|}_{<\varepsilon \ (f_j \to f)} + \underbrace{|f(x_l) - f_k(x_l)|}_{<\varepsilon \ (f_k \to f)} + \underbrace{|f_k(x_l) - f_k(y)|}_{<\varepsilon \text{ (equicontinuity)}} < 4\varepsilon$$

This allows us to proceed inductively as follows:

- Choosing $\varepsilon = 1$, there is a subsequence $f_k^{(1)}$ such that for all $k, l \in \mathbb{N}$ holds:

$$\left\| f_k^{(1)} - f_l^{(1)} \right\|_{C^0(K)} < 4$$

- Choosing $\varepsilon = \frac{1}{2}$, there is a subsequence $f_k^{(2)}$ such that for all $k, l \in \mathbb{N}$ holds:

$$\left\| f_k^{(2)} - f_l^{(2)} \right\|_{C^0(K)} < 2$$

- ...
- Choosing $\varepsilon = \frac{1}{n}$, there is a subsequence $f_k^{(n-1)}$ such that for all $k, l \in \mathbb{N}$ holds:

$$\left\| f_k^{(n)} - f_l^{(n)} \right\|_{C^0(K)} < \frac{4}{n}$$

Then the diagonal sequence $\tilde{f}_k := f_k^{(k)}$ is a Cauchy sequence in $C^0(K)$. $\square_{10.3}$

The Arzelá-Ascoli Theorem can immediately be applied to Hölder continuous functions in a bounded domain $\Omega \subseteq \mathbb{R}^n$. Morrey's inequality states:

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C(\Omega)\|u\|_{H^{1,p}(\Omega)}$$

Let $u_k \in H^{1,p}(\Omega)$ be a bounded sequence, i.e. for all $k \in \mathbb{N}$ holds $\|u_k\|_{H^{1,p}(\Omega)} < c$. Then for all $k \in \mathbb{N}$ follows:

$$\|u_k\|_{C^{0,\gamma}(\overline{\Omega})} \leq C(\Omega) \cdot c =: \tilde{c}$$

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |u(x)| + \sup_{x \neq y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{\|x - y\|^\gamma}$$

$$\Rightarrow \qquad |u_k(x)| < \tilde{c} \qquad \forall_{k \in \mathbb{N}, x \in \overline{\Omega}}$$

Moreover the equicontinuity condition is satisfied if we choose $\delta = \left(\frac{\varepsilon}{\tilde{c}}\right)^{\frac{1}{\gamma}}$.

$$|u(x) - u(y)| \leq \tilde{c}\|x - y\|^\gamma$$

$$\|x - y\| < \delta \quad \Rightarrow \quad |u(x) - u(y)| \leq \tilde{c}\delta^\gamma = \varepsilon$$

By the Arzelá-Ascoli Theorem, there is a subsequence $u_k \to u$ converging in $C^0\left(\overline{\Omega}\right)$. Moreover $\|u_k - u\|_{C^{0,\gamma}}$ also converges.

10.4 Theorem (Rellich-Kondrachov)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with C^1-boundary and $1 \leq p < n$. Then the imbedding

$$H^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for all $1 \leq q < p^* = \frac{np}{n-p}$.

Proof

The Gagliardo-Nirenberg inequalities imply $H^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ and by Hölder's theorem holds $L^{p^*} \subseteq L^q(\Omega)$:

$$\|u\|_{L^q(\Omega)} \overset{\text{Hölder}}{\leq} c\|u\|_{L^{p^*}(\Omega)} \leq \tilde{c}\|u\|_{H^{1,p}(\Omega)}$$

So we have:

$$H^{1,p}(\Omega) \subseteq L^{p^*}(\Omega) \subseteq L^q(\Omega)$$

The boundedness of the imbedding is clear. To show compactness, let $u_k \in H^{1,p}(\Omega)$ be a bounded sequence in $H^{1,p}(\Omega)$. Due to our extension theorem, we can extend the u_k to functions in $H_0^{1,p}(V)$ with $\Omega \Subset V \Subset \mathbb{R}^n$ and $\|u_k\|_{H_0^{1,p}(V)} < M$.

We proceed in several steps:

1. Let $\eta_\varepsilon = \varepsilon^{-n}\eta\left(\frac{x}{\varepsilon}\right)$ be a convolution kernel, i.e. $\eta \in C_0^\infty\left(B_1\left(0\right)\right)$, $\eta\left(x\right) \geq 0$ and:

$$\int_{\mathbb{R}^n} \eta\left(x\right) \mathrm{d}^n\left(x\right) = 1$$

Then $\operatorname{supp}\left(\eta_\varepsilon\right) \subseteq B_\varepsilon\left(0\right)$ and $\int_{\mathbb{R}^n} \eta_\varepsilon\left(x\right) \mathrm{d}^n x = 1$.

$$u_k^\varepsilon\left(x\right) := \left(\eta_\varepsilon * u_k\right)\left(x\right) = \int_{\mathbb{R}^n} \eta_\varepsilon\left(x-y\right) u_k\left(y\right) \mathrm{d}^n y$$

Claim: $u_k^\varepsilon \xrightarrow{\varepsilon \searrow 0} u_k$ converges uniformly in k in the space $L^q\left(V\right)$.
Proof: It holds:

$$\begin{aligned}
\left(u_k^\varepsilon - u_k\right)\left(x\right) &= \int_{\mathbb{R}^n} \eta_\varepsilon\left(y\right)\left(u_k\left(x-y\right) - u_k\left(x\right)\right) \mathrm{d}^n y = \\
&= \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\eta\left(\frac{y}{\varepsilon}\right)\left(u_k\left(x-y\right) - u_k\left(x\right)\right) \mathrm{d}^n y = \\
&\overset{\tilde{y}:=\frac{y}{\varepsilon}}{=} \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n}\eta\left(\tilde{y}\right)\left(u_k\left(x-\varepsilon\tilde{y}\right) - u_k\left(x\right)\right) \varepsilon^n \mathrm{d}^n \tilde{y} = \\
&= \int_{B_1(0)} \eta\left(y\right)\left(u_k\left(x-\varepsilon y\right) - u_k\left(x\right)\right) \mathrm{d}^n y = \\
&= \int_{B_1(0)} \eta\left(y\right) \int_0^1 \underbrace{\frac{\mathrm{d}}{\mathrm{d}\tau} u_k\left(x - \varepsilon\tau y\right)}_{=-\varepsilon \mathrm{D}u_k\left(x-\varepsilon\tau y\right)\cdot y} \mathrm{d}\tau \mathrm{d}^n y
\end{aligned}$$

To be precise, here we need to assume $u_k \in C_0^1\left(V\right)$ to be able to differentiate. Afterwards one can use an approximation argument for $u_k \in H_0^{1,p}\left(V\right)$. We get:

$$\begin{aligned}
\left|u_k^\varepsilon - u_k\right|\left(x\right) &\leq \int_{B_1(0)} \eta\left(y\right) \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}\tau} u_k\left(x-\varepsilon\tau y\right)\right| \mathrm{d}\tau \mathrm{d}^n y \leq \\
&\leq \int_{B_1(0)} \eta\left(y\right) \int_0^1 \left\|\mathrm{D}u_k\left(x-\varepsilon\tau y\right)\right\| \varepsilon \left\|y\right\| \mathrm{d}\tau \mathrm{d}^n y \leq \\
&\leq \varepsilon \int_{B_1(0)} \eta\left(y\right) \int_0^1 \left\|\mathrm{D}u_k\left(x-\varepsilon\tau y\right)\right\| \mathrm{d}\tau \mathrm{d}^n y
\end{aligned}$$

Integrating over x gives:

$$\begin{aligned}
\int_V \left|u_k^\varepsilon - u_k\right|\left(x\right) \mathrm{d}^n x &\leq \varepsilon \int_V \mathrm{d}^n x \int_{B_1(0)} \mathrm{d}^n y \underbrace{\eta\left(y\right)}_{\leq c} \int_0^1 \mathrm{d}\tau \left\|\mathrm{D}u_k\left(x-\varepsilon\tau y\right)\right\| \leq \\
&\leq \varepsilon c \int_{B_1(0)} \mathrm{d}^n y \int_0^1 \mathrm{d}\tau \underbrace{\int_V \mathrm{d}^n x \left\|\mathrm{D}u_k\left(x-\varepsilon\tau y\right)\right\|}_{\leq \int_V \mathrm{d}^n x \left\|\mathrm{D}u_k\right\|} \leq \\
&\leq \varepsilon c \underbrace{\int_{B_1(0)} \mathrm{d}^n y}_{=\omega_n} \underbrace{\int_0^1 \mathrm{d}\tau}_{=1} \int_V \mathrm{d}^n x \left\|\mathrm{D}u_k\right\| = \varepsilon \underbrace{c\omega_n}_{=:\tilde{c}} \int_V \mathrm{d}^n x \left\|\mathrm{D}u_k\right\|
\end{aligned}$$

This yields:

$$\left\|u_k^\varepsilon - u_k\right\|_{L^1(V)} \leq \varepsilon\tilde{c}\left\|\mathrm{D}u_k\right\|_{L^1(V)} \overset{\text{Hölder}}{\leq} \varepsilon\tilde{c}\left\|\mathrm{D}u_k\right\|_{L^q(V)} \leq \varepsilon\tilde{c}\left\|\mathrm{D}u_k\right\|_{H^{1,p}(V)}$$

Moreover holds:

$$\|u_k^\varepsilon - u_k\|_{L^{p^*}(V)} \leq \underbrace{\|u_k^\varepsilon\|_{L^{p^*}(V)}}_{\leq C\|u_k\|_{L^{p^*}(V)}} + \|u_k\|_{L^{p^*}(V)} \overset{C\geq 1}{\leq} 2C\,\|u_k\|_{L^{p^*}(V)}$$

Now the L^q-norm of $u_k^\varepsilon - u_k$ can be estimated by the *interpolation inequality*:

$$\|u_k^\varepsilon - u_m\|_{L^q(V)} \leq \|u_k^\varepsilon - u_k\|_{L^1(V)}^{\Theta} \cdot \|u_k^\varepsilon - u_k\|_{L^{p^*}(V)}^{1-\Theta}$$

Here we have $\Theta \in (0,1)$ with:

$$\frac{1}{q} = \Theta + \frac{1-\Theta}{p^*}$$

Idea of proof:

$$\int |v|^q \,\mathrm{d}^n x = \int |v|^{\alpha q}\,|v|^{(1-\alpha)q}\,\mathrm{d}^n x \overset{\text{Hölder}}{\leq} \|v^{\alpha q}\|_{L^r}\,\left\|v^{(1-\alpha)q}\right\|_{L^{\tilde r}}$$

We get:

$$\|u_k^\varepsilon - u_k\|_{L^q} \leq c\varepsilon^{\Theta}\,\|u_k\|_{H^{1,p}(V)} \xrightarrow{\varepsilon\searrow 0} 0$$

$\square_{\text{1. Claim}}$

2. **Claim:** For any $\varepsilon \in \mathbb{R}_{>0}$, the u_k^ε are uniformly bounded and equicontinuous in k.
   Proof: It holds:

$$|u_k^\varepsilon(x)| = \left|\int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)\,|u_k(y)|\,\mathrm{d}^n y\right| \leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \underbrace{\|u_k\|_{L^1(V)}}_{\leq C\|u_k\|_{H^{1,p}(V)}} \leq \frac{c}{\varepsilon^n}$$

$$\|\mathrm{D}u_k^\varepsilon(x)\| = \left|\int_{B_\varepsilon(x)} \mathrm{D}\eta_\varepsilon(x-y)\,u_k(y)\,\mathrm{d}^n y\right| \leq \|\mathrm{D}\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\,\|u_k\|_{L^1(V)} \leq \frac{\tilde c}{\varepsilon^{n+1}}$$

$\square_{\text{2. Claim}}$

3. **Claim:** For all $\delta \in \mathbb{R}_{>0}$ there is a subsequence u_{k_j} such that for all $j, l \in \mathbb{N}$ holds:

$$\left\|u_{k_j} - u_{k_l}\right\|_{L^q(V)} < \delta$$

   Proof: Choose $\varepsilon \in \mathbb{R}_{>0}$ so small that for all $k \in \mathbb{N}$ holds:

$$\|u_k^\varepsilon - u_k\|_{L^q(V)} < \frac{\delta}{4}$$

According to 2. and the Arzelà-Ascoli Theorem, a subsequence of u_k^ε converges in C^0, i.e.:

$$\sup_V \left|u_{k_j}^\varepsilon - u_{k_l}^\varepsilon\right| \xrightarrow{j,l\to\infty} 0$$

Then follows:

$$\left\|u_{k_j}^\varepsilon - u_{k_l}^\varepsilon\right\|_{L^q(V)} = \int_V \left|u_{k_j}^\varepsilon(x) - u_{k_l}^\varepsilon(x)\right|^q \mathrm{d}^n x \leq \mu(V)\left(\sup_V \left|u_{k_j}^\varepsilon - u_{k_l}^\varepsilon\right|\right)^q \xrightarrow{j,l\to\infty} 0$$

Choose $N \in \mathbb{N}$ such that for all $j, l \in \mathbb{N}_{\geq N}$ holds:

$$\left\| u_{k_j}^{\varepsilon} - u_{k_l}^{\varepsilon} \right\|_{L^q(V)} < \frac{\delta}{2}$$

We remove the first N elements from the sequence, so that this holds for all $j, l \in \mathbb{N}$.

$$\left\| u_{k_j} - u_{k_l} \right\|_{L^q(V)} \leq \underbrace{\left\| u_{k_j} - u_{k_j}^{\varepsilon} \right\|_{L^q(V)}}_{<\frac{\delta}{4}} + \underbrace{\left\| u_{k_j}^{\varepsilon} - u_{k_l}^{\varepsilon} \right\|_{L^q(V)}}_{<\frac{\delta}{2}} + \underbrace{\left\| u_{k_l}^{\varepsilon} - u_{k_l} \right\|_{L^q(V)}}_{<\frac{\delta}{4}} < \delta$$

$\square_{\text{3. Claim}}$

4. Now we choose inductively subsequences for $\delta = 1, \frac{1}{2}, \ldots, \frac{1}{n}$. The resulting diagonal sequence $u_k^{(k)}$ is a $L^q(V)$-Cauchy sequence (just as in the proof of the Arzelà-Ascoli Theorem).

$\square_{10.4}$

11 Regularity Theory for linear elliptic equations

Let $u \in H^{1,2}(\Omega)$ be a weak solution, i.e. for all $\eta \in C_0^\infty$:

$$\int_\Omega \Big(a^{jk}(x)\, \partial_j u(x)\, \partial_k \eta(x) + c(x)\, u(x)\, \eta(x) - f(x)\, \eta(x) \Big) = 0$$

By approximation follows this equation directly for all $\eta \in H_0^{1,2}(\Omega)$.
Goal: Show that $u \in H^{2,2}(\Omega)$.
Interior regularity: Consider $\Omega' \Subset \Omega$ and show $u \in H^{2,2}(\Omega')$.

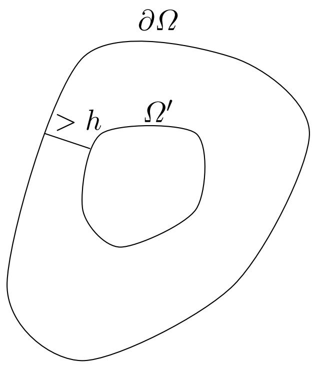


Figure 11.1: $\Omega' \Subset \Omega$

Boundary regularity:

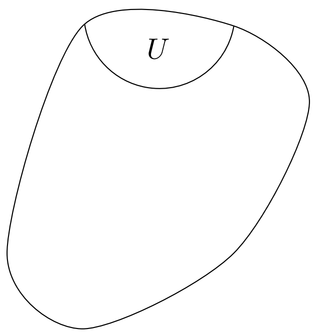


Figure 11.2: Boundary Regularity

Is $u \in H^{2,2}(U)$? Here one needs to know about the regularity of the boundary and the boundary condition.

We mainly consider the Poisson equation:

$$-\Delta u = f$$

For all $\eta \in H_0^{1,2}(\Omega)$ holds:

$$\int_\Omega (\nabla u)(\nabla \eta) + f\eta \mathrm{d}^n x = 0$$

11.1 Interior Regularity

Fix a $\Omega' \Subset \Omega$ and define the difference quotient with the standard basis vectors e_i of $\mathbb{R}^n$:

$$\left(\Delta_i^h u\right)(x) := \frac{u(x + he_i) - u(x)}{h}$$

11.1.1 Lemma

Consider $u \in H^{1,2}(\Omega)$ and choose $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. Then holds $\Delta_i^h u \in L^2(\Omega')$ and:

$$\left\|\Delta_i^h u\right\|_{L^2(\Omega')} \leq \|\mathrm{D}_i u\|_{L^2(\Omega)}$$

Proof

Since continuously differentiable functions are dense in $H^{1,2}(\Omega)$, it suffices to prove the inequality for $u \in C^1(\Omega) \cap H^{1,2}(\Omega)$. By the fundamental theorem of calculus holds:

$$\left(\Delta_i^h u\right)(x) = \frac{1}{h}\int_0^h \partial_i u\left(x^1, \ldots, x^i + \xi, x^{i+1}, \ldots, x^n\right) \mathrm{d}\xi$$

By Hölder's inequality (in the special case of the Schwarz inequality) follows:

$$\begin{aligned}\left|\left(\Delta_i^h u\right)(x)\right|^2 &\leq \frac{1}{h^2}\left(\int_0^h 1 \mathrm{d}\xi\right)\left(\int_0^h \left|\partial_i u\left(x^1, \ldots, x^i + \xi, x^{i+1}, \ldots, x^n\right)\right|^2 \mathrm{d}\xi\right) \leq \\ &\leq \frac{1}{h}\left(\int_0^h \left|\partial_i u\left(x^1, \ldots, x^i + \xi, x^{i+1}, \ldots, x^n\right)\right|^2 \mathrm{d}\xi\right)\end{aligned}$$

$$\left\|\Delta_i^h u\right\|_{L^2(\Omega')}^2 = \int_{\Omega'} \left|\left(\Delta_i^h u\right)(x)\right|^2 \mathrm{d}^n x \leq \frac{1}{h}\int_0^h \mathrm{d}\xi \int_{\Omega'} \mathrm{d}^n x \left|\partial_i u\left(x^1, \ldots, x^i + \xi, x^{i+1}, \ldots, x^n\right)\right|^2$$

Due to $\Omega + \xi e_i \subseteq \Omega$ follows

$$\int_{\Omega'} \mathrm{d}^n x \left|\partial_i u\left(x^1, \ldots, x^i + \xi, x^{i+1}, \ldots, x^n\right)\right|^2 \leq \int_\Omega \mathrm{d}^n x \left|\partial_i u\left(x^1, \ldots, x^n\right)\right|^2$$

and thus:

$$\|\Delta_i u\|_{L^2(\Omega')}^2 \leq \frac{1}{h}\underbrace{\int_0^h \mathrm{d}\xi}_{=h} \int_\Omega \mathrm{d}^n x \left|\partial_i u\right|^2 = \int_\Omega \mathrm{d}^n x \left|\partial_i u\right|^2 = \|\partial_i u\|_{L^2(\Omega)}^2$$

$\square_{11.1.1}$

11.1.2 Lemma

Let $u \in L^2(\Omega)$ and assume that there is a $K \in \mathbb{R}$ such that for all $h \in \mathbb{R}_{>0}$ and all domains $\Omega' \Subset \Omega$ with $\operatorname{dist}(\Omega', \partial\Omega) > h$ the following inequality holds:

$$\left\| \Delta_i^h u \right\|_{L^2(\Omega')} \leq K$$

Then u is weakly differentiable in the coordinate i and $\|\mathrm{D}_i u\|_{L^2(\Omega)} \leq K$.

Proof

Let $\eta \in C_0^1(\Omega)$ and choose h so small that $h < \operatorname{dist}(\operatorname{supp}(\eta), \partial\Omega)$. Then $\Delta_i^h u$ is well-defined on $\operatorname{supp}(\eta)$. For the difference quotient one can use *discrete integration by parts*:

$$\int_\Omega \left(\Delta_i^h u\right) \eta \mathrm{d}^n x = -\int_\Omega u \left(\Delta_i^{-h} \eta\right) \mathrm{d}^n x$$

This is possible due to:

$$\begin{aligned}
\int_\Omega \left(\Delta_i^h u\right) \eta \mathrm{d}^n x &= \int_\Omega \frac{u(x + he_i) - u(x)}{h} \eta(x) \mathrm{d}^n x = \\
&= \frac{1}{h} \int_\Omega u(x + he_i) \eta(x) \mathrm{d}^n x - \int_\Omega u(x) \eta(x) \mathrm{d}^n x = \\
&\overset{\tilde{x} = x + he_i}{=} \frac{1}{h} \int_\Omega u(\tilde{x}) \eta(\tilde{x} - he_i) \mathrm{d}^n \tilde{x} - \int_\Omega u(x) \eta(x) \mathrm{d}^n x = \\
&= \frac{1}{h} \int_\Omega u(x) \left(\eta(x - he_i) - \eta(x)\right) \mathrm{d}^n x = \\
&= -\int_\Omega u(x) \frac{\eta(x) - \eta(x - he_i)}{h} \mathrm{d}^n x = -\int_\Omega u \left(\Delta_i^{-h} \eta\right) \mathrm{d}^n x
\end{aligned}$$

In the limit $h \to 0$ converges:

$$-\int_\Omega u \left(\Delta_i^{-h} \eta\right) \mathrm{d}^n x \xrightarrow{h \to 0} -\int_\Omega u \left(\partial_i \eta\right) \mathrm{d}^n x$$

By assumption, we know:

$$\left| \int_\Omega \left(\Delta_i^h u\right) \eta \mathrm{d}^n x \right| \leq \underbrace{\left\| \Delta_i^h u \right\|_{L^2(\operatorname{supp}(\eta))}}_{\leq K} \|\eta\|_{L^2(\Omega)} \leq K \|\eta\|_{L^2(\Omega)}$$

It follows:

$$\left| \int_\Omega u \left(\partial_i \eta\right) \mathrm{d}^n x \right| \leq K \|\eta\|_{L^2(\Omega)}$$

Thus the functional

$$\begin{aligned}
\Phi : C_0^1(\Omega) \subseteq L^2(\Omega) &\to \mathbb{R} \\
\eta &\mapsto -\int_\Omega u \left(\partial_i \eta\right) \mathrm{d}^n x
\end{aligned}$$

is densely defined and bounded:

$$|\Phi(\eta)| \le K \|\eta\|_{L^2(\Omega)}$$

Φ has a unique extension to $L^2(\Omega)$: For $u \in L^2(\Omega)$ there is a sequence $u_k \in C_0^1(\Omega)$ with $u_k \to u$ in $L^2(\Omega)$ due to the denseness of $C_0^1(\Omega)$ in $L^2(\Omega)$. Then follows:

$$|\Phi(u_k) - \Phi(u_l)| = |\Phi(u_k - u_l)| \le K \|u_k - u_l\|_{L^2(\Omega)} \xrightarrow{k,l\to\infty} 0$$

Hence the limit $\lim_{k\to\infty} \Phi(u_k)$ exists. Since $\Phi(u_k)$ converges for any such sequence u_k, the limit is independent of the choice of (u_k). Define:

$$\Phi(u) := \lim_{k\to\infty} \Phi(u_k)$$

Thus we have a linear bounded functional $\Phi \in L(H,\mathbb{R}) = H^*$. The Fréchet-Riesz Theorem yields a $v \in L^2(\Omega)$ such that holds:

$$-\int_\Omega u(\partial_i \eta)\,\mathrm{d}^n x = \Phi(\eta) = \langle v, \eta\rangle = \int_\Omega v\eta \mathrm{d}^n x$$

Thus (by definition) u is weakly differentiable and $\mathrm{D}_i u = v$. $\square_{11.1.2}$

11.1.3 Theorem

Let $u \in H^{1,2}(\Omega)$ be a weak solution of $-\Delta u = f$. Then for every $\Omega' \Subset \Omega$ holds $u \in H^{2,2}(\Omega')$ and it holds:

$$\|u\|_{H^{2,2}(\Omega')} \le c(\Omega') \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

Motivation

To explain the inequality, we first assume $u \in H^{2,2}(\Omega')$. A weak solution means that for all $v \in H_0^{1,2}(\Omega)$ holds:

$$\int_\Omega (\nabla u)(\nabla v)\,\mathrm{d}^n x = -\int_\Omega f v \mathrm{d}^n x$$

For $v = u$ would follow

$$\int (\nabla u)(\nabla v)\,\mathrm{d}^n x = \int (\nabla u)^2\,\mathrm{d}^n x$$

and for $v = \Delta u$:

$$\int (\nabla u)(\nabla(\Delta u))\,\mathrm{d}^n x = \int (\partial_i u)\,\partial^i \partial_j \partial^j u \mathrm{d}^n x = \underset{\text{by parts}}{\overset{\text{integration}}{=}} -\int \left(\partial^j \partial_i u\right) \partial^i \partial_j u \mathrm{d}^n x = -\int \left\|\mathrm{D}^2 u\right\|^2 \mathrm{d}^n x$$

Problem: These choices for v are *not* in $H_0^{1,2}(\Omega)$.
Choose $\eta \in C_0^\infty(\Omega)$ with $\eta\big|_{\Omega'} = 1$ and $\eta(\Omega) \subseteq [0,1]$. Then we have $v := \eta^2 u \in H_0^{1,2}(\Omega)$ and:

$$-\int f\left(\eta^2 u\right)\mathrm{d}^n x = \int_\Omega (\nabla u)\,\nabla\left(\eta^2 u\right)\mathrm{d}^n x = \int_\Omega \eta^2 \left\|\nabla^2 u\right\|^2 \mathrm{d}^n x + 2\int_\Omega (\nabla u)\,\eta\,(\nabla \eta)\,u \mathrm{d}^n x$$

$$\int_{\Omega}\eta^2\left\|\nabla^2 u\right\|^2\mathrm{d}^n x = 2\int_{\Omega}\eta\left(\nabla u\right)\left(\nabla\eta\right)u\mathrm{d}^n x + \int_{\Omega}\eta^2 f u\mathrm{d}^n x \le$$

$$\underset{\text{inequality}}{\overset{\text{Schwarz}}{\le}} 2\underbrace{\|\eta\nabla u\|_{L^2(\Omega)}}_{=:a}\underbrace{\|u\nabla\eta\|_{L^2(\Omega)}}_{=:b} + \|\eta f\|_{L^2(\Omega)}\|\eta u\|_{L^2(\Omega)}$$

For $\kappa\in\mathbb{R}_{>0}$ holds:

$$2ab = 2\left(\frac{a}{\kappa}\right)(\kappa b)\le\frac{a^2}{\kappa^2}+\kappa^2 b^2$$

$$\|\eta\nabla u\|^2_{L^2(\Omega)}\le\frac{1}{\kappa^2}\|\eta\nabla u\|^2_{L^2(\Omega)}+\kappa^2\|u\nabla\eta\|^2_{L^2(\Omega)}+\|\eta f\|_{L^2(\Omega)}\|\eta u\|_{L^2(\Omega)}$$

Choose $\kappa=\sqrt{2}$ to get:

$$\begin{aligned}\|\eta\nabla u\|^2_{L^2(\Omega)}&\le\frac{1}{2}\|\eta\nabla u\|^2_{L^2(\Omega)}+2\|u\nabla\eta\|^2_{L^2(\Omega)}+\|\eta f\|_{L^2(\Omega)}\|\eta u\|_{L^2(\Omega)}\\ \frac{1}{2}\|\eta\nabla u\|^2_{L^2(\Omega)}&\le 2\|u\nabla\eta\|^2_{L^2(\Omega)}+\|\eta f\|_{L^2(\Omega)}\|\eta u\|_{L^2(\Omega)}\end{aligned}$$

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega')}&\le\|\eta\nabla u\|_{L^2(\Omega)}\\ \|\eta f\|_{L^2(\Omega)}&\le\|f\|_{L^2(\Omega)}\\ \|\eta u\|_{L^2(\Omega)}&\le\|u\|_{L^2(\Omega)}\end{aligned}$$

Choose η such that for $\delta:=\operatorname{dist}(\Omega',\partial\Omega)$ holds:

$$\sup_{x\in\Omega}\left(\|\nabla\eta(x)\|\right)\le\frac{2}{\delta}$$

We thus obtain:

11.1.4 Lemma (Cacciopoli inequality)

For $\Omega'\Subset\Omega$ with $\operatorname{dist}(\Omega',\Omega)>\delta>0$ and a weak solution $u\in H^{1,2}(\Omega)$ of $-\Delta u=f$ holds:

$$\int_{\Omega'}\|\mathrm{D}u\|^2\mathrm{d}^n x\le\frac{17}{\delta^2}\|u\|^2_{L^2(\Omega)}+\delta^2\|f\|^2_{L^2(\Omega)}$$

Proof

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega')}\le\|\eta\nabla u\|^2_{L^2(\Omega)}&\le 4\|u\nabla\eta\|^2_{L^2(\Omega)}+2\|\eta f\|_{L^2(\Omega)}\|\eta u\|_{L^2(\Omega)}\le\\ &\le\frac{16}{\delta^2}\|u\|^2_{L^2(\Omega)}+\underbrace{2\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}}_{\le\frac{1}{\delta^2}\|u\|^2_{L^2(\Omega)}+\delta^2\|f\|^2_{L^2(\Omega)}}\le\\ &\le\frac{17}{\delta^2}\|u\|^2_{L^2(\Omega)}+\delta^2\|f\|^2_{L^2(\Omega)}\end{aligned}$$

$\square_{11.1.4}$

Now explain the method for the second derivatives: Suppose $u \in H^{3,2}(\Omega)$ then holds for all $v \in H^{1,2}(\Omega)$:

$$\int_\Omega (\nabla u)(\nabla v)\,\mathrm{d}^n x = -\int_\Omega f v \mathrm{d}^n x = -\int (\Delta u)\, v \mathrm{d}^n x$$

$$\begin{aligned}\int_{\Omega'} \left\|\mathrm{D}^2 u\right\|^2 \mathrm{d}^n x &= \int_{\Omega'} (\partial_i\partial_j u)(\partial_i\partial_j u)\,\mathrm{d}^n x = \\ &= \int_{\Omega'} (\Delta u)(\Delta u)\,\mathrm{d}^n x + \text{boundary terms} = \\ &\leq \|\Delta u\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} + \ldots \leq \\ &\leq \frac{1}{4C}\left\|\mathrm{D}^2 u\right\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(\Omega)} + \ldots\end{aligned}$$

$$\Rightarrow \qquad \left\|\mathrm{D}^2 u\right\|_{L^2(\Omega)} \leq c\|f\|^2_{L^2(\Omega)} + \ldots$$

Proof of 11.1.3

We know for all $v \in H^{1,2}_0(\Omega)$:

$$\int_\Omega (\mathrm{D}u)(\mathrm{D}v) = -\int_\Omega f v$$

Choose $\Omega'' \Subset \Omega'$ with $\operatorname{dist}(\Omega'', \partial\Omega') \geq \delta$ for $\delta = \operatorname{dist}(\Omega', \partial\Omega)$, $h \in \mathbb{R}_{>0}$ with $h < \delta$ and $v \in H^{1,2}_0(\Omega')$.

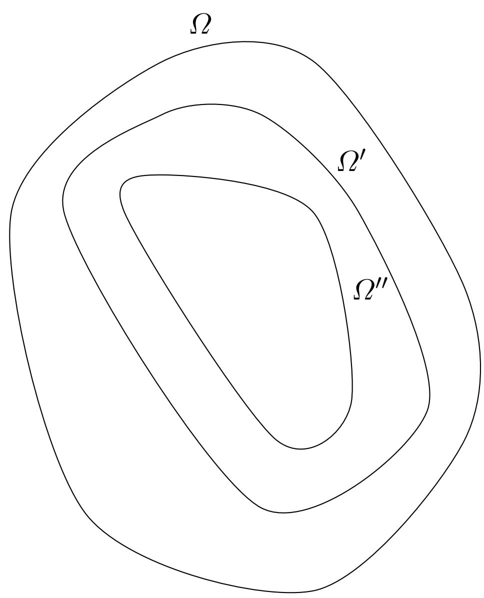


Figure 11.3: $\Omega'' \Subset \Omega' \Subset \Omega$

$$\int_\Omega \left(\mathrm{D}\Delta_i^h u\right)(\mathrm{D}v) = -\int_\Omega (\mathrm{D}u)(\mathrm{D}\underbrace{\Delta_i^{-h} v}_{\in H^{1,2}_0(\Omega)}) = -\int_\Omega f \Delta_i^{-h} v$$

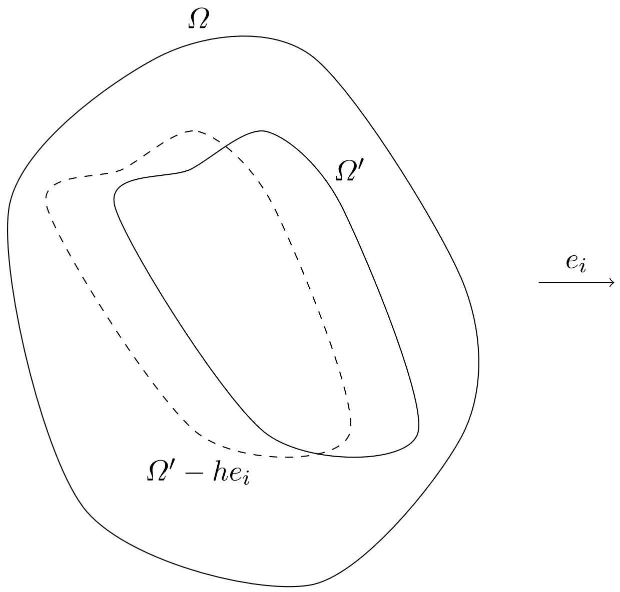


Figure 11.4: $\Omega' - he_i \subseteq \Omega$

Now follows:

$$\left|\int_\Omega \left(\mathrm{D}\Delta_i^h u\right)(\mathrm{D}v)\right| \leq \|f\|_{L^2(\Omega)} \left\|\Delta_i^{-h} v\right\|_{L^2(\Omega')} \overset{11.1.1}{\leq} \|f\|_{L^2(\Omega)} \|\mathrm{D}v\|_{L^2(\Omega')}$$

We cannot choose $v = \Delta_i^h u$, because it is not in $H_0^{1,2}(\Omega)$ in general and thus cannot be used for testing.

Instead we take $v = \eta^2 \Delta_i^h u$ where $\eta \in C_0^\infty(\Omega')$ with $\eta(\Omega') \subseteq [0,1]$, $\eta\big|_{\Omega''} = 1$, and $|\mathrm{D}\eta| < \frac{8}{\delta}$. $\operatorname{supp}(\eta) \subseteq \Omega'$ is needed for $\eta^2 \Delta_i^h u$ to be well-defined.

$$\mathrm{D}v = \mathrm{D}\left(\eta^2 \Delta_i^h u\right) = \eta^2 \mathrm{D}\Delta_i^h u + 2\eta(\mathrm{D}\eta)\left(\Delta_i^h u\right)$$

$$\begin{aligned}
\int_{\Omega'} \left(\mathrm{D}\Delta_i^h u\right)(\mathrm{D}v) &= \int_{\Omega'} \underline{\eta^2 \left(\mathrm{D}\Delta_i^h u\right)^2} + 2\int_{\Omega'} \underline{(\eta \mathrm{D}\Delta_i^h u)}\underline{(\mathrm{D}\eta)\left(\Delta_i^h u\right)} \\
\int_{\Omega'} \left(\mathrm{D}\Delta_i^h u\right)(\mathrm{D}v) &\leq \|f\|_{L^2(\Omega)} \|\mathrm{D}v\|_{L^2(\Omega')} = \\
&= \underline{\|f\|_{L^2(\Omega)}} \left\|\eta^2 \underline{\mathrm{D}\Delta_i^h u} + 2\eta\underline{(\mathrm{D}\eta)\left(\Delta_i^h u\right)}\right\|_{L^2(\Omega')}
\end{aligned}$$

The green term is the square of the second derivative that we want to bound. The red terms are bad, because they contain also second derivatives, but luckily only in first order. The yellow terms do no harm, since they contain only lower derivatives. We can estimate as follows:

$$\begin{aligned}
2\left|\int_{\Omega'} \left(\eta \mathrm{D}\Delta_i^h u\right)(\mathrm{D}\eta)\left(\Delta_i^h u\right)\right| &\leq 2\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega')} \|\mathrm{D}\eta\|_{C^0(\Omega')} \left\|\Delta_i^h u\right\|_{L^2(\Omega')} \leq \\
&\leq \frac{16}{\delta}\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega')} \|\mathrm{D}u\|_{L^2(\Omega')} \leq \\
&\overset{c:=\frac{16}{\delta}}{\leq} c\left(\frac{1}{4c}\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega')}^2 + c\|\mathrm{D}u\|_{L^2(\Omega')}^2\right) \leq
\end{aligned}$$

$$\leq \frac{1}{4}\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega')}^2 + c^2\left\|\mathrm{D}u\right\|_{L^2(\Omega')}^2$$

For the other side holds

$$\|f\|_{L^2(\Omega)}\left\|\eta^2 \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega')} \leq \frac{1}{4}\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega')}^2 + \|f\|_{L^2(\Omega)}^2$$

and:

$$\begin{aligned}\|f\|_{L^2}\left\|2\eta\left(\mathrm{D}\eta\right)\left(\Delta_i^h u\right)\right\|_{L^2(\Omega')} &\leq \frac{16}{\delta}\|f\|_{L^2(\Omega)}\cdot\|\mathrm{D}u\|_{L^2(\Omega)} \leq \\ &\overset{c:=\frac{16}{\delta}}{\leq} c\left(\frac{1}{4c}\|\mathrm{D}u\|_{L^2(\Omega)}^2 + c\,\|f\|_{L^2(\Omega)}^2\right) = \\ &= \frac{1}{4}\|\mathrm{D}u\|_{L^2(\Omega)}^2 + c^2\|f\|_{L^2(\Omega)}^2\end{aligned}$$

We get:

$$\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega)}^2 \leq \frac{1}{4}\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega)}^2 + \frac{1}{4}\left\|\eta \mathrm{D}\Delta_i^h u\right\|_{L^2(\Omega)}^2 + C\left(\|\mathrm{D}u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2\right)$$

$$\Rightarrow \qquad \left\|\Delta_i^h \mathrm{D}u\right\|_{L^2(\Omega'')}^2 = \left\|\eta\Delta_i^h \mathrm{D}u\right\|_{L^2(\Omega'')}^2 \leq \left\|\eta\Delta_i^h \mathrm{D}u\right\|_{L^2(\Omega)}^2 \leq 2C\left(\|\mathrm{D}u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2\right)$$

This is uniformly in h. After Lemma 11.1.2 the weak derivatives $\partial_i \mathrm{D}u$ exists and it holds:

$$\begin{aligned}\int_{\Omega''}\left\|\mathrm{D}^2 u\right\|^2 &\leq 2C\left(\|f\|_{L^2(\Omega)}^2 + \|\mathrm{D}u\|_{L^2(\Omega)}^2\right) \leq \\ &\overset{\text{Cacciopoli}}{\leq} K\left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)\end{aligned}$$

$\square_{11.1.3}$

Now we move to inner regularity for linear elliptic equations:

$$\partial_i\left(a^{ij}\partial_j u\right) + b^i\partial_i u + cu = f$$

Assumptions:

- Uniform ellipticity: For all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ holds:

$$a^{ij}(x)\,\xi_i\xi_j \geq \kappa\|\xi\|^2$$

- Uniform boundedness: For all $x \in \Omega$ holds:

$$\left|a^{ij}(x)\right|, \left|b(x)\right|, \left|c(x)\right| \leq C$$

- Regularity: $a^{ij} \in C^1(\Omega)$, $b^i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $\left|\partial_k a^{ij}\right| < C$

Let $u \in H^{1,2}(\Omega)$ be a weak solution.
First step: We absorb the lower order terms into the inhomogeneity, i.e.:

$$\partial_i\left(a^{ij}\partial_j u\right) = \tilde{f} := f - b^i\partial_i u - cu \in L^2(\Omega)$$

Second step: We apply Cacciopoli's lemma: For $v \in H_0^{1,2}(\Omega)$ holds:

$$\int_\Omega a^{ij} (\partial_j u) \partial_i v = - \int_\Omega \tilde{f} v$$

Choose $v = \eta^2 u$ with $\eta(\Omega) \in [0,1]$ and $\eta\big|_{\Omega'} = 1$ for a $\Omega' \Subset \Omega$.

$$\partial_i v = \eta^2 (\partial_i u) + 2\eta (\partial_i \eta) u$$

$$\int_\Omega \eta^2 a^{ij} (\partial_i u)(\partial_j u) \leq C \|\eta \mathrm{D} u\| \|u\| + C \|f\| \|u\| + C \left| \int b^i (\partial_i u) \eta^2 u \right|$$

$$\begin{aligned} \left| \int b^i (\partial_i u) \eta^2 u \right| &\leq C \|\eta \mathrm{D} u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \left(\frac{1}{4C} \|\eta \mathrm{D} u\|^2_{L^2(\Omega)} + C \|u\|_{L^2(\Omega)} \right) \leq \\ &\leq \frac{1}{4} \|\eta \mathrm{D} u\|^2_{L^2(\Omega)} + C^2 \|u\|_{L^2(\Omega)} \end{aligned}$$

Due to uniform ellipticity we get:

$$\int_\Omega \eta^2 a^{ij} (\partial_i u)(\partial_j u) \geq \kappa \int_\Omega \eta^2 \|\mathrm{D} u\|^2$$

Now we can compensate the bad terms just as before to obtain:

$$\|\mathrm{D} u\|_{L^2(\Omega')} \leq c(\Omega', \kappa, C) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)$$

Third step:

$$\int_{\Omega''} a^{ij} \left(\partial_j \Delta_l^h u \right) (\partial_i v) = - \int_{\Omega''} (\partial_j u) \Delta_l^{-h} \left(a^{ij} \partial_i v \right)$$

$$\begin{aligned} \Delta_i^h (f \cdot g) &= \frac{1}{h} \left(f(x + he_i) g(x + he_i) - f(x) g(x) - f(x) g(x + e_i) + f(x) g(x + he_i) \right) = \\ &= g(x + he_i) \frac{1}{h} \left(f(x + he_i) - f(x) \right) + f(x) \frac{1}{h} \left(g(x + he_i) - g(x) \right) = \\ &= g(x + he_i) \left(\Delta_i^h f \right)(x) + f(x) \left(\Delta_i^h g \right)(x) \end{aligned}$$

We get:

$$\int_{\Omega''} a^{ij} \left(\partial_j \Delta_l^h u \right) (\partial_i v) = - \int_{\Omega''} a^{ij} (\partial_j u) \left(\Delta_l^{-h} \partial_i v \right) - \int_{\Omega''} (\partial_j u) \underbrace{\left(\Delta_l^{-h} a^{ij} \right)}_{<C} (\partial_i v)(x - he_i)$$

Now we choose $v = \Delta_l^h \left(\eta^2 u \right)$ with $\eta \in C_0^\infty(\Omega'')$. We thus obtain:

11.1.5 Theorem

For all $\Omega' \Subset \Omega$ holds $u \in H^{2,2}(\Omega')$.

Proof

Analogous to proof of 11.1.3. $\square_{11.1.5}$

Starting from here, there is a simple iteration method to show $u \in H^{k,2}(\Omega')$, provided that the coefficients are sufficiently regular.

For example, assume $f \in H^{1,2}(\Omega)$, $b_1, c \in C^1(\Omega)$ and $a^{ij} \in C^2(\Omega)$. Moreover, we know $u \in H^{2,2}_{\text{loc}}(\Omega)$.

$$\underbrace{\partial_i(\underbrace{a^{ij}}_{\in C^2}\underbrace{\partial_j u}_{\in H^{1,2}})}_{\in L^2} + \underbrace{\underbrace{b^i}_{\in C^1}\underbrace{\partial_i u}_{\in H^{1,2}}}_{\in H^{1,2}} + \underbrace{\underbrace{c}_{\in C^1}\underbrace{u}_{\in H^{1,2}}}_{\in H^{1,2}} = \underbrace{f}_{\in H^{1,2}}$$

Since all other terms are in $H^{1,2}$, also $\partial_i\left(a^{ij}\partial_j u\right) \in H^{1,2}$ must hold and thus can be differentiated.

$$\partial_i\left(a^{ij}\partial_j\left(\partial_l u\right)\right) = \hat{f} \in L^2_{\text{loc}}(\Omega)$$

With Theorem 11.1.5 follows $\partial_l u \in H^{2,2}_{\text{loc}}(\Omega)$ and thus $u \in H^{3,2}(\Omega)$.

11.1.6 Theorem

For $f \in C^\infty(\Omega)$, $a^{ij}, b^i \in C^\infty(\Omega)$ follows $u \in H^{k,2}_{\text{loc}}(\Omega)$ for all $k \in \mathbb{N}$.

There is a $k_0 = k_0(n)$ such that for all $\Omega' \Subset \Omega$ holds:

$$H^{k_0,2}\left(\Omega'\right) \hookrightarrow C^{0,\gamma}\left(\Omega'\right)$$
$$H^{k_0+p,2}\left(\Omega'\right) \hookrightarrow C^{p,\gamma}\left(\Omega'\right)$$

Since $k_0 + p$ can be chosen arbitrarily large, we conclude $u \in C^\infty(\Omega')$.

11.2 Boundary regularity for linear elliptic equations

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\partial\Omega$ be C^k with $k \in \mathbb{N}_{\geq 2}$. Consider the same equation as before:

$$\partial_i\left(a^{ij}\partial_j u\right) + b^i\partial_i u + cu = f$$

Suppose $u \in H^{1,2}(\Omega)$ is a weak solution.

1. We need to impose a certain regularity of $u\big|_{\partial\Omega}$. To this end assume that there is a $g \in H^{2,2}(\Omega)$ such that holds:

$$\tilde{u} := u - g \in H^{1,2}_0(\Omega)$$

   As an example, one can consider the Dirichlet problem $u\big|_{\partial\Omega} = u_0 \in C^2(\partial\Omega)$, then u_0 can be extended to a function $g \in C^2(\Omega)$, i.e. $g\big|_{\Omega} = u_0$. Then holds $g \in H^{2,2}(\Omega)$. For $\tilde{u} \in H^{1,2}_0(\Omega)$ follows:

$$\partial_i\left(a^{ij}\partial_j\tilde{u}\right) + b^i\left(\partial_i\tilde{u}\right) + c\tilde{u} = \tilde{f} := f - \partial_i\left(a^{ij}\partial_j g\right) - b^i\left(\partial_i g\right) - cg$$

2. Definition of C^k boundary:

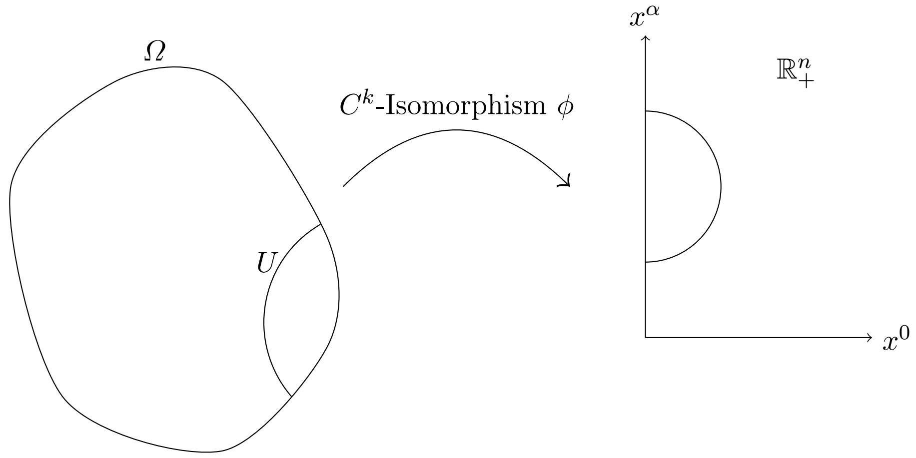


Figure 11.5: Chart maps C^k-boundary by C^k-Isomorphism to $\mathbb{R}^n_0$.

After the coordinate transformation

$$y = \phi(x)$$
$$\frac{\partial y^i}{\partial x^l} = \frac{\partial \phi^i}{\partial x^l}$$

the equation is still of the same type.
The discrete derivative Δ_0^h causes problems near the boundary for $h < 0$, but Δ_α^h for $\alpha \in \{1, \ldots, n-1\}$ can be used even near the boundary.
Now we proceed just as for the interior regularity. $\left\|\Delta_\alpha^h \partial_i u\right\|_{L^2}$ is bounded uniformly in h and thus holds $\partial_\alpha \partial_i u \in L^2(\Omega)$ for $i \in \{0, \ldots, n-1\}$. Then follows:

$$\partial_0 \partial_\alpha u = \partial_\alpha \partial_0 u \in L^2(\Omega)$$

In order to estimate $\partial_0 \partial_0 u$, we use the equation:

$$a^{00}\partial_{00}u + 2a^{0\alpha}\partial_{0\alpha}u + a^{\alpha\beta}\partial_{\alpha\beta}u + b^i\partial_i u + cu + \left(\partial_i a^{ij}\right)\partial_j u = \tilde{f}$$

All the terms, except for the first one, are in $L^2(\Omega)$ and thus follows $a^{00}\partial_{00}u \in L^2(\Omega)$. We know $a^{00} \in C^2(\Omega)$ and $a^{00}(x) \geq \kappa > 0$ for all $x \in \Omega$.

$$\partial_{00}u = \underbrace{\frac{1}{a^{00}}}_{\in C^2} \underbrace{\left(a^{00}\partial_{00}u\right)}_{\in L^2} \in L^2(\Omega)$$

11.2.1 Theorem

For $u \in H^{1,2}(\Omega)$ holds $u \in H^{2,2}(U)$ for all $U \Subset \Omega$.

12 Regularity for Nonlinear Equations

Consider the variational principle for the action

$$S = \int \left(\frac{1}{p} \left(\|\mathrm{D}u\|^2 \right)^{\frac{p}{2}} + fu \right) \mathrm{d}^n x$$

on $H_0^{1,p}(\Omega)$ for $p \in \mathbb{R}_{>2}$ and $f \in L^q(\Omega)$ with $q \geq q_0$ and $q_0^{-1} + (p^*)^{-1} = 1$.
Compute the Euler-Lagrange equations with $v \in H_0^{1,p}(\Omega)$:

$$u_\tau := u + \tau v$$

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}\tau} S(u + \tau v) \Big|_{v=0} &= \int \left(\frac{1}{p} \cdot \frac{p}{2} \|\mathrm{D}u\|^{p-2} \cdot 2 (\mathrm{D}u)(\mathrm{D}v) + fv \right) \mathrm{d}^n x = \\ &= \int \left(\|\mathrm{D}u\|^{p-2} (\mathrm{D}u)(\mathrm{D}v) + fv \right) \mathrm{d}^n x = \int \left(-\partial_j \left(\|\mathrm{D}u\|^{p-2} \partial^j u \right) + f \right) v \mathrm{d}^n x \end{aligned}$$

So the p-Laplacian equals f:

$$\begin{aligned} \partial_j \left(\|\mathrm{D}u\|^{p-2} \partial^j u \right) &= f \\ \partial_i \left(a^{ij} \partial_j u \right) &= f \\ a^{ij} &= \underbrace{\|\mathrm{D}u\|^{p-2}}_{\in L^{\frac{p}{p-2}}} \delta^{ij} \end{aligned}$$

If we already have a solution for the nonlinear equation, we can consider u as a solution for the linear equation with the above a^{ij}. Here the coefficients a^{ij} do not even need to be bounded, which makes proving regularity difficult!
A somewhat simpler problem is the action

$$S = \int_\Omega \left(g \left(\|\nabla u\|^2 \right) + fu \right) \mathrm{d}^n x$$

with bounded g'.

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} S(u + \tau v) \big|_{\tau=0} = \int \left(g' \left(\|\nabla u\| \right)^2 \right) \cdot 2 (\nabla u)(\nabla v) + fv$$

$$\begin{aligned} 2\partial_i \left(g' \left(\|\nabla u\|^2 \right) \partial^i u \right) &= f \\ \partial_i \left(a^{ij} \partial_j u \right) &= f \\ a^{ij} &= 2g' \left(\|\nabla u\|^2 \right) \delta^{ij} \end{aligned}$$

Choose $g \in C^\infty(\mathbb{R})$ with $0 < \lambda \leq g' \leq \Lambda < \infty$. Then holds $a^{ij} \in L^\infty(\Omega)$ and $a^{ij}\xi_i\xi_j \geq \lambda \|\xi\|^2$, thus the equation is uniformly elliptic.

The goal is to analyze regularity for the linear system. This method is called *freezing of coefficients* (*Einfrieren der Koeffizienten*).

Mathematicians involved in solving this problem are:

- De Giorgi (1928-1996) is one of the most known mathematician of the 20th century.
- John Nash (1928-): Nash imbedding theorem (and Nobel Prize in Economic Sciences)
- Jürgen Moser (1928-1999) worked at the Courant institut, then at the ETH Zürich. He is known for the KAM-Theorem (Kolmogoroff, Arnold, Moser).

We follow mainly the ideas of Moser.

$$\int a^{ij}\left(\partial_i \Delta_l^h u\right)(\partial_j v) = -\int a^{ij}\partial_i u\left(\Delta_l^{-h}\partial_j v\right) - \int \underbrace{\left(\Delta_l^{-h} a^{ij}\right)}_{\text{cannot be controlled}} (\partial_i u)(\partial_j v)$$

12.1 Hölder-Continuity for Weak Solutions with Bounded Coefficients

Let $\Omega \subseteq \mathbb{R}^n$ be open.

$$Lu := \partial_i\left(a^{ij}\partial_j u\right)$$

Assume $a^{ij} \in L^\infty$, i.e. $\left|a^{ij}(x)\right| \leq \Lambda$, and uniform ellipticity, i.e. $a^{ij}(x)\xi_i\xi_j \geq \lambda\|\xi\|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Furthermore $a^{ij}(x)$ shall be measurable. Consider the homogeneous equation

$$Lu = 0$$

and a weak solution $u \in H^{1,2}(\Omega)$ of it. We are interested in the inner regularity in Ω.

12.1.1 Theorem (De Giorgi-Nash)

For all $\Omega' \Subset \Omega$ holds $u \in C^{0,\alpha}(\Omega')$.

12.1.2 Theorem (Moser)

Let $u \in H^{1,2}(\Omega)$ be a weak solution of $Lu = 0$, then holds $\sup_{B_R(x_0)}(u) < \infty$. For $u \geq 0$ in $B_{4R}(x_0)$ follows the *Harnack inequality*:

$$\sup_{B_R(x_0)}(u) \leq C\left(n, \frac{\Lambda}{\lambda}\right)\inf_{B_R(x_0)}(u)$$

Note that the constant C is independent of R.

Proof of 12.1.1, assuming Moser's Harnack inequality

Consider $x_0 \in \Omega$ and $R \in \mathbb{R}_{>0}$ with $B_{4R}(x_0) \subseteq \Omega$.

$$M(R) := \sup_{B_R(x_0)}(u) \geq m(r) := \inf_{B_R(x_0)}(u)$$

From Theorem 12.1.2 we know $M(R) < \infty$ and also $m(R) > -\infty$, since $-u$ is another solution:

$$L(-u) = -L(u) = 0$$

In $B_R(x_0)$ holds:

$$M(R) - u(x) \geq 0 \qquad\qquad u(x) - m(R) \geq 0$$

These are non-negative weak solutions.

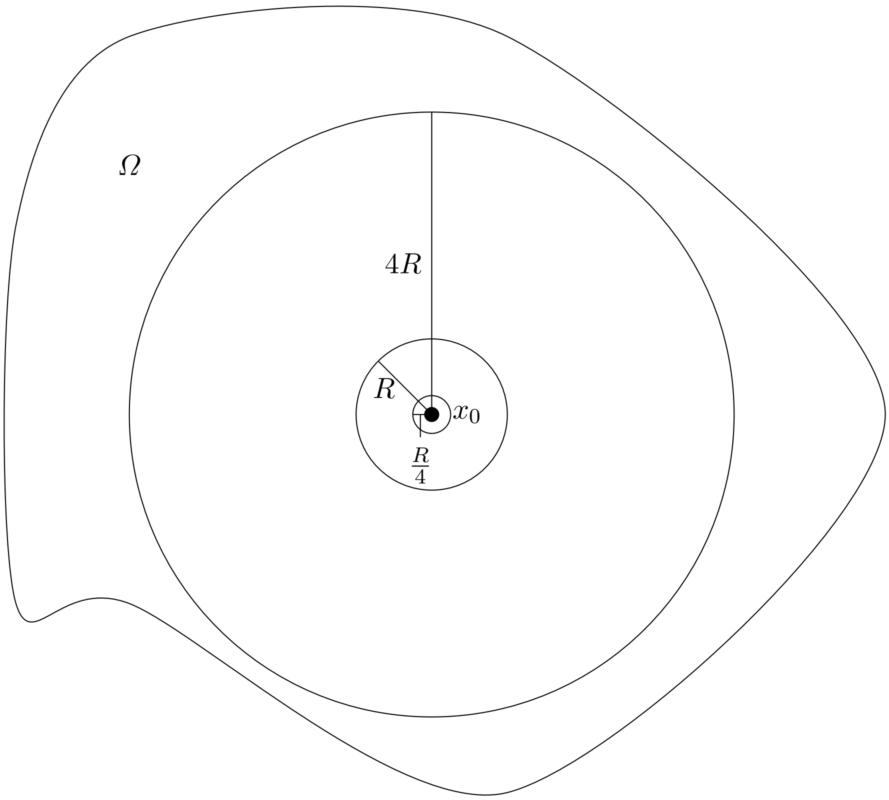


Figure 12.1: $B_{\frac{R}{4}}(x_0) \subseteq B_R(x_0) \subseteq B_{4R}(x_0) \subseteq \Omega$

$$\begin{aligned} M(R) - m\left(\frac{R}{4}\right) &= \sup_{B_{\frac{R}{4}}(x_0)} \left(M(R) - u(x)\right) \leq \\ &\leq C \inf_{B_{\frac{R}{4}}(x_0)} \left(M(R) - u(x)\right) = C\left(M(R) - M\left(\frac{R}{4}\right)\right) \end{aligned}$$

$$\begin{aligned} M\left(\frac{R}{4}\right) - m(R) &= \sup_{B_{\frac{R}{4}}(x_0)} \left(u(x) - m(R)\right) \leq \\ &\leq C \inf_{B_{\frac{R}{4}}(x_0)} \left(u(x) - m(R)\right) = C\left(m\left(\frac{R}{4}\right) - m(R)\right) \end{aligned}$$

Adding the inequalities yields:

$$(1+C)\left(M\left(\frac{R}{4}\right)-m\left(\frac{R}{4}\right)\right)\leq (C-1)\left(M(R)-m(R)\right)$$
$$\underbrace{M\left(\frac{R}{4}\right)-m\left(\frac{R}{4}\right)}_{\geq 0}\leq \underbrace{\frac{C-1}{C+1}}_{0\leq .<1}\underbrace{\left(M(R)-m(R)\right)}_{\geq 0}$$

$$w(r):=M(r)-m(r)$$

is called *oscillation* of u on $B_r(x_0)$. The second term of the Hölder-Norm is:

$$\sup_{x\neq y}\frac{u(x)-u(y)}{|x-y|^{\alpha}}$$

This means that, if u is Hölder-continuous, it holds:

$$\omega(r)\leq cr^{\alpha}$$

The oscillation gets smaller as the ball gets smaller. With $\vartheta:=\frac{C-1}{C+1}<1$ (independent of R) we know:

$$w\left(\frac{R}{4}\right)\leq \vartheta w(R)$$
$$w\left(\frac{R}{16}\right)\leq \vartheta w\left(\frac{R}{4}\right)$$
$$w\left(\frac{R}{4^n}\right)\leq \vartheta^n w(R)$$

For $r\in(0,R)$ choose $n\in\mathbb{N}$ such that holds:

$$\frac{R}{4^{n+1}}\leq r<\frac{R}{4^n}$$

Because w is by definition monotonically increasing in r we have:

$$w(r)<w\left(\frac{R}{4^n}\right)\leq \vartheta^n w(R)=\left(\frac{1}{4^n}\right)^{\alpha}w(R)$$

$$\alpha:=\frac{\ln(\vartheta)}{\ln\left(\frac{1}{4}\right)}>0$$

Thus we get:

$$w(r)<\left(\frac{4}{R}\frac{R}{4^{n+1}}\right)^{\alpha}w(R)\leq 4^{\alpha}\left(\frac{r}{R}\right)^{\alpha}w(R)$$

So for all $x\in B_R(x_0)$ follows with $r=|x-x_0|$:

$$|u(x)-u(x_0)|\leq w(r)\leq\left(\frac{4}{R}\right)^{\alpha}w(R)\,r^{\alpha}=\left(\frac{4}{R}\right)^{\alpha}w(R)\,|x-x_0|^{\alpha}$$

Thus u is Hölder-continuous in x_0 with exponent α. $\square_{12.1.2}$

12.2 The Moser Iteration

Recall for $r'' > r' > r$:

$$\int_{B_r} \|\mathrm{D}u\|^2 \leq C(r, r') \int_{B_{r'}} \|u\|^2$$
$$\int_{B_r} \left\|\mathrm{D}^2 u\right\|^2 \leq C(r, r') \int_{B_{r'}} \|\mathrm{D}u\|^2 \leq C(r, r')\, C(r', r'') \int_{B_{r''}} \|u\|^2$$

The idea is to iterate this process controlling the constants and the use the imbedding theorems. In preparation for this we need to introduce weak super- and subsolutions.

12.2.1 Definition (weak super-/subsolution)

A function $u \in H^{1,2}(\Omega)$ is a *weak subsolution*, $Lu \geq 0$ (in the weak sense), in Ω if for all $\eta \in H_0^{1,2}$ with $\eta \geq 0$ holds:

$$\int a^{ij} (\partial_i u)(\partial_j \eta) \leq 0$$

u is a supersolution, if $-u$ is a subsolution.
Explanation: For a (strong) solution $u \in C^2$ holds:

$$\int a^{ij} (\partial_i u)(\partial_i \eta) = -\int \left(\partial_j a^{ij} (\partial_i u)\right) \eta = -\int \underbrace{(Lu)}_{\geq 0} \underbrace{\eta}_{\geq 0} \leq 0$$

Consider for $f \in C^2(\mathbb{R})$ the composition $(f \circ u)(x)$. The weak derivative is:

$$\mathrm{D}(f \circ u)(x) = f'(u) \cdot \mathrm{D}u(x)$$
$$\mathrm{D}^2 (f \circ u)(x) = f''(u) \cdot \underbrace{(\mathrm{D}u(x))^2}_{\geq 0} + f'(u)\, \mathrm{D}^2 u(x)$$

Assuming $\mathrm{D}^2 u \geq 0$, $f'' \geq 0$ and $f' \geq 0$ we get $\mathrm{D}^2(f \circ u)(x) \geq 0$. So a subsolution composited with a monotonically increasing function is again a subsolution.
Later on we will use $f(u) = \|u\|^p$.

12.2.2 Lemma

Let $f \in C^2(\mathbb{R})$ and $\sup|f'|, \sup|f''| < \infty$.

i) If $u \in H^{1,2}(\Omega)$ is a subsolution, f is convex and $f' \geq 0$, then $f \circ u$ is a subsolution.
ii) If $u \in H^{1,2}(\Omega)$ is a solution and f is convex, then $f \circ u$ is a subsolution.

Proof

Since f is convex, it holds $f'' \geq 0$. First assume $u \in C^2(\Omega)$ to get:

$$\partial_j (f \circ u) = f'(u)\, \partial_j u$$
$$L(f \circ u) = \partial_i \left(a^{ij} \left(f'(u)\, \partial_j u\right)\right) =$$

$$= \underbrace{a^{ij} f''(u)(\partial_i u)(\partial_j u)}_{\geq 0} + f'(u) \underbrace{\partial_i \left(a^{ij} \partial_j u\right)}_{=Lu}$$

If u is a subsolution, it holds $Lu \geq 0$ and due to $f'(u) \geq 0$ also $L(f \circ u) \geq 0$. Thus follows i).
If u is a solution, it holds $Lu = 0$ and thus follows ii).
For $u \in H^{1,2}(\Omega)$ we choose $u_n \in C^2(\Omega)$ with $u_n \to u$ in $H^{1,2}(\Omega)$ and $\eta \in C_0^\infty(\Omega)$.

$$\int_\Omega ((f \circ u_n) - (f \circ u)) \xrightarrow{n \to \infty \ (?)} 0$$

$$\int_\Omega (f \circ u_n)(-\partial_j \eta) = \int f'(u_n)(\partial_j u_n) \eta \tag{12.1}$$

We assumed $|f'|, |f''| \leq C < \infty$ and thus $f'(u_n)$ is uniformly bounded by C.

$$|(f \circ u_n)(x) - (f \circ u)(x)| \underset{\text{Theorem}}{\overset{\text{mean value}}{\leq}} \sup\left(|f'|\right) \cdot |u_n(x) - u(x)|$$

This gives:

$$\left| \int_\Omega (f \circ u_n - f \circ u)(-\partial_j \eta) \right| \leq \sup\left(|f'|\right) \int_\Omega |u_n - u| \cdot |\partial_j \eta| \leq$$
$$\leq \|u_n - u\|_{L^2} \|\partial_j \eta\|_{L^2} \to 0$$

Thus the left side of (12.1) converges.

$$\left|f'(u_n)(\partial_j u_n) - f'(u)(\partial_j u)\right| \leq \left|f'(u_n)(\partial_j u_n) - f'(u_n)(\partial_j u)\right| + \left|f'(u_n)(\partial_j u) - f'(u)(\partial_j u)\right| =$$
$$= \underbrace{\left|f'(u_n)\right|}_{\leq C} \underbrace{\left|\partial_j u_n - \partial_j u\right|}_{\to 0 \text{ in } L^2} + \left|f'(u_n) - f'(u)\right| \cdot \underbrace{\left|\partial_j u\right|}_{\in L^2}$$

$$\left|f'(u_n) - f'(u)\right| \leq \underbrace{\sup\left(|f''|\right)}_{\leq C} \cdot \underbrace{|u - u_n|}_{\to 0 \text{ in } L^2}$$

So we get:

$$\int_\Omega f'(u_n)(\partial_j u_n) \eta \xrightarrow{n \to \infty} \int_\Omega f'(u)(\partial_j u) \eta$$

So we can use the chain rule for weak derivatives:

$$\partial_j (f \circ u) = f'(u) \partial_j u$$

We want to show that $f \circ u$ is a subsolution, i.e. for all $\eta \in C_0^2(\Omega)$ (or $\eta \in H_0^{1,2}(\Omega)$) with $\eta \geq 0$ must hold:

$$0 \overset{!}{\geq} \int_\Omega a^{ij} \partial_i (f \circ u)(\partial_j \eta) = \int_\Omega a^{ij} f'(u)(\partial_i u)(\partial_j \eta) =$$
$$= \underbrace{\int_\Omega a^{ij}(\partial_i u)\left(\partial_j f'(u) \eta\right)}_{\substack{\leq 0 \text{ if } u \text{ is subsolution} \\ =0 \text{ if } u \text{ is solution}}} \underbrace{- \int_\Omega a^{ij}(\partial_i u) f''(u)(\partial_j u) \eta}_{\leq 0}$$

$\square_{12.2.2}$

12.2.3 Lemma

Let $u \in H^{1,2}(\Omega)$ be a weak subsolution and $k \in \mathbb{R}$. Then $v := \max(u,k)$ is also a subsolution.

Proof

Approximate $f(x) := \max(x,k)$ by convex $f_n \in C^2$ with $f_n' \geq 0$, so that $f_n \rightrightarrows f$ converges. The previous Lemma applies to $f_n \circ u$. We need to arrange:

$$\int_\Omega |f_n \circ u - f \circ u| \to 0 \tag{12.2}$$

$$\int_\Omega |f_n \circ u - f \circ u| \, \partial_j \eta \to 0 \tag{12.3}$$

$$\int_\Omega \left|\left(f_n'(u) - f'(u)\right) \partial_i u\right|^2 \to 0 \tag{12.4}$$

(12.2) and (12.3) are guaranteed by $f_n \rightrightarrows f$.
For (12.4) we know that for $x \neq k$ converges

$$f_k'(x) \to f'(x)$$

point-wise. Thus follows by dominated convergence:

$$\int_\Omega \left|\left(f_n'(u) - f'(u)\right) \partial_i u\right|^2 \leq \int_\Omega \left|f_n'(u) - f'(u)\right|^2 \cdot |\partial_i u|^2 \xrightarrow[\text{dom. con.}]{n\to\infty} 0$$

$\square_{12.2.3}$

12.2.4 Theorem (Moser)

Let $u \in H^{1,2}(\Omega)$ be a subsolution in $B_{4R}(x_0)$ and $p \in \mathbb{R}_{>1}$. Then holds

$$\sup_{B_R(x_0)} (u) \leq C\left(n, \frac{\Lambda}{\lambda}\right) \left(\frac{p}{p-1}\right)^{\frac{2}{p}} \left(\fint_{B_{2R}(x_0)} (\max(0,u))^p\right)^{\frac{1}{p}}$$

$$\fint_{B_R(x_0)} v := \frac{1}{\mu(B_R(x_0))} \int_{B_R(x_0)} v \mathrm{d}^n x$$

12.2.5 Theorem

Let $u \geq 0$ be a subsolution in $B_{4R}(x_0)$ and $n \in \mathbb{N}_{\geq 3}$.

$$\left(\fint_{B_{2R}(x_0)} u^p\right)^{\frac{1}{p}} \leq \frac{C_2\left(p, n, \frac{\lambda}{\Lambda}\right)}{\left(\frac{n}{n-2} - p\right)^2} \inf_{B_R(x_0)} (u)$$

If $u \geq 0$ is a weak solution choose $p > 1$ such that $u \in L^p$. Then holds:

$$\sup_{B_R(x_0)} (u) \leq C\left(p, n, \frac{\lambda}{\Lambda}\right) \inf_{B_R(x_0)} (u)$$

We now enter the proof of Theorem 12.2.4:
Fix x_0 and let $u \in H^{1,2}(\Omega)$ be a non-negative function. For $p \in \mathbb{R} \setminus \{0\}$ define:

$$\phi(p, R) := \left(\fint_{B_R(x_0)} u^p \right)^{\frac{1}{p}}$$

12.2.6 Lemma

$$\begin{aligned} \lim_{p \to \infty} \phi(p, R) &= \sup_{B_R(x_0)} \operatorname{ess}(u) = \|u\|_{L^\infty(B_R(x_0))} =: \phi(\infty, R) \\ \lim_{p \to -\infty} \phi(p, R) &= \inf_{B_R(x_0)} \operatorname{ess}(u) =: \phi(-\infty, R) \end{aligned}$$

Proof

Hölder's inequality gives:

$$\begin{aligned} \fint_{B_R(x_0)} u^p &= \frac{1}{\mu(B_R(x_0))} \int_{B_R(x_0)} 1 \cdot u^p \mathrm{d}^n x \leq \\ &\overset{\frac{1}{r}+\frac{1}{r'}=1}{\leq} \frac{1}{\mu(B_R(x_0))} \underbrace{\|1\|_r}_{=(\mu(B_R(x_0)))^{\frac{1}{r}}} \|u^p\|_{r'} = \\ &= \frac{1}{(\mu(B_R(x_0)))^{1-\frac{1}{r}}} \left(\int u^{p \cdot r'} \mathrm{d}^n x \right)^{\frac{1}{r'}} = \left(\frac{1}{\mu(B_R(x_0))} \int u^{p \cdot r'} \mathrm{d}^n x \right)^{\frac{1}{r'}} \end{aligned}$$

Thus follows:

$$\left(\fint u^p \right)^{\frac{1}{p}} \leq \left(\fint u^{p \cdot r'} \right)^{\frac{1}{p \cdot r'}}$$

Due to $r' \geq 1$ follows, that $\left(\fint u^p \right)^{\frac{1}{p}}$ is monotonically increasing in p.

$$\begin{aligned} \left(\fint u^p \right)^{\frac{1}{p}} &\leq \left(\sup \ \operatorname{ess} \left(u^p \fint 1 \right) \right)^{\frac{1}{p}} = \sup \ \operatorname{ess}(u) \\ \Rightarrow \qquad \phi(p, R) &\xrightarrow{p \to \infty} C \leq \phi(\infty, R) \end{aligned}$$

By definition of the essential supremum holds for all $\varepsilon \in \mathbb{R}_{>0}$ and all $V \subseteq \Omega$ with $\mu(V) > 0$ and $u\big|_V \geq \sup(u) - \varepsilon$:

$$\begin{aligned} \phi(p, R) &= \left(\frac{1}{\mu(B_R(x_0))} \int_{B_R(x_0)} u^p \right)^{\frac{1}{p}} \geq \left(\frac{1}{\mu(B_R(x_0))} \int_V u^p \right)^{\frac{1}{p}} \geq \\ &\geq \left(\frac{\mu(V)}{\mu(B_R(x_0))} (\sup(u) - \varepsilon)^p \right)^{\frac{1}{p}} = \\ &= \underbrace{\left(\frac{\mu(V)}{\mu(B_R(x_0))} \right)^{\frac{1}{p}}}_{>0} (\sup(u) - \varepsilon) \xrightarrow{p \to \infty} \sup(u) - \varepsilon \end{aligned}$$

Since ε is arbitrary, the result follows.

The limit $p \to -\infty$ can be treated as follows: Let $p \in \mathbb{R}_{<0}$.

$$\phi(p, R) = \left(\fint_{B_R(x_0)} u^p \right)^{\frac{1}{p}} = \frac{1}{\left(\fint_{B_R(x_0)} \left(\frac{1}{u}\right)^{|p|} \right)^{\frac{1}{|p|}}} \xrightarrow{|p|\to\infty} \frac{1}{\sup\ \operatorname{ess}\left(\frac{1}{u}\right)} = \inf\ \operatorname{ess}(u)$$

$\square_{12.2.6}$

12.2.7 Lemma

Let $u \in H^{1,2}(\Omega)$ with $u \geq 0$ be a weak subsolution and assume

$$v := u^q \in L^2(\Omega)$$

for some $q > \frac{1}{2}$. Then for all $\eta \in H_0^{1,2}(\Omega)$ holds:

$$\int_\Omega \eta^2 \|\mathrm{D}v\|^2 \leq \frac{\Lambda^2}{\lambda^2} \left(\frac{2q}{2q-1}\right)^2 \int_\Omega \|\mathrm{D}\eta\|^2 v^2$$

Proof

Consider $f_\delta(\xi) := \xi^{2q} e^{-\delta\xi}$, in particular $f \in C^2\left(\mathbb{R}_0^+\right)$. The derivative

$$f_\delta'(\xi) = (2q)\,\xi^{2q-1} e^{-\delta\xi} - \delta \xi^{2q} e^{-\delta\xi}$$

is bounded. Due to $q > \frac{1}{2}$ follows $2q - 1 > 0$. Choose $\varphi = f_\delta'(u) \cdot \eta^2$ with $\eta \in H_0^{1,2}$ as the test function, which is possible since $f_\delta'(u)$ is bounded. The weak equation is:

$$\int a^{ij}(\partial_i u)(\partial_j \varphi) \leq 0 \qquad \underset{\varphi \in H_0^{1,2}(\Omega)}{\forall}$$

This gives for our test function:

$$0 \geq \int_\Omega a^{ij}(\partial_i u)\left(f_\delta''(u)(\partial_j u)\,\eta^2 + f_\delta'(u) \cdot 2\eta \partial_j \eta\right)$$

$$f_\delta''(u) = \left((2q)(2q-1)\,u^{2q-2} - 2 \cdot 2q\delta u^{2q-1} + \delta^2 u^{2q}\right) e^{-\delta u}$$

We use:

$$\partial_j v = q u^{q-1} \partial_j u$$

$$\begin{aligned} a^{ij}(\partial_i u)(\partial_j u)\, f_\delta''(u) &= \frac{2q(2q-1)}{q^2} a^{ij}(\partial_i v)(\partial_j v)\, e^{-\delta u} + \delta(\ldots)\, e^{-\delta u} \geq \\ &\geq 2\lambda \frac{2q-1}{q} \|\mathrm{D}v\|^2 e^{-\delta u} + \delta(\ldots)\, e^{-\delta u} \end{aligned}$$

$$(\partial_i u)\, f_\delta'(u) = (\partial_i u)\left(2qu^{2q-1} - \delta u^{2q}\right) e^{-\delta u} = 2(\partial_i v)\, v e^{-\delta u} - \delta(\ldots)\, e^{-\delta u}$$

Leave out the correction terms $\delta\left(\ldots\right)e^{-\delta u}$.

$$2\lambda\frac{2q-1}{q}\int_{\Omega}\left\|\mathrm{D}v\right\|^2\eta^2\leq\int a^{ij}2\left(\partial_i v\right)v\cdot 2\eta\partial_j\eta\leq$$
$$\leq 2\Lambda\left(\frac{1}{\kappa}\int\left\|\mathrm{D}v\right\|^2\eta^2+\kappa\int v^2\left\|\mathrm{D}\eta\right\|^2\right)$$

Choose:

$$\frac{4\Lambda}{\kappa}:=2\lambda\frac{2q-1}{q}$$
$$\kappa=\frac{\Lambda}{\lambda}\frac{2q}{2q-1}$$

$$\lambda\frac{2q-1}{q}\int_{\Omega}\left\|\mathrm{D}v\right\|^2\eta^2\leq 2\Lambda\kappa\int v^2\left\|\mathrm{D}\eta\right\|^2$$
$$\int_{\Omega}\left\|\mathrm{D}v\right\|^2\eta^2\leq\left(\frac{\Lambda}{\lambda}\frac{2q}{2q-1}\right)^2\int v^2\left\|\mathrm{D}\eta\right\|^2$$

$\square_{12.2.7}$

Proceed with the proof of Theorem 12.2.4:
Choose $R=1$ and $x_0=0$ without loss of generality, since otherwise we can scale by a factor $\frac{1}{R}$ and shift by $-x_0$. Moreover, we can assume $u>0$, because otherwise we consider $\tilde{u}:=\max\left(u,\varepsilon\right)$ for a $\varepsilon\in\mathbb{R}_{>0}$. Then $\tilde{u}$ is again a subsolution by Lemma 12.2.3. In the end we can take the limit $\varepsilon\to 0$.
Thus we need to show:

$$\sup_{B_1(0)}\left(u\right)\leq C\left(\frac{p}{p-1}\right)^{\frac{2}{p}}\left(\fint_{B_2(0)}u^p\right)^{\frac{1}{p}}$$

Choose $0<r'<r<2r'<2$ and a test function $\eta\in C_0^\infty\left(B_2\left(0\right)\right)$ such that $\eta\big|_{B_{r'}}=1$, $\sup\left(\eta\right)\subseteq B_r$ and:

$$\left\|\mathrm{D}\eta\right\|\leq\frac{2}{r-r'}$$

Assume $u\in L^{2q}$. Then holds $v:=u^q\in L^2$ and Lemma 12.2.7 yields:

$$\int_{B_{r'}}\left\|\mathrm{D}v\right\|^2\leq\int_{\Omega}\left\|\mathrm{D}v\right\|^2\eta^2\leq C\left(\frac{2q}{2q-1}\right)^2\int_{\Omega}\left\|\mathrm{D}\eta\right\|^2v^2$$
$$\int_{B_{r'}}\left\|\mathrm{D}v\right\|^2\leq C\left(\frac{2q}{2q-1}\right)^2\frac{4}{\left(r-r'\right)^2}\int_{B_r}v^2$$

The Sobolev imbedding theorem gives $H^{1,2}\left(B_{r'}\right)\hookrightarrow L^{p^*}\left(B_{r'}\right)$ with $\frac{1}{p^*}=\frac{1}{2}-\frac{1}{n}$, i.e. $p^*=\frac{2n}{n-2}$.

$$\left(\fint_{B_{r'}}v^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}\cdot 2}\leq\frac{1}{\left(r'\right)^n}\left\|v\right\|_{H^{1,2}\left(B_{r'}\right)}^2=\frac{1}{\left(r'\right)^n}\int_{B_{r'}}\left\|\mathrm{D}v\right\|^2+\frac{1}{\left(r'\right)^n}\int_{B_{r'}}v^2\leq$$
$$\leq C\left(\frac{2q}{2q-1}\right)^2\frac{4}{\left(r-r'\right)^2}\frac{1}{\left(r'\right)^n}\int_{B_r}v^2\mathrm{d}^nx+\frac{1}{\left(r'\right)^n}\int_{B_{r'}}v^2$$

$$
\begin{aligned}
\left(\fint_{B_{r'}} v^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} &\leq \overline{C} \int_{B_r} v^2 \\
\overline{C} &= C\left(n, \frac{\lambda}{\Lambda}\right)\left(\left(\frac{r'}{r-r'}\right)^2\left(\frac{2q}{2q-1}\right)^2 + 1\right)
\end{aligned}
$$

Appendix

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