

# Partial Differential Equations I

*lecture by*

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*during the summer semester 2013*

*revision and layout in  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  by*

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Last changed: June 14, 2013

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## Literature

Elliptic and parabolic partial differential equations:

- JÜRGEN JOST: *Partial Differential Equations*; Springer, 2007;  
ISBN: 978-0-387-49318-3; doi: 10.1007/978-0-387-49319-0  
(good book, but not all details, small errors)
- LAWRENCE C. EVANS: *Partial Differential Equations*; American Mathematical Society, 2010; ISBN: 978-0-8218-4974-3  
(part of the lecture follows this book, lots of details)
- DAVID GILBARG, NEIL S. TRUDINGER: *Elliptic Partial Differential Equations of second order*; Springer, 2001; ISBN: 3-540-41160-7  
(classic textbook, complete treatment)
- ROBERT A. ADAMS, JOHN J. F. FOURNIER: *Sobolev Spaces*; Academic Press, 2009; ISBN: 978-0-12-044143-3
- RICHARD COURANT, DAVID HILBERT: *Methods of Mathematical Physics: Partial Differential Equations, Volume II*; Wiley, 2008; ISBN: 978-0471504399; doi: 10.1002/9783527617234  
(Lebesgue spine)
- WALTER RUDIN: *Real and Complex Analysis*; McGraw-Hill, 2009; ISBN: 978-0-07-054234-1  
(Urysohn's Lemma)

Hyperbolic partial differential equations (for the lecture “Partial Differential Equations II”):

- FRITZ JOHN: *Partial Differential Equations*; Springer, 1999  
ISBN: 0-387-90609-6
- MICHAEL E. TAYLOR: *Partial Differential Equations I - III*; Springer, 1997  
ISBN: 0-387-94653-5, 0-387-94651-9, 0-387-94652-7  
(nice detailed text books)
- JOEL SMOLLER: *Shock waves and reaction-diffusion equations*; Springer, 1994  
ISBN: 3-540-94259-9  
(nicely presented, good motivations, covers most of the material)
- FRIEDRICH SAUVIGNY: *Partial Differential Equations I-II*; Springer, 2012  
ISBN: 978-1-4471-2981-3, 978-1-4471-2984-4;  
doi: 10.1007/978-1-4471-2981-3, 10.1007/3-540-27540-1
- and many more ...

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# 1 A Brief Introduction

An ordinary differential equation (ODE) can be written as:

$$\frac{d}{dt}u(t) = \dot{u}(t) = v(t, u)$$

$$u : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^N$$

This equation involves only derivatives with respect to *one* variable  $t$ .

$$\frac{\partial}{\partial x}f(x, y) + \frac{\partial}{\partial y}f(x, y) = 0$$

This is an example for a partial differential equation.

## 1.1 Definition (Partial Differential Equation)

A *partial differential equation* (PDE) is a (scalar) equation, which involves partial derivatives of an unknown function  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ . We always assume that  $\Omega \subseteq \mathbb{R}^n$  is open.

More generally, a *system of partial differential equations* is a system of equations involving partial derivatives of a function  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ .

Similarly one can define partial differential equations on manifolds.

For ordinary differential equations we considered the initial-value problem:

$$\dot{u}(t) = v(t, u) \qquad u(t_0) = u_0$$

For partial differential equations one considers

- the initial-value problem and
- the boundary-value problem.

## 1.2 Examples

1. Cauchy-Riemann equations: Let

$$f : \Omega \stackrel{\text{open}}{\subseteq} \mathbb{C} \rightarrow \mathbb{C}$$

be holomorphic.

$$f = a + \mathbf{i}b \qquad a := \operatorname{Re}(f) \qquad b := \operatorname{Im}(f)$$

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad \frac{\partial b}{\partial x} = -\frac{\partial a}{\partial y}$$

This is a system of two partial differential equations.

$$u := \begin{pmatrix} a \\ b \end{pmatrix} \quad u : \Omega \subseteq \mathbb{C} = \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = \frac{\partial b}{\partial x \partial y} - \frac{\partial b}{\partial y \partial x} = 0$$

$$\Rightarrow \underbrace{\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}_{=: \Delta} a = 0 \quad \Delta b = 0$$

This is the *Laplace equation* with the *Laplace operator* (or *Laplacian*)  $\Delta$ . Solutions of the Laplace equation are called *harmonic functions*.

2. Let  $(M, g)$  be a Riemannian manifold. Here exists the Laplace-Beltrami operator  $\Delta$ .

– In the special case  $M = \mathbb{R}^n$  we have:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

$$\Delta \varphi = 0$$

– With the Riemannian metric  $g_{ij}$  we can define:

$$\Delta \varphi = g^{ij} \nabla_i \nabla_j \varphi = \operatorname{div}(\operatorname{grad}(\varphi)) = \frac{1}{\sqrt{\det(g)}} \partial_j \left( \sqrt{\det(g)} g^{jk} \partial_k \varphi \right)$$

This gives an elliptic equation.

3. Newton's gravitational law: Let  $\varrho(x)$  be the mass density and  $\varphi(x)$  the Newtonian potential.

$$\Delta \varphi = \underbrace{-4\pi \varrho}_{\text{inhomogeneity}}$$

Such an inhomogeneous Laplace equation is usually referred to as Poisson equation and it is elliptic.

4. Heat flow equation (Wärmeleitungsgleichung): Let  $\varphi(t, x)$  be the temperature at time  $t \in \mathbb{R}$  and position  $x \in \mathbb{R}^n$ .

$$\partial_t \varphi(t, x) = \Delta \varphi(t, x)$$

This is a parabolic equation.

5. The Schrödinger equation is a parabolic equation:

$$\mathbf{i} \hbar \partial_t \psi(t, x) = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \psi(t, x)$$

Additionally to the the heat flow equation there is the potential  $V$ , but more important there is a factor of  $\mathbf{i}$  in front of the partial derivative. The time-independent Schrödinger equation is:

$$E \psi(x) = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \psi(x)$$

This is similar to the Poisson equation and also elliptic.



6. The wave equation

$$(\partial_t^2 - \Delta_x) \psi(t, x) = 0$$

is hyperbolic. We will consider it in the lecture “Partial Differential Equations II”.

7. Maxwell’s equations:  $E(t, x)$  is the electric field and  $B(t, x)$  the magnetic field.

$$\begin{array}{ll} \operatorname{div}(E) = 4\pi\varrho & \text{Gauss law} \\ \operatorname{rot}(E) = -\partial_t B & \text{Maxwell} \\ \operatorname{div}(B) = 0 & \\ \operatorname{rot}(B) = 4\pi j - \partial_t E & \text{Faraday} \end{array}$$

This is a system of 8 partial differential equation.

8. Einstein’s field equation:

$$R_{ij} - \frac{1}{2} R g_{ij} = 4\pi\kappa T_{ij}$$

This is a geometric partial differential equation.  $R_{ij}$  is the Ricci curvature,  $R$  the scalar curvature and  $T_{ij}$  the energy-momentum tensor. It is a system of 10 partial differential equations.

9. Equations of relativistic quantum mechanics:

$$(-\partial_t^2 + \Delta) \psi = m^2 \psi$$

This is the Klein-Gordon equation with the mass  $m$ .

$$\mathbf{i}\gamma^j \partial_j \psi = m\psi$$

This is the Dirac equation, a system of 4 complex-valued or 8 real-valued partial differential equations for a particle with spin  $\frac{1}{2}$ .

10. Water waves can be described by the Korteweg-de Vries equation:

$$\partial_t u + u \partial_x u + \partial_x^3 u = 0$$

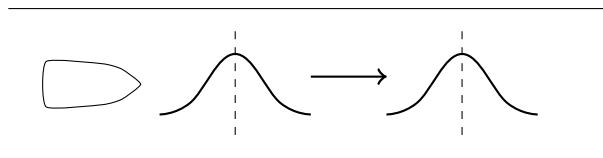


Figure 1.1: Solitons (discovered by John Russel in 1834): When the ship suddenly stops, the water flows on along the channel. This wave moves with a constant speed and its shape stays the same.

11. Shock waves: Burger’s equation

$$\partial_t u + u \partial_x u = 0$$

is hyperbolic.

12. Turbulence can be described by the incompressible Navier-Stokes equations for the velocity  $v : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

$$\begin{array}{l} \operatorname{div}(v) = 0 \\ \rho \partial_t v^j + \rho v^i \partial_i v^j - \eta \Delta v^j = -\partial_j P \end{array}$$

Here  $\varrho$  is the gas density,  $P$  the pressure and  $\eta$  the viscosity.

### 1.3 Classification

I) The *order* of a partial differential equation is the highest order of the derivatives in it.

$$\begin{array}{ll} \Delta u = f & \text{second order} \\ \partial_t \varphi = \Delta \varphi & \text{second order} \\ \partial_t u + u \partial_x u + \partial_x^3 u = 0 & \text{third order} \end{array}$$

II) Algebraic classification:

a) *Linear* equations: The unknown function  $u$  and its derivatives appear only linearly.

$$\begin{array}{ll} \partial_t u = u & \text{linear} \\ \partial_t u + u \partial_x u = 0 & \text{non-linear} \end{array}$$

b) Linear *homogeneous* equations: If  $u$  is a solution, then  $\lambda u$  for  $\lambda \in \mathbb{R}$  is also a solution.

$$\begin{array}{ll} \Delta u = 0 & \text{linear homogeneous} \\ \Delta u = \varrho & \text{linear inhomogeneous} \\ \Delta u = 0 & \text{linear homogeneous} \end{array}$$

c) Linear with *constant coefficients*:

$$\begin{array}{ll} \Delta u = \rho & \text{linear with constant coefficients} \\ \Delta u = \varrho & \text{in general non-constant coefficients} \end{array}$$

III) Classification by type: elliptic, parabolic, hyperbolic

Here we only consider scalar second order equations with  $x \in \Omega \subseteq \mathbb{R}^n$ .

$$\begin{array}{l} F(x, u, Du, D^2u) = 0 \\ F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \end{array}$$

$$A_{ij} := \frac{\partial F(x, u, p_i, p_{ij})}{\partial p_{ij}}$$

is a symmetric  $n \times n$  matrix.

- If  $A$  is positive definite, the equation is called *elliptic*.
  - If  $A$  has  $n - 1$  positive and one negative eigenvalue, the equation is called *hyperbolic*.
  - If  $A$  has  $n - 1$  positive eigenvalues and a non-trivial kernel, the equation is called *parabolic*.
  - If all eigenvalues are negative or  $n - 1$  are negative, then we replace  $F$  by  $-F$ .
- All other case of *mixed type* are difficult and we do not consider them in this lecture.

### 1.4 Examples

Consider the Poisson equation:

$$\Delta u = \varrho$$

$$\begin{aligned}
 F(x, u, Du, D^2u) &= -\varrho(x) + \delta^{ij} \partial_{ij} u \\
 F(x, u, p_i, p_{ij}) &= -\varrho(x) + \delta^{ij} p_{ij} \\
 A_{ij} &= \frac{\partial F(x, u, p_i, p_{ij})}{\partial p_{ij}} = \delta_{ij}
 \end{aligned}$$

So we have  $A = \mathbb{1}$  and thus the equation is elliptic.

Now consider the inhomogeneous wave equation:

$$(\partial_t^2 - \Delta) \phi(t, x_1, x_2, x_3) = \varrho$$

$$F(x, u, p_i, p_{ij}) = \varrho(x) + \eta^{ij} p_{ij} \quad \eta = \text{diag}(-1, 1, 1, 1)$$

So  $A = \eta$  has one negative and three positive eigenvalues which means that the equation is hyperbolic.

$$\partial_t \phi = \Delta \phi$$

$$\begin{aligned}
 F(x, u, D\phi, D^2\phi) &= -\partial_0 \phi + \sum_{i,j=1}^3 \delta^{ij} \partial_{ij} \phi \\
 A_{ij} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

Therefore the equation is parabolic.

## 2 Distributions and Fourier Transform

### Motivation

We want to solve partial differential equations with constant coefficients in  $\Omega = \mathbb{R}^n$ , for example:

$$(-\partial_t^2 + \Delta) \phi = 0$$

Now we make a “plane wave ansatz” with  $t, \omega \in \mathbb{R}$  and  $k, x \in \mathbb{R}^{n-1}$ :

$$\begin{aligned}\phi(t, x) &= e^{-i\omega t + i\langle k, x \rangle} \\ \partial_t \phi(t, x) &= -i\omega \phi(t, x) \\ \partial_j \phi(t, x) &= ik_j \phi(t, x)\end{aligned}$$

This gives an algebraic equation:

$$\begin{aligned}(-(-i\omega)^2 + (ik)^2) \phi &= 0 \\ \Leftrightarrow \quad \omega^2 &= k^2\end{aligned}$$

We also want to differentiate non-smooth functions, e.g.:

$$\Delta_{\mathbb{R}^3} \frac{1}{|x|} = -4\pi \delta(x)$$

$\delta(x)$  is called Dirac  $\delta$ -distribution.

### 2.1 The Schwartz Space and Distributions

Laurent Schwartz was the first to investigate distributions systematically. He was awarded the fields medal for his research.

#### 2.1.1 Definition (Multi-Index)

For  $\mathbb{R}^n$  we denote indices by  $i, j, k \in \{1, \dots, n\}$ . We call  $\alpha = (i_1, \dots, i_k)$  with  $i_l \in \{1, \dots, n\}$  a *multi-index*.  $|\alpha| := k$  is called the *order* or *absolute value* of the multi-index.

With this we can write differentials of order  $k$  as

$$D^\alpha := \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} \tag{2.1}$$

and homogeneous polynomials of degree  $k$  in the components of a vector  $x = (x^1, \dots, x^n)$  as:

$$x^\alpha := x^{i_1} \cdots x^{i_k} \tag{2.2}$$

For  $f \in C^\infty(\mathbb{R}^n)$  and  $r, s \in \mathbb{N}$  we define the *Schwartz norm*:

$$\|f\|_{r,s} := \sum_{\alpha, |\alpha| \leq r} \sum_{\beta, |\beta| \leq s} \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \quad (2.3)$$

For example for  $r = 0 = s$  we have:

$$\|f\|_{0,0} = \sup_{x \in \mathbb{R}^n} |f(x)| = \|f\|_{C^0}$$

### 2.1.2 Definition (Schwartz Space)

The *Schwartz space*  $\mathcal{S}(\mathbb{R}^n)$  is the vector space of all  $f \in C^\infty(\mathbb{R}^n)$  for which all Schwartz norms are finite, i.e. for all  $r, s \in \mathbb{N}$  holds:

$$\|f\|_{r,s} < \infty$$

This space is an infinite-dimensional vector space.

On a normed space  $(E, \|\cdot\|)$ , the topology is given by the open sets.

$\Omega \subseteq E$  is defined as *open* if holds:

$$\forall x \in \Omega \quad \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq \Omega$$

A subset  $\Omega \subseteq \mathcal{S}(\mathbb{R}^n)$  is called *open* if for every  $f \in \Omega$  there is a  $\varepsilon > 0$  and  $r, s \in \mathbb{N}$  such that holds:

$$\left\{ g \in \mathcal{S} \mid \|g - f\|_{r,s} < \varepsilon \right\} \subseteq \Omega \quad (2.4)$$

**Note:** This topology is fine, because it involves many open sets, since the condition for open only involves the statement “there are  $r, s \in \mathbb{N}$ ”.

Convergence  $f_n \rightarrow f$  in  $\mathcal{S}$  means that *every* open neighborhood  $U$  of  $f$  contains almost all  $f_n$ . For a finer topology, the condition for a sequence to converge is stronger.

### 2.1.3 Theorem (Criterion for Convergence)

Convergence  $f_n \rightarrow f$  in  $\mathcal{S}$  is equivalent to the convergence  $\|f_n - f\|_{r,s} \rightarrow 0$  for all  $r, s \in \mathbb{N}$ .

#### Proof

“ $\Rightarrow$ ”: Suppose that  $f_n \rightarrow f$  converges. By definition of the convergence, every open neighborhood of  $f$  contains almost all  $f_n$ . For all  $r, s \in \mathbb{N}$  the sets  $U_\varepsilon^{r,s} := \{g \mid \|g - f\|_{r,s} < \varepsilon\}$  are open by definition. So the inequality

$$\|f_n - f\|_{r,s} < \varepsilon$$

holds for almost all  $f_n$  and thus converges  $\|f_n - f\|_{r,s} \rightarrow 0$ .

“ $\Leftarrow$ ”: Assume that  $\|f_n - f\|_{r,s} \rightarrow 0$  converges for all  $r, s \in \mathbb{N}$ . Let  $A$  be an open neighborhood of  $f$ . This means by definition that there exist  $r, s \in \mathbb{N}$  and  $\varepsilon \in \mathbb{R}_{>0}$  with  $U_\varepsilon^{r,s} \subseteq A$ . For this  $(r, s)$  we know that  $\|f_n - f\|_{r,s} \rightarrow 0$  converges. Hence there exists a  $N \in \mathbb{N}$  such that  $\|f_n - f\|_{r,s} < \varepsilon$  holds for all  $n \in \mathbb{N}_{>N}$ , in other words  $f_n \in U_\varepsilon^{r,s} \subseteq A$ . So  $f_n \rightarrow f$  converges in  $\mathcal{S}$ .  $\square_{2.1.3}$

A vector space with a topology generated by a family of norms or semi-norms is a *uniform space* and is called *topological vector space*.

### 2.1.4 Definition (Tempered Distribution)

Let  $\mathcal{S}^*(\mathbb{R}^n)$  be the dual space of  $\mathcal{S}(\mathbb{R}^n)$ . It is called the space of *tempered distributions* (temperierte Distributionen).

In linear algebra for a finite-dimensional vector space  $V$ , the dual space  $V^* = L(V, \mathbb{R})$  is the space of linear functionals.  $V^*$  is again a vector space with  $\dim(V^*) = \dim(V)$ .

Here  $\mathcal{S}(\mathbb{R}^n)$  is an infinite-dimensional vector space with a topology.  $\mathcal{S}^*(\mathbb{R}^n)$  is the space of all *continuous* linear functionals.

In a Banach space  $(E, \|\cdot\|)$  holds: A linear functional  $A : E \rightarrow \mathbb{R}$  is continuous if and only if  $A$  is bounded, i.e.  $|Au| \leq c \|U\|$  for all  $u \in E$ .

### 2.1.5 Lemma (Criterion for Continuity)

A linear functional  $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is continuous if and only if there are  $r, s \in \mathbb{N}$  and a  $c \in \mathbb{R}_{>0}$  such that holds:

$$|Tf| \leq c \|f\|_{r,s} \quad (2.5)$$

#### Proof

“ $\Leftarrow$ ”: Assume that (2.5) holds for some  $r, s \in \mathbb{N}$ . We want to show that  $T$  is continuous. To this end, let  $f_n \rightarrow f$  be a convergent series in  $\mathcal{S}$ . Our task is to show that  $Tf_n \rightarrow Tf$  converges.

The convergence  $f_n \rightarrow f$  implies  $\|f_n - f\|_{r',s'} \rightarrow 0$  for all  $r', s' \in \mathbb{N}$  and thus in particular for  $r, s$  satisfying the inequality (2.5). By linearity follows:

$$|Tf_n - Tf| = |T(f_n - f)| \leq c \|f_n - f\|_{r,s} \xrightarrow{n \rightarrow \infty} 0$$

So  $T$  maps convergent sequences to convergent sequences and is thus continuous.

“ $\Rightarrow$ ”: Assume that  $T$  is continuous. Then the preimage of open sets is open, in particular  $T^{-1}(B_1(0)) \subseteq \mathcal{S}(\mathbb{R}^n)$  is open. So there exist  $r, s \in \mathbb{N}$  and a  $\varepsilon \in \mathbb{R}_{>0}$  such that holds:

$$T^{-1}(B_1(0)) \supseteq U_{\varepsilon}^{r,s} := \{g \mid \|g\|_{r,s} < \varepsilon\}$$

This implies:

$$\|g\|_{r,s} < \varepsilon \quad \Rightarrow \quad g \in T^{-1}(B_1(0))$$

Now  $g \in T^{-1}(B_1(0))$  means  $|Tg| < 1$ . For any  $f \in \mathcal{S}(\mathbb{R}^n)$  apply this to  $g = \frac{f}{\lambda}$  with  $\lambda \in \mathbb{R}_{>0}$ .

$$\begin{aligned} \frac{1}{\lambda} \|f\|_{r,s} < \varepsilon &\Rightarrow \frac{1}{\lambda} |Tf| < 1 & / \cdot \lambda \\ \|f\|_{r,s} < \lambda \varepsilon &\Rightarrow |Tf| < \lambda \end{aligned}$$

Now choose  $\lambda = \frac{2}{\varepsilon} \|f\|_{r,s}$ , so the left side holds, which implies:

$$|Tf| < \frac{2}{\varepsilon} \|f\|_{r,s} \quad \forall_{f \in \mathcal{S}(\mathbb{R}^n)}$$

□<sub>2.1.5</sub>

### 2.1.6 Example ( $\delta$ -Distribution)

a) Consider the following functional:

$$\begin{aligned}\delta : \mathcal{S}(\mathbb{R}) &\rightarrow \mathbb{R} \\ f &\mapsto f(0)\end{aligned}\tag{2.6}$$

This is obviously linear.

$$|\delta(f)| = |f(0)| \leq \sup_{\mathbb{R}} |f| = \|f\|_{0,0}$$

Hence  $\delta$  is continuous, which means that  $\delta \in \mathcal{S}^*(\mathbb{R})$  is a tempered distribution. A convenient *notation* with  $f \in \mathcal{S}(\mathbb{R})$  is:

$$\delta(f) = \int_{\mathbb{R}} f(x) \delta(x) dx$$

b) In higher dimension  $n \in \mathbb{N}$  we define:

$$\begin{aligned}\delta : \mathcal{S}(\mathbb{R}^n) &\rightarrow \mathbb{R} \\ f &\mapsto f(0)\end{aligned}$$

Again holds  $|\delta(f)| \leq \|f\|_{0,0}$ . The physicists' notation for this is:

$$\begin{aligned}\delta(f) &= \int_{\mathbb{R}^n} f(x) \delta^{(n)}(x) dx \\ \delta^{(n)}(x) &= \delta(x^1) \cdots \delta(x^n)\end{aligned}$$

#### Remark

$\delta$  can also be introduced as a *measure* on  $\mathbb{R}^n$ , the *Dirac measure*. For  $A \subseteq \mathbb{R}^n$  define:

$$\delta(x) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{otherwise} \end{cases}\tag{2.7}$$

Then for  $f \in C^0(\mathbb{R}^n)$  the expression

$$\int_{\mathbb{R}^n} f(x) d\delta(x) = f(0)$$

makes mathematical sense as an integral.

This is useful because convergence theorems and so on from measure theory are available. The problem is, that this does not work for every distribution and thus is not general enough for most purposes, e.g. the derivative  $\delta'(x)$  is a distribution, but cannot be written as a measure.

### 2.1.7 Example (Integral Operator)

a) Consider  $g \in C^\infty(\mathbb{R}^n)$  with at most polynomial growth, i.e. there are  $c \in \mathbb{R}_{>0}$  and  $r \in \mathbb{N}$  such that holds:

$$|g(x)| \leq c(1 + |x|^r)$$

Now define:

$$T_g : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$f \mapsto \int_{\mathbb{R}^n} g(x) f(x) d^n x$$

The integral here is just the Lebesgue integral and it exists:

For  $f \in \mathcal{S}(\mathbb{R}^n)$  holds  $\|f\|_{r,s} < \infty$  for all  $r, s \in \mathbb{N}$ .

$$\sup_{\mathbb{R}} |f| + \sup_{x \in \mathbb{R}} (|x|^{\tilde{r}} \cdot |f(x)|) \leq \|f\|_{\tilde{r},0} < \infty$$

$$\Rightarrow \sup_{x \in \mathbb{R}} \left( (1 + |x|^{\tilde{r}}) |f(x)| \right) \leq \|f\|_{\tilde{r},0}$$

$$\Rightarrow |f(x)| \leq \frac{\|f\|_{\tilde{r},0}}{1 + |x|^{\tilde{r}}} \quad \forall \tilde{r} \in \mathbb{N}$$

So we get:

$$T_g f = \int g(x) f(x) d^n x \leq \int c(1 + |x|^r) \frac{\|f\|_{\tilde{r},0}}{(1 + |x|^{\tilde{r}})} d^n x =$$

$$\stackrel{\substack{\text{polar coordinates} \\ \rho := |x|}}{=} c \|f\|_{\tilde{r},0} \underbrace{\mu(S^{n-1})}_{\text{volume of unit sphere}} \int_0^\infty \rho^{n-1} \frac{1 + \rho^r}{1 + \rho^{\tilde{r}}} d\rho \stackrel{\tilde{r} > r+n}{<} \infty$$

This is finite if and only if the integrand decays faster than  $\rho^{-1}$ , i.e.  $n - 1 + r - \tilde{r} < -1$  and thus  $\tilde{r} > n + r$ . Since  $\tilde{r} \in \mathbb{N}$  is arbitrary, the integral exists.

Continuity: The previous estimate implies with  $\tilde{r} = n + r + 1$ :

$$|T_g f| \leq C(g, n) \|f\|_{\tilde{r},0}$$

Thus  $T_g \in \mathcal{S}^*(\mathbb{R}^n)$  is a tempered distribution.

b) Chose  $g(x) = e^x$  and define:

$$T_g f := \int_{-\infty}^{\infty} f(x) e^x dx$$

This is *not* a well-defined tempered distribution. Namely, choose:

$$f(x) = \frac{1}{\cosh\left(\frac{x}{2}\right)}$$

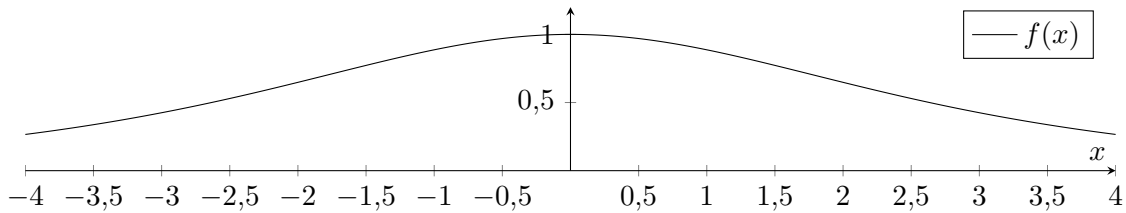


Figure 2.1:  $f(x)$  decays rapidly.

$f(x)$  and all its derivatives decay rapidly (exponentially fast  $\sim e^{-\frac{x}{2}}$ ) at  $\pm\infty$ , so  $f \in \mathcal{S}$  is a Schwartz function. But  $T_g f$  diverges:

$$T_g f = \int_{-\infty}^{\infty} \frac{e^x}{\cosh\left(\frac{x}{2}\right)} dx = +\infty$$



**2.1.8 Remark** (Schwartz Functions as Distributions)

The mapping

$$\begin{aligned} T : \mathcal{S}(\mathbb{R}^n) &\rightarrow \mathcal{S}^*(\mathbb{R}^n) \\ f &\mapsto T_f \end{aligned}$$

is injective.

**Proof**

If  $T$  was injective, there were  $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$  with  $T_{f_1} = T_{f_2}$  and  $f_1 \neq f_2$ . By linearity this would imply  $T_g = 0$  with  $g = f_1 - f_2 \neq 0$  and we could choose a  $y \in \mathbb{R}^n$  with  $g(y) \neq 0$ . By continuity follows  $g > 0$  or  $g < 0$  in a neighborhood  $U$  of  $y$ . Now choose a test function  $h$  with  $\text{supp}(h) \subseteq U$  and  $h \geq 0$ . Then follows the contradiction:

$$0 = T_g(h) = \int_{\mathbb{R}^n} g(x) h(x) \, d^n x \neq 0$$

□<sub>2.1.8</sub>

Thus we can regard distributions as “generalized functions”. Namely we identify a function  $g$  with  $T_g$ . (Later on many people often do not distinguish between  $g$  and  $T_g$ .)

**Operations on Schwartz Functions and Distributions**

- $\mathcal{S}^*(\mathbb{R}^n)$  is a vector space with addition  $T+S$  and scalar multiplication  $\alpha \cdot f$  for distributions  $T, S \in \mathcal{S}^*(\mathbb{R}^n)$  and  $\alpha \in \mathbb{R}$ .
- Multiplication of a distribution by a Schwartz function is defined for  $T \in \mathcal{S}^*(\mathbb{R}^n)$  and  $f, g \in \mathcal{S}(\mathbb{R}^n)$  as:

$$(fT)(g) := T(f \cdot g) \tag{2.8}$$

This is well defined, because  $f \cdot g$  is again a Schwartz function and for  $h \in \mathcal{S}(\mathbb{R}^n)$  holds:

$$\begin{aligned} (fT_h)(g) &\stackrel{\text{definition of } fT_h}{=} T_h(f \cdot g) \stackrel{\text{definition of } T_h}{=} \int_{\mathbb{R}^n} h(x) (f \cdot g)(x) \, d^n x = \\ &= \int_{\mathbb{R}^n} (f \cdot h)(x) g(x) \, d^n x = T_{fh}(g) \end{aligned}$$

So this definition extends the multiplication of Schwartz functions to distributions. But we still have to show, that this operation gives a continuous functional.

**2.1.9 Definition** (regular/singular distribution)

A tempered distribution  $T$  is called *regular* distribution if there is a  $g \in L^1_{\text{loc}}(\mathbb{R}^n)$  (locally integrable, i.e. integrable on every compact interval) with  $T = T_g$ . Otherwise,  $T$  is called *singular*.

For example  $\delta(x)$  is singular.

**2.1.10 Lemma** (Multiplication of a Distribution by a Schwartz Function)

Let  $f \in \mathcal{S}(\mathbb{R}^n)$  be a Schwartz function and  $T \in \mathcal{S}^*(\mathbb{R}^n)$  a distribution. Then  $fT$  is a *continuous* linear functional on  $\mathcal{S}(\mathbb{R}^n)$ , in other words  $fT \in \mathcal{S}^*(\mathbb{R}^n)$  is also a distribution.

**Proof**

According to Lemma 2.1.5, our task is to show that there are  $r, s \in \mathbb{N}$  and a  $C \in \mathbb{R}_{>0}$  with:

$$|(fT)(g)| \leq C \|g\|_{r,s}$$

Since  $T$  is continuous, there exist  $r, s \in \mathbb{N}$  and a  $\tilde{C} \in \mathbb{R}_{>0}$  with:

$$|T(fg)| \leq \tilde{C} \|fg\|_{r,s}$$

Thus it remains to show that there is a  $c(r, s) \in \mathbb{R}_{>0}$  such that for all  $f, g \in \mathcal{S}(\mathbb{R}^n)$  holds:

$$\|fg\|_{r,s} \leq c(r, s) \|f\|_{r,s} \cdot \|g\|_{r,s}$$

This inequality can be proven by induction in  $s$ .

Induction basis  $s = 0$ :

$$\begin{aligned} \|fg\|_{r,0} &= \sum_{|\alpha| \leq r} \sup_{x \in \mathbb{R}^n} (|x^\alpha f(x) g(x)|) \leq \\ &\leq \sup_{y \in \mathbb{R}^n} |g(y)| \sum_{|\alpha| \leq r} \sup_{x \in \mathbb{R}^n} (|x^\alpha f(x)|) = \\ &= \|g\|_{0,0} \cdot \|f\|_{r,0} \leq \|g\|_{r,0} \cdot \|f\|_{r,0} \end{aligned}$$

Induction step  $s \rightsquigarrow s+1$ : Assume that the statement holds for a  $s \in \mathbb{N}$  for all  $r \in \mathbb{N}$ . Let  $\beta$  be a multi-index with  $|\beta| = s+1$ , i.e.  $\beta = (i_1, \dots, i_{s+1})$ . Now set:

$$\hat{\beta} := (i_1, \dots, i_s) \quad j := i_{s+1} \quad D^\beta = D^{\hat{\beta}} \frac{\partial}{\partial x^j}$$

It holds:

$$D^\beta(fg) = D^{\hat{\beta}} \frac{\partial}{\partial x^j} (fg) = D^{\hat{\beta}} \left( \left( \frac{\partial}{\partial x^j} f \right) g + f \left( \frac{\partial}{\partial x^j} g \right) \right)$$

$$\begin{aligned} \|fg\|_{r,s+1} &= \|f \cdot g\|_{r,s} + \sum_{\substack{|\alpha| \leq r \\ |\beta|=s+1}} \sup_{x \in \mathbb{R}} \left| x^\alpha D^\beta (f \cdot g)(x) \right| = \\ &= \|fg\|_{r,s} + \sum_{\substack{|\alpha| \leq r \\ |\hat{\beta}|=s}} \sum_{j=1}^n \sup_{x \in \mathbb{R}} \left| x^\alpha D^{\hat{\beta}} \left( \left( \frac{\partial}{\partial x^j} f(x) \right) g(x) + f(x) \left( \frac{\partial}{\partial x^j} g(x) \right) \right) \right| \leq \\ &\stackrel{\substack{\text{induction} \\ \leq \\ \text{hypothesis}}}{\leq} c(r, s) \|f\|_{r,s} \|g\|_{r,s} + \sum_{j=1}^n c(r, s) \left( \left\| \frac{\partial}{\partial x^j} f \right\|_{r,s} \|g\|_{r,s} + \|f\|_{r,s} \left\| \frac{\partial}{\partial x^j} g \right\|_{r,s} \right) \leq \\ &\leq c(r, s) \|f\|_{r,s} \|g\|_{r,s} + n \cdot c(r, s) \left( \|f\|_{r,s+1} \|g\|_{r,s} + \|f\|_{r,s} \|g\|_{r,s+1} \right) \leq \\ &\leq \underbrace{(2n+1) c(r, s)}_{=c(r,s+1)} \|f\|_{r,s+1} \|g\|_{r,s+1} \end{aligned}$$

□2.1.10

**2.1.11 Example** (Derivative of the  $\delta$ -Distribution)

We make a formal computation:

$$\int_{\mathbb{R}} \delta'(x) f(x) dx = \int_{\mathbb{R}} \left( \frac{d}{dx} \delta(x) \right) f(x) dx \stackrel{\text{integration by parts}}{=} - \int_{\mathbb{R}} \delta(x) f'(x) dx = -f'(0)$$

This motivates us to *define*:

$$\begin{aligned} \delta' : \mathcal{S}(\mathbb{R}) &\rightarrow \mathbb{R} \\ f &\mapsto -f'(0) \end{aligned} \tag{2.9}$$

This is obviously linear and it is continuous, because for all  $f \in \mathcal{S}(\mathbb{R})$  holds:

$$|\delta'(f)| = |f'(0)| \leq \|f\|_{0,1}$$

Hence we have  $\delta' \in \mathcal{S}^*(\mathbb{R})$ .

**Remark**

In contrast to  $\delta$ , the distribution  $\delta'$  cannot be introduced as a measure:

$$\begin{aligned} \delta(\Omega) &= \begin{cases} 1 & \text{if } 0 \in \Omega \\ 0 & \text{otherwise} \end{cases} \\ \delta'(\Omega) &=? \end{aligned}$$

**2.1.12 Definition** (Distributional Derivative, Convolution)

- For a tempered distribution  $T \in \mathcal{S}^*(\mathbb{R}^n)$  we define the *distributional derivative*  $D^\alpha T$  by:

$$(D^\alpha T)(f) := (-1)^{|\alpha|} T(D^\alpha f) \tag{2.10}$$

$D^\alpha T$  is a distribution, since it is a continuous functional:

$$|(D^\alpha T)(f)| = |T(D^\alpha f)| \stackrel{T \text{ continuous}}{\leq} C \|D^\alpha f\|_{r,s} \leq C \|f\|_{r,s+|\alpha|}$$

So we have a mapping  $D^\alpha : \mathcal{S}^*(\mathbb{R}^n) \rightarrow \mathcal{S}^*(\mathbb{R}^n)$ .

- The convolution (Faltung) for  $f, g \in \mathcal{S}(\mathbb{R}^n)$  is defined as:

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y) g(y) dy \tag{2.11}$$

**2.1.13 Lemma** (Commutativity and Associativity of the Convolution)

The convolution is commutative and associative, i.e. for  $f, g, h \in \mathcal{S}(\mathbb{R}^n)$  holds:

$$\begin{aligned} f * g &= g * f & f * (g * h) &= (f * g) * h \end{aligned} \tag{2.12}$$

**Proof**

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) d^n y \stackrel{z:=x-y}{\underset{d^n z = d^n y}{=}} \int_{\mathbb{R}^n} f(z) g(x-z) d^n z = (g * f)(x)$$

Associativity follows analogously using Fubini's theorem, which can be applied, since the function  $(y, z) \mapsto f(x-y) g(y-z) h(z)$  is an element of  $\mathcal{S}(\mathbb{R}^{2n}) \subseteq L^1(\mathbb{R}^{2n})$ :

$$\begin{aligned} f * (g * h)(x) &= \int_{\mathbb{R}^n} f(x-y) (g * h)(y) d^n y = \\ &= \int_{\mathbb{R}^n} f(x-y) \left( \int_{\mathbb{R}^n} g(y-z) h(z) d^n z \right) d^n y = \\ &\stackrel{\tilde{y}:=y-z}{\underset{d^n \tilde{y} = d^n y}{=}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-\tilde{y}-z) g(\tilde{y}) h(z) d^n \tilde{y} d^n z = \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x-z-\tilde{y}) g(\tilde{y}) d^n \tilde{y} \right) h(z) d^n z = \\ &= \int_{\mathbb{R}^n} (f * g)(x-z) h(z) d^n z = \\ &= (f * g) * h(x) \end{aligned}$$

□<sub>2.1.13</sub>

### 2.1.14 Proposition (Convolution is Continuous)

The convolution  $*$  :  $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  is continuous.

**Proof**

$$\|f * g\|_{r,s} = \sum_{\substack{|\alpha| \leq r \\ |\beta| \leq s}} \sup_{x \in \mathbb{R}^n} \left| x^\alpha D_x^\beta \int_{\mathbb{R}^n} f(x-y) g(y) d^n y \right|$$

The derivative may be commuted with the integral:

$$\begin{aligned} \frac{\partial}{\partial x_j} (f * g)(x) &= \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} ((f * g)(x + \varepsilon e_j) - (f * g)(x)) = \\ &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} \frac{f(x + \varepsilon e_j - y) - f(x - y)}{\varepsilon} g(y) d^n y \end{aligned}$$

Using the estimate

$$\begin{aligned} \frac{f(x + \varepsilon e_j - y) - f(x - y)}{\varepsilon} &= \frac{1}{\varepsilon} \int_0^1 \frac{d}{d\tau} (f(x + \varepsilon \tau e_j - y)) d\tau = \\ &= \frac{1}{\varepsilon} \int_0^1 (\partial_j f)(x + \varepsilon \tau e_j - y) \varepsilon d\tau \\ \Rightarrow \left| \frac{f(x + \varepsilon e_j - y) - f(x - y)}{\varepsilon} \right| &\leq \sup_{\mathbb{R}^n} |\partial_j f| \leq \|f\|_{0,1} \end{aligned}$$

we get:

$$\left| \frac{f(x + \varepsilon e_j - y) - f(x - y)}{\varepsilon} g(y) \right| \leq \|f\|_{0,1} \cdot |g(y)| \in L^1(\mathbb{R}^n)$$

Thus the dominated convergence theorem implies:

$$\frac{\partial}{\partial x_j} (f * g)(x) = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} f(x - y) g(y) d^n y$$

By induction follows:

$$x^\alpha D^\beta \int_{\mathbb{R}^n} f(x - y) g(y) d^n y = x^\alpha \int_{\mathbb{R}^n} (D^\beta f)(x - y) g(y) d^n y$$

Now we treat the  $x^\alpha$ :

$$x^\alpha = ((x - y) + y)^\alpha = \sum_{\gamma, \delta \text{ with } \alpha = \gamma + \delta} c_{\gamma\delta} (x - y)^\gamma \cdot y^\delta$$

Here holds  $|\gamma|, |\delta| \leq |\alpha|$ . Now we can estimate:

$$\left| (x - y)^\gamma D^\beta f(x) \right| \leq \|f\|_{r,s}$$

Hence we get:

$$\begin{aligned} \|f * g\|_{r,s} &\leq c(r, s) \|f\|_{r,s} \sum_{\delta} \int_{\mathbb{R}^n} |y^\delta g(y)| \cdot \frac{(1 + |y|)^{n+1}}{(1 + |y|)^{n+1}} d^n y \leq \\ &\leq c(r, s) \|f\|_{r,s} \|g\|_{n+1+r,0} \underbrace{\int_{\mathbb{R}^n} \frac{d^n y}{(1 + |y|)^{n+1}}}_{< \infty} \end{aligned}$$

□<sub>2.1.14</sub>

### 2.1.15 Definition (Convolution with Distribution)

How can we define the convolution of  $T \in \mathcal{S}^*(\mathbb{R}^n)$  with  $f \in \mathcal{S}(\mathbb{R}^n)$ ?  $f * T_g$  should be equal to  $T_{f*g}$ . For  $h \in \mathcal{S}(\mathbb{R}^n)$  holds:

$$\begin{aligned} T_{f*g}(h) &= \int_{\mathbb{R}^n} (f * g)(x) \cdot h(x) d^n x = \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x - y) g(y) d^n y \right) \cdot h(x) d^n x \end{aligned}$$

By Fubini's theorem we may interchange the order of integration to get:

$$\begin{aligned} T_{f*g}(h) &= \int_{\mathbb{R}^n} g(y) \left( \int_{\mathbb{R}^n} f(x - y) \cdot h(x) d^n x \right) d^n y = \\ &\stackrel{\tilde{f}(z) := f(-z)}{=} \int_{\mathbb{R}^n} g(y) \left( \int_{\mathbb{R}^n} \tilde{f}(y - x) \cdot h(x) d^n x \right) d^n y = \\ &= \int_{\mathbb{R}^n} g(y) (\tilde{f} * h)(y) d^n y = T_g(\tilde{f} * h) \end{aligned}$$

So for a distribution  $T \in \mathcal{S}^*(\mathbb{R}^n)$  we define the *convolution* as:

$$\begin{aligned} * : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}^*(\mathbb{R}^n) &\rightarrow \mathcal{S}^*(\mathbb{R}^n) \\ (f * T)(h) &:= T(\tilde{f} * h) \end{aligned} \tag{2.13}$$

For  $S, T \in \mathcal{S}^*(\mathbb{R}^n)$ ,  $S * T$  and  $S \cdot T$  are ill-defined in general. For example  $\delta(x) \cdot \delta(x)$  makes no sense, as well as  $T_f * T_f$  for  $f = 1$ .

## 2.2 The Fourier Transform

First consider the Fourier transform on  $\mathcal{S}(\mathbb{R}^n)$  and later on  $\mathcal{S}'(\mathbb{R}^n)$ .

### 2.2.1 Definition (Fourier Transform)

Define linear functionals  $\mathcal{F}$  and  $\overline{\mathcal{F}}$  on  $\mathcal{S}$ :

$$(\mathcal{F}f)(p) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ipx} f(x) \, d^n x \quad (2.14)$$

$$(\overline{\mathcal{F}}f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ipx} f(p) \, d^n p \quad (2.15)$$

The integrals are well-defined and finite, because  $f$  has suitable decay properties at infinity. An alternative convention, which is not convenient here, because it has less symmetry, is:

$$(\mathcal{F}f)(p) := \int_{\mathbb{R}^n} e^{-ipx} f(x) \, d^n x$$

$$(\overline{\mathcal{F}}f)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ipx} f(p) \, d^n p$$

### 2.2.2 Proposition (Fourier Transform)

$\mathcal{F}$  and  $\overline{\mathcal{F}}$  are well-defined linear operators from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}(\mathbb{R}^n)$ .

#### Proof

The linearity is clear. We still have to show, that all norms  $\|\mathcal{F}f\|_{r,s}$  are finite. First consider the norm  $\|\cdot\|_{0,0}$ :

$$\begin{aligned} |(\mathcal{F}f)(p)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)| \cdot \frac{(1+|x|)^{n+1}}{(1+|x|)^{n+1}} \, d^n x \leq \\ &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_{n+1,0} \underbrace{\int_{\mathbb{R}^n} \frac{d^n x}{(1+|x|)^{n+1}}}_{<\infty} \leq c \|f\|_{n+1,0} \end{aligned}$$

Now we consider  $|p^\alpha D^\beta (\mathcal{F}f)(p)|$ .

$$\begin{aligned} \frac{\partial}{\partial p^j} (\mathcal{F}f)(p) &= \frac{\partial}{\partial p^j} \int_{\mathbb{R}^n} e^{-ipx} f(x) \, d^n x = \dots = \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial p^j} e^{-ipx} \right) f(x) \, d^n x = \\ &= \int_{\mathbb{R}^n} (-ix^j) e^{-ipx} f(x) \, d^n x \end{aligned}$$

That the derivative and the integral can be interchanged is shown as follows:

$$\begin{aligned} \frac{\partial}{\partial p^j} \int_{\mathbb{R}^n} e^{-ipx} f(x) \, d^n x &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} \frac{e^{-i(p+\varepsilon e_j)x} - e^{-ipx}}{\varepsilon} f(x) \, d^n x \\ \frac{e^{-i(p+\varepsilon e_j)x} - e^{-ipx}}{\varepsilon} &= \frac{1}{\varepsilon} \int_0^1 \frac{d}{d\tau} e^{-i(p+\varepsilon \tau e_j)x} \, d\tau = -ie_j x \int_0^1 e^{-i(p+\varepsilon \tau e_j)x} \, d\tau \end{aligned}$$

$$\Rightarrow \left| \frac{e^{-i(p+\varepsilon e_j)x} - e^{-ipx}}{\varepsilon} \right| \leq \|x\| \cdot \underbrace{\int_0^1 \left| e^{-i(p+\varepsilon \tau e_j)x} \right| d\tau}_{=1} = \|x\|$$

$$\left| \frac{e^{-i(p+\varepsilon e_j)x} - e^{-ipx}}{\varepsilon} f(x) \right| \leq \|x\| \cdot |f(x)| \leq \frac{\|x\|}{1 + \|x\|^{n+2}} \|f\|_{n+2,0} =: h(x) \in L^1(\mathbb{R}^n)$$

This allows us to apply the dominated convergence theorem to take the limit  $\varepsilon \rightarrow 0$  inside the integral. Iteration of this process gives:

$$\begin{aligned} D^\beta(\mathcal{F}f)(p) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}} (-i)^{|\beta|} x^\beta f(x) e^{-ipx} d^n x \\ p^\alpha D^\beta(\mathcal{F}f)(p) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}} (-i)^{|\beta|} x^\beta f(x) p^\alpha e^{-ipx} d^n x \\ p^j e^{-ipx} &= i \frac{\partial}{\partial x^j} e^{-ipx} \\ \Rightarrow p^\alpha e^{-ipx} &= i^{|\alpha|} D_x^\alpha e^{-ipx} \\ p^\alpha D^\beta(\mathcal{F}f)(p) &= \frac{(-i)^{|\beta|} i^{|\alpha|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \underbrace{(x^\beta f(x))}_{\text{rapid decay}} (D_x^\alpha e^{-ipx}) d^n x = \\ &\stackrel{\substack{\text{integration} \\ \text{by parts}}}{=} \frac{(-i)^{|\beta|} i^{|\alpha|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (-1)^{|\alpha|} D_x^\alpha (x^\beta f(x)) e^{-ipx} d^n x = \\ &= \frac{(-i)^{|\alpha|+|\beta|}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} D_x^\alpha (x^\beta f(x)) e^{-ipx} d^n x \end{aligned}$$

From the computation we did earlier in the proof we know:

$$|D_x^\alpha(\mathcal{F}f)(p)| \leq C \left\| D_x^\alpha (x^\beta f) \right\|_{n+1,0} \leq \tilde{C}(\alpha, \beta) \|f\|_{|\beta|+n+1,|\alpha|}$$

$$\|\mathcal{F}f\|_{r,s} \leq \tilde{c}(s, r, n) \|f\|_{s+n+1,r}$$

Therefore  $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$  is well-defined. The same follows analogously for  $\overline{\mathcal{F}}$ .  $\square_{2.2.2}$

So we have the following correspondence:

$$-ix^j \leftrightarrow \frac{\partial}{\partial p^j} \tag{2.16}$$

$$-i \frac{\partial}{\partial x^j} \leftrightarrow p^j \tag{2.17}$$

Here the derivatives always act on  $f$  or  $\mathcal{F}f$  and not on  $e^{-ipx}$ .

$$x^\alpha D^\beta f \leftrightarrow i^{|\alpha|+|\beta|} D_p^\alpha p^\beta (\mathcal{F}f)(p) = i^{|\alpha|+|\beta|} p^\beta D^\alpha (\mathcal{F}f)(p) + \text{lower order terms}$$

Suppose we had worked with  $\|f\|_{0,k} = |f|_{C^k}$  as family of norms. Then the norms of the Fourier transform of a function with finite norms would not necessarily be finite.

### 2.2.3 Theorem (Plancherel, Convergence Generating Factor)

$\mathcal{F}$  and  $\overline{\mathcal{F}}$  are inverse to each other:

$$\overline{\mathcal{F}}\mathcal{F} = \mathcal{F}\overline{\mathcal{F}} = \mathbb{1} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n) \tag{2.18}$$

**Proof**

$$\begin{aligned}
 (\mathcal{F}\overline{\mathcal{F}}f)(p) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ipx} \underbrace{\left( \int_{\mathbb{R}^n} e^{iqx} f(q) d^n q \right)}_{(\overline{\mathcal{F}}f)(x)} d^n x \stackrel{?}{=} f(p) \\
 &\stackrel{?}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(q) \left( \int_{\mathbb{R}^n} e^{-i(p-q)x} d^n x \right) d^n q
 \end{aligned}$$

The problem here is, that  $e^{-i(p-q)x}$  does not decay at infinity, so the integral is not well-defined. Instead we have to introduce a *convergence generating factor*  $e^{-\varepsilon x^2}$  and, after integrating, calculate the limes  $\varepsilon \rightarrow 0$ .

$$\begin{aligned}
 (\mathcal{F}\overline{\mathcal{F}}f)(p) &= \frac{1}{(2\pi)^n} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} e^{-ipx} e^{-\varepsilon x^2} \left( \int_{\mathbb{R}^n} e^{iqx} f(q) d^n q \right) d^n x = \\
 &\stackrel{\text{Fubini}}{=} \frac{1}{(2\pi)^n} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} f(q) \left( \int_{\mathbb{R}^n} e^{-i(p-q)x} e^{-\varepsilon x^2} d^n x \right) d^n q
 \end{aligned}$$

The resulting Gaussian integral can be computed in closed form. In one dimension it is:

$$\begin{aligned}
 \int_{\mathbb{R}} e^{-i\lambda x} e^{-\varepsilon x^2} dx &= \int_{\mathbb{R}} e^{-\varepsilon \left(x + \frac{i\lambda}{2\varepsilon}\right)^2 - \frac{\lambda^2}{4\varepsilon}} dx = e^{-\frac{\lambda^2}{4\varepsilon}} \int_{\mathbb{R}} e^{-\varepsilon \left(x + \frac{i\lambda}{2\varepsilon}\right)^2} dx = \\
 &\stackrel{z = \sqrt{\varepsilon} \left(x + \frac{i\lambda}{2\varepsilon}\right)}{=} e^{-\frac{\lambda^2}{4\varepsilon}} \int_{\mathbb{R} + \frac{i\lambda}{2\sqrt{\varepsilon}}} e^{-z^2} \frac{dz}{\sqrt{\varepsilon}} = \\
 &\stackrel{\text{contour deformation}}{=} \frac{e^{-\frac{\lambda^2}{4\varepsilon}}}{\sqrt{\varepsilon}} \underbrace{\int_{\mathbb{R}} e^{-z^2} dz}_{=\sqrt{\pi}} = \sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\lambda^2}{4\varepsilon}}
 \end{aligned}$$

So we get:

$$(\mathcal{F}\overline{\mathcal{F}}f)(p) = \frac{1}{(2\pi)^n} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n} f(q) \left( \frac{\pi}{\varepsilon} \right)^{\frac{n}{2}} e^{-\frac{(p-q)^2}{4\varepsilon}} d^n q$$

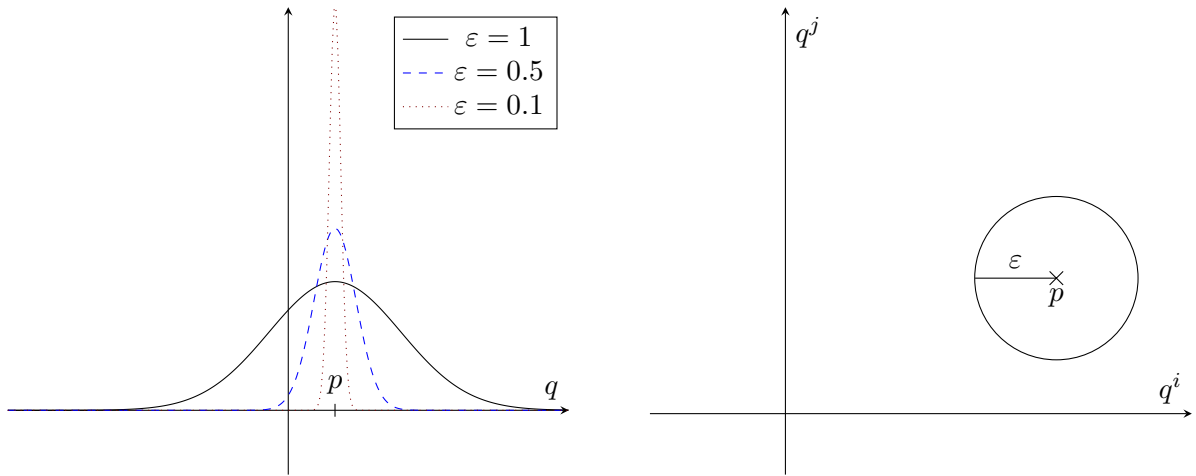


Figure 2.2: The Gaussian gets very narrow and very high as  $\varepsilon$  decreases.



Estimate the integral as follows:

$$(\mathcal{F}\overline{\mathcal{F}}f)(p) = \lim_{\varepsilon \searrow 0} \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}} \left( \int_{\mathbb{R}^n} f(p) e^{-\frac{(p-q)^2}{4\varepsilon}} d^n q + \int_{\mathbb{R}^n} (f(q) - f(p)) e^{-\frac{(p-q)^2}{4\varepsilon}} d^n q \right)$$

The first integral gives:

$$\int_{\mathbb{R}^n} e^{-\frac{(p-q)^2}{4\varepsilon}} d^n q \stackrel{\substack{z=\frac{p-q}{2\sqrt{\varepsilon}} \\ d^n z = \frac{d^n q}{(4\varepsilon)^{\frac{n}{2}}}}}{=} (4\varepsilon)^{\frac{n}{2}} \underbrace{\int_{\mathbb{R}^n} e^{-z^2} d^n q}_{=\pi^{\frac{n}{2}}} = (4\pi\varepsilon)^{\frac{n}{2}}$$

So we get:

$$(\mathcal{F}\overline{\mathcal{F}}f)(p) = f(p) + \lim_{\varepsilon \searrow 0} \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (f(q) - f(p)) e^{-\frac{(p-q)^2}{4\varepsilon}} d^n q$$

It remains to show that the second summand goes to zero for  $\varepsilon \rightarrow 0$ . We use the following scaling argument:

$$\begin{aligned} \frac{1}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} (f(q) - f(p)) e^{-\frac{(p-q)^2}{4\varepsilon}} d^n q &\stackrel{\substack{u=\frac{p-q}{\sqrt{\varepsilon}} \\ d^n q = \varepsilon^{\frac{n}{2}} d^n u}}{=} \frac{1}{\varepsilon^{\frac{n}{2}}} \int_{\mathbb{R}^n} (f(p - \sqrt{\varepsilon}u) - f(p)) e^{-\frac{u^2}{4}} \varepsilon^{\frac{n}{2}} d^n u = \\ &= \int_{\mathbb{R}^n} \underbrace{(f(p - \sqrt{\varepsilon}u) - f(p)) e^{-\frac{u^2}{4}}}_{\xrightarrow{\varepsilon \searrow 0} 0 \text{ pointwise}} d^n u \end{aligned}$$

For the integrand holds:

$$\underbrace{(f(p - \sqrt{\varepsilon}u) - f(p)) e^{-\frac{u^2}{4}}}_{\xrightarrow{\varepsilon \searrow 0} 0 \text{ pointwise}} \leq \|f\|_{0,0} e^{-\frac{u^2}{4}} \in L^1(\mathbb{R}^n)$$

So the dominated convergence theorem can be applied to get:

$$(\mathcal{F}\overline{\mathcal{F}}f)(p) = f(p) + \lim_{\varepsilon \searrow 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \underbrace{\int_{\mathbb{R}^n} \lim_{\varepsilon \searrow 0} (f(p - \sqrt{\varepsilon}u) - f(p)) e^{-\frac{u^2}{4}} d^n u}_{=0} = f(p)$$

$\overline{\mathcal{F}}\mathcal{F} = \mathbb{1}$  follows analogously. □<sub>2.2.3</sub>

We want to generalize the Fourier transform to  $\mathcal{S}^*(\mathbb{R}^n)$ . We begin with the case  $T_g$  with  $g \in \mathcal{S}(\mathbb{R}^n)$ . We want:

$$\mathcal{F}(T_g) = T_{\mathcal{F}g}$$

$$T_{\mathcal{F}g}(f) = \int_{\mathbb{R}^n} (\mathcal{F}g)(p) f(p) d^n p = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-ipx} g(x) d^n x \right) f(p) d^n p$$

Fubini's theorem allows us to interchange the order of integration:

$$T_{\mathcal{F}g}(f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(x) \underbrace{\left( \int_{\mathbb{R}^n} e^{-ipx} f(p) d^n p \right)}_{=\mathcal{F}f(x)} d^n x = \int_{\mathbb{R}^n} g(x) (\mathcal{F}f)(x) d^n x = T_g(\mathcal{F}f)$$

Since we want  $(\mathcal{F}T_g)(f) = T_{\mathcal{F}g} = T_g(\mathcal{F}f)$ , this motivates the following general definition:

### 2.2.4 Definition (Fourier Transform of Distributions)

$\mathcal{F}, \overline{\mathcal{F}} : \mathcal{S}^*(\mathbb{R}^n) \rightarrow \mathcal{S}^*(\mathbb{R}^n)$  are defined by their action on a test function  $f \in \mathcal{S}(\mathbb{R}^n)$ :

$$(\mathcal{F}T)(f) := T(\mathcal{F}f) \quad (2.19)$$

$$(\overline{\mathcal{F}}T)(f) := T(\overline{\mathcal{F}}f) \quad (2.20)$$

It holds:

$$|(\mathcal{F}T)(f)| = |T(\mathcal{F}f)| \stackrel{T \in \mathcal{S}^*(\mathbb{R}^n)}{\leq} \underset{\Rightarrow \exists r, s \in \mathbb{N}, c \in \mathbb{R}_{>0}}{c} \|\mathcal{F}f\|_{r,s} \leq \tilde{c} \|f\|_{s+n+1,r}$$

Thus  $\mathcal{F}T$  is indeed a tempered distribution.

### 2.2.5 Theorem (Plancherel for Distributions)

Plancherel's theorem holds on  $\mathcal{S}^*(\mathbb{R}^n)$  as well:

$$\mathcal{F}\overline{\mathcal{F}} = \overline{\mathcal{F}}\mathcal{F} = \mathbb{1}_{\mathcal{S}^*(\mathbb{R}^n)} \quad (2.21)$$

**Proof**

$$(\mathcal{F}\overline{\mathcal{F}}T)(f) \stackrel{\text{Definition 2.2.4}}{=} (\overline{\mathcal{F}}T)(\mathcal{F}f) = T(\overline{\mathcal{F}}\mathcal{F}f) \stackrel{\text{Plancherel 2.2.3}}{=} T(f)$$

Since this holds for all  $f \in \mathcal{S}(\mathbb{R}^n)$  and all  $T \in \mathcal{S}^*(\mathbb{R}^n)$  it follows:

$$\mathcal{F}\overline{\mathcal{F}} = \mathbb{1}_{\mathcal{S}^*(\mathbb{R}^n)}$$

The same follows for  $\overline{\mathcal{F}}\mathcal{F}$ .

□<sub>2.2.5</sub>

### 2.2.6 Examples

1. The Fourier transform of the  $\delta$ -Distribution can be calculated as follows:

$$\begin{aligned} (\mathcal{F}\delta)(f) &= \delta(\mathcal{F}f) = (\mathcal{F}f)(0) = \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ipx} f(x) \, d^n x \Big|_{p=0} = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} f(x) \, d^n x = T_{(2\pi)^{-\frac{n}{2}}}(f) \end{aligned}$$

This means:

$$\mathcal{F}\delta = T_{(2\pi)^{-\frac{n}{2}}}$$

Or, in a more computational manner, one can write this as:

$$\mathcal{F}\delta = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ipx} \delta(x) \, d^n x = \frac{1}{(2\pi)^{\frac{n}{2}}}$$

This is not satisfying from a mathematical point of view, because one does not know, that the usual formula for the Fourier transform also works for distributions.

2. Consider  $T_f$  with  $f(p) = e^{ipy}$  for a given  $y \in \mathbb{R}^n$ .

$$\begin{aligned} (\mathcal{F}T_f)(h) &= T_f(\mathcal{F}h) = T_f\left(\frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-ipx} h(x) \, d^n x\right) = \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ipy} \left(\int e^{-ipx} h(x) \, d^n x\right) d^n p = \\ &= (2\pi)^{\frac{n}{2}} (\overline{\mathcal{F}\mathcal{F}}) h(y) = (2\pi)^{\frac{n}{2}} h(y) \end{aligned}$$

So we get:

$$\begin{aligned} \mathcal{F}T_f h &= (2\pi)^{\frac{n}{2}} h(y) \\ (\mathcal{F}T_f)(x) &= (2\pi)^{\frac{n}{2}} \delta^{(n)}(x - y) \end{aligned}$$

Formally one can write:

$$(\mathcal{F}f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ipx} e^{ipy} d^n p = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ip(x-y)} d^n p$$

This is ill-defined, but physicists use the formal relation:

$$\int_{\mathbb{R}^n} e^{-ip(x-y)} d^n p = (2\pi)^n \delta^{(n)}(x - y)$$

3. Consider  $T = T_g$  with  $g(p) = p^2 e^{ipy}$  for a given  $y \in \mathbb{R}$ .

$$\begin{aligned} (\mathcal{F}T_g)(f) &= T_g(\mathcal{F}f) = \int_{-\infty}^{\infty} g(p) (\mathcal{F}f)(p) \, dp = \\ &= \int g(p) \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} \, dx \right) dp \end{aligned}$$

One cannot interchange the integrals, because the integral

$$\int_{-\infty}^{\infty} \underbrace{g(p) e^{-ip\alpha}}_{\notin L^1} \, dp$$

does not exist. Therefore we work again with a convergence generating factor, which we can due to the dominated convergence theorem:

$$\begin{aligned} (\mathcal{F}T_g)(f) &= \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} g(p) e^{-\varepsilon|p|} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} \, dx \right) dp = \\ &= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) \left( \int_{-\infty}^{\infty} g(p) e^{-\varepsilon|p|} e^{-ipx} \, dp \right) dx = \\ &= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) \left( \int_{-\infty}^{\infty} p^2 e^{ipy} e^{-\varepsilon|p|} e^{-ipx} \, dp \right) dx = \\ &= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) \left( \int_{-\infty}^{\infty} (\mathbf{i}\partial_x)^2 e^{-ip(x-y)-\varepsilon|p|} \, dp \right) dx = \\ &\stackrel{\text{Lebesgue theorem}}{=} \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) (\mathbf{i}\partial_x)^2 \left( \int_{-\infty}^{\infty} e^{-ip(x-y)-\varepsilon|p|} \, dp \right) dx \end{aligned}$$

Now one can decompose the integral into integrals from  $-\infty$  to 0 and from 0 to  $\infty$  and calculate the result.

$$\begin{aligned}
(\mathcal{F}T_g)(f) &= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) (\mathbf{i}\partial_x)^2 \left( \int_{-\infty}^0 e^{-\mathbf{i}p(x-y)+\varepsilon p} dp + \int_0^{\infty} e^{-\mathbf{i}p(x-y)-\varepsilon p} dp \right) dx = \\
&= \frac{-1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) \partial_x^2 \left( \frac{1}{-\mathbf{i}(x-y)+\varepsilon} - \frac{1}{-\mathbf{i}(x-y)-\varepsilon} \right) dx = \\
&= \frac{-1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) \partial_x^2 \left( \frac{-\mathbf{i}(x-y)-\varepsilon - (-\mathbf{i}(x-y)+\varepsilon)}{-(x-y)^2 - \varepsilon^2} \right) dx = \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f(x) \partial_x^2 \left( \frac{2\varepsilon}{(x-y)^2 + \varepsilon^2} \right) dx = \\
&\stackrel{\text{integration by parts}}{=} \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f''(x) \cdot \frac{\varepsilon}{(x-y)^2 + \varepsilon^2} dx = \\
&\stackrel{z:=\frac{x-y}{\varepsilon}}{=} \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f''(\varepsilon z + y) \cdot \frac{\varepsilon}{\varepsilon^2 z^2 + \varepsilon^2} \varepsilon dz = \\
&\stackrel{dz=\frac{dx}{\varepsilon}}{=} \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} f''(\varepsilon z + y) \cdot \frac{1}{z^2 + 1} dz = \\
&= \sqrt{\frac{2}{\pi}} f''(y) \underbrace{\int_{-\infty}^{\infty} \frac{1}{z^2 + 1} dz}_{=\pi} = \sqrt{2\pi} f''(y)
\end{aligned}$$

So we get:

$$(\mathcal{F}T_g)(x) = \sqrt{2\pi} \delta''(x - y)$$

4. Consider the following Hilbert space:

$$L^2(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \mid f \text{ measurable with } \int |f|^2 d^n x < \infty \right\}$$

For  $f \in L^2(\mathbb{R}^n)$  define:

$$T_f(g) := \int_{\mathbb{R}^n} f(x) g(x) d^n x$$

$$|T_f(g)| \stackrel{\text{Schwarz}}{\leq} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \cdot c(n) \|g\|_{n+1,0}$$

Here we used:

$$|g(x)| \leq \tilde{c}(n) \frac{\|g\|_{n+1,0}}{(1+|x|)^{n+1}}$$

So we have  $T_f \in \mathcal{S}^*(\mathbb{R}^n)$ .

$$\begin{aligned}
L^2(\mathbb{R}^n) &\hookrightarrow \mathcal{S}^*(\mathbb{R}^n) \xrightarrow{\mathcal{F}} \mathcal{S}^*(\mathbb{R}^n) \\
f &\mapsto T_f \mapsto \mathcal{F}T_f
\end{aligned}$$

### 2.2.7 Theorem (Fourier Transform is isometry)

The mappings  $\mathcal{F}, \overline{\mathcal{F}} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  are isometries, i.e.:

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} \quad (2.22)$$

Due to  $\mathcal{F}\overline{\mathcal{F}} = 1$  they are even unitary transformations.

**Proof**

Consider first  $f, g \in \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ .

$$\begin{aligned} \langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \overline{(\mathcal{F}f)(x)} (\mathcal{F}g)(x) \, dx \stackrel{?}{=} \int \bar{f}(x) g(x) \, dx \\ &= T_{\bar{\mathcal{F}}\bar{f}}(\mathcal{F}g) = \left( \bar{\mathcal{F}} T_{\bar{f}} \right) (\mathcal{F}g) = T_{\bar{f}}(\bar{\mathcal{F}}\mathcal{F}g) = \\ &\bar{\mathcal{F}}^{\mathcal{F}=1} T_{\bar{f}}(g) = \int \bar{f}(x) g(x) \, dx \end{aligned}$$

$\mathcal{S}(\mathbb{R}^n)$  is dense in  $L^2(\mathbb{R}^n)$ . Because  $\mathcal{F}$  is continuous,  $\mathcal{F}$  is also isometric on  $L^2(\mathbb{R}^n)$ . For  $f \in L^2(\mathbb{R}^n)$  choose  $f_n \in \mathcal{S}(\mathbb{R}^n)$  with  $f_n \rightarrow f$  converging in  $L^2(\mathbb{R}^n)$ . Then holds for all  $n \in \mathbb{N}$ :

$$\|f_n\|_{L^2} = \|\mathcal{F}f_n\|$$

In the limit  $n \rightarrow \infty$  we get, since  $\mathcal{F}$  is continuous:

$$\|f\| = \|\mathcal{F}f\|$$

□<sub>2.2.7</sub>

## 2.3 Applications to Partial Differential Equations with Constant Coefficients

Consider as example the Poisson equation

$$\Delta u = f$$

in  $\mathbb{R}^n$  with a given  $f$  and assume for simplicity  $f \in \mathcal{S}(\mathbb{R}^n)$ . After a Fourier transform and defining  $\hat{u} := \mathcal{F}u$  and  $\hat{f} = \mathcal{F}f$  we get:

$$\begin{aligned} \left( -\|p\|^2 \right) \hat{u}(p) &= \hat{f}(p) \\ \Rightarrow \hat{u}(p) &= -\frac{\hat{f}(p)}{\|p\|^2} \end{aligned}$$

Then  $\bar{\mathcal{F}}\hat{u}$  is a solution of the Poisson equation.

- For  $\|p\|^{-2} \hat{f}(p) \in \mathcal{S}(\mathbb{R}^n)$  the method works directly and one gets a  $u \in \mathcal{S}(\mathbb{R}^n)$ .
- In the case of  $n \geq 3$ ,  $\hat{u} = -\|p\|^{-2} \hat{f}(p)$  is a regular distribution. (If  $n < 3$  the integral does not necessarily converge.)

$$(T_{\hat{u}})(g) := \int \left( -\frac{\hat{f}(p)}{\|p\|^2} \right) g(p) \, d^n p$$

Therefore  $u := \bar{\mathcal{F}}T_{\hat{u}}$  is a distributional solution of the Poisson equation, so we get  $\Delta u = T_f$ .

**Problem:** The distributional solution is not unique, because e.g.

$$\hat{u}(p) = -\frac{\hat{f}(p)}{\|p\|^2} + c\delta(p)$$

is also a solution. Therefore we have to specify the behavior of  $u(x)$  in the limit  $\|x\| \rightarrow \infty$ .

## 3 The Laplace Equation in $\Omega \subseteq \mathbb{R}^n$

In this chapter we always consider an open subset  $\Omega \subseteq \mathbb{R}^n$  with boundary  $\partial\Omega$ . With the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad (3.1)$$

the Laplace equation can be written as:

$$\Delta u = 0$$

The inhomogeneous Laplace equation is called Poisson equation:

$$\Delta u = f$$

For both one needs to specify boundary conditions on  $\partial\Omega$  to get a unique solution.

### 3.0 Reminder

#### 3.0.1 Theorem (Gauss's Theorem)

If  $Y$  is a smooth vector field on the closure  $\overline{\Omega}$  and  $\partial\Omega$  is smooth with outer normal  $\nu$ , then holds:

$$\int_{\Omega} \operatorname{div}(Y) \underbrace{\mathrm{d}^n x}_{=\mathrm{d}\mu} = \int_{\partial\Omega} \langle Y, \nu \rangle \mathrm{d}\mu_{\partial\Omega} \quad (3.2)$$

(Proof omitted)

#### 3.0.2 Theorem (Green's Identities)

For  $u, w \in C^\infty(\overline{\Omega})$  holds:

$$\int_{\Omega} w \Delta u \mathrm{d}\mu_{\Omega} = \int_{\partial\Omega} w (\nabla_{\nu} u) \mathrm{d}\mu_{\partial\Omega} - \int_{\Omega} \langle \nabla w, \nabla u \rangle \mathrm{d}\mu_{\Omega} \quad (3.3)$$

$$\int_{\Omega} (w (\Delta u) - (\Delta w) u) \mathrm{d}\mu_{\Omega} = \int_{\partial\Omega} (w (\nabla_{\nu} u) - (\nabla_{\nu} w) u) \mathrm{d}\mu_{\partial\Omega} \quad (3.4)$$

### Proof

We use  $\Delta = \operatorname{div} \operatorname{grad}$  and integrate by parts:

$$\int_{\Omega} w \Delta u \, d\mu_{\Omega} = \int_{\Omega} w \cdot \operatorname{div} (\nabla u) \, d\mu_{\Omega} = \int_{\Omega} \operatorname{div} (w \nabla u) - \langle \nabla w, \nabla u \rangle \, d\mu_{\Omega}$$

Then one can use Gauss's theorem to get the first identity. Now one subtracts the identity with  $w$  and  $u$  commuted:

$$\begin{aligned} w \Delta u - u \Delta w &= \operatorname{div} (w \nabla u) - \langle \nabla w, \nabla u \rangle - (\operatorname{div} (u \nabla w) - \langle \nabla u, \nabla w \rangle) = \\ &\stackrel{\langle \cdot, \cdot \rangle \text{ symmetric}}{=} \operatorname{div} (w \nabla u) - \operatorname{div} (u \nabla w) \end{aligned}$$

Using Gauss's theorem the second identity follows. □<sub>3.0.2</sub>

## 3.1 Representation Formulas for Harmonic Functions

### 3.1.1 Definition (Harmonic Functions)

A function  $u \in C^2(\overline{\Omega})$  is called *harmonic*, if the Laplacian vanishes:

$$\Delta u = 0$$

The harmonic functions form a vector space.

### Examples

- Constant or linear functions
- $u(x_1, x_2) = x_1^2 - x_2^2$  is harmonic on  $\mathbb{R}^2$ .
- Holomorphic functions on  $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$

Consider now *spherically symmetric harmonic functions*. For this we choose polar coordinates  $(r, \vartheta, \varphi)$  in  $\mathbb{R}^3$ :

$$\begin{aligned} x &= r \cos(\vartheta) \\ y &= r \sin(\vartheta) \cos(\varphi) \\ z &= r \sin(\vartheta) \sin(\varphi) \end{aligned}$$

More general in  $\mathbb{R}^n$  with  $n \in \mathbb{N}_{\geq 2}$  we choose  $r = \|x\|$  and  $\omega \in S^{n-1}$ . Regard  $\mathbb{R}^n$  with the Euclidian metric as a Riemannian manifold  $(M, g)$ . Polar coordinates give a special chart on  $\Omega \subseteq M$  with  $0 \notin \Omega$ . Then we can calculate  $\Delta = \Delta_{\omega}$  as Laplace-Beltrami operator in polar coordinates. The metric is:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g_{S^{n-1}} \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} g_{S^{n-1}}^{-1} \end{pmatrix}$$

$$\begin{aligned} \det(g) &= r^{2(n-1)} g_{S^{n-1}} \\ \sqrt{\det(g)} &= r^{n-1} \sqrt{\det(g_{S^{n-1}})} \end{aligned}$$

Now the Laplace-Beltrami operator can be calculated with the *Koszul formula*:

$$\begin{aligned}
 \Delta u &= \nabla_j \nabla^j u = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^j} \left( \sqrt{\det(g)} g^{jk} \frac{\partial}{\partial x^k} u \right) = \\
 &= \frac{1}{r^{n-1} \sqrt{\det(g_{S^{n-1}})}} \frac{\partial}{\partial x^j} \left( r^{n-1} \sqrt{\det(g_{S^{n-1}})} g^{jk} \frac{\partial}{\partial x^k} u \right) = \\
 &= \frac{1}{r^{n-1} \sqrt{\det(g_{S^{n-1}})}} \frac{\partial}{\partial r} \left( r^{n-1} \sqrt{\det(g_{S^{n-1}})} \frac{\partial}{\partial r} u \right) + \\
 &\quad + \sum_{j,k=2}^n \frac{1}{r^{n-1} \sqrt{\det(g_{S^{n-1}})}} \frac{\partial}{\partial x^j} \left( r^{n-1} \sqrt{\det(g_{S^{n-1}})} \frac{1}{r^2} (g_{S^{n-1}})^{jk} \frac{\partial}{\partial x^k} u \right) = \\
 &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} u \right) + \frac{1}{r^2} \Delta_{S^{n-1}} u
 \end{aligned} \tag{3.5}$$

The important formula is:

$$\Delta = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \cdot \right) + \frac{1}{r^2} \Delta_{S^{n-1}} \tag{3.6}$$

For spherically symmetric solutions  $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ , the Laplace equation reads:

$$\Delta \Gamma = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \Gamma(r) \right) = 0$$

This gives:

$$\begin{aligned}
 r^{n-1} \frac{\partial}{\partial r} \Gamma(r) &= c \\
 \frac{\partial}{\partial r} \Gamma(r) &= cr^{1-n}
 \end{aligned}$$

$$\Gamma(r) = a + C \int^r \tau^{1-n} d\tau = \begin{cases} a + \tilde{c} \ln(r) & \text{if } n = 2 \\ a + \frac{c}{r^{n-2}} & \text{if } n > 2 \end{cases}$$

Now we choose specific values for  $a$  and  $c$  or  $\tilde{c}$ .

### 3.1.2 Definition (Fundamental Solution of Laplace Equation)

The *fundamental solution*  $\Gamma$  of the Laplace equation in  $\mathbb{R}^n$  is defined by:

$$\Gamma(x, y) = \Gamma(\|x - y\|) := \begin{cases} \frac{1}{2\pi} \ln(\|x - y\|) & \text{if } n = 2 \\ \frac{1}{n(2-n)\omega_n} \|x - y\|^{2-n} & \text{if } n > 2 \end{cases} \tag{3.7}$$

Here  $\omega_n := \mu(B_1(0))$  is the Lebesgue measure of the unit ball.

For example for  $n = 3$  we have  $\omega_3 = \frac{4\pi}{3}$  and thus:

$$\Gamma(x, y) = \frac{1}{3(-1)\frac{4\pi}{3}} \cdot \frac{1}{\|x - y\|} = -\frac{1}{4\pi} \frac{1}{\|x - y\|} \tag{3.8}$$



### 3.1.3 Theorem (Green's representation)

Let  $\Omega \subseteq \mathbb{R}^n$  be open with smooth boundary and  $u \in C^2(\overline{\Omega})$ . Then for any  $y \in \Omega$  holds:

$$u(y) = \int_{\partial\Omega} (u(x) \nabla_\nu \Gamma(x, y) - \Gamma(x, y) \nabla_\nu u(x)) d\mu_{\partial\Omega}(x) + \int_{\Omega} \Gamma(x, y) (\Delta u)(x) d\mu(x) \quad (3.9)$$

For harmonic functions with  $\Delta u = 0$ , this simplifies to:

$$u(y) = \int_{\partial\Omega} (u(x) \nabla_\nu \Gamma(x, y) - \Gamma(x, y) \nabla_\nu u(x)) d\mu_{\partial\Omega}(x) \quad (3.10)$$

Thus  $u$  has an explicit representation in terms of its boundary values on  $\partial\Omega$ .

#### Proof

Choose  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_\varepsilon(y) \subseteq \Omega$ .

$$\int_{\Omega \setminus B_\varepsilon(y)} \underbrace{\Gamma(x, y)}_{\text{smooth}} \underbrace{(\Delta u)(x)}_{\text{continuous}} d\mu(x)$$

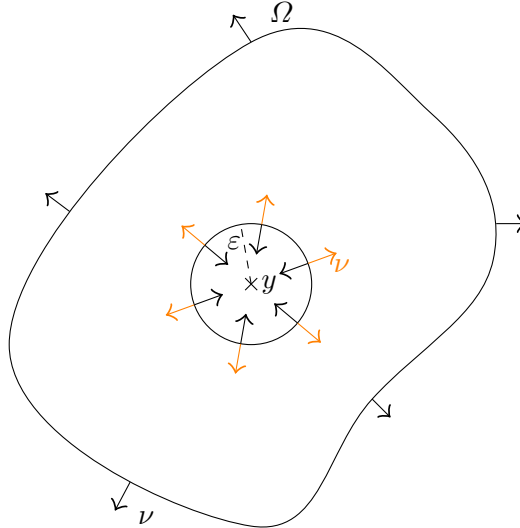


Figure 3.1: Outer normal  $\nu$  of  $\Omega \setminus B_\varepsilon(y)$  (black) and of  $B_\varepsilon(y)$  (orange)

We apply the second Green's identity to obtain with  $v(x) := \Gamma(x, y)$ :

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon(y)} (v \cdot \Delta u - \underbrace{(\Delta v) u}_{=0}) d\mu &= \int_{\partial(\Omega \setminus B_\varepsilon(y))} (v(\nabla_\nu u) - (\nabla_\nu v) u) d\mu_{\partial\Omega} = \\ &= \int_{\partial\Omega} (\Gamma(x, y) \nabla_\nu u(x) - (\nabla_\nu \Gamma)(x, y) u(x)) d\mu_{\partial\Omega} + \\ &\quad - \int_{\partial B_\varepsilon(y)} (\Gamma(x, y) \nabla_\nu u(x) - \nabla_\nu \Gamma(x, y) u(x)) d\mu_{\partial B_\varepsilon(y)} \end{aligned}$$

The minus in front of  $\int_{\partial B_\varepsilon(y)}$  comes from the fact, that the outer normal of  $\partial(\Omega \setminus B_\varepsilon(y))$  shows in the opposite direction of the outer normal of  $\partial B_\varepsilon(y)$ . The left side of the integral gives in the limit  $\varepsilon \searrow 0$ :

$$\lim_{\varepsilon \searrow 0} \int_{\Omega \setminus B_\varepsilon(y)} v \cdot \Delta u d\mu = \int_{\Omega} v \cdot \Delta u d\mu = \int_{\Omega} \Gamma(x, y) \cdot (\Delta u)(x) d\mu(x)$$

We estimate the integral over  $\partial B_\varepsilon(y)$  in the limit  $\varepsilon \searrow 0$ :

$$\begin{aligned} \left| \int_{\partial B_\varepsilon(y)} \Gamma(x, y) \nabla_\nu u(x) d\mu_{\partial B_\varepsilon(y)} \right| &\leq \Gamma(\varepsilon) \sup_{B_\varepsilon(y)} \|\nabla u\| \cdot \underbrace{n\omega_n}_{\text{surface area of the unit sphere}} \varepsilon^{n-1} \sim \\ &\sim \begin{cases} \varepsilon & \text{if } n > 2 \\ \varepsilon \ln(\varepsilon) & \text{if } n = 2 \end{cases} \xrightarrow{\varepsilon \searrow 0} 0 \end{aligned}$$

Now we expand the second part around  $\varepsilon = 0$ :

$$\begin{aligned} \int_{\partial B_\varepsilon(y)} (\nabla_\nu \Gamma)(x, y) u(x) d\mu_{\partial B_\varepsilon(y)} &= \underbrace{\frac{\partial}{\partial \varepsilon} \Gamma(\varepsilon)}_{=\frac{1}{n\omega_n} \varepsilon^{1-n}} \underbrace{\int_{\partial B_\varepsilon(y)} u(x) d\mu_{\partial B_\varepsilon(y)}}_{=u(y)n\omega_n \varepsilon^{n-1} + o_0(\varepsilon^{n-1})} = \\ &= u(y) + o_0(\varepsilon^0) \xrightarrow{\varepsilon \searrow 0} u(y) \end{aligned}$$

This gives:

$$\begin{aligned} \int_{\Omega} \Gamma(x, y) (\Delta u)(x) d\mu(x) &= \int_{\partial \Omega} (\Gamma(x, y) \nabla_\nu u(x) - (\nabla_\nu \Gamma)(x, y) u(x)) d\mu_{\partial \Omega} + u(y) \\ \Rightarrow u(y) &= \int_{\partial \Omega} (u(x) \nabla_\nu \Gamma(x, y) - \Gamma(x, y) \nabla_\nu u(x)) d\mu_{\partial \Omega}(x) + \int_{\Omega} \Gamma(x, y) (\Delta u)(x) d\mu(x) \end{aligned}$$

□<sub>3.1.3</sub>

### 3.1.4 Corollary (Laplacian of Fundamental Solution is Delta Distribution)

For any  $\varphi \in C_0^\infty(\Omega)$  holds:

$$\varphi(y) = \int_{\Omega} \Gamma(x, y) (\Delta \varphi)(x) d\mu(x) \quad (3.11)$$

This can also be expressed in terms of distributions as:

$$\Delta_x \Gamma(x, y) = \delta^{(n)}(x - y) \quad (3.12)$$

More correctly, for fixed  $y \in \mathbb{R}^n$ ,  $T(x) := T_{\Gamma(x, y)}$  defines a regular distribution. Equation (3.11) means, that for all  $\varphi \in C_0^\infty(\Omega) \subseteq \mathcal{S}(\mathbb{R}^n)$ :

$$\begin{aligned} (\Delta T)(\varphi) &= \varphi(y) \\ \Leftrightarrow \Delta T &= \delta_y \end{aligned}$$

### Proof

Since the support of  $\varphi$  lies inside of  $\Omega$ , the first term in Green's representation vanishes:

$$\begin{aligned}\varphi(y) &= \int_{\partial\Omega} (\varphi(x) \nabla_\nu \Gamma(x, y) - \Gamma(x, y) \nabla_\nu \varphi(x)) d\mu_{\partial\Omega}(x) + \int_{\Omega} \Gamma(x, y) (\Delta \varphi)(x) d\mu(x) = \\ &= \int_{\Omega} \Gamma(x, y) (\Delta \varphi)(x) d\mu(x)\end{aligned}$$

□<sub>3.1.4</sub>

Next we investigate the existence of solutions.

$$\Delta u = 0 \quad u|_{\partial\Omega} = u_0 \quad \text{Dirichlet problem} \quad (3.13)$$

$$\Delta u = 0 \quad \nabla_\nu u|_{\partial\Omega} = u_1 \quad \text{Neumann problem} \quad (3.14)$$

The problem is, that the representation

$$u(y) = \int_{\partial\Omega} (u(x) \nabla_\nu \Gamma(x, y) - \Gamma(x, y) \nabla_\nu u(x)) d\mu_{\partial\Omega}(x)$$

needs both  $u$  and  $\nabla_\nu u$  on the boundary. Suppose we want to solve the Dirichlet problem. Then  $u_0 = u|_{\partial\Omega}$  is given, but  $\nabla_\nu u|_{\partial\Omega}$  is unknown.

## 3.2 The Green's Function

Consider the Dirichlet problem for the Poisson equation:

$$\Delta u(x) = f(x) \quad \forall_{x \in \Omega} \quad u|_{\partial\Omega} = \varphi \quad (3.15)$$

Assume  $u \in C^2(\overline{\Omega})$ ,  $f \in C^0(\overline{\Omega})$  and  $\varphi \in C^2(\partial\Omega)$ .

### 3.2.1 Definition (Green's function)

A function  $G(x, y)$  defined for  $x, y \in \overline{\Omega}$  with  $x \neq y$  is called *Green's function* for the domain  $\Omega$  if the following conditions are satisfied:

- i) For all  $x \in \partial\Omega$  and  $y \neq x$  holds  $G(x, y) = 0$ .
- ii)  $h(x, y) := G(x, y) - \Gamma(x, y)$  is in  $C^2(\Omega)$ , even for  $x = y$ , and harmonic in  $x \in \Omega$ .

### 3.2.2 Proposition (Solution of Dirichlet Problem)

For a solution  $u$  of the Dirichlet problem for the Poisson equation holds:

$$u(y) = \int_{\partial\Omega} u(x) \nabla_\nu G(x, y) d\mu_{\partial\Omega}(x) + \int_{\Omega} f(x) G(x, y) d\mu(x) \quad (3.16)$$

**Proof**

$$\begin{aligned} \int_{\Omega} h(x, y) (\Delta u)(x) d\mu(x) &= \int_{\Omega} h(x, y) f(x) d\mu(x) = \\ &\stackrel{\substack{2. \text{ Green's} \\ \text{identity}}}{=} \int_{\partial\Omega} (h(x, y) (\nabla_{\nu} u)(x) - (\nabla_{\nu} h(x, y)) \cdot u(x)) d\mu_{\partial\Omega}(x) \end{aligned}$$

Now we add the Green's representation

$$\int_{\Omega} \Gamma(x, y) f(x) d\mu(x) = \int_{\partial\Omega} (\Gamma(x, y) (\nabla_{\nu} u)(x) - (\nabla_{\nu} \Gamma(x, y)) u(x)) d\mu_{\partial\Omega}(x) + u(y)$$

to get:

$$\int_{\Omega} G(x, y) f(x) d\mu(x) = \int_{\partial\Omega} (G(x, y) (\nabla_{\nu} u)(x) - (\nabla_{\nu} G(x, y)) u(x)) d\mu_{\partial\Omega}(x) + u(y)$$

Since  $G(x, y) = 0$  on  $\partial\Omega$ , the proposition follows.  $\square_{3.2.2}$

### 3.2.3 Theorem (Symmetry of the Green's Function)

For all  $x, y \in \Omega$  with  $x \neq y$  holds:

$$G(x, y) = G(y, x) \quad (3.17)$$

**Proof**

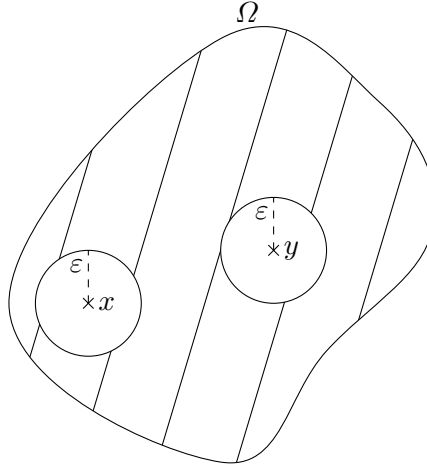


Figure 3.2:  $B_{\varepsilon}(x) \subseteq \Omega$ ,  $B_{\varepsilon}(y) \subseteq \Omega$ ,  $B_{\varepsilon}(x) \cap B_{\varepsilon}(y) = \emptyset$

Choose  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_{\varepsilon}(x)$  and  $B_{\varepsilon}(y)$  are disjoint subsets of  $\Omega$ .

$$u(z) := G(z, x) \qquad v(z) := G(z, y)$$

It holds  $u, v \in C^2(\Omega \setminus (B_{\varepsilon}(x) \cup B_{\varepsilon}(y)))$  and  $u, v$  are harmonic. Moreover for  $z \in \partial\Omega$  holds:

$$u(z) = G(z, x) = 0 \qquad v(z) = G(z, y) = 0$$

Apply again the second Green's identity:

$$\begin{aligned} 0 &= \int_{\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))} \underbrace{(v(\Delta u))}_{=0} - \underbrace{(\Delta v)u}_{=0} d\mu = \\ &= \int_{\partial\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))} (v(\nabla_\nu u) - (\nabla_\nu v)u) d\mu_{\partial\Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))} \end{aligned}$$

Moreover, the boundary values on  $\partial\Omega$  vanish. We conclude:

$$0 = \int_{\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)} (v(\nabla_\nu u) - (\nabla_\nu v)u) d\mu_{\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)}$$

Consider the integral over  $\partial B_\varepsilon(x)$  first. Since  $G = \Gamma + h$  and  $h \in C^2$  is bounded, we know for all  $z \in \partial B_\varepsilon(x)$ :

$$u(z) = G(z, x) \sim \Gamma(z, x) \sim \begin{cases} \ln(\varepsilon) & \text{if } n = 2 \\ \varepsilon^{2-n} & \text{if } n > 2 \end{cases}$$

Since  $\nabla_\nu v$  is also bounded due to  $v \in C^2(B_\varepsilon(x))$ , we get:

$$\int_{\partial B_\varepsilon(x)} (\nabla_\nu v)u d\mu_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon \searrow 0} 0$$

The other term gives:

$$\int_{\partial B_\varepsilon(x)} v(\nabla_\nu u) d\mu_{\partial B_\varepsilon(x)} \stackrel{\nabla_\nu u = \frac{\partial}{\partial \varepsilon} \Gamma(\varepsilon) + o_0(\varepsilon^0) = \tilde{c}\varepsilon^{1-n} + o_0(\varepsilon^0)}{=} cv(x)\varepsilon^{1-n} \underbrace{\int_{\partial B_\varepsilon(x)} 1 d\mu_{\partial B_\varepsilon(x)}}_{=n\omega_n\varepsilon^{n-1}} + o_0(\varepsilon) \xrightarrow{\varepsilon \searrow 0} cv(x)$$

Here the constant  $c \neq 0$  does not vanish. Now follows:

$$\begin{aligned} &\int_{\partial B_\varepsilon(x)} ((\nabla_\nu v)u - v(\nabla_\nu u)) d\mu_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon \searrow 0} cv(x) \\ &\int_{\partial B_\varepsilon(y)} ((\nabla_\nu v)u - v(\nabla_\nu u)) d\mu_{\partial B_\varepsilon(y)} \xrightarrow{\varepsilon \searrow 0} cu(y) \end{aligned}$$

Adding these two integrals gives:

$$\begin{aligned} 0 &= c(v(x) - u(y)) \\ \Rightarrow \quad G(x, y) &= v(x) = u(y) = G(y, x) \end{aligned}$$

□<sub>3.2.3</sub>

Any solution  $u$  of the Dirichlet problem

$$\Delta u = f \qquad u|_{\partial\Omega} = \varphi$$

has the representation:

$$u(y) = \int_{\partial\Omega} u(x) \nabla_\nu G(x, y) d\mu_{\partial\Omega}(x) + \int_{\Omega} G(x, y) f(x) d\mu(x)$$

We define:

$$u(y) := \int_{\partial\Omega} \varphi(x) \nabla_\nu G(x, y) d\mu_{\partial\Omega}(x) + \int_{\Omega} G(x, y) f(x) d\mu(x)$$

Since  $G$  is symmetric,  $G(x, y) = G(y, x)$ , the equation  $\Delta_x G(x, y) = \delta^{(n)}(x - y)$  implies  $\Delta_y G(x, y) = \delta^{(n)}(x - y)$  as well. As a consequence, a formal computation gives:

$$\Delta_y u(y) = \int_{\partial\Omega} \varphi(x) \nabla_\nu \delta^{(n)}(x - y) d\mu_{\partial\Omega}(x) + \int_{\Omega} \delta^{(n)}(x - y) f(x) d\mu(x)$$

For  $y \in \Omega$  the first term vanishes, so we get:

$$\Delta_y u(y) = f(y)$$

This formal calculation can be made rigorous, once an explicit Green's function is given. Then one can also check, whether the boundary conditions are satisfied.

### 3.2.4 Example (Green's function for $B_R(0)$ )

Now we want to construction the Green's function for  $\Omega := B_R(0) \subseteq \mathbb{R}^n$ . Consider an electric charge inside an earthed, electrically conducting sphere, that screens the electric field, so that it vanishes outside the sphere. The electric field inside the sphere can be calculated using the concept of a mirror charge, which ensures, that the electric field is perpendicular to the sphere.

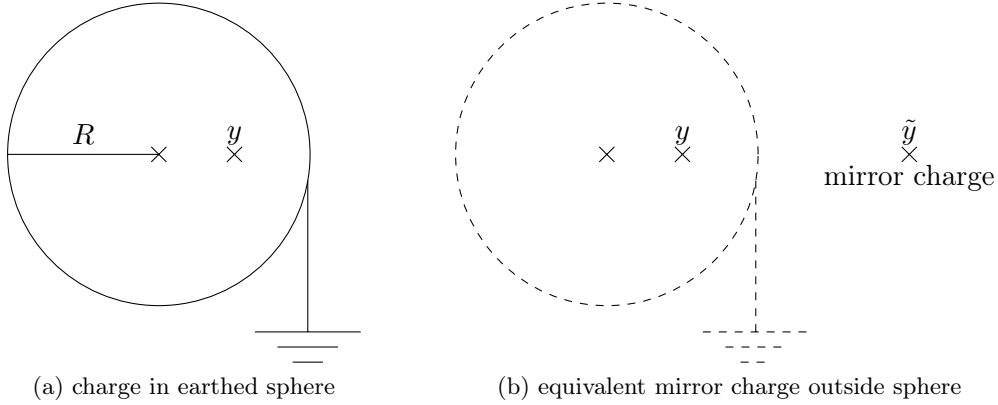


Figure 3.3: The mirror charge replicates the boundary conditions of the sphere.

The position of the mirror charge must be:

$$\begin{aligned} \tilde{y} &= \frac{R^2}{y^2} \cdot y \\ \Rightarrow \|y\| \cdot \|\tilde{y}\| &= R^2 \end{aligned}$$

Compare this with the inversion with respect to the unit circle (Spiegelung am Einheitskreis) in the complex plane:

$$z \mapsto \frac{1}{\bar{z}} = \frac{z}{|z|^2}$$

This motivates the ansatz:

$$G(x, y) = \begin{cases} \Gamma(\|x - y\|) - \Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) & \text{if } y \neq 0 \\ \Gamma(\|x\|) - \Gamma(R) & \text{if } y = 0 \end{cases}$$

Let us verify that  $G$  has all the required properties:

$$G(x, y) - \Gamma(x, y) = \begin{cases} -\Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) & \text{if } y \neq 0 \\ -\Gamma(R) & \text{if } y = 0 \end{cases}$$

For  $x, y \in B_R(0)$  follows  $\tilde{y} \notin B_R(0)$  and thus  $x \neq \tilde{y}$ . So  $\Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right)$  is smooth if  $y \neq 0$ . To see smoothness in the case  $y = 0$ , we rewrite:

$$\begin{aligned} \Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) &= \Gamma\left(\sqrt{\frac{y^2}{R^2} (x^2 + \tilde{y}^2 - 2\langle x, \tilde{y} \rangle)}\right) = \\ &= \Gamma\left(\sqrt{\frac{y^2}{R^2} \left(x^2 + \frac{R^4}{y^2} - 2\frac{R^2}{y^2} \langle x, y \rangle\right)}\right) = \\ &= \Gamma\left(\sqrt{\frac{x^2 y^2}{R^2} + R^2 - 2\langle x, y \rangle}\right) \end{aligned} \quad (3.18)$$

The argument is smooth at  $y = 0$  and it holds:

$$\lim_{y \rightarrow 0} \Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) = \Gamma(R)$$

So  $G(x, y) - \Gamma(x, y)$  is in  $C^2$  for  $x, y \in \Omega$ .

For  $x \in \partial\Omega$ , i.e.  $\|x\| = R$ , holds:

$$\begin{aligned} G(x, y) &= \Gamma(\|x - y\|) - \Gamma\left(\sqrt{\frac{\|x\|^2 \|y\|^2}{R^2} + R^2 - 2\langle x, y \rangle}\right) = \\ &\stackrel{\|x\|=R}{=} \Gamma(\|x - y\|) - \Gamma\left(\sqrt{\|y\|^2 + \|x\|^2 - 2\langle x, y \rangle}\right) = \\ &= \Gamma(\|x - y\|) - \Gamma\left(\sqrt{\|x - y\|^2}\right) = 0 \end{aligned}$$

Thus  $G(x, y)$  satisfies the boundary condition  $G(x, y) = 0$  for  $x \in \partial\Omega$ .

Now we show that  $G(x, y) - \Gamma(x, y)$  is harmonic:

$$\begin{aligned} G(x, y) - \Gamma(x, y) &= -\Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) \\ \nabla \Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) &= \Gamma'\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) \frac{\|y\|}{R} \nabla \|x - \tilde{y}\| \\ \Delta \Gamma\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) &= \Gamma''\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) \left(\frac{\|y\|}{R}\right)^2 (\nabla \|x - \tilde{y}\|)^2 + \\ &\quad + \Gamma'\left(\frac{\|y\|}{R} \|x - \tilde{y}\|\right) \frac{\|y\|}{R} \Delta \|x - \tilde{y}\| \end{aligned}$$

We know:

$$\begin{aligned} 0 &= \Delta \Gamma = \frac{1}{r^{n-1}} \partial_r (r^{n-1} \partial_r \Gamma) \\ \Rightarrow \quad 0 &= \Gamma''(r) + \frac{n-1}{r} \Gamma'(r) \end{aligned}$$

With this follows:

$$\begin{aligned} \Delta \Gamma \left( \frac{\|y\|}{R} \|x - \tilde{y}\| \right) &= -(n-1) \frac{R}{\|y\| \|x - \tilde{y}\|} \Gamma' \left( \frac{\|y\|}{R} \|x - \tilde{y}\| \right) \left( \frac{\|y\|}{R} \right)^2 (\nabla \|x - \tilde{y}\|)^2 + \\ &\quad + \Gamma' \left( \frac{\|y\|}{R} \|x - \tilde{y}\| \right) \frac{\|y\|}{R} \Delta \|x - \tilde{y}\| = \\ &= \frac{\|y\|}{R} \Gamma' \left( \frac{\|y\|}{R} \|x - \tilde{y}\| \right) \left( \frac{-(n-1)}{\|x - \tilde{y}\|} (\nabla \|x - \tilde{y}\|)^2 + \Delta \|x - \tilde{y}\| \right) = \\ &= \frac{\|y\|}{R} \Gamma' \left( \frac{\|y\|}{R} \|x - \tilde{y}\| \right) \left( \frac{-(n-1)}{\|x - \tilde{y}\|} \left( \frac{x - \tilde{y}}{\|x - \tilde{y}\|} \right)^2 + \frac{n-1}{\|x - \tilde{y}\|} \right) = 0 \end{aligned}$$

Thus  $G(x, y)$  is the desired Green's function. From (3.18) one sees explicitly:

$$G(x, y) = G(y, x)$$

We hope that the solution of the Dirichlet problem

$$\Delta u = f \quad u|_{\partial B_R(0)} = \varphi$$

is given by the Green's representation:

$$u(y) = \int_{B_R(0)} G(x, y) f(x) d\mu(x) + \int_{\partial B_R(0)} \nabla_\nu G(x, y) \varphi(x) d\mu_{\partial B_R(0)}(x)$$

Computing  $\nabla_\nu G(x, y)$  for  $x \in \partial B_R(0)$  gives:

$$\nabla_\nu G(x, y) = \frac{R^2 - y^2}{n\omega_n R} \cdot \frac{1}{\|x - y\|^n}$$

### 3.2.5 Theorem (Poisson representation)

The function

$$u(y) := \begin{cases} \frac{R^2 - y^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{\varphi(x)}{\|x - y\|^n} d\mu_{\partial B_R(0)} & \text{if } y \in B_R(0) \\ \varphi(y) & \text{if } y \in \partial B_R(0) \end{cases} \quad (3.19)$$

with  $\varphi \in C^0(\partial B_R(0))$  has the following properties:

- $u \in C^0(\overline{B_R(0)})$
- $u \in C^2(B_R(0))$
- $u$  is harmonic in  $B_R(0)$ .

#### Proof

This can be shown using Green's representation, computing the boundary values and justifying that the  $y$ -derivative may be taken inside the integral.  $\square_{3.2.5}$



### 3.3 The Mean Value Theorem and the Maximum Principle for Harmonic Functions

#### 3.3.1 Theorem (Mean Value Formulas)

A continuous function  $u : \Omega \rightarrow \mathbb{R}$  with  $\Omega \subseteq \mathbb{R}^n$  is harmonic, i.e.  $u \in C^2(\Omega)$  and  $\Delta u = 0$ , if and only if for all  $x_0 \in \Omega$  and all balls  $B_r(x_0) \subseteq \Omega$  one of the following mean value formulas holds:

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(x) d\mu_{\partial B_r}(x) \quad (\text{spherical mean}) \quad (3.20)$$

$$u(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u(x) d\mu(x) \quad (\text{mean over ball}) \quad (3.21)$$

In this case, the other formula holds as well.

#### Proof

“ $\Rightarrow$ ”: Let  $u \in C^2(\Omega)$  be harmonic and  $B_r(x_0) \subseteq \Omega$ . Then  $B_r(y) \subseteq \Omega$  holds also for all  $y$  in a neighborhood of  $x_0$ . The function

$$H(x, y) = \Gamma(x, y) - \Gamma(r)$$

coincides up to a constant with the fundamental solution and it holds  $H(x, y) = 0$  for  $x \in \partial B_r(y)$ . Using Green's representation gives:

$$\begin{aligned} u(y) &= \int_{B_r(y)} \underbrace{(\Delta u)(x)}_{=0} \cdot H(x, y) d\mu(x) + \\ &\quad + \int_{\partial B_r(y)} \left( u(x) \nabla_\nu H(x, y) - \underbrace{H(x, y)}_{=0} \nabla_\nu u(x) \right) d\mu_{\partial B_r(y)}(x) = \\ &= \int_{\partial B_r(y)} u(x) \nabla_\nu H(x, y) d\mu_{\partial B_r(y)}(x) \end{aligned}$$

$$\nabla_\nu H(x, y) = \nabla_{\nu, x} \Gamma(\|x - y\|) \stackrel{r:=\|x-y\|}{=} \frac{\partial}{\partial r} \Gamma(r) = \frac{1}{n\omega_n r^{n-1}}$$

For  $y := x_0$  follows the spherical mean formula:

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(x) d\mu_{\partial B_r}(x)$$

Now follows:

$$\begin{aligned} \int_{B_r(y)} u(x) d\mu(x) &\stackrel{\text{Fubini}}{=} \int_0^r d\rho \int_{\partial B_\rho(y)} u(x) d\mu_{\partial B_\rho}(x) = \int_0^r u(y) n\omega_n \rho^{n-1} d\rho = \\ &= u(y) n\omega_n \cdot \frac{1}{n} r^n = \omega_n r^n u(y) \end{aligned} \quad (3.22)$$

“ $\Leftarrow$ ”: Let  $u \in C^0(\Omega)$  be continuous. If the formula for the spherical mean holds, the computation (3.22) gives the formula for means over balls. Let us show that the formula for means over balls implies the formula for spherical means. So assume for fixed  $y$  and all  $r < r_0$ :

$$u(y) = \frac{1}{\omega_n r^n} \int_{B_r(y)} u(x) d\mu(x)$$

Thus using Fubini's theorem follows:

$$r^n u(y) = \frac{1}{\omega_n} \int_0^r d\rho \underbrace{\int_{\partial B_\rho(y)} u(x) d\mu_{\partial B_\rho(y)}(x)}_{\text{continuous in } \rho} \quad \overbrace{\quad}^{C^1 \text{ in } r}$$

Differentiation on both sides with respect to  $r$  gives the formula for spherical means:

$$nr^{n-1}u(y) = \frac{1}{\omega_n} \int_{\partial B_r(y)} u(x) d\mu_{\partial B_r(y)}(x)$$

Next we show, that  $u$  is smooth in  $\Omega$ . The idea is to mollify by convolution. Choose:

$$\varrho(t) := \begin{cases} c_n e^{\frac{1}{t^2-1}} & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases} \in C_0^\infty(\mathbb{R})$$

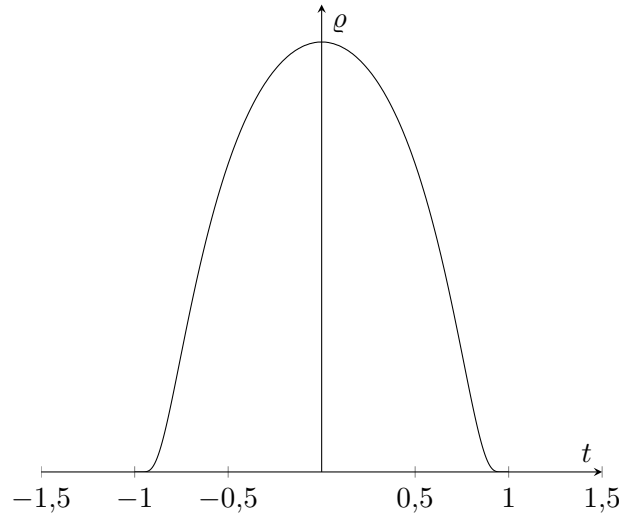


Figure 3.4:  $\varrho(t)$  is smooth and has compact support  $[-1, 1]$ .

Then holds  $\varrho(\|x\|) \in C_0^\infty(\mathbb{R}^n)$ . Choose  $c_n$  such that holds:

$$1 = \int_{\mathbb{R}^n} \varrho(\|x\|) d^n x$$

Now for  $\varepsilon \in \mathbb{R}_{>0}$  set:

$$\varrho_\varepsilon(x, y) := \frac{1}{\varepsilon^n} \varrho\left(\frac{\|x - y\|}{\varepsilon}\right)$$

Then still holds:

$$1 = \int_{\mathbb{R}^n} \varrho_\varepsilon(x, y) d^n x$$

Also we know  $\varrho_\varepsilon(x, \cdot) \in C_0^\infty(B_{2\varepsilon}(x))$ . Choose  $\varepsilon$  so small that  $B_{2\varepsilon}(y) \subseteq \Omega$  and use  $\varrho_\varepsilon$  as our convolution kernel to define:

$$u_\varepsilon(y) := \int_{\Omega} \varrho_\varepsilon(x, y) u(x) d\mu(x)$$

Now  $u_\varepsilon$  is a smooth function, because  $\varrho_\varepsilon(x, \cdot)$  is smooth and any derivative with respect to  $y$  can be exchanged with the integral, since the integration volume is compact.

$$u_\varepsilon(y) = \frac{1}{\varepsilon^n} \int_{\Omega} \varrho\left(\frac{\|x - y\|}{\varepsilon}\right) u(x) d\mu(x)$$

Choose polar coordinates around  $y$  and use Fubini to get:

$$\begin{aligned} u_\varepsilon(y) &= \frac{1}{\varepsilon^n} \int_0^{2\varepsilon} dr \int_{\partial B_r(y)} \varrho\left(\frac{r}{\varepsilon}\right) u(x) d\mu_{\partial B_r(y)}(x) = \\ &\stackrel{\text{spherical mean}}{=} \frac{1}{\varepsilon^n} \int_0^{2\varepsilon} dr \varrho\left(\frac{r}{\varepsilon}\right) u(y) n\omega_n r^{n-1} = \\ &= u(y) \frac{1}{\varepsilon^n} \int_0^{2\varepsilon} dr \int_{\partial B_r(y)} 1 \cdot \varrho\left(\frac{r}{\varepsilon}\right) d\mu_{\partial B_r(y)}(x) = \\ &= u(y) \int_{\mathbb{R}^n} \varrho_\varepsilon(x, y) d^n x = u(y) \end{aligned}$$

Thus  $u$  is smooth.

Compute  $\Delta u$  using the theorem of Gauss:

$$\int_{B_r(y)} (\Delta u(x)) d\mu(x) = \int_{\partial B_r(y)} \nabla_\nu u d\mu_{\partial B_r(y)}$$

With

$$\omega := \frac{x - y}{\|x - y\|} \in S^{n-1}$$

we get:

$$\begin{aligned} \int_{B_r(y)} (\Delta u(x)) d\mu(x) &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial}{\partial r} u(y + r\omega) d\mu_{\partial B_1(0)}(\omega) = \\ &= r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1}} \int_{\partial B_r(y)} u(x) d\mu_{\partial B_r(y)}(x) \right) = \\ &\stackrel{\text{mean value property}}{=} r^{n-1} \frac{\partial}{\partial r} \left( \frac{1}{r^{n-1}} u(y) n\omega_n r^{n-1} \right) = 0 \end{aligned}$$

In the limit  $r \rightarrow 0$  follows  $\Delta u(y) = 0$  since  $u$  is smooth. Thus  $u$  is harmonic. □<sub>3.3.1</sub>

### 3.3.2 Theorem (Strong Maximum Principle)

Let  $u \in C^2(\Omega)$  be a harmonic function on an open and connected subset  $\Omega \subseteq \mathbb{R}^n$ . If a point  $x_0 \in \Omega$  with

$$u(x_0) = \sup_{x \in \Omega} (u)$$

or

$$u(x_0) = \inf_{x \in \Omega} (u)$$

exists, then  $u$  is constant.

**Proof**

Since  $-u$  is also harmonic, it suffices to consider the case  $u(x_0) = \sup_{\Omega} (u)$ , so assume

$$u(x_0) = \sup_{\Omega} (u) =: M$$

and define:

$$\Omega_M := \{x \in \Omega \mid u(x) = M\}$$

Clearly,  $\Omega_M$  is not empty, because  $x_0 \in \Omega_M$ . By continuity,  $\Omega_M$  is closed in  $\Omega$  with respect to the relative topology. Thus it suffices to show that  $\Omega_M$  is open. Consider  $y \in \Omega_M$ , i.e.  $u(y) = M$ , and choose  $r \in \mathbb{R}_{>0}$  with  $B_r(y) \subseteq \Omega$ .

$$0 = u(y) - M \stackrel{\substack{\text{mean value} \\ \text{property}}}{=} \frac{1}{\omega_n r^n} \int_{B_r(y)} \underbrace{(u(x) - M)}_{\leq 0} d\mu(x)$$

Since  $u(x) \leq M$ , the integrand is non-positive and continuous. So  $u(x) = M$  for all  $x \in B_r(y)$  and thus follows  $B_r(y) \subseteq \Omega_M$ . This means that  $\Omega_M$  is open. Since it is also non-empty, closed and connected, the only possibility is  $\Omega_M = \Omega$ .  $\square_{3.3.2}$

**3.3.3 Theorem (Weak Maximum Principle)**

Let  $\Omega \subseteq \mathbb{R}^n$  be open and bounded and  $u \in C^0(\overline{\Omega})$  be harmonic. Then holds for all  $x \in \Omega$ :

$$\min_{y \in \partial\Omega} (u(y)) \leq u(x) \leq \max_{y \in \partial\Omega} (u(y))$$

**Proof**

If the statement were false, there would be an interior maximum or minimum at point  $x_0 \in \Omega$ , since  $\overline{\Omega}$  is compact, which ensures that the maximum and minimum of  $u$  are attained in  $\overline{\Omega}$ . Let  $\tilde{\Omega} \subseteq \Omega$  be the connected component of  $\Omega$  which contains  $x_0$ .

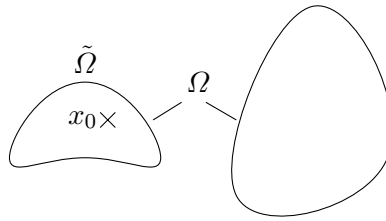


Figure 3.5:  $\tilde{\Omega}$  is the connected component of  $\Omega$  that contains  $x_0$ .

The strong maximum principle implies  $u = u(x_0)$  on  $\tilde{\Omega}$ . Thus follows:

$$\begin{aligned} \min_{\partial\tilde{\Omega}} (u) &= u(x_0) = \max_{\partial\tilde{\Omega}} (u) \\ \Rightarrow \min_{\partial\Omega} (u) &\leq \min_{\partial\tilde{\Omega}} (u) = u(x_0) = \max_{\partial\tilde{\Omega}} (u) \leq \max_{\partial\Omega} (u) \end{aligned}$$

Therefore  $x_0$  cannot be an interior maximum or minimum in contradiction to our assumption.  $\square_{3.3.3}$

### 3.3.4 Corollary (Uniqueness of Solutions of the Poisson Equation)

Let  $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$  be solutions of the Poisson equation:

$$\Delta u = f = \Delta v$$

If moreover holds  $u \leq v$  on  $\partial\Omega$ , then follows  $u \leq v$  in  $\Omega$ .

In particular,  $u = v$  on  $\partial\Omega$  implies  $u = v$  in  $\Omega$ .

#### Proof

The function  $h := u - v$  is harmonic

$$\Delta h = \Delta u - \Delta v = f - f = 0$$

and on  $\partial\Omega$  holds  $h \leq 0$ . The weak maximum principle gives  $h \leq 0$  in  $\Omega$ .

□<sub>3.3.4</sub>

### 3.3.5 Corollary (Liouville's Theorem)

If  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is harmonic and bounded, then  $u$  is constant.

#### Proof

Choose  $y_1, y_2 \in \mathbb{R}^n$ . Then for any  $r \in \mathbb{R}_{>0}$  holds for  $i \in \{1, 2\}$ .

$$u(y_i) = \frac{1}{\omega_n r^n} \int_{B_r(y_i)} u(x) \, d\mu(x)$$

The difference is:

$$u(y_1) - u(y_2) = \frac{1}{\omega_n r^n} \left( \int_{B_r(y_1)} u(x) \, d\mu(x) - \int_{B_r(y_2)} u(x) \, d\mu(x) \right)$$

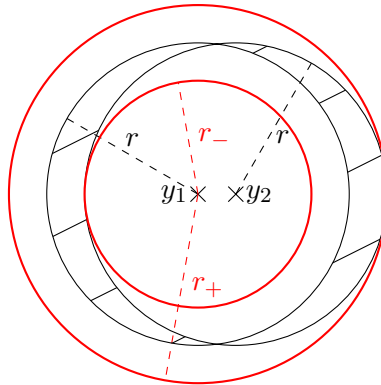


Figure 3.6:  $r_- := r - \|y_1 - y_2\|$ ,  $r_+ := r + \|y_1 - y_2\|$

So we get:

$$u(y_1) - u(y_2) \leq \frac{1}{\omega_n r^n} \underbrace{\sup_{\mathbb{R}^n} (|u|)}_{< \infty} \mu(B_r(y_1) \setminus B_r(y_2) \cup B_r(y_2) \setminus B_r(y_1)) \leq$$

$$\begin{aligned}
 &\leq \frac{1}{\omega_n r^n} \sup_{\mathbb{R}^n} (|u|) \left( \mu(B_{r+\|y_2-y_1\|}(y_1)) - \mu(B_{r-\|y_2-y_1\|}(y_1)) \right) \leq \\
 &\leq \frac{1}{\omega_n r^n} \sup_{\mathbb{R}^n} (|u|) (\omega_n (r + \|y_2 - y_1\|)^n - \omega_n (r - \|y_2 - y_1\|)^n) = \\
 &= \sup_{\mathbb{R}^n} (|u|) \frac{1}{r^n} (2n \|y_2 - y_1\| r^{n-1} + \dots) \xrightarrow{r \rightarrow \infty} 0
 \end{aligned}$$

This gives  $u(y_1) = u(y_2)$  and thus  $u$  is constant.  $\square_{3.3.5}$

### 3.4 The Harnack Inequality for Harmonic Functions

#### 3.4.1 Theorem (Harnack inequality)

Let  $\Omega \subseteq \mathbb{R}^n$  be open and  $u : \Omega \rightarrow \mathbb{R}$  harmonic and non-negative. Then for every compact, (for simplicity) connected subset  $\Omega' \subseteq \Omega$  there is a constant  $c = c(\Omega, \Omega') \in \mathbb{R}$ , which is independent of  $u$ , such that holds:

$$\sup_{\Omega'} (u) \leq c \cdot \inf_{\Omega'} (u)$$

#### Proof

We begin with the case  $\Omega' = B_r(x_0)$  and  $B_{4r}(x_0) \subseteq \overline{\Omega}$ . Choose  $y_1, y_2 \in \Omega'$ .

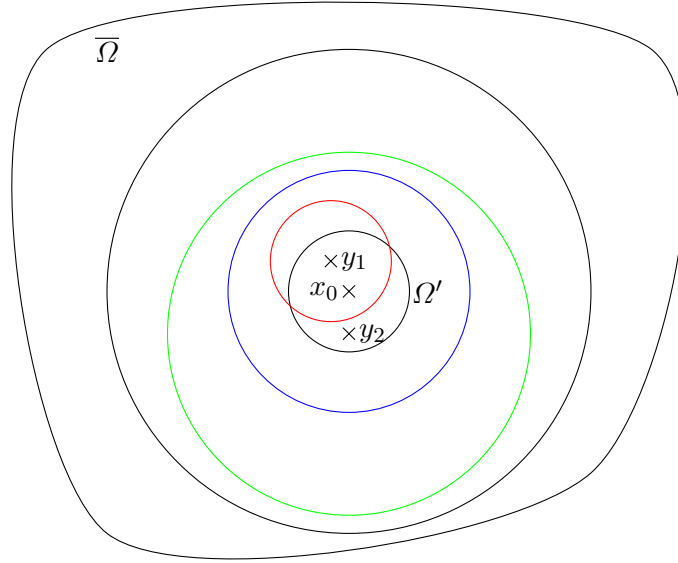


Figure 3.7:  $B_r(y_1) \subseteq B_{2r}(x_0) \subseteq B_{3r}(y_2) \subseteq B_{4r}(x_0) \subseteq \overline{\Omega}$

It holds:

$$\begin{aligned}
 u(y_1) &= \frac{1}{\omega_n r^n} \int_{B_r(y_1)} u(x) d\mu(x) \stackrel{u \geq 0}{\leq} \frac{1}{\omega_n r^n} \int_{B_{2r}(x_0)} u(x) d\mu(x) \leq \\
 &\leq \frac{1}{\omega_n r^n} \int_{B_{3r}(y_2)} u(x) d\mu(x) \leq \frac{3^n}{\omega_n (3r)^n} \int_{B_{3r}(y_2)} u(x) d\mu(x) = 3^n u(y_2)
 \end{aligned}$$

Taking the supremum over  $y_1$  and the infimum over  $y_2$  we get:

$$\sup_{B_r(x_0)} (u) \leq 3^n \inf_{B_r(x_0)} (u)$$

Now consider a general compact subset  $\Omega' \subseteq \Omega$ . If  $\Omega'$  is empty, we have nothing to show, so consider a non-empty  $\Omega'$ . We want to find a  $r \in \mathbb{R}_{>0}$  such that for every  $x \in \Omega'$  holds  $B_{4r}(x) \subseteq \Omega$ . For  $\Omega = \mathbb{R}^n$  we can choose any  $r$ . Otherwise  $\partial\Omega \subseteq \mathbb{R}^n$  is non-empty and closed. Since  $\Omega'$  is also non-empty and even compact, the distance  $\text{dist}(\Omega', \partial\Omega)$  is attained. This must be larger than zero, because from  $\partial\Omega \cap \Omega = \emptyset$  and  $\Omega' \subseteq \Omega$  follows  $\partial\Omega \cap \Omega' = \emptyset$ . So we can choose any  $r \in \mathbb{R}_{>0}$  with:

$$r < \frac{1}{4} \text{dist}(\Omega', \partial\Omega)$$

We cover  $\Omega'$  by a finite number  $\Omega_1, \dots, \Omega_N$  of balls of radius  $r$ . On each of the sets  $\Omega_l$  holds:

$$\sup(u) \leq 3^n \inf(u)$$

One can proceed inductively as follows: Without loss of generality holds  $\sup_{\Omega'}(u) \leq \sup_{\Omega_1}(u)$ , otherwise change the numbering of the  $\Omega_i$ . Since  $\Omega'$  is connected, one can choose a finite number  $\Omega_2, \dots, \Omega_k$  from the balls of the covering such that for  $i \in \{1, \dots, k-1\}$  holds

$$\Omega_i \cap \Omega_{i+1} \neq \emptyset$$

and  $\inf_{\Omega_k}(u) \leq \inf_{\Omega'}(u)$ . If  $\inf_{\Omega_1}(u) \leq \inf_{\Omega'}(u)$ , we can choose  $k = 1$ . It holds:

$$\sup_{\Omega_i}(u) \leq 3^n \inf_{\Omega_i}(u) \stackrel{z_i \in \Omega_i \cap \Omega_{i+1}}{\leq} 3^n u(z_i) \leq 3^n \sup_{\Omega_{i+1}}(u)$$

Inductively one gets after  $k-1$  steps:

$$\sup_{\Omega'}(u) \leq \sup_{\Omega_1}(u) \leq 3^{(k-1)n} \sup_{\Omega_k}(u) \leq 3^{kn} \inf_{\Omega_k}(u) \leq 3^{kn} \inf_{\Omega'}(u)$$

In the worst case of  $u$ , we have  $k = N$  and therefore for any  $u$  holds:

$$\sup_{\Omega'}(u) \leq 3^{Nn} \inf_{\Omega'}(u)$$

Thus the theorem is proven. □<sub>3.4.1</sub>

### 3.4.2 Corollary (Harnack's Convergence Theorem)

Let  $u_n : \Omega \rightarrow \mathbb{R}^n$  be a monotonically increasing sequence of harmonic functions. Assume that there is a  $y \in \Omega$ , where  $u_n(y)$  is a bounded sequence. Then the functions  $u_n$  converge locally uniformly, i.e. uniformly in a small neighborhood of  $y$ , to a harmonic function  $u$ .

#### Proof

Since the sequence  $u_n(y)$  is bounded and monotonically increasing, it is convergent, i.e. for all  $\varepsilon \in \mathbb{R}_{>0}$  there exists a  $N \in \mathbb{N}$  such that for all  $k > m > N$  holds:

$$u_k(y) - u_m(y) < \varepsilon$$

The function  $u_k - u_m$  is non-negative and harmonic. Now choose a bounded and connected neighborhood  $\Omega'$  of  $y$  with  $\overline{\Omega'} \subseteq \Omega$ . According to the Harnack inequality holds:

$$\sup_{\overline{\Omega'}} (u_k - u_m) \leq c(\Omega, \overline{\Omega'}) \cdot \inf_{\overline{\Omega'}} (u_k - u_m) \leq c(u_k(y) - u_m(y)) \xrightarrow{k, m \rightarrow \infty} 0$$

Hence  $u_k$  converges *uniformly* in  $\Omega'$  to a function  $u \in C^0(\Omega')$ . The function  $u$  is harmonic, because for all  $x_0 \in \Omega'$  and  $r \in \mathbb{R}_{>0}$  such that  $B_r(x_0) \subseteq \Omega'$  holds:

$$u_k(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u_k(x) \, d\mu(x)$$

From  $u_k(x_0) \rightarrow u(x_0)$  and  $u_k(x) \rightrightarrows u(x)$  follows:

$$u(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u(x) \, d\mu(x)$$

So  $u$  has the mean-value property and is thus harmonic. □<sub>3.4.2</sub>



## 4 Perron Method of Sub- and Supersolutions

Consider the Dirichlet problem on a bounded  $\Omega \subseteq \mathbb{R}^n$  with  $\varphi \in C^2(\partial\Omega)$ :

$$\Delta u = 0 \qquad u|_{\partial\Omega} = \varphi$$

The Perron method, named after Oskar Perron (1880-1972), yields the existence of a solution if  $\partial\Omega$  is sufficiently smooth. This is a classical method, in the sense that it works always with continuous functions and no weak solutions, but only classical solutions. The idea is to find the solution as a supremum of subsolutions:

$$u(x) := \sup_{v \text{ subsolution}} (v(x))$$

For any domain this gives a harmonic function, but  $\partial\Omega$  has to be sufficiently smooth, for the boundary condition to be satisfied.

### 4.1 Definition (Sub- and Superharmonic Functions, Compactly Contained, Subsolution)

A function  $u \in C^0(\Omega)$  is called *subharmonic* (*superharmonic*) in  $\Omega$ , if for every ball  $B \Subset \Omega$  (*compactly contained* in  $\Omega$ , i.e.  $\overline{B} \subseteq \Omega$ ) and every harmonic function  $h \in C^0(\overline{B})$ ,  $h|_B \in C^2(B)$  with  $u \leq h$  (respectively  $u \geq h$ ) in  $\partial B$  follows  $u \leq h$  (respectively  $u \geq h$ ) in all of  $B$ .

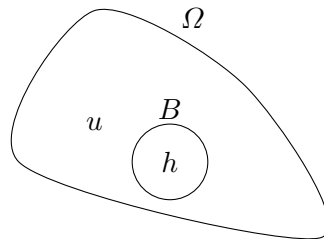


Figure 4.1:  $h \in C^0(\overline{B})$

Note that every harmonic function is subharmonic and superharmonic by the weak maximum principle.

For  $\varphi \in C^2(\partial\Omega)$  we consider the family  $S_\varphi$  of functions defined as:

$$S_\varphi := \{v \in C^0(\overline{\Omega}) \mid v \text{ subharmonic}, v|_{\partial\Omega} \leq \varphi|_{\partial\Omega}\}$$

Because the constant  $v := \min_{\partial\Omega} \varphi$  is subharmonic and satisfies  $v|_{\partial\Omega} \leq \varphi|_{\partial\Omega}$ , we know  $S_\varphi \neq \emptyset$ . The constant is bigger than minus infinity, because  $\Omega$  is bounded and thus  $\partial\Omega$  is compact. A

function  $v \in S_\varphi$  is called *subsolution*. Now define the function  $u$  by:

$$u(x) := \sup_{v \in S_\varphi} (v(x)) \in \mathbb{R} \cup \{\infty\}$$

We hope, that  $u$  is the solution of the Dirichlet problem.

## 4.2 Lemma

If  $u_1$  and  $u_2$  are subharmonic then  $\max(u_1, u_2)$  is also subharmonic.

### Proof

Let  $B \Subset \Omega$  and  $h \in C^2(\Omega)$  be harmonic.

$$\max(u_1, u_2)|_{\partial B} \leq h|_{\partial B}$$

Then follows  $u_1|_{\partial B} \leq h|_{\partial B}$  and  $u_2|_{\partial B} \leq h|_{\partial B}$  in  $B$ , since  $u_1$  and  $u_2$  are subharmonic. Therefore holds also for the maximum in  $B$ :

$$\max(u_1, u_2)|_B \leq h|_B$$

□<sub>4.2</sub>

## 4.3 Theorem (Supremum of Subsolutions is Harmonic)

The function

$$u(x) := \sup_{v \in S_\varphi} (v(x)) \in \mathbb{R} \cup \{\infty\}$$

is a harmonic function in  $\Omega$ .

After proving this theorem, we still need to verify that  $u$  satisfies the boundary condition.

## 4.4 Definition (Harmonic Lift)

Let  $u \in C^0(\Omega)$  be subharmonic in  $\Omega$  and  $B \Subset \Omega$  a ball. The function  $\bar{v} \in C^0(\Omega)$  defined by

$$\bar{v}|_{\Omega \setminus B} = v|_{\Omega \setminus B} \qquad \Delta \bar{v}|_B = 0$$

is referred to as the *harmonic lift* of  $v$  in  $B$ .

## 4.5 Lemma (Harmonic Lift Exists)

The harmonic lift exists.

**Proof**

Consider the Dirichlet problem on the ball  $B$ :

$$\Delta u|_B = 0 \qquad u|_{\partial B} = v|_{\partial B}$$

This Dirichlet problem has an explicit solution  $u$  given by the Poisson representation. Set:

$$\bar{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \setminus B \\ u(x) & \text{if } x \in B \end{cases}$$

□<sub>4.5</sub>

Note that  $\bar{v} \geq v$  follows by the definition of subharmonic, because  $u \geq v$  holds in  $B$  and thus  $\bar{v} \geq v$  holds in  $\Omega$ .

### 4.6 Lemma (Harmonic Lift is Subharmonic)

The harmonic lift is again subharmonic.

**Proof**

Let  $\bar{v}$  be the harmonic lift of  $v$  in  $B$ .

We want to prove, that for all balls  $B' \Subset \Omega$  and for all harmonic functions  $u$  with  $u \geq \bar{v}$  on  $\partial B'$  holds  $u \geq \bar{v}$  in  $B'$ .

If  $B' \cap B = \emptyset$  this follows, because  $\bar{v}|_{\Omega \setminus B} = v$  and  $v$  is subharmonic.

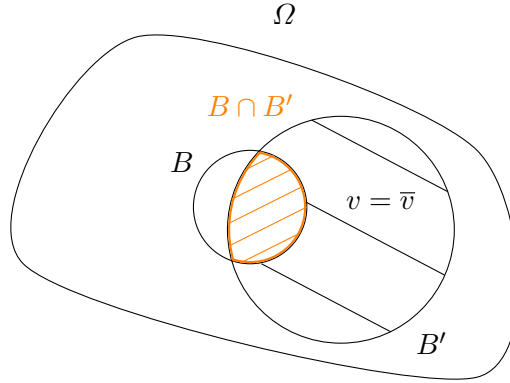


Figure 4.2:  $B' \cap B \neq \emptyset$

Otherwise let  $u \in C^2(B')$  be harmonic and  $u \geq \bar{v}$  on  $\partial B'$ . We know  $v \leq \bar{v}$  in  $\Omega$  thus follows:

$$\begin{aligned} v|_{\partial B'} &\leq \bar{v}|_{\partial B'} \leq u|_{\partial B'} \\ \stackrel{\text{subharmonic}}{\Rightarrow} v|_{B'} &\leq u|_{B'} \end{aligned}$$

Therefore we have  $\bar{v} = v \leq u$  in  $B' \setminus B$  and thus  $\bar{v} \leq u$  on  $\partial(B \cap B')$ . Since the functions  $\bar{v}$  and  $u$  are both harmonic in  $B \cap B'$ , the weak maximum principle implies  $\bar{v} \leq u$  in  $B \cap B'$ . □<sub>4.6</sub>

## 4.7 Proposition (Maximum Principle for Subharmonic Functions)

Let  $\Omega \subseteq \mathbb{R}^n$  be open and connected and  $u \in C^0(\overline{\Omega})$  subharmonic.

- a) Strong maximum principle: If there is a  $x_0 \in \Omega$  with  $u(x_0) = \sup_{\Omega} (u)$ , then  $u$  is constant.
- b) Weak maximum principle:  $u(x) \leq \sup_{\partial\Omega} (u)$

### Proof

b) follows from a) just as in the proof of Theorem 3.3.3.

To prove a), suppose that  $u(x_0) = \sup_{\Omega} (u)$  holds, but  $u$  is *not* constant. Then there is a  $y \in \Omega$  and a  $r \in \mathbb{R}_{>0}$  such that  $u(y) = \sup_{x \in \Omega} u(x)$ , but  $u|_{\partial B_r(y)}$  is not constant.

Let  $c(\tau)$  be a curve joining  $c(0) = x_0$  with  $c(1) = z$ .

$$\tilde{\tau} := \sup \{ \tau \mid u(c(\tau)) = u(x_0) \} < 1$$

Choose  $y := c(\tilde{\tau})$  and  $r$  small enough, so that  $B_r(y) \subseteq \Omega$ .

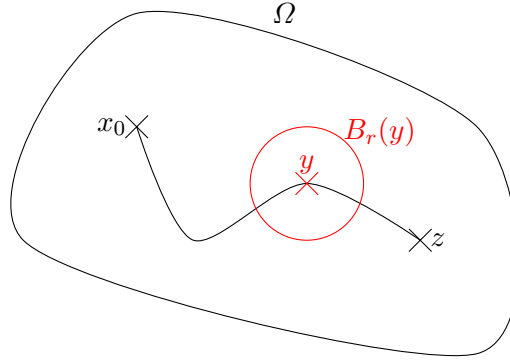


Figure 4.3:  $u(z) < u(x_0) = u(y)$ ;  $u|_{\partial B_r(y)}$  is not constant.

Let  $\bar{u}$  be the harmonic lift of  $u$  in  $B_r$ . Then holds:

$$\bar{u}(y) \geq u(y) \geq \sup_{\partial B_r(y)} (u) = \sup_{\partial B_r(y)} (\bar{u})$$

By the weak maximum principle, which states  $\bar{u}(y) \leq \sup_{\partial B_r(y)} (\bar{u})$ , follows that  $\bar{u}$  is constant in  $B_r(y)$ . This implies that  $\bar{u}|_{\partial B_r} = u|_{\partial B_r}$  is constant, which is a contradiction to  $u$  not being constant on  $\partial B_r$ .  $\square_{4.7}$

## 4.8 Proposition

If  $v$  is subharmonic and  $u$  superharmonic in  $\Omega$  and  $v \leq u$  holds on  $\partial\Omega$ , then in  $\Omega$  holds either  $v < u$  or  $v = u$ , in which case  $u$  and  $v$  are harmonic.

### Proof

This is proven in the exercises.  $\square_{4.8}$

### Proof of Theorem 4.3

Define  $u(x) := \sup_{v \in S_\varphi} (v(x))$ . According to Proposition 4.8 we know, since the constant function  $\sup_{\partial\Omega}(\varphi)$  is superharmonic:

$$\begin{aligned} v(x) &\leq \sup_{\partial\Omega}(\varphi) \quad \forall_{v \in S_\varphi} \\ \Rightarrow \quad u(x) &\leq \sup_{\partial\Omega}(\varphi) \end{aligned}$$

From  $\inf_{\partial\Omega}(\varphi) \in S_\varphi$  and because  $u$  is the supremum of the functions in  $S_\varphi$ , we also know:

$$\inf_{\partial\Omega}(\varphi) \leq u(x)$$

Hence  $u$  is a bounded function on  $\Omega$ . Consider  $y \in \Omega$ . By definition of the supremum, there is a sequence  $v_n \in S_\varphi$  such that  $v_n(y) \rightarrow u(y)$  converges. We can assume that the functions  $v_n$  are bounded, because  $v_n \leq u$  bounds from above and if  $v_n$  is not bounded from below, we can replace it by  $\max(v_n, \inf_{\partial\Omega}(\varphi))$ . This is again a subsolution.

Choose  $r \in \mathbb{R}_{>0}$  such that  $B_r(y) \Subset \Omega$  and let  $\bar{v}_n$  be the harmonic lift of  $v_n$  in  $B_r(y)$ . Then  $v_n \leq \bar{v}_n$  holds in  $\Omega$  and  $\bar{v}_n(y) \leq u(y)$ , since  $\bar{v}_n$  is again a subsolution. Hence follows:

$$\underbrace{v_n(y)}_{\rightarrow u(y)} \leq \bar{v}_n(y) \leq u(y)$$

Thus we have  $\bar{v}_n(y) \rightarrow u(y)$ . Possibly after choosing a subsequence, the sequence  $\bar{v}_n(y)$  is monotonically increasing. Moreover, we can assume that the sequence  $v_n$  is monotonically increasing in  $\Omega$ . Namely, we can replace  $v_n$  by  $\tilde{v}_n$  defined by:

$$\begin{aligned} \tilde{v}_1 &= v_1 \\ \tilde{v}_2 &:= \max(v_1, v_2) \\ &\vdots \\ \tilde{v}_n &:= \max(v_1, \dots, v_n) \end{aligned}$$

These are again subharmonic and they are monotonically increasing. Then the  $\bar{v}_n$  are also monotonically increasing in  $B_r(y)$  using the maximum principle:

$$(\bar{v}_{n+1} - \bar{v}_n)|_{\partial B_r(y)} = (v_{n+1} - v_n)|_{\partial B_r(y)} \geq 0$$

The weak maximum principle implies  $\bar{v}_{n+1} - \bar{v}_n \geq 0$  in  $B_r(y)$ .

Applying Harnack's convergence theorem, we conclude

$$\bar{v}_n \rightrightarrows \tilde{v} \text{ in } B_r(y)$$

and  $\tilde{v}$  is harmonic.

Now we show  $u = \tilde{v}$  in  $B_r(y)$ . Obviously holds  $\tilde{v} \leq u$ , because  $\bar{v}_n \leq u$  holds for all  $n \in \mathbb{N}$ .

Thus assume that there is a point  $x \in B_r(y)$  with  $\tilde{v}(x) < u(x)$ . Then there exists a subsolution  $s \in S_\varphi$  with  $\tilde{v}(x) < s(x) < u(x)$ , because  $u(x) = \sup_{v \in S_\varphi} (v(x))$ . Thus we also have:

$$v_n(x) \leq \tilde{v}(x) < s(x)$$

Now form the sequence:

$$w_n := \max(s, v_n)$$

These are again subharmonic, from  $v_n(x) < s(x)$  follows  $w_n(x) = s(x)$ , and for the harmonic lifts holds  $\bar{w}_n \geq \bar{v}_n$ . Then repeating the above construction for  $v_n$  replaced by  $w_n$ , we find  $\bar{w}_n \rightrightarrows \tilde{w}$  converges uniformly to a harmonic  $\tilde{w}$ . Then follows  $\tilde{v} \leq \tilde{w} \leq u$  just as before and thus  $\tilde{v} - \tilde{w} \leq 0$  in  $B$ . Additionally  $\tilde{v}$  and  $\tilde{w}$  are harmonic and we know:

$$\tilde{v}(y) = \tilde{w}(y) = u(y)$$

So  $\tilde{v} - \tilde{w}$  attains its maximum in  $y \in B$  and the strong maximum principle gives  $\tilde{v} = \tilde{w}$  in  $B$ . From  $w_n(x) > s(x)$  follows  $\tilde{w}(x) \geq s(x) > \tilde{v}(x)$ . This is a contradiction.  $\square_{4.3}$

Question: Is  $u$  also a solution of the Dirichlet problem?

## 4.9 Proposition (Existing Solution is Perron Solution)

If the Dirichlet problem has a solution, then

$$u(x) = \sup_{v \in S_\varphi} (v(x))$$

equals this solution.

### Proof

Let  $\tilde{u}$  be a solution of the Dirichlet problem. Then  $\tilde{u}$  is also a subsolution, i.e.  $\tilde{u} \in S_\varphi$ . Therefore holds  $u(x) \geq \tilde{u}(x)$  for all  $x \in \Omega$ . Hence  $u - \tilde{u}$  is harmonic and  $(u - \tilde{u})|_{\partial\Omega} = 0$ . By the weak maximum principle follows  $u = \tilde{u}$  in  $\Omega$ .  $\square_{4.9}$

## 4.10 Definition (Barrier, Local Barrier)

A function  $w \in C^0(\overline{\Omega})$  is called *barrier* at  $\xi \in \partial\Omega$ , if holds:

- i)  $w$  is superharmonic.
- ii) In  $\overline{\Omega} \setminus \{\xi\}$  holds  $w > 0$  and  $w(\xi) = 0$ .

We show first that the barrier is a local concept.

A *local barrier* in  $\xi \in \partial\Omega$  is a function  $w \in C^0(\overline{\Omega \cap B_\varepsilon(\xi)})$  for  $\varepsilon \in \mathbb{R}_{>0}$  with:

- i)  $w$  is superharmonic in  $\Omega \cap B_\varepsilon(\xi)$ .
- ii) In  $\overline{\Omega \cap B_\varepsilon(\xi)} \setminus \{\xi\}$  holds  $w > 0$  and  $w(\xi) = 0$ .

## 4.11 Theorem (Barrier is Local Concept)

If there is a local barrier, there is also a barrier.

### Proof

Let  $w \in C^0(\overline{\Omega \cap B_\varepsilon(\xi)})$  be a local barrier. With

$$m := \inf_{B_\varepsilon(\xi) \setminus B_{\frac{\varepsilon}{2}}(\xi)} (w) > 0$$

we choose:

$$\bar{w}(x) := \begin{cases} \min(w(x), m) & x \in \overline{\Omega \cap B_{\frac{\varepsilon}{2}}(\xi)} \\ m & x \notin \overline{\Omega \cap B_{\frac{\varepsilon}{2}}(\xi)} \end{cases}$$

This is a barrier. □<sub>4.11</sub>

#### 4.12 Definition (regular point)

A border-point  $\xi \in \partial\Omega$  is called *regular* if there is a barrier at  $\xi$ .

#### 4.13 Theorem (Solution of Dirichlet Problem with Regular Border)

If  $\xi$  is a regular border-point, then holds for the function  $u(x) := \sup_{v \in S_\varphi} (v(x))$ :

$$\lim_{\Omega \ni x \rightarrow \xi} u(x) = \varphi(\xi)$$

So if every border point is regular, the Perron solution is the solution of the Dirichlet problem.

#### Proof

We construct a supersolution. Because  $\xi$  is a regular border-point, there is a barrier  $w$  at  $\xi$ . Consider  $\varepsilon \in \mathbb{R}_{>0}$  and define:

$$M := \sup_{\partial\Omega} (|\varphi|)$$

Now choose  $\delta, k \in \mathbb{R}_{>0}$  such that holds:

$$\begin{aligned} |\varphi(x) - \varphi(\xi)| &< \varepsilon & \forall x \in B_\delta(\xi) \cap \partial\Omega \\ kw(x) &\geq 2M & \forall x \in \partial\Omega \setminus B_\delta(\xi) \end{aligned}$$

Then

$$u_+(x) := \varphi(\xi) + \varepsilon + kw(x)$$

is a supersolution and

$$u_-(x) := \varphi(\xi) - \varepsilon - kw(x)$$

a subsolution. (These are sub-/superharmonic, because  $w(x)$  is superharmonic and thus  $-w(x)$  subharmonic and constants are harmonic.)

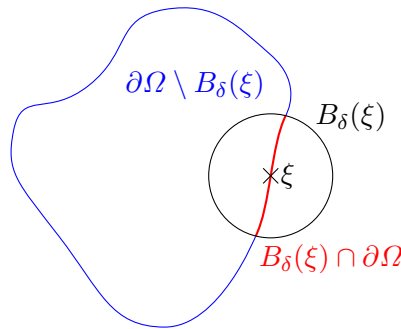


Figure 4.4: Construction of  $u_+$  and  $u_-$

The  $\varepsilon$  ensures  $u_-(x) \leq \varphi(x) \leq u_+(x)$  for  $x \in B_\delta(\xi) \cap \partial\Omega$  and for  $x \in \partial\Omega \setminus B_\delta(\xi)$  this follows from  $kw \geq 2M$ . Now the maximum principle implies:

$$\varphi(\xi) - \varepsilon - kw(x) \leq u(x) \leq \varphi(\xi) + \varepsilon + kw(x)$$

Due to  $\lim_{x \rightarrow \xi} (w(x)) = 0$  and because  $\varepsilon$  can be chosen arbitrarily, follows:

$$\lim_{x \rightarrow \xi} u(x) = \varphi(\xi)$$

□<sub>4.13</sub>

#### 4.14 Theorem (Existence of Solution for Dirichlet Problem)

The following statements are equivalent:

1. The Dirichlet problem has for every  $\varphi \in C^0(\partial\Omega)$  a solution in  $C^2(\Omega) \cap C^0(\overline{\Omega})$ .
2. Every border-point of  $\Omega$  is regular.

##### Proof

“2.  $\Rightarrow$  1.”: If every border-point is regular, then the Perron solution  $u(x)$  is the solution of the Dirichlet problem after Theorem 4.13.

“1.  $\Rightarrow$  2.”: For  $\xi \in \partial\Omega$  let  $u$  be the solution of the following Dirichlet problem:

$$\begin{aligned} \Delta u|_\Omega &= 0 \\ u(x)|_{\partial\Omega} &= \|x - \xi\| |_{\partial\Omega} \end{aligned}$$

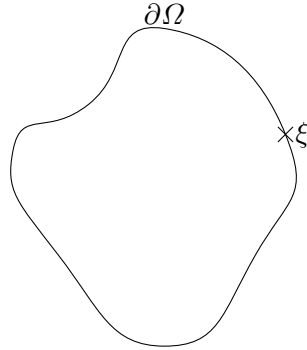


Figure 4.5:  $u(\xi) = 0$ ,  $u|_{\partial\Omega \setminus \{\xi\}} > 0$

Were  $u(y) = 0$  for a  $y \in \Omega$ , we had an interior minimum, and thus the maximum principle would give the contradiction  $u = 0$ . So this  $u$  is a barrier. □<sub>4.14</sub>

How can one construct barriers? Under what geometrical conditions do barriers exist?

Consider the dimension  $n = 2$ . Without loss of generality we assume  $\xi = 0$ . We consider this as a problem in  $\mathbb{C} \cong \mathbb{R}^2$  and define:

$$w(z) = -\operatorname{Re} \left( \frac{1}{\ln(z)} \right)$$



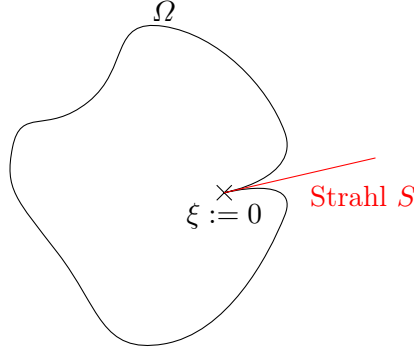


Figure 4.6:  $\Omega$  under consideration

Choose the branch cut of the logarithm on  $S$ . Such an  $S$  with  $\Omega \cap S = \emptyset$  and  $\xi \in S$  can always be chosen locally. Thus  $\ln(z)$  is holomorphic in  $\mathbb{C} \setminus S$  and therefore  $(\ln(z))^{-1}$  is holomorphic on  $\mathbb{C} \setminus S$ , which means that  $w$  is harmonic on  $\mathbb{C} \setminus S$  and especially continuous on  $\mathbb{C} \setminus S$ .

To study the behaviour at the origin, we choose the polar representation:

$$z = re^{i\varphi}$$

$$\ln(z) = \ln(r) + i\varphi + 2\pi in$$

Here  $n \in \mathbb{Z}$  depends on the branch of the logarithm.

$$-\frac{1}{\ln(z)} = -\frac{1}{\ln(r) + i\varphi + 2\pi in}$$

$$-\operatorname{Re}\left(\frac{1}{\ln(z)}\right) = -\frac{\ln(r)}{(\ln(r))^2 + (\varphi + 2\pi n)^2} \xrightarrow{r \rightarrow 0} 0$$

This converges locally uniformly in  $\varphi$ .

Alternatively, this follows from the Riemann mapping theorem.

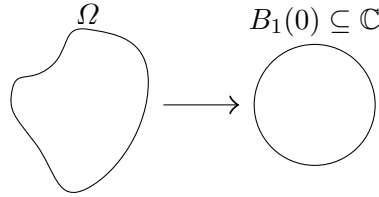


Figure 4.7: Mapping from  $\Omega$  to  $B_1(0)$

Just transform from any simply connected  $\Omega$  to  $B_1(0) \subseteq \mathbb{C}$ , solve the Dirichlet problem in the ball and transform back.

Consider now the case  $n \geq 3$ : One gets barriers if additional conditions are satisfied.

#### 4.15 Definition (Exterior Sphere/Cone Condition)

An open subset  $\Omega \subseteq \mathbb{R}^n$  satisfies in  $\xi \in \partial\Omega$  the *exterior sphere condition* if there is a closed ball  $K \subseteq \mathbb{R}^n$  with radius  $R$  such that  $K \cap \overline{\Omega} = \{\xi\}$ .

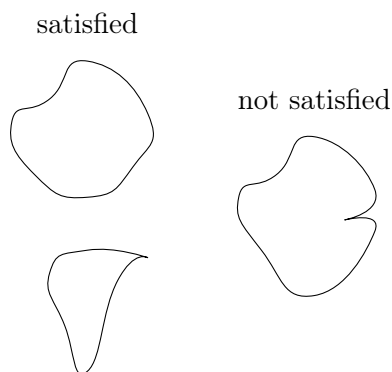


Figure 4.8: Exterior Sphere Condition

Now choose:

$$w(x) = -\Gamma(\|x - y\|) + \Gamma(\|\xi - y\|)$$

This function is harmonic and non-negative. Furthermore  $w(x) = 0$  implies  $x = \xi$ . So  $w$  is a barrier and we can solve the Dirichlet problem.

An improvement is the so called *exterior cone condition*.

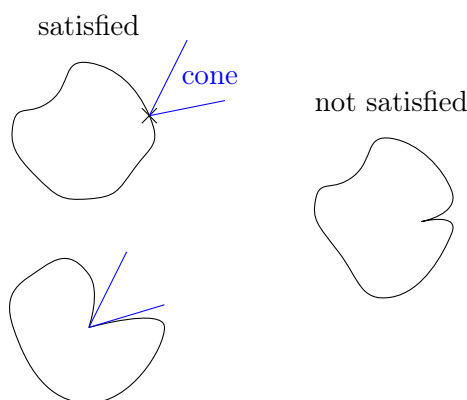


Figure 4.9: Exterior Cone Condition

Here one can also construct a barrier.

#### 4.16 Example (Dirichlet Problem without Solution, Lebesgue Spine)

A counter example for the solvability of the Dirichlet problem with continuous boundary values was found in 1912 by Lebesgue. The details can be found in the book of Courant and Hilbert.

We use the Ansatz:

$$u(x) = \int \Gamma(\|x - y\|) \varrho(y) \, d^n y$$

Here  $\varrho(y)$  can be viewed as “charge density”. Then follows:

$$\Delta u(x) = \varrho(x)$$

$$\Delta u(x) = 0 \quad \text{if } \varrho(x) = 0$$

Consider the charge distribution on the  $x$ -axis:

$$\varrho(x, y, z) = \begin{cases} \delta(y) \delta(z) (-4\pi x) & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

This is the so-called Lebesgue spine in  $\mathbb{R}^3$ .

$$\begin{aligned} u(x, y, z) &= \int_0^1 \Gamma(\|(\xi, 0, 0) - (x, y, z)\|) (-4\pi\xi) d\xi = \\ &= \int_0^1 \frac{\xi d\xi}{\sqrt{(\xi - x)^2 + y^2 + z^2}} d\xi \end{aligned}$$

Define  $\zeta := \sqrt{y^2 + z^2}$ . The integral gives:

$$u(x, \zeta) = A(x, \zeta) - 2x \ln(\zeta)$$

$$A(x, \zeta) = \sqrt{(1-x)^2 + \zeta^2} - \sqrt{x^2 + \zeta^2} + x \ln \left( \left| \left( 1 + x + \sqrt{(1-x)^2 + \zeta^2} \right) \left( x + \sqrt{x^2 + \zeta^2} \right) \right| \right)$$

$A$  is continuous at the origin:

$$\begin{aligned} 1 + x + \sqrt{(1-x)^2 + \zeta^2} &\rightarrow 2 \\ \lim_{(x, \zeta) \rightarrow 0} \left( x + \sqrt{x^2 + \zeta^2} \right) &= \lim_{x \rightarrow 0} (2x) \\ \lim_{x \rightarrow 0} x \ln(x) &= 0 \\ \Rightarrow \lim_{(x, \zeta) \rightarrow 0} A(x, \zeta) &= 1 \end{aligned}$$

But  $x \ln(\zeta)$  is not continuous at the origin. Consider for example:

$$\begin{aligned} \zeta &= |x|^\alpha & x \ln(\zeta) &= x\alpha \ln(|x|) \xrightarrow{x \rightarrow 0} 0 \\ \zeta &= e^{-\frac{c}{x}} & x \ln(\zeta) &= x \frac{-c}{x} \xrightarrow{x \rightarrow 0} -c \end{aligned}$$

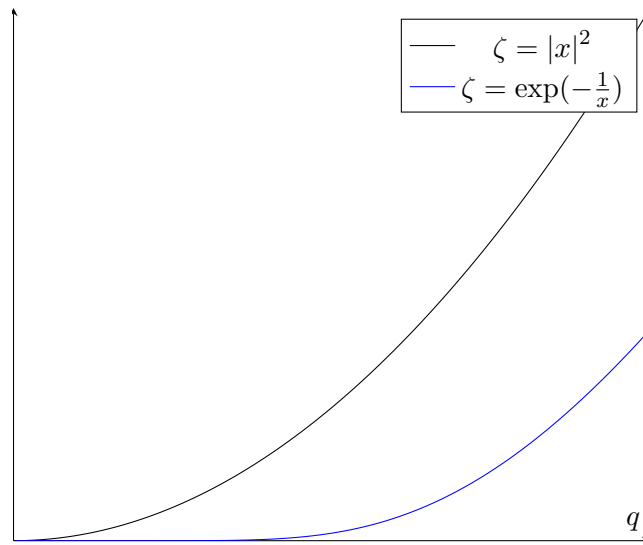


Figure 4.10:  $\lim_{(x, \zeta) \rightarrow 0} A(x, \zeta)$  has different values for different path.

So the function  $-2x \ln(\zeta)$  is constant along the curve  $\left\{ \zeta = e^{-\frac{c}{2x}} \mid x > 0 \right\}$ , but is not continuous at the origin. Consider the level set  $u(x, \zeta) = 1 + C$  for  $C \in \mathbb{R}_{>0}$ .

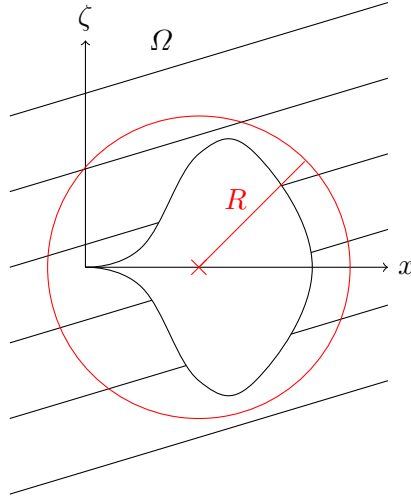


Figure 4.11: The level set  $u(x, \zeta) = 1 + C$  defines the region  $\Omega$ .

We have:

$$\begin{aligned} \Delta u|_{\Omega} &= 0 \\ u|_{\partial\Omega} &= 1 + C \end{aligned}$$

But  $u$  is *not* continuous at  $(x, y, z) = 0$ . Let  $B$  be a ball such that  $\partial B \cap \partial\Omega = \emptyset$ . Now perform an inversion:

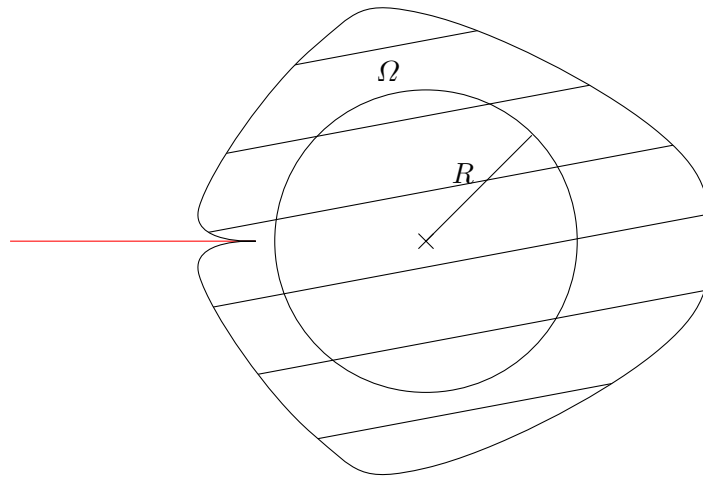


Figure 4.12:  $\Omega$  inverted

For formulas see the book of Courant and Hilbert.

$$\begin{aligned} \Delta u|_{\Omega} &= 0 \\ u|_{\partial\Omega} &= 1 + C \end{aligned}$$

$u$  is not continuous in  $\overline{\Omega}$ .

## 5 The Maximum Principle for Linear Elliptic Equations

Again  $\Omega \subseteq \mathbb{R}^n$  is an open domain and  $u \in C^2(\Omega, \mathbb{R})$  a scalar function. Let  $L$  be a differential operator of second order, linear and elliptic.

$$Lu(x) = \sum_{i,j=1}^n a^{ij}(x) \partial_{ij}u(x) + \sum_{i=1}^n b^i(x) \partial_i u(x) + c(x)u(x)$$

Usually we use the summation convention to omit the sums. We make the following assumptions:

- i) *Uniform ellipticity*: There exists a  $\lambda \in \mathbb{R}_{>0}$  such that for all  $\xi \in \mathbb{R}^n$  and all  $x \in \Omega$  holds:

$$a^{ij}(x) \xi_i \xi_j \geq \lambda \cdot |\xi|^2$$

- ii) The coefficients are *uniformly bounded*: There exists a  $K \in \mathbb{R}_{>0}$  such that for all  $x \in \Omega$  and  $i, j \in \{1, \dots, n\}$  holds:

$$|a^{ij}(x)|, |b^i(x)|, |c(x)| \leq K$$

Consider solutions of the homogeneous equation  $Lu = 0$ . Our goal is to derive a maximum principle.

### 5.1 Example (One-Dimensional)

We begin with a one-dimensional example.

$$u''(x) + cu(x) = 0$$

- a)  $c < 0$ : There are two fundamental solutions, i.e. every solution is a linear combination:

$$\begin{aligned} u_1(x) &= e^{\sqrt{|c|x}} \\ u_2(x) &= e^{-\sqrt{|c|x}} \end{aligned}$$

Consider  $\Omega = (0, 1)$  and impose Dirichlet boundary conditions  $u(0) = 0 = u(1)$ .

$$u = \alpha u_1 + \beta u_2$$

$$\begin{aligned} u(0) = 0 & \Rightarrow \alpha = -\beta \Rightarrow u = \alpha \left( e^{\sqrt{|c|x}} - e^{-\sqrt{|c|x}} \right) \\ u(1) = 0 & \Rightarrow \alpha \left( e^{\sqrt{|c|}} - e^{-\sqrt{|c|}} \right) = 0 \Rightarrow \alpha = 0 \end{aligned}$$

Thus the Dirichlet problem has a unique solution  $u = 0$ , in accordance with the maximum principle.

b)  $c > 0$ : The functions

$$\begin{aligned} u_1(x) &= \sin(\sqrt{c}x) \\ u_2(x) &= \cos(\sqrt{c}x) \end{aligned}$$

are fundamental solutions.

$$\begin{aligned} u &= \alpha u_1 + \beta u_2 \\ u(0) &= 0 \quad \Rightarrow \quad \beta = 0 \\ u(1) &= 0 = \alpha \sin(\sqrt{c}) \end{aligned}$$

In the case of  $\sin(\sqrt{c}) = 0$ , the parameter  $\alpha$  is arbitrary, otherwise follows  $\alpha = 0$  and thus  $u = 0$ .

For example for  $c = \pi^2$  with  $\alpha = 1$  we have:

$$u(x) = \sin(\pi x)$$

This is a non-trivial solution of the Dirichlet problem and it has an interior maximum. So the maximum principle is violated.

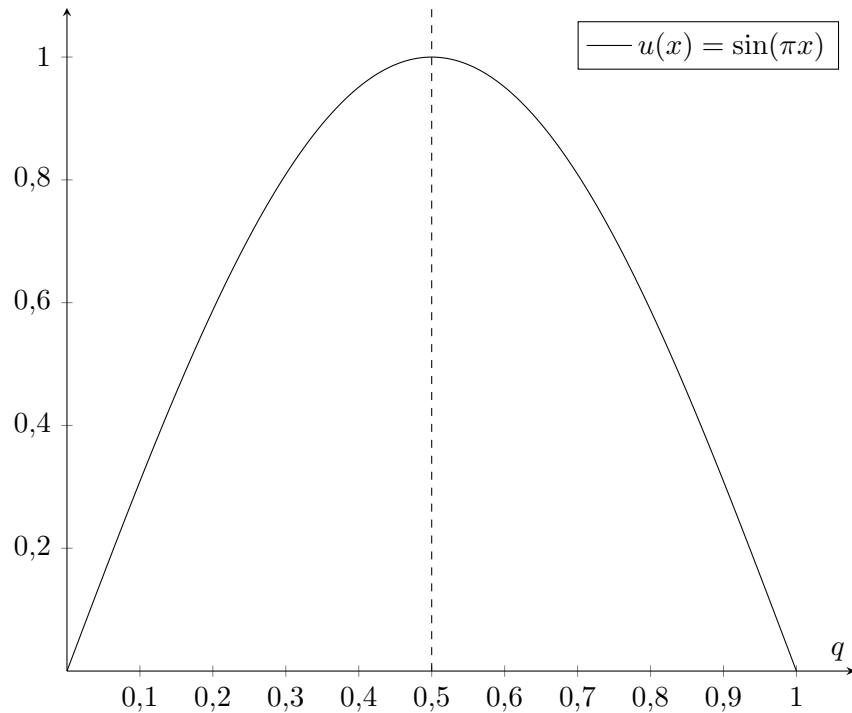


Figure 5.1:  $u(x)$  has an interior maximum.

## 5.2 Theorem (Weak Maximum Principle)

Let  $\Omega \subseteq \mathbb{R}^n$  be bounded and  $c(x) = 0$ . Assume  $Lu \geq 0$  (or  $Lu \leq 0$ ) in all of  $\Omega$  and  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ . Then follows:

$$\sup_{\Omega} (u) = \sup_{\partial\Omega} (u) \qquad \left( \inf_{\Omega} (u) = \inf_{\partial\Omega} (u) \right)$$

**Proof**

Only consider the case  $Lu \geq 0$ , since for  $Lu \leq 0$  holds  $L(-u) \geq 0$ .

- a) Assume  $Lu > 0$  and  $\sup_{\Omega}(u) > \sup_{\partial\Omega}(u)$ . Then the function  $u$  has a maximum at  $x_0 \in \Omega$ . (Here we use that  $\Omega$  is bounded and thus  $\overline{\Omega}$  is compact.) This implies

$$\partial_i u(x_0) = 0$$

and that  $\partial_{ij}u(x_0)$  is negative semi-definite.

$$Lu(x_0) = \underbrace{a^{ij}(x_0)}_{\text{pos. def.}} \underbrace{\partial_{ij}u(x_0)}_{\text{neg. sem.-def.}} \leq 0$$

This is a contradiction to  $Lu > 0$ .

- b) Assume only  $Lu \geq 0$ . The idea is to modify  $u$  such that a) applies.

$$Le^{\gamma x_1} = a^{11}\partial_{x_1}\partial_{x_1}e^{\gamma x_1} + b^1\partial_{x_1}e^{\gamma x_1} = (\gamma^2 a^{11} + \gamma b^1)e^{\gamma x_1}$$

We know  $a^{11}(x) > \lambda$  and  $|b^1| < K$ . For  $\gamma > \frac{K}{\lambda}$  we have:

$$Le^{\gamma x_1} > 0$$

Now consider for  $\varepsilon \in \mathbb{R}_{>0}$ :

$$L(u + \varepsilon e^{\gamma x_1}) = Lu + \varepsilon Le^{\gamma x_1} > 0$$

Thus one can apply a) to conclude:

$$\sup_{\Omega}(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial\Omega}(u + \varepsilon e^{\gamma x_1})$$

Since this holds for all  $\varepsilon \in \mathbb{R}_{>0}$ , we get also:

$$\sup_{\Omega}(u) = \sup_{\partial\Omega}(u)$$

□<sub>5.2</sub>

### 5.3 Corollary

For  $c(x) \leq 0$  and  $Lu \geq 0$  (or  $Lu \leq 0$ ) holds with  $u^+ := \max(u, 0)$  (and  $u^- := \min(u, 0)$ ):

$$\sup_{\Omega}(u) \leq \sup_{\partial\Omega}(u^+) \quad \left( \inf_{\Omega}(u) \leq \inf_{\partial\Omega}(u) \right)$$

**Proof**

We show this only for  $Lu \geq 0$ , since for  $Lu \leq 0$  one can consider  $-u$ .  $L$  without the zero order term is:

$$L_0 := a^{ij}(x)\partial_{ij} + b^i(x)\partial_i$$

Now consider the domain:

$$\Omega^+ := \{x \in \Omega \mid u(x) > 0\}$$

$\Omega^+$  is again an open, bounded domain and in  $\Omega^+$  holds:

$$L_0 u \geq L_0 u + \underbrace{cu}_{\leq 0} = Lu \geq 0$$

Then 5.2 yields:

$$\sup_{\Omega^+} (u) = \sup_{\partial\Omega^+} (u) \leq \sup_{\partial\Omega} (u^+)$$

If in  $\Omega$  holds  $u \leq 0$ , there is nothing to prove. Otherwise holds:

$$\sup_{\Omega} (u) = \sup_{\Omega^+} (u) = \sup_{\partial\Omega^+} (u) \leq \sup_{\partial\Omega} (u^+)$$

□<sub>5.3</sub>

## 5.4 Theorem

Let  $c \in \mathbb{R}_{\leq 0}$  and  $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$  satisfy the inequalities  $Lu \geq Lv$  in  $\Omega$  and  $u \leq v$  on  $\partial\Omega$ . Then follows  $u \leq v$  in  $\Omega$ .

**Proof**

$$\begin{aligned} L(u - v)|_{\Omega} &\geq 0 \\ (u - v)|_{\partial\Omega} &\leq 0 \end{aligned}$$

Apply Corollary 5.3 to obtain:

$$\sup_{\Omega} (u - v) \leq 0 \quad \Rightarrow \quad u|_{\Omega} \leq v|_{\Omega}$$

□<sub>5.4</sub>

Now we want to show the strong maximum principle, i.e.  $u(x) = \max_{\Omega} (u)$  for  $x \in \Omega$  implies that  $u$  is constant.

## 5.5 Definition (Interior Sphere Condition)

A boundary point  $x_0 \in \partial\Omega$  satisfies the *interior sphere condition* if there is a ball  $B \subseteq \Omega$  such that  $x_0 = \partial B$ .

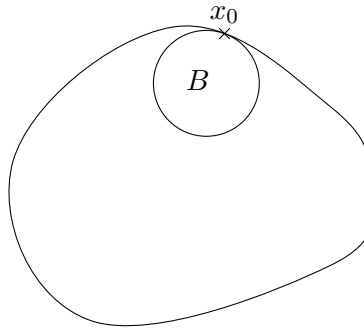


Figure 5.2: Interior sphere condition



## 5.6 Lemma

Assume  $c = 0$  and  $Lu \geq 0$  in  $\Omega$ . Moreover, at the boundary point  $x_0 \in \partial\Omega$  the following conditions should hold:

- i)  $u$  is continuous at  $x_0$ .
- ii)  $u(x_0) > u(x)$  for all  $x \in \Omega$ .
- iii) The interior sphere condition is satisfied.

Then the normal derivative of  $u$  at  $x_0$ , if it exists, satisfies the inequality:

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

We take the *outer* normal.

*Note:* The statement  $\frac{\partial u}{\partial \nu}(x_0) \geq 0$  is obvious. The point is to prove the *strict* inequality.

### Proof

Let  $B = B_R(y)$  be the sphere of the interior sphere condition.

$$v(x) := e^{-\alpha r^2} - e^{-\alpha R^2} \quad r := \|x - y\| \quad r^2 = (x - y)_i (x - y)^i$$

$$\partial_i r^2 = 2(x - y)_i$$

$$\partial_{ij} r^2 = 2\delta_{ij}$$

$$\partial_i e^{-\alpha r^2} = e^{-\alpha r^2} (-2\alpha) (x - y)_i$$

$$\partial_{ij} e^{-\alpha r^2} = e^{-\alpha r^2} 4\alpha^2 (x - y)_i (x - y)_j + e^{-\alpha r^2} (-2\alpha) \delta_{ij}$$

$$Lv = e^{-\alpha r^2} \left( \underbrace{4\alpha^2 a^{ij}(x) (x - y)_i (x - y)_j}_{\leq \lambda \|x - y\|^2} - 2\alpha \underbrace{(a^{ij} \delta_{ij} + b^i (x - y)_i)}_{|\dots| \leq n^2 K + nKR} \right)$$

For  $\varrho < R$  consider  $B_R(y) \setminus B_\varrho(y)$ .

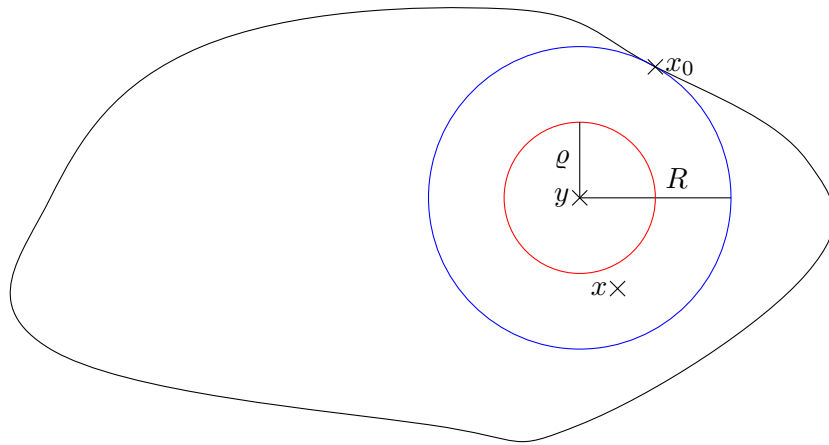


Figure 5.3:  $Lv \geq 0$  in  $B_R(y) \setminus B_\varrho(y)$

By choosing  $\alpha$  sufficiently large, we can arrange that  $Lv \geq 0$  in  $B_R(y) \setminus B_\varrho(y)$ . It holds  $u(x) - u(x_0) < 0$  on  $\partial B_\varrho(y)$ .

On  $\partial B_R(y)$  holds  $v = 0$  and  $u(x) - u(x_0) \leq 0$ . Thus follows

$$u - u(x_0) + \varepsilon v \leq 0$$

on  $\partial(B_R(y) \setminus B_\varrho(y))$  for small enough  $\varepsilon \in \mathbb{R}_{>0}$  and:

$$L(u - u(x_0) + \varepsilon v) \geq 0$$

The weak maximum principle implies

$$u - u(x_0) + \varepsilon v \leq 0$$

in  $B_R(y) \setminus B_\varrho(y)$ . Now we compute the normal derivative.

$$0 \leq \frac{\partial}{\partial \nu} (u - u(x_0) + \varepsilon v) \Big|_{x_0} = \frac{\partial u}{\partial \nu} \Big|_{x_0} + \varepsilon \frac{\partial v}{\partial \nu}$$

$$\frac{\partial v}{\partial \nu} = v'(r) = -2\alpha r e^{-\alpha r^2} \Big|_{r=R} < 0$$

Therefore follows:

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

□<sub>5.6</sub>

## 5.7 Theorem (Hopf's Maximum Principle)

Assume that  $L$  is uniformly elliptic with uniformly bounded coefficients and  $c = 0$ . Furthermore assume  $Lu \geq 0$  (or  $Lu \leq 0$ ) in  $\Omega$ , but not necessarily bounded. If  $u$  attains a maximum (respectively minimum) in  $\Omega$ , then  $u$  is constant.

For  $c \leq 0$ , the function  $u$  cannot attain a non-negative maximum (respectively a non-positively minimum) in  $\Omega$ , unless  $u$  is constant.

This theorem can be viewed as a generalization of Liouville's theorem to  $\mathbb{R}^n$ .

There were two important mathematicians Hopf:

- Heinz Hopf (1894-1971): Hopf fibration, Hopf algebra, ...
- Ebehard Hopf (1902-1983): Hopf's maximum principle is named after him.  
1936 he was Professor in Leipzig, in 1944 he went to Munich as a successor of Caratheodory.  
After the war, he went to Bloomington in the USA.

### Proof

Assume that  $Lu \geq 0$  and that  $u$  attains a maximum  $M$  in  $\Omega$ , but  $u$  is not constant. Define:

$$\Omega^- := \{x \in \Omega \mid u(x) < M\} \neq \emptyset$$

Since  $\Omega^- \subsetneq \Omega$  is a proper subset,  $\partial\Omega^- \cap \Omega$  is not empty. For  $y \in \Omega^-$  let  $B_R(y)$  be the largest ball, which does not intersect  $\Omega \setminus \Omega^-$ . By choosing a  $x_0 \in \Omega^-$  sufficiently close to  $\partial\Omega^- \cap \Omega$ , one can arrange  $B := B_R(x_0) \subseteq \Omega$ .

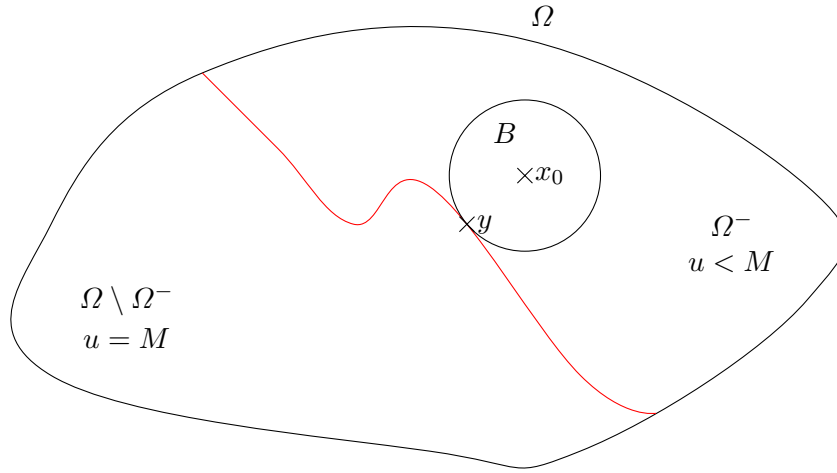


Figure 5.4:  $B_R(x_0) \subseteq \Omega$

It holds  $\overline{B} \cap \partial\Omega^- \neq \emptyset$ , since otherwise the radius would not be maximal. Choose  $y \in \overline{B} \cap \partial\Omega^-$ , i.e.  $u(y) = M$ . We want to apply the previous Lemma to  $\Omega^-$  and the boundary point  $y$ . We check the conditions:

1.  $u$  is continuous in  $y$ , because we have  $u \in C^2(\Omega)$ .
2. We have  $u(y) = M$ , but  $u < M$  in  $\Omega^-$ .
3. The interior sphere condition holds by construction.

So we get:

$$\left. \frac{\partial u}{\partial \nu} \right|_y > 0$$

On the other hand,  $Du|_y$  vanishes, because  $u$  has a maximum at  $y$ . This is a contradiction.  $\square_{5.7}$

## 6 Weak Solutions, Sobolev Spaces

We can write the elliptic operator

$$L = a^{ij} \partial_{ij} + b^i \partial_i + c$$

in divergence form:

$$\begin{aligned} 0 &= Lu = a^{ij} \partial_{ij} u + b^i \partial_i u + cu = \\ &= \partial_i (a^{ij} \partial_j u) + \underbrace{(-\partial_i a^{ij} + b^j)}_{=: \tilde{b}^j} \partial_j u + cu = \\ &= \partial_i (a^{ij} \partial_j u) + \tilde{b}^j \partial_j u + cu \end{aligned}$$

We consider  $u \in C^2(\Omega)$ . Let  $\eta \in C_0^\infty(\Omega)$  be a “test function”. We evaluate weakly:

$$\int_{\Omega} (Lu) \eta \, d^n x = \int_{\Omega} \left( \partial_i (a^{ij} \partial_j u) + \tilde{b}^j \partial_j u + cu \right) \eta \, d^n x$$

Now we apply Gauß’ divergence theorem. The boundary terms vanish due to  $\eta|_{\partial\Omega} = 0$ .

$$\int_{\Omega} (Lu) \eta \, d^n x = \int_{\Omega} \left( -a^{ij} (\partial_j u) (\partial_i \eta) + \left( \tilde{b}^j \partial_j u \right) \eta + cu \eta \right) \, d^n x$$

Now  $Lu = 0$  is equivalent to  $\int_{\Omega} (Lu) \eta \, d^n x = 0$ , since  $(Lu) \eta$  is continuous for any  $\eta \in C_0^\infty(\Omega)$ .

$$0 = \int_{\Omega} \left( -a^{ij} (\partial_j u) (\partial_i \eta) + \left( \tilde{b}^j \partial_j u \right) \eta + cu \eta \right) \, d^n x$$

This works just as well for  $u \in C^1(\Omega)$ . One can do even better, by only demanding  $\partial_i u \in L^1(\Omega)$ . The partial derivative is defined weakly. For  $u \in C^1(\Omega)$  assume:

$$\partial_i u = v$$

Equivalently holds:

$$\int_{\Omega} v \eta \, d^n x = \int_{\Omega} (\partial_i u) \eta \, d^n x = - \int_{\Omega} u (\partial_i \eta) \, d^n x \quad \forall \eta \in C_0^\infty(\Omega)$$

The last condition can be stated for  $u \in L^2(\Omega)$ .

What are the resulting function spaces?

These are Sobolev space spaces  $H^{n,2}(\Omega) = W^{n,2}(\Omega)$  (Hilbert spaces, in this case  $H^{1,2}(\Omega)$ ) or more general the Sobolev spaces of  $n$ -times weakly differentiable functions  $W^{n,p}(\Omega)$  with  $D^\beta f \in L^p$  for all multi-indices  $\beta$  with  $|\beta| \leq n$  (Banach spaces).

Writing the equation in divergence form is also of advantage for using *variational methods*. For simplicity for  $\Delta u = 0$  in  $\Omega$  holds equivalently:

$$\int_{\Omega} (\partial_i u) (\partial^i \eta) \, d^n x = 0 \quad \forall_{\eta \in C_0^\infty(\Omega)}$$

The Dirichlet energy is defined as:

$$E(u) := \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 \, d^n x$$

Minimize  $E$  in a suitable Sobolev space  $(H^{1,2}(\Omega) \cap \dots)$ . Suppose a minimizer  $u$  exists.

$$E(u + \tau \eta) \geq E(u) \quad \forall_{\eta \in C_0^\infty} \quad \forall_{\tau \in \mathbb{R}}$$

If  $E(u + \tau \eta)$  is differentiable, now follows:

$$0 = \left. \frac{d}{d\tau} E(u + \tau \eta) \right|_{\tau=0}$$

For the Dirichlet energy this gives for all  $\eta \in C_0^\infty$ :

$$\begin{aligned} 0 &= \left. \frac{d}{d\tau} \frac{1}{2} \int_{\Omega} \partial_i (u + \tau \eta) \partial^i (u + \tau \eta) \, d^n x \right|_{\tau=0} = \\ &= \left. \int_{\Omega} \partial_i (u + \tau \eta) \partial^i \eta \, d^n x \right|_{\tau=0} = \\ &= \int_{\Omega} (\partial_i u) (\partial^i \eta) \, d^n x \end{aligned}$$

Thus holds  $\Delta u = 0$  in the weak sense.

With regularity theory one can proof, that  $u$  is smooth.

## 6.1 Hölder Spaces

Let  $\Omega \subseteq \mathbb{R}^n$  be open.

### 6.1.1 Definition (Hölder Continuous, Hölder Norm)

A function  $u : \Omega \rightarrow \mathbb{R}^m$  is called *Hölder continuous* with exponent  $\alpha \in \mathbb{R}_{\geq 0}$ , if the *Hölder norm* is finite:

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{C^0(\Omega)} + \sup_{x \neq y \in \Omega} \left( \frac{\|u(x) - u(y)\|}{\|x - y\|^\alpha} \right) < \infty$$

This implies:

$$\|u(x) - u(y)\| \leq \sup_{x \neq y \in \Omega} \left( \frac{\|u(x) - u(y)\|}{\|x - y\|^\alpha} \right) \cdot \|x - y\|^\alpha$$

We often use the short notation:

$$\|\cdot\|_{C^{0,\alpha}(\Omega)} =: \|\cdot\|_{0,\alpha}$$

Obviously holds  $C^{0,\alpha}(\Omega) \subseteq C^0(\Omega)$ .

$C^{0,0}(\Omega)$  are bounded continuous functions.

$C^{0,1}(\Omega)$  are bounded, Lipschitz continuous functions.

This can also be defined locally:  $f \in C_{\text{loc}}^{0,\alpha}(\Omega)$  means that for all  $x \in \Omega$  exists a neighborhood  $U \subseteq \Omega$  such that  $f|_U \in C^{0,\alpha}(U)$ .

A typical example is  $u(x) = \sqrt{|x|}$  on  $[-1, 1]$  with  $f \in C^{0,\frac{1}{2}}(\Omega)$ .

$$C^{k,\alpha}(\Omega) := \left\{ f \in C^k \mid D^\beta f \in C^{0,\alpha} \quad \forall \text{ multi-index } \beta, |\beta| \leq k \right\}$$

$$\|f\|_{k,\alpha} = \sum_{\beta, |\beta| \leq k} \|D^\beta f\|_{0,\alpha}$$

### 6.1.2 Proposition (Hölder Continuous Functions are Complete)

$(C^{k,\alpha}(\Omega), \|\cdot\|_{k,\alpha})$  is a Banach space.

#### Proof

Let  $u_n \in C^{k,\alpha}(\Omega)$  be a Cauchy sequence. Then  $u_n$  is also a Cauchy sequence in  $C^k(\Omega)$ . Since  $C^k(\Omega)$  is complete,  $D^\beta u_n \rightrightarrows D^\beta u$  converges uniformly for all multi-indices  $\beta$  with  $|\beta| \leq k$ .

Next we have by uniform convergence:

$$\frac{D^\beta u_n(x) - D^\beta u_n(y)}{\|x - y\|^\alpha} \xrightarrow{n \rightarrow \infty} \frac{D^\beta u(x) - D^\beta u(y)}{\|x - y\|^\alpha}$$

Therefore the Hölder norms also converge. □<sub>6.1.2</sub>

## 6.2 $L^p$ -Spaces, Approximate Theorems

$$L^p(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} \mid \int_\Omega |f(x)|^p dx < \infty \right\}$$

$$\|f\|_p := \left( \int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

### 6.2.1 Definition and Lemma (Mollifier)

A *mollifier* is a function  $\eta \in C_0^\infty(\mathbb{R}^n)$  with  $\eta(x) \in \mathbb{R}_{\geq 0}$  and  $\int_{\mathbb{R}^n} \eta(x) dx = 1$ . Example:

$$\eta(x) = \begin{cases} ce^{-\frac{1}{\|x\|^2-1}} & \text{if } \|x\| < 1 \\ 0 & \text{otherwise} \end{cases}$$

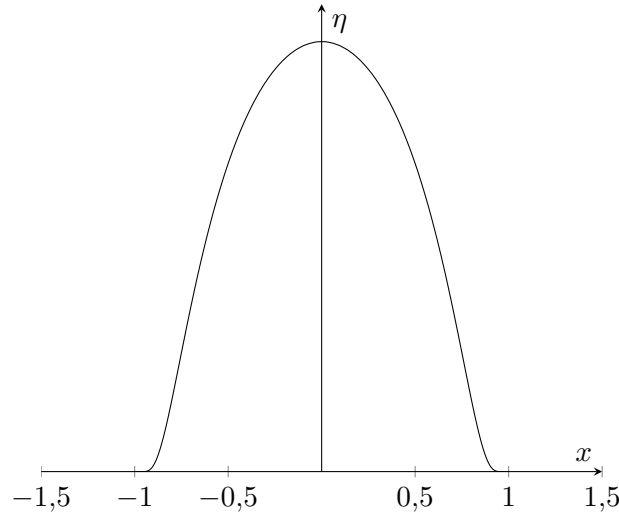


Figure 6.1:  $\eta(x)$  is smooth and has compact support  $[-1, 1]$ .

For  $u \in L^1_{\text{loc}}(\Omega)$  and  $\varepsilon \in \mathbb{R}_{>0}$  define the mollified function as:

$$u_\varepsilon(x) = \int_{\Omega} \underbrace{\frac{1}{\varepsilon^n} \eta\left(\frac{x-y}{\varepsilon}\right)}_{=: \eta_\varepsilon(x-y)} u(y) \, d^n y = (\eta_\varepsilon * u)(x)$$

Now holds  $u_\varepsilon \in C^\infty(\Omega')$  for every  $\Omega' \Subset \Omega$ .

### Proof

The support satisfies  $\text{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(0)$ . Define  $\varepsilon_0 := \text{dist}(\Omega', \partial\Omega) > 0$ . For  $\varepsilon < \varepsilon_0$ , we have for  $x \in \Omega'$ :

$$u_\varepsilon(x) = \int_{\Omega'} \eta_\varepsilon(x-y) u(y) \, d^n y$$

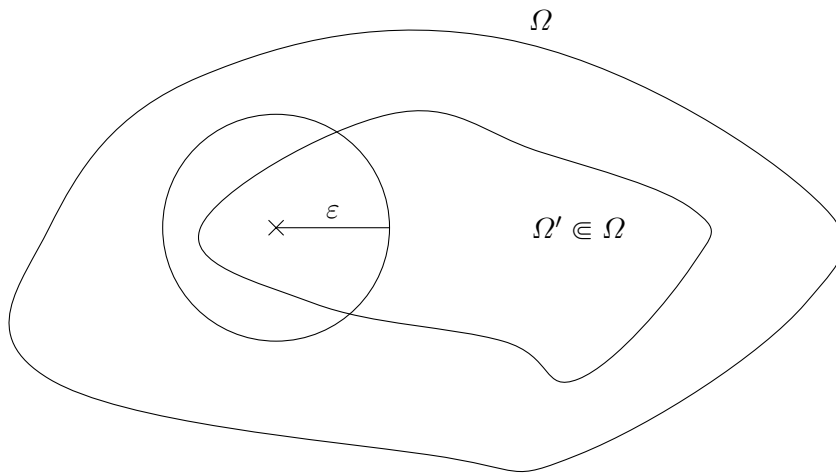


Figure 6.2:  $\text{supp}(\eta_\varepsilon) \subseteq B_\varepsilon(0) \subseteq \Omega$

$$\frac{u_\varepsilon(x + he_i) - u_\varepsilon(x)}{h} = \int_{\Omega'} \underbrace{\frac{1}{h} (\eta_\varepsilon(x + he_i - y) - \eta_\varepsilon(x - y))}_{\xrightarrow{h \rightarrow 0} \partial_i \eta_\varepsilon(x-y)} u(y) d^n y$$

The limit  $h \searrow 0$  can be taken inside the integral by Lebesgue's dominated convergence theorem, since both  $|u|$  and  $\eta_\varepsilon$  are integrable.  $\square_{6.2.1}$

### 6.2.2 Lemma (Mollified Function Converges Locally Uniformly)

For  $u \in C^0(\Omega)$  the functions  $u_\varepsilon \xrightarrow{\text{loc}} u$  converge locally uniformly.

#### Proof

$$u_\delta(x) = \frac{1}{\delta^n} \int_{\Omega} \eta\left(\frac{x-y}{\delta}\right) u(y) d^n y$$

For  $x_0 \in \Omega$  we choose  $\varrho \in \mathbb{R}_{>0}$  such that  $B_{2\varrho}(x_0) \subseteq \Omega$ . Let us show the uniform convergence  $u_\delta \rightrightarrows u$  in  $B_\varrho(x_0)$ . We always assume  $\delta < \varrho$ . Then for all  $x \in B_\varrho(x_0)$  holds:

$$\begin{aligned} u_\delta(x) &= \frac{1}{\delta^n} \int_{\Omega} \eta\left(\frac{x-y}{\delta}\right) u(y) d^n y = \frac{1}{\delta^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{\delta}\right) u(y) d^n y = \\ &\stackrel{z := \frac{x-y}{\delta}}{=} \frac{1}{\delta^n} \int_{\mathbb{R}^n} \eta(z) u(x - \delta z) \delta^n d^n z = \\ &\stackrel{d^n z = \frac{d^n y}{\delta^n}}{=} \int_{\mathbb{R}^n} \eta(z) u(x - \delta z) d^n z \end{aligned}$$

$u$  is uniformly continuous on the compact set  $\overline{B_\delta(x)}$ . For all  $\varepsilon \in \mathbb{R}_{>0}$ , there exists a  $\delta \in \mathbb{R}_{>0}$  such that for all  $y, y' \in \overline{B_\delta(x)}$  holds:

$$\|y - y'\| < \delta \quad \Rightarrow \quad |u(y) - u(y')| < \varepsilon$$

$$\begin{aligned} \Rightarrow \quad (u_\delta - u)(x) &= \int_{\mathbb{R}^n} \eta(z) (u(x - \delta z) - u(x)) d^n z \\ |(u_\delta - u)(x)| &\leq \int_{\mathbb{R}^n} \eta(z) |u(x - \delta z) - u(x)| d^n z \leq \varepsilon \int_{\mathbb{R}^n} \eta = \varepsilon \end{aligned}$$

$\square_{6.2.2}$

### 6.2.3 Lemma (Mollified Function Converges)

For  $u \in L^p_{\text{loc}}(\Omega)$  (or  $u \in L^p(\Omega)$ ) converges  $u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u$  in  $L^p_{\text{loc}}(\Omega)$  (respectively in  $L^p(\Omega)$ ).



**Proof**

For  $\Omega' \Subset \Omega$  holds for small enough  $\varepsilon \in \mathbb{R}_{>0}$ :

$$\text{dist}(\Omega', \partial\Omega) > 2\varepsilon$$

The definition of  $u_\varepsilon(x)$  for  $x \in \Omega'$  is:

$$u_\varepsilon(x) = \int_{\Omega} \frac{1}{\varepsilon^n} \eta\left(\frac{x-y}{\varepsilon}\right) u(y) \, d^n y$$

By a variable transformation  $z := \frac{x-y}{\varepsilon}$  follows due to  $\text{supp}(\eta) = B_1(0)$ :

$$u_\varepsilon(x) = \int_{\|z\| \leq 1} \eta(z) u(y + \varepsilon z) \, d^n z$$

Now define the measure  $d\rho := \eta(z) \, dz$  and apply Hölder's inequality:

$$|u_\varepsilon(x)| = \left| \int_{\|z\| \leq 1} u(x + \varepsilon z) \, d\rho \right| \stackrel{\text{Hölder}}{\leq} \underbrace{\left( \int_{\|z\| \leq 1} 1 \, d\rho \right)^{\frac{1}{q}}}_{=1} \left( \int_{\|z\| \leq 1} |u|^p(x + \varepsilon z) \, d\rho \right)^{\frac{1}{p}}$$

This implies:

$$|u_\varepsilon(x)|^p \leq \int_{\|z\| \leq 1} |u|^p(x + \varepsilon z) \eta(z) \, dz$$

Integrating over  $x$  yields:

$$\begin{aligned} \int_{\Omega'} |u_\varepsilon(x)|^p \, dx &\leq \int_{\Omega'} dx \int_{\|z\| \leq 1} |u|^p(x + \varepsilon z) \eta(z) \, dz = \\ &\stackrel{\text{Fubini}}{=} \int_{\|z\| \leq 1} dz \eta(z) \int_{\Omega'} dx |u|^p(x + \varepsilon z) \leq \\ &\leq \underbrace{\int_{\|z\| \leq 1} dz \eta(z)}_{=1} \int_{\Omega''} dx |u|^p(x) = \int_{\Omega''} dx |u|^p(x) \end{aligned}$$

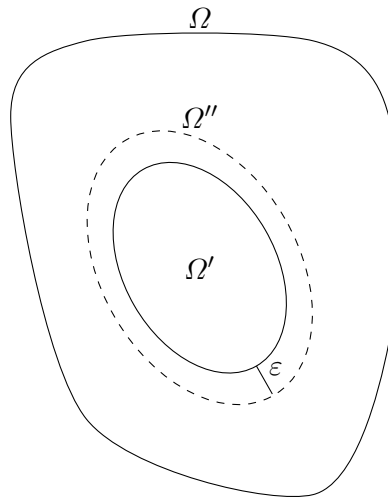


Figure 6.3:  $\Omega'' := \{x + \varepsilon z \mid x \in \Omega', \|z\| \leq 1\}$

This means:

$$\|u_\varepsilon\|_{L^p(\Omega')} \leq \|u\|_{L^p(\Omega'')}$$

We want to apply the Lemma 6.2.2. For  $u \in L^p_{\text{loc}}(\Omega)$  and  $\varepsilon \in \mathbb{R}_{>0}$  choose  $\omega \in C^0(\Omega)$  with:

$$\|u - \omega\|_{L^p(\Omega'')} \leq \varepsilon$$

This is possible after Urysohn's lemma (cf. the book of Rudin). With

$$\|\omega_\varepsilon - u_\varepsilon\|_{L^p(\Omega')} \leq \|\omega - u\|_{L^p(\Omega'')} \leq \varepsilon$$

this gives:

$$\begin{aligned} \|u - u_\varepsilon\|_{L^p(\Omega')} &\leq \|u - \omega\|_{L^p(\Omega'')} + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} + \|\omega_\varepsilon - u_\varepsilon\|_{L^p(\Omega')} \leq \\ &\leq \|u - \omega\|_{L^p(\Omega'')} + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} + \|\omega - u\|_{L^p(\Omega'')} \leq \\ &\leq 2\|u - \omega\|_{L^p(\Omega'')} + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} \leq \\ &\leq 2\varepsilon + \|\omega - \omega_\varepsilon\|_{L^p(\Omega')} \end{aligned}$$

By Lemma 6.2.2 holds, since  $\omega$  is continuous:

$$\|\omega - \omega_\varepsilon\|_{L^p(\Omega')} \xrightarrow{\varepsilon \rightarrow 0} 0$$

Therefore converges  $u_\varepsilon \rightarrow u$  in  $L^p_{\text{loc}}(\Omega)$ .

To get the statement without “loc”, extend  $u$  by 0 to all of  $\mathbb{R}^n$ . If  $\Omega$  is bounded,  $u_\varepsilon \rightarrow u$  converges in  $L^p(\Omega)$  as above, as  $\overline{\Omega}$  is compact, so  $\overline{\Omega} \Subset \mathbb{R}^n$ . If  $\Omega$  is not bounded, we choose a ball  $B_R(0)$  such that  $\|u\|_{L^p(\Omega \setminus B_R(0))} \leq \varepsilon$ . This is possible, since  $u$  is integrable on  $\Omega$ .  $\square_{6.2.3}$

### 6.2.4 Corollary (Dense Subsets of Integrable Functions)

$C^\infty(\Omega)$  and  $C^{k,\alpha}(\Omega)$  are dense in  $L^p_{\text{loc}}(\Omega)$  and  $C_0^\infty(\Omega)$  is dense in  $L^p(\Omega)$  for  $1 \leq p < \infty$ .

## 6.3 Sobolev Spaces

### 6.3.1 Definition (Weak Derivative)

Consider  $u, v \in L^1_{\text{loc}}(\Omega)$  and a multi-index  $\alpha$ . The function  $u$  is the  $\alpha$ -th *weak derivative* of  $v$  if and only if for all test functions  $\eta \in C_0^\infty(\Omega)$  holds:

$$\int_{\Omega} v(x) (-1)^{|\alpha|} D^\alpha \eta(x) dx = \int_{\Omega} u(x) \eta(x) dx$$

This reminds of the definition of the distributional derivative  $T_u = D^\alpha T_v$  by

$$T_u(\eta) = (D^\alpha T_v)(\eta) := (-1)^{|\alpha|} T_v(D^\alpha \eta)$$

for all Schwartz functions  $\eta$ .

Here it is more general, because we have less test functions  $\eta \in C_0^\infty$ , but on the other hand less general due to  $v \in L^1_{\text{loc}}$ , so  $T_v$  is a kind of “regular distribution”.

### 6.3.2 Definition (Sobolev Space)

The *Sobolev space*  $H^{k,p}(\Omega)$  for  $k \in \mathbb{N}_{\geq 0}$  and  $1 \leq p \leq \infty$  consists of  $L^1_{\text{loc}}$  functions  $u : \Omega \rightarrow \mathbb{C}$  such that for all multi-indices  $\alpha$  with  $|\alpha| \leq k$ , the weak derivative  $D^\alpha u$  exists and is in  $L^p(\Omega)$ .

$$\|u\|_{H^{k,p}(\Omega)} := \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^p dx \right)^{\frac{1}{p}}$$

A special case of interest is  $p = 2$ , where we have an associated scalar product.

$$\langle u, v \rangle_{H^{k,2}(\Omega)} := \sum_{|\alpha| \leq k} \int_{\Omega} \overline{D^\alpha u} \cdot D^\alpha v dx$$

### 6.3.3 Theorem (Sobolev Spaces are Complete)

The spaces  $H^{k,p}(\Omega)$  are Banach spaces and Hilbert spaces for  $p = 2$ .

#### Proof

We have to show completeness. Let  $u_n$  be a Cauchy sequence in  $H^{k,p}(\Omega)$ . Completeness of  $L^p(\Omega)$  implies that  $u_n^\alpha := D^\alpha u_n \xrightarrow{n \rightarrow \infty} v^\alpha$  converges in  $L^p(\Omega)$ . Then holds

$$v := \lim_{n \rightarrow \infty} u_n \in H^{k,p}(\Omega)$$

and  $\partial^\alpha v = v^\alpha$ , because for all  $\eta \in C_0^\infty(\Omega)$  holds:

$$\int_{\text{supp}(\eta)} v (-1)^{|\alpha|} D^\alpha \eta dx \leftarrow \int u_n (-1)^{|\alpha|} D^\alpha \eta dx = \int u_n^\alpha \eta dx \rightarrow \int_{\text{supp}(\eta)} v^\alpha \eta dx$$

This is true, because convergence in  $L^p$  implies convergence in  $L^1_{\text{loc}}$ . □<sub>6.3.3</sub>

What is the relation between the weak derivative and mollifying?

### 6.3.4 Lemma (Weak Derivative and Mollifying Commute)

Let  $u \in L^1_{\text{loc}}(\Omega)$  be  $\alpha$ -times *weakly differentiable*, i.e.  $D^\alpha u$  exists for the multi-index  $\alpha$ . Let  $\Omega' \Subset \Omega$  be a compact subset with  $\text{dist}(\Omega', \partial\Omega) > \varepsilon \in \mathbb{R}_{>0}$ . Then holds:

$$(D^\alpha u)_\varepsilon = D^\alpha u_\varepsilon$$

#### Proof

$$\begin{aligned} D^\alpha u_\varepsilon(x) &= \int_{\mathbb{R}^n} D_x^\alpha \eta_\varepsilon(x-y) u(y) dy = \int_{\mathbb{R}^n} (-1)^{|\alpha|} D_y^\alpha \eta_\varepsilon(x-y) u(y) dy = \\ &\stackrel{\text{integration by parts}}{=} \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) D^\alpha u(y) dy = (D^\alpha u)_\varepsilon \end{aligned}$$

□<sub>6.3.4</sub>

### 6.3.5 Theorem (Dense Subset of Sobolev Space)

$C^\infty(\Omega) \cap H^{k,p}(\Omega)$  is dense in  $H^{k,p}(\Omega)$ .

#### Proof

Let  $(\Omega_n)_{n \in \mathbb{N}}$  be an exhaustion of  $\Omega$  by relatively compact sets  $\Omega_n$ , i.e.  $\Omega_n \Subset \Omega_{n+1} \Subset \Omega$  and  $\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$ , with  $\Omega_0 \neq \emptyset$ . Define  $A_n := \Omega_{n+1} \setminus \overline{\Omega_n}$ . The  $A_n$  form a locally finite open covering.

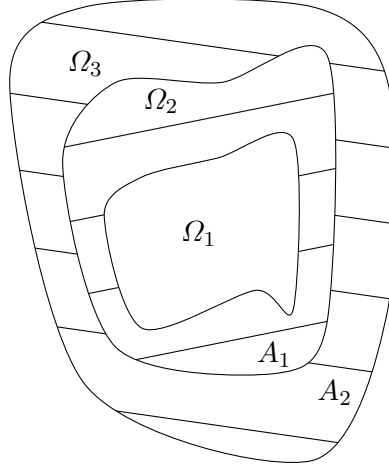


Figure 6.4:  $A_n$  form locally finite open covering

Let  $(\varphi_n)_{n \in \mathbb{N}}$  be a subordinate partition of unity, i.e.  $\varphi_n \in C_0^\infty(A_n)$  and  $\sum_n \varphi_n = 1$ . For  $u \in H^{k,p}(\Omega)$  holds  $\varphi_n u \in H^{k,p}(A_n)$ .

Now  $(\varphi_n u)_{\varepsilon_n} \xrightarrow{\varepsilon_n \rightarrow 0} \varphi_n u$  converges in  $L^p(A_n)$  after Lemma 6.2.3 due to  $\varphi_n u \in L^p(A_n)$ . Since weak derivative and mollifying commute (Lemma 6.3.4), follows in  $L^p(A_n)$ :

$$D^\alpha((\varphi_n u)_{\varepsilon_n}) = (D^\alpha \varphi_n u)_{\varepsilon_n} \xrightarrow{\varepsilon_n \rightarrow 0} D^\alpha \varphi_n u$$

So  $(\varphi_n u)_{\varepsilon_n} \xrightarrow{\varepsilon_n \rightarrow 0} \varphi_n u$  converges in  $H^{k,p}(\Omega)$ . Therefore one can choose  $\varepsilon_n \in \mathbb{R}_{>0}$  such that for every  $\delta \in \mathbb{R}_{>0}$  holds:

$$\|(\varphi_n u)_{\varepsilon_n} - \varphi_n u\|_{H^{k,p}(\Omega)} \leq \frac{\delta}{2^n}$$

Define:

$$\tilde{u} = \sum_{n=1}^{\infty} (\varphi_n u)_{\varepsilon_n} \in C^\infty(\Omega)$$

This  $\tilde{u}$  is smooth, because the sum is locally finite and the  $(\varphi_n u)_{\varepsilon_n}$  are smooth. So we get:

$$\|u - \tilde{u}\|_{H^{k,p}(\Omega)} \leq \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta$$

□<sub>6.3.5</sub>

Warning:  $C_0^\infty(\Omega)$  is in general *not* dense in  $H^{k,p}(\Omega)$ . This motivates the following definition:

**6.3.6 Definition** (Sobolev Spaces for Dirichlet Boundary)

The *Sobolev spaces for Dirichlet boundary* condition are defined as:

$$H_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{H^{k,p}} \subseteq H^{k,p}$$

We have:

$$\begin{aligned} C^\infty(\Omega) \cap H^{k,p}(\Omega) &\stackrel{\text{dense}}{\subseteq} H^{k,p}(\Omega) \subseteq L^p(\Omega) \\ \cup & \quad \cup \quad (\text{in general not dense}) \\ C_0^\infty(\Omega) &\stackrel{\text{dense}}{\subseteq} H_0^{k,p}(\Omega) \end{aligned}$$

Is  $C^\infty(\overline{\Omega}) \cap H^{k,p}(\Omega)$  dense in  $H_0^{k,p}(\Omega)$ ? In general it is not, but for smooth boundary of  $\Omega$ , it is.

**6.3.7 Theorem** (Elementary Properties of Sobolev Functions)

- i) For  $u \in H^{k,p}(\Omega)$  holds  $D^\alpha u \in H^{k-|\alpha|,p}(\Omega)$ .
- ii) For  $\Omega' \subseteq \Omega$  holds  $H^{k,p}(\Omega) \subseteq H^{k,p}(\Omega')$ .
- iii) Leibniz rule: For  $u \in H^{k,p}(\Omega)$ ,  $v \in C_0^\infty$  and  $|\alpha| \leq k$  holds:

$$vu \in H^{k,p}$$

$$D^\alpha(vu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta v D^{\alpha-\beta} u$$

For multi-indices  $\alpha = (i_1, \dots, i_r)$  and  $\beta = (j_1, \dots, j_s)$  with  $i_l, j_l \in \{1, \dots, n\}$  let  $\alpha_m$  be the number indices  $i_l = m$  and analog  $\beta_m$ . The Binomial coefficient is defined as follows:

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \binom{\alpha_2}{\beta_2} \cdot \dots \cdot \binom{\alpha_n}{\beta_n}$$

- iv) Chain rule: For  $f \in C^1$ ,  $f' \in L^\infty$  and  $u$  weakly differentiable holds:

$$D(f \circ u) = f'(u) Du$$

**Proof**

i) and ii) are obvious.

- iii) For brevity consider the first derivatives only.

$$\begin{aligned} \int (vu) (-1) (D\eta) dx &\stackrel{\text{Leibniz}}{\underset{\text{for } C^\infty}{=}} \int u (-1) (D(\eta v) - \eta (Dv)) dx = \\ &\stackrel{\text{integration}}{\underset{\text{by parts}}{=}} \int ((Du) \eta v + u \eta Dv) dx = \int (v Du + u Dv) \eta dx \\ \Rightarrow \quad D(uv) &= (Du) v + u (Dv) \end{aligned}$$

Be careful: A product of two  $H^{k,p}$  functions need not be in  $H^{k,p}$  in general!

- iv) We use an approximation argument: Consider  $\Omega' \Subset \Omega$ ,  $u_n \in C^\infty(\Omega')$  with  $u_n \rightarrow u$  and  $Du_n \rightarrow Du$  converging in  $L^1(\Omega')$ . The chain rule in  $C^1$  gives  $D(f \circ u_n) = f'(u_n) Du_n$ . Now evaluate in the weak sense with a test function  $\eta \in C_0^\infty(\Omega)$ :

$$\int (f \circ u_n) (-D\eta) dx = \int f'(u_n) (Du_n) \eta dx$$

We have to show, that we can take the limit on both sides.

$$\int_{\Omega'} |f(u_n) - f(u)| |D\eta| dx \leq \sup |D\eta| \sup |f'| \int_{\Omega'} (u_n - u) dx \xrightarrow{n \rightarrow \infty} 0$$

The general technique

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n - a| |b_n| + |a| |b_n - b|$$

gives for the other side:

$$\begin{aligned} \left| \int_{\Omega'} f'(u_n) Du_n - f'(u) (Du) \eta dx \right| &\leq \\ &\leq \int_{\Omega'} |f'(u_n) - f'(u)| |Du| |\eta| dx + \sup |f'| \cdot \int_{\Omega'} |Du_n - Du| |\eta| dx \end{aligned}$$

The second term converges to zero, because  $Du_n \rightarrow Du$  converges in  $L^1(\Omega')$ .

The first term converges to zero, because as  $u_n \rightarrow u$  converges almost everywhere,  $f'(u_n) \rightarrow f'(u)$  converges almost everywhere and due to  $f' \in C^0$  we can apply the dominated convergence theorem to take the limit under the integral.  $\square_{6.3.7}$

## 6.4 Traces of Sobolev Functions at the Boundary

Note: These traces have nothing to do with traces in linear algebra.

Consider  $u \in H^{k,p}(\Omega)$ .

Questions: What is  $u|_{\partial\Omega}$ ? In which space is it?

Assume for simplicity that  $\partial\Omega$  is  $C^1$ , i.e.  $\overline{\Omega}$  is a manifold with  $C^1$ -boundary  $\partial\Omega$ .

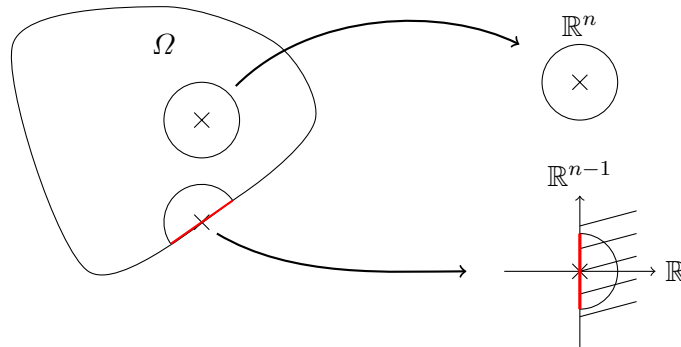


Figure 6.5:  $\Omega$  is a manifold with  $C^1$ -boundary.

Thus for every  $p \in \partial\Omega$  exists a neighborhood  $p \in U \subseteq \overline{\Omega}$  open in the relative topology and a mapping

$$\varphi : U \rightarrow \mathbb{R}^n$$

with  $\varphi(U) \subseteq \mathbb{R}_+^n \setminus \{x \in \mathbb{R}^n \mid x^n \geq 0\}$  and  $\varphi(\partial\Omega \cap U) \subseteq \{x \in \mathbb{R}^n \mid x^n = 0\}$  such that the mapping  $\varphi : U \rightarrow \varphi(U)$  is a  $C^1$ -Diffeomorphism.

Moreover we assume that  $\Omega$  is bounded. Thus  $\partial\Omega$  is compact and therefore we can work with a finite number of charts  $(\varphi_i, U_i)$  with  $i \in \{1, \dots, N\}$  and  $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$ .

### 6.4.1 Theorem and Definition (Trace)

Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded domain with  $C^1$ -boundary. Then there is a unique bounded operator  $T : H^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ , called *trace*, such that for all  $u \in H^{1,p}(\Omega) \cap C^0(\overline{\Omega})$  holds for all  $x \in \partial\Omega$ :

$$(Tu)(x) = u(x)$$

#### Proof

First consider  $u \in C^1(\overline{\Omega})$ . ( $C^1(\overline{\Omega}) \subseteq H^{1,p}(\Omega)$  holds, because  $\Omega$  is bounded.) We show that  $T$  is bounded, i.e.

$$\|Tu\|_{L^p(\partial\Omega)} \leq K \|u\|_{H^{1,p}(\Omega)}$$

with a  $C \in \mathbb{R}_{>0}$  independent of  $u$ . Then  $T$  has a unique continuation to  $\overline{C^1(\overline{\Omega})}^{H^{1,p}} = H^{1,p}(\Omega)$ , because  $C^1$  is dense in  $H^{1,p}$ .

1. Let  $u \in C^1(\overline{\Omega})$ ,  $y \in \partial\Omega$  and assume that  $\partial\Omega$  is flat in a neighborhood  $U$  of  $y$ , i.e. a hyperplane. By a rotation in  $\mathbb{R}^n$ , we can arrange that  $U \cap \Omega = U \cap \mathbb{R}_+^n$ . Set  $\tilde{x} = (x^1, \dots, x^{n-1})$ .

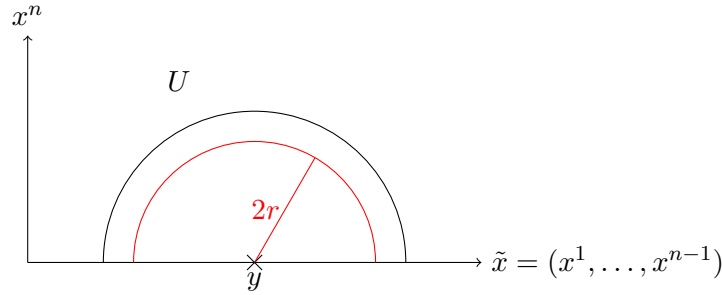


Figure 6.6:  $U \cap \Omega = U \cap \mathbb{R}_+^n$

Choose  $r \in \mathbb{R}_{>0}$  with  $B_{2r}(y) \cap \mathbb{R}_+^n \subseteq U$  and  $\eta \in C_0^\infty(B_{2r}(y))$  with  $\eta|_{B_r(y)} = 1$  and  $\eta \geq 0$ . Define  $\Gamma := B_r(y) \cap \partial\Omega$ .

$$\begin{aligned} \int_{\Gamma} |u(\tilde{x}, 0)|^p d^{n-1}\tilde{x} &\leq \int_{\mathbb{R}^{n-1}} \underbrace{\eta(\tilde{x}, 0) |u(\tilde{x}, 0)|^p}_{\geq 0} d^{n-1}\tilde{x} = \\ &= \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty -\frac{\partial}{\partial x^n} (\eta(x) |u(x)|^p) dx^n \right) d^{n-1}\tilde{x} = \\ &\stackrel{\text{Fubini}}{=} - \int_{\mathbb{R}_+^n} \frac{\partial}{\partial x^n} (\eta(x) |u(x)|^p) d^n x = \\ &= \int_{\mathbb{R}_+^n} -\frac{\partial \eta}{\partial x^n} |u(x)|^p - \text{sgn}(u(x)) \eta p u^{p-1} \left( \frac{\partial}{\partial x^n} u \right) d^n x \end{aligned}$$

The derivative of  $|u(x)|^p$  does not exist for  $p = 1$  and  $u = 0$ , but any point with  $u = 0$  does not contribute to the original integral. We estimate:

$$\int_{\Gamma} |u(\tilde{x}, 0)|^p d^{n-1}\tilde{x} \leq \left| \frac{\partial \eta}{\partial x^n} \right|_{C^0} \int_{\mathbb{R}_+^n \cap \Omega} |u|^p d^n x + |\eta|_{C^0} p \cdot \int_{\mathbb{R}_+^n \cap \Omega} |u|^{p-1} |\partial_n u| d^n x$$

Now we use Young's inequality for  $1 \leq p \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ :

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

This yields for  $q = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$ :

$$|\partial_n u| |u|^{p-1} \leq \frac{|\partial_n u|^p}{p} + \frac{|u|^{q(p-1)}}{q} = \frac{|\partial_n u|^p}{p} + \frac{|u|^p}{q}$$

So we get:

$$\begin{aligned} \int_{\Gamma} |u(\tilde{x}, 0)|^p d^{n-1}\tilde{x} &\leq \left| \frac{\partial \eta}{\partial x^n} \right|_{C^0} \int_{\mathbb{R}_+^n \cap \Omega} |u|^p d^n x + |\eta|_{C^0} p \cdot \int_{\mathbb{R}_+^n \cap \Omega} \left( \frac{|\partial_n u|^p}{p} + \frac{|u|^p}{q} \right) d^n x \leq \\ &\leq \left| \frac{\partial \eta}{\partial x^n} \right|_{C^0} \int_{\mathbb{R}_+^n \cap \Omega} (|u|^p + |\partial_n u|^p) d^n x + \\ &\quad + |\eta|_{C^0} \left( 1 + \frac{p}{q} \right) \int_{\mathbb{R}_+^n \cap \Omega} (|\partial_n u|^p + |u|^p) d^n x = \\ &= \left( \left| \frac{\partial \eta}{\partial x^n} \right|_{C^0} + |\eta|_{C^0} \left( 1 + \frac{p}{q} \right) \right) \int_{\mathbb{R}_+^n \cap \Omega} (|u|^p + |\partial_n u|^p) d^n x \leq \\ &\leq C \|u\|_{H^{1,p}(\Omega)}^p \end{aligned}$$

All in all follows:

$$\|Tu\|_{L^p(\Gamma)} \leq C^{\frac{1}{p}} \|u\|_{H^{1,p}(\Omega)}$$

2. Now consider a general  $\Omega$ . We cover the boundary by a finite number  $N \in \mathbb{N}$  of balls with charts  $(\varphi_i, U_i)$ . Let  $\eta_i(\tilde{x})$  be a subordinate partition of unity to get:

$$\begin{aligned} \int_{\partial \Omega} |u|^p d\mu_{\partial \Omega} &= \sum_{i=1}^N \int_{\varphi_i(U_i)} \underbrace{\eta_i(\tilde{x})}_{\leq C(\Omega)} \underbrace{|(u \circ \varphi_i^{-1})(\tilde{x})|^p}_{=|u(\tilde{y})|^p} \underbrace{\left| \det \left( \frac{\partial \varphi}{\partial \tilde{x}} \right) \right|}_{=d^{n-1}\tilde{y}} d^{n-1}\tilde{x} \leq \\ &\leq C(\Omega) \sum_{i=1}^N \int_{\Gamma_i} |u(\tilde{y})|^p d^{n-1}\tilde{y} \end{aligned}$$

With the first step follows:

$$\begin{aligned} \int_{\partial \Omega} |u|^p d\mu_{\partial \Omega} &\leq C(\Omega) \sum_{i=1}^N \int_{\Gamma_i} (|u|^p + |\partial_n u|^p) d^{n-1}\tilde{x} \leq \\ &\leq \tilde{C}(\Omega) \int_{\Omega} (|u|^p + |\partial_n u|^p) d^n x \end{aligned}$$



This gives for all  $u \in C^1(\overline{\Omega})$ :

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \leq \underbrace{\left(\tilde{C}(\Omega)\right)^{\frac{1}{p}}}_{=:K} \|u\|_{H^{1,p}(\Omega)}$$

3. Now we use an approximation argument for  $u \in H^{1,p}(\Omega)$ . Since  $C^1(\overline{\Omega})$  is dense in  $H^{1,p}(\Omega)$ , we can take a sequence  $u_n \in C^1(\overline{\Omega})$  with  $u_n \rightarrow u$  converging in  $H^{1,p}(\Omega)$ . Thus holds:

$$\|T(u_n - u_m)\|_{L^p(\partial\Omega)} = \|u_n|_{\partial\Omega} - u_m|_{\partial\Omega}\|_{L^p(\partial\Omega)} \leq K \|u_n - u_m\|_{H^{1,p}(\Omega)} \xrightarrow{n,m \rightarrow \infty} 0$$

Thus  $Tu_n$  converges in  $L^p(\partial\Omega)$ . So we define:

$$Tu := \lim_{n \rightarrow \infty} Tu_n$$

This is determined uniquely by the demand that  $T$  be continuous. Then the inequality also holds in the limit:

$$\|Tu\|_{L^p(\partial\Omega)} \leftarrow \|Tu_n\|_{L^p(\partial\Omega)} \leq K \|u_n\|_{H^{1,p}(\Omega)} \rightarrow K \|u\|_{H^{1,p}(\Omega)}$$

□<sub>6.4.1</sub>

### 6.4.2 Theorem

Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded domain,  $1 \leq p < \infty$ . Then for every  $u \in H^{1,p}(\Omega)$  holds  $u \in H_0^{1,p}(\Omega)$  if and only if  $T(u) = 0$ .

#### Proof

“ $\Rightarrow$ ”: For  $u \in H_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{H^{1,p}(\Omega)}$  exists a sequence  $u_n \in C_0^\infty(\Omega)$  with  $u_n \rightarrow u$  in  $H^{1,p}$  and  $Tu_n = 0$ . So we get:

$$0 = Tu_n \rightarrow Tu$$

“ $\Leftarrow$ ”: The proof is quite technical and can be found in the book of Evans.

□<sub>6.4.2</sub>

## 6.5 The Gagliardo-Nirenberg Inequalities

Question: Consider  $u \in C_0^1(\mathbb{R}^n)$ . For which  $p$  and  $q$  does the inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq c \|Du\|_{L^p(\mathbb{R}^n)} \quad \forall_{u \in C_0^1(\mathbb{R}^n)}$$

hold? This inequality does not hold for  $u \in C^1(\mathbb{R}^n)$ , because  $u$  can be an arbitrarily large constant with vanishing derivative.

We consider the scaling behavior, i.e. the behavior under the substitution  $x \rightsquigarrow \lambda x$  with  $\lambda \in \mathbb{R}_{>0}$ .

$$u_\lambda(x) := u(\lambda x) \quad Du_\lambda(x) = \lambda Du|_{\lambda x}$$

$$\begin{aligned}\|Du_\lambda\|_{L^p(\mathbb{R}^n)} &= \left( \int_{\mathbb{R}^n} \|Du_\lambda\|^p dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^n} \|\lambda Du|_{\lambda x}\|^p dx \right)^{\frac{1}{p}} = \lambda \left( \int_{\mathbb{R}^n} \|Du|_{\lambda x}\|^p dx \right)^{\frac{1}{p}} = \\ &\stackrel{y:=\lambda x}{=} \lambda \left( \int_{\mathbb{R}^n} \|Du(y)\|^p \frac{d^n y}{\lambda^n} \right)^{\frac{1}{p}} = \lambda^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left( \int |u(\lambda x)|^q dx \right)^{\frac{1}{q}} \stackrel{y=\lambda x}{=} \left( \int |u(x)|^q \frac{d^n y}{\lambda^n} \right)^{\frac{1}{q}} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$$

The inequality can only be true, if both scalings are the same:

$$\begin{aligned}1 - \frac{n}{p} &\stackrel{!}{=} -\frac{n}{q} \\ \frac{1}{q} &= \frac{1}{p} - \frac{1}{n} \\ q &= \frac{np}{n-p}\end{aligned}$$

$q$  is called the *Sobolev conjugate*, usually denoted by  $p^*$ .

### 6.5.1 Theorem (Gagliardo-Nirenberg-Sobolev Inequality)

For  $1 \leq p < n$  there is a constant  $c \in \mathbb{R}_{>0}$  such that holds:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_0^1(\mathbb{R}^n)$$

One can choose:

$$c = \frac{p(n-1)}{n-p}$$

### 6.5.2 Lemma (Generalized Hölder Inequalities)

Consider  $1 \leq p_1, \dots, p_m < \infty$  with:

$$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} = 1$$

Then for any  $u_i \in L^{p_i}(\Omega)$  holds:

$$\int_{\Omega} |u_1 \cdots u_m| dx \leq \|u_1\|_{L^{p_1}(\Omega)} \cdots \|u_m\|_{L^{p_m}(\Omega)}$$

#### Proof

We proceed inductively:  $\frac{1}{p_1} + \frac{1}{q_1} = 1$

$$\begin{aligned}\int |u_1| \cdot |u_2 \cdots u_m| dx &\leq \|u_1\|_{p_1} \cdot \left( \int |u_2 \cdots u_m|^{q_1} dx \right)^{\frac{1}{q_1}} \leq \\ &\leq \|u_1\|_{p_1} \cdot \left( \int |u_2|^{q_1} |u_3 \cdots u_m|^{q_1} dx \right)^{\frac{1}{q_1}}\end{aligned}$$

Applying Hölder's inequality with exponents  $r, s$  with  $\frac{1}{r} + \frac{1}{s} = 1$  such that  $rq_1 = p_2$  gives:

$$\begin{aligned} \int |u_2|^{q_1} |u_3 \cdots u_m|^{q_1} dx &\leq \underbrace{\|u_2^{q_1}\|_r}_{=\|u_2\|_{rq_1}^{q_1}} \cdot \left( \int |u_3|^{q_1 s} |u_4 \cdots u_m|^{q_1 s} dx \right)^{\frac{1}{s}} \\ \int |u_1| \cdot |u_2 \cdots u_m| dx &\leq \|u\|_{p_1} \cdot \|u_2\|_{p_2} \left( \int |u_3|^{q_1 s} |u_4 \cdots u_m|^{q_1 s} dx \right)^{\frac{1}{q_1 s}} \end{aligned}$$

Iteratively this gives the the generalized Hölder Inequality. □<sub>6.5.2</sub>

### Proof of Theorem 6.5.1

We need to find a constant  $c \in \mathbb{R}_{>0}$  such that for  $u \in C_0^1(\mathbb{R}^n)$  holds:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du\|_{L^p(\mathbb{R}^n)}$$

1. Consider the case  $p = 1$ , i.e.  $p^* = \frac{n-1}{n-2}$ . We have to show:

$$\|u\|_{L^{\frac{n}{n-1}}} \leq \|Du\|_{L^1}$$

For any  $i \in \{1, \dots, n\}$  the fundamental theorem of calculus yields:

$$\begin{aligned} u(x) &= \int_{-\infty}^{x_i} \frac{\partial}{\partial x_i} u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) dy_i \\ |u(x)| &\leq \int_{-\infty}^{\infty} \|Du(x_1, \dots, y_i, \dots, x_n)\| dy_i \\ |u(x)|^{\frac{n}{n-1}} &\leq \prod_{i=1}^n \left( \int_{-\infty}^{\infty} \|Du(x_1, \dots, y_i, \dots, x_n)\| dy_i \right)^{\frac{1}{n-1}} \end{aligned}$$

We proceed inductively. First we integrate over  $x_1$ :

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \int_{-\infty}^{\infty} \prod_{i=1}^n \left( \int_{-\infty}^{\infty} \|Du(x_1, \dots, y_i, \dots, x_n)\| dy_i \right)^{\frac{1}{n-1}} dx_1 = \\ &= \underbrace{\left( \int_{-\infty}^{\infty} \|Du(y_1, x_2, \dots, x_n)\| dy_1 \right)^{\frac{1}{n-1}}}_{\text{independent of } x_1} \cdot \int_{-\infty}^{\infty} \prod_{i=2}^n \left( \int_{-\infty}^{\infty} \|Du(x_1, \dots, y_i, \dots, x_n)\| dy_i \right)^{\frac{1}{n-1}} dx_1 \end{aligned}$$

The generalized Hölder's inequality for the  $n-1$  exponents

$$\begin{aligned} p_2 = \dots = p_n &= n-1 \\ \Rightarrow \frac{1}{p_2} + \dots + \frac{1}{p_n} &= 1 \end{aligned}$$

gives:

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \left( \int_{-\infty}^{\infty} \|Du(y_1, x_2, \dots, x_n)\| dy_1 \right)^{\frac{1}{n-1}}.$$

$$\cdot \prod_{i=2}^n \left( \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_i \|Du(x_1, \dots, y_i, \dots, x_n)\| \right)^{\frac{1}{n-1}}$$

Next, we integrate over  $x_2$  (and don't write the boundary of the integral any more):

$$\begin{aligned} \int |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 &\leq \underbrace{\left( \int dx_1 \int dy_i \|Du(x_1, y_2, x_3, \dots, x_n)\| \right)^{\frac{1}{n-1}}}_{\text{independent of } x_2} \\ &\cdot \int dx_2 \left( \int dy_i \|Du(y_1, x_2, \dots, x_n)\| \right)^{\frac{1}{n-1}} \\ &\cdot \prod_{i=3}^n \left( \int dx_1 \int dy_i \|Du(x_1, \dots, y_i, \dots, x_n)\| \right) \end{aligned}$$

We again have  $n-1$  factors, so we can apply the generalized Hölder inequality with exponents  $n-1$  to get:

$$\begin{aligned} \int |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 &\leq \left( \int dx_1 \int dy_i \|Du(x_1, y_2, x_3, \dots, x_n)\| \right)^{\frac{1}{n-1}} \\ &\cdot \left( \int dx_2 \int dy_i \|Du(y_1, x_2, \dots, x_n)\| \right)^{\frac{1}{n-1}} \\ &\cdot \prod_{i=3}^n \left( \int dx_2 \int dx_1 \int dy_i \|Du(x_1, \dots, y_i, \dots, x_n)\| \right)^{\frac{1}{n-1}} \end{aligned}$$

After also integrating over  $x_3, \dots, x_n$  and then renaming  $y_i \rightarrow x_i$  we get:

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} d^n x \leq \prod_{i=1}^n \left( \int_{\mathbb{R}^n} d^n x \|Du(x)\| \right)^{\frac{1}{n-1}} = \left( \int_{\mathbb{R}^n} d^n x \|Du(x)\| \right)^{\frac{n}{n-1}}$$

$$\|u\|_{L^{\frac{n}{n-1}}} = \left( \int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} d^n x \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} d^n x \|Du(x)\| = \|Du\|_{L^1}$$

2. The general case  $1 < p < n$  follows from the first case by using Hölder's inequality:  
We have to prove for  $p^* = \frac{np}{n-p}$ :

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u(x)|^{p^*} dx \right)^{\frac{1}{p^*}} \leq c \left( \int_{\mathbb{R}^n} |Du(x)|^p dx \right)^{\frac{1}{p}} = c \|Du\|_{L^p(\mathbb{R}^n)}$$

Now introduce  $v := |u|^c$  such that  $|u|^{p^*} = v^{\frac{n}{n-1}}$ :

$$\begin{aligned} c \frac{n}{n-1} &= p^* = \frac{np}{n-p} \\ \Rightarrow c &= \frac{(n-1)p}{n-p} \end{aligned}$$

Now follows:

$$\left( \int |u|^{p^*} d^n x \right)^{\frac{n-1}{n}} = \left( \int v^{\frac{n}{n-1}} d^n x \right)^{\frac{n-1}{n}} = \|v\|_{L^{\frac{n}{n-1}}} \leq$$

$$\stackrel{1.}{\leq} \int |\mathrm{D}v| \, \mathrm{d}^n x \leq c \int |u|^{c-1} |\mathrm{D}u| \, \mathrm{d}^n x$$

Applying Hölder's inequality with  $\tilde{q}$  and  $\tilde{p}$  gives:

$$\left( \int |u|^{p^*} \, \mathrm{d}^n x \right)^{\frac{n-1}{n}} \leq c \left( \int |u|^{\tilde{q}(c-1)} \, \mathrm{d}^n x \right)^{\frac{1}{\tilde{q}}} \left( \int |\mathrm{D}u|^{\tilde{p}} \, \mathrm{d}^n x \right)^{\frac{1}{\tilde{p}}}$$

We choose  $\tilde{q}$  such that  $\tilde{q}(c-1) = p^*$  holds.

$$\begin{aligned} \tilde{q} &= \frac{p^*}{c-1} = \frac{\frac{np}{n-p}}{\frac{(n-1)p}{n-p} - 1} = \frac{np}{(n-1)p - (n-p)} = \frac{np}{np - n} = \frac{p}{p-1} \\ \tilde{p} &= \frac{1}{1 - \frac{1}{\tilde{q}}} = \frac{1}{1 - \frac{p-1}{p}} = \frac{p}{p - (p-1)} = p \end{aligned}$$

We get:

$$\begin{aligned} \left( \int |u|^{p^*} \, \mathrm{d}^n x \right)^{\frac{n-1}{n}} &\leq c \left( \int |u|^{p^*} \, \mathrm{d}^n x \right)^{\frac{p-1}{p}} \left( \int |\mathrm{D}u|^p \, \mathrm{d}^n x \right)^{\frac{1}{p}} \\ \left( \int |u|^{p^*} \, \mathrm{d}^n x \right)^{\frac{n-1}{n} - \frac{p-1}{p}} &\leq c \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{p(n-1) - n(p-1)}{np} = \frac{np - p - np + n}{np} = \frac{n-p}{np} = \frac{1}{p^*}$$

Finally follows:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} = \left( \int |u|^{p^*} \, \mathrm{d}^n x \right)^{\frac{1}{p^*}} \leq c \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$$

□<sub>6.5.1</sub>

### 6.5.3 Theorem

Let  $\Omega \subseteq \mathbb{R}^n$  be an open, bounded domain with  $C^1$ -boundary. Then there is a constant  $C \in \mathbb{R}_{>0}$  such that for all  $u \in H^{1,p}(\Omega)$  holds:

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{H^{1,p}(\Omega)}$$

Recall:

$$\|u\|_{H^{1,p}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^p + \|\mathrm{D}u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

#### Proof

Consider  $u \in H^{1,p}(\Omega)$  and choose  $\Omega'$  with  $\Omega \Subset \Omega'$ . Then  $u$  can be extended to a function  $\bar{u} \in H_0^{1,p}(\Omega')$  with:

$$\|\bar{u}\|_{H^{1,p}(\Omega')} \leq \tilde{c} \|u\|_{H^{1,p}(\Omega)}$$

Namely choose  $u_n \in C^1(\Omega)$  with  $u_n \rightarrow u$  in  $H^{1,p}(\Omega)$  and extend  $u_n$  to  $C_0^1(\Omega')$  to obtain functions  $\bar{u}_n$  with:

$$\|\bar{u}_n\|_{H^{1,p}(\Omega')} \leq \tilde{c} \|u_n\|_{H^{1,p}(\Omega)}$$

A subsequence of the  $\bar{u}_n$  converges to  $\bar{u}$  in  $H_0^{1,p}(\Omega')$ . The constant  $\tilde{c}$  can be chosen independently of  $u$ , because one can construct a linear mapping

$$A : C^1(\Omega) \rightarrow C_0^1(\Omega')$$

with:

$$\|Au\|_{H^{1,p}(\Omega')} \leq c \|u\|_{H^{1,p}(\Omega)} \quad \forall u \in C^1(\Omega)$$

Extending  $\bar{u}$  by zero to  $\mathbb{R}^n$  we get:

$$\|\bar{u}\|_{H^{1,p}(\mathbb{R}^n)} \leq \tilde{c} \|u\|_{H^{1,p}(\Omega)}$$

Now follows:

$$\begin{aligned} \|u\|_{L^{p^*}(\Omega)} &\leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \stackrel{6.5.1}{\leq} c \|\mathrm{D}\bar{u}\|_{L^p(\mathbb{R}^n)} \leq \\ &\leq c \|\bar{u}\|_{H^{1,p}(\mathbb{R}^n)} \leq \underbrace{c\tilde{c}}_{=:C} \|u\|_{H^{1,p}(\Omega)} \end{aligned}$$

□<sub>6.5.3</sub>

#### 6.5.4 Theorem (Poincaré's inequality)

Let  $\Omega$  be a bounded open domain,  $1 \leq p < \infty$  and  $1 \leq q \leq p^*$ . Then there is a  $C(\Omega) \in \mathbb{R}$  such that for all  $u \in H_0^{1,p}(\Omega)$  holds:

$$\|u\|_{L^q(\Omega)} \leq C(\Omega) \|Du\|_{L^p(\Omega)}$$

#### Proof

For  $u \in H_0^{1,p}(\Omega)$  there is by definition a sequence  $u_k \in C_0^1(\Omega)$  with  $u_k \rightarrow u$  converging in  $H_0^{1,p}(\Omega)$ . We extend  $u_n$  by zero to  $\mathbb{R}^n$  and apply the Gagliardo-Nirenberg inequality:

$$\|u_k\|_{L^{p^*}(\Omega)} = \|u_k\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du_k\|_{L^p(\mathbb{R}^n)} = c \|Du_k\|_{L^p(\Omega)} \xrightarrow{k \rightarrow \infty} c \|Du\|_{L^p(\Omega)}$$

For the left sides holds:

$$\|u_k - u_l\|_{L^{p^*}(\Omega)} \leq c \|\mathrm{D}(u_k - u_l)\|_{L^p(\mathbb{R}^n)} \leq c \|u_k - u_l\|_{H^{1,p}(\Omega)} \xrightarrow{k,l \rightarrow \infty} 0$$

Therefore  $u_k \rightarrow u$  converges in  $L^{p^*}$ . In the limit we get:

$$\|u\|_{L^{p^*}(\Omega)} \leq c \|Du\|_{L^p(\Omega)}$$

Now we have:

$$\|u\|_{L^q(\Omega)} = \left( \int 1 \cdot |u|^q \, \mathrm{d}^n x \right)^{\frac{1}{q}}$$

Finally, we choose Hölder exponents  $r, s$  with

$$\begin{aligned} sq &= p^* \\ s &= \frac{p^*}{q} \geq 1 \end{aligned}$$

to get:

$$\|u\|_{L^q(\Omega)} \leq (\mu(\Omega))^{\frac{1}{r}} \|u\|_{L^{p^*}(\Omega)} \leq \underbrace{c(\mu(\Omega))^{\frac{1}{r}}}_{=:C(\Omega)} \|Du\|_{L^p(\Omega)}$$

□<sub>6.5.4</sub>

### 6.5.5 Corollary

Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded domain and  $1 \leq p < n$ . Then  $\|Du\|_{L^p(\Omega)}$  is a norm on  $H_0^{1,p}(\Omega)$  which is equivalent to the  $H^{1,p}$ -norm.

#### Proof

The  $H^{1,p}$ -norm

$$\|u\|_{H^{1,p}} = (\|u\|_{L^p}^p + \|Du\|_{L^p}^p)^{\frac{1}{p}}$$

is equivalent to:

$$\|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}$$

It holds:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\begin{aligned} \|Du\|_{L^p(\Omega)} &\leq \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)} \stackrel{p < p^*}{\leq} \tilde{c} \underbrace{\|u\|_{L^{p^*}(\Omega)}}_{\leq c \|Du\|_{L^p(\Omega)}} + \|Du\|_{L^p(\Omega)} \leq (1 + \tilde{c}c) \|Du\|_{L^p(\Omega)} \end{aligned}$$

□<sub>6.5.5</sub>

## 6.6 The Morrey Inequalities

This traces back to Charles Morrey (1907-1984).

### 6.6.1 Theorem (Morrey Inequalities)

For  $n < p \leq \infty$  exists a constant  $C \in \mathbb{R}_{>0}$  (depending only on  $n$  and  $p$ ), such that for  $u \in C^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$  and  $\gamma := 1 - \frac{n}{p}$  holds:

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{H^{1,p}(\mathbb{R}^n)}$$

The exponent  $\gamma$  can be found with a scaling argument.

**Proof**

We define:

$$\oint_U f(y) \, d^n y := \frac{1}{\mu(U)} \int_U f(y) \, d^n y$$

1. **Claim:**

$$\oint_{B_r(x)} |u(x) - u(y)| \, d^n y \leq C_1(n) \cdot \int \frac{\|Du(y)\|}{\|x - y\|^{n-1}} d^n y$$

**Proof:** Choose  $B_r(x)$ ,  $s \in (0, r)$  and a unit vector  $w \in \partial B_1(0)$ .

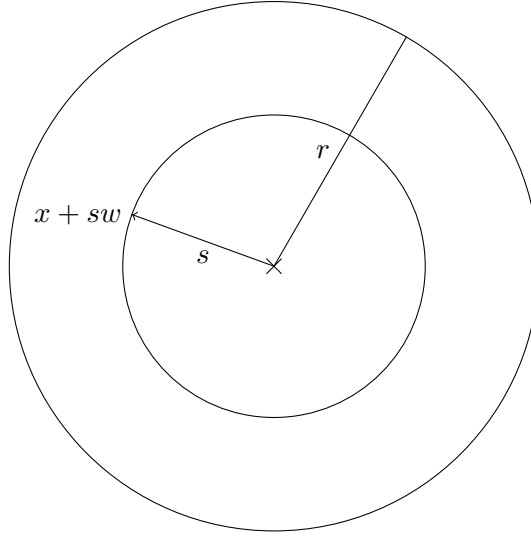


Figure 6.7:  $x + sw \in B_r(x)$

$$|u(x + sw) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x + tw) \, dt \right| \leq \int_0^s \|Du(x + tw)\| \cdot \underbrace{\|w\|}_{=1} \, dt$$

Integration gives:

$$\begin{aligned} \int_{\partial B_1(0)} |u(x + sw) - u(x)| \, d\mu_{\partial B_1(0)}(w) &\leq \\ &\leq \int_0^s dt \int_{\partial B_1(0)} \|Du(x + tw)\| \, d\mu_{\partial B_1(0)}(w) = \\ &\stackrel{z:=x+tw}{=} \int_0^s \frac{dt}{t^{n-1}} \int_{\partial B_t(x)} \|Du(z)\| \, d\mu_{\partial B_t(x)}(z) = \\ &\stackrel{t=\|x-z\|}{=} \int_0^s dt \int_{\partial B_t(x)} \frac{\|Du(z)\|}{\|x - z\|^{n-1}} d\mu_{\partial B_t(x)}(z) = \\ &\stackrel{\text{Fubini}}{=} \int_{B_s(x)} \frac{\|Du(y)\|}{\|x - y\|^{n-1}} d\mu_{B_s(x)}(y) \leq \int_{B_r(x)} \frac{\|Du(y)\|}{\|x - y\|^{n-1}} d^n y \end{aligned}$$



Multiplying by  $s^{n-1}$  and integrating over  $s$  gives:

$$\begin{aligned} \int_0^r ds \cdot s^{n-1} \int_{\partial B_1(0)} |u(x+sw) - u(x)| d\mu_{\partial B_1(0)}(w) &\leq \int_0^r ds \cdot s^{n-1} \int_{B_r(x)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} d^n y \\ &\stackrel{\text{Fubini}}{\Rightarrow} \int_{\partial B_r(x)} |u(y) - u(x)| d\mu_{B_r(x)}(y) \leq \frac{r^n}{n} \int_{B_r(x)} \frac{\|Du(y)\|}{\|x-y\|^{n-1}} d^n y \end{aligned}$$

Dividing by  $r^n$  gives the claim.  $\square_1$ .

2. Now estimate  $u$  point-wise:

$$\begin{aligned} |u(x)| &= |u(x)| \underbrace{\int_{B_1(x)} d^n y}_{=1} \leq \int_{B_1(x)} |u(x) - u(y)| d^n y + \int_{B_1(x)} |u(y)| d^n y \leq \\ &\stackrel{1.}{\leq} C_1 \cdot \int \frac{\|Du(y)\|}{\|x-y\|^{n-1}} d^n y + \frac{1}{\omega_n} \int_{B_1(x)} |u(y)| d^n y \leq \\ &\stackrel{\text{H\"older}}{\leq}_{p^{-1}+q^{-1}=1} C_1 \|Du\|_{L^q(B_1(x))} \left( \int_{B_1(x)} \left( \frac{1}{\|x-y\|^{n-1}} \right)^q d^n y \right)^{\frac{1}{q}} + \\ &\quad + \|u\|_{L^p(B_1(x))} \underbrace{\frac{1}{\omega_n} \left( \int_{B_1(x)} 1^q d^n y \right)^{\frac{1}{q}}}_{=:C_2} \leq \\ &\leq C_1 \|Du\|_{L^q(\mathbb{R}^n)} \left( \int_{B_1(x)} \left( \frac{1}{\|x-y\|^{n-1}} \right)^q d^n y \right)^{\frac{1}{q}} + C_2 \|u\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

Is the pole integrable? The order of the pole is:

$$\begin{aligned} (n-1)q &= (n-1) \frac{p}{p-1} \stackrel{?}{<} n \\ (n-1)p &\stackrel{?}{<} n(p-1) \\ -p &\stackrel{?}{<} -n \\ n &< p \end{aligned}$$

This holds by assumption, thus the pole is integrable and we get:

$$\left| \int_{B_1(x)} \frac{1}{\|x-y\|^{(n-1)q}} \right| \leq C_3 < \infty$$

Thus holds:

$$|u(x)| \leq C_1 C_3 \|Du\|_{L^q(B_1(x))} + C_2 \|u\|_{L^p(\mathbb{R}^n)} \leq C_4 \|u\|_{H^{1,p}(\mathbb{R}^n)}$$

3. Now estimate  $|u(x) - u(y)|$ . For  $x, y \in \mathbb{R}^n$  define  $r := \|x - y\|$  and  $W := B_r(x) \cap B_r(y)$ .

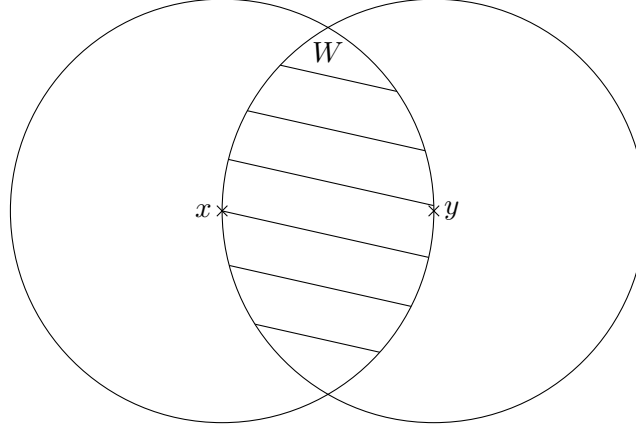


Figure 6.8: Intersection of spheres

Thus we get:

$$\begin{aligned}
 |u(x) - u(y)| &= \int_W |u(x) - u(y)| \, d^n z \leq \int_W |u(x) - u(z)| \, d^n z + \int_W |u(z) - u(y)| \, d^n z \\
 \int_W |u(x) - u(z)| \, d^n z &\leq \frac{1}{\mu(W)} \int_{B_r(x)} |u(x) - u(z)| \, d^n z = C_5(n) \int_{B_r(x)} |u(x) - u(z)| \, d^n z
 \end{aligned} \tag{6.1}$$

The constant

$$C_5(n) = \frac{\mu(B_r)}{\mu(W)}$$

is independent of  $r$ , due to the proportionalities  $\mu(B_r) \sim r^n \sim \mu(W)$ . With 1. follows:

$$\begin{aligned}
 \int_W |u(x) - u(z)| \, d^n z &\leq C_5(n) \int_{B_r(x)} |u(x) - u(z)| \, d^n z \leq \\
 &\leq C_5(n) C_1(n) \cdot \int_{B_r(x)} \frac{\|Du(z)\|}{\|x - z\|^{n-1}} \, d^n z \leq \\
 &\stackrel{\text{Hölder}}{\leq} C_6(n) \|Du\|_{L^p(B_r(x))} \left( \int_{B_r(x)} \frac{1}{\|x - z\|^{(n-1)\frac{p}{p-1}}} \, d^n z \right)^{\frac{p-1}{p}} \\
 \int_{B_r(x)} \frac{1}{\|x - z\|^{(n-1)\frac{p}{p-1}}} \, d^n z &= \int_0^r ds \, (n\omega_n) s^{n-1} \frac{1}{s^{(n-1)\frac{p}{p-1}}} = C_7(n) r \cdot r^{n-1} \frac{1}{r^{(n-1)\frac{p}{p-1}}} = \\
 &= C_7(n) r^{n - \frac{(n-1)p}{p-1}} = C_7(n) r^{\frac{np - n - np + p}{p-1}} = C_7(n) r^{\frac{p-n}{p-1}}
 \end{aligned}$$

Thus follows:

$$\int_W |u(x) - u(z)| \, d^n z \leq \underbrace{C_6(n) (C_7(n))^{\frac{p-1}{p}}}_{=: C_8(n)} \|Du\|_{L^p(B_r(x))} r^{\frac{p-n}{p-1} \cdot \frac{p-1}{p}}$$

Estimating the other integral in (6.1) in the same way, we obtain:

$$|u(x) - u(y)| \leq \underbrace{2C_8(n)}_{=: C_9(n)} r^{1 - \frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

With  $\gamma := 1 - \frac{n}{p}$  follows for all  $x, y \in \mathbb{R}^n$  due to  $r = \|x - y\|$ :

$$\frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \leq C_9(n) \|Du\|_{L^p(\mathbb{R}^n)}$$

4. Combining 2. and 3., we get:

$$\begin{aligned} \|u\|_{C^{0,\gamma}(\mathbb{R}^n)} &= \|u\|_{C^0(\mathbb{R}^n)} + \sup_{x \neq y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \leq \\ &\leq C_4(n) \|u\|_{H^{1,p}(\mathbb{R}^n)} + C_9(n) \|Du\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{H^{1,p}(\mathbb{R}^n)} \end{aligned}$$

□<sub>6.6.1</sub>

We use a denseness argument to extend the statement to  $H^{1,p}$ -functions and  $C^{0,\alpha}$ -functions. For  $u \in H^{1,p}(\mathbb{R}^n)$ , weak derivatives  $D^{(i)}u$  exist and are in  $L^p$ . Note that  $u \in L^p$  stands for an equivalence class of function which differ on a set of measure zero. The inequality

$$|u(x)| \leq c \|u\|_{H^{1,p}(\mathbb{R}^n)}$$

makes no sense on these classes.

### 6.6.2 Theorem (Hölder Continuous Representative)

Let  $\Omega \subseteq \mathbb{R}^n$  be bounded with  $C^1$ -boundary. Assume  $n < p \leq \infty$  and  $u \in H^{1,p}(\Omega)$ . Then there is a representative  $u^*$  of  $u$  in  $H^{1,p}(\Omega)$ , which is Hölder continuous  $u \in C^{0,\gamma}(\overline{\Omega})$  with  $\gamma = 1 - \frac{n}{p}$  and:

$$\|u^*\|_{C^{0,\gamma}} \leq c \|u\|_{H^{1,p}(\Omega)}$$

In what follows, we always choose this representative and omit the star.

#### Proof

For  $u \in H^{1,p}(\Omega)$  there are  $u_m \in C_0^1(\mathbb{R}^n)$  with  $u_m|_\Omega \rightarrow u$  converging in  $H^{1,p}(\Omega)$  and

$$\|u_m\|_{H^{1,p}(\mathbb{R}^n)} \leq c \|u_m\|_{H^{1,p}(\Omega)}$$

just as in the proof of Theorem 6.5.3.

According to Theorem 6.6.1, the Morrey inequalities hold:

$$\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u_m - u_l\|_{H^{1,p}(\mathbb{R}^n)} \leq cC \|u_m - u_l\|_{H^{1,p}(\Omega)} \xrightarrow{m,l \rightarrow \infty} 0$$

Thus  $u_l$  is a Cauchy sequence in  $C^{0,\gamma}(\overline{\Omega})$ . Hence converges  $u_l \rightarrow u^* \in C^{0,\gamma}(\overline{\Omega})$ .

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \xleftarrow{l \rightarrow \infty} \|u_l\|_{C^{0,\gamma}(\overline{\Omega})} \leq c \|u_l\|_{H^{1,p}(\Omega)} \xrightarrow{l \rightarrow \infty} c \|u\|_{H^{1,p}(\Omega)}$$

□<sub>6.6.2</sub>

## 7 Construction of Weak Solutions of Linear Elliptic PDEs

Let  $\Omega \subseteq \mathbb{R}^n$  be a open domain with smooth boundary and  $n \in \mathbb{N}_{>2}$ . Consider the linear partial differential operator in divergence form:

$$Lu = -\frac{\partial}{\partial x^i} \left( a^{ij}(x) \frac{\partial}{\partial x^j} u(x) \right) + c(x) u(x)$$

Assume that the  $a^{ij}$  are uniformly elliptic, i.e. for all  $\xi \in \mathbb{R}^n$  and all  $x \in \Omega$  holds:

$$a^{ij}(x) \xi_i \xi_j \geq c \|\xi\|^2$$

Also assume that the coefficients are smooth and uniformly bounded  $|a^{ij}|, |c| \leq K$ . The first order term is missing!

We consider the Dirichlet problem:

$$\begin{aligned} Lu &= f \in C^\infty(\overline{\Omega}) \quad \text{in } \Omega \\ u|_{\partial\Omega} &= u_0 \in C^\infty(\partial\Omega) \end{aligned}$$

Examples:

- Poisson equation in  $\Omega \subseteq \mathbb{R}^n$
- Let  $(M, g)$  be a Riemannian manifold and  $\Omega \subseteq M$  be contained in a coordinate chart  $(x, U)$ .

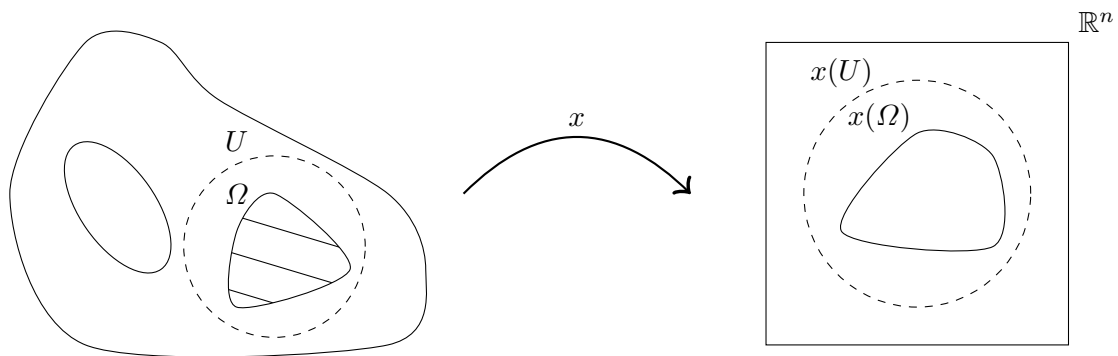


Figure 7.1: The chart maps  $\Omega$  to  $\mathbb{R}^n$ .

The Laplace-Beltrami-Operator is:

$$\Delta u = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i} \left( \sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x^j} u \right)$$

$$\Delta u = f \quad \Leftrightarrow \quad \frac{\partial}{\partial x^i} \left( \sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x^j} u \right) = \sqrt{\det(g)} f = \tilde{f}$$

This is in the desired divergence form.

Reduction to the case  $u_0 = 0$ : Let  $v \in C^\infty(\overline{\Omega})$  with  $v|_{\partial\Omega} = u_0$ . Set  $\tilde{u} := u - v$ .

$$\begin{aligned} L\tilde{u} &= f - Lv =: \tilde{f} \\ \tilde{u}|_{\partial\Omega} &= 0 \end{aligned}$$

A solution  $\tilde{u}$  of this Dirichlet problem yields a solution  $u = \tilde{u} + v$  of the original problem.

Therefore, from now on we consider the problem:

$$\begin{aligned} Lu &= f \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

## 7.1 The Dirichlet Principle

Define the *action*:

$$S := \int \left( a^{ij}(x) \left( \frac{\partial}{\partial x^i} u(x) \right) \left( \frac{\partial}{\partial x^j} u(x) \right) + c(x) u^2(x) - 2f(x) u(x) \right) d^n x$$

Suppose  $u$  is a critical point (for example a minimizer), then holds for  $\eta \in C_0^\infty(\Omega)$  formally:

$$\begin{aligned} 0 &= \frac{d}{dt} S(u + \tau\eta) \Big|_{\tau=0} = \\ &= \int_{\Omega} \left( 2a^{ij}(x) \left( \frac{\partial}{\partial x^i} u(x) \right) \left( \frac{\partial}{\partial x^j} \eta(x) \right) + 2c(x) u(x) \eta(x) - 2f(x) \eta(x) \right) d^n x = \\ &\stackrel{\text{integration by parts}}{=} 2 \int_{\Omega} \eta(x) \underbrace{\left( -\frac{\partial}{\partial x^j} \left( a^{ij}(x) \left( \frac{\partial}{\partial x^i} u(x) \right) \right) + c(x) u(x) - f(x) \right)}_{=Lu-f} d^n x \end{aligned}$$

This has to hold for any  $\eta$  and thus follows  $Lu - f = 0$ .

Assume  $u \in H^{1,2}(\Omega)$  for the integral over the two partial derivatives in the action being well defined. To ensure the boundary value  $u|_{\partial\Omega} = 0$  we assume even  $u \in H_0^{1,2}(\Omega)$ . Then holds:

$$\begin{aligned} \int_{\Omega} \underbrace{a^{ij}(\partial_i u)(\partial_j u)}_{\geq 0} d^n x &\leq \sup_{\Omega} (|a^{ij}|) \int_{\Omega} \|Du\|^2 d^n x \leq \sup_{\Omega} (|a^{ij}|) \|u\|_{H_0^{1,2}(\Omega)}^2 < \infty \\ \int_{\Omega} |c(x) u^2(x)| d^n x &\leq \sup_{\Omega} (|c|) \int_{\Omega} u^2(x) d^n x \leq \sup_{\Omega} (|c|) \|u\|_{H_0^{1,2}(\Omega)}^2 < \infty \\ \int_{\Omega} 2|f(x) u(x)| d^n x &\leq 2\|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq 2\|f\|_{L^2(\Omega)} \|u\|_{H_0^{1,2}(\Omega)} < \infty \end{aligned}$$

We always assume that  $\Omega$  is bounded. Then  $f \in C^\infty(\overline{\Omega})$  implies  $f \in L^2(\Omega)$ .

Furthermore, we know from the Gagliardo-Nirenberg estimate

$$H_0^{1,2}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

with  $p = 2$  and:

$$\frac{1}{p^*} = \frac{1}{2} - \frac{1}{n} < \frac{1}{2}$$

$$p^* = \frac{2n}{n-2} > 2$$

This is an *imbedding* (*embedding*), i.e. an injective mapping.

If  $\Omega$  has finite volume, which it has in our case, since we assume that  $\Omega$  is bounded, it follows:

$$\|u\|_{L^2(\Omega)} = \left( \int_{\Omega} u^2(x) \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} u^{2\tilde{p}}(x) \, dx \right)^{\frac{1}{2\tilde{p}}} (\mu(\Omega))^{\frac{1}{2\tilde{q}}} \leq \tilde{c} \|u\|_{L^{2\tilde{p}}(\Omega)}$$

We chose  $\tilde{p}$  such that  $2\tilde{p} = p^*$  and use Poincaré's inequality:

$$\|u\|_{L^2(\Omega)} \leq \tilde{c} \|u\|_{L^{p^*}(\Omega)} \leq \underbrace{\tilde{c}C}_{=:c} \|Du\|_{L^2(\Omega)}$$

Thus the action can be viewed as a mapping:

$$S : H_0^{1,2}(\Omega) \rightarrow \mathbb{R}$$

### 7.1.1 Theorem (Action Bounded from Below)

$S$  is bounded from below, i.e.  $S(u) \geq -K \in \mathbb{R}_{\leq 0}$  for all  $u \in H_0^{1,2}(\Omega)$ .

#### Proof

The uniform ellipticity means:

$$a^{ij}(\partial_i u)(\partial_j u) \geq \underbrace{\lambda}_{>0} \|Du\|^2$$

Moreover holds  $C(x) \geq 0$ . Therefore follows:

$$\begin{aligned} S(u) &\geq \int_{\Omega} \lambda \|Du\|^2 \, dx - \|f\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} \geq \\ &\stackrel{\text{Poincaré}}{\geq} \frac{\lambda}{c^2} \|u\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} = \\ &= \frac{\lambda}{c^2} \left( \|u\|_{L^2(\Omega)} - \frac{c^2}{2\lambda} \|f\|_{L^2(\Omega)} \right)^2 - \frac{c^2}{4\lambda} \|f\|_{L^2(\Omega)}^2 \geq \\ &\geq -\frac{c^2}{4\lambda} \|f\|_{L^2(\Omega)}^2 =: -K \end{aligned}$$

□<sub>7.1.1</sub>

We now choose a *minimizing sequence* (Minimalfolge), i.e. a sequence  $u_n \in H_0^{1,2}(\Omega)$  with:

$$S(u_n) \rightarrow \inf_{H_0^{1,2}(\Omega)} (S)$$

Hope:

- A subsequence of the  $u_n$  converges (weakly?) in  $H_0^{1,2}(\Omega)$  to  $u \in H_0^{1,2}(\Omega)$ .
- $S(u) = \lim_{n \rightarrow \infty} S(u_n) = \inf_{H_0^{1,2}(\Omega)} S$

Such a minimizer satisfies the following Euler-Lagrange equation:

### 7.1.2 Theorem (Euler-Lagrange equation)

If  $u \in H^{1,2}(\Omega)$  satisfies  $S(u) = \inf_{H_0^{1,2}(\Omega)}(S)$ , then holds for all  $\eta \in C_0^\infty(\Omega)$ :

$$\int_{\Omega} \left( u \left( -\frac{\partial}{\partial x^i} a^{ij} \frac{\partial}{\partial x^j} \eta + c\eta \right) - f\eta \right) d^n x = 0$$

Thus  $u$  is a weak solution of the equation  $Lu = f$ .

For  $u \in C^2(\Omega)$ , we can integrate by parts twice, which would give:

$$\int (Lu - f) \eta d^n x = 0$$

#### Proof

$$\begin{aligned} S(u + \tau\eta) &= \int_{\Omega} a^{ij} (\partial_i (u + \tau\eta)) (\partial_j (u + \tau\eta)) + c(u + \tau\eta)^2 - 2f(u + \tau\eta) d^n x = \\ &= \tau^2 \int_{\Omega} (a^{ij} (\partial_i \eta) (\partial_j \eta) + c\eta^2) d^n x + 2\tau \int_{\Omega} (a^{ij} (\partial_i u) (\partial_j \eta) + cu\eta - f\eta) d^n x + \\ &\quad + \int_{\Omega} (a^{ij} (\partial_i u) (\partial_j u) + cu^2 - 2fu) d^n x \end{aligned}$$

$S(\tau)$  has a minimum at  $\tau = 0$ . Therefore holds:

$$\begin{aligned} 0 &= \frac{d}{d\tau} S(u + \tau\eta) \big|_{\tau=0} = 2 \int_{\Omega} (a^{ij} (\partial_i u) (\partial_j \eta) + cu\eta - f\eta) d^n x \\ 0 &= \int_{\Omega} (u \partial_i (-a^{ij} (\partial_j \eta)) + cu\eta - f\eta) d^n x \end{aligned}$$

□<sub>7.1.2</sub>

## 7.2 Existence of Minimizers

Let  $(u_n)$  be a minimizing sequence. To show is  $u_n \rightarrow u$  and:

$$S(u) = \kappa := \inf_{H_0^{1,2}(\Omega)}(S)$$

More generally, let  $\mathcal{H}$  be a Hilbert space. (In our case we have  $\mathcal{H} = H_0^{1,2}(\Omega)$ .)

$$A(u, v) := \int_{\Omega} (a^{ij} (\partial_i u) (\partial_j v) + cuv) d^n x$$

$$l(u) := -2 \int_{\Omega} f u d^n x$$

$$S(u) = A(u, u) + l(u)$$

The bilinear map  $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  and the linear map  $l : \mathcal{H} \rightarrow \mathbb{R}$  are continuous:

$$\begin{aligned} |A(u, v)| &\leq \|A\| \|u\| \cdot \|v\| \\ |l(u)| &\leq \|l\| \|u\| \end{aligned}$$

Moreover  $A$  is symmetric:

$$A(u, v) = A(v, u)$$

### 7.2.1 Definition (Coercive)

$A$  is called coercive (koerziv, manchmal auch elliptisch) if there is a  $\lambda \in \mathbb{R}_{>0}$  such that for all  $u \in \mathcal{H}$  holds:

$$A(u, u) \geq \lambda \|u\|_{\mathcal{H}}^2$$

In our problem, this condition is fulfilled due to the uniform ellipticity:

$$\begin{aligned} A(u, u) &= \int_{\Omega} a^{ij} (\partial_i u) (\partial_j u) \, dx \stackrel{\text{uniform ellipticity}}{\geq} \lambda \int_{\Omega} \|Du\|^2 \, dx \geq \\ &\stackrel{\text{Poincaré}}{\geq} \frac{\lambda}{2} \int_{\Omega} \|Du\|^2 \, dx + \frac{\lambda}{2c^2} \int_{\Omega} \|u\|^2 \, dx \geq \tilde{c} \|u\|_{H_0^{1,2}(\Omega)}^2 \end{aligned}$$

### 7.2.2 Theorem (Unique Minimizer)

Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space with norm  $\|\cdot\|_{\mathcal{H}}$ ,  $V \subseteq \mathcal{H}$  closed and convex,

$$A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$$

bilinear, symmetric, continuous and coercive and

$$l : \mathcal{H} \rightarrow \mathbb{R}$$

linear and continuous.

Then the action  $S(u) := A(u, u) + l(u)$  has a unique minimizer in  $V$ .

#### Proof

The action is bounded from below:

$$S(v) \geq \lambda \|v\|_{\mathcal{H}}^2 - \|l\| \|v\|_{\mathcal{H}} \stackrel{\text{completing the square}}{\geq} -\frac{\|l\|^2}{2\lambda}$$

Let  $\kappa := \inf_V (S)$ . We choose a minimizing sequence  $u_n \in V$  with  $S(u_n) \rightarrow \kappa$ . We want to show  $u_n \rightarrow u$ . Then by continuity follows:

$$S(u) = \lim_{n \rightarrow \infty} (S(u_n)) = \kappa$$

Since  $\mathcal{H}$  is a Hilbert space and thus complete and  $V$  is closed, we only have to verify the Cauchy condition, to show the existence. Because  $V$  is convex holds  $\frac{1}{2}(u_n + u_m) \in V$ . So we get:

$$\begin{aligned} \kappa &= \inf_V (S) \leq S\left(\frac{u_n + u_m}{2}\right) = A\left(\frac{u_n + u_m}{2}, \frac{u_n + u_m}{2}\right) + l\left(\frac{u_n + u_m}{2}\right) = \\ &= \frac{1}{4}A(u_n, u_n) + \frac{1}{4}A(u_m, u_m) + \frac{1}{2}A(u_n, u_m) + \frac{1}{2}l(u_n) + \frac{1}{2}l(u_m) = \\ &= \frac{1}{2}S(u_n) + \frac{1}{2}S(u_m) - \frac{1}{4}A(u_n, u_n) - \frac{1}{4}A(u_m, u_m) + \frac{1}{2}A(u_n, u_m) = \\ &= \underbrace{\frac{1}{2}S(u_n)}_{\rightarrow \frac{\kappa}{2}} + \underbrace{\frac{1}{2}S(u_m)}_{\rightarrow \frac{\kappa}{2}} - \underbrace{\frac{1}{4}A(u_n, u_n) - \frac{1}{4}A(u_m, u_m) + \frac{1}{2}A(u_n, u_m)}_{\leq 0} \end{aligned}$$



Thus converges  $A(u_n - u_m, u_n - u_m) \xrightarrow{n,m \rightarrow \infty} 0$ .

$$\|u_n - u_m\|^2 \leq \frac{1}{\lambda} A(u_n - u_m, u_n - u_m) \xrightarrow{n,m \rightarrow \infty} 0$$

Hence  $(u_n)$  is a Cauchy sequence and  $u_n \rightarrow u$ .

*Uniqueness:* Let  $u_1, u_2$  be minimizers, i.e.  $S(u_1) = \kappa = S(u_2)$ . Then holds:

$$\kappa \leq S\left(\frac{u_1 + u_2}{2}\right) = \underbrace{\frac{1}{2}S(u_1) + \frac{1}{2}S(u_2)}_{=\kappa} - \underbrace{\frac{1}{4}A(u_1 - u_2, u_1 - u_2)}_{\leq 0}$$

$$\begin{aligned} \Rightarrow A(u_1 - u_2, u_1 - u_2) &= 0 \\ \Rightarrow u_1 &= u_2 \end{aligned}$$

□<sub>7.2.2</sub>

### 7.2.3 Corollary

Under the conditions of Theorem 7.2.2 and if in addition  $V$  is a closed linear subspace of  $\mathcal{H}$ , there is a unique  $u \in V$  such that for all  $\varphi \in V$  holds:

$$2A(u, \varphi) + l(\varphi) = 0$$

#### Proof

For a minimizer  $u$ ,  $S(u + \tau v)$  is minimal at  $\tau = 0$ . This is a polynomial in  $\tau$  and thus holds:

$$0 = \frac{d}{d\tau} S(u + \tau v) \Big|_{\tau=0} = \frac{d}{d\tau} (A(u + \tau v, u + \tau v) + l(u + \tau v)) \Big|_{\tau=0} = 2A(u, v) + l(v)$$

□<sub>7.2.3</sub>

# Appendix

# Acknowledgements

My special thanks goes to Professor Finster, who gave this lecture and allowed me to publish this script of the lecture.

I would also like to thank all those, who found errors by careful reading and told me of them.

Andreas Völklein