A Semantics for Weak, Question-Sensitive Belief

$\sim$ 10700 words

Abstract

Recent work in epistemology defends the unorthodox theses that (1) belief is an evidentially weak, (2) question-sensitive attitude, and that (3) rationally permitted belief is sometimes a matter of guessing. These theses fit together naturally to form a unified account of weak, question-sensitive belief. A formal account of weak, question-sensitive belief as a coherent phenomenon is still forthcoming, however.

The main aim of this paper is to develop a formal account that captures belief's weakness and question-sensitivity in the setting of epistemic logic. We introduce a class of models in which the points of evaluation are situations, or world-evidence pairs, with evidence understood liberally to include sets of live possibilities, measures of uncertainty, and QUDs. A proposition is believed at a situation just in case it is implied by the most informative probabilistically dominant answer to the QUD, on some way of specifying the threshold of probabilistic dominance.

The second aim of the paper is to explore two sets of epistemological implications in our formal setting. First, we consider whether beliefs are preserved between situations upon shifting the QUD parameter; specifically, we consider whether beliefs are preserved upon updating with learned information, under refining and coarsening questions, and whether belief is closed under conjunction. Second, we consider the interaction of knowledge and belief; specifically, we consider whether the principles governing the interaction of knowledge and belief in Stalnaker's **KD45** also hold in our setting.

Keywords: weak belief; question-sensitivity; epistemic logic; belief dynamics; knowledge and belief;

1 Introduction

Recent work in epistemology defends the unorthodox theses that (1) belief is an evidentially weak, (2) question-sensitive attitude, and that (3) rationally permitted belief is sometimes a matter of guessing. (1) Belief is weak: the evidential requirements for believing a proposition are undemanding; particularly, one can believe p without knowing p and without being highly confident in p (Hawthorne, Rothschild, & Spectre, 2016; Holguin, 2022; Rothschild, 2020). (2) Belief is question-sensitive: agents believe propositions against the backdrop of salient partitions of the logical space provided by a question under discussion (QUD, Roberts (2012)) (Drucker, 2020; Hoek, 2022, 2024; Yalcin, 2018). (3) Beliefs are best guesses: in situations of uncertainty, one's belief just is one's best guess, where one's guess concerns a QUD, and a particular guess can dominate other guesses despite one's evidence assigning it low probability (Dorst & Mandelkern, 2022; Holguin, 2022). These theses fit together naturally to form a unified account of weak, question-sensitive belief.[1] A formal account of this phenomenon is still forthcoming, however.[2]

The first aim of the paper is to outline a model for belief that predicts its weakness and question-sensitivity (SECTION 3). We construct models in which world-evidence pairs, or situations, take centre stage, with evidence understood liberally to include (i) a set of worlds non-excluded by one's evidence, (ii) a measure of uncertainty over worlds, and (iii) a partition. We stipulate that the strongest proposition believed relative to a situation is the probabilistically dominant proposition relevant to the question (though see the discussion for details). As standard in epistemic logic, ϕ is then believed at a situation just in case ϕ is implied by the situation's strongest believed proposition.

We also argue that this treatment offers a perspicuous way of studying the properties of weak, question-sensitive belief in a formal setting. The second aim of the paper is to explore two sets of implications of the foregoing semantics for issues in epistemology. First (SECTION 4), we consider whether beliefs are preserved between situations upon shifting the QUD parameter; specifically, we consider whether beliefs are preserved upon updating with learned information (SECTION 4.2), under refining and coarsening questions (SECTION 4.3), and whether belief is closed under conjunction (SECTION 4.4). Second (SECTION 5), we consider the interaction of knowledge and belief; specifically, we consider whether the principles governing the interaction of knowledge and belief in Stalnaker's **KD45** also hold in our setting.

[1] But not necessarily so; see Leitgeb (2017) for a partition-sensitive account of belief that is stronger than our target notion.

[2] In what follows, we sometimes speak simply of belief when talking about our target notion of weak, question-sensitive belief. We stay neutral on the issue of how weak, question-sensitive belief relates to notions such as full belief.

2 ... Weak, Question-Sensitive Belief

We introduce the theses that belief is weak, question-sensitive, and a matter of guessing, suggest some *desiderata* for our semantic account, and illustrate the theses by a paradigmatic example. The following is not meant as an exhaustive defence of the theses, but rather as a survey of our target notion of belief.

Belief is **weak**: the evidential requirements for believing a proposition are undemanding, in the sense that believing p is more akin to thinking that p is likely than to being certain that p. If belief is weak, believing p is rationally compatible with not knowing p, with not being certain that p, and not being confident that p. A formal model of an agent's beliefs should predict that beliefs are not discounted for falling short of certainty, confidence, or some other strong epistemic attitude, and should also minimally incorporate considerations of subjective (un)certainty to capture the closeness of believing p and thinking p likely.

Belief is **question-sensitive**: agents believe propositions against the backdrop of contextually salient questions, in the sense that the alternatives determined by a given question make a difference to what the agent believes. Different questions determine different ways of partitioning the space of possibilities, with different partitions inducing different sets of alternatives to be considered by the agent. Relative to one set of alternatives, an agent may rationally believe that p, while relative to another the same agent may rationally disbelieve that p. A formal model of an agent's beliefs should predict that the same evidence may induce different beliefs relative to different partitions of the space of possibilities, and should thus incorporate QUDs to capture the question-sensitivity of belief.

The foregoing claims together motivate an account of belief on which beliefs are akin to **guesses**. In situations of uncertainty, forming a guess is a matter of choosing between options in line with one's evidence, where an option can be chosen as long as it is more supported than some alternatives. This account of guessing is immediately applicable to the foregoing account of belief, once forming a belief is seen to be akin to forming a guess. Since making a guess is sensitive to alternatives provided by questions and does not require that the guess meet some context-invariant threshold of support, beliefs formed by this procedure are expected to have traits of weakness and question-sensitivity in the above sense.

To illustrate these claims, consider the simple example of an urn:

The Urn consists of a 100 marbles, with 45 blue, 30 green, 15 red, and 10 yellow marbles inside. A marble is randomly chosen from the urn. What should you believe?

Suppose that we are in a guessing context which requires maximally specific guesses (this assumption is made for convenience and relaxed in the models below). Relative to Q, (1) appears permissible and (2) appears impermissible:

Q: What is the colour of the chosen marble?

(1) I {believe}{think} that the chosen marble is blue.
(2) I {believe}{think} that the chosen marble is non-blue.

Since Q asks for a complete guess concerning the colour of the chosen marble, only (1) delivers on this while respecting the chances of picking each colour. Relative to Q', however, the situation is reversed:

Q': Is the chosen marble blue or not?

(1) I {believe}{think} that the chosen marble is blue.
(2) I {believe}{think} that the chosen marble is non-blue.

Since Q' asks whether the chosen marble is blue, in the reversed scenario only (2) delivers on this in accordance with the chances.

Such judgments of rational permissibility illustrate the weakness and question-sensitivity of belief in a simplified case:

WEAKNESS: Relative to both Q and Q', some proposition is believed despite the fact that, given the setup of the case, it is neither known nor certain; relative to Q, moreover, the proposition believed is probabilistically supported to a degree less than 0.5.

QUESTION-SENSITIVITY: A proposition p is believed relative to Q, while its negation is believed relative to Q' — and so as a function of the alternatives made salient by Q and Q', respectively.

GUESSING: In both cases, the belief is arguably formed by making a probabilistically-informed guess relative to a set of options (alternatives).

In what follows, we outline a model for belief that captures these judgments of doxastic, rational permissibility.

3 A Semantics for ...

This section outlines a class of models capturing the weakness and question-sensitivity of belief. We first survey the proposal informally, from a bird's eye perspective.

3.1 Informal Overview

The main components of our models are world-evidence pairs, called *situations*, in which 'worlds' are points and 'evidence' stands for a triple of (i) a set of worlds s, (ii) a probability distribution over s, and (iii) a partition over s. Intuitively, a situation's world specifies how the situation is *factually* speaking, while a situation's evidence specifies how the situation is *informationally* speaking. Evidence is here construed liberally so as to include a set of live possibilities, a measure of uncertainty, and a QUD. We constrain the space of admissible situations so that (i) there are no situations in which a world discounts itself as a live possibility; so that (ii) if w considers v live in situation c, there is a situation c' with the same evidence in which v considers w live; and so that (iii) if w considers v live in c and v considers u live in c', w considers u live in c.

Each evidence triple determines the strongest proposition (in the sense of a set of possible worlds) supported by the evidence. The procedure can be introduced, in broad strokes, as follows. A proposition is relevant to a question if it is an answer to the question, and it is probabilistically dominant relative to a question if it is t-times more probable than any of the answers that are equally ranked in terms of informativity (with t an independently specified threshold). It can be shown (Fact 1 below) that, relative to each evidence triple, there exists *at most one* probabilistically dominant proposition for each informativity rank. Relative to some evidence triple, we take the *most informative* probabilistically dominant proposition as the strongest proposition supported by the evidence.

Finally, we define evidential and doxastic accessibility relations over situations in the model. A situation c *evidentially* accesses situation c' iff the evidence triple in c and c' is the same; intuitively, one's evidence is thus taken to be transparent, in the sense that one's evidence includes what one's evidence is. A situation c *doxastically* accesses situation c' iff c evidentially accesses c' *and* the world in situation c' makes the strongest proposition supported by the evidence in c true. A proposition (now construed as a set of situations) is believed at a situation iff it is implied by the situation's set of doxastically accessible situations.

3.2 Formal Treatment

In what follows, we formalise this proposal and discuss it further as we flesh out the formal machinery. Let P be a set of atomic sentences, and $\mathcal{L}$ be a modal propositional language generated by the following grammar (where B is the operator for belief):

$$\mathcal{L} ::= \phi \in P \mid \neg\phi \mid \phi \vee \psi \mid B\phi$$

We build the components needed for the semantics step by step. Let W be a set of points and $v(\cdot)$ be the function $P \longrightarrow \mathscr{P}(W)$; we abbreviate $v(p)$ by $\mathbf{p}$. We introduce some definitions:

Definition 1. An INFORMATION STATE s is a set of points ($s \subseteq W$). Let $S = \mathscr{P}(W)$ be the set of all information states on W.

Definition 2. QUESTION, QUESTION SPACE, and ALTERNATIVES

1. A QUESTION on s is a partition of s, i.e., a division of $s \in S$ into subsets $\{X_i\}_{i \in I}$ such that (a) $\bigcup_{i \in I} X_i = s$, (b) $X_i \cap X_j = \emptyset$ for all $i \neq j \in I$, and (c) $X_i \neq \emptyset$ for all $i \in I$. (Groenendijk & Stokhof, 1984; Hamblin, 1973; Karttunen, 1977)
2. A QUESTION SPACE on s, $\mathcal{Q}(s)$ is the set of all admissible questions on s.[3]
3. The set of ANSWERS to a question is $S(Q) = \{A \in \mathscr{P}(s) \mid A \subseteq Q\}$.
4. The INFORMATIVITY of an answer A to Q is $n(A)^Q = |A|$ if $A \in S(Q)$. Say that A is of RANK k if $n(A)^Q = k$. The set of k-ranked answers to Q is $[A]_k^Q = \{B \in S(Q) \mid n(A)^Q = n(B)^Q = k\}$.
5. The set of k-RANKED ALTERNATIVES to X_i in $Q, \mathcal{A}_{X_i}^{Q,k} = \{X \in [A]_k^Q \mid X \cap X_i = \emptyset\}$, if $X \in [A]_k^Q$, and $\emptyset$ otherwise.

Definition 3. PROBABILITY DISTRIBUTION and PROBABILITY SPACE

1. A PROBABILITY DISTRIBUTION on s is a function $\pi : \mathscr{P}(s) \longrightarrow [0, 1]$ such that (a) $\pi(s) = 1$ and (b) for any $s', s'' \in \mathscr{P}(s)$ such that $s' \cap s'' = \emptyset$, $\pi(s' \cup s'') = \pi(s') + \pi(s'')$.
2. A PROBABILITY SPACE on s, $\Pi(s)$ is the set of probability distributions on s.

Definition 4. A body of EVIDENCE is a triple $i \in I \subseteq S \times \Pi(S) \times \mathcal{Q}(S)$ (cf. sharp information states in Yalcin (2012)).

A body of evidence is here taken to be a triple consisting of a set of worlds s, a probability distribution on s, and a partition over s. Intuitively, a body of evidence represents the set of worlds not excluded by some evidence, a measure of uncertainty over these worlds, and a question under discussion. Before outlining the models, we sketch the main idea behind believing a proposition relative to some evidence, on the target notion of belief.

A natural thought is that a proposition forms the belief set if it probabilistically dominates some alternatives and is relevant to the QUD. In other words, the strongest proposition believed must be more certain than its alternatives (as specified by the question) and address a contextually salient question. We can capture this formally.

First, probabilistic dominance. Take an arbitrary $i = \langle s, \pi, Q \rangle$. Where $t \in \mathbb{R}^{\geq 1}$, say that a proposition $X \in [A]_k^Q$ probabilistically t-dominates its k-ranked alternatives if:

$$\pi(X) > \max_{X' \in \mathcal{A}_X^{Q,k}} t \cdot \pi(X') \qquad (\pi\text{-Dom.})$$

[3] This set can be characterised formally as follows. There exists a bijection between the set of partitions of s and the set of equivalence classes on s. For any equivalence relation $\approx_i$, the set of all equivalence classes in s generated by $\approx_i$ is $\{x \in s | [x]_{\approx_i}\}$, or the quotient set of s for $\approx_i$, $s/\approx_i$. Where $\mathcal{E}(s)$ is the set of all equivalence relations on s, $\mathcal{Q}(s) = \{s/\approx_i \mid \approx_i \in \mathcal{E}(s)\}$ is the set of all quotient sets on s. $\mathcal{Q}(s)$ just is the set of possible partitions on s. See Hulstijn (1997) for a similar account.

Intuitively, if X is the probabilistically t-dominant proposition in its rank, X is t-times as probable as any of the alternatives with the same degree of informativity. In what follows, we consider models without specifying the value of t, in the interest of generality.[4] Second, relevance. Say that a proposition X is relevant to Q if:[5]

$$X \in S(Q) \tag{Rel.}$$

Schematically, we want to predict:

$$X \text{ is the belief set relative to } i \text{ iff } X \text{ is } \pi\text{-Dominant and Relevant at } i \tag{Bel.}$$

This is a good start. However, the conditions of π-Dominance and Relevance are not sufficient to determine a unique belief set. The reason is, roughly speaking, that these conditions do not specify how the informativity of an answer is valued in determining the belief set.[6] To see this, note that (π-Dom.) restricts the set of alternatives against which a proposition is compared to the set of k-ranked alternatives.[7] With this restriction in place, it is possible to get different probabilistically t-dominant answers for each rank k. But then being π-Dominant and Relevant is not sufficient for determining what is believed relative to an information state. As such, a tie-breaking heuristic is needed.

The heuristic we propose is that the more informative probabilistically t-dominant answers get privileged over less informative probabilistically t-dominant answers, as long as they exist. Roughly speaking, we want to predict that *the most informative* t-dominant answer is the strongest proposition believed. It is possible to show that this set is unique, providing both necessary and sufficient conditions for determining a belief set based on some i. To unpack this, first note that:

Fact 1. If a proposition $X_i \in [A]^Q_k$ such that $\pi(X_i) > \max_{X_j \in \mathcal{A}^{Q,k}_{X_i}} t \cdot \pi(X_j)$ exists (for some t), it is unique.

Proof. Suppose $X_i \in [A]^Q_k$ is such that $\pi(X_i) > \max_{X_j \in \mathcal{A}^{Q,k}_{X_i}} t \cdot \pi(X_j)$. Suppose, for contradiction, that some $X_l \in [A]^Q_k$ is such that $X_l \neq X_i$ and that $\pi(X_l) > \max_{X_j \in \mathcal{A}^{Q,k}_{X_l}} t \cdot \pi(X_j)$. By the

[4]Roughly speaking, a low value of t indicates the agent's relative boldness, while a higher value indicates more caution. For instance, for $t = 1$, a proposition can be treated as the core belief set merely on account of its being more probable than any alternative, regardless of the magnitude of difference. It is to be expected that t's value depends on various contextual factors: whether the relevant agent is bold (low t) or cautious (high t), whether the number of alternatives considered is high (high t) or low (low t), etc.

[5]While this may not be an exhaustive account of relevance (see Hartzell (2025); Roberts (2012) for discussion), being an answer to a question is plausibly a necessary condition on any account.

[6]Cf. Dorst and Mandelkern (2022) for an extensive discussion of the value of informativity (and accuracy) in the context of guessing.

[7]Removing this restriction and considering the whole set of answers $S(Q)$ implies that, for any $i = \langle s, \pi, Q \rangle$, the probabilistically k-dominant answer is s itself, or the minimally informative (trivial) answer.

generalisation of Definition 2, $X_l \in \mathcal{A}^{Q,k}_{X_i}$ and $X_i \in \mathcal{A}^{Q,k}_{X_j}$. But then $\pi(X_i) > t \cdot \pi(X_l)$ and $\pi(X_l) > t \cdot \pi(X_i)$. Contradiction. So: X_i is unique, if it exists. $\square$

Fact 1 says that, for any informativity rank k, if a probabilistically t-dominant answer of rank k exists, it is unique. As such, for any informativity rank, there is at most one probabilistically t-dominant answer. Let $\boldsymbol{X}^{i,t}_k$ denote this answer, where k is the answer's rank, i is the corresponding $i = \langle s, \pi, Q\rangle$, and t the relevant threshold.

Generalising over ranks, we determine, for any i and threshold t, the least k for which $\boldsymbol{X}^{i,t}_k$ is non-empty (i.e., for which a probabilistically t-dominant k-ranked answer exists). We capture this as follows:

$$\boldsymbol{X}^{i,t} = \begin{cases} \arg\min_{k\geq 1}\left(\left\{X \in [A]^Q_k \mid \pi(X) > \max_{X_j \in \mathcal{A}^{Q,k}_X} t \cdot \pi(X_j)\right\}\right) & \text{if the set is non-empty} \\ s & \text{if the set is empty for all } k \end{cases}$$

If successful, this procedure yields a set $\boldsymbol{X}^{i,t}_k$ such that $\boldsymbol{X}^{i,t}_j = \emptyset$ for all $1 \leq j < k$. Intuitively, the procedure determines the lowest rank k for which $\boldsymbol{X}^{i,t}_k$ is non-empty and returns the (unique) probabilistically t-dominant set of rank k. In case $\boldsymbol{X}^{i,t}_k$ is empty for all k, the procedure outputs just the information state and does not allow any non-trivial belief (with respect to i). We propose that $\boldsymbol{X}^{i,t}$ be treated as the strongest proposition believed at i. This set-of-possible-worlds propositions will play a central role in what follows.[8]

With this background in place, we define models for $\mathcal{L}$.

Definition 5. A MODEL $\mathcal{M}$ is a tuple $\langle \mathcal{W}, [\![]\!], t\rangle$, where $\mathcal{W} \subseteq W \times I$, and $t \in \mathbb{R}^{\geq 1}$.

Formally, some $c \in \mathcal{W}$ is a pair of a world and a body of evidence. Call any $c \in \mathcal{W}$ a *situation*. Intuitively, a situation consists of a specification of what the world is like (what is true at some w) *and* a specification of what one's information about the world is like. In our models, sentences are true at situations conceived as world-evidence pairs.

A significant question to be settled by the models is whether any constraints should be imposed on admissible world-evidence pairs (i.e., admissible situations). As the main component of one's evidence is the set of worlds s, this question can be approached by considering constraints on admissible situation pairs in terms of the relation between w and s in some $i = \langle w, \langle s, \pi, Q\rangle\rangle$. Where I is the set of bodies of evidence, we impose the following constraints:

(i) If $c = \langle w, \langle s, \pi, Q\rangle\rangle \in \mathcal{W}$, then $w \in s$;
(ii) If $c = \langle w, \langle s, \pi, Q\rangle\rangle \in \mathcal{W}$, then $\forall v \in s : c' = \langle v, \langle s, \pi, Q\rangle\rangle \in \mathcal{W}$ for all $v \in s$;
(iii) If $c = \langle w, \langle s, \pi, Q\rangle\rangle \in \mathcal{W}$ with $v \in s$ and $c' = \langle v, \langle s', \pi', Q'\rangle\rangle \in \mathcal{W}$, then $\forall u \in s'$: $u \in s$.

[8]Note that $\boldsymbol{X}^{i,t}$ only determines the *strongest* proposition believed (or the belief set) relative to a body of evidence. Although we imposed the constraint that this set be relevant to the QUD, not every implication of the belief set is relevant to the QUD. Speaking loosely, although the core of of one's beliefs will be relevant to the QUD, one may believing logical consequences of this core that are not relevant in this sense.

(i) corresponds to the reflexivity condition on the accessibility relation in epistemic logic; intuitively, situations in which a world excludes itself as a live possibility (i.e., in which $w \notin s$ for some $c = \langle w, \langle s, \pi, Q\rangle\rangle$) are inadmissible.[9] (ii) and (iii) in turn correspond to the availability of reflexive and transitive closures of any situation in $\mathcal{W}$.

We now define the evidential and doxastic accessibility relations for situations:[10]

$$R_E(\langle w, i\rangle) = \{\langle v, i'\rangle \in \mathcal{W} \mid i' = i\}$$
$$R_D(\langle w, i\rangle) = \{\langle v, i'\rangle \in R_E(\langle w, i\rangle) \mid v \in \boldsymbol{X}^{i,t}\}$$

Situation c evidentially accesses all situations with the same evidence. Situation c doxastically accesses all situations that, besides having the same evidence, also make $\boldsymbol{X}^{i,t}$ true. Intuitively, a situation doxastically accesses only situations with identical evidence whose worlds verify the probabilistically t-dominant proposition (relative to i). As is immediate, although we imposed a reflexivity constraint on admissible situations, this constraint can fail on $R_D(\cdot)$.

The semantics for $\mathcal{L}$ is as follows (recall that $\boldsymbol{\phi} = v(\phi)$ for any atomic ϕ):

$$[\![\phi]\!] = \{\langle w, i\rangle \in \mathcal{W} \mid w \in \boldsymbol{\phi}\}$$
$$[\![\neg\phi]\!] = \mathcal{W} \setminus [\![\phi]\!]$$
$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$$
$$[\![B\phi]\!] = \{\langle w, i\rangle \in \mathcal{W} \mid R_D(\langle w, i\rangle) \subseteq [\![\phi]\!]\}$$

As anticipated above, the set of situations at which ϕ holds is a set of situations whose world parameter makes ϕ true. Moreover, $B\phi$ holds at any situation c such that ϕ holds throughout the set of situations doxastically accessible from c. Entailment is standardly defined on our models:

$$\Gamma \models_{\mathcal{M}} \phi \text{ iff } \bigcap_{\psi \in \Gamma} [\![\psi]\!] \subseteq [\![\phi]\!]$$

To illustrate the semantics, consider a model of **The Urn** from Section 2.

The Urn (Model). Let $W = \{b, g, r, y\}$. Let $Q = \{\{b\}, \{g\}, \{r\}, \{y\}\}$ and $Q' = \{\{b\}, \{g, r, y\}\}$; intuitively, Q is expressed by 'What is the colour of the marble?' and Q by 'Is the marble blue or non-blue?'. Let $\pi(b) = 0.45$, $\pi(g) = 0.3$, $\pi(r) = 0.15$, and $\pi(y) = 0.1$. Let $i = \langle W, \pi, Q\rangle$ and $i' = \langle W, \pi, Q'\rangle$. For illustrations **I** and **II**, let $t = 1$.

[9] Note that no corresponding restriction is imposed on the relevant measure of uncertainty, so that a situation may consider its world to be arbitrarily improbable.

[10] See Goodman and Salow (2024) for a similar framework of evidential and doxastic accessibility relations in a different setting.

I. Consider the situation $c \in \mathcal{W}$, $c = \langle y, i\rangle$ (with y chosen arbitrarily). We want to determine what is believed at c, relative to parameter i and $t = 1$. As can be checked, $\boldsymbol{X}^{i,1} = \boldsymbol{X}^{i,1}_{k=1} = \{b\}$ since $\{b\} \in [A]^Q_1$ and $\pi(b) > \max_{X \in \mathcal{A}^{Q,k}_{\{b\}}} \pi(X)$ (i.e., $0.45 > 0.3$). Then it holds that $R_D(c) = \{\langle v, i\rangle \in R_E(c) \mid v \in \{b\}\}$. Then: $R_D(c) = \langle b, i\rangle$. Supposing p is the sentence 'The chosen marble is blue', so that $v(p) = \{b\}$, we get that $R_D(c) \subseteq [\![p]\!]$ since $\langle b, i\rangle \subseteq \{b\} \times I$. As such, $[\![Bp]\!]^c$ is true. (As $c \notin [\![p]\!]$, it is immediate that this belief is false.)

II. Consider, now, the situation $c' \in \mathcal{W}$, $c' = \langle b, i'\rangle$ (with b chosen arbitrarily). Intuitively, Q' is the question: 'Is the chosen marble blue or non-blue?' In this alternative situation, $\boldsymbol{X}^{i',1} = \boldsymbol{X}^{i',1}_{k=1} = \{g, r, y\}$ since $\{g, r, y\} \in [A]^Q_1$ and $\pi(g \cup r \cup y) > \max_{X \in \mathcal{A}^{Q,k}_{\{g,r,y\}}} \pi(X)$ (i.e., $0.55 > 0.45$). We get that $R_D(c') = \{\langle g, i'\rangle, \langle r, i'\rangle, \langle y, i'\rangle\}$. Since $R_D(c') \subseteq \mathcal{W} \setminus [\![p]\!]$, we get that $[\![B\neg p]\!]^{c'}$ is true.

III. Consider, finally, a slightly altered model where $t = 1.5$. Intuitively, the agent can now be seen as more cautious about her beliefs than in **I** and **II**. Take $c = \langle y, i\rangle$ again. Contrary to **I**, $\boldsymbol{X}^{i,1.5}$ is not $\{b\}$ since the condition of probabilistic 1.5-dominance is not met ($0.45 \not> 1.5 \cdot 0.3$). This equally means, moreover, that $\boldsymbol{X}^{i,1}_{k=1.5} = \emptyset$, and that $\boldsymbol{X}^{i,1.5}$ will be of rank $k > 1$. As can be checked, $\boldsymbol{X}^{i,1.5} = \boldsymbol{X}^{i,1.5}_{k=2} = \{b \cup g\}$ since $\{b \cup g\} \in [A]^Q_2$ and $\pi(b \cup g) > \max_{X \in \mathcal{A}^{Q,2}_{\{b \cup g\}}} 1.5 \cdot \pi(X)$ (i.e., $0.75 > 0.675$). Supposing q is the sentence 'The chosen marble is green', so that $v(\psi) = \{g\}$, we get that $[\![p \vee q]\!] = \{b\} \times I \cup \{g\} \times I$. As can be checked, when $t = 1.5$, $[\![B(p \vee q)]\!]^c$ is true.

This concludes our account of the class of models for weak, question-sensitive belief. The formal account of our paradigmatic example suggests that the models capture our target notion of belief. The rest of this paper pursues some predictions of our models on the score of (A) belief preservation between situations (and, specifically, QUD shifts) (Section 4) and (B) the interaction of knowledge and belief (Section 5). Besides their independent interest, such applications may also be used as further illustrations of our models.

4 The Question-Sensitivity of Belief

An important aspect of our target notion of belief is its question-sensitivity, or the notion that salient alternatives make a difference to what is believed. In the foregoing account, we cashed out question-sensitivity formally by enforcing that the belief set at a situation be relevant to the QUD. However, besides ensuring relevance in this way, we stayed largely silent on the extent to which belief is question-sensitive on our account.

A significant worry about making belief question-sensitive (or context-sensitive, broadly) is that the attitude thereby captured is too sensitive to extraneous parameters. Broadly speaking, one may worry that many beliefs, on our account, do not get preserved once the question-parameter is shifted. It is thus important to survey some predictions of our account on this score and consider whether they are motivated.

4.1 Stage Setting

We pursue this issue by considering whether beliefs are preserved between situations when the QUD parameter is shifted. As it stands, the issue is not clearly stated. (A) How do the situations under consideration relate? (B) What does it mean for a QUD parameter to be shifted, and which kind of shifts do we consider? (C) Finally, when considering whether beliefs are preserved, which beliefs are relevant? We provide clarifications on these questions in order.

(A) In considering whether beliefs are preserved between situations, we should specify how the situations relate. It is implausible to expect, for instance, that situations that are entirely evidentially dissimilar relate non-trivially. We should thus identify a criterion of similarity between situations under comparison. Here's one way of doing this. Note that, for any body of evidence (in our sense), we may distinguish between evidence *proper*, which includes a set of live possibilities and a measure of uncertainty, and *zetetic* 'evidence', which includes the QUD. The main criterion of similarity we propose is equivalence of evidence *proper*, which we define as follows:

Definition 6. Situations c and c are INFORMATIONALLY EQUIVALENT ($c \approx_{inf} c'$) if $s = s'$ and $\pi(\cdot) = \pi'(\cdot)$.

Thus, when considering whether beliefs are preserved between situations, we chiefly mean situations that are *informationally equivalent*.

Although informational equivalence is one natural way in which situations are similar, we also consider another. Intuitively, it is possible for situation c' to exclude more possibilities than c, and yet that ask 'the same' question as c (albeit with respect to a reduced set of live possibilities) and have the same judgments of comparative uncertainty as c. Suppose one is in situation c, receives evidence X (that is consistent with the set of live possibilities), and updates on X. Although the initial situation c and the updated situation c_X are informationally non-equivalent, they are similar in the sense that one can be derived from the other by updating. We propose to call situation c_X the X-reduct of c.

We capture this formally as follows:

Definition 7. REDUCTS. Where $X \subseteq s$,[11]

1. s' is an X-reduct of s if $s' = s_X = s \cap X$.
2. $\pi'(\cdot)$ is an X-reduct of $\pi(\cdot)$ if $\pi'(\cdot) = \pi(\cdot \mid X) = \frac{\pi(\ \cdot\ \cap X)}{\pi(X)}$.

[11] We restrict the definitions to s rather than W since we only consider consistent (i.e., non-empty) reducts.

3. Q' is an X-reduct of Q if $Q' = Q_X = \{Y \in Q \mid Y \cap X \text{ and } Y \cap X \neq \emptyset\}$.
4. i' is an X-reduct of $i = \langle s, \pi, Q\rangle$ if $i' = i_X = \langle s_X, \pi(\cdot \mid X), Q_X\rangle$.
5. where $c = \langle w, i\rangle$, c' is an X-reduct of c if $c' = c_X = \langle v, i_X\rangle$.

The second criterion of similarity we consider, then, is one situation being an X-reduct of another.[12]

This answers the question of how the situations under consideration relate. When considering whether beliefs are preserved between situations, we only consider whether they are preserved between:

- situations c and c' such that $c \approx_{inf} c'$, and
- a situation c and any of its reducts c_X.

In brief, we discuss the preservation of beliefs under informational equivalence and reductions.

(B) We now proceed to specify what QUD shifts amount to, as well as some restrictions on QUD shifts under consideration. The first thing to note is that QUD shifts should not bring to mind dynamic notions akin to those from belief revision. While we may loosely speak of 'shifting' the QUD from one situation to another, our models are strictly speaking static. A QUD shift is thus best seen as a pair of situations $\langle c, c'\rangle$, in which c specifies the QUD 'prior' to the shift, and c' specifies the QUD 'after' the shift. When asking whether beliefs are preserved relative to shifts, we are asking about the relation between the belief set at c and the belief set at c'.

In line with the discussion above, we can distinguish

- a shift $\langle c, c'\rangle$ where $c \approx_{inf} c'$, and
- a shift $\langle c, c'\rangle$ where $c' = c_X$ for some reduct of c.

The former shift corresponds to changing the QUD while not acquiring new information, and the latter corresponds to considering 'the same' QUD after acquiring new information. Different questions about belief preservation are natural relative to these contexts. The former case allows us to consider which constraints hold between beliefs held at informationally equivalent situations. Most naturally construed, predictions on this score concern the

[12] A few points should be noted. First: the reducts of some c are in general non-unique, as the world parameter in any c_X is underdetermined by the definition above. In other words, $c' = \langle v, i'\rangle$ and $c'' = \langle u, i''\rangle$ may be distinct X-reducts of $c = \langle w, i\rangle$ when $v \neq u$, as long as $i' = i'' = i_X$. Second: although we used the notion of updating in introducing reducts, this should not bring to mind dynamic notions from belief revision. Although c_X may be loosely described as the result of updating c with X, our models are strictly speaking static and do not involve update or revision procedures in the object language. Third: the constraints on situations we introduced in Section 3.2 hold of a situation and its reduct(s) alike; for instance, that if $c_X = \langle w, i_X\rangle \in \mathcal{W}$, it holds that $w \in s_X$ (i.e., that $w \in X$). If a reduct c_X is loosely taken to be a result of learning X in c, this can be understood as a constraint to the effect that falsehoods cannot be *learned*.

agent's cross-question coherence. The latter case, on the other hand, allows us to consider the preservation of beliefs upon learning new information.

The last thing to consider when it comes to QUD shifts is the relation between the QUD 'prior' to the shift and 'after' the shift. In the case of reducts, the relation is relatively straightforward: for every pair $\langle c, c' \rangle$ where $c' = c_X$, the QUD at c' is the unique X-reduct of QUD Q in c. With informationally equivalent situations, the relation is less straightforward: in principle, it is possible to identify many relations between the QUDs in an arbitrary pair $\langle c, c' \rangle$ where $c \approx_{inf} c'$. For simplicity, in Section 4.3 we only focus on the relation of *refinement* – and its dual, *coarsening* – between questions in informationally equivalent situations. We define refinement and its dual as follows:

Definition 8. Where Q and Q' are partitions of s, Q REFINES Q', $Q' \sqsubseteq Q$ iff for all $X \in Q'$, $X = Y_1 \cup \cdots \cup Y_n$ for some $Y_1 \cup \cdots \cup Y_n \in Q$. If $Q' \sqsubseteq Q$, we also say that Q' COARSENS Q.[13]

The main question that we pursue, then, is whether beliefs are preserved between situations c and c' if the QUD at c refines (coarsens) the QUD at c'.

(C) Having specified the bulk of the background, we specify which kinds of belief we consider. On this score, we may be comparatively brief. With ϕ an atomic sentence or a Boolean combination thereof, we say that a belief of the form $B\phi$ is descriptive, and introspective otherwise. Descriptive beliefs are, intuitively, about the world, while introspective beliefs concern one's attitudes about the world. The following only discusses the preservation of descriptive beliefs under QUD shifts. As such, whenever we state that beliefs are (not) preserved, the intended meaning is that *descriptive* beliefs are (not) preserved. We now move onto results about belief preservation under QUD shifts as defined above.

4.2 Belief Preservation and Reducts

First, we discuss belief preservation between a situation and its reduct. Intuitively, an X-reduct of situation c is the result of updating each of c's evidential components with information X. What kind of information does X bring to a situation c? Our stipulation (Definition 7) ensures that X must induce a non-empty reduct, i.e., that X must be consistent with s. There are three ways in which such a proposition may relate to the belief set $\boldsymbol{X}^{i,t}$:

(i) X is entailed by $\boldsymbol{X}^{i,t}$ (and hence believed at c), i.e., $\boldsymbol{X}^{i,t} \subseteq X$;
(ii) X is consistent with $\boldsymbol{X}^{i,t}$, i.e., $\boldsymbol{X}^{i,t} \cap X \neq \emptyset$;
(iii) X is inconsistent with $\boldsymbol{X}^{i,t}$, i.e., $\boldsymbol{X}^{i,t} \cap X = \emptyset$;

Should one's beliefs change as a result of learning X, for any of these patterns? Here are some initial judgments. (i) When X is believed at the prior situation, it is plausible that

[13]Note that $\sqsubseteq$ induces an ordering on $\mathcal{Q}(s)$, thus forming a lattice of partitions over s with $\{\{x\} \mid x \in s\}$ as the $\mathbf{1}$ and s as the $\mathbf{0}$ (see Ellerman (2010) for an extended study). An alternative way of framing our question, which we will not explicitly pursue, is by tracking the shifts of belief sets across such lattices.

one's weak beliefs are preserved upon learning X – learning something one believes to be true should normally not change one's beliefs.[14] (ii) Going further, when X is consistent but not entailed by what one weakly believes, it is plausible that one's beliefs are preserved but that one may come to believe something stronger as a result – learning something consistent with one's weak beliefs can only strengthen one's beliefs. (iii) Finally, when X is inconsistent with one's beliefs, it is to be expected that one's beliefs will change as a result of learning. In what follows, we focus on cases (i) and (ii).[15]

First, we capture our initial judgments in formal terms and see if the principles hold on our class of models. First, when X is entailed by $\boldsymbol{X}^{i,t}$, we capture the idea that beliefs are preserved upon learning X. Where $\mathrm{Con}(\boldsymbol{X}^{i,t}) = \{Y \in S \mid \boldsymbol{X}^{i,t} \subseteq Y\}$, call the principle Trivial Information:

TI. Where $\langle c, c' \rangle$ is such that $c' = c_X$ for $X \in \mathrm{Con}(\boldsymbol{X}^{i,t})$:

(i) if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true; and
(ii) if $[\![B\phi]\!]^{c'}$ is true, then $[\![B\phi]\!]^c$ is true.

In a word, upon learning that a believed proposition is true, beliefs at c and c' are identical.

Second, when X is consistent with $\boldsymbol{X}^{i,t}$, we capture the idea that beliefs are preserved and possibly strengthened upon learning X. Where $\mathrm{Sat}(\boldsymbol{X}^{i,t}) = \{Y \in S \mid \boldsymbol{X}^{i,t} \cap Y \neq \emptyset\}$, call the principle Non-Trivial Information:

NTI. Where $\langle c, c' \rangle$ is such that $c' = c_X$ for $X \in \mathrm{Sat}(\boldsymbol{X}^{i,t})$:

(i) if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true; and
(ii) it is not the case that if $[\![B\phi]\!]^{c'}$ is true, then $[\![B\phi]\!]^c$ is true.

In a word, whatever is believed at c is believed at c' upon learning something consistent with one's beliefs, but not vice versa.

Our initial judgments above suggest that TI and NTI should both hold on a natural account of learning and updating with new information, even for weak, question-sensitive belief. We can show that (see Appendix A):

Fact 2. TI and **NTI** hold.

We submit that TI and NTI are desirable consequences of our models when it comes to belief dynamics in the context of consistent information.

Before we move on, we note a relation that provably holds on our models between the belief set at a situation c and the belief set at any of its X-reducts c_X (where X is consistent

[14] We are here putting to the side beliefs based on inductive evidence, which can provide counterexamples to this principle (see Goodman and Salow (2024)).

[15] We omit pattern (iii) from consideration since the set $\boldsymbol{X}^{i_X,t}$ will plausibly not be a function of the set $\boldsymbol{X}^{i,t}$ whenever $\boldsymbol{X}^{i,t} \cap X = \emptyset$. In other words, it is impossible to determine $\boldsymbol{X}^{i_X,t}$ solely via $\boldsymbol{X}^{i,t}$ and X. (Cf. Fact 3 below for how this can be done in patterns (i) and (ii).)

with what is believed at c). The relation in question is that of *identity* between the belief set at an X-reduct of c and the intersection of the belief set at c with X. More formally:

Fact 3. With $c = \langle w, i\rangle$ and $c' = \langle v, i'\rangle$, if $c' = c_X$ for $X \in \mathrm{Sat}(\boldsymbol{X}^{i,t})$, then $\boldsymbol{X}^{i',t} = \boldsymbol{X}^{i,t} \cap X$

Thus, as long as the information with which we update is consistent with the belief set, the belief set after learning is a simple function of the belief set before learning and the learned information.

This equality may, moreover, be seen as an analog of a consequence of AGM belief revision (Alchourrón, Gärdenfors, & Makinson, 1985; Gärdenfors, 1988) for expansive updates. On AGM, an update of a set of sentences K with A is *expansive* if K is such that $A, \neg A \notin K$; then $K * A$ (read K expanded by A) is $Cn(K \cup \{A\})$ (where $Cn(A)$ is the logical closure of A). Put in terms of propositions: an update of belief set K with A is *expansive* if $\bigcap_{\phi \in K} [\![\phi]\!] \cap [\![A]\!] \neq \emptyset$; then $K * A = \bigcap_{\phi \in K} [\![\phi]\!] \cap [\![A]\!]$. Substituting in the terms of our foregoing account, where $\boldsymbol{X}^{i,t}$ is the belief set at some situation c, Fact 3 states that the belief set at situation c_X just is $\boldsymbol{X}^{i,t} \cap X$. Fact 3 can thus be read as showing that weak, question-sensitive is well-behaved in terms of updating via expansion (i.e. updating with entailed propositions).[16]

4.3 Belief Preservation upon Coarsening and Refining

We now discuss belief preservation between informationally equivalent situations. This type of belief preservation is more variegated than in the case of reducts. To see why, note that, for any situation c with QUD Q, the QUD parameter for any X-reduct of c is the (unique) QUD reduct Q_X. On the contrary, the relation between QUDs for informationally equivalent situations is underdetermined, since the relation of informational equivalence groups together any c and c' with the same s and π, as long as their QUD parameters are in $\mathcal{Q}(s)$.

The refinement relation $\sqsubseteq$ given in Definition 8 will helps us pursue the question systematically by examining whether beliefs are preserved under (a) refinements and (b) coarsenings. Since our target notion of belief is question-sensitive, it is expected that beliefs fail to be preserved under both arbitrary refining and arbitrary coarsening. However, it is also interesting to consider whether weaker principles imposing additional restrictions on situations lead to beliefs being preserved. The bulk of this section investigates this latter question by considering which types of refinements and coarsenings suffice for belief preservation.

The first prediction of our models is that neither arbitrary refinements nor arbitrary coarsenings preserve beliefs between informationally equivalent situations. We consider the

[16] Discussion of other types of updating discussed by AGM (e.g.contraction) is a more complicated matter even on usual models of belief, and are so beyond the scope of this discussion.

following pair of statements – Belief Preservation under Refining and Belief Preservation under Coarsening:

> **BPR** Where $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$ and $Q \sqsubseteq Q'$: if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true.
>
> **BPC** Where $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$ and $Q' \sqsubseteq Q$: if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true.

BPR states that beliefs are preserved when the QUD shifts to a refined QUD, while BPC states that beliefs are preserved when the QUD shifts to a coarsened QUD. We can show that (see Appendix A):

Fact 4. BPR and BPC fail.

Although formal countermodels are reserved for Appendix A, note that failures of BPR and BPC are suggested by informal considerations of our paradigmatic example, **The Urn**. Take situation c to be one where it is believed that the chosen marble is blue relative to the question Q : 'What is the colour of the chosen marble?' and situation c' to be one where it is believed that the chosen marble is non-blue relative to the question Q': 'Is the chosen marble blue or non-blue?'. It is clear that beliefs about the marble's colour are preserved neither upon coarsening the question (i.e., when the pair is $\langle c, c' \rangle$), nor upon refining the question (i.e., when the pair is $\langle c', c \rangle$). In the former case, one loses the belief that the chosen marble is blue, while in the latter case one loses the belief that the chosen marble is non-blue.

BPR and BPC are principles that our account was arguably set up to invalidate. Different sets of relevant alternatives may induce different (and even contradictory) beliefs. Refining and coarsening the QUD results in changing the relevant alternatives, and so refining and coarsening induces different beliefs. BPR and BPC resist this by imposing belief invariance across such QUD shifts. Moreover, BPR and BPC *together* impose belief invariance across all informationally equivalent situations, since for any $Q, Q' \in \mathcal{Q}(s)$, it either holds that $Q \sqsubseteq Q'$ or that $Q' \sqsubseteq Q$. Thus, if beliefs are to be question-sensitive at all, one of BPR and BPC – and plausibly both – should fail.

Although BPR and BPC are objectionably strong for our target notion of belief, it is nevertheless interesting to see whether weaker principles can be vindicated. Such weakenings suggest, broadly speaking, that beliefs are preserved under a specific kind of refinement (coarsening).

One constraint that may be considered, for $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ such that $Q \sqsubseteq Q'$, is that the belief set $\boldsymbol{X}^{i,t}$ have the same k-rank in Q' as it does in Q. We say that:

$$X \text{ is } \textsc{rank-constant} \text{ in } \langle c, c' \rangle \text{ just in case: } n(X)^Q = i \text{ iff } n(X)^{Q'} = i$$

That X is rank-constant in $\langle c, c' \rangle$ means, roughly speaking, that it retains the informativity rank it has in Q when the QUD is shifted to Q'. When Q' refines Q, that X is rank-constant can also be read as saying that X is not itself refined by Q', or that only the alternatives to X in Q get partitioned in Q'. The proposed weakening of BPR, then, is that $\boldsymbol{X}^{i,t}$ be rank-constant in $\langle c, c' \rangle$.

Although refinements do not in general ensure rank-constancy for $\boldsymbol{X}^{i,t}$,[17] rank-constancy is a minimal and simple constraint on the relation between Q and Q'. Moreover, it appears intuitively plausible that beliefs should be preserved under refinements that preserve rank-constancy for the belief set. Consider **The Urn**, again, for illustration. Suppose the relevant question is 'Is the chosen marble (blue or green), or (red or yellow)?' ($Q_1 = \{\{b, g\}, \{r, y\}\}$), and that one believes the first answer is true ($\boldsymbol{X}^{i_1,t} = \{b, g\}$). It is plausible that, when the slightly refined question 'Is the chosen marble (blue or green), or red, or yellow?' ($Q_2 = \{\{b, g\}, \{r\}, \{y\}\}$) is posed, one should hold onto one's belief that the first answer is true ($\boldsymbol{X}^{i_2,t} = \{b, g\}$). If the first answer to Q_1 dominates the second answer, it would be unusual if it the same answer were not dominant over 'parts' of the second answer separately. In a word, refining alternatives to what is believed should not lead to belief change.

With this in mind, we may formulate the following weakening of BPR:

> **BPR$^+$** Where $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$, $Q \sqsubseteq Q'$, and $\boldsymbol{X}^{i,t}$ is rank-constant in $\langle c, c' \rangle$: if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true.

As opposed to BPR, which just ensures that the QUD Q shifts to one of its refinements Q', BPR$^+$ also ensures that the most informative probabilistically dominant answer to Q is itself not partitioned by Q'. Broadly speaking, BPR$^+$ maintains that beliefs at c are preserved upon refining, provided the refinement does not partition the belief set at c into more alternatives.

Above, we saw that the idea behind BPR$^+$ has some intuitive plausibility, since refining alternatives to what is believed should not lead to belief change. For symmetry, we also consider a weakening of BPC, where $\boldsymbol{X}^{i,t}$ is rank-constant between a partition and *its coarsening*:

> **BPC$^+$** Where $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$, $Q' \sqsubseteq Q$, and $\boldsymbol{X}^{i,t}$ is rank-constant: if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true.

[17] Consider the pair $Q = \{\{a, b, c\}, \{d\}\}$ and $Q' = \{\{a\}, \{b, c\}, \{d\}\}$. Clearly, $Q \sqsubseteq Q'$. Suppose $\boldsymbol{X}^{i,t} = \{a, b, c\}$ at some c. As can be checked, $n(\boldsymbol{X}^{i,t})^Q = 1$ and $n(\boldsymbol{X}^{i,t})^{Q'} = 2$. As such, Q' is a refinement of Q that does not ensure rank-constancy for the belief set at c.

Is BPC$^+$ intuitively plausible? We suggest not. The plausibility of BPR$^+$ stems from the idea that *refining* just the alternatives does not lead to belief change since subsets of dominated alternatives are dominated whenever the set is (by standard probability theory). However, the converse reasoning does not hold: if some alternatives are separately dominated by the belief set, there is no guarantee that their union is not dominant over the belief set.[18]

Appealing to our models, we can show that (see Appendix A):

Fact 5. BPR$^+$ holds, while BPC$^+$ fails.

As such, ensuring that the belief set at c is rank-constant upon refining yields the result that beliefs are preserved, while the same condition for coarsening fails to preserve beliefs. In brief: while rank-constant refining ensures the preservation of beliefs, rank-constant coarsening does not.

Rank-constancy for the belief set in $\langle c, c' \rangle$ is thus sufficient for the preservation of beliefs upon refining, but not sufficient for the preservation of beliefs upon coarsening. We saw above that belief fails to be preserved under coarsening because the aggregate of alternatives to the belief set may be more probable than the belief set, even when the latter is more probable than each of the aggregated alternatives alone. In such a situation one may, particularly, believe the negation of what one believed with respect to the (relatively) finer question (cf. **The Urn**). Is there a class of coarsenings that excludes such belief reversals?

A simple way of ensuring that beliefs are not reversed under coarsening, we suggest, is to ensure that the belief set itself is included in the (only) coarsened cell when the QUD is shifted, or that sets that are disjoint from the belief set are not aggregated when the QUD is shifted.

This suggestion deserves some unpacking. We focus on an arbitrary pair $\langle c, c' \rangle$ and $\langle Q, Q' \rangle$ (with $Q' \sqsubseteq Q$) for ease of exposition. The target condition ensures that if a complete Q-answer that is disjoint from $\boldsymbol{X}^{i,t}$ is also in the coarsened question Q', then its rank in Q' is the same as its rank in Q. Alternatively, whatever coarsening Q' brings with respect to Q, the condition ensures (i) that the coarsened cell in Q' is either a superset of $\boldsymbol{X}^{i,t}$ (when $\boldsymbol{X}^{i,t}$ is of rank 1) or a subset of $\boldsymbol{X}^{i,t}$ (when $\boldsymbol{X}^{i,t}$ is of rank > 1); and (ii) that all the sets disjoint from $\boldsymbol{X}^{i,t}$ that are also in Q' are not aggregated in Q'. The proposed condition thus delineates a class of coarsenings of Q in which $\boldsymbol{X}^{i,t}$ must either be included in the coarsened cell of Q' or include a coarsened cell, and no set disjoint from $\boldsymbol{X}^{i,t}$ may be included in a disjoint coarsened cell of Q'.

[18]For illustration, take the two questions considered above (we use formal terms for brevity here). Suppose the relevant question is Q_2 and that $\boldsymbol{X}^{i_2,t} = \{b, g\}$. Contrary to the case above, however, there is no guarantee that $\{b, g\}$ will probabilistically dominate $\{r, y\}$ whenever it dominates $\{r\}$ and $\{y\}$ separately.

A schematic example may illustrate the difference between coarsenings that meet the target condition and those that do not. Consider set $s = \{a, b, c, d, e\}$ and partition $Q = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$, and suppose that $\boldsymbol{X}^{i,t} = \{a, b\}$ relative to Q. Consider two coarsenings of Q:

- $Q_1 = \{\{a, b, e\}, \{c\}, \{d\}\}$
- $Q_2 = \{\{a, b, e\}, \{c, d\}\}$

It can be checked that, while Q_1 meets the target condition, Q_2 does not. To see this, note that all complete answers to Q that are disjoint from $\boldsymbol{X}^{i,t}$ preserve their ranks provided they are in Q_1; in other words, $\{c\}$ and $\{d\}$ are both disjoint from $\boldsymbol{X}^{i,t}$ and their rank $= 1$ in both Q and Q_1. On the contrary, Q_2 does not meet this condition: $\{c, d\}$ is disjoint from $\boldsymbol{X}^{i,t}$, and yet $n(\{c, d\})^Q = 2 \neq 1 = n(\{c, d\})^{Q_2}$.

With all this in mind we may formulate the weakening of BPC that prevents belief reversals:

> **BPC^{++}** Where $\langle c, c' \rangle$ with QUDs $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$, $Q' \sqsubseteq Q$, and any Y such that $Y \in S(Q')$ and $Y \subseteq \neg\boldsymbol{X}^{i,t}$ is rank-constant: if $[\![B\phi]\!]^c$ is true, then $[\![\neg B \neg \phi]\!]^{c'}$ is true.

Appealing to our models, we can indeed show that (see Appendix A):

Fact 6. BPC^{++} holds.

As such, ensuring that all subsets of the complement of the belief set at c be rank-constant upon coarsening yields the result that beliefs cannot be reversed in the sense that ϕ is believed relative to Q and $\neg\phi$ is believed relative to Q'.

In sum: our models predict that informationally equivalent situations may fail to preserve beliefs under both arbitrary refining and coarsening, but that additional constraints nevertheless ensure more continuity. In both cases, specifically, we established that species of refinement and coarsening establish belief preservation and non-reversal, respectively.

There are significant parallels between the weakened principles we explored. BPR^{+} and BPC^{++} both impose conditions on $\langle Q, Q' \rangle$ as a function of the first parameter in $\langle c, c' \rangle$; as for the former, the condition is that Q' be a refinement that respects the informativity rank of $\boldsymbol{X}^{i,t}$; as for the latter, the condition is that Q' be a coarsening that respects the informativity rank of any subset of $\neg\boldsymbol{X}^{i,t}$. In case of refinement, belief preservation is ensured since increasing the number of alternatives to what is believed only decreases the probability of the alternatives. In the case of coarsening, belief non-reversal is ensured since aggregating over answers one believes only increases their probability.

4.4 Beliefs and Closure under Conjunction

One putative prediction of theories of weak, question-sensitive belief is that belief is not closed under conjunction. Lottery scenarios are standard illustrations of this prediction. For any ticket t_i in some lottery, the polar question 'Did t_i win?' induces a partition into the proposition that t_i won and the proposition that t_i did not win. It is commonly thought that, for any t_i, it is rationally permissible to believe that t_i did not win relative to its corresponding polar question. A conjunction of the propositions that t_i did not win for all i in some I, however, is equivalent to the proposition that no ticket won, and this proposition is not believed. Thus, on pain of inconsistency, the conjunction of beliefs for each ticket t_i in I is not predicted to be believed, and so belief is not closed under conjunction.

Suppose briefly that belief sets are sets of sentences and that an agent believes whichever propositions are expressed by the sentences in her belief set. Although not specified formally, the reasoning above arguably assumes that, for any propositions p and q believed relative to questions Q and Q', respectively, sentences 'P' and 'Q' are added to *the same* belief set. The failure of closure under conjunction consists in the fact that 'P' and 'Q' are in some belief set, and yet the conjunctive sentence '$P \wedge Q$' is not. In brief, the assumption is that, while propositions are believed relative to QUDs, the agent's underlying belief set 'equalises' between these beliefs by 'forgetting' the QUDs relative to which they are held.

This assumption should be made explicit when discussing closure under conjunction, since the type of belief set employed determines whether closure holds. To see this, note the distinction between a QUD-insensitive and a QUD-sensitive belief set. For the former, although beliefs depend on contextually salient QUDs, the sum of what is believed is presented by a QUD-insensitive belief set. For the latter, beliefs equally depend on contextually salient QUDs, but the sum of what is believed is constrained to some QUD, in the sense that the QUD is an independent parameter in specifying the belief set.

To formally capture a QUD-sensitive belief set, we need not go further than our foregoing account. The belief set $\boldsymbol{X}^{i,t}$ at situation $c = \langle w, i \rangle$ is (in part) a function of the Q specified in i; moreover, it is possible that $\boldsymbol{X}^{i,t} \neq \boldsymbol{X}^{i',t}$ for situations c and c' such that $c \approx_{inf} c'$. But if the belief set is a function of the QUD, and different QUDs induce different belief sets even between informationally equivalent situations, then the notion of belief set we employ is highly QUD-dependent.

To capture the notion of QUD-insensitivity, we can employ our notion of informational equivalence. With $\mathcal{W}$ the set of situations as above, $[c]_{\approx_{inf}}$ is the equivalence class for any c induced by $\approx_{inf}$, and $\mathcal{W} \backslash \approx_{inf}$ is the resulting quotient set. Now, take a set E, corresponding to 'purely evidential' situations, such that $f : \mathcal{W} \backslash \approx_{inf} \longrightarrow E$ is a bijection. For any $c \in [c]_{\approx_{inf}}$, let $f([c]_{\approx_{inf}}) = e$ be called c's evidential situation. We can then specify the belief set at some purely evidential situation, $\boldsymbol{X}^e$, as the strongest proposition believed at all situations $c \in [c]_{\approx_{inf}}$. Where $\mathrm{Con}(\boldsymbol{X}^{i,t}) = \{Y \in S \mid \boldsymbol{X}^{i,t} \subseteq Y\}$, and each $\boldsymbol{X}^{i,t}$ corresponds to some c as above, we write:

$$\boldsymbol{X}^e = \bigcap_{c\ \in\ [c]_{inf}} Cn(\boldsymbol{X}^{i,t})$$

The difference between the belief set at a situation c and a belief set at a purely evidential situation $f([c]_{\approx_{inf}})$ allows us to formulate two principles of Closure under Conjunction:

CC If $[\![B\phi]\!]^c$ and $[\![B\psi]\!]^c$ is true, then $[\![B(\phi \wedge \psi)]\!]^c$ is true.

CC$^+$ Where $c \approx_{inf} c'$ and $f([c]_{\approx_{inf}}) = e$, if $[\![B\phi]\!]^c$ is true and $[\![B\psi]\!]^{c'}$ is true, then $\boldsymbol{X}^e \subseteq [\![\phi \wedge \psi]\!]$.

CC just states that beliefs are closed under conjunction relative to some situation c, i.e., relative to a QUD-sensitive belief set determined at c. CC$^+$ is the stronger principle that beliefs are closed under conjunction relative to informationally equivalent situations, in the sense that the QUD-insensitive belief set $\boldsymbol{X}^e$ entails the conjunction of any two propositions believed at any situations c, c' such that $c \approx_{inf} c'$. We can show that (see Appendix A):

Fact 7. CC holds, while CC$^+$ fails.

In a sense, both halves of this result are unsurprising. First, note why CC holds: a belief set X at a situation is modelled as a set of worlds, and for ϕ to be believed just is for ϕ to be entailed by the evidentially accessible situations whose worlds make X true; thus, if any ϕ and ψ are so entailed, it is set-theoretically immediate that $\phi \wedge \psi$ is also so entailed, guaranteeing closure under conjunction (in the sense of CC).

Second, note why CC$^+$ fails: the principle requires, for any two ϕ and ψ believed at (possibly) different situations c, c', that ϕ *and* ψ be entailed by *all* informationally equivalent situations; this just amounts to the requirement that all situations in some $[c]_{\approx_{inf}}$ have the same belief set; however, since this has immediate failures on our models, closure under conjunction (in the sense of CC$^+$) is not guaranteed.

The result allows us to draw more interesting conclusions about closure under conjunction, however. It may be assumed that closure under conjunction is the (informal) principle:

If ϕ is believed and ψ is believed, $\phi \wedge \psi$ is believed.

This formulation is underspecified, however, since the parameter relevant for belief is left tacit. Once we formalise and make the parameter explicit – i.e., specify whether the belief set is QUD-sensitive or -insensitive – the resulting principles are non-trivially different.

What Fact 7 suggests is that, when question-sensitive theories of belief are imputed with invalidating closure under conjunction, what they actually invalidate is a principle stronger than CC, and arguably a principle akin to CC$^+$. But CC$^+$, we showed, imposes a constraint to the effect that the belief sets of any two informationally equivalent situations are equivalent.

Since informationally equivalent situations differ at most as to the QUD parameter, this constraint amounts to the QUD parameter being irrelevant for determining the belief set at a situation. However, this is clearly not meant to be a consequence of question-sensitive theories of belief. Thus, CC^+ is objectionably strong for our target notion of belief.

4.5 Overview

This concludes our discussion of (descriptive) belief dynamics in the context of question-sensitivity. Question-sensitive accounts of belief are often thought to make the attitudes of rational agents overly sensitive to questions, in the sense that beliefs fail to be preserved across different questions. A formal treatment of these issues is needed, however, in order to adequately approach the question of whether, and to which extent, failures of preservation are problematic.

The foregoing account made headway on this issue by outlining some predictions on cross-question belief dynamics in the context of learning new information (4.2), changing the relevant QUD (4.3), and conjoining beliefs (4.4). Summarising:

(1) Learning new information (Fact 2).

  a. beliefs are preserved upon learning that a believed propositions is true
  b. beliefs may be strengthened upon learning that a consistent proposition is true

(2) Refining and Coarsening.

  a. beliefs are not in general preserved under refining and coarsening (Fact 4)
  b. alternative-increasing refinements ensure belief preservation (Fact 5), and
  c. alternative-stable coarsenings ensure belief consistency (Fact 6).

(3) Closure under conjunction.

  a. beliefs are closed under conjunction relative to a single situation (Fact 7);
  b. beliefs fail to be closed under conjunction relative to any two informationally equivalent situations (Fact 7);

We note some general points:

STRONG PRINCIPLES. As our discussion above indicates, the failure of strong principles is unsurprising in our setting. Specifically, since BPR, BPC, CC^+ impose the constraint that informationally equivalent situations have the same belief set, and this in turn amounts to QUD-insensitivity. (2a) and (3b) thus presumably hold on any question-sensitive account of belief. Otherwise put, if we are to accept question-sensitivity at all, such principles as BPR, BPC, CC^+ should fail.

WEAKENINGS. Although principles implying QUD-insensitivity fail, this is not to say that plausible weakenings of such principles fail to hold. Specifically, we singled out principles

BPR^+ and BPC^{++}, and showed that they hold. We submit that, in so far as some belief preservation is desirable even in our setting, these principles are plausible candidates for weaker forms of preservation under refining, coarsening, and conjunctive closure. Further work may show, of course, that other plausible weakenings of BPR, BPC, CC^+ also hold.

LEARNING AND UPDATING. Although our target notion is seldom discussed in the context of learning new information, our setting allowed us to consider standard questions about learning consistent information. Our discussion shows that the belief dynamics is as expected: learning believed propositions preserves descriptive beliefs, and learning consistent propositions can strengthen beliefs.

5 Knowledge and Belief

Knowledge and belief are non-trivially related. Some relations that can be proposed include: knowing that p implies believing that p, believing that p implies believing that one knows p, and so on. The foregoing semantics allows us to formally study whether such claims hold of weak, question-sensitive belief. This section minimally extends the foregoing semantics to capture knowledge and then goes on to outline some predictions regarding the interaction of knowledge and weak, question-sensitive belief.

A prominent strategy for accounting for the relation between knowledge and belief in a formal setting is the one proposed by Stalnaker, on which belief is reduced to knowledge (Baltag, Bezhanishvili, Özgün, & Smets, 2019; Stalnaker, 2006). Stalnaker defends a unimodal logic with an operator K for knowledge and proposes that the operator B for belief be defined as follows (where $\langle K \rangle$ is K's dual):

$$B\phi \leftrightarrow \langle K \rangle K\phi \tag{$*$}$$

Since $\langle K \rangle \phi$ intuitively corresponds to it being open that ϕ (in the sense that there is at least one epistemically accessible world at which ϕ holds), believing ϕ corresponds to it being open that one knows ϕ. Contraposing, we get that $\neg B\phi \leftrightarrow K\neg K\phi$. Roughly speaking, knowledge of ignorance suffices to defeat belief.

It can be shown that $(*)$ is a theorem of Stalnaker's bimodal logic (with operators K and B),[19] which is meant to capture properties commonly attributed to knowledge and belief. Particularly, the logic is meant to capture a 'strong' notion of belief on which belief amounts to "subjective certainty" (p. 179). The logic can be characterised by the following table:

Substituting each occurrence of B in the axioms in Table 1 with $\langle K \rangle K$ results in a unimodal account of belief via knowledge. Substituting B with $\langle K \rangle K$ in **CB** in particular results in the modal axiom **0.2**: $\langle K \rangle K\phi \rightarrow K\langle K \rangle \phi$. Adding **0.2** to **S4** (i.e., **K + T + 4**) results in

[19]For a proof of $(*)$ in **KD45**, see (Ozgün, 2013, p. 28)

Abbr.	Axiom	Condition
K	$\vdash K(\phi \to \psi) \to (K\phi \to K\psi)$	K for K
T	$\vdash K\phi \to \phi$	Factivity
4	$\vdash K\phi \to KK\phi$	Pos. introspection for K
PI	$\vdash B\phi \to KB\phi$	Pos. introspection for B
NI	$\vdash \neg B\phi \to K\neg B\phi$	Neg. introspection for B
KB	$\vdash K\phi \to B\phi$	Knowledge implies belief
CB	$\vdash B\phi \to \neg B\neg\phi$	Consistency of belief
SB	$\vdash B\phi \to BK\phi$	Strong belief

Table 1: Stalnaker's Logic for K and B (**KD45**)

S4.2. Substituting B with $\langle K\rangle K$ in all the other axioms results in theorems of **S4.2**. Thus, Stalnaker accounts for belief within the **S4.2** logic for knowledge, where $B\phi$ is defined as $\langle K\rangle K\phi$.

Even a cursory inspection of Table 1 suggests that Stalnaker's proposed axioms are too strong for our target notion of belief. Indeed, the target notion for Stalnaker is that of *strong* belief, in the sense that belief is tantamount to subjective certainty. However, our target notion does *not* imply subjective certainty. The particularly objectionable axiom, from this perspective, is the axiom of 'Strong Belief', which intuitively states that whatever is believed is believed to be known. Weak, question-sensitive belief should invalidate this implication. **The Urn** is an intuitive illustration of this: although believing that the marble is blue relative to the question 'What is the colour of the marble?' is permissible, this belief is consistent with believing that one is ignorant about the colour of the marble.

Although 'Strong Belief' might have intuitive failures in our setting, other Stalnakerian axioms governing the interaction of knowledge and belief might hold even on our semantics for weak, question-sensitive belief. It is thus interesting to study the interaction more formally in the foregoing setting.

Let L^+ be just like L except for the addition of an operator K for knowledge. Let the semantics be as above, and let:

$$[\![K\phi]\!] = \{\langle w, i\rangle \in \mathcal{W} \mid R_E(\langle w, i\rangle) \subseteq [\![\phi]\!]\}$$

Intuitively, ϕ is known in situation c just in case every situation evidentially accessed by c is a situation in which ϕ holds. The Stalnakerian axioms of particular interest for studying (a) the properties of belief and (b) the interaction of knowledge and belief are (PI), (NI), (KB), (CB), and (SB). Although these are introduced as axiom schemas in Table 1, we may study the following semantic analogs in our setting:

$$B\phi \models_{\mathcal{M}} KB\phi \tag{PI}$$

$$\neg B\phi \models_{\mathcal{M}} K\neg B\phi \tag{NI}$$

$$K\phi \models_{\mathcal{M}} B\phi \tag{KB}$$

$$B\phi \models_{\mathcal{M}} \neg B\neg\phi \tag{CB}$$

$$B\phi \models_{\mathcal{M}} BK\phi \tag{SB}$$

It can be shown that, in our models, (PI), (NI), (KB) and (CB) hold, while (SB) fails. For the positive: when one (dis)believes that p, one knows that one (dis)believes that p (Facts 2, 3); everything known is believed (Fact 4); it is not the case that both p and $\neg p$ are believed in one situation (Fact 5). For the negative: believing p does *not* imply believing that one knows p (Fact 7). Furthermore, although belief is consistent within a situation, it can be cross-situation inconsistent, even when the situations differ at most with respect to which QUD is selected in i (Fact 6). The proofs of these Facts are reserved for Appendix B.

Most immediately, this shows that weak, question-sensitive belief, as treated in Section 3, is not as formally ill-behaved in its relation to knowledge as it might be thought. In fact, all but one of the entailments motivating Stalnaker's formal account of the interaction of knowledge and belief hold on the foregoing semantics. The semantics invalidates only the analog of the axiom of Strong Belief, on which believing that p entails believing that one knows p, and validates analogs of the other axioms governing the interaction of knowledge and belief. As such, at least with respect to this interaction, the foregoing proposal is a conservative restriction of one prominent treatment of belief in epistemic logic.

This conservative restriction also has philosophical motivation on the foregoing proposal. Our main goal in this paper was to develop formal models of rational belief that bear out the predictions of our guiding claims (as surveyed in Section 2). The formal models should not, nevertheless, overgenerate by making predictions that are not motivated or entailed by the theses that belief is weak and question-sensitive.

This may be illustrated in two ways. Here's one. While we took it as a *desideratum* that belief be consistent with ignorance, this *desideratum* is itself plausibly consistent with knowing what one's beliefs are and believing everything one knows. In a word: although beliefs may be uncertain, it does not immediately follow that one is uncertain about one's beliefs or that there is a significant disconnect between what one knows and believes. If this is so, the formal models should not predict failures of principles over and above those that immediately conflict with the consistency of belief and ignorance. In this regard, our predictions (cf. Facts 2, 3, 4, and 7 in Appendix B) strike a good balance: while belief is consistent with ignorance, it is well-behaved in other respects.

Here's another. A second *desideratum* for our account was that belief be question-sensitive. An important and oft-contested consequence of question-sensitivity is that, with respect to the same evidence, an agent can believe p and not-p as a function of how the space of possibilities is structured. Given this paper's exploratory spirit, here is not the place to adjudicate whether such inconsistency is permissible, or to discuss the ways it can be made less glaring (see Borgoni, Kindermann, and Onofri (2021); Lewis (1982) for some answers on the latter score). It should be noted, nevertheless, that even proponents of question-sensitivity are not prepared to accept inconsistency with respect to a single question, but rather locate inconsistency in *cross*-question inconsistency. With this in mind, it is clear that successful formal models should predict inconsistency only when questions are not held fixed. In this regard, our predictions (cf. Facts 5, 6 in Appendix B) equally strike a good balance: permissible belief is single-question consistent, and *may* be cross-question inconsistent.

Before concluding, we consider the issue of a reductive account of belief. One motivation for adopting **KD45** is its promise of a reductive account of belief in terms of knowledge via the equivalence $B\phi \leftrightarrow \langle K\rangle K\phi$ $(*)$. A proof of $(*)$ in **KD45** due to (Ozgün, 2013, p. 28) shows that:

- deriving $B\phi \rightarrow \langle K\rangle K\phi$ involves SB, and
- deriving $\langle K\rangle K\phi \rightarrow B\phi$ does not involve SB

Consider the following semantic analogs of the two directions of the biconditional:

$$B\phi \models_{\mathcal{M}} \langle K\rangle K\phi \tag{$\Rightarrow *$}$$

$$\langle K\rangle K\phi \models_{\mathcal{M}} B\phi \tag{$\Leftarrow *$}$$

Given that our semantics invalidates (SB), it is to be expected that $(\Rightarrow *)$ fails and that $(\Leftarrow *)$ holds. This can indeed be shown (Facts 8, 9 in Appendix B). Intuitively put, this means that (a) believing p is compatible with knowing that one is ignorant of p, but that (b) if one leaves it open that one knows p, one believes that p. From a formal perspective, since only $(\Rightarrow *)$ requires assuming that belief is strong, our semantics provides a principled departure from $(*)$ in retaining only $(\Leftarrow *)$.

It is useful to put these predictions into context. Above, we showed that weakly believing p does not imply believing that one knows p. This result was taken to formally track the claim that believing p is compatible with being ignorant about p or lacking subjective certainty that p. The fact that $(\Rightarrow *)$ also fails shows that, besides being compatible with ignorance, belief is compatible with *knowledge of* ignorance. This failure is, moreover, philosophically motivated: in many cases where one forms beliefs by guessing, one equally knows that one's guess does not amount to knowledge (i.e., that one is ignorant).

$(\Leftarrow *)$, on the other hand, is not as intuitively problematic as its converse, since it only indicates that p is believed whenever there is an accessible situation at which p is known. Given that one's evidence is here taken to be transparent, it is clear that accessing such a situation should indicate that one's own situation is the same. In this sense, if there is a prospect of knowing p, one should also take oneself to know, and thereby believe p as well.

As such, $[\![B\phi]\!]$ and $[\![\langle K\rangle K\phi]\!]$ are not generally equivalent. The foregoing suggests, moreover, that a reductive account of weak, question-sensitive belief in terms of knowledge is not forthcoming on our account. This can be informally motivated by considering the fact that believing p at a situation does not in general entail having any knowledge-related attitude towards p: as we saw, believing p is compatible both with (i.a) knowing that p and (i.b) being ignorant of p, with (ii) believing one does not know p and (ii.b) believing that one knows p, and finally both with (iii.a) knowing that one is ignorant of p and (iii.a) not knowing that one is ignorant of p. Although a reductive account of strong belief is possible, then, the same is plausibly not true for weak, question-sensitive belief.

6 Conclusion

This paper proposed a semantic account of weak, question-sensitive belief. We introduced a class of models in which the points of evaluation are situations, or world-evidence pairs, with evidence understood liberally to include sets of live possibilities, measures of uncertainty, and QUDs. A proposition is believed at a situation iff it is implied by the most informative probabilistically dominant answer to the QUD, on some way of specifying the threshold of probabilistic dominance.

We also argued that the class of models can be fruitfully applied to epistemological issues, such as belief dynamics and the interaction of knowledge and belief. On the former score, we considered whether descriptive beliefs are preserved between informationally equivalent situations. Although beliefs do not obey conjunctive closure and fail to be preserved under arbitrary refinements and coarsenings, we showed that plausible weakenings of some considered principles hold in our setting.

On the latter score, we considered which principles governing the interaction of knowledge and belief in Stalnaker's **KD45** hold also in our setting. On our models, belief is positively and negatively introspective, implied by knowledge, and consistent (albeit only with respect to a single situation); furthermore, believing that p is consistent both with not believing that one knows p and with knowing that one is ignorant of p.

Before concluding, we note one direction in which the account might be extended. Agents typically hold beliefs not only about how the world is, but also about how the world is conditional on some supposition. Such beliefs are usually reported via conditional beliefs ('If ϕ, I {believe}{think} that ψ') and beliefs in conditionals ('I {believe}{think} that if ϕ, then ψ'). The hypothesis that beliefs *simpliciter* are weak and question-sensitive naturally extends to beliefs in conditionals and conditional beliefs (see Pearson (2024)). Immediate extensions of the foregoing proposal cannot account for this extended class of beliefs, notably due to issues with probabilities of conditionals and their non truth-conditional behaviour. As such, accounting for this extended class of beliefs would plausibly require departures from our proposal. Nevertheless, the paper's core proposal might still serve as a basis for such extensions in future work.

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A The Question-Sensitivity of Belief

Fact 2. **TI** and **NTI** hold.

TI holds; (i) and (ii).

Proof. We show that $[\![B\phi]\!]^c$ is true iff $[\![B\phi]\!]^{c'}$ is true by showing that $\boldsymbol{X}^{i_X,t} = \boldsymbol{X}^{i,t}$. (To see that this implication holds, note that whenever $[\![B\phi]\!]^c$ is true, it holds that $R_D(c) \subseteq [\![\phi]\!]$, and that for any descriptive ϕ, $[\![\phi]\!] = \boldsymbol{\phi} \times I$. For a set of situations $X \times \{i\}$ to entail a descriptive ϕ, it suffices that $\{w \in W \mid \langle w, i\rangle \in X\} \subseteq \boldsymbol{\phi}$. If it holds that $\boldsymbol{X}^{i_X,t} \subseteq \boldsymbol{X}^{i,t}$, then $\{w \in W \mid \langle w, i\rangle \in \boldsymbol{X}^{i_X,t}\} \subseteq \{w \in W \mid \langle w, i\rangle \in \boldsymbol{X}^{i,t}\}$, and so if $\{w \in W \mid \langle w, i\rangle \in \boldsymbol{X}^{i,t}\} \subseteq [\![\phi]\!]$, then $\{w \in W \mid \langle w, i\rangle \in \boldsymbol{X}^{i_X,t}\} \subseteq [\![\phi]\!]$. The reasoning is parallel in the other direction.) Since $\text{Con}(\boldsymbol{X}^{i,t}) \subseteq \text{Sat}(\boldsymbol{X}^{i,t})$, from Fact 3 it follows that $\boldsymbol{X}^{i_X,t} = \boldsymbol{X}^{i,t} \cap X$, and from the fact that $X \in \text{Con}(\boldsymbol{X}^{i,t})$ it follows that $\boldsymbol{X}^{i_X,t} = \boldsymbol{X}^{i,t}$. But then: $\boldsymbol{X}^{i_X,t} \subseteq \boldsymbol{X}^{i,t}$. Hence, if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$. □

NTI holds.

(i) *Proof.* The proof is parallel to the proof of TI above: we show that if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true by showing that $\boldsymbol{X}^{i_X,t} \subseteq \boldsymbol{X}^{i,t}$. Since $X \in \text{Sat}(\boldsymbol{X}^{i,t})$, it holds by Fact 3 that $\boldsymbol{X}^{i_X,t} = \boldsymbol{X}^{i,t} \cap X \subseteq \boldsymbol{X}^{i,t}$. Hence, if $[\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$. □

(ii) We show that it is not the case that if $[\![B\phi]\!]^{c'}$ is true, then $[\![B\phi]\!]^c$ is true by providing a **countermodel** in which $[\![Bp]\!]^{c'}$ is true and $[\![Bp]\!]^c$ is false for some p. **The Urn** can be extended into such a countermodel. Take situation $c = \langle b, i\rangle$ and let $t = 1.5$ as in **The Urn** (Model) **III**. We have that $\boldsymbol{X}^{i,1.5} = \{b \cup g\}$. Since $\{b\} \in \text{Sat}(\boldsymbol{X}^{i,t})$, we may consider a b-reduct of c: $c_b = \langle b, \langle s_b, \pi(\cdot \mid b), Q_b\rangle\rangle$. As can be checked, $\boldsymbol{X}^{i_X,1.5} = \{b\}$; with $[\![p]\!] = \{b\}$, it holds that $[\![Bp]\!]^{c_b}$ is true and $[\![Bp]\!]^c$ is false.

Fact 3. With $c = \langle w, i\rangle$ and $c' = \langle v, i'\rangle$, if $c' = c_X$ for $X \in \text{Sat}(\boldsymbol{X}^{i,t})$, then $\boldsymbol{X}^{i',t} = \boldsymbol{X}^{i,t} \cap X$

Proof. We go over cases for determining the value of $\boldsymbol{X}^{i',t}$:

(i) First, the case where $\boldsymbol{X}^{i,t} = \{X_i \in [A]^Q_k \mid \pi(X_i) > \max_{X_j \in \mathcal{A}^{Q,k}_{X_i}} t \cdot \pi(X_j)\} \neq \emptyset$ for some k. Let $k = n$ be the least such k, and denote the corresponding set with $\boldsymbol{X}^{i,t}_n$. Then we have that $\boldsymbol{X}^{i,t}_j = \emptyset$ for all $j < n$, i.e., for all Y of rank $j < n$, $\pi(Y) \not> \max_{Z \in \mathcal{A}^{Q,j}_Y} t \cdot \pi(Z)$. (a) We first check that all answers to Q_X of rank $j < k$ also meet this condition. Since $\pi(\cdot \mid X) = \frac{1}{\pi(W \setminus X)} \cdot \pi(\cdot)$, we have that no set $Y \cap X$ of rank j in Q_X is such that $\pi(Y \cap X \mid X) > \max_{Z \in \mathcal{A}^{Q_X,j}_{Y \cap X}} t \cdot \pi(Z)$. But then $\boldsymbol{X}^{i',t} = \emptyset$ for all $j < n$ in Q_X. (b) Next, verify that the answer $\boldsymbol{X}^{i,t}_n \cap X$ is such that $\pi(\boldsymbol{X}^{i,t}_n \cap X \mid X) > \max_{Z \in \mathcal{A}^{Q_X,k}_{\boldsymbol{X}^{i,t}_n \cap X}} t \cdot \pi(Z \mid X)$. For the same reason as in (a), this condition is immediately met. But then $\boldsymbol{X}^{i',t} = \boldsymbol{X}^{i,t}_n \cap X$, as desired.

(ii) Second, the case where $\{X_i \in [A]^Q_k \mid \pi(X_i) > \max_{X_j \in \mathcal{A}^{Q,k}_{X_i}} t \cdot \pi(X_j)\} = \emptyset$ for all k. We then set $\boldsymbol{X}^{i,t} = s$. If the set is empty for all k, then it is empty for all k with Q_X substituted for Q and $\pi(\cdot \mid X)$ substituted for $\pi(\cdot)$ (since $[A]^{Q_X}_k \subseteq [A]^Q_k$, $\mathcal{A}^{Q,k}_{X_i} \subseteq \mathcal{A}^{Q_X,k}_{X_i \cap X}$, and $\pi(\cdot \mid X) = \pi(\cdot) \cdot \frac{1}{\pi(W \setminus X)}$). But then $\boldsymbol{X}^{i',t} = s_X = s \cap X = \boldsymbol{X}^{i,t} \cap X$.

Thus, both when $\boldsymbol{X}^{i,t} \subset s$ and when $\boldsymbol{X}^{i,t} = s$, it holds that $\boldsymbol{X}^{i_X,t} = \boldsymbol{X}^{i,t} \cap X$ for $X \in \mathrm{Sat}(\boldsymbol{X}^{i,t})$. $\square$

Fact 4. BPR and **BPC** fail.

The Urn is a countermodel for both claims (with $t = 1$). Let $c = \langle y, \langle s, \pi(\cdot), Q \rangle \rangle$ with $Q = \{\{b\}, \{g\}, \{r\}, \{y\}\}$ and $c' = \langle b, \langle s, \pi(\cdot), Q' \rangle \rangle$ with $Q = \{\{b\}, \{g \cup r \cup y\}\}$.

BPR fails.

Countermodel. The pair relevant for BPR is $\langle c', c \rangle$ with QUD parameters $\langle Q', Q \rangle$. It holds that $c \approx_{inf} c'$ and $Q' \sqsubseteq Q$. Take the proposition $\neg b$. $[\![B\neg b]\!]^{c'}$ is true (see **The Urn** (Model)). However, $[\![B\neg b]\!]^{c}$ is not true, contrary to BPR.

BPC fails.

Countermodel. The pair relevant for BPC is $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$. As above, it holds that $c \approx_{inf} c'$ and $Q' \sqsubseteq Q$. Take the proposition b. $[\![Bb]\!]^{c}$ is true (see **The Urn** (Model)). However, $[\![Bb]\!]^{c'}$ is not true, contrary to BPC.

Fact 5. BPR$^+$ holds while **BPC$^+$** fails.

BPR$^+$ holds.

Proof. Suppose $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$ and $Q \sqsubseteq Q'$. Suppose further that $\boldsymbol{X}^{i,t}$ is rank-constant, so that $n(\boldsymbol{X}^{i,t})^Q = n(\boldsymbol{X}^{i',t})^{Q'}$.

(i) Consider first the case where $i = 1$, so that $\boldsymbol{X}^{i,t}$ is of rank 1. Since $Q \sqsubseteq Q'$, there is some $X \in Q$ such that $X = Y_1 \cup \cdots \cup Y_n$ for some $Y_1 \cup \cdots \cup Y_n \in Q'$. Since $n(\boldsymbol{X}^{i,t})^Q = 1$, $n(\boldsymbol{X}^{i,t})^{Q'} = 1$, and so $\boldsymbol{X}^{i,t} \in Q'$. But then some $X \in \mathcal{A}^{Q,1}_{\boldsymbol{X}^{i,t}}$ is such that $X = Y_1 \cup \cdots \cup Y_n$ for some $Y_1 \cup \cdots \cup Y_n \in Q'$. But if $\pi(\boldsymbol{X}^{i,t}) > \max_{X \in \mathcal{A}^{Q,1}_{\boldsymbol{X}^{i,t}}} t \cdot \pi(X)$, then $\pi(\boldsymbol{X}^{i,t}) > \max_{X \in \mathcal{A}^{Q',1}_{\boldsymbol{X}^{i,t}}} t \cdot \pi(X)$, since for any $X \in \mathcal{A}^{Q',1}_{\boldsymbol{X}^{i,t}}$ it holds that $\pi(X) \leq \pi(Y)$ where Y is such that $X \subseteq Y$ and $Y \in Q$. But then $\boldsymbol{X}^{i,t} = \boldsymbol{X}^{i',t}$. From this it follows (see proof of Fact 2) that, if $[\![B\phi]\!]^{c}$ is true, $[\![B\phi]\!]^{c'}$ is true.

(ii) Then, consider the case where $i > 1$, so that $\boldsymbol{X}^{i,t}$ is of rank > 1. It then holds that $\boldsymbol{X}^{i,t} = X_1 \cup \cdots \cup X_n$ for some $X_1, \ldots, X_n$ of rank 1. Since $Q \sqsubseteq Q'$, there is some $X_i \in Q$ such that $X_i = Y_1 \cup \cdots \cup Y_n$ for some $Y_1 \cup \cdots \cup Y_n \in Q'$. Since $\boldsymbol{X}^{i,t}$ is rank-constant, it holds of any $X \subseteq \boldsymbol{X}^{i,t}$ such that $X \in S(Q)$ that $n(X)^Q = n(X)^{Q'}$. Thus for any $X_i = Y_1 \cup \cdots \cup Y_n$ (with $Y_1 \cup \cdots \cup Y_n \in Q'$), it holds that $X_i \cap \boldsymbol{X}^{i,t} = \emptyset$. We consider two cases: (i) the case where, of some $X \in S(Q')$ such that $X \subseteq \boldsymbol{X}^{i,t}$, it holds that $\pi(X) > \max_{Z \in \mathcal{A}^{Q',k}_X} t \cdot \pi(Z)$ where $k < i$; and (ii) the case where for all $X \in S(Q')$ such that $X \subseteq \boldsymbol{X}^{i,t}$, it holds that $\pi(X) \not> \max_{Z \in \mathcal{A}^{Q',k}_X} t \cdot \pi(Z)$. (Cases (i) and (ii) are exhaustive since $\mathcal{A}^{Q',k}_X \subseteq \mathcal{A}^{Q,k}_X$ whenever $Q \sqsubseteq Q'$.) It can be checked that, in both cases, $\boldsymbol{X}^{i',t} \subseteq \boldsymbol{X}^{i,t}$; in case (i): since $X \subseteq \boldsymbol{X}^{i,t}$, whenever X meets the required condition $\boldsymbol{X}^{i',t} = X \subseteq \boldsymbol{X}^{i,t}$; in case (ii): $X = \boldsymbol{X}^{i,t}$, and it holds that $\boldsymbol{X}^{i',t} = X = \boldsymbol{X}^{i,t} \subseteq \boldsymbol{X}^{i,t}$. In both cases it follows that if $[\![B\phi]\!]^{c}$ is true, then $[\![B\phi]\!]^{c'}$ is true.

Thus, both when $i = 1$ and when $i > 1$, it holds that $if [\![B\phi]\!]^c$ is true, then $[\![B\phi]\!]^{c'}$ is true. $\square$

BPC$^+$ fails.

Countermodel. BPC$^+$ is invalidated by **The Urn** (Model), as in the Countermodel above. We just check that $\boldsymbol{X}^{i,t} = \{b\}$ is rank-constant in Q and Q', i.e., that $n(\{b\})^Q = i$ and $n(\{b\})^{Q'} = i$. Indeed: $n(\{b\})^Q = 1 = n(\{b\}^{Q'}$. But then $\{b\}$ is rank constant, and yet $[\![Bb]\!]^c$ is true and $[\![Bb]\!]^{c'}$ is false.

Fact 6. BPC^{++} holds.

Proof. Suppose $\langle c, c' \rangle$ with QUD parameters $\langle Q, Q' \rangle$ is such that $c \approx_{inf} c'$ and $Q' \sqsubseteq Q$. Suppose further that for any Y such that $Y \in S(Q')$ and $Y \subseteq \neg \boldsymbol{X}^{i,t}$, Y is rank-constant.

(i) Consider first the base case where $i = 1$, so that $\boldsymbol{X}^{i,t}$ is of rank 1. Since $Q' \sqsubset Q$, there is some $X \in Q'$ such that $X = Y_1 \cup \cdots \cup Y_n$ for some $Y_1 \cup \cdots \cup Y_n \in Q$. As it holds, of any Y s.t. $Y \in Q'$ and $Y \subseteq \neg \boldsymbol{X}^{i,t}$ that $n(Y)^Q = 1$ iff $n(Y)^{Q'} = 1$, it holds of any such Y that $Y \not\subseteq X$. It holds then that (i) $\boldsymbol{X}^{i,t} \subseteq X$ and (ii) for at least one Z s.t. $Z \in Q$ and $Z \notin Q'$ that $Z \subseteq X$. Since $\pi(\boldsymbol{X}^{i,t}) > t \cdot \max_{X' \in \mathcal{A}^{Q,1}_{\boldsymbol{X}^{i,t}}} \pi(X')$ and $\boldsymbol{X}^{i,t} \subseteq X$, it follows that $\pi(X) > t \cdot \max_{X' \in \mathcal{A}^{Q',1}_{X}} \pi(X')$. It follows that $X = \boldsymbol{X}^{i',t}$, and as such that $\boldsymbol{X}^{i,t} \subseteq \boldsymbol{X}^{i',t}$. Hence, since $\boldsymbol{X}^{i,t} \cap \boldsymbol{X}^{i',t} \neq \emptyset$, if $[\![B\phi]\!]^c$ is true, $[\![\neg B \neg \phi]\!]^{c'}$ is true.

(ii) Then, consider the case where $i > 1$, so that $\boldsymbol{X}^{i,t}$ is of rank > 1. It then holds that that $\boldsymbol{X}^{i,t} = X_1 \cup \cdots \cup X_n$ for some $X_1 \cup \cdots \cup X_n$ of rank 1. Since $Q' \sqsubseteq Q$, there is some $Y \in Q'$ such that $Y = X_1 \cup \cdots \cup X_n$ for some $Y = X_1 \cup \cdots \cup X_n \in Q$. It then holds that $Y \subseteq \boldsymbol{X}^{i,t}$ for such (unique) Y and that for any $X \subseteq \neg \boldsymbol{X}^{i,t}$, $X \cap Y = \emptyset$. We consider two cases: (i) the case where, of some $X \in S(Q')$ such that $X \subseteq \boldsymbol{X}^{i,t}$, it holds that $\pi(X) > \max_{Z \in \mathcal{A}^{Q',k}_{X}} t \cdot \pi(Z)$ where $k < i$; and (ii) the case where for all $X \in S(Q')$ such that $X \subseteq \boldsymbol{X}^{i,t}$, it holds that $\pi(X) \not> \max_{Z \in \mathcal{A}^{Q',k}_{X}} t \cdot \pi(Z)$. In case (i), since $X \subseteq \boldsymbol{X}^{i,t}$, whenever X meets the condition $\boldsymbol{X}^{i',t} = X \subseteq \boldsymbol{X}^{i,t}$; in case (ii), since $\boldsymbol{X}^{i,t} \subseteq X$, it holds that $\pi(X) > \boldsymbol{X}^{i,t}$, and hence that $X = \boldsymbol{X}^{i',t}$, from which it follows that $\boldsymbol{X}^{i,t} \subseteq \boldsymbol{X}^{i',t}$. In both (i) and (ii), then, it holds that $\boldsymbol{X}^{i,t} \cap \boldsymbol{X}^{i',t} \neq \emptyset$, and so: if $[\![B\phi]\!]^c$ is true, $[\![\neg B \neg \phi]\!]^{c'}$ is true.

Thus, both when $i = 1$ and when $i > 1$, it holds that if $[\![B\phi]\!]^c$ is true, then $[\![\neg B \neg \phi]\!]^{c'}$ is true. $\square$

Fact 7. CC holds, while CC$^+$ fails.

CC holds.

Proof. Suppose $[\![B\phi]\!]^c$ and $[\![B\psi]\!]^c$ are true. Then $R_D(c) \subseteq [\![\phi]\!]$ and $R_D(c) \subseteq [\![\psi]\!]$. Since our semantics is compositional, $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$. But then $R_D(c) \subseteq [\![\phi \wedge \psi]\!]$, from which it follows that $[\![B(\phi \wedge \psi)]\!]^c$ is true. $\square$

CC$^+$ fails.

Countermodel. The Urn is a countermodel for CC$^+$. We have that $[\![Bp]\!]^c$ and $[\![B\neg p]\!]^{c'}$ are true, and that $c \approx_{inf} c'$. Let e be such that $f([c]_{\approx_{inf}}) = e$. We show that $\boldsymbol{X}^e \not\subseteq [\![p \wedge \neg p]\!]$, i.e., that $\boldsymbol{X}^e \neq \emptyset$. For simplicity, assume $I = \{i, i'\}$, so that $\boldsymbol{X}^e = Cn(\boldsymbol{X}^{i,t}) \cap Cn(\boldsymbol{X}^{i',t})$. By the definition of $Cn(\cdot)$, it holds that $\boldsymbol{X}^e = \{Y \in S \mid \boldsymbol{X}^{i,t} \subseteq Y\} \cap \{Y \in S \mid \boldsymbol{X}^{i',t} \subseteq Y\}$. As can be checked, $s = \{b, g, r, y\}$ is the smallest set in $\mathscr{P}(W)$ such that $\boldsymbol{X}^{i,t} \subseteq s$ and $\boldsymbol{X}^{i',t} \subseteq s$. But then $X^e = s$ and so $X^e \neq \emptyset$. But since $X \subseteq [\![p \wedge \neg p]\!]$ only if $X = \emptyset$, it follows that $X^e \not\subseteq \emptyset$.

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Fact 8. (PI) holds.

Proof. Take some $c = \langle w, i\rangle = \langle w, \langle s, \pi, Q\rangle\rangle$. Suppose $[\![B\phi]\!]^c$ is true. Then $R_D(c) \subseteq [\![\phi]\!]$. By the definition of $R_E(\cdot)$ and $R_D(\cdot)$, it holds that $R_D(c) \subseteq R_E(c)$. Take any $c' = \langle v, i'\rangle = \langle v, \langle s', \pi', Q'\rangle\rangle \in R_E(c)$. By the definition of $R_E(\cdot)$, if cR_Ec', then $i' = i$. But then, symmetrically, $c'R_Ec$. Since $X^{i,t} = X^{i',t}$ whenever $i = i'$, $R_D(c) = R_D(c')$. But then $R_D(c') \subseteq [\![\phi]\!]$. Since c' was arbitrary, $R_D(c'') \subseteq [\![\phi]\!]$ for any $c'' \in R_E(c)$. For any $c'' \in R_E(c)$, then, $[\![B\phi]\!]^{c''}$ is true. But then $R_E(c) \subseteq [\![B\phi]\!]$. As such, $[\![KB\phi]\!]^c$ is true. □

Fact 9. (NI) holds.

Proof. Immediate from proof of (PI). □

Fact 10. (KB) holds.

Proof. Take some $c = \langle w, i\rangle = \langle w, \langle s, \pi, Q\rangle\rangle$. Suppose $[\![K\phi]\!]^c$ is true. Then $R_E(c) \subseteq [\![\phi]\!]$. By the definition of $R_D(\cdot)$, $R_D(c) \subseteq R_E(c)$. But then $R_D(c) \subseteq [\![\phi]\!]$. As such, $[\![B\phi]\!]^c$ is true. □

Fact 11. (CB) holds.

Proof. Take some $c = \langle w, i\rangle = \langle w, \langle s, \pi, Q\rangle\rangle$. Suppose $[\![B\phi]\!]^c$ is true. Then $R_D(c) \subseteq [\![\phi]\!]$. Note that (i) $X^{i,t} \neq \emptyset$ if $s \neq \emptyset$, and that (ii) if $\langle w, \langle s, \pi, Q\rangle\rangle \in \mathcal{W}$, then $w \in s$; so, since $s \neq \emptyset$, $X^{i,t} \neq \emptyset$. Since $R_D(c) = \{\langle v, i'\rangle \in R_E(c) \mid v \in X^{i,t}\}$, $R_D(c) \neq \emptyset$. But then, since $[\![\neg\phi]\!] = \mathcal{W} \setminus [\![\phi]\!]$, it holds that $R_D(c) \not\subseteq [\![\neg\phi]\!]$. As such, $[\![B\neg\phi]\!]^c$ is not true, and $[\![\neg B\neg\phi]\!]^c$ is true. □

Fact 12. Where $c = \langle w, \langle s, \pi, Q\rangle\rangle$ and $c' = \langle w, \langle s, \pi, Q'\rangle\rangle$, it is consistent that $[\![B\phi]\!]^c$ and $[\![B\neg\phi]\!]^{c'}$ are true.

Proof. **The Urn** (Model) is an example of such a case. Relative to $t = 1$, we saw that $[\![B\phi]\!]^c$ and $[\![B\neg\phi]\!]^{c'}$ are both true, while c and c' differ only with respect to the Q parameter. □

Fact 13. (SB) fails.

Countermodel. **The Urn** (Model) is a countermodel to (SB). ϕ is the sentence 'The chosen marble is blue'. We saw that $[\![B\phi]\!]^c$ is true, with $c = \langle y, i\rangle$. We show that $[\![BK\phi]\!]^c$ is false, or that $R_D(c) \not\subseteq [\![K\phi]\!]$, by showing that for some c' (i) $c' \in R_D(c)$ and (ii) $c' \notin [\![K\phi]\!]$. (i) $c' \in R_D(c)$: $R_E(\langle y, i\rangle) = \{\langle v, i'\rangle \in \mathcal{W} \mid i = i'\} = \{\langle b, i\rangle, \langle g, i\rangle, \langle r, i\rangle, \langle y, i\rangle\}$; since $\boldsymbol{X}^{i,t} = \{b\}$, $R_D(c) = \{\langle v, i\rangle \in R_E \mid v \in \boldsymbol{X}^{i,t}\} = \{\langle b, i\rangle\}$. Let $c' = \langle b, i\rangle$. (ii) $c' \notin [\![K\phi]\!]$: By hypothesis, $[\![\phi]\!] = \{b\} \times I$; since $R_E(c') \not\subseteq [\![\phi]\!]$, $[\![\neg K\phi]\!]^{c'}$ is true, and hence $c' \notin [\![K\phi]\!]$. Since $R_D(c) \not\subseteq [\![K\phi]\!]$, it follows that $[\![BK\phi]\!]^c$ is false. Thus: $[\![B\phi]\!]^c$ and $[\![\neg BK\phi]\!]^c$ are both true, contrary to (SB).

Fact 14. $(\Rightarrow *)$ fails.

Countermodel. **The Urn** (Model) is a countermodel to $(\Rightarrow *)$. We saw that $[\![B\phi]\!]^c$ is true, with $c = \langle y, i\rangle$. We show that $[\![K\neg K\phi]\!]^c$ is true by showing that $R_E(c) \subseteq [\![\neg K\phi]\!]$ by showing that for all $c' \in R_E(c), [\![\neg K\phi]\!]^{c'}$ is true. Note that, if $c' \in R_E(c)$, then $R_E(c) = R_E(c)$. It thus follows that if $[\![\neg K\phi]\!]^c$ is true, then $[\![\neg K\phi]\!]^{c'}$ is true for all $c' \in R_E(c)$. Thus it suffices to show that $[\![\neg K\phi]\!]^c$ is true, or that $R_E(c) \not\subseteq [\![\phi]\!]$. We know from the countermodel above that $R_E(c) = \{\langle b, i\rangle, \langle g, i\rangle, \langle r, i\rangle, \langle y, i\rangle\}$, and that $\phi = \{b\} \times I$. But then $R_E(c) \not\subseteq [\![\phi]\!]$, as desired. Thus: $[\![B\phi]\!]^c$ and $[\![K\neg K\phi]\!]^c$ are true, contrary to $(\Rightarrow *)$.

Fact 15. $(\Leftarrow *)$ holds.

Proof. Take some $c = \langle w, i\rangle = \langle w, \langle s, \pi, Q\rangle\rangle$. Suppose $[\![\langle K\rangle K\phi]\!]^c$ is true. Then there is a $c' = \langle v, i'\rangle \in R_E(c)$ such that $[\![K\phi]\!]^{c'}$ is true. But then $R_E(c') \subseteq [\![\phi]\!]$. Since $\langle v, i'\rangle \in R_E(\langle w, i\rangle)$, it holds that$i = i'$. By the definition of $R_E(\cdot)$, $R_E(c) = R_E(c')$ whenever $i = i'$. But then $R_E(c) \subseteq [\![\phi]\!]$. As such, $[\![K\phi]\!]^c$ is true. By (KB), $[\![B\phi]\!]^c$ is true. $\square$