Introduction to Random Effects of Time and Model Estimation
Hoffman Ch. 5

Estimation in Mplus with Extended Examples

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$
$$\boldsymbol{\eta}_i = \boldsymbol{\mu}_\eta + \boldsymbol{\Gamma}\mathbf{w}_i + \boldsymbol{\zeta}_i$$

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\mu}_\eta + \boldsymbol{\Lambda}\boldsymbol{\Gamma}\mathbf{w}_i + \boldsymbol{\Lambda}\boldsymbol{\zeta}_i + \boldsymbol{\varepsilon}_i$$

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{bmatrix} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (T-1)^1 & \cdots & (T-1)^P \end{bmatrix} \quad \boldsymbol{\eta}_i = \begin{bmatrix} \alpha_i \\ \beta_{1i} \\ \vdots \\ \beta_{Pi} \end{bmatrix} \quad \boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{1i} \\ \vdots \\ \varepsilon_{Ti} \end{bmatrix}$$

$$\boldsymbol{\eta}_i = \begin{bmatrix} \alpha_i \\ \beta_{1i} \\ \vdots \\ \beta_{Pi} \end{bmatrix} \quad \boldsymbol{\mu}_\eta = \begin{bmatrix} \mu_\alpha \\ \mu_{\beta 1} \\ \vdots \\ \mu_{\beta P} \end{bmatrix} \quad \boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{\alpha 1} & \gamma_{\alpha 2} & \cdots & \gamma_{\alpha K} \\ \gamma_{\beta 11} & \gamma_{\beta 12} & \cdots & \gamma_{\beta 1K} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{\beta P1} & \gamma_{\beta P1} & \cdots & \gamma_{\beta PK} \end{bmatrix} \quad \mathbf{w}_i = \begin{bmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{Ki} \end{bmatrix} \quad \boldsymbol{\zeta}_i = \begin{bmatrix} \zeta_{\alpha i} \\ \zeta_{\beta 1i} \\ \vdots \\ \zeta_{\beta Pi} \end{bmatrix}$$

$$\begin{bmatrix} \zeta_{\alpha i} \\ \zeta_{\beta 1i} \\ \vdots \\ \zeta_{\beta Pi} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_{\alpha\alpha} & & & \\ \psi_{\alpha\beta 1} & \psi_{\beta 1\beta 1} & & \\ \vdots & \vdots & \ddots & \\ \psi_{\alpha\beta P} & \psi_{\beta 1\beta P} & \cdots & \psi_{\beta P\beta P} \end{bmatrix} \right)$$

$\mathbf{y}_i$ - A vector of responses of individual i for times T

$\boldsymbol{\Lambda}$ - Matrix of weights for P functions of time

$\boldsymbol{\eta}_i$ -Vector of person-specific weights for P time effects

$\boldsymbol{\mu}_\eta$ - Vector of $(P+1)$ fixed effect estimates

$\boldsymbol{\Gamma}$ - Matrix of fixed effect estimates for $\mathbf{w}_i$ with K predictors

$\mathbf{w}_i$ - Time invariant, fixed predictors of $\boldsymbol{\eta}_i$

$\boldsymbol{\zeta}_i$ - Random effect estimates

$\boldsymbol{\varepsilon}_i$ - Residual variance

T - Number of time points

P - Number of time effects estimated beyond intercept

K - Number of predictors $\mathbf{w}_i$ used to predict time effects $\boldsymbol{\eta}_i$

↑ Bollen & Curran (2006)

↓ Snijders & Bosker (2011)

$$y_{ti} = \beta_{0i} + \beta_{1i}TVC_{1ti} + \beta_{2i}TVC_{2ti} + \ldots + \beta_{Pi}TVC_{Pti} + \varepsilon_{ti}$$
$$\beta_{0i} = \gamma_{00} + \gamma_{01}TIC_{1i} + \gamma_{02}TIC_{2i} + \ldots + \gamma_{0K}TIC_{Ki} + u_{0i}$$
$$\beta_{1i} = \gamma_{10} + \gamma_{11}TIC_{1i} + \gamma_{12}TIC_{2i} + \ldots + \gamma_{1K}TIC_{Ki} + u_{1i}$$
$$\vdots \quad \vdots \quad \vdots$$
$$\beta_{Pi} = \gamma_{P0} + \gamma_{P1}TIC_{1i} + \gamma_{P2}TIC_{2i} + \ldots + \gamma_{PK}TIC_{Ki} + u_{Ki}$$

$$\varepsilon_{ti} \sim N\left([0], [\sigma^2]\right)$$

$$\begin{bmatrix} u_{0i} \\ u_{1i} \\ \vdots \\ u_{Pi} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & & & \\ \tau_{10} & \tau_{11} & & \\ \vdots & \vdots & \ddots & \\ \tau_{P0} & \tau_{P1} & \cdots & \tau_{PP} \end{bmatrix} \right)$$

General Specification of a Multilevel Model for Longitudinal Data

Fundamental questions

1. **What** kind of change occurs on average?

- What kind of **population model** generated the observed trajectories?
  - Linear/nonlinear? One/multiple processes? Continuous/discontinuous?

- What is the most **appropriate metric of time**?
  - Time in study (with predictors of BP differences in age)
  - Time since birth (chronological age)? Time before death? Time to an event (marriage; diagnosis)?
  - *Note:* Measurement occasions need not need be equally spaced or the same across individuals (i.e., TSCORES in Mplus)

Fundamental questions

2. **Do people differ** in their change parameters?

- In **Level**?
  - Do you expect individual differences in level?

- In **Rate of change**?
  - Do you expect individual differences in magnitude and/or direction of change?

Fundamental questions

3. **Why** do people differ from each other in terms of change parameters?

- What **person-level variables** predict individual differences in aspects of change?
  → Why are the lines different for different people?

- What **time-level variables** predict intraindividual deviation from predicted change?
  → Why are individuals functioning above/below their predicted slope?

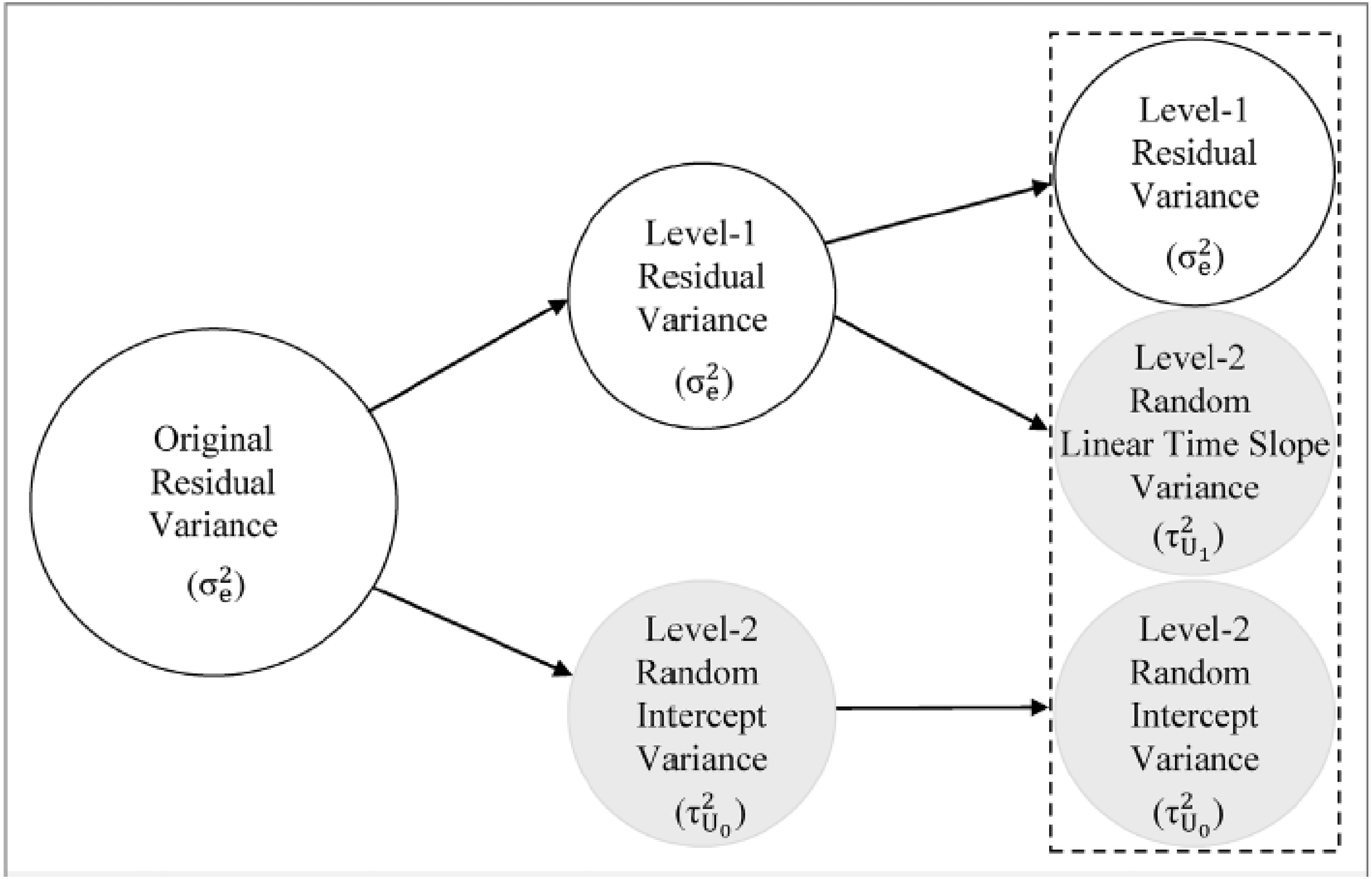

Figure 5.3 Partitioning of variance across models for the four-occasion example data.

Constructing a Growth Curve Model: 1. Estimate an empty model

- Useful statistical baseline model – partitions variance into between- and within-person variance
- Calculate **ICC** = between / (between + within)

  = Average correlation between observations within a person

  = Proportion of variance that is between persons
- Tells you where the action is:
  - If most of the variance is **between-persons (level 2)**, you will need **person-level predictors** to reduce that variance (i.e., to account for **inter-individual differences**)
  - If most of the variance is **within-persons (level 1)**, you will need **time-level predictors** to reduce that variance (i.e., to account for **intra-individual differences**)

LCM / MLM Specification

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$
$$\boldsymbol{\eta}_i = \boldsymbol{\mu}_\eta$$

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$\mathbf{y}_i$ - A vector of responses of individual i for times T

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$\boldsymbol{\eta}_i$ -Vector of person-specific weights for P time effects

$\boldsymbol{\mu}_\eta$ - Vector of $(P+1)$ fixed effect estimates

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{\eta}_i = [\alpha_i] \quad \boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \end{bmatrix}$$

$\boldsymbol{\varepsilon}_i$ - Residual variance

$T = 4$ - Number of time points

$P = 0$ - Number of time effects estimated beyond intercept

$$\boldsymbol{\eta}_i = [\alpha_i] \qquad \boldsymbol{\mu}_\eta = [\mu_\alpha]$$

Bollen & Curran (2006)

(3.1)

Snijders & Bosker (2011)

$$y_{ti} = \beta_{0i} + \varepsilon_{ti}$$
$$\beta_{0i} = \gamma_{00}$$

$$\varepsilon_{ti} \sim N\left([0], [\sigma^2]\right)$$

$$y_{ti} = (\beta_{0i}) + \varepsilon_{ti}$$

$$y_{ti} = (\gamma_{00}) + \varepsilon_{ti}$$

$$\mathbf{y}_i = \mathbf{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$
$$\boldsymbol{\eta}_i = \boldsymbol{\mu}_\eta + \boldsymbol{\zeta}_i$$

$$\mathbf{y}_i = \mathbf{\Lambda}\boldsymbol{\mu}_\eta + \mathbf{\Lambda}\boldsymbol{\zeta}_i + \boldsymbol{\varepsilon}_i$$

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$$\boldsymbol{\eta}_i = [\alpha_i] \quad \boldsymbol{\mu}_\eta = [\mu_\alpha] \quad \boldsymbol{\zeta}_i = [\zeta_{\alpha i}]$$

Bollen & Curran (2006)

$$[\zeta_{\alpha i}] \sim N\left([0], [\psi_{\alpha\alpha}]\right)$$

(5.1)

Snijders & Bosker (2011)

$$y_{ti} = \beta_{0i} + \varepsilon_{ti}$$
$$\beta_{0i} = \gamma_{00} + u_{0i}$$

$$\varepsilon_{ti} \sim N\left([0], [\sigma^2]\right)$$

$$[u_{0i}] \sim N\left([0], [\tau_{00}]\right)$$

$$y_{ti} = (\beta_{0i}) + \varepsilon_{ti}$$

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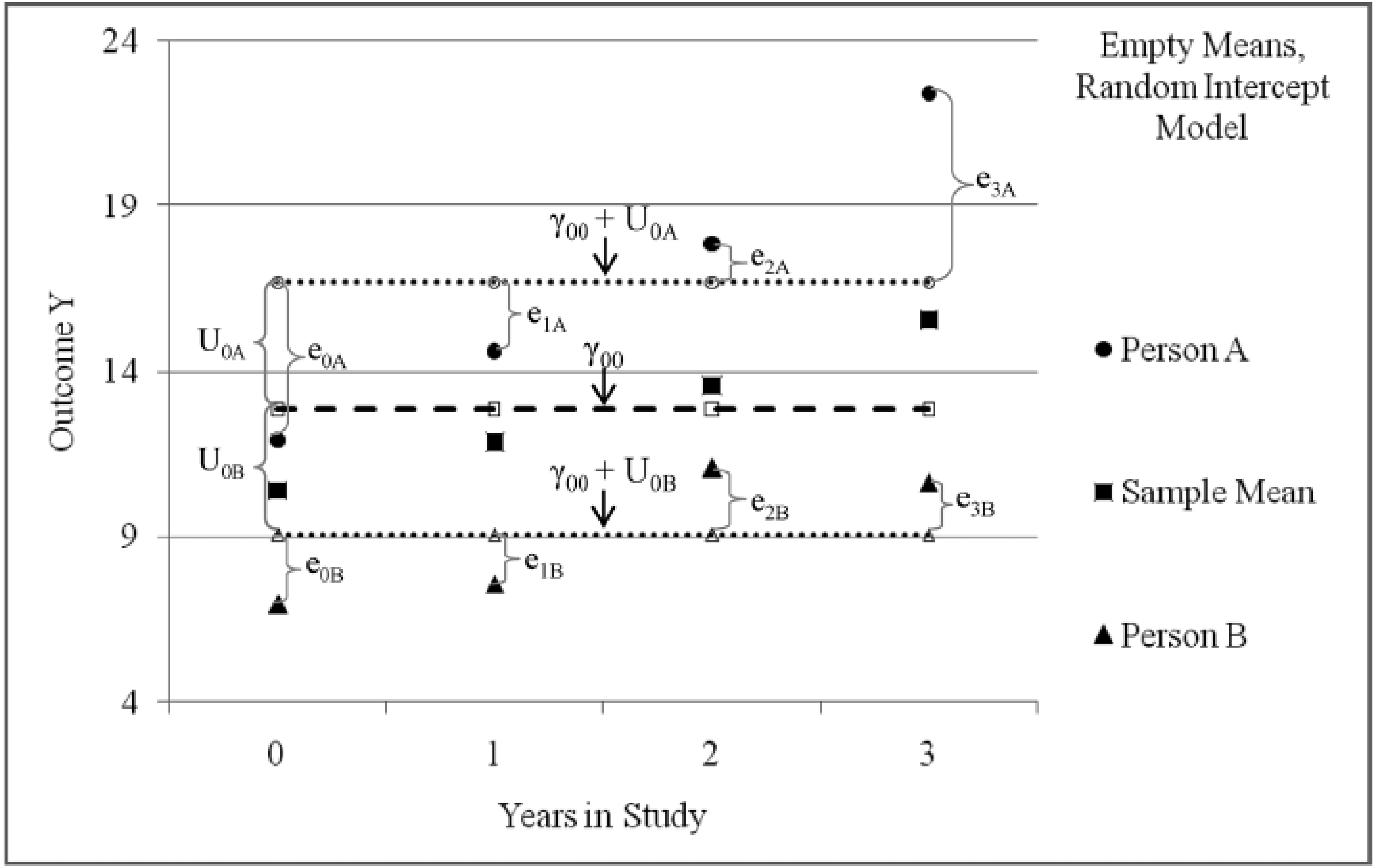

Figure 5.3 Partitioning of variance across models for the four-occasion example data.

Exercise: Fit Empty Model to ELSA Data

- What does the Intraclass correlation tell us about the “cdwlnum” variable?

Constructing a Growth Curve Model: 2. Decide on a centering point

- Where do you want your intercept?
  - Re-code time such that the centering point (intercept) = 0
- Different centerings of time will produce statistically equivalent models with somewhat different parameters
  - i.e., expected level and rate of change *at the centering point*
- How to choose: At what point would you like a snap-shot of inter-individual differences?
  - Intercept variance represents inter-individual differences at that particular time point (that you can later predict!)

Constructing a Growth Curve Model: 3. Evaluate fixed effects of time

- How many polynomials, pieces, discontinuities, or combinations thereof are needed to parsimoniously represent the observed means across time points?
- Test your hypotheses about the fixed effects of time
- Are Wald tests (**p-values** for fixed effects) significant? Alternatively, is the **ML** deviance significantly lower after adding the **fixed effects**?
- How many parameters do you need to sufficiently account for the trajectory of change?

Mplus MLM syntax:
Random Intercept and Slope Model

(aka “Unconditional Growth”)

```
VARIABLE: Names are id time outcome;
   USEVARIABLES ARE time outcome;
   CLUSTER = id;
   WITHIN = time ;                  ! Linear Slope model;
   BETWEEN =  ;                     ! No BP predictors yet;

ANALYSIS: TYPE IS TWOLEVEL RANDOM;
MODEL:
     ! Level-1, time-level model;
      %WITHIN%
         linear | outcome ON time;   ! Fixed Linear Slope
    ! Level-2, person-level model;
      %BETWEEN%
         outcome*;                  ! Random Intercept;
         linear*;                   ! Random Linear Slope;
         cdwr WITH linear;          ! Intercept-Slope Covariance;
```

Which variables are predictors

How to use the predictors

LCM / MLM Specification

$$\mathbf{y}_i = \mathbf{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$
$$\boldsymbol{\eta}_i = \boldsymbol{\mu}_\eta + \boldsymbol{\zeta}_i$$

$$\mathbf{y}_i = \mathbf{\Lambda}\boldsymbol{\mu}_\eta + \mathbf{\Lambda}\boldsymbol{\zeta}_i + \boldsymbol{\varepsilon}_i$$

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix} \quad \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \boldsymbol{\eta}_i = \begin{bmatrix} \alpha_i \\ \beta_{1i} \end{bmatrix} \quad \boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \end{bmatrix}$$

$$\boldsymbol{\eta}_i = \begin{bmatrix} \alpha_i \\ \beta_{1i} \end{bmatrix} \quad \boldsymbol{\mu}_\eta = \begin{bmatrix} \mu_\alpha \\ \mu_{\beta 1} \end{bmatrix} \quad \boldsymbol{\zeta}_i = \begin{bmatrix} \zeta_{\alpha i} \\ 0 \end{bmatrix}$$

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Bollen & Curran (2006)

$$\begin{bmatrix} \zeta_{\alpha i} \\ 0 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_{\alpha\alpha} & \\ 0 & 0 \end{bmatrix} \right)$$

(5.3)

Snijders & Bosker (2011)

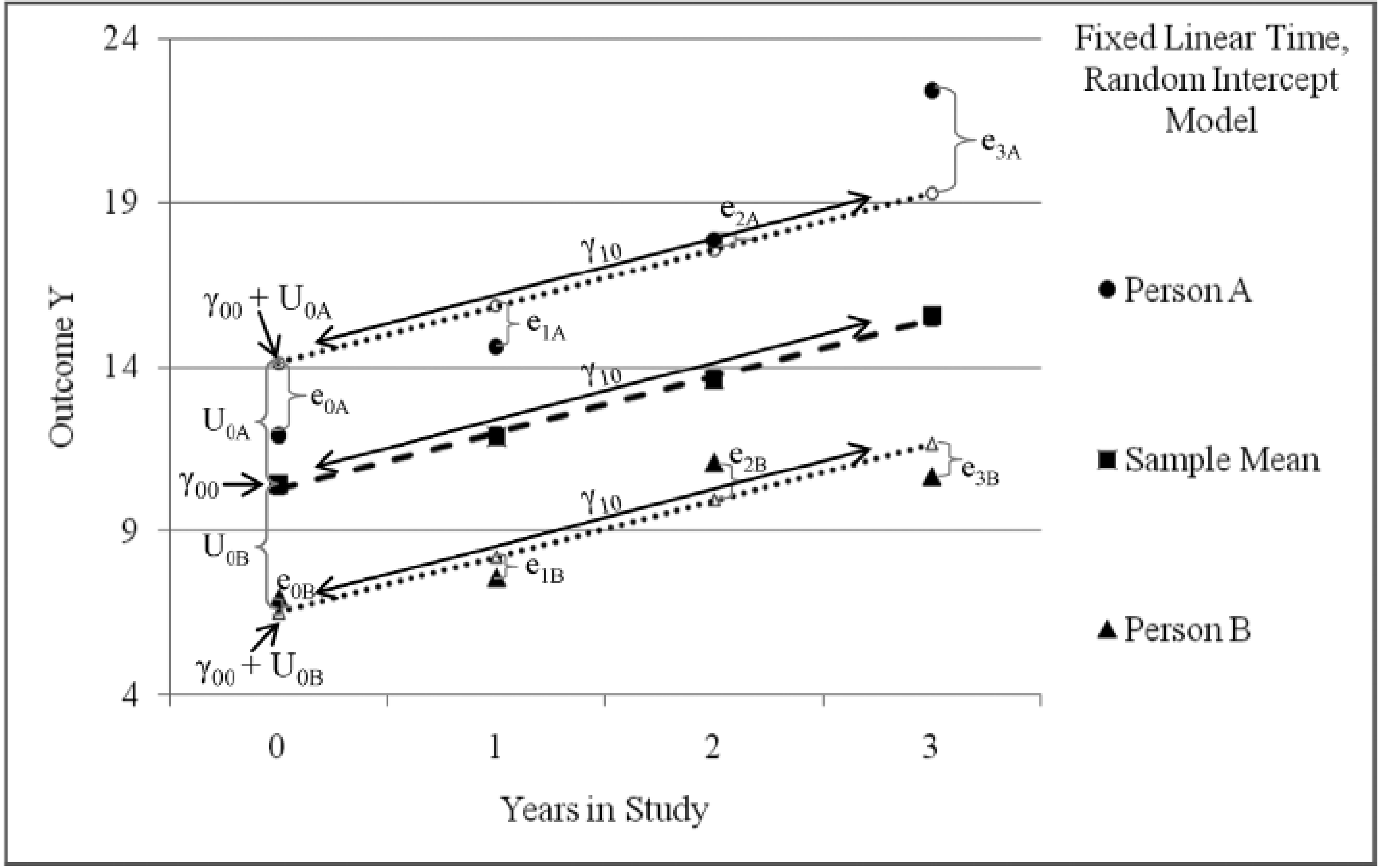
$$y_{ti} = \beta_{0i} + \beta_{1i} TVC_{1ti} + \varepsilon_{ti}$$
$$\beta_{0i} = \gamma_{00} + u_{0i}$$
$$\beta_{1i} = \gamma_{10}$$

$$\varepsilon_{ti} \sim N\left([0], [\sigma^2]\right)$$

$$[u_{0i}] \sim N\left([0], [\tau_{00}]\right)$$

$$y_{ti} = (\beta_{0i}) + (\beta_{1i} TVC_{1ti}) + \varepsilon_{ti}$$

$$y_{ti} = (\gamma_{00}) + u_{0i} + (\gamma_{10} TVC_{1ti}) + \varepsilon_{ti}$$



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Exercise: Fit Fixed Time Model to ELSA Data

- Is the rate of change over time-in-study significant (i.e., non-zero)?
- What is the interpretation of the linear slope in "cdwlnum" units?
- Compare the intercept variance to the "empty" model. Is there a difference and why?

Constructing a Growth Curve Model: 4. Evaluate random effects of time

- Which fixed effects of time need to have random variation around them to sufficiently account for the observed data?
- Test hypotheses about inter-individual differences
- Compare nested models with **same fixed effects** and **different random effects** using **ML** deviance tests in Mplus
- Compare intraclass correlations (IC) when choosing among non-nested models
  - Differ in Fixed effects? Compare IC
  - Differ in Random effects? Compare IC

LCM / MLM Specification

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$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\mu}_\eta + \boldsymbol{\Lambda}\boldsymbol{\zeta}_i + \boldsymbol{\varepsilon}_i$$

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \boldsymbol{\eta}_i = \begin{bmatrix} \alpha_i \\ \beta_{1i} \end{bmatrix} \quad \boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \end{bmatrix}$$

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Bollen & Curran (2006)

$$\begin{bmatrix} \zeta_{\alpha i} \\ \zeta_{\beta 1 i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_{\alpha\alpha} & \\ 0 & \psi_{\beta 1 \beta 1} \end{bmatrix} \right)$$

(5.5)

Snijders & Bosker (2011)

$$y_{ti} = \beta_{0i} + \beta_{1i} TVC_{1ti} + \varepsilon_{ti}$$

$$\beta_{0i} = \gamma_{00} + u_{0i}$$

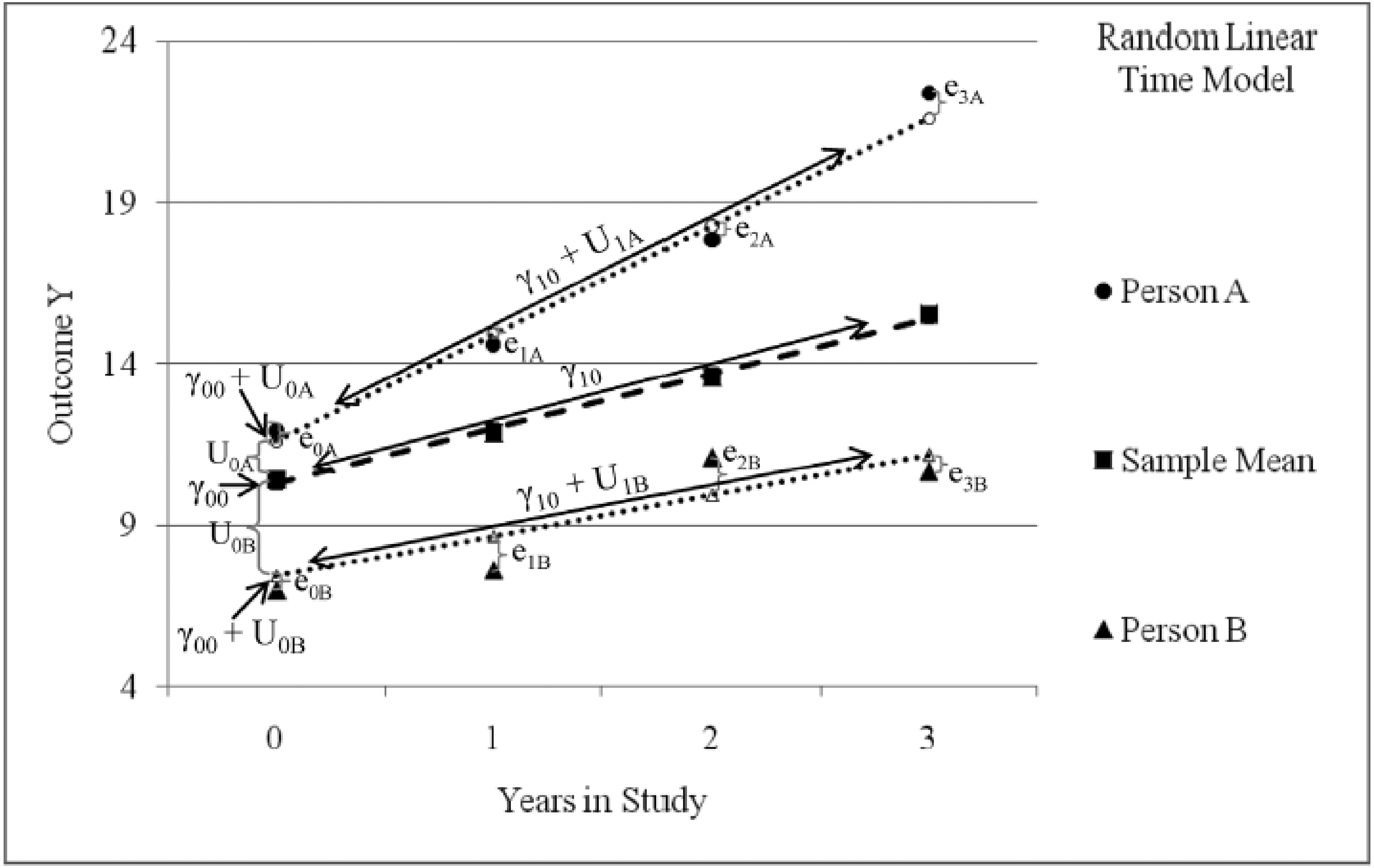
$$\beta_{1i} = \gamma_{10} + u_{1i}$$

$$\varepsilon_{ti} \sim N\left([0], [\sigma^2] \right)$$

$$\begin{bmatrix} u_{0i} \\ u_{1i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} & \\ 0 & \tau_{11} \end{bmatrix} \right)$$

$$y_{ti} = (\beta_{0i}) + (\beta_{1i} TVC_{1ti}) + \varepsilon_{ti}$$

$$y_{ti} = (\gamma_{00}) + u_{0i} + (\gamma_{10} TVC_{1ti}) + u_{1i} + \varepsilon_{ti}$$



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Exercise: Fit Random Time Model to ELSA Data

- Are there individual differences in linear rate of change in the “cdwlnum” variable?
- What is the interpretation of the intercept-slope covariance?

Constructing a Growth Curve Model: 5. Fitting the error model

- Longitudinal mixed models have 2 sides to worry about:
  - Model for the means (fixed effects) AND model for the variances
- Residuals from same person are correlated
  - Addressed in RM ANOVA as CS ≈ a Random intercept only model
- Model for the variances must ALSO address possibility that:
  - Variances of residuals may change/differ over time
  - Residuals closer together in time may be more correlated
  - If this is not addressed, p-values for fixed effects may be wrong
- Two ways of dealing with these possibilities:
  - Random Slopes for Time (preferred)
  - Direct modeling of observed pattern of variances and covariances using a variety of possible alternative structures

What about the covariances/correlations between random effects?

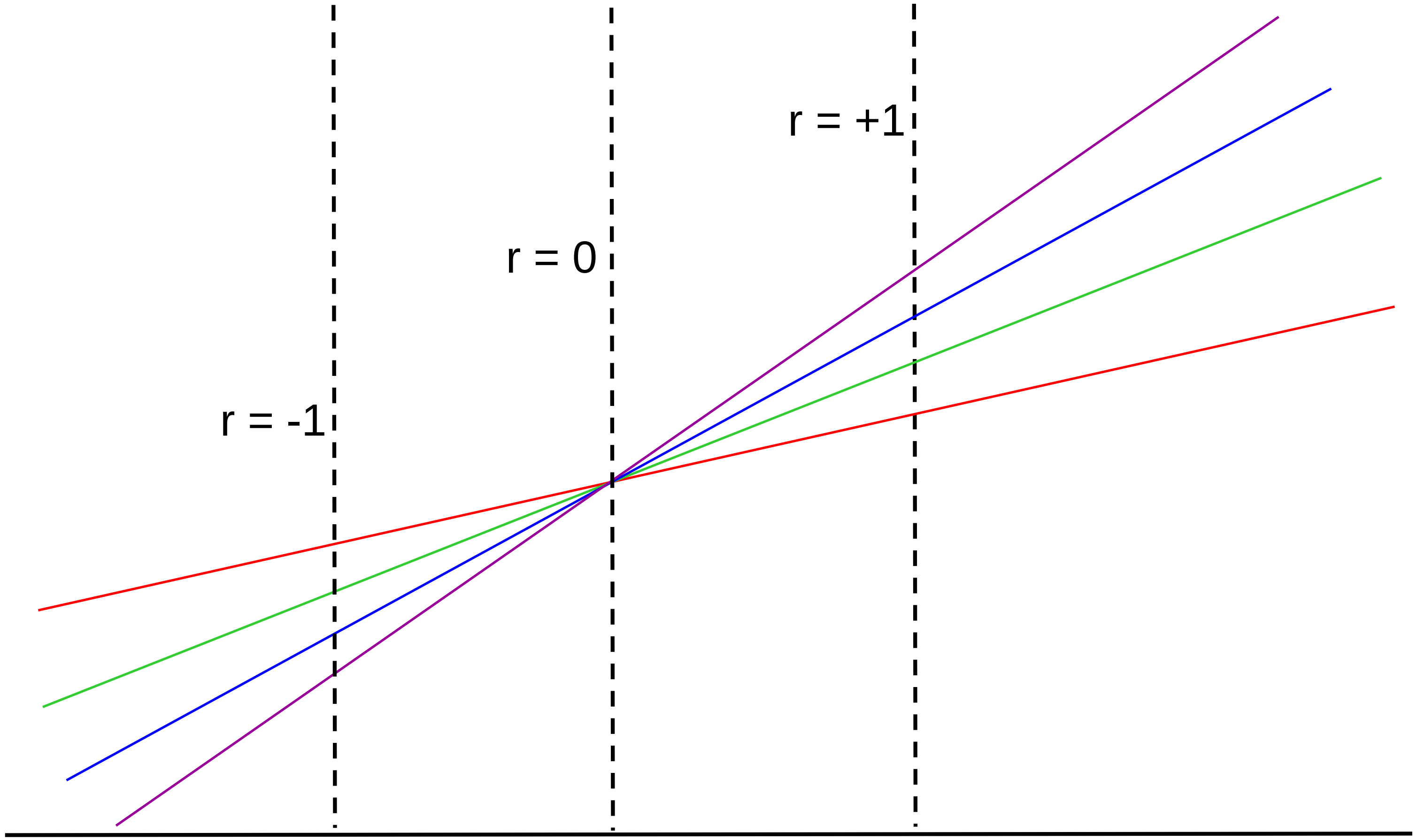
<u>Should covariances always be **included**?</u>

- For longitudinal models, **YES**

<u>Should L-S covariances always be **interpreted**?</u>

- For longitudinal models, **not necessarily**
- Correlation between random effects of time will change with centering (as will intercept variance)
- Correlation is NOT interpretable unless:
  - Time 1 *really is* the beginning of the process under study
  - Intercept is coded as the mean across time points
    - Even then only within the range of time measured

Correlation Between Intercept & Slope



Nonparallel lines will eventually cross.