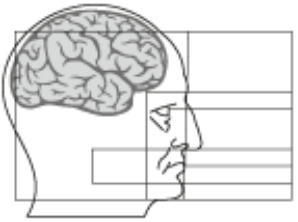

MAX PLANCK UCL CENTRE
for Computational Psychiatry and Ageing Research

UCL

Introduction to hierarchical probabilistic inference

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Zurich Computational Psychiatry Course
December 12, 2015

A spectacular piece of information

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Does chocolate make you clever?

By Charlotte Pritchard
BBC News

Eating more chocolate improves a nation's chances of producing Nobel Prize winners - or at least that's what a recent study appears to suggest. But how much chocolate do Nobel laureates eat, and how could any such link be explained?

A spectacular piece of information

- Messerli, F. H. (2012). Chocolate Consumption, Cognitive Function, and Nobel Laureates. *New England Journal of Medicine*, *367*(16), 1562–1564.

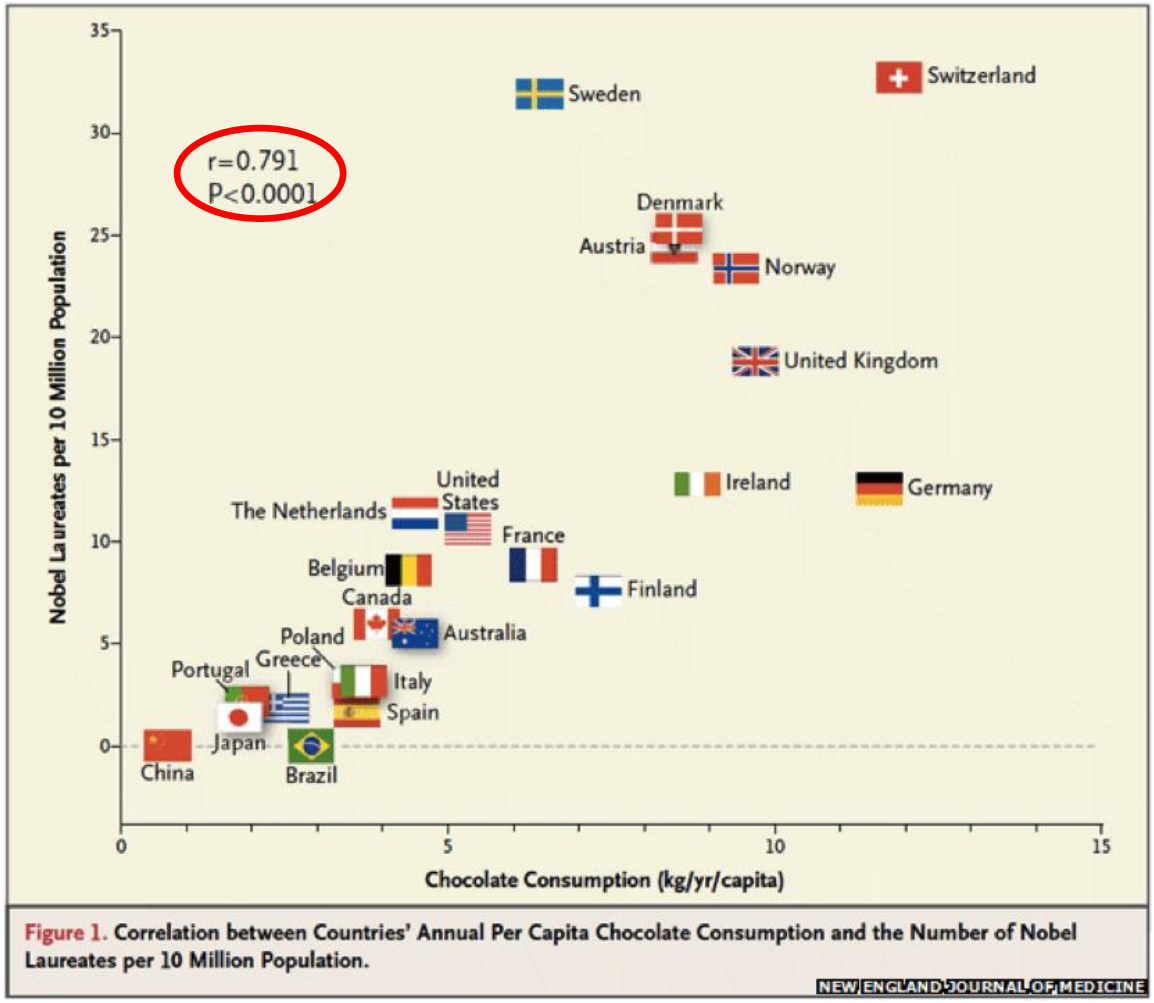


Figure 1. Correlation between Countries' Annual Per Capita Chocolate Consumption and the Number of Nobel Laureates per 10 Million Population.

NEW ENGLAND JOURNAL OF MEDICINE

So will I win the Nobel prize if I eat lots of chocolate?

- This is a question referring to uncertain quantities. Like almost all scientific questions, it cannot be answered by deductive logic. Nonetheless, quantitative answers can be given – but they can only be given in terms of probabilities.

- Our question here can be rephrased in terms of a conditional probability:

$$p(Nobel \mid lots\ of\ chocolate) = ?$$

- To answer it, we have to learn to calculate such quantities. The tool for this is Bayesian inference.

«Bayesian» = logical and logical = probabilistic

«The actual science of logic is conversant at present only with things either certain, impossible, or entirely doubtful, none of which (fortunately) we have to reason on. Therefore the true logic for this world is the calculus of probabilities, which takes account of the magnitude of the probability which is, or ought to be, in a reasonable man's mind.»

— James Clerk Maxwell, 1850

«Bayesian» = logical and logical = probabilistic

But in what sense is probabilistic reasoning (i.e., reasoning about uncertain quantities according to the rules of probability theory) «logical»?

R. T. Cox showed in 1946 that the rules of probability theory can be derived from three basic desiderata:

1. Representation of degrees of plausibility by real numbers
2. Qualitative correspondence with common sense (in a well-defined sense)
3. Consistency

The rules of probability

By mathematical proof (i.e., by deductive reasoning) the three desiderata as set out by Cox imply the rules of probability (i.e., the rules of inductive reasoning).

This means that anyone who accepts the desiderata must accept the following rules:

1. $\sum_a p(a) = 1$ (Normalization)
2. $p(b) = \sum_a p(a, b)$ (Marginalization – also called the **sum rule**)
3. $p(a, b) = p(a|b)p(b) = p(b|a)p(a)$ (Conditioning – also called the **product rule**)

«Probability theory is nothing but common sense reduced to calculation.»

— Pierre-Simon Laplace, 1819

Conditional probabilities

The probability of ***a* given *b*** is denoted by

$$p(a|b).$$

In general, this is different from the probability of a alone (the *marginal* probability of a), as we can see by applying the sum and product rules:

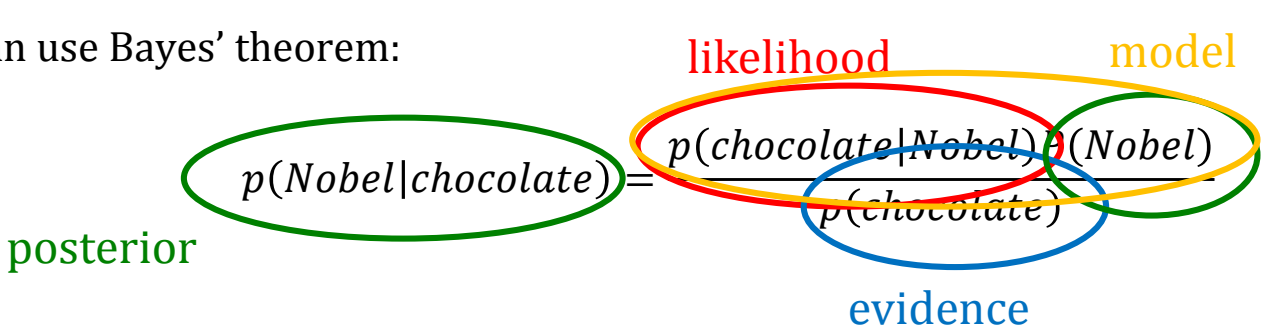
$$p(a) = \sum_b p(a,b) = \sum_b p(a|b)p(b)$$

Because of the product rule, we also have the following rule (**Bayes' theorem**) for going from $p(a|b)$ to $p(b|a)$:

$$p(b|a) = \frac{p(a|b)p(b)}{p(a)} = \frac{p(a|b)p(b)}{\sum_{b'} p(a|b')p(b')}$$

The chocolate example

In our example, it is immediately clear that $P(Nobel|chocolate)$ is very different from $P(chocolate|Nobel)$. While the first is hopeless to determine directly, the second is much easier to find out: ask Nobel laureates how much chocolate they eat. Once we know that, we can use Bayes' theorem:



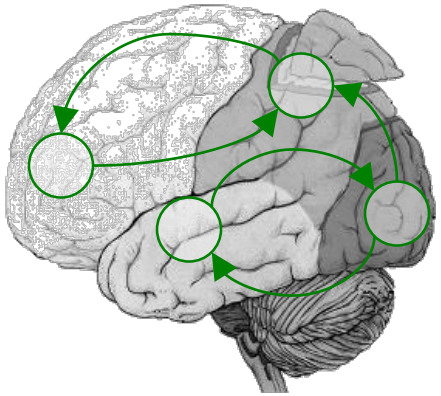
Inference on the quantities of interest in neuroscientific studies has exactly the same general structure.

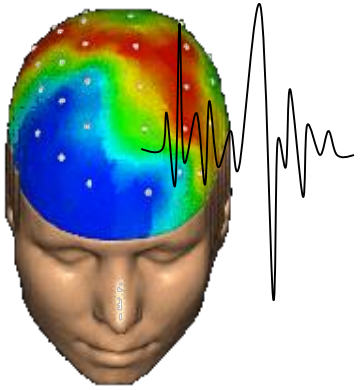
Inference in computational psychiatry

forward problem

$p(y|\vartheta, m)$

likelihood



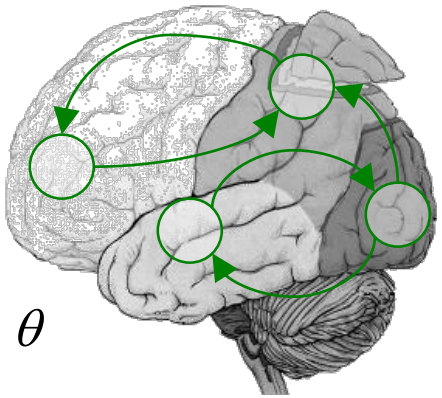


posterior distribution

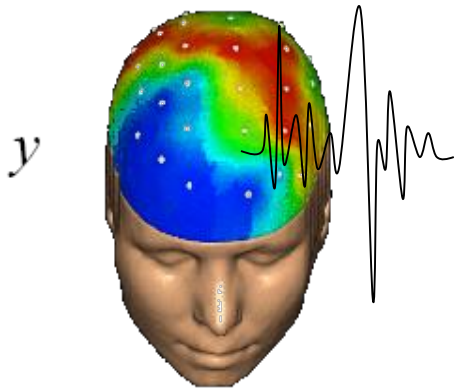
$p(\vartheta|y, m)$

inverse problem

Inference on neural processes



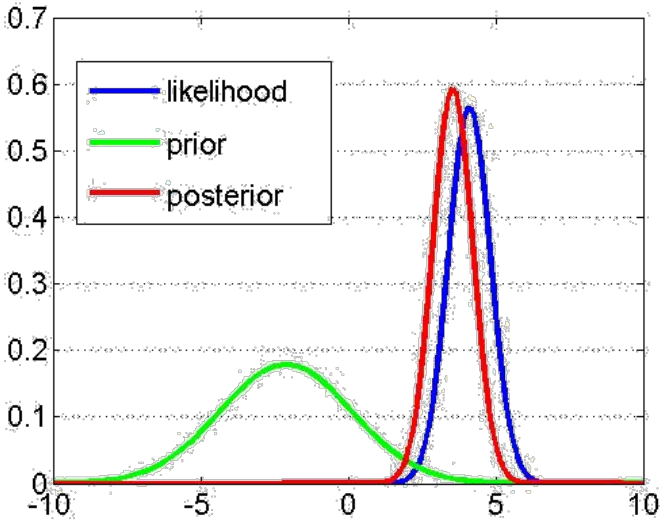
generative model m



Likelihood: $p(y|\vartheta, m)$

Prior: $p(\vartheta|m)$

Bayes' theorem: $p(\vartheta|y, m) = \frac{p(y|\vartheta, m)p(\vartheta|m)}{p(y|m)}$



A simple example of Bayesian inference

(adapted from Jaynes (1976))

Two manufacturers, A and B, deliver the same kind of components that turn out to have the following lifetimes (in hours):

A:

59.5814
37.3953
47.5956
40.5607
48.6468
36.2789
31.5110
31.3606
45.6517

B:

48.8506
48.7296
59.1971
51.8895

Assuming prices are comparable, from which manufacturer would you buy?

A simple example of Bayesian inference

How do we compare such samples?

- By comparing their arithmetic means

Why do we take means?

- If we take the mean as our estimate, the error in our estimate is the mean of the errors in the individual measurements
- Taking the mean as maximum-likelihood estimate implies a **Gaussian error distribution**
- A Gaussian error distribution appropriately reflects our **prior** knowledge about the errors whenever we know nothing about them except perhaps their variance

A simple example of Bayesian inference

What next?

- Let's do a t-test (but first, let's compare variances with an F-test):

```
>> [fh,fp,fci,fstats] = vartest2(xa,xb)

fh =          fp =          fci =          fstats =

     0        0.3297         0.2415          fstat: 3.5114
                            19.0173            df1: 8
                                               df2: 3
```

Variances not significantly different!

```
>> [h, p, ci, stats]= ttest2(xa,xb)

h =           p =           ci =           stats =

     0        0.0665       -21.0191          tstat: -2.0367
                             0.8151             df: 11
                                                sd: 8.2541
```

Means not significantly different!

Is this satisfactory? No, so what can we learn by turning to probability theory (i.e., Bayesian inference)?

A simple example of Bayesian inference

The procedure in brief:

- Determine your question of interest («What is the probability that...?»)
- Specify your model (likelihood and prior)
- Calculate the full posterior using Bayes' theorem
- [Pass to the uninformative limit in the parameters of your prior]
- Integrate out any nuisance parameters
- Ask your question of interest of the posterior

All you need is the rules of probability theory.

(Ok, sometimes you'll encounter a nasty integral – but that's a technical difficulty, not a conceptual one).

A simple example of Bayesian inference

The question:

- What is the probability that the components from manufacturer B have a longer lifetime than those from manufacturer A?
- More specifically: given how much more expensive they are, how much longer do I require the components from B to live.
- Example of a decision rule: if the components from B live 3 hours longer than those from A with a probability of at least 80%, I will choose those from B.

A simple example of Bayesian inference

The model (bear with me, this **will** turn out to be simple):

- likelihood (Gaussian):

$$p(\{x_i\}|\mu,\lambda) = \prod_{i=1}^{n}\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{\lambda}{2}(x_i-\mu)^2\right)$$

- prior (Gaussian-gamma):

$$p(\mu,\lambda|\mu_0,\kappa_0 a_0,b_0) = \mathcal{N}(\mu|\mu_0,(\kappa_0\lambda)^{-1})\mathrm{Gam}(\lambda|a_0,b_0)$$

A simple example of Bayesian inference

The posterior (Gaussian-gamma):

$$p(\mu, \lambda | \{x_i\}) = \mathcal{N}(\mu | \mu_n, (\kappa_n \lambda)^{-1}) \mathrm{Gam}(\lambda | a_n, b_n)$$

Parameter updates:

$$\mu_n = \mu_0 + \frac{n}{\kappa_0 + n}(\bar{x} - \mu_0), \qquad \kappa_n = \kappa_0 + n, \qquad a_n = a_0 + \frac{n}{2}$$

$$b_n = b_0 + \frac{n}{2}\left(s^2 + \frac{\kappa_0}{\kappa_0 + n}(\bar{x} - \mu_0)^2\right)$$

with

$$\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i\,, \qquad s^2 := \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

A simple example of Bayesian inference

The limit for which the prior becomes uninformative:

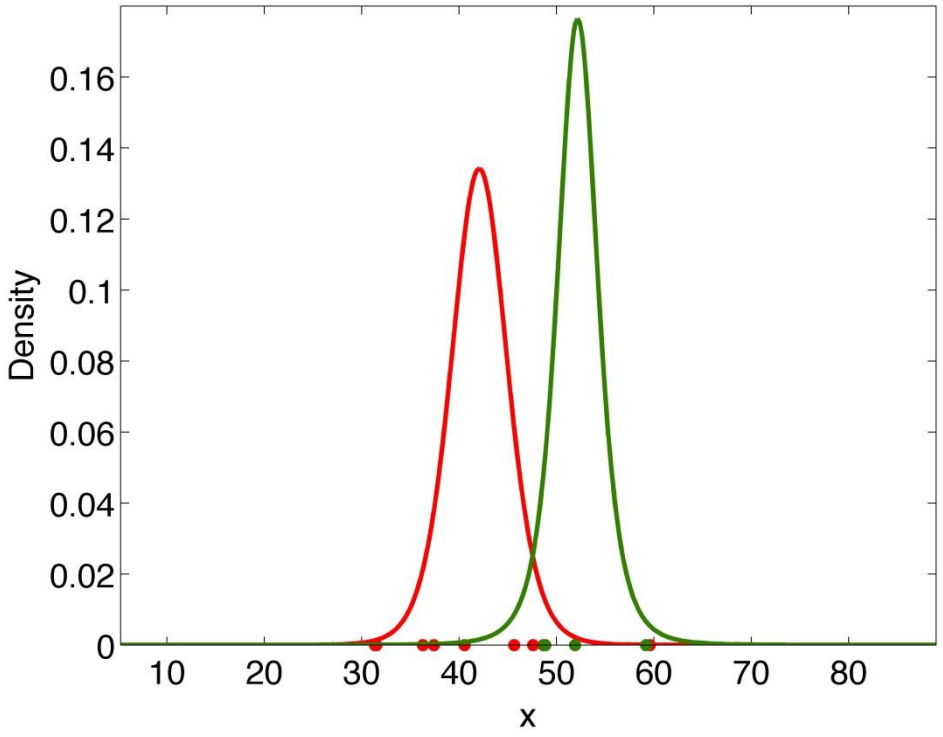
- For $\kappa_0 = 0, a_0 = 0, b_0 = 0$, the updates reduce to:

$$\mu_n = \bar{x} \qquad \kappa_n = n \qquad a_n = \frac{n}{2} \qquad b_n = \frac{n}{2} s^2$$

- As promised, this is really simple: **all you need is n, the number of datapoints; $\bar{x}$, their mean; and s^2, their variance.**
- This means that only the data influence the posterior and all influence from the parameters of the prior has been eliminated.
- The uninformative limit should only ever be taken **after** the calculation of the posterior using a proper prior.

A simple example of Bayesian inference

Integrating out the nuisance parameter λ gives rise to a t-distribution:



A simple example of Bayesian inference

The joint posterior $p(\mu_A, \mu_B | \{x_i\}_A, \{x_k\}_B)$ is simply the product of our two independent posteriors $p(\mu_A | \{x_i\}_A)$ and $p(\mu_B | \{x_k\}_B)$. It will now give us the answer to our question:

$$p(\mu_B - \mu_A > 3) = \int_{-\infty}^{\infty} \mathrm{d}\mu_A \, p(\mu_A | \{x_i\}_A) \int_{\mu_A+3}^{\infty} \mathrm{d}\mu_B \, p(\mu_B | \{x_k\}_B) = 0.9501$$

Note that the t-test told us that there was «no significant difference» even though there is a >95% probability that the parts from B will last at least 3 hours longer than those from A.

Bayesian inference

The procedure in brief:

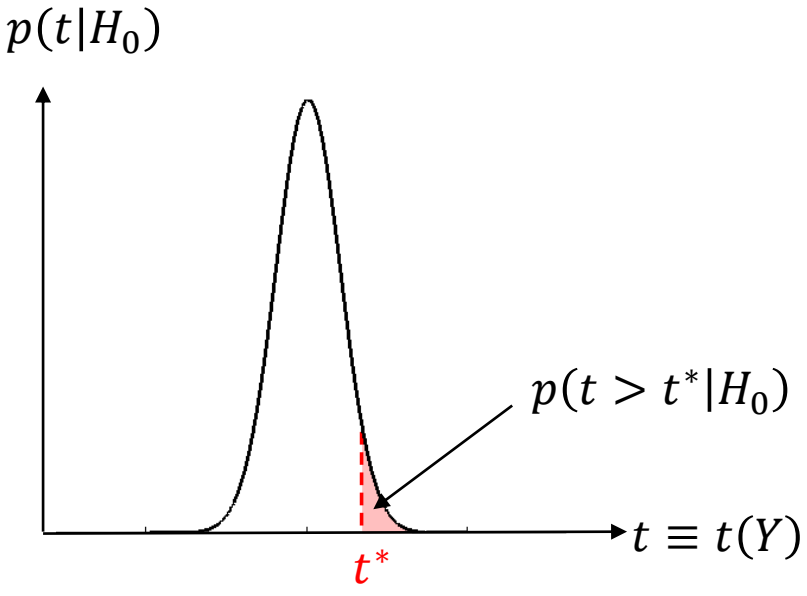
- Determine your question of interest («What is the probability that...?»)
- Specify your model (likelihood and prior)
- Calculate the full posterior using Bayes' theorem
- [Pass to the uninformative limit in the parameters of your prior]
- Integrate out any nuisance parameters
- Ask your question of interest of the posterior

All you need is the rules of probability theory.

Frequentist (or: orthodox, classical) versus Bayesian inference: hypothesis testing

Classical

- define the null, e.g.: $H_0: \vartheta = 0$

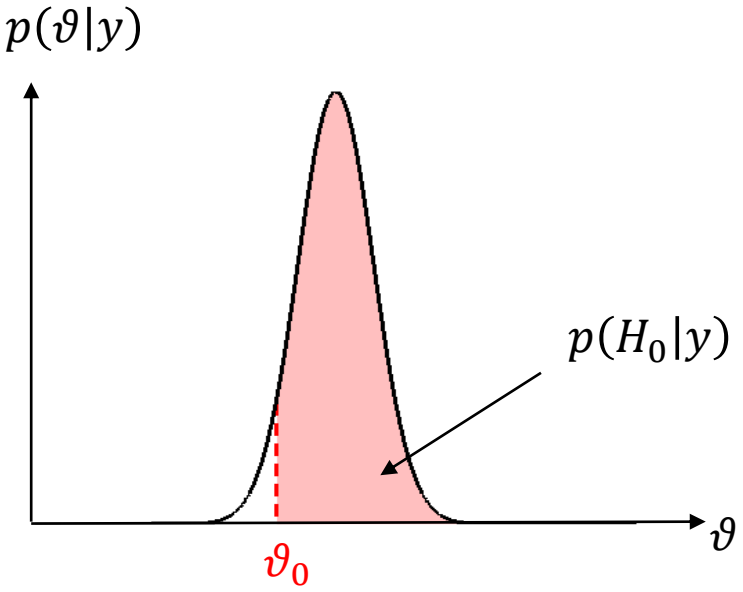


- estimate parameters (obtain test stat. t^*)
- apply decision rule, i.e.:

if $p(t > t^*|H_0) \leq \alpha$ then reject H_0

Bayesian

- invert model (obtain posterior pdf)

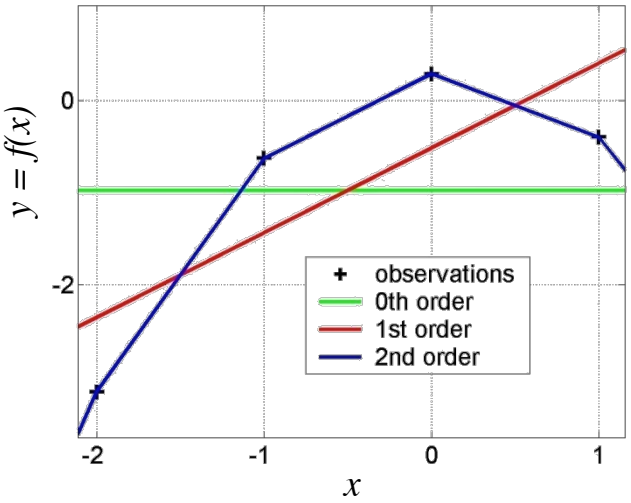


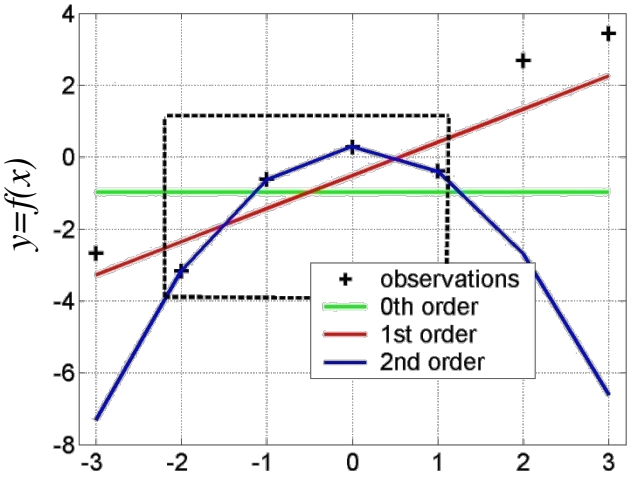
- define the null, e.g.: $H_0: \vartheta > \vartheta_0$
- apply decision rule, i.e.:

if $p(H_0|y) \geq \alpha$ then accept H_0

Model comparison: general principles

- *Principle of parsimony*: «plurality should not be assumed without necessity»
- Automatically enforced by Bayesian model comparison



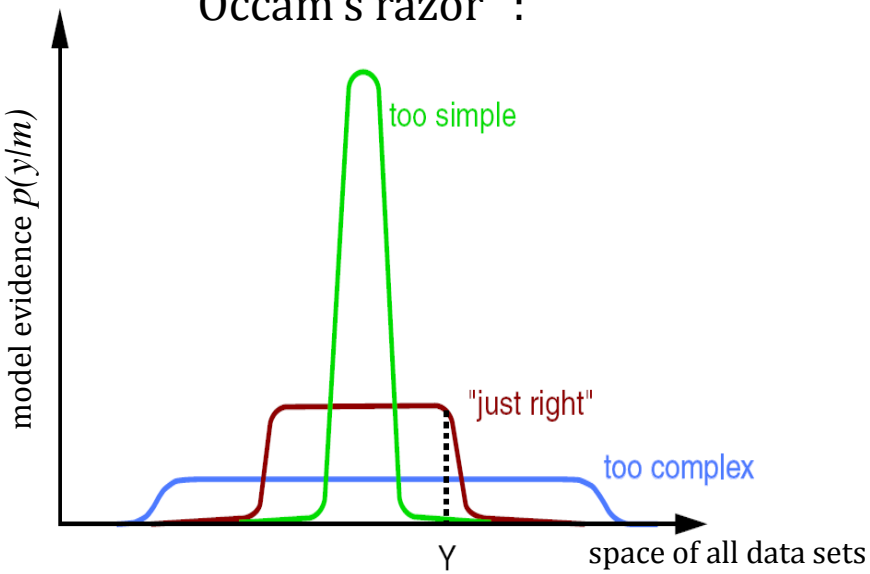


Model evidence:

$$p(y|m) = \int p(y|\vartheta, m)p(\vartheta|m)\mathrm{d}\vartheta$$

$$\approx \exp(accuracy - complexity)$$

"Occam's razor" :



Model comparison: negative variational free energy F

$$\textbf{log}-\textbf{model evidence} := \log p(y|m)$$

$$= \log \int p(y,\vartheta|m)\,\mathrm{d}\vartheta$$

sum rule

$$= \log \int q(\vartheta)\frac{p(y,\vartheta|m)}{q(\vartheta)}\,\mathrm{d}\vartheta$$

multiply by $1 = \frac{q(\vartheta)}{q(\vartheta)}$

$$\geq \int q(\vartheta)\log\frac{p(y,\vartheta|m)}{q(\vartheta)}\,\mathrm{d}\vartheta$$

Jensen's inequality

a lower bound on the log-model evidence

$$=: \boldsymbol{F} = \textbf{negative variational free energy}$$

$$F := \int q(\vartheta)\log\frac{p(y,\vartheta|m)}{q(\vartheta)}\,\mathrm{d}\vartheta$$

$$= \int q(\vartheta)\log\frac{p(y|\vartheta,m)p(\vartheta|m)}{q(\vartheta)}\,\mathrm{d}\vartheta$$

product rule

Kullback-Leibler divergence

$$= \underbrace{\int q(\vartheta)\log p(y|\vartheta,m)\,\mathrm{d}\vartheta}_{\textbf{Accuracy (expected log-likelihood)}} - \underbrace{KL[q(\vartheta),p(\vartheta|m)]}_{\textbf{Complexity}}$$

Model comparison: *F* in relation to Bayes factors, AIC, BIC

$$\textbf{Bayes factor} := \frac{p(y|m_1)}{p(y|m_0)} = \exp\left(\log\frac{p(y|m_1)}{p(y|m_0)}\right) = \exp(\log p(y|m_1) - \log p(y|m_0))$$

$$\approx \exp(F_1 - F_0)$$

[Meaning of the Bayes factor:

Bayes factor

Posterior odds

Prior odds

$$\frac{p(m_1|y)}{p(m_0|y)} = \frac{p(y|m_1)}{p(y|m_0)}\frac{p(m_1)}{p(m_0)}$$

$$\boldsymbol{F} = \int q(\vartheta)\log p(y|\vartheta, m)\,\mathrm{d}\vartheta - KL[q(\vartheta), p(\vartheta|m)]$$

$$= \text{Accuracy} - \text{Complexity}$$

$$\textbf{AIC} := \text{Accuracy} - p$$

Number of parameters

$$\textbf{BIC} := \text{Accuracy} - \frac{p}{2}\log N$$

Number of data points

A note on informative priors

- Any model consists of two parts: likelihood and prior.
- The choice of likelihood requires as much justification as the choice of prior because it is just as «subjective» as that of the prior.
- The data never speak for themselves. They only acquire meaning when seen through the lens of a model. However, this does not mean that all is subjective because models differ in their validity.
- In this light, the widespread concern that informative priors might bias results (while the form of the likelihood is taken as a matter of course requiring no justification) is misplaced.
- Informative priors are an important tool and their use can be justified by establishing the validity (face, construct, and predictive) of the resulting model as well as by model comparison.

A note on uninformative priors

- Using a flat or «uninformative» prior doesn't make you more «data-driven» than anybody else. It's a choice that requires just as much justification as any other.
- For example, if you're studying a small effect in a noisy setting, using a flat prior means assigning the same prior probability mass to the interval covering effect sizes -1 to +1 as to that covering effect sizes +999 to +1001.
- Far from being unbiased, this amounts to a bias in favor of implausibly large effect sizes. Using flat priors is asking for a replicability crisis.
- One way to address this is to collect enough data to swamp the inappropriate priors. A cheaper way is to use more appropriate priors.
- Disclaimer: if you look at my papers, you will find flat priors. I'll try to do better in future. So do as I say, not as I do.

A note on model comparison

- In theory, model comparison is a solved problem: just compute the model evidences using one of our great pieces of software and then compare them using another of our great pieces of software.
- But it's important not to get hung up on the results of formal model comparison. Here's why:
- Before we do model comparison, we define a model space. Usually, we place a flat prior on all models included in the comparison (while effectively placing a prior of zero to all the other conceivable models). This leads to two problems:
- First, we get the same overfitting issues we always get when we use flat priors. We're now overfitting in model space.
- Second, even if we don't use a flat prior, it is often hard to formalize the different plausibility of models in terms of a prior.
- For example, some models are biologically more plausible than others, or they address a question of interest better than another.
- Summa summarum: there is no way around arguing qualitatively for your modeling approach and convincing your audience.

Now for the applications: a shamelessly artificial example

Imagine the following situation:

You're on a boat, you're lost in a storm and trying to get back to shore. A lighthouse has just appeared on the horizon, but you can only see it when you're at the peak of a wave. Your GPS etc., has all been washed overboard, but what you can still do to get an idea of your position is to measure the angle between north and the lighthouse. These are your measurements (in degrees):

76, 73, 75, 72, 77

What number are you going to base your calculation on?

Right. The mean: 74.6. How do you calculate that?

Updates to the mean

The usual way to calculate the mean $\bar{x}$ of $x_1, x_2, \ldots, x_n$ is to take

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

This requires you to remember all x_i, which can become inefficient. Since the measurements arrive sequentially, we would like to update $\bar{x}$ sequentially as the x_i come in – without having to remember them.

It turns out that this is possible. After some algebra (see next slide), we get

$$\bar{x}_{n+1} = \bar{x}_n + \frac{1}{n+1}(x_{n+1} - \bar{x}_n)$$

Updates to the mean

Proof of sequential update formula:

$$\bar{x}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i = \frac{x_{n+1}}{n+1} + \frac{1}{n+1}\sum_{i=1}^{n} x_i = \frac{x_{n+1}}{n+1} + \frac{n}{n+1}\underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i}_{=\bar{x}_n} =$$

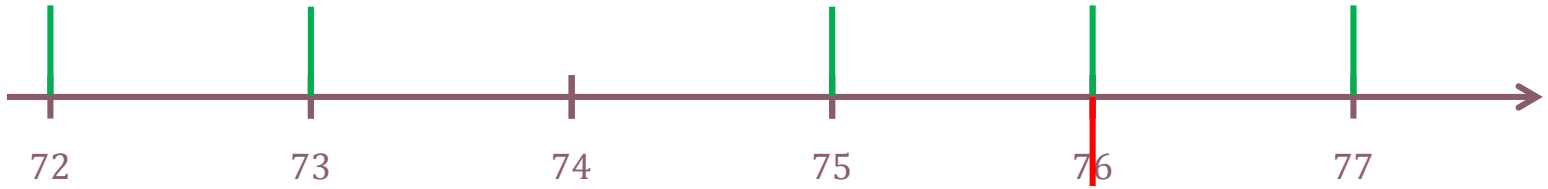
$$= \frac{x_{n+1}}{n+1} + \frac{n}{n+1}\bar{x}_n = \bar{x}_n + \frac{x_{n+1}}{n+1} + \frac{n}{n+1}\bar{x}_n - \frac{n+1}{n+1}\bar{x}_n =$$

$$= \bar{x}_n + \frac{1}{n+1}(x_{n+1} + (n - n - 1)\bar{x}_n) = \bar{x}_n + \frac{1}{n+1}(x_{n+1} - \bar{x}_n)$$

q.e.d.

Uncertainty: updates to the mean

The seqential updates in our example now look like this:



$$\bar{x}_1 = 76$$

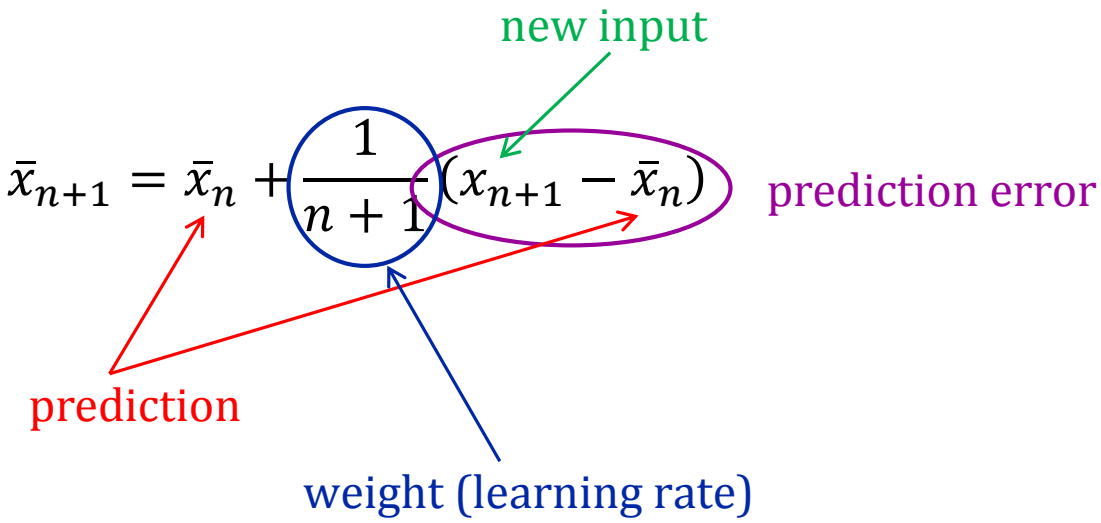
$$\bar{x}_2 = 76 + \frac{1}{2}(73 - 76) = 74.5$$

$$\bar{x}_3 = 74.5 + \frac{1}{3}(75 - 74.5) = 74.\overline{6}$$

$$\bar{x}_4 = 74.\overline{6} + \frac{1}{4}(72 - 74.\overline{6}) = 74$$

$$\bar{x}_5 = 74 + \frac{1}{5}(77 - 74) = 74.6$$

What are the building blocks of the updates we've just seen?



Is this a general pattern?

- More specifically, does it generalize to Bayesian inference?
- «Bayesian inference» simply means inference on uncertain quantities according to the rules of probability theory (i.e., according to logic).
- Agents who use Bayesian inference will make better predictions (provided they have a good model of their environment), which will give them an evolutionary advantage.
- We may therefore assume that evolved biological agents use Bayesian inference, or a close approximation to it.
- So is Bayesian inference based on predictions that are updated using uncertainty-weighted prediction errors?

Updates in a simple Gaussian model

- Think boat, lighthouse, etc., again, but now we're doing Bayesian inference.

- Before we make the next observation, our belief about the true angle ϑ can be described by a Gaussian prior:

$$p(\vartheta) \sim \mathcal{N}(\mu_\vartheta, \pi_\vartheta^{-1})$$

- The likelihood of our observation is also Gaussian, with precision π_ε :

$$p(x|\vartheta) \sim \mathcal{N}(\vartheta, \pi_\varepsilon^{-1})$$

- Bayes' rule now tells us that the posterior is Gaussian again:

$$p(\vartheta|x) = \frac{p(x|\vartheta)p(\vartheta)}{\int p(x|\vartheta')p(\vartheta')\mathrm{d}\vartheta'} \sim \mathcal{N}\left(\mu_{\vartheta|x}, \pi_{\vartheta|x}^{-1}\right)$$

Updates in a simple Gaussian model

- Here's how the updates to the sufficent statistics μ and π describing our belief look like:

$$\pi_{\vartheta|x} = \pi_\vartheta + \pi_\varepsilon$$

$$\mu_{\vartheta|x} = \mu_\vartheta + \frac{\pi_\varepsilon}{\pi_{\vartheta|x}}(x - \mu_\vartheta)$$

prediction error

prediction

weight (learning rate)$=\frac{\text{how much we're learning here}}{\text{how much we already know}}$

- So it's the same story all over again: the mean is updated by an uncertainty-weighted (more specifically: precision-weighted) prediction error.

- The size of the update is proportional to the likelihood precision and inversely proportional to the posterior precision.

- This pattern is not specific to the univariate Gaussian case, but generalizes to Bayesian updates for all exponential families of likelihood distributions with conjugate priors (i.e., to all formal descriptions of inference you are ever likely to need).

The analogy with simple mean updating goes further

- Reminder (Gaussian update):

$$\mu_{\vartheta|x} = \mu_\vartheta + \frac{\pi_\varepsilon}{\pi_{\vartheta|x}}(x - \mu_\vartheta) = \mu_\vartheta + \frac{\pi_\varepsilon}{\pi_\vartheta + \pi_\varepsilon}(x - \mu_\vartheta)$$

- Reducing by $\pi_\varepsilon$the fraction of precisions that make the learning rate, we get

$$\mu_{\vartheta|x} = \mu_\vartheta + \frac{1}{\frac{\pi_\vartheta}{\pi_\varepsilon} + 1}(x - \mu_\vartheta)$$

- This is again our equation for updating an arithmetic mean, but with n replaced by $\frac{\pi_\vartheta}{\pi_\varepsilon}$.

- This shows that Bayesian inference on the mean of a Gaussian distribution entails nothing more than updating the arithmetic mean of observations with $\frac{\pi_\vartheta}{\pi_\varepsilon} =: \nu$ as a proxy for the number of prior observations, i.e. for the **weight of the prior relative to the observation**.

Generalization to all exponential families of distributions

- Many of the most widely used probability distributions are families of exponential distributions.

- For example, the Gaussian distribution is an exponential family of distributions (and so are the beta, gamma, binomial, Bernoulli, multinomial, categorical, Dirichlet, Wishart, Gaussian-gamma, log-Gaussian, multivariate Gaussian, Poisson, and exponential distributions, among others). This means it can be written the following way:

$$p(\boldsymbol{x}|\boldsymbol{\vartheta}) = h(\boldsymbol{x}) \exp(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{T}(\boldsymbol{x}) - A(\boldsymbol{\vartheta})) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right)$$

with

$$\boldsymbol{x} = x, \qquad \boldsymbol{\vartheta} = (\mu, \sigma)^{\mathrm{T}}, \qquad h(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}, \qquad \boldsymbol{\eta}(\boldsymbol{\vartheta}) = \left(\frac{\mu}{\sigma}, -\frac{1}{2\sigma}\right)^{\mathrm{T}}, \qquad \boldsymbol{T}(\boldsymbol{x}) = (x, x^2)^{\mathrm{T}}, \qquad A(\boldsymbol{\vartheta}) = \frac{\mu^2}{\sigma} + \frac{\ln \sigma}{2}$$

- This allows us to look at Bayesian belief updating in a very general way for all exponential families of distributions.

Generalization to all exponential families of distributions

- Our likelihood is an exponential family in its general form:

$$p(\boldsymbol{x}|\boldsymbol{\vartheta}) = h(\boldsymbol{x}) \exp(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{T}(\boldsymbol{x}) - A(\boldsymbol{\vartheta}))$$

- The vector $\boldsymbol{T}(\boldsymbol{x})$ (a function of the observation $\boldsymbol{x}$) is called the sufficient statistic.

- For the prior, we may assume that we have made ν observations with sufficient statistic $\boldsymbol{\xi}$:

$$p(\boldsymbol{\vartheta}|\boldsymbol{\xi}, \nu) = z(\boldsymbol{\xi}, \nu) \exp\big(\nu(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\xi} - A(\boldsymbol{\vartheta}))\big) \quad \text{(where } z(\xi, \nu) \text{ is a normlization constant)}$$

- It then turns out that the posterior has the same form, but with an updated $\boldsymbol{\xi}$ and ν replaced with $\nu + 1$:

$$p(\boldsymbol{\vartheta}|\boldsymbol{x}, \boldsymbol{\xi}, \nu) = z(\boldsymbol{\xi}', \nu + 1) \exp\big((\nu + 1)(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\xi}' - A(\boldsymbol{\vartheta}))\big)$$

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + \frac{1}{\nu + 1}(\boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{\xi})$$

Proof of the update equation

$$\overbrace{p(\boldsymbol{\vartheta}|\boldsymbol{x},\boldsymbol{\xi},\nu)}^{\text{posterior}} \propto \overbrace{p(\boldsymbol{x}|\boldsymbol{\vartheta})}^{\text{likelihood}}\ \overbrace{p(\boldsymbol{\vartheta}|\boldsymbol{\xi},\nu)}^{\text{prior}}$$

$$= h(\boldsymbol{x})\exp(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot\boldsymbol{T}(\boldsymbol{x}) - A(\boldsymbol{\vartheta}))z(\boldsymbol{\xi},\nu)\exp\big(\nu(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot\boldsymbol{\xi} - A(\boldsymbol{\vartheta}))\big)$$

$$\propto \exp\big(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot(\boldsymbol{T}(\boldsymbol{x}) + \nu\boldsymbol{\xi}) - (\nu+1)A(\boldsymbol{\vartheta})\big)$$

$$= \exp\left((\nu+1)\left(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot\frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) + \nu\boldsymbol{\xi}) - A(\boldsymbol{\vartheta})\right)\right)$$

$$= \exp\left((\nu+1)\left(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot\left(\boldsymbol{\xi} + \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) + \nu\boldsymbol{\xi} - (\nu+1)\boldsymbol{\xi})\right) - A(\boldsymbol{\vartheta})\right)\right)$$

$$= \exp\left((\nu+1)\left(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot\left(\underbrace{\boldsymbol{\xi} + \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{\xi})}_{=:\boldsymbol{\xi}'}\right) - A(\boldsymbol{\vartheta})\right)\right)$$

$$\Longrightarrow \quad p(\boldsymbol{\vartheta}|\boldsymbol{x},\boldsymbol{\xi},\nu) = z(\boldsymbol{\xi}',\nu')\exp\Big(\nu'\big(\boldsymbol{\eta}(\boldsymbol{\vartheta})\cdot\boldsymbol{\xi}' - A(\boldsymbol{\vartheta})\big)\Big)$$

$$\text{with } \nu' \coloneqq \nu+1, \qquad \boldsymbol{\xi}' \coloneqq \boldsymbol{\xi} + \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{\xi})$$

q.e.d.

Some examples

- **Univariate Gaussian** model with unkown mean but **known precision** (our example from the beginning):

$$\boldsymbol{T}(x) = x$$

- This means updating beliefs about the mean simply requires tracking the mean of observations

- **Univariate Gaussian** model with unkown mean and unkown precision:

$$\boldsymbol{T}(x) = (x, x^2)^{\mathrm{T}}$$

- Updating beliefs about both mean and precision of a Gaussian requires tracking the means of observations and squared observations; this amounts to the first and second moments by which a Gaussian distribution is fully characterized.

- In the **multivariate Gaussian** case we have $\boldsymbol{T}(\boldsymbol{x}) = \left(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\right)^{\mathrm{T}}$

Some examples

- **Bernoulli** model (one out of two possible outcomes, coded as 0 and 1; e.g., coin flipping):

$$\boldsymbol{T}(x) = x$$

- The prior here turns out to be a **beta distribution** corresponding to ν pseudo-observations with mean ξ. All we need to do to get the posterior (i.e., to update our belief) is to update the mean as new observations come in.

- **Categorical** model (one out of several possible outcomes, with the observed outcome coded as 1, the rest as 0)

$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{x}$$

- The prior and posterior here are **Dirichlet distributions**, and again, all we need to do to update beliefs that have a Dirichlet form is to track the means of observed succeses (1) and failures (0).

Some examples

- **Beta** model (an outcome bounded between 0 and 1):

$$\boldsymbol{T}(x) = (\ln x\,, \ln(1-x))^{\mathrm{T}}$$

- **Gamma** model (an outcome bounded below at 0):

$$\boldsymbol{T}(x) = (\ln x\,, x)^{\mathrm{T}}$$

- Now that we have dealt with beliefs about states that are binary (Bernoulli), categorical, bounded on both sides (beta), bounded on one side (gamma), and unbounded (Gaussian), we have most kinds of states we can have beliefs about.

- **All Bayesian (i.e., probabilistic, rational) updates of such beliefs take the form of precision-weighted prediction errors.**

Limitations

- Examples of distributions that are not exponential families: Student's *t*, Cauchy
- These distributions are popular because of their «fat tails». However, fat tails can also be achieved with appropriate hierarchies of Gaussians (cf. the hierarchical Gaussian filter, HGF)
- A further kind of distributions that are not exponential families are found in mixture models.
- Such models are popular because of they provide multimodal distributions. But again, appropriate hierarchies of distributions may save the day.

"Every good regulator of a system must be a model of that system" (Conant & Ashby, 1970)

Abstract:

«*The design of a complex regulator often includes the making of a model of the system to be regulated. The making of such a model has hitherto been regarded as optional, as merely one of many possible ways.*

In this paper a theorem is presented which shows, under very broad conditions, that any regulator that is maximally both successful and simple must *be isomorphic with the system being regulated. (The exact assumptions are given.) Making a model is thus necessary.*

The theorem has the interesting corollary that ***the living brain****, so far as it is to be successful and efficient as a regulator for survival,* **must *proceed****, in learning,* ***by the formation of a model (or models) of its environment.***»

Systems theory as the conceptual bridge between clinical phenomena and neuronal pathophysiology

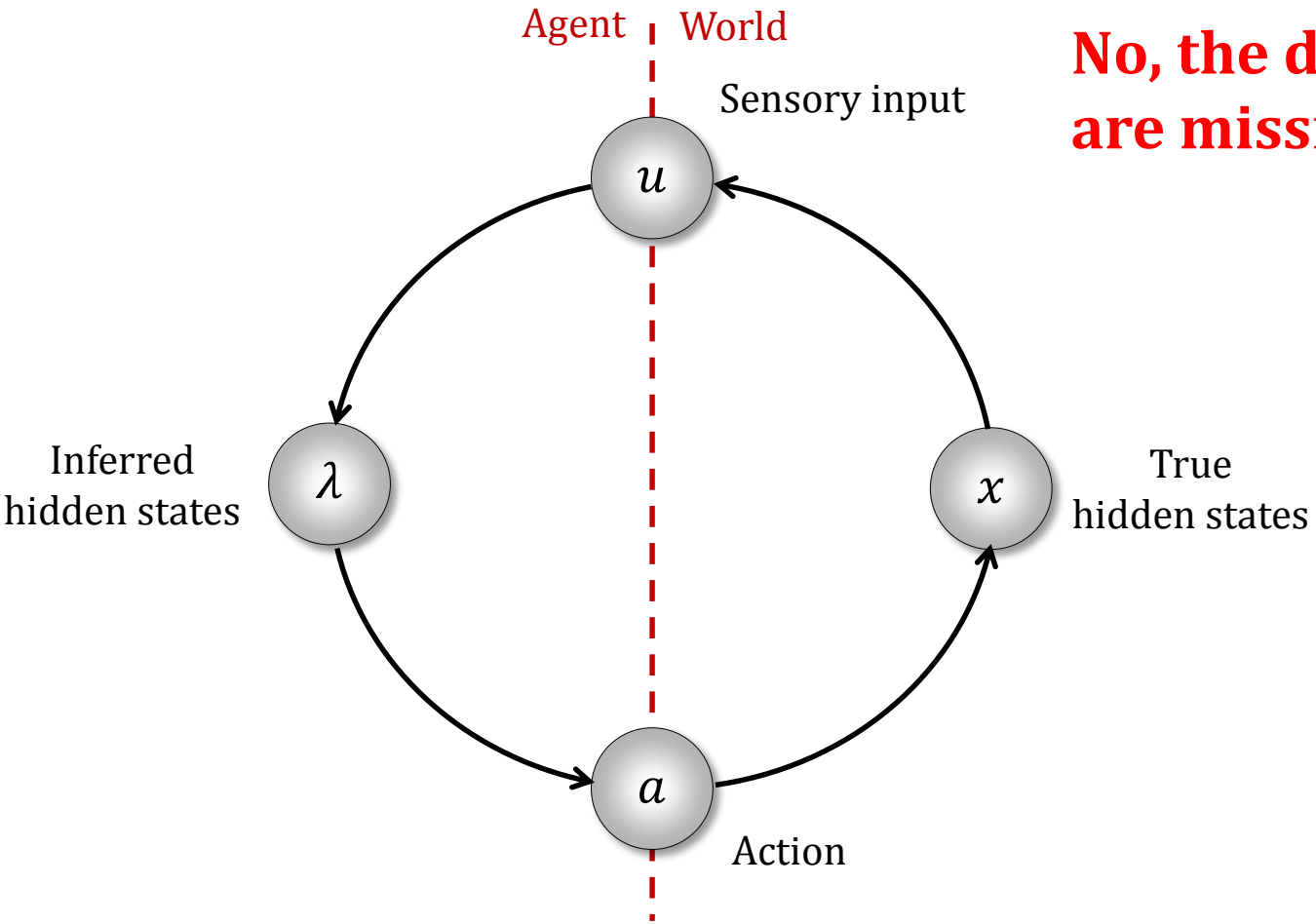
- Belief updating by precision-weighted prediction errors provides a conceptual framework in which both clinical phenomena and neurobiological findings can be interpreted.

- For examples of this approach, see Adams et al. (2013) (psychosis), or Lawson et al. (2014), Quattrocki & Friston (2014) (autism).

- Summary: the mind needs to be a model of its environment $\Longrightarrow$ needs to perform Bayesian inference $\Longrightarrow$ needs to use precision-weighting of prediction errors $\Longrightarrow$ if that's all the mind does, it's also all that can go wrong $\Longrightarrow$ both clinical manifestations and the neurobiology of psychiatric disorders must be interpretable in these terms.

- Now that we have this conceptual framework, we can start filling it with content.

Does inference as we've described it adequately describe the situation of actual biological agents?



No, the dynamics are missing!

What about dynamics?

- Up to now, we've only looked at inference on static quantities, but biological agents live in a continually changing world.
- In our example, the boat's position changes and with it the angle to the lighthouse.
- How can we take into account that old information becomes obsolete? If we don't, our learning rate becomes smaller and smaller because our eqations were derived under the assumption that we're accumulating information about a stable quantity.

What's the simplest way to keep the learning rate from going too low?

- Keep it constant!
- So, taking the update equation for the mean of our observations as our point of departure...

$$\bar{x}_n = \bar{x}_{n-1} + \frac{1}{n}(x_n - \bar{x}_{n-1}),$$

- ... we simply replace $\frac{1}{n}$ with a constant α:

$$\mu_n = \mu_{n-1} + \alpha(x_n - \mu_{n-1}).$$

- This is called *Rescorla-Wagner learning* [although it wasn't this line of reasoning that led Rescorla & Wagner (1972) to their formulation].

Does a constant learning rate solve our problems?

- Partly: it implies a certain rate of forgetting because it amounts to taking only the $n = \frac{1}{\alpha}$ last data points into account. But...
- ... if the learning rate is supposed to reflect uncertainty in Bayesian inference, then how do we
- (a) know that α reflects the right level of uncertainty at any one time, and
- (b) account for changes in uncertainty if α is constant?
- What we really need is an adaptive learning that accurately reflects uncertainty.

Needed: an adaptive learning rate that accurately reflects uncertainty

- This requires us to think a bit more about what kinds of uncertainty we are dealing with.

- A possible taxonomy of uncertainty is (cf. Yu & Dayan, 2003; Payzan-LeNestour & Bossaerts, 2011):

- (a) **outcome uncertainty** that remains unaccounted for by the model, called *risk* by economists (π_ε in our Bayesian example); this uncertainty remains even when we know all parameters exactly,

- (b) **informational** or *expected* uncertainty about the value of model parameters ($\pi_{\vartheta|x}$ in the Bayesian example),

- (c) **environmental** or *unexpected* uncertainty owing to changes in model parameters (not accounted for in our Bayesian example, hence unexpected).

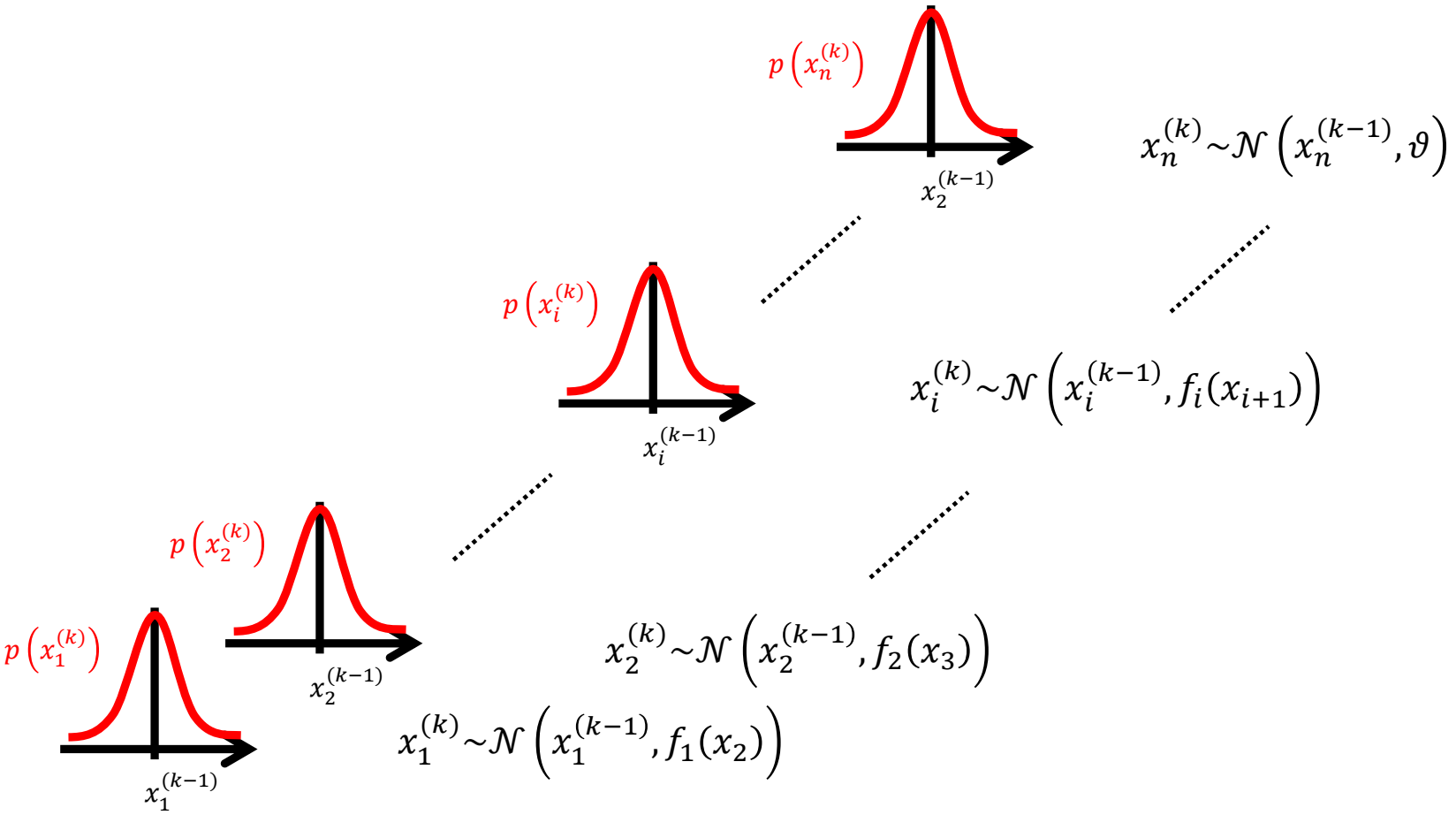
An adaptive learning rate that accurately reflects uncertainty

- Various efforts have been made to come up with an adaptive learning rate:
  - Kalman (1960)
  - Sutton (1992)
  - Nassar et al. (2010)
  - Payzan-LeNestour & Bossaerts (2011)
  - Mathys et al. (2011)
  - Wilson et al. (2013)

- The Kalman filter is optimal for linear dynamical systems, but realistic data usually require non-linear models.

- Mathys et al. use a generic non-linear hierarchical Bayesian model that allows us to derive update equations that are optimal in the sense that they minimize surprise.

The hierarchical Gaussian filter (HGF)



The hierarchical Gaussian filter (HGF)

- At the outcome level (i.e., at the very bottom of the hierarchy), we have

$$u^{(k)} \sim \mathcal{N}\left(x_1^{(k)}, \hat{\pi}_u^{-1}\right)$$

- This gives us the following update for our belief on x_1 (our quantity of interest):

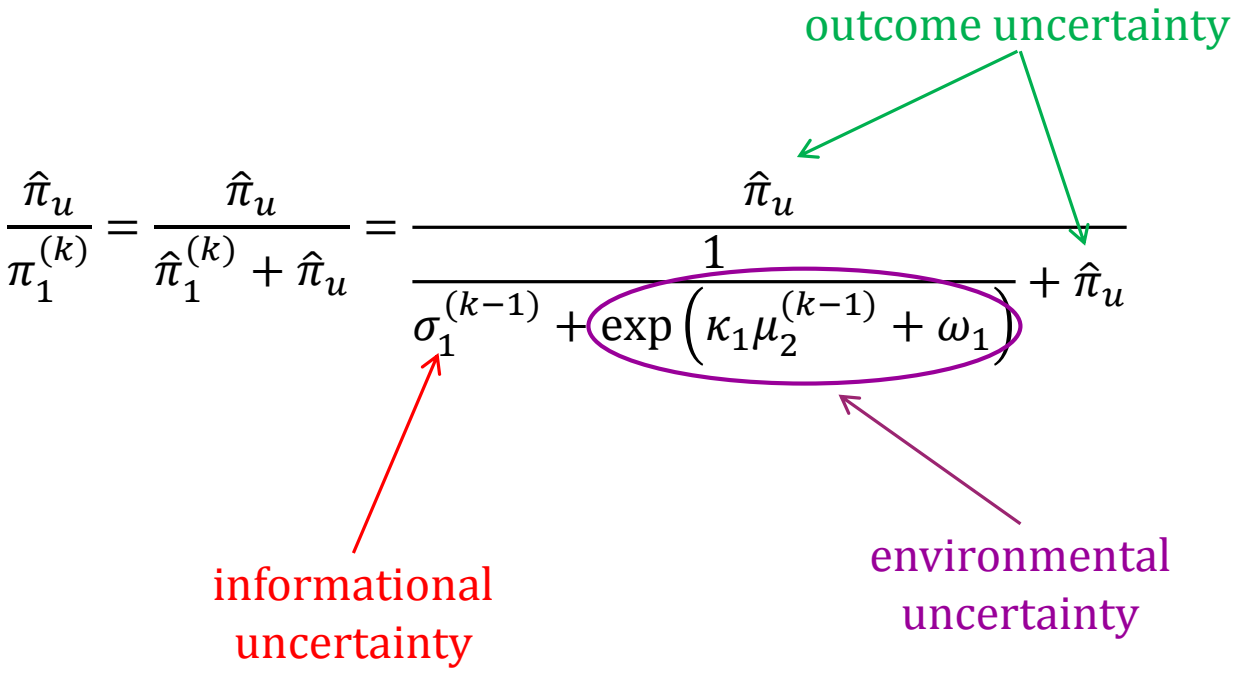
$$\pi_1^{(k)} = \hat{\pi}_1^{(k)} + \hat{\pi}_u$$

$$\mu_1^{(k)} = \mu_1^{(k-1)} + \frac{\hat{\pi}_u}{\pi_1^{(k)}}\left(u^{(k)} - \mu_1^{(k-1)}\right)$$

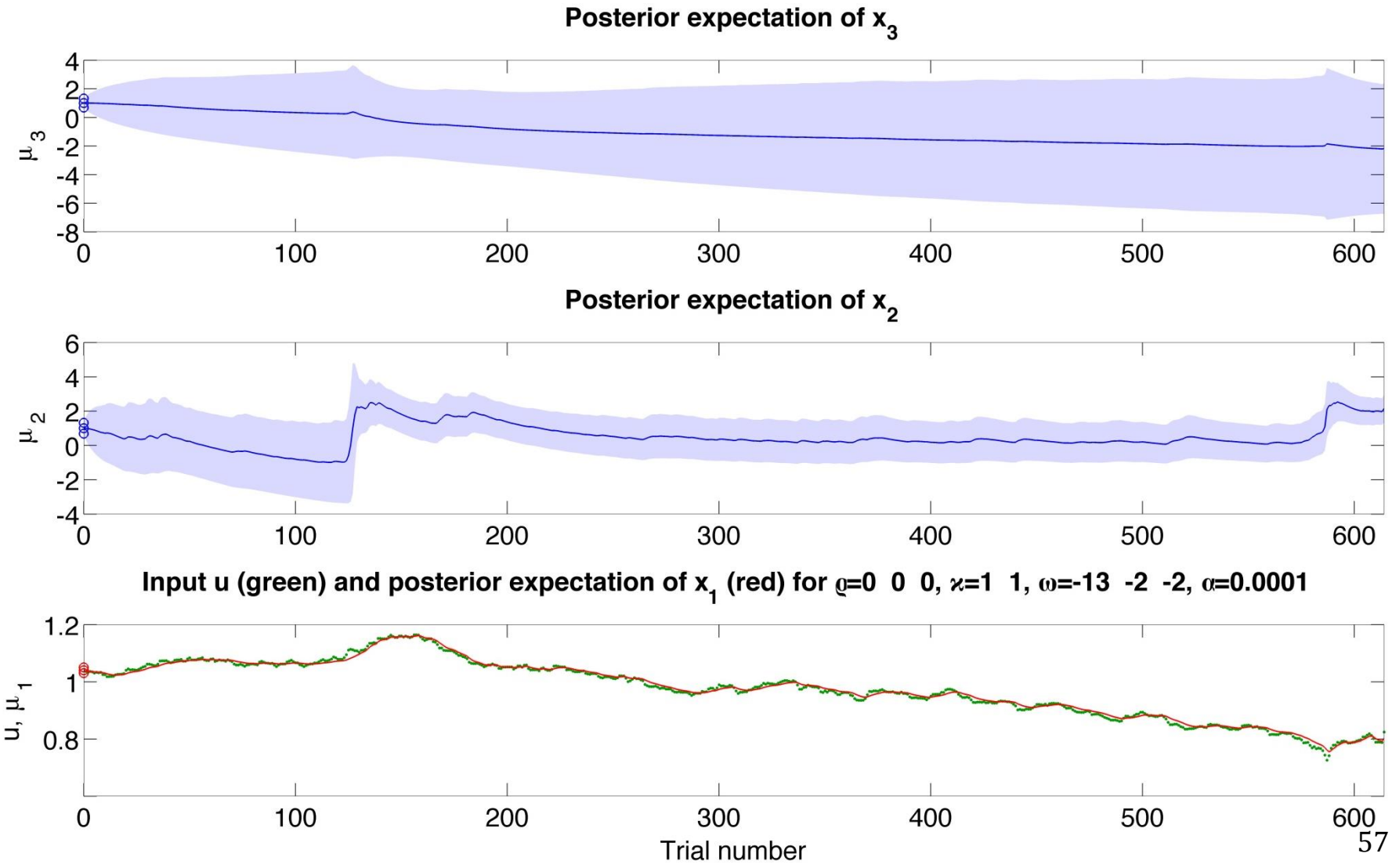
- The familiar structure again – but now with a learning rate that is responsive to all kinds of uncertainty, including environmental (unexpected) uncertainty.

The learning rate in the HGF

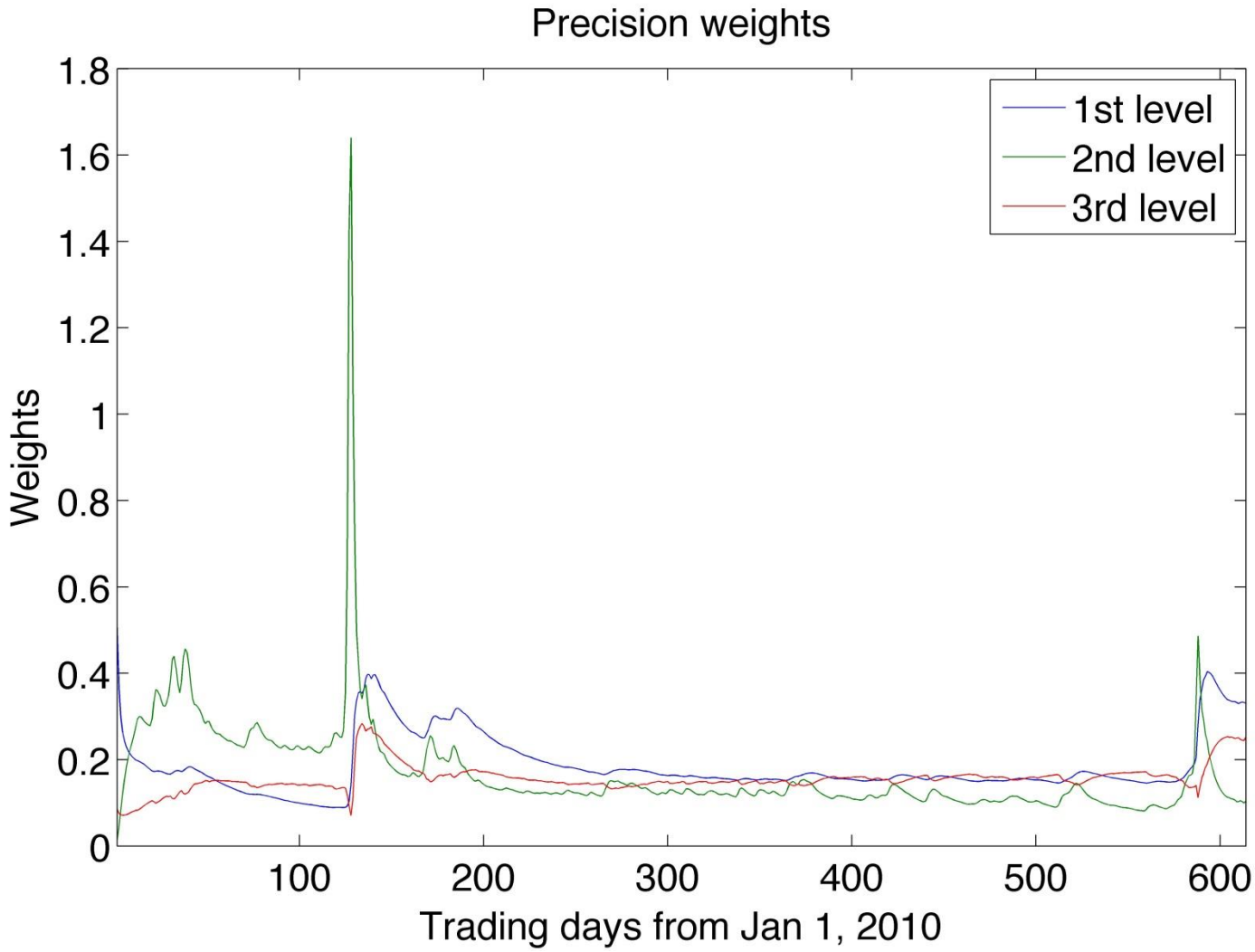
Unpacking the learning rate, we see:



3-level HGF for continuous observations



3-level HGF for continuous observations



VAPEs and VOPEs

The updates of the belief on x_1 are driven by value prediction errors (VAPEs)

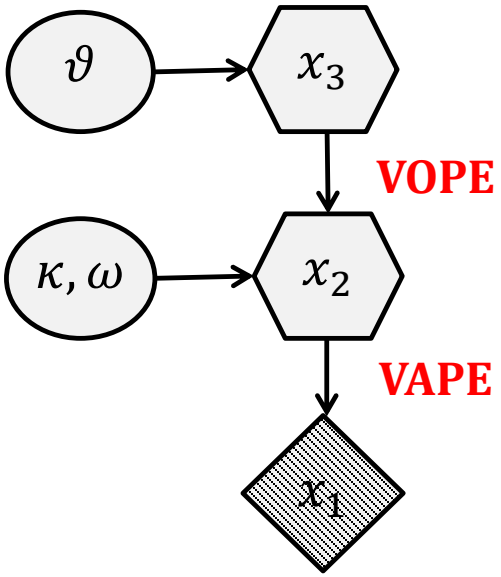
$$\mu_1^{(k)} = \mu_1^{(k-1)} + \frac{\hat{\pi}_u}{\pi_1^{(k)}}\left(u^{(k)} - \mu_1^{(k-1)}\right), \quad \textbf{VAPE}$$

while the x_2-updates are driven by volatility prediction errors (VOPEs)

$$\mu_2^{(k)} = \mu_2^{(k-1)} + \frac{1}{2}\kappa_1 v_1^{(k)} \frac{\hat{\pi}_1^{(k)}}{\pi_2^{(k)}} \delta_1^{(k)} \quad \textbf{VOPE}$$

$$\delta_1^{(k)} \stackrel{\text{def}}{=} \frac{\sigma_1^{(k)} + \left(\mu_1^{(k)} - \mu_1^{(k-1)}\right)^2}{\sigma_1^{(k-1)} + \exp\left(\kappa_1 \mu_2^{(k-1)} + \omega_1\right)} - 1$$

3-level HGF for binary observations



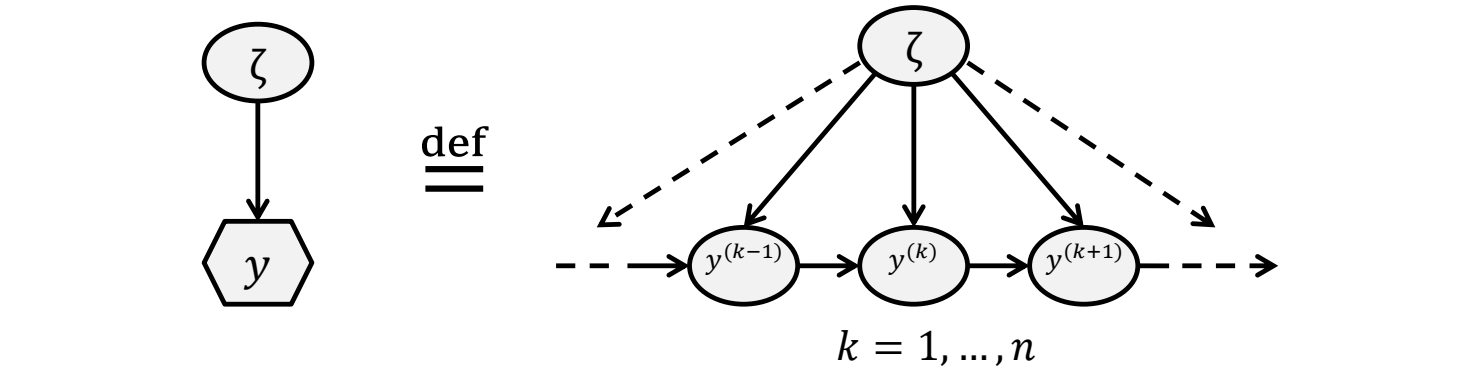
$$x_3^{(k)} \sim \mathcal{N}\left(x_3^{(k-1)}, \vartheta\right)$$

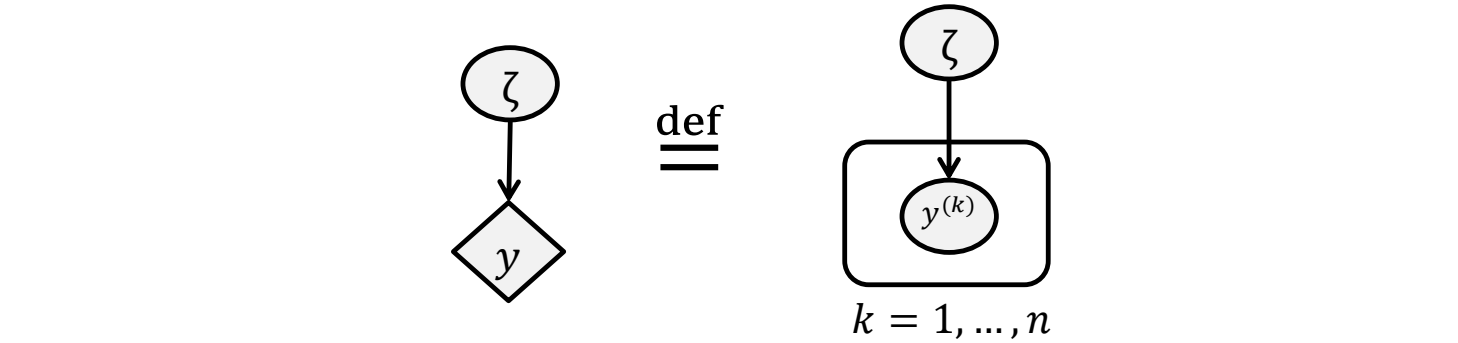
$$x_2^{(k)} \sim \mathcal{N}\left(x_2^{(k-1)}, \exp\left(\kappa x_3^{(k)} + \omega\right)\right)$$

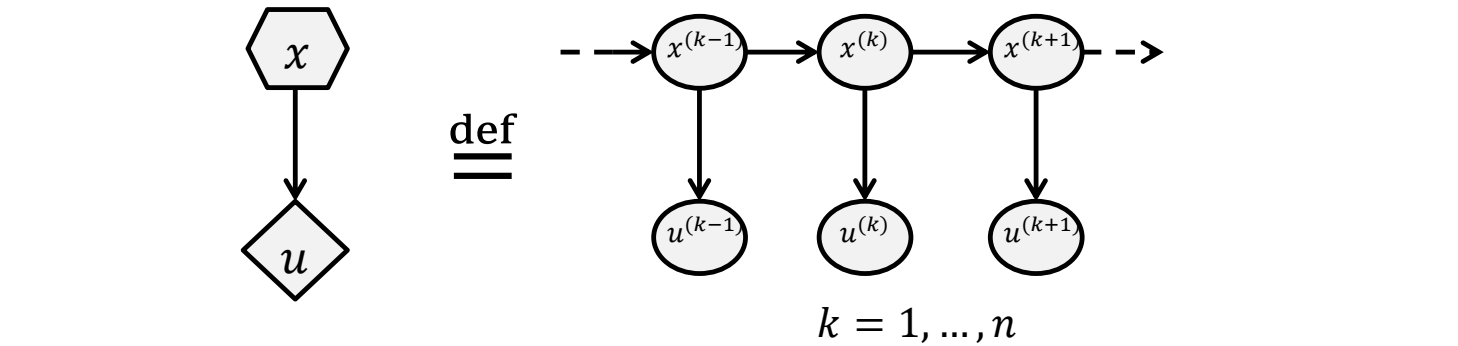
$$x_1^{(k)} \sim \text{Bern}\left(x_2^{(k)}\right)$$

Mathys et al., 2011; Iglesias et al., 2013; Vossel et al., 2014a; Hauser et al., 2014; Diaconescu et al., 2014; Vossel et al., 2014b; ...

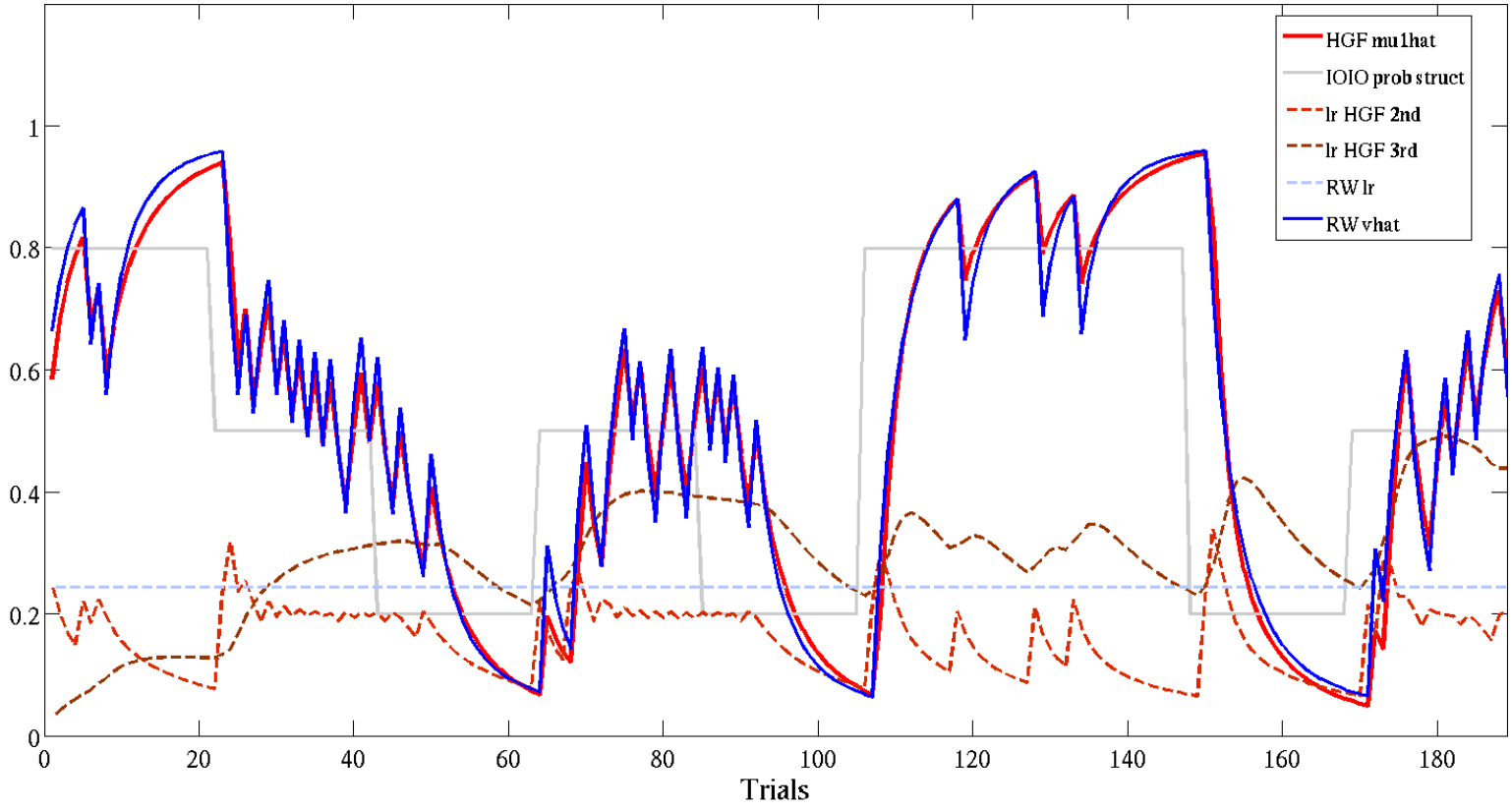
Taking it all together: notation





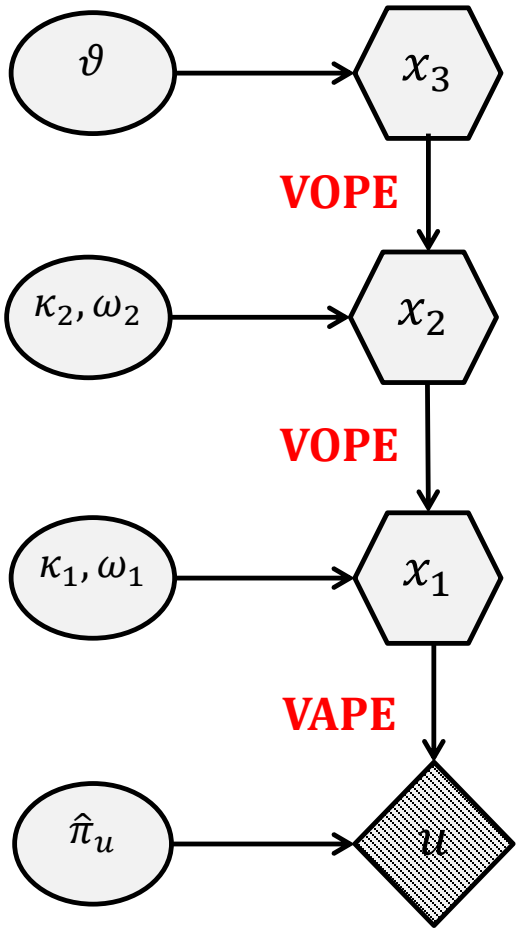


The learning rate in the HGF



Andreea Diaconescu

3-level HGF for continuous observations



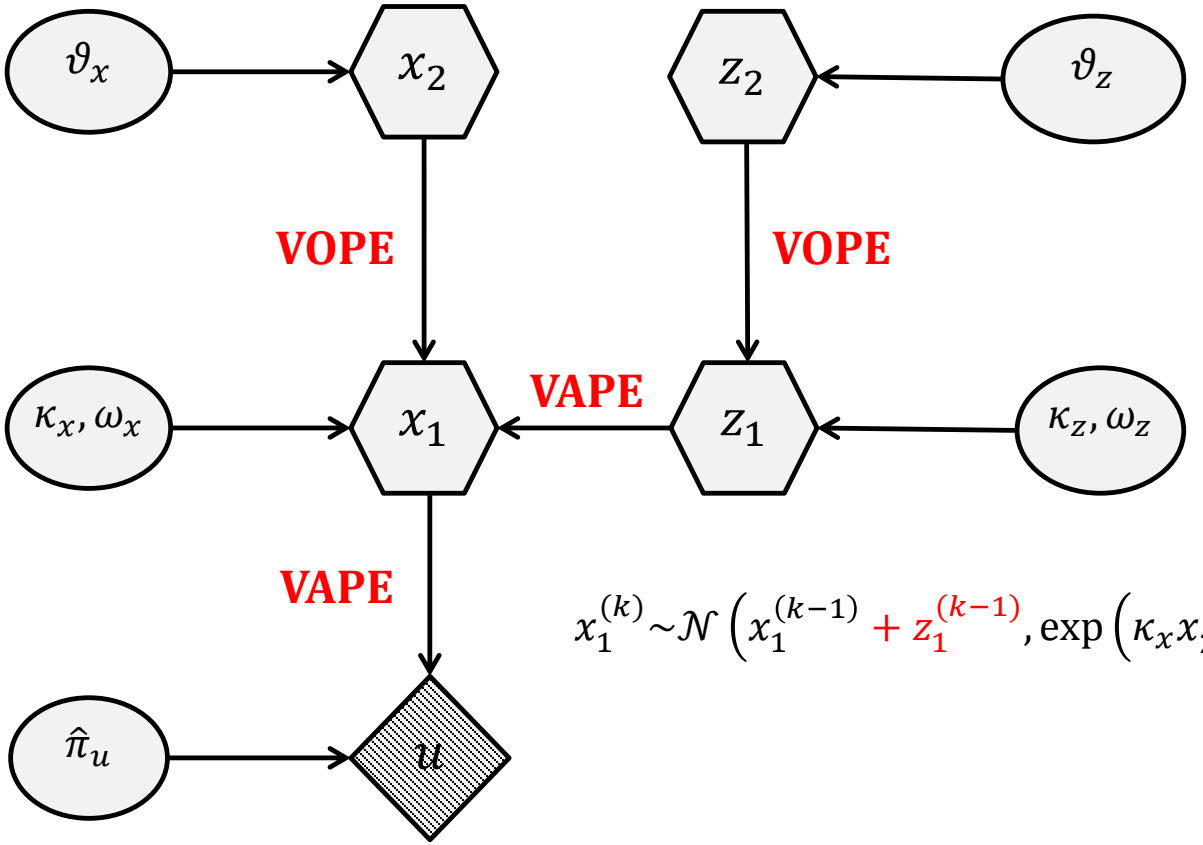
$$x_3^{(k)} \sim \mathcal{N}\left(x_3^{(k-1)}, \vartheta\right)$$

$$x_2^{(k)} \sim \mathcal{N}\left(x_2^{(k-1)}, \exp\left(\kappa_2 x_3^{(k)} + \omega_2\right)\right)$$

$$x_1^{(k)} \sim \mathcal{N}\left(x_1^{(k-1)}, \exp\left(\kappa_1 x_2^{(k)} + \omega_1\right)\right)$$

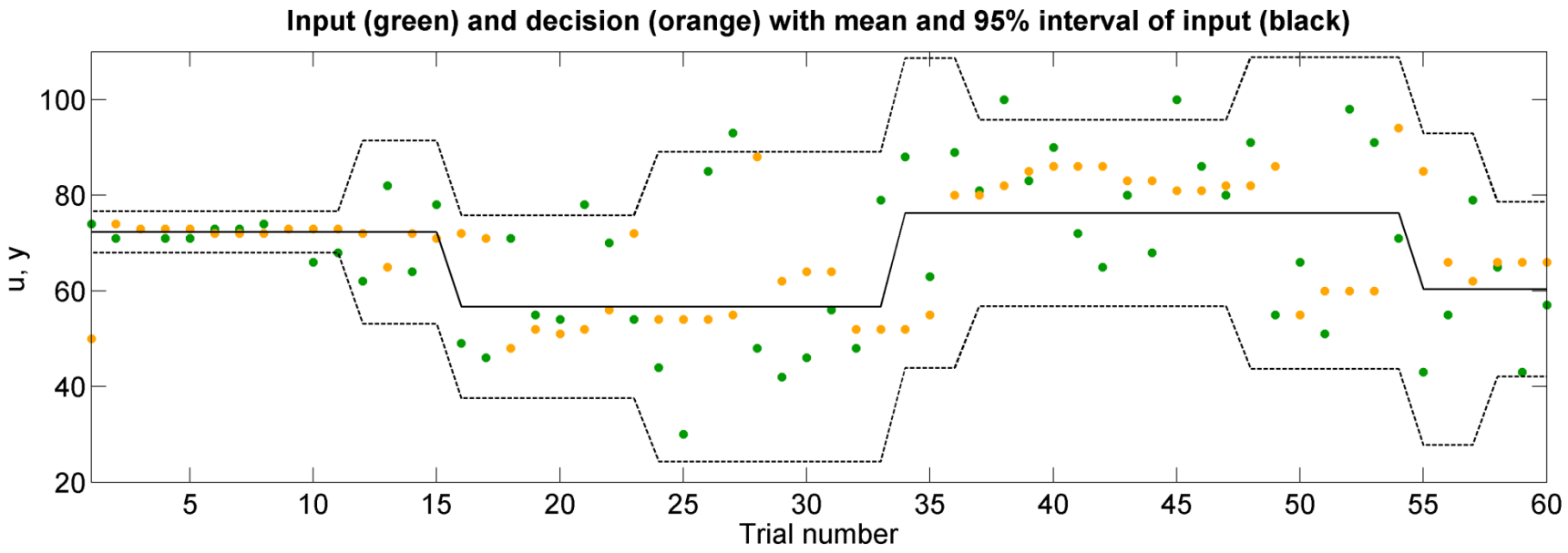
$$u^{(k)} \sim \mathcal{N}\left(x_1^{(k)}, \hat{\pi}_u^{-1}\right)$$

Variable drift



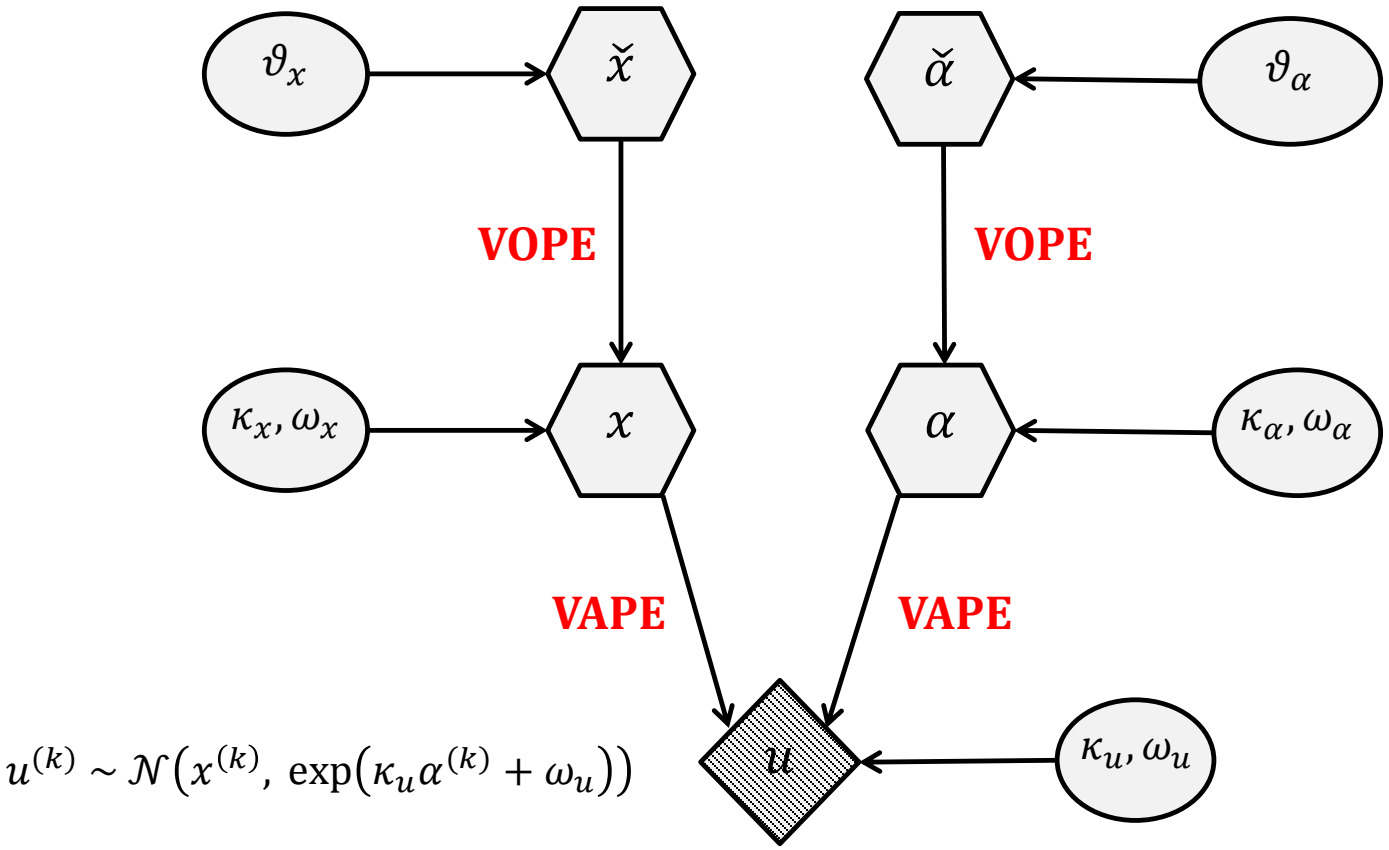
$$x_1^{(k)} \sim \mathcal{N}\left(x_1^{(k-1)} + z_1^{(k-1)}, \exp\left(\kappa_x x_2^{(k)} + \omega_x\right)\right)$$

Jumping Gaussian estimation task

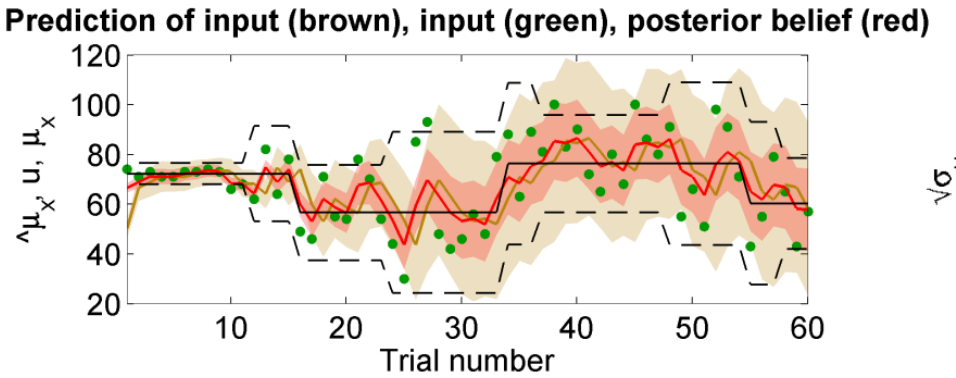


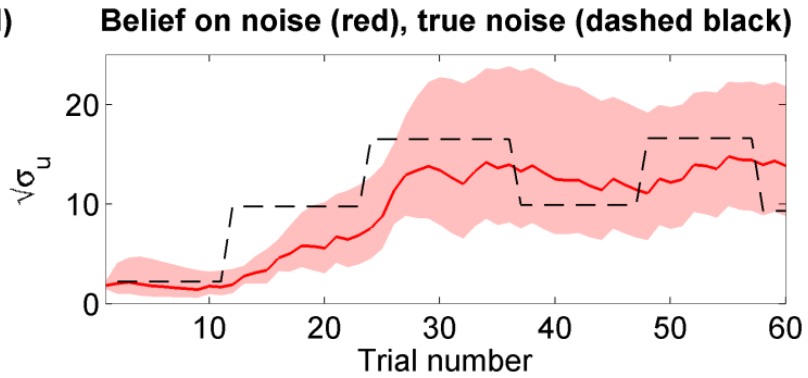
Chaohui Guo

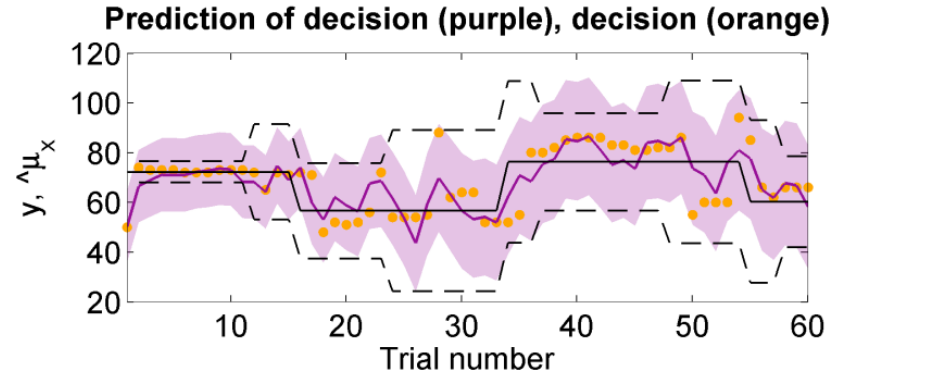
Independent mean and variance model

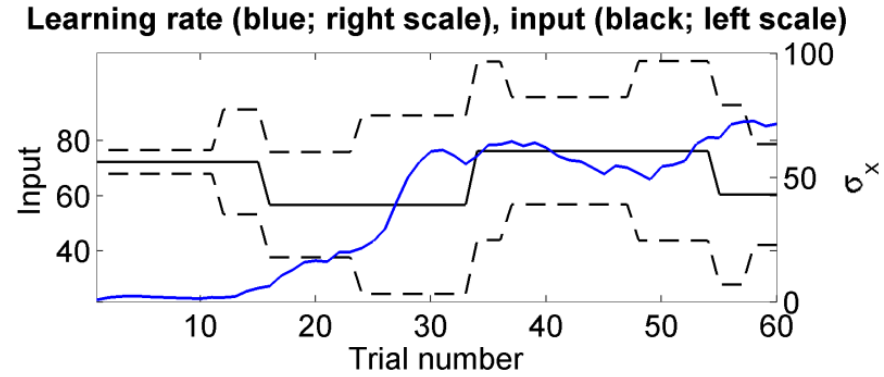


Jumping Gaussian estimation task





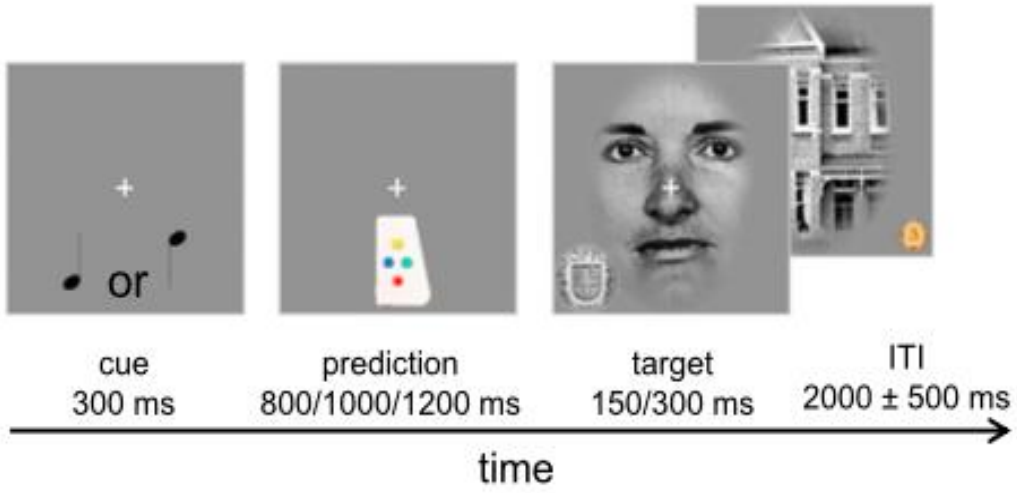


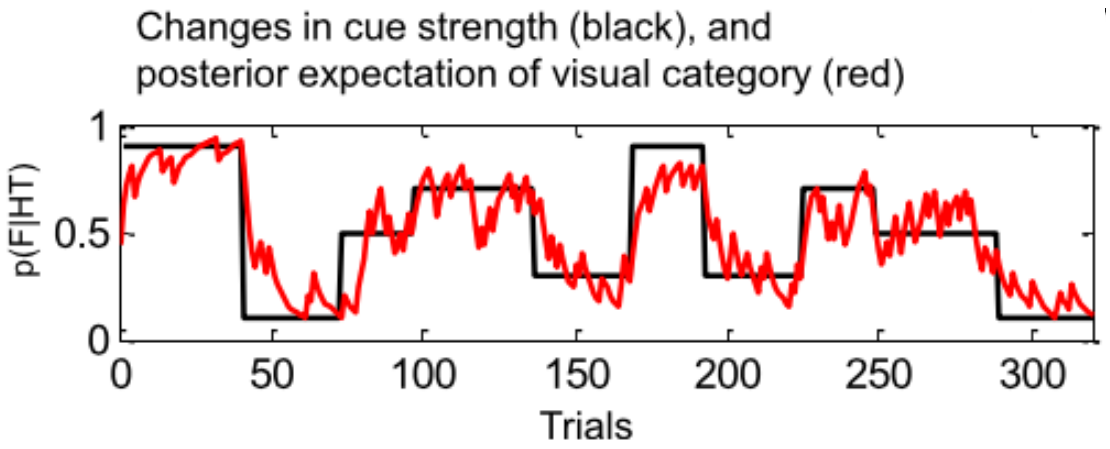


How to reveal the precision-weighting of prediction errors when simple exponential-family likelihoods will not do

- Formulate the problem hierarchically (i.e., imitate evolution: when it built a brain that supports a mind which is a model of its environment, it came up with a (largely) hierarchical solution)
- Separate levels using a mean-field approximation
- Derive update equations

HGF: empirical evidence (Iglesias et al., 2013)





HGF: empirical evidence (Iglesias et al., 2013)

Model comparison:

	Behavioral study		fMRI study 1		fMRI study 2	
BMS results	PP	XP	PP	XP	PP	XP
HGF1	0.8435	1	0.7422	1	0.7166	1
HGF2	0.0259	0	0.0200	0	-	-
HGF3	0.0361	0	0.1404	0	0.1304	0
Sutton	0.0685	0	0.0710	0	0.0761	0
RW	0.0260	0	0.0264	0	0.0769	0

HGF: empirical evidence (Iglesias et al., 2013)

Model comparison:

	Behavioral study		fMRI study 1		fMRI study 2	
BMS results	PP	XP	PP	XP	PP	XP
HGF1	0.8435	1	0.7422	1	0.7166	1
HGF2	0.0259	0	0.0200	0	-	-
HGF3	0.0361	0	0.1404	0	0.1304	0
Sutton	0.0685	0	0.0710	0	0.0761	0
RW	0.0260	0	0.0264	0	0.0769	0

HGF: empirical evidence (Iglesias et al., 2013)

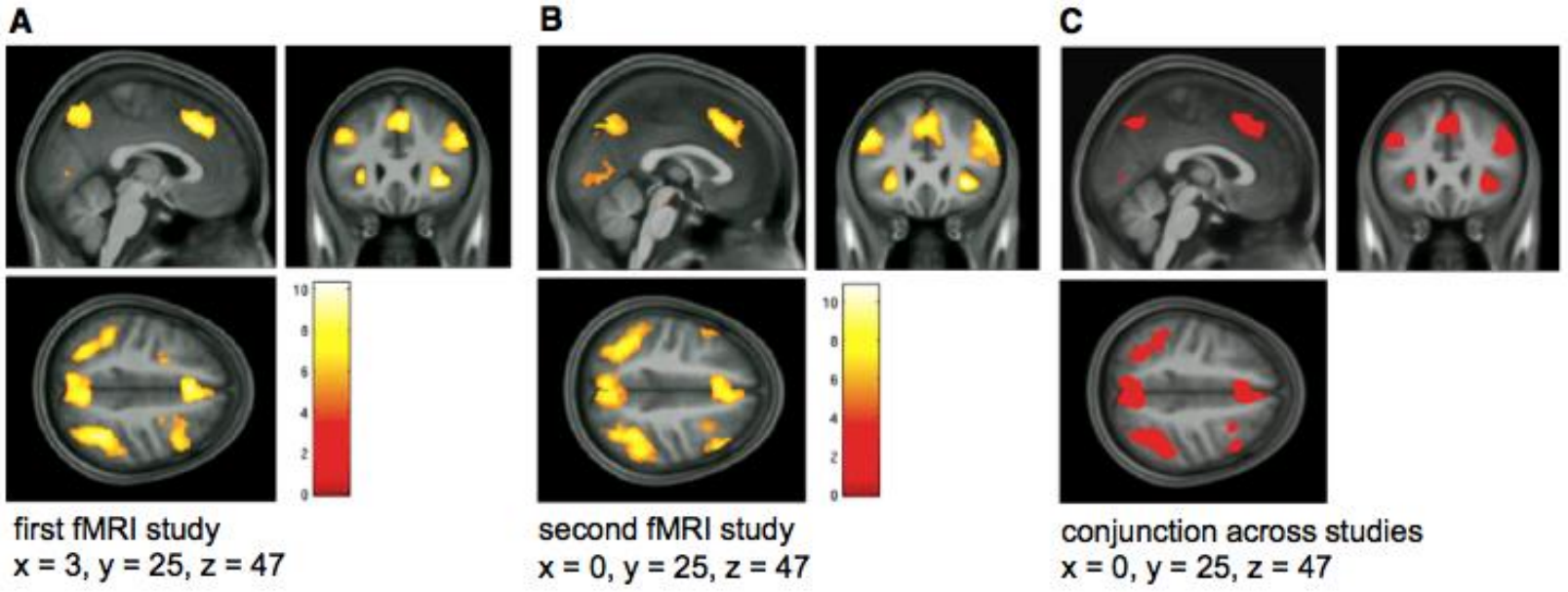


Figure 2. Whole-Brain Activations by ε_2

Activations by precision-weighted prediction error about visual stimulus outcome, ε_2, in the first fMRI study (A) and the second fMRI study (B). Both activation maps are shown at a threshold of $p < 0.05$, FWE corrected for multiple comparisons across the whole brain. To highlight replication across studies, (C) shows the results of a "logical AND" conjunction, illustrating voxels that were significantly activated in both studies.

HGF: empirical evidence (Iglesias et al., 2013)

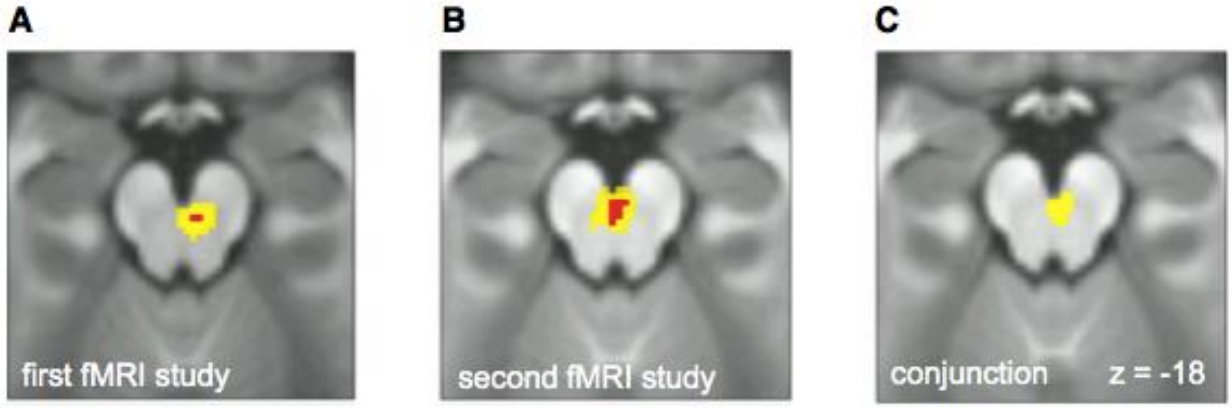


Figure 3. Midbrain Activation by ε_2
Activation of the dopaminergic VTA/SN associated with precision-weighted prediction error about stimulus category, ε_2. This activation is shown both at $p < 0.05$ FWE whole-brain corrected (red) and $p < 0.05$ FWE corrected for the volume of our anatomical mask comprising both dopaminergic and cholinergic nuclei (yellow).
(A) Results from the first fMRI study.
(B) Second fMRI study.
(C) Conjunction (logical AND) across both studies.

HGF: empirical evidence (Iglesias et al., 2013)

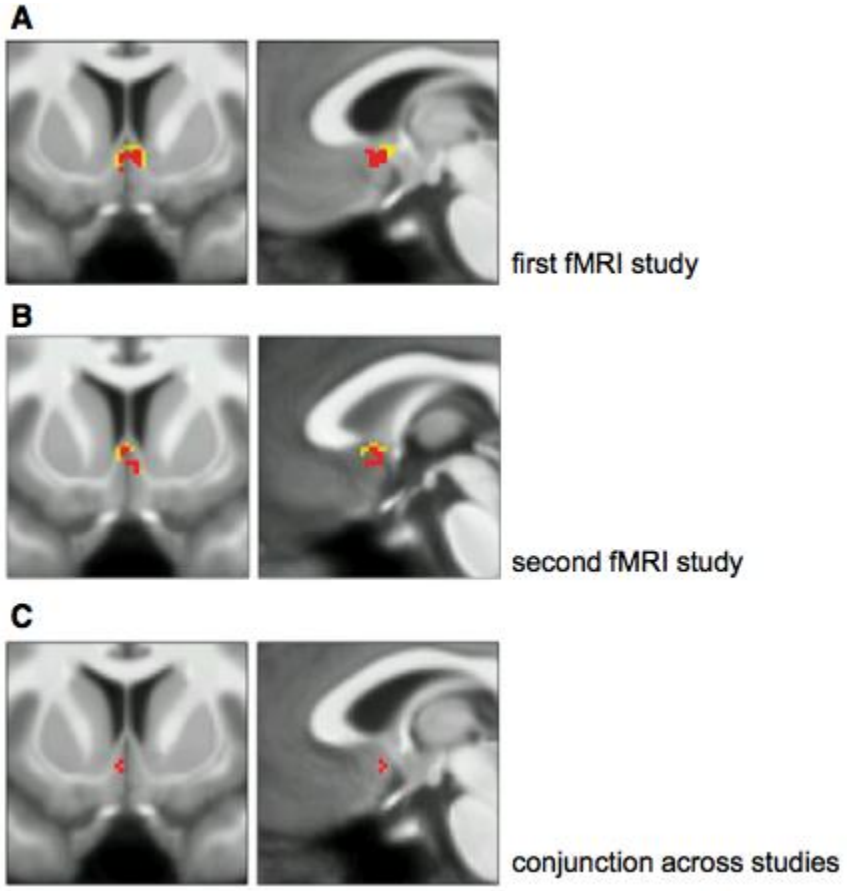


Figure 6. Basal Forebrain Activations by ε_3
Activation of the cholinergic basal forebrain associated with precision-weighted prediction error about stimulus probabilities ε_3 within the anatomically defined mask. For visualization of the activation area we overlay the results thresholded at $p < 0.05$ FWE corrected for the entire anatomical mask (red) on the results thresholded at $p < 0.001$ uncorrected (yellow) in the first (A: $x = 3, y = 9, z = -8$) and the second fMRI study (B: $x = 0, y = 10, z = -8$). (C) The conjunction analysis ("logical AND") across both studies ($x = 2, y = 11, z = -8$).

HGF: empirical evidence (Diaconescu et al., in preparation)

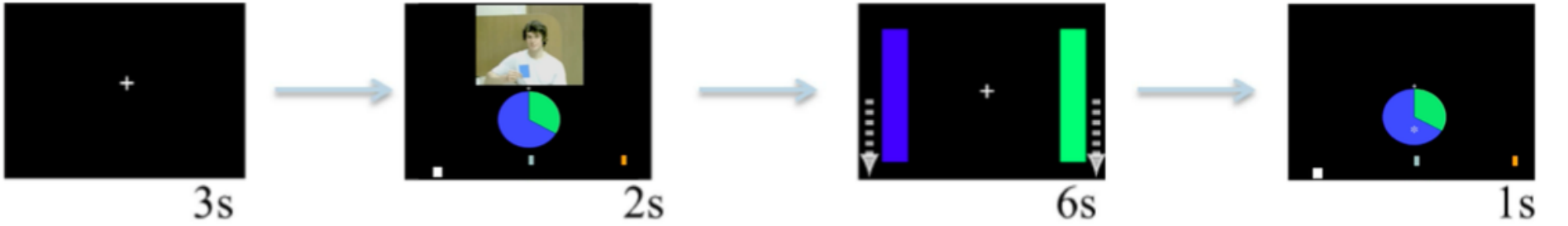
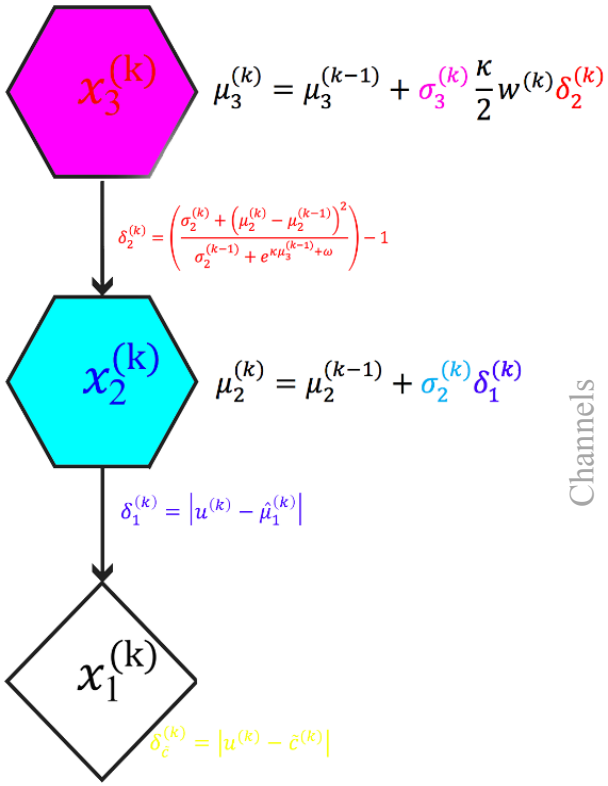
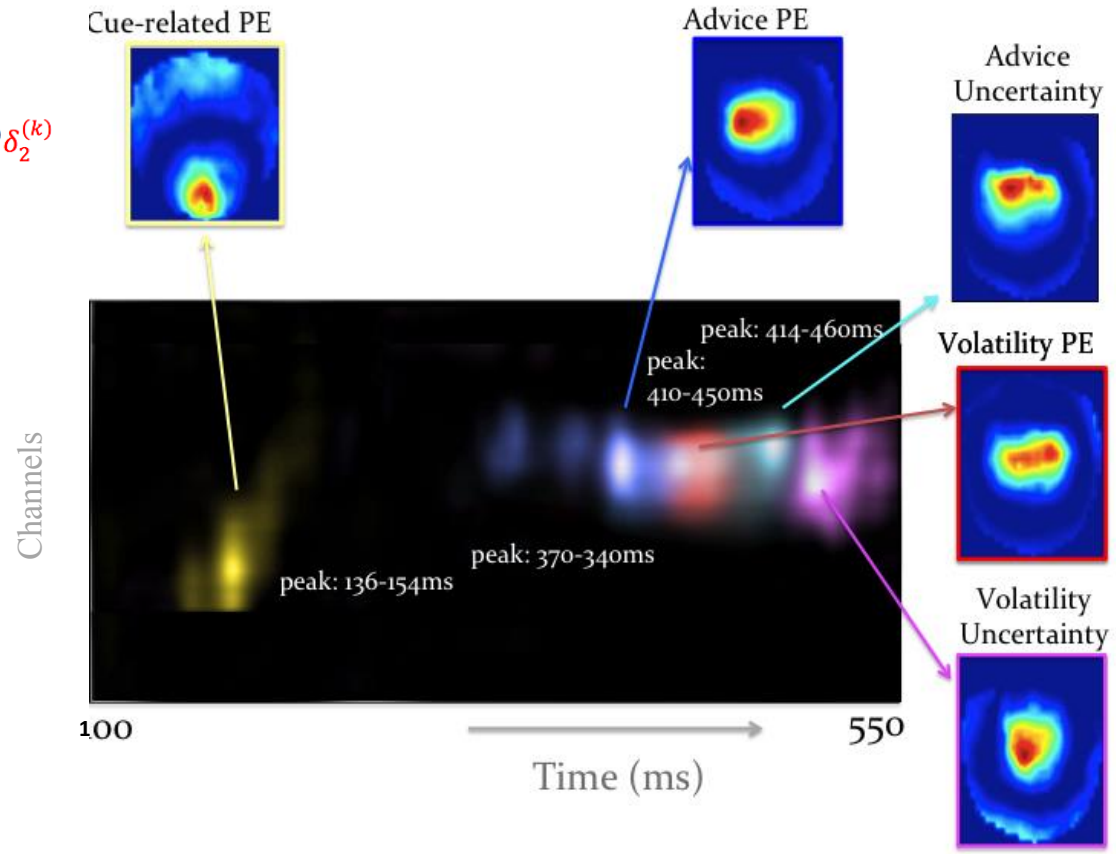


Fig. 1. Experimental Paradigm: 100 male volunteers played a binary lottery task and received advice about which option to choose from a more informed agent who was also incentivized to influence the participants' choices. To decide whether to take his advice into account, participants also inferred on the other's intentions and how they changed in time.

HGF: empirical evidence (Diaconescu et al., in preparation)





$N = 100$

How to estimate and compare models: the HGF Toolbox

- Available at

  http://www.translationalneuromodeling.org/tapas

- Start with README, manual, and interactive demo
- Modular, extensible
- Matlab-based

Summary

- We have to make good predictions to avoid surprise and survive, that is we have to use probabilistic (i.e., Bayesian) inference based on a good model of our environment.

- Bayesian inference means updating beliefs by uncertainty- (i.e., precision-) weighted prediction errors.

- Precision-weighting has to take account of all forms of uncertainty.

- A breakdown in this may be the root of many psychopathological phenomena.

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