Problem 1

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The random variable ξ has Poisson distribution with the parameter λ. If ξ = k
we perform k Bernoulli trials with the probability of success p. Let us define
the random variable η as the number of successful outcomes of Bernoulli trials.
Prove that η has Poisson distribution with the parameter pλ.
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Definitions

1. The function e^x can be expressed as:

   $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. A random variable X with Poisson distribution and parameter λ is defined by:

   $$\Pr(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

3. If we perform k Bernoulli trials with the probability of success p, the probability of having exactly s succesful outcomes is given by:

   $$\frac{k!}{s!(k-s)!} p^s (1-p)^{(k-s)}$$

4. The relation between a joint probability and a conditional probability of two events is given as:

   $$\Pr(A, B) = \Pr(A \mid B) \Pr(B)$$

Proof

By definition (2), the probability that $\xi = k$ is given by:

$$\Pr(\xi = k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad (5)$$

By definition (3), the conditional probability of having $\eta = s$ for a given value of k is:

$$\Pr(\eta = s \mid \xi = k) = \frac{k!}{s!(k-s)!} p^s (1-p)^{(k-s)} \qquad (6)$$

The actual probability of having $\eta = s$ can be obtained by marginalizing out the random variable ξ from the joint probability:

$$\Pr(\eta = s) = \sum_{k=s}^{\infty} \Pr(\eta = s, \xi = k) \qquad (7)$$

Note that, since we can't have more successful outcomes than trials, we start the summation from $k = s$ on equation (7).

$$\Pr(\eta = s) = \sum_{k=s}^{\infty} \Pr(\eta = s \mid \xi = k) \Pr(\xi = k) \qquad (8)$$

$$\Pr(\eta = s) = \sum_{k=s}^{\infty} \left(\frac{k!}{s!(k-s)!} p^s (1-p)^{k-s}\right) \left(e^{-\lambda} \frac{\lambda^k}{k!}\right) \qquad (9)$$

$$\Pr(\eta = s) = e^{-\lambda} \frac{p^s}{s!} \sum_{k=s}^{\infty} \left(\frac{k!}{(k-s)!} (1-p)^{k-s}\right) \left(\frac{\lambda^k}{k!}\right) \qquad (10)$$

$$\Pr(\eta = s) = e^{-\lambda} \frac{p^s}{s!} \sum_{k=s}^{\infty} \frac{\lambda^k (1-p)^{k-s}}{(k-s)!} \qquad (11)$$

$$\Pr(\eta = s) = e^{-\lambda} \frac{p^s}{s!} \sum_{k=s}^{\infty} \frac{\lambda^s}{\lambda^s} \frac{\lambda^k (1-p)^{k-s}}{(k-s)!} \qquad (12)$$

$$\Pr(\eta = s) = e^{-\lambda} \frac{p^s}{s!} \sum_{k=s}^{\infty} \lambda^s \frac{\lambda^{k-s} (1-p)^{k-s}}{(k-s)!} \qquad (13)$$

Removing λ^s from the summation and making $n = k - s$, we can rewrite the equation as:

$$\Pr(\eta = s) = e^{-\lambda} \frac{p^s}{s!} \lambda^s \sum_{n=0}^{\infty} \frac{\lambda^n (1-p)^n}{n!} \qquad (14)$$

$$\Pr(\eta = s) = e^{-\lambda} \frac{p^s}{s!} \lambda^s \sum_{n=0}^{\infty} \frac{\lambda^n (1-p)^n}{n!} \qquad (15)$$

$$\Pr(\eta = s) = e^{-\lambda} \frac{(p\lambda)^s}{s!} \sum_{n=0}^{\infty} \frac{(\lambda(1-p))^n}{n!} \qquad (16)$$

By definition (1), this leads to:

$$\Pr(\eta = s) = e^{-\lambda} \frac{(p\lambda)^s}{s!} e^{\lambda(1-p)} \qquad (17)$$

$$\Pr(\eta = s) = e^{-\lambda + \lambda(1-p)} \frac{(p\lambda)^s}{s!} \qquad (18)$$

$$\Pr(\eta = s) = e^{-(p\lambda)} \frac{(p\lambda)^s}{s!} \qquad (19)$$

Which is equivalent to having parameter $p\lambda$ in the definition (2), using $\eta = s$, instead of $X = x$.

Thus, equation (19), shows us that η has Poisson distribution with the parameter $p\lambda$.

QED

Problem P2

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A strict reviewer needs t1 minutes to check assigned application to Deep|Bayes
summer school, where t1 has normal distribution with parameters μ1 = 30, σ1 =
10. While a kind reviewer needs t2 minutes to check an application, where t2
has normal distribution with parameters μ2 = 20, σ2 = 5. For each application
the reviewer is randomly selected with 0.5 probability. Given that the time of
review t = 10, calculate the conditional probability that the application was
checked by a kind reviewer.
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Definitions

1. Bayes' rule:

$$\Pr(A \mid B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

2. The relation between a joint probability and a conditional probability of two events is given as:

$$\Pr(A, B) = \Pr(A \mid B)\Pr(B)$$

3. If random variable X is normally distributed ($X \sim \mathcal{N}(\mu, \sigma)$):

$$\Pr(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Solution

- Two random variables:
  - Time to check the application: T
  - Assigned reviewer: R
- Reviewer R:
  - Strict reviewer $R = 1$:
    - $\Pr(R = 1) = 0.5$
  - Kind reviewer $R = 2$:
    - $\Pr(R = 2) = 0.5$
- $t_1 \sim \mathcal{N}(\mu_1 = 30, \sigma_1 = 10)$
- $t_2 \sim \mathcal{N}(\mu_2 = 20, \sigma_2 = 5)$
- Question: "what's the conditional probability that the application was checked by a kind reviewer, given that the time of review was t = 10?"
  - $\Pr(R = 2 \mid T = 10) = ?$

By definition (1):

$$\Pr(R = 2 \mid T = 10) = \frac{\Pr(T=10|R=2)\Pr(R=2)}{\Pr(T=10)} \qquad (4)$$

By enumerating the possibilites for the random variable R, we have:

$\Pr(T=10) = \Pr(T=10, R=1) + \Pr(T=10, R=2)$ (5)

Which, by definition (2) is equivalent to:

- $\Pr(T=10) = \Pr(T=10 \mid R=1)\Pr(R=1) + \Pr(T=10 \mid R=2)\Pr(R=2)$ (6)
- $\Pr(T=10) = \Pr(T=10 \mid R=1)(0.5) + \Pr(T=10 \mid R=2)(0.5)$ (7)
- $\Pr(T=10) = \frac{1}{2}(\Pr(T=10 \mid R=1) + \Pr(T=10 \mid R=2))$ (8)

From (4) and (8), we have:

- $\Pr(R=2 \mid T=10) = \frac{\Pr(T=10|R=2)\Pr(R=2)}{\frac{1}{2}(\Pr(T=10|R=1)+\Pr(T=10|R=2))}$ (9)
- $\Pr(R=2 \mid T=10) = \frac{2\Pr(T=10|R=2)\Pr(R=2)}{\Pr(T=10|R=1)+\Pr(T=10|R=2)}$ (10)
- $\Pr(R=2 \mid T=10) = \frac{2\Pr(T=10|R=2)(0.5)}{\Pr(T=10|R=1)+\Pr(T=10|R=2)}$ (11)
- $\Pr(R=2 \mid T=10) = \frac{\Pr(T=10|R=2)}{\Pr(T=10|R=1)+\Pr(T=10|R=2)}$ (12)

By definition (3):

- $\Pr(T=10 \mid R=1) = \frac{1}{\sqrt{2\pi 10^2}}e^{-\frac{(10-30)^2}{2\cdot 10^2}} = \frac{1}{10\sqrt{2\pi}}e^{-\frac{(-20)^2}{200}} = \frac{1}{10\sqrt{2\pi}}e^{-\frac{400}{200}} = \frac{1}{10\sqrt{2\pi}}e^{-2}$ (13)
- $\Pr(T=10 \mid R=2) = \frac{1}{\sqrt{2\pi 5^2}}e^{-\frac{(10-20)^2}{2\cdot 5^2}} = \frac{1}{5\sqrt{2\pi}}e^{-\frac{(-10)^2}{2\cdot 25}} = \frac{1}{5\sqrt{2\pi}}e^{-\frac{100}{50}} = \frac{1}{5\sqrt{2\pi}}e^{-2}$ (14)
- $\Pr(T=10 \mid R=2) = 2 \cdot \Pr(T=10 \mid R=1)$ (15)

From (12) and (15), we have:

- $\Pr(R=2 \mid T=10) = \frac{2\cdot\Pr(T=10|R=1)}{\Pr(T=10|R=1)+2\cdot\Pr(T=10|R=1)}$ (16)
- $\Pr(R=2 \mid T=10) = \frac{2\cdot\Pr(T=10|R=1)}{3\cdot\Pr(T=10|R=1)}$ (17)
- $\Pr(R=2 \mid T=10) = \frac{2}{3}$ (18)

Thus, the conditional probability that the application was checked by a kind reviewer, given that the time of review t = 10 is $2/3 \approx 67\%$.