14.661 Recitation 5

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Today we will derive the *fundamental law of labor demand*, which allows us to express the long run elasticity of labor demand in terms of (1) the elasticity of substitution between inputs (σ), (2) the elasticity of demand for the final product (η), (3) the labor's share of income (s_L).

Fundamental Law of Labor Demand: *Suppose that (i) there is a perfectly competitive market for a final good with demand $D(p)$ and (ii) producers have access to a CRTS production function $F(K,L)$. Then the elasticity of labor demand (η_L) is:*

$$\eta_L \equiv \frac{\partial \log L(w,r)}{\partial \log w} = \underbrace{-\sigma(1-s_L)}_{\text{substitution effect}} \underbrace{-\eta s_L}_{\text{scale effect}} \tag{1}$$

where

- σ is the elasticity of substitution between labor and capital, defined to be positive $\sigma \equiv -\frac{\partial \log(K/L)}{\partial \log(r/w)}$
- s_L is the labor share: $s_L \equiv \frac{wL}{PY}$
- η is the final good demand elasticity $\eta \equiv D'(p)\frac{p}{D(p)}$

We will derive this result by building on several individual results, we show in the first sections.

Review of Properties of Homogeneous Functions

A function $f(x_1,...,x_N)$ defined for all nonnegative values $(x_1,...,x_N) \geq 0$ is said to be homogeneous of degree k, if for every $\alpha > 0$,

$$f(\alpha x_1,...,\alpha x_N) = \alpha^k f(x_1,...,x_N). \tag{2}$$

Homogeneous functions' derivatives have two interesting properties, that we can get using the implicit function theorem. Recall the IFT. For an implicit function for $y(x)$ defined so that:

$$G(x,y) = 0,$$

[*]These notes draw from the ones prepared by Clémence Idoux, who drew on Christopher Walters' (former TA, now Berkeley Professor), which in turn are based on Hamermesh's (not a former TA) HOLE chapter on labor demand.

as long as the function is differentiable in x and y and the derivative wrt y is non-zero, we have:

$$\rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

Now consider:

$$G(\boldsymbol{x}, f(\boldsymbol{x})) = f(\alpha x_1, ..., \alpha x_N) - \alpha^k f(x_1, ..., x_N)$$

Then:

$$\alpha \frac{\partial f(\alpha x_1, ..., \alpha x_N)}{\partial x_1} \mathrm{dx} - \alpha^k \mathrm{d} f(x_1, ..., x_N) = 0,$$
$$\frac{\partial f(\alpha x_1, ..., \alpha x_N)}{\partial x_1} = \alpha^{k-1} \frac{\partial f(x_1, ..., x_N)}{\partial \mathrm{x}},$$

so the derivative is homogenous of degree $k-1$!

Another cool trick (aka Euler's homogeneous function theorem) by using the IFT wrt α:

$$f(\alpha x_1, ..., \alpha x_N) - \alpha^k f(x_1, ..., x_N)$$
$$\sum_{i=1}^{N} x_1 \frac{\partial f(\alpha x_1, ..., \alpha x_N)}{\partial x} \mathrm{d}\alpha - k\alpha^{k-1} f(x_1, ..., x_N) \mathrm{d}\alpha$$

Setting $\alpha = 1$, gives Euler's formula

$$\sum_{i=1}^{N} x_i \frac{\partial f(\alpha x_1, ..., \alpha x_N)}{\partial x_i} = k f(x_1, ..., x_N)$$

Formal Recap

Property 1 (HF1): If $f(x_1, ..., x_N)$ is homogeneous of degree k, for $k = ... -1, 0, 1....$ then $\frac{\partial f(x_1, ..., x_N)}{\partial x_i}$ is homogeneous of degree $k-1$ for all $i = 1, ...N$

Property 2 (Euler's homogeneous function theorem): If $f(x_1, ..., x_N)$ is homogeneous of degree k, for $k = ... -1, 0, 1....$ then $\sum_{i=1}^{N} \frac{\partial f(x_1, ..., x_N)}{\partial x_i} x_i = k f(x_1, ..., x_N)$

Applications to Cost Minimization

Dual of profit maximization is cost minimization:

$$c(w, r, q) = \min_{K,L} wL + rK \quad \text{s.t.} \quad F(K, L) \geq q$$

Just like the expenditure minimization from consumer theory, with same properties.

First and foremost, Shephard's Lemma (aka Envelope Theorem).

Second, The cost function is homogeneous of degree 1 in *input prices* (double all prices, I do not change my optimal relative choices but cost doubles):

$$\begin{aligned} c(\alpha w, \alpha r, q) &= \min_{K,L} \alpha wL + \alpha rK \quad \text{s.t.} \quad F(K,L) \geq q \\ &= \alpha \min_{K,L} wL + rK \quad \text{s.t.} \quad F(K,L) \geq q \\ &= \alpha c(w,r,q) \end{aligned}$$

Third, the cost function is homogeneous in the *quantity to produce, q,* **assuming constant returns to scale** (homogeneity of degree 1 of the production function). In this case, the optimal capital/labor ratio is fixed regardless of q, and only depends on (w,r). In math:

$$\begin{aligned} c(w,r,\alpha q) &= \min_{K,L} wL + rK \quad \text{s.t.} \quad F(K,L) \geq \alpha q, \\ &= \alpha \min_{K,L} w\frac{L}{\alpha} + r\frac{K}{\alpha} \quad \text{s.t.} \quad F\left(\frac{K}{\alpha}, \frac{L}{\alpha}\right) \geq q, \\ &= \alpha c(w,r,q). \end{aligned}$$

In particular:

$$c(w,r,q) = qc(w,r,1) \equiv qc^U(w,r),$$

where c^U denotes the *unit cost function.*

Formal Recap

Result 1 (Shephard's Lemma):

$$\frac{\partial c(w,r,q)}{\partial w} = L^c(w,r,q), \quad \frac{\partial c(w,r,q)}{\partial r} = K^c(w,r,q),$$

Where $\left(L^c(w,r,q), K^c(w,r,q)\right) = \arg\min_{(L,K)} wL + rK \quad \text{s.t.} \quad F(K,L) \geq q$

Result 2: *For $\alpha \neq 0$,*

$$c(\alpha w, \alpha r, q) = \alpha c(w,r,q)$$

Result 3: *For $\alpha \neq 0$,*

$$c(w,r,\alpha q) = \alpha c(w,r,q).$$

In particular,

$$c(w,r,q) = qc^U(w,r)$$

Elasticity of Substitution

The elasticity of substitution σ for a production function is defined as[1]

$$\sigma \equiv -\frac{\mathrm{d}\log\left(K^c/L^c\right)}{\mathrm{d}\log\left(r/w\right)} = \frac{\mathrm{d}\log\left(K^c/L^c\right)}{\mathrm{d}\log\left(w/r\right)}.$$

This is just *minus* the elasticity of the capital-labor ratio to relative factor prices r/w. We put a minus because the elasticity is clearly negative (also properties of the cost minimization problem). When the production function has constant returns to scale (CRS):

Result 4:

$$\sigma = \frac{c^U c^U_{wr}}{c^U_r c^U_w}.$$

This result relies heavily on the homogeneity of the cost function. By what we have see above:

1. $c(w,r,q) = wc_w + rc_r$, from Shephard' Lemma.
2. $c_w(w,r,q) = c_w + wc_{ww} + rc_{rw} \Rightarrow wc_{ww} + rc_{rw} = 0$.
3. c is homogeneous of degree $1 \Rightarrow c_w, c_r$ are homogeneous of degree $0 \Rightarrow c_{ww}, c_{rw}$ are homogeneous of degree -1.

Combining these fact:

$$\begin{aligned}\log\left(K^c/L^c\right) &= \log c_r(w,r,q) - \log c_w(w,r,q)\\ &= \log c_r\left(\frac{w}{r},1,q\right) - \log c_w\left(\frac{w}{r},1,q\right)\end{aligned}$$

where the second line uses the fact that c_w, c_r, are homogeneous of degree 0 in factor prices.

Note:

$$\begin{aligned}\sigma &= \frac{\mathrm{d}\log\left(K^c/L^c\right)}{\mathrm{d}\log\left(w/r\right)} = \frac{\partial\log\left(K^c/L^c\right)}{\partial\left(w/r\right)}\cdot\frac{w}{r}\\ &= \frac{w}{r}\left(\frac{c_{rw}\left(\frac{w}{r},1,q\right)}{c_r\left(\frac{w}{r},1,q\right)} - \frac{c_{ww}\left(\frac{w}{r},1,q\right)}{c_w\left(\frac{w}{r},1,q\right)}\right)\\ &= \frac{w}{r}\left(\frac{rc_{rw}(w,r,q)}{c_r(w,r,q)} - \frac{rc_{ww}(w,r,q)}{c_w(w,r,q)}\right)\end{aligned}$$

[1] Trivia: the elasticity of substitution was first defined in Hicks' *Theory of Wages*, which came out in 1932. This was the same year of too many horrible and tragic events: the Nazi party won the elections in Germany, Japan occupies Manchuria and restore the last Chinese Emperor Pu Yi as a puppet, the start of the Great Famine in Ukraine, the start of the fascist dictatorship in Portugal... But some good things happened too, in addition to Hicks' books: Aldous Axley published Brave New World, FDR won the election, and Goofy made his first appearance! Also, Australia declared war on Emus (check this one out!)

where the last line uses the fact that c_{ww}, c_{rw}, are homogenous of degree -1 in (w,r).

By $wc_{ww} + rc_{wr} = 0$:

$$\frac{w}{r} = -\frac{c_{wr}}{c_{ww}} \quad \text{and} \quad wc_{ww} = -rc_{wr}.$$

Thus, finally:

$$\begin{aligned}
\sigma &= -\frac{c_{wr}}{c_{ww}}\left(-\frac{wc_{ww}}{c_r} - \frac{rc_{ww}}{c_w}\right) \\
&= c_{wr}\left(\frac{wc_w + rc_r}{c_r c_w}\right) \\
&= \frac{c \cdot c_{wr}}{c_r c_w} \\
&= \frac{c^U \cdot c^U_{wr}}{c^U_r c^U_w},
\end{aligned}$$

where the third line uses Euler's HFT and the last the fact that $c$is homogeneous of degree 1, c_{wr} of degree -1, and c_r, c_w of degree 0.

Formal Recap

Result 4:

$$-\frac{\mathrm{d}\log\left(K^c/L^c\right)}{\mathrm{d}\log\left(r/w\right)} \equiv \sigma = \frac{c^U c^U_{wr}}{c^U_r c^U_w}.$$

Market Demand

Result 5:

$$L = Qc^U_w \text{and} \quad K = Qc^U_r$$

Assuming all firms have the same production function, we can thus write the market's demand for labor as:

$$\begin{aligned}
L(w,r) &= \sum_j L^c_j(w,r,q_j) \\
&= \sum_j c_w(w,r,q_j) && \text{By R1} \\
&= \sum_j q_j c^U_w(w,r) && \text{By R4} \\
&= Qc^U_w(w,r)
\end{aligned}$$

where Q is the total quantity produced in the market. A similar type of reasoning leads to the equivalent result for capital demand.

The main result in this section gives us a convenient expression for the derivative of the market demand for labor $L(w,r)$, in terms of substitution and a scale effect.

Result 6:

$$\frac{\partial L(w,r)}{\partial w} = D(p)\, c^U_{ww} + D'(p)\,(c^U_w)^2$$

Begin by noticing that in equilibrium Q will have to be equal to $D(p)$, so that

$$L(w,r) = D(p)\, c^U_w(w,r)$$

Furthermore, in competitive equilibrium it must be the case that the product price is equal to marginal cost for each firm, $p = c^u(w,r)$, so that we can write the previous expression as follows:

$$L(w,r) = D(c^U(w,r))c^U_w(w,r)$$

We can now differentiate this expression to obtain the result above.

Finally: The Fundamental Law of Factor Demand

Let's put everything together:

$$\begin{aligned}
\frac{\partial L(w,r)}{\partial w} &= D(p)\, c^U_{ww} + D'(p)\,(c^U_w)^2 \\
&= -D(c^U(w,r))\frac{r}{w}c^U_{wr} + D'(p)\,(c^U_w)^2 && \text{By Euler's HFT} \\
&= -D(p)\cdot\frac{r}{w}\cdot\sigma\cdot\frac{c^U_r c^U_w}{c^U} + D'(p)\,(c^U_w)^2 && \text{By R4} \\
&= -D(p)\cdot\frac{r}{w}\cdot\frac{\sigma}{c^U}\cdot\frac{LK}{Q^2} + D'(p)\cdot\frac{L^2}{Q^2} && \text{By R5} \\
&= -\frac{rK}{pQ}\cdot\frac{\sigma L}{w} + D'(p)\cdot\frac{L^2}{D(p)}
\end{aligned}$$

were we used that $c^U = p$ and $D(p) = Q$ in the last step.

We now write this as an elasticity:

$$
\begin{aligned}
\eta_L &= \frac{\partial L}{\partial w} \cdot \frac{w}{L} \\
&= -\frac{rK}{pQ} \cdot \frac{\sigma L}{w} \cdot \frac{w}{L} + D'(p) \cdot \frac{L^2}{D(p)} \cdot \frac{w}{L} \\
&= -s_K \cdot \sigma + \frac{D'(p)}{D(p)} \cdot \frac{wL}{D(p)} && \text{Since } s_K = \frac{rK}{pQ} \\
&= = -s_K \cdot \sigma + D'(p) \cdot \frac{p}{D(p)} \cdot \frac{wL}{pQ} \\
&= -s_K \cdot \sigma - \eta \cdot s_L && \text{Since } s_L = \frac{wL}{pQ} \\
&= -(1 - s_L) \cdot \sigma - \eta \cdot s_L
\end{aligned}
$$

In the last step we used $s_K + s_L = 1$ since there are no profits (or, alternatively, it is implied by Euler's HFT along with $p = c^U$).

Comparative Statics

1. $\sigma \uparrow \implies |\eta_L| \uparrow$: When substitution to other factors is easier, the market demand curve for labor is more elastic. Intuition: use more capital in response to higher wages

2. $\eta \uparrow \implies |\eta_L| \uparrow$: When product demand is more elastic, labor demand is more elastic. Intuition: I loose more demand when I have to pass on higher wages to prices, so I also produce less.

3. $\left(s_L \uparrow \implies |\eta_L| \uparrow\right) \iff \left(\sigma < \eta\right)$: When labor's share of income increases, labor demand becomes more elastic iff the elasticity of product demand is larger than the elasticity of substitution. Intuition: if the labor share is high, it means WL is relatively high, that is I employ lots of labor. If the elasticity of product demand is very large, I loose lots of demand (labor) when the wage increases. However, if I have a high elasticity of substitution, I can trade off (relatively little) labor to save demand and charge not so high prices, which result in a lower decrease in labor demand.

These are three of the "Hicks-Marshall Laws of Derived Demand." The remaining one requires looking at the elasticity of *supply* of other factors, which here we did not cover, which would affect how much easy it is to scale up with capital relative to labor. That is:

4. $|\eta_L| \uparrow$ if the supply of other factors is more elastic. Intuition: it is easier to scale up the use of alternative factors.