

Exam #3 Practice Problems*Jonathan Cohen (jpcohen@mit.edu)*

This is a non-exhaustive list of the types of longer problems that can be on the exam. For example, the graphs that you've seen are not covered here but are also fair game. I also included a long question about regression discontinuity because a lot of people had trouble with that in the in-class quiz. Good luck!

Lotteries, Expected Utility, and Risk Aversion**Examples - Literal Lotteries**

1. State Lotteries sold \$53 billion of lottery tickets in 2010. Suppose the cost of playing is \$1 and the expected value of playing is 50 cents (i.e. $\sum_i p_i W_i = \$0.50$ where W_i is payout i , and p_i is probability of that payout). Mr. Jo, a VNM expected utility maximizer, is indifferent between playing the lottery or not. Is Mr. Jo risk-averse, risk-loving, neither, or is there not enough information to say? Explain.

Solution. Mr. Jo's certainty equivalent of playing the lottery is \$1 (since he is indifferent between playing or not), so $CE[\text{Lottery}] > EV[\text{Lottery}]$ and the risk premium is -50 cents. This makes Mr. Jo a risk-lover (even though he may not actually play the lottery).

2. Consider the following lotteries:

- (a) \$100 with certainty
- (b) \$200 with probability 0.7; \$0 with probability 0.3
- (c) \$110 with certainty
- (d) \$1200 with probability 0.1; \$0 with probability 0.9
- (e) \$200 with probability 0.35; \$100 with probability 0.5; \$0 with probability 0.15
- (f) \$1200 with probability 0.05; \$110 with probability 0.5; \$0 with probability 0.45

Suppose Bob is indifferent between lottery (a) and lottery (b). He is also indifferent between lottery (c) and lottery (d). Is Bob risk-averse, risk-loving, risk-neutral or is it impossible to say? Explain your answer.

Solution. The expected value to lottery (b) is \$140. Bob is indifferent between lottery (a), that provides \$100 with certainty and lottery (b), that provides a higher amount, \$140, in expectation but with uncertainty. This must mean that he is risk-averse in this choice. The same conclusion follows if we examine his choice between lottery (c) and (d).

3. Cynthia has same choices as Bob. She is indifferent between lotteries (a) and (b) and between lotteries (c) and (d). If Cynthia is an expected utility maximizer and her utility is increasing in money/wealth, which lottery will she pick when faced with a choice between lotteries (e) and (f)?

Solution. Note that lottery (e) is a convex combination of lotteries (a) and (b): It offers lottery (a) with 50% probability and lottery (b) with 50% probability. Since Cynthia is indifferent between lotteries (a) and (b), lottery (e) is equivalent to lottery (a). Similarly, lottery (f) is a convex combination of lotteries (c) and (d) and is therefore equivalent to lottery (c). As Cynthia's utility is increasing in wealth, she prefers lottery (c) to (a). Therefore, she will pick lottery (f) to lottery (e). More formally, Cynthia's

choice between lotteries (a) and (b) and lotteries (c) and (d) imply that (normalizing Cynthia's initial wealth level to 0)

$$\begin{aligned} u(100) &= 0.7u(200) + 0.3u(0) \\ \implies 0.5u(100) &= 0.35u(200) + 0.15u(0) \end{aligned}$$

and

$$\begin{aligned} u(110) &= 0.1u(1200) + 0.9u(0) \\ \implies 0.5u(110) &= 0.05u(1200) + 0.45u(0) \end{aligned}$$

Now, the expected utility from lottery (e) is given by

$$\begin{aligned} &0.35u(200) + 0.5u(100) + 0.15u(0) \\ &= 0.5u(100) + 0.5u(100) \\ &= u(100) \end{aligned}$$

The expected utility from lottery (f) is given by

$$\begin{aligned} &0.05u(1200) + 0.5u(110) + 0.45u(0) \\ &= 0.5u(110) + 0.5u(110) \\ &= u(110) \end{aligned}$$

Since Cynthia's utility is increasing in wealth

$$u(110) > u(100)$$

so she will pick lottery (f).

Example - Startup Insurance

Following the increase in the number of MIT graduates who start new companies and given the inherent risks of starting a company, Cambridge Savings Bank offers insurance to startup companies with more than 5 employees. Before the startup hires any employees, the startup's founder has to invest \$100,000 in equipment. Throughout the question, assume that the initial wealth of startup founders is given by $w_0 = \$150,000$ and that they are risk-averse with a VNM utility function over wealth that is given by $u(w) = \ln(w)$. A startup becomes the "next Uber" with probability $\pi = .05$, resulting in a net gain of \$19,850,000 to its founder (so the total wealth of the founder if the startup is successful will be 20 million dollars); with probability of $1 - \pi$, the startup goes bust, resulting in the loss of the \$100,000 invested in equipment.

1. The startup insurance offered by Cambridge Savings Bank looks as follows: The bank offers to pay the startup \$1 if it goes bust in exchange for receiving \$ S dollars if the startup is successful. What should the value of S be for the insurance to be considered actuarially fair?

Solution. The expected return to the bank of this contract is given by $\pi S - (1 - \pi)$. For the insurance to be actuarially fair, this expected return must be equal to zero; that is,

$$S = \frac{1 - \pi}{\pi} = \frac{.95}{.05} = 19.$$

2. How much startup insurance do startup founders buy if S is chosen as in the previous part?

Solution. Suppose the startup founder buys x units of insurance. Then his/her expected utility is given by

$$\pi \ln(20M - 19x) + (1 - \pi) \ln(50,000 + x).$$

The startup founder chooses x to maximize this expression. We can solve this problem by taking the FOC with respect to x :

$$-\frac{19\pi}{20M - 19x} + \frac{(1 - \pi)}{50,000 + x} = 0.$$

Noting that $19\pi = 1 - \pi$, we can immediately see that it must be that $20M - 19x = 50,000 + x$, which implies that $x = 997,500$. Regardless of whether the startup succeeds, the founder will have $20M - 19 \cdot 997,500 = 50,000 + 997,500 = 1,047,500$ dollars in wealth. This is a more general principle: risk-averse agents always purchase full insurance if they are offered insurance at the actuarially fair price.

- Suppose that every MIT graduate who starts a company also buys startup insurance (to the extent that you found in the previous part). Are MIT graduates better off starting companies (and buying startup insurance) or not starting a company at all (and keeping their \$150,000 to themselves)?

Solution. The payoff to the founder from starting a company and buying full insurance is given by $\ln(1,047,500)$, whereas if they don't start a company their payoff will be $\ln(150,000)$. So MIT graduates are better off starting companies and buying insurance.

- A new deep learning algorithm can analyze a startup immediately after the \$100,000 investment in equipment has been made to determine whether the startup will be a success or not. The algorithm's output is posted on a publicly-accessible online database. Who will buy insurance in equilibrium after the introduction of the algorithm? [Hint: The output of the algorithm will be publicly visible *before* the startup hires any employees and so before it is eligible for insurance. Assume that Cambridge Savings Bank cannot reject any customers but it can adjust S in response to changes in the set of customers who buy insurance.]

Solution. Since all the uncertainty is resolved after the output of the algorithm is observed, there are no insurance opportunities left: as long as $S > 0$, only startups founders whose companies will not be successful will buy insurance. But if that's the case, then no contract (with finite S) would be actuarially fair.

- How does your answer to part (3) change as a result of the advent of the algorithm?

Solution. Given that there are no longer any opportunities to buy insurance, the expected utility of an MIT graduate from starting a company is given by $\pi \ln(20M) + (1 - \pi) \ln(50,000) \approx 11.120$, whereas their expected payoff from not starting a company is given by $\ln(150,000) \approx 11.918$. Therefore, MIT graduates are better off not starting a company. This problem shows that more information can lead to a decrease in welfare if it is provided at an interim stage (when investments are made but the risky output has not yet to realize).

Regression Discontinuity

Example - Saving Babies

Medical expenditures in the United States are high and increasing. A key policy question is, therefore, whether the benefits of additional medical expenditures exceed their costs.¹

¹Note: this question is based on the paper "Estimating marginal returns to medical care: evidence from at-risk newborns," by Douglas Almond, Joseph Doyle, Amanda Kowalski, and Heidi Williams, *Quarterly Journal of Economics*, 2011. It is not

1. Let Y be a health outcome of newborns (for example, one-year mortality), and (Y_0, Y_1) be the potential outcomes when the newborn does not receive intensive medical treatment ($T = 0$) and when the newborn receives intensive medical treatment ($T = 1$). Explain in words and show mathematically why comparing $E[Y|T = 1]$ and $E[Y|T = 0]$ would not be a good idea to estimate the causal effect of intensive medical treatment on health outcomes of newborns.

Solution. We might have a problem of omitted variables that are correlated with both intensive treatment and health outcomes. In this example, newborns that have a higher risk probably receive more intensive treatment. Mathematically:

$$E[Y|T = 1] - E[Y|T = 0] = E[Y_1|T = 1] - E[Y_0|T = 0] = E[Y_1 - Y_0|T = 1] + \{E[Y_0|T = 1] - E[Y_0|T = 0]\}.$$

In a 2011 paper, Douglas Almond, Joseph Doyle, Amanda Kowalski, and Heidi Williams estimate the causal effect of intensive medical treatment on the health outcomes of low birth weight newborns using a regression discontinuity approach. When a newborn weighs less than 1,500g, he/she is classified as “very low birth weight” (VLBW). Newborns classified as VLBW are more likely to receive Intensive Medical Treatment (IMT), such as admission to neonatal intensive care, ventilation, incubation, etc.

2. Let Z be a dummy variable equal to 1 if a newborn is classified as VLBW, and 0 otherwise. Let w_i be the birth weight of a newborn. Hence, $Z = 1$ if $w_i < 1,500$ and $Z = 0$ otherwise. Would it be a good idea to use Z as an instrument for T to estimate the causal effect of intensive medical treatment on health outcomes? Which IV identification assumption might be invalid?

Solution. This would not be a good idea. Z would satisfy one IV condition, because it is correlated with IMT. However, newborns with $T = 1$ would have lower birth weight than newborns with $T = 0$. Since birth weight might have an independent effect on health outcomes, the exclusion restriction would not be satisfied.

3. Instead, the authors use the discontinuity at 1,500g to estimate the causal effect of interest.

- (a) Explain the rationale of this strategy. What are the identification assumptions?
- (b) Write the appropriate RD estimator using causal notation.

Solution. (a) The rationale of this strategy is that the only difference between newborns just below and just above the cutoff is that the one classified as VLBW are more likely to receive IMT. The key assumptions are (1) no manipulation of the running variable (weight), (2) the outcome variable evolves smoothly in the absence of treatment, i.e.

$$\lim_{\varepsilon \downarrow 0} E[Y_0|w = 1500 + \varepsilon] = \lim_{\varepsilon \uparrow 0} E[Y_0|w = 1500 + \varepsilon],$$

$$\lim_{\varepsilon \downarrow 0} E[Y_1|w = 1500 + \varepsilon] = \lim_{\varepsilon \uparrow 0} E[Y_1|w = 1500 + \varepsilon].$$

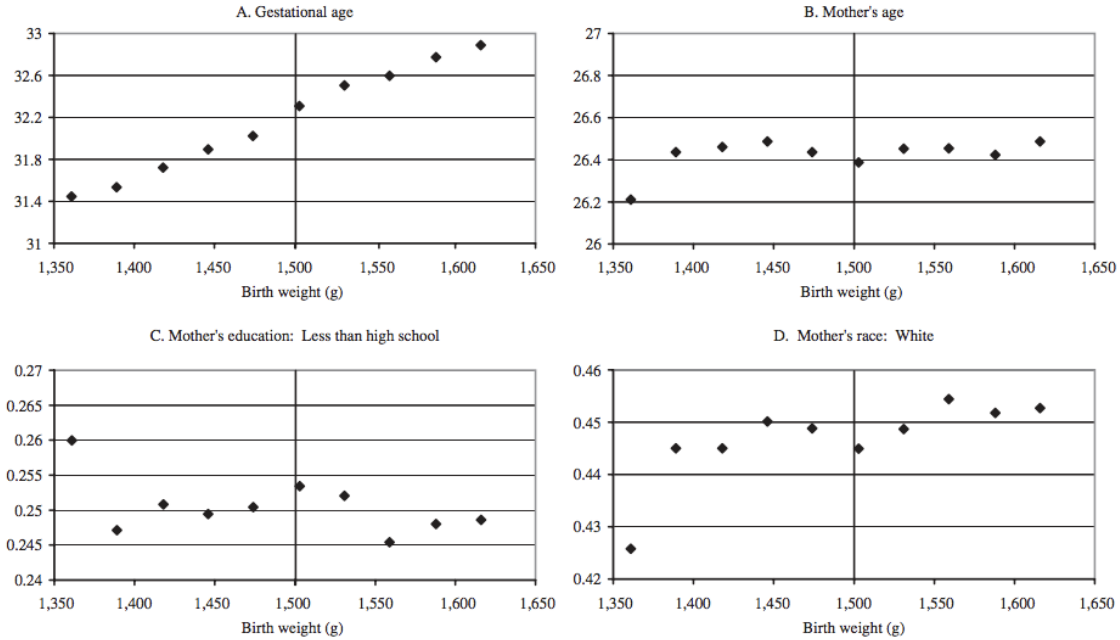
and (3) intensive medical treatment is the only thing that changes when a baby crosses the 1500g threshold.

- (b) The RD estimator would be:

$$\hat{T}_{RD} = \lim_{\varepsilon \downarrow 0} E[Y|w = 1500 + \varepsilon] - \lim_{\varepsilon \uparrow 0} E[Y|w = 1500 + \varepsilon] = E[Y_1 - Y_0|w = 1500].$$

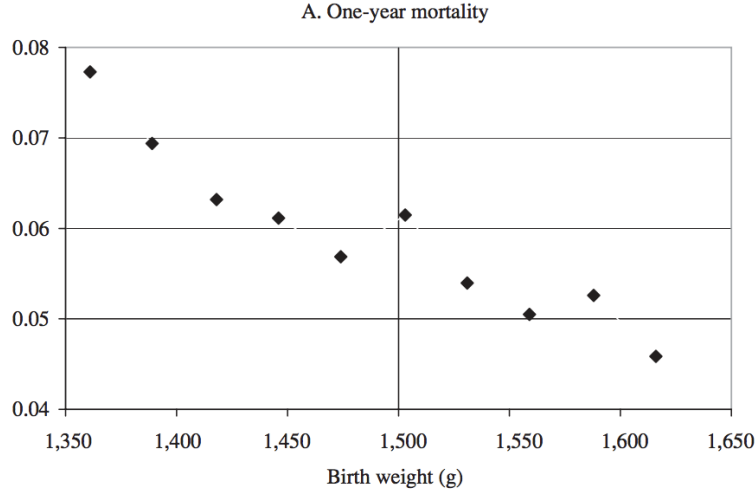
necessary to read the paper to answer this question. But it is a terrific paper to read.

4. The graphs in Figure V plot the relationship between several covariates and birth weight. Are these graphs consistent with the identification assumptions? Explain. What pattern(s) would tell you that the graphs are *not* consistent with the identification assumptions?



Solution. These graphs show that there are not any discontinuous change in these covariates around the threshold, which is consistent with the identification assumption. If there were a discontinuous change in some covariates around the threshold, we should worry that there are factors that change around the threshold other than IMT.

5. The graph in Figure II.A plots the relationship between one-year mortality and birth weight. Based on this graph and on the graph from III.A, what would you conclude about the causal effect of Intensive Medical Care on one-year mortality of newborns?



Solution. This graph shows that newborns just below the cutoff have a lower mortality rate. Under the assumptions that the only difference between newborns just below and just above the cutoff is that the ones just below were more likely to receive IMT, this suggests that IMT reduces mortality.

Signaling

Example - Signaling with Two Types

Setup:

- There are two types of ability. Half of agents are type H with $a_H = 1$ and half of agents are type L with $a_L = 0$.
- Productivity is given by $Y(a) = a$
- Before entering the workforce, agents can choose to obtain schooling $S \in \{0, 1\}$. Schooling is costly, but less costly if you are high ability. Let the cost be given by

$$c(a) = k(1 - a)$$

where $k \geq 0$.

- The labor market is competitive, so firms pay employees their expected productivity. Wages are given by $W(S = 0)$ and $W(S = 1)$.
- The agents' utility is given by

$$u(W, c) = W - c.$$

Separating Equilibrium

Let $k = \frac{3}{2}$. We are looking for an equilibrium that consists of:

- Agents with a_H get schooling and agents with a_L don't get schooling

- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by assuming that we have a separating equilibrium. Then,

$$\begin{aligned} W(S = 0) &= E[a|S = 0] = 0 \\ W(S = 1) &= E[a|S = 1] = 1. \end{aligned}$$

Is this a truly a separating equilibrium, or will one type want to deviate? Compare the payoffs from school and no school for both types of agents. For the H type:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= 0 - 0 = 0 \\ \text{school: } u(W(S = 1), c(a_H)) &= 1 - \frac{3}{2}(1 - a_H) = 1 \end{aligned}$$

For the L type:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= 0 - 0 = 0 \\ \text{school: } u(W(S = 1), c(a_L)) &= 1 - \frac{3}{2}(1 - a_L) = -\frac{1}{2}. \end{aligned}$$

So nobody will want to deviate. H prefers school, L prefers no school.

Pooling Equilibrium

Let $k = \frac{1}{4}$. We are looking for an equilibrium that consists of:

- All agents want to get schooling
- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by assuming that we have a pooling equilibrium. Then,²

$$\begin{aligned} W(S = 0) &= 0 \\ W(S = 1) &= E[a|S = 1] = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2} \end{aligned}$$

Is this a truly a pooling equilibrium, or will one type want to deviate? Compare the payoffs from school and no school for both types of agents. For the H type:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= 0 - 0 = 0 \\ \text{school: } u(W(S = 1), c(a_H)) &= \frac{1}{2} - \frac{1}{4}(1 - a_H) = \frac{1}{2} \end{aligned}$$

For the L type:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= 0 - 0 = 0 \\ \text{school: } u(W(S = 1), c(a_L)) &= \frac{1}{2} - \frac{1}{4}(1 - a_L) = \frac{1}{4}. \end{aligned}$$

So nobody will want to deviate. H and L both prefer school.

²Why is $W(S = 0) = 0$ in a pooling equilibrium when in a pooling equilibrium everyone selects $S = 1$? We have to assume that the employer will make some assumption about an agent's type if they go "off-equilibrium." It seems more likely that somebody who goes off equilibrium in this case is the L type. This is formalized by a notion called the "Intuitive Criterion" which is beyond the scope of this course.

Example - Signaling with a Continuum of Types

[Note: This will not be on exam, but it was asked about on Piazza and (might) clarify intuition]

Setup:

- There is a continuum of ability: $a \sim U[0, 1]$ where $U[\cdot]$ is the uniform distribution.
- Productivity is given by $Y(a) = a$
- Before entering the workforce, agents can choose to obtain schooling $S \in \{0, 1\}$. Schooling is costly, but less costly if you are high ability. Let the cost be given by

$$c(a) = k(1 - a)$$

where $k \geq 0$.

- The labor market is competitive, so firms pay employees their expected productivity. Wages are given by $W(S = 0)$ and $W(S = 1)$.
- The agents' utility is given by

$$u(W, c) = W - c.$$

Separating Equilibrium

Let $k = \frac{3}{4}$. We are looking for an equilibrium that consists of:

- A cutoff value a^* , above which agents will choose $S = 1$.
- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by just assuming that we have a separating equilibrium, with a cutoff of a^* . Then,

$$\begin{aligned} W(S = 0) &= E[a|S = 0] = E[a|a \leq a^*] = \frac{a^*}{2} \\ W(S = 1) &= E[a|S = 1] = E[a|a > a^*] = \frac{1 + a^*}{2}. \end{aligned}$$

Next, solve for the cutoff a^* . We want to find the agent who is indifferent between schooling and no schooling:

$$\begin{aligned} \underbrace{W(S = 0) - 0}_{\text{utility from no school}} &= \underbrace{W(S = 1) - c(a^*)}_{\text{utility from school}} \\ \frac{a^*}{2} - 0 &= \frac{1 + a^*}{2} - \frac{3}{4}(1 - a^*) \\ 0 &= \frac{1}{2} - \frac{3}{4} + \frac{3}{4}a^* \\ \frac{3}{4}a^* &= \frac{1}{4} \\ a^* &= \frac{1}{3}. \end{aligned}$$

Plugging in a^* to $W(S = 0)$ and $W(S = 1)$ gives us

$$\begin{aligned} W(S = 0) &= \frac{1}{6} \\ W(S = 1) &= \frac{2}{3}. \end{aligned}$$

Finally, we need to check whether this is a separating equilibrium. Will anyone want to deviate? Compare the payoffs from school and no school:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= \frac{1}{6} - 0 = \frac{1}{6} \\ \text{school: } u(W(S = 1), c(a)) &= \frac{2}{3} - \frac{3}{4}(1 - a) \end{aligned}$$

If $a = \frac{1}{3} - \epsilon$ (i.e. someone just below the cutoff), then the utility from school is

$$\begin{aligned} \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} + \epsilon) &= \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} + \epsilon) \\ &= \frac{2}{3} - \frac{3}{4}(\frac{2}{3} + \epsilon) \\ &= \frac{1}{6} - \frac{3}{4}\epsilon < \frac{1}{6} \end{aligned}$$

so this person will prefer $S = 0$. If $a = \frac{1}{3} + \epsilon$ (i.e. someone just above the cutoff), then the utility from school is

$$\begin{aligned} \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} - \epsilon) &= \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} - \epsilon) \\ &= \frac{2}{3} - \frac{3}{4}(\frac{2}{3} - \epsilon) \\ &= \frac{1}{6} + \frac{3}{4}\epsilon > \frac{1}{6} \end{aligned}$$

so this person will prefer $S = 1$. So nobody will want to deviate. H prefers school, L prefers no school.

Pooling Equilibrium

Let $k = \frac{1}{4}$. We are looking for an equilibrium that consists of:

- All agents want to get schooling.
- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by just assuming that we have a pooling equilibrium³

$$\begin{aligned} W(S = 0) &= E[a|S = 0] = 0 \\ W(S = 1) &= E[a|S = 1] = E[a] = \frac{1}{2}. \end{aligned}$$

³Again, we are assuming that $E[a|S = 0] = 0$ even though choosing $s = 0$ is off the equilibrium.

Finally, we need to check whether this is a separating equilibrium. Will anyone want to deviate? Compare the payoffs from school and no school:

$$\text{no school: } u(W(S=0), 0) = 0 - 0 = 0$$

$$\text{school: } u(W(S=1), c(a)) = \frac{1}{2} - \frac{1}{4}(1-a)$$

The utility of school is positive for all values of $a \in [0, 1]$, so nobody will want to deviate. H and L both prefer school.