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Recitation 5

Andrea Manera
(PhD Student, MIT)

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The yield curve
Formulas and definitions
Application to Problem set 6.2

The No Arbitrage condition (without risk)

- The main principle underlying asset pricing is the **No Arbitrage condition**;
- A condition that *assets traded in the market* should satisfy;
- Simplest, informal definition:
  "Any trading strategy that is expected to pay at the same time t in the future the same value F should cost the same";
- (Simple because it excludes risk and uncertainty, but accurate in our context);
- *Why should this hold?* Only the strategy that costs less can be traded, otherwise people could *buy the cheap* strategy and *sell the expensive* one (i.e. they could emit bonds), and make infinite amounts of money at no risk.
- Without this condition, there will never be equilibrium in the market (as per above, everyone wants infinite quantities of the cheap asset and negative quantities of the expensive one).

The No Arbitrage condition (without risk) in Formulas

- Simplest, informal definition:
  "Any trading strategy that is expected to pay at the same time t in the future the same value F should cost the same";
- Price of two strategies: P_1, P_2, both pay F in **one year**. **NA** reads:

$$\frac{P_1}{F} = \frac{P_2}{F}$$

- Equivalent to:
  "If I invest a dollar in any strategy that pays *up to* time t I should get the same overall return at time t";
- By the previous relation:

$$\frac{1}{1+i_1} = \frac{1}{1+i_2}$$

Yields to maturity

Definition of yield: The *constant discount rate* y_i, t such that the discounted present value of the payments from a strategy equals the price.
Examples:

- 2-year bond: $y_{2,t}$ $s.t.$ $P_{2,t} = \dfrac{F}{(1+y_{2,t})^2}$
- 1-year bond: $y_{1,t}$ $s.t.$ $P_{1,t} = \dfrac{F}{(1+y_{1,t})}$

Note: for *riskless bonds*, and **without term premia**, yields and interest rates coincide, so $y_{1,t} = i_{1,t}$, $y_{2,t} = i_{2,t}$

Yields to maturity: General formula

Definition of yield: The *constant discount rate* y_i, t such that the discounted present value of the payments from a strategy equals the price of said strategy. Discounted present value for generic interest rate path and payments (equals the price, or no market equilibrium):

$$P_{n,t} = V_{n,t} = \sum_{j=1}^{n} \frac{z^e_{t+j}}{\prod_{i=1}^{j}(1 + i^e_{1,t+i-1})} = \frac{z^e_{t+1}}{(1 + i_{1,t})} + \frac{z^e_{t+2}}{(1 + i_{1,t})(1 + i^e_{1,t+1})} + \ldots$$

The **yield-to-maturity** is then defined as

$$y_{n,t} \quad \text{such that} \quad P_{n,t} = \sum_{j=1}^{n} \frac{z^e_{t+j}}{\prod_{i=1}^{j}(1 + y_{n,t})} = \sum_{j=1}^{n} \frac{z^e_{t+j}}{(1 + y_{n,t})^j}$$

Application: 1.b in the Problem set

The yield curve

- Plots the **nominal** yields of ZCB at each maturity;
- Usually *upward-sloping* as there is a **term premium** (what Blanchard calls x);
- Can be used to extract information on investors' expectations thanks to the **NA** condition.
- Example: expected 1-y interest rate. By **NA**:

$$(1 + i_{2,t})^2 = (1 + i_{1,t})(1 + i^e_{1,t+1})$$

- The yield curve tells us $i_{1,t}$,$i_{2,t}$, so plug in to solve for i^e_{t+1}. Example from question **1.b**

The *inverted* yield curve

When does this occur?

- *Common case*: investors expect *falling interest rates*: overshadows recession as the rates fall when the Fed wants to stimulate the economy. This typically occurs in recessions;
- *Less common case*: Investors expect a *sovereign default* (The Euro debt crisis saw many inverted yield curves).

General Formula Bonanza: Question **2.II**

(d) Suppose that Italy offers a one-year and a two-year zero-coupon bond (ZCB) that pay \$100 at maturity. Moreover, assume that one-year interest rates are not expected to change and equal to $i_{1,t} = 1\%$ for all t, but there is a small maturity premium $x = 0.5\%$.

1.-2. What are the prices of the bonds?

3.-4. What are the bond yields? What is the shape of the yield curve?

General Formula Bonanza: Question **2.II**

(f)-(g) Now suppose that investors are afraid that Italy might default in exactly one year. In that case, they anticipate that, with probability $p = 0.5$ they will only recover a fraction $\delta = 0.1$ of the face value of the each bond (100). That is, *regardless of the maturity of the bond*, w.p. $p = 0.5$ investors will receive a payment $\delta \times \$100$ *in year 1*. With the remaining probability of $p = 0.5$ investors will be paid at maturity in full. Further assume that the interest rate and the risk premium stay unchanged at the values in subpoint (d).

f.1-f.2 Compute the price of the two bonds;

g.1-g.2 Compute the yield to maturity of the two bonds;

Extra
Derivation of NA for two ZCBs

Example: two-year ZCB (no term premium) and one-year ZCB are traded, interest rates with usual notation. Face value F for both.
Two strategies available to get F dollar in two years:

1. Buy a two-year bond at price $P_{2,t}$, get F in $t = 2$;
2. Buy a one-year bond in one year for $P^e_{1,t+1}$ to get F in $t = 2$. To do so, buy today a quantity $\frac{P^e_{1,t+1}}{F}$ of bonds to get $P^e_{1,t+1}$ in one year.

The overall cost of the strategies is:

Strategy	Cost
1. 2-y bonds	$P_{2,t}$
2. 1-y bonds	$P_{1,t}\frac{P^e_{1,t+1}}{F}$

Thus **No Arbitrage**, diving both sides by F:

$$\frac{P_{2,t}}{F} = \frac{P_{1,t}}{F}\frac{P^e_{1,t+1}}{F}$$

Back to the familiar formulation of NA using the yield definition (I)

Definition of yield: The *discount rate* y_i, t such that the discounted present value of the strategy equals the price.
In our case:

- 2-year bond: $y_{2,t}$ $s.t.$ $P_{2,t} = \dfrac{F}{(1+y_{2,t})^2}$
- 1-year bond: $y_{1,t}$ $s.t.$ $P_{1,t} = \dfrac{F}{(1+y_{1,t})}$

Note: for *riskless bonds*, and without term premia, yields and interest rates coincide, so $y_{1,t} = i_{1,t}$, $y_{2,t} = i_{2,t}$

Back to the familiar formulation of NA using the yield definition (II)

We had found that no arbitrage required:

$$\frac{P_{2,t}}{F} = \frac{P_{1,t}}{F}\frac{P^e_{1,t+1}}{F}$$

Replacing the definition of yields:

$$\frac{1}{(1+y_{2,t})^2} = \frac{1}{(1+y_{1,t})(1+y^e_{1,t+1})}$$

so NA can be expressed as the following relation for $y_{2,t}$:

$$y_{2,t} = \sqrt{(1+y_{1,t})(1+y^e_{1,t+1})} - 1$$

Back to the familiar formulation of NA using the yield definition (III)

If there is no risk and we have ZCB, yield and interest rates coincide (it's a good exercise to show that), so finally:

$$i_{2,t} = \sqrt{(1 + i_{1,t})(1 + i^e_{1,t+1})} - 1$$