14.03/14.003: Microeconomic Theory and Public Policy	12/14/18

Exam #3 Review

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Lotteries, Expected Utility, and Risk Aversion

- A **lottery** is a list L of probabilities and associated payoffs:

$$L = \begin{cases} \pi_1 & \text{with probability } p_1 \\ \pi_2 & \text{with probability } p_2 \\ ... & ... \\ \pi_n & \text{with probability } p \end{cases}$$

  where p_i denotes a probability and π_i denotes a monetary payoff. $\sum_{i=1}^{n} p_i = 1$ (probabilities sum to 1).

- The von Neuman Morgenstern **expected utility property** says that utility is linear in probabilities. For example, the expected utility of lottery L can be written as:

$$u(L) = p_1 u(\pi_1) + p_2 u(\pi_2) + ... + p_n u(\pi_n).$$

  Alternatively, if we had two lotteries (L and L') and some $\beta \in [0, 1]$, then

$$u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)u(L').$$

- Agents are risk-averse if their utility functions are concave (they display diminishing marginal returns). They are risk-loving if their utility functions are convex (they display increasing marginal returns). They are risk-neutral if their utility functions are linear (they display constant marginal returns)

- Two key concepts:

  1. **The utility of expected wealth**. This is the utility if you gave the agent the expected value of the bet *for sure*. Let w be the agent's wealth after the bet. Then the utility of expected wealth is

$$u\left(E[w]\right) = u\left(p_1 w_1 + p_2 w_2 + ... + p_n w_n\right).$$

  2. **The expected utility of wealth**. This is the expected utility if the agent takes the bet. Again, let w be the agent's wealth after the bet. Then the expected utility of wealth is

$$E\left[u(w)\right] = p_1 u(w_1) + p_2 u(w_2) + ... + p_n u(w_n).$$

- The following inequalities are true:
  - If the agent is risk-averse, then $u\left(E[w]\right) > E\left[u(w)\right]$
  - If the agent is risk-loving, then $u\left(E[w]\right) < E\left[u(w)\right]$
  - If the agent is risk-neutral, then $u\left(E[w]\right) = E\left[u(w)\right]$

- The certainty equivalent is the amount of money you would accept for sure instead of taking the bet. For a risk-averse agent, the certainty equivalent is less than the expected value of the bet, i.e. $CE < E[w]$. For a risk-loving agent, the certainty equivalent is more than the expected value of the bet, i.e. $CE > E[w]$. For a risk-neutral agent, they are equal.

Examples - Short Answer (2014 Exam #3 and 2014 Problem Set #5)

1. State Lotteries sold \$53 billion of lottery tickets in 2010. Suppose the cost of playing is \$1 and the expected value of playing is 50 cents (i.e. $\Sigma_i p_i W_i = \$0.50$ where W_i is payout i, and p_i is probability of that payout). Mr. Jo, a VNM expected utility maximizer, is indifferent between playing the lottery or not. Is Mr. Jo risk-averse, risk-loving, neither, or is there not enough information to say? Explain.

   Solution. *Mr. Jo's certainty equivalent of playing the lottery is \$1 (since he is indifferent between playing or not), so $CE[Lottery] > EV[Lottery]$ and the risk premium is -50 cents. This makes Mr. Jo a risk-lover (even though he may not actually play the lottery).*

2. Consider the following lotteries:

   (a) \$100 with certainty
   (b) \$200 with probability 0.7; \$0 with probability 0.3
   (c) \$110 with certainty
   (d) \$1200 with probability 0.1; \$0 with probability 0.9
   (e) \$200 with probability 0.35; \$100 with probability 0.5; \$0 with probability 0.15
   (f) \$1200 with probability 0.05; \$110 with probability 0.5; \$0 with probability 0.45

   Suppose Bob is indifferent between lottery (a) and lottery (b). He is also indifferent between lottery (c) and lottery (d). Is Bob risk-averse, risk-loving, risk-neutral or is it impossible to say? Explain your answer.

   Solution. *The expected value to lottery (b) is \$140. Bob is indifferent between lottery (a), that provides \$100 with certainty and lottery (b), that provides a higher amount, \$140, in expectation but with uncertainty. This must mean that he is risk-averse in this choice. The same conclusion follow if we examine his choice between lottery (c) and (d).*

3. Cynthia has same choices as Bob. She is indifferent between lotteries (a) and (b) and between lotteries (c) and (d). If Cynthia is an expected utility maximizer and her utility is increasing in money/wealth, which lottery will she pick when faced with a choice between lotteries (e) and (f)?
   Solution. *Note that lottery (e) a convex combination of lotteries (a) and (b): It offers lottery (a) with 50% probability and lottery (b) with 50% probability. Since Cynthia is indifferent between lotteries (a) and (b), lottery (e) is equivalent to lottery (a). Similarly, lottery (f) is a convex combination of lotteries (c) and (d) and is therefore equivalent to lottery (c). As Cynthia's utility is increasing in wealth, she prefers lottery (c) to (a). Therefore, she will pick lottery (f) to lottery (e). More formally, Cynthia's choice between lotteries (a) and (b) and lotteries (c) and (d) imply that (normalizing Cynthia's initial wealth level to 0)*

$$\begin{aligned} u(100) &= 0.7u(200) + 0.3u(0) \\ \implies 0.5u(100) &= 0.35u(200) + 0.15u(0) \end{aligned}$$

   and

$$\begin{aligned} u(110) &= 0.1u(1200) + 0.9u(0) \\ \implies 0.5u(110) &= 0.05u(1200) + 0.45u(0) \end{aligned}$$

   Now, the expected utility from lottery (e) is given by

$$\begin{aligned} &0.35u(200) + 0.5u(100) + 0.15u(0) \\ =&0.5u(100) + 0.5u(100) \\ =&u(100) \end{aligned}$$

The expected utility from lottery (f) is given by

$$\begin{aligned} & 0.05u(1200) + 0.5u(110) + 0.45u(0) \\ =& 0.5u(110) + 0.5u(110) \\ =& u(110) \end{aligned}$$

Since Cynthia's utility is increasing in wealth

$$u(110) > u(100)$$

so she will pick lottery (f).

Insurance

- If most people are risk averse, they will pay money to avoid risk. This is the key idea behind insurance.
- Three key concepts:
  1. Risk pooling: defrays the risk across independent events by exploiting the law of large numbers. Example: most insurance companies work this way - they pool many independent risks.
  2. Risk spreading: because of the concavity of the utility function, taking a little bit of money away from many people incurs lower social costs than taking a lot of money from a few people. Example: disaster relief - tax money (taken from everyone) funds disaster victims (a few people who were hit very hard).
  3. Risk transfer: if the utility cost of risk is declining in wealth, then less wealthy people can pay wealthy people to bear their risks, and everyone will be better off. Example: Llyods of London used to pool wealthy peoples' money to take on large idiosyncratic risks (insuring the Titanic, for example).

Example - Startup Insurance (2017 Problem Set #5)

Following the increase in the number of MIT graduates who start new companies and given the inherent risks of starting a company, Cambridge Savings Bank offers insurance to startup companies with more than 5 employees. Before the startup hires any employees, the startup's founder has to invest $\$100,000$ in equipment. Throughout the question, assume that the initial wealth of startup founders is given by $w_0 = \$150,000$ and that they are risk-averse with a VNM utility function over wealth that is given by $u(w) = \ln(w)$. A startup becomes the "next Uber" with probability $\pi = .05$, resulting in a net gain of $\$19,850,000$ to its founder (so the total wealth of the founder if the startup is successful will be 20 million dollars); with probability of $1-\pi$, the startup goes bust, resulting in the loss of the $\$100,000$ invested in equipment.

1. The startup insurance offered by Cambridge Savings Bank looks as follows: The bank offers to pay the startup \$1 if it goes bust in exchange for receiving $\$S$ dollars if the startup is successful. What should the value of S be for the insurance to be considered actuarially fair?

   Solution. The expected return to the bank of this contract is given by $\pi S - (1-\pi)$. For the insurance to be actuarially fair, this expected return must be equal to zero; that is,

   $$S = \frac{1-\pi}{\pi} = \frac{.95}{.05} = 19.$$

2. How much startup insurance do startup founders buy if S is chosen as in the previous part?

   Solution. Suppose the startup founder buys x units of insurance. Then his/her expected utility is given by

   $$\pi \ln(20M - 19x) + (1-\pi)\ln(50{,}000 + x).$$

   The startup founder chooses x to maximize this expression. We can solve this problem by taking the FOC with respect to x:

   $$-\frac{19\pi}{20M - 19x} + \frac{(1-\pi)}{50{,}000 + x} = 0.$$

   Noting that $19\pi = 1 - \pi$, we can immediately see that it must be that $20M - 19x = 50{,}000 + x$, which implies that $x = 997{,}500$. Regardless of whether the startup succeeds, the founder will have $20M - 19 \cdot 997{,}500 = 50{,}000 + 997{,}500 = 1{,}047{,}500$ dollars in wealth. This is a more general principle: risk-averse agents always purchase full insurance if they are offered insurance at the actuarially fair price.

3. Suppose that every MIT graduate who starts a company also buys startup insurance (to the extent that you found in the previous part). Are MIT graduates better off starting companies (and buying startup insurance) or not starting a company at all (and keeping their \$150,000 to themselves)?

   Solution. The payoff to the founder from starting a company and buying full insurance is given by $\ln(1{,}047{,}500)$, whereas if they don't start start a company their payoff will be $\ln(150{,}000)$. So MIT graduates are better off starting companies and buying insurance.

4. A new deep learning algorithm can analyze a startup immediately after the \$100,000 investment in equipment has been made to determine whether the startup will be a success or not. The algorithm's output is posted on a publicly-accessible online database. Who will buy insurance in equilibrium after the introduction of the algorithm? [Hint: The output of the algorithm will be publicly visible *before* the startup hires any employees and so before it is eligible for insurance. Assume that Cambridge Savings Bank cannot reject any customers but it can adjust S in response to changes in the set of customers who buy insurance.]

   Solution. Since all the uncertainty is resolved after the output of the algorithm is observed, there are no insurance opportunities left: as long as $S > 0$, only startups founders whose companies will not be successful will buy insurance. But if that's the case, then no contract (with finite S) would be actuarially fair.

5. How does your answer to part (3) change as a result of the advent of the algorithm?

   Solution. Given that there are no longer any opportunities to buy insurance, the expected utility of an MIT graduate from starting a company is given by $\pi \ln(20M) + (1-\pi)\ln(50{,}000) \approx 11.120$, whereas their expected payoff from not starting a company is given by $\ln(150{,}000) \approx 11.918$. Therefore, MIT graduates are better off not starting a company. This problem shows that more information can lead to a decrease in welfare if it is provided at an interim stage (when investments are made but the risky output has not yet to realize).

Regression Discontinuity

- As always, we'd like to estimate the causal effect of treatment:

  $$T = E[Y_{i1} - Y_{i0} | D_i = 1]$$

  but we run into the Fundamental Problem of Causal Inference - we can't observe both Y_{i1} and Y_{i0} for a single unit.

- The idea of regression discontinuity (RD) is to find some arbitrary cutoff, where there is no treatment on one side of the cutoff ($D_i = 0$) and treatment on the other side of the cutoff ($D_i = 1$). The idea is that right on either side of the cutoff, units are similar. Therefore, contrasting outcomes on either side of the cutoff is a fairly "apples to apples" comparison.

- Let X_i denote the "running variable" and let $X_i = x^*$ denote the "cutoff." If $X_i \leq x^*$, then unit i does not receive treatment. If $X_i = x^*$, then unit i does receive treatment. The RD estimator is given by:

$$\hat{T}_{RD} = \lim_{\epsilon \downarrow 0} E[Y_i | X_i = x^* + \epsilon] - \lim_{\epsilon \uparrow 0} E[Y_i | X_i = x^* + \epsilon].$$

- What is this estimating? It's not the ATT for the full sample - rather it's the ATT for the subsample with values of $X_i \approx x^*$.

- For the RD estimator to be valid, we need three assumptions:

  1. There is no manipulation of the running variable. If people know about the rule and they are able to manipulate X, then they may selectively change the recorded value of X to ensure that a given individual does or does not receive the treatment. How would we try to check this assumption?
  2. In the absence of the discontinuity, the outcome Y is changing smoothly as a function of the running variable. In other words, in the absence of the cutoff x^*, for both untreated and treated units, we have:

$$\lim_{\epsilon \downarrow 0} E[Y_{i0} | X_i = x + \epsilon] = \lim_{\epsilon \uparrow 0} E[Y_{i0} | X_i = x + \epsilon]$$
$$\lim_{\epsilon \downarrow 0} E[Y_{i1} | X_i = x + \epsilon] = \lim_{\epsilon \uparrow 0} E[Y_{i1} | X_i = x + \epsilon].$$

  3. Nothing else changes at the discontinuity (besides the policy of interest). Otherwise, we don't know whether the policy of interest or something else caused the effect we observe.

Example - Saving Babies (2017 Problem Set #6)

Medical expenditures in the United States are high and increasing. A key policy question is, therefore, whether the benefits of additional medical expenditures exceed their costs.[1]

1. Let Y be a health outcome of newborns (for example, one-year mortality), and (Y_0, Y_1) be the potential outcomes when the newborn does not receive intensive medical treatment ($T = 0$) and when the newborn receives intensive medical treatment ($T = 1$). Explain in words and show mathematically why comparing $E[Y|T = 1]$ and $E[Y|T = 0]$ would not be a good idea to estimate the causal effect of intensive medical treatment on health outcomes of newborns.

   Solution. We might have a problem of omitted variables that are correlated with both intensive treatment and health outcomes. In this example, newborns that have a higher risk probably receive more intensive treatment. Mathematically:

$$E[Y|T = 1] - E[Y|T = 0] = E[Y_1|T = 1] - E[Y_0|T = 0] = E[Y_1 - Y_0|T = 1] + \{E[Y_0|T = 1] - E[Y_0|T = 0]\}.$$

[1] Note: this question is based on the paper "Estimating marginal returns to medical care: evidence from at-risk newborns," by Douglas Almond, Joseph Doyle, Amanda Kowalski, and Heidi Williams, *Quarterly Journal of Economics*, 2011. It is not necessary to read the paper to answer this question. But it is a terrific paper to read.

In a 2011 paper, Douglas Almond, Joseph Doyle, Amanda Kowalski, and Heidi Williams estimate the causal effect of intensive medical treatment on the health outcomes of low birth weight newborns using a regression discontinuity approach. When a newborn weighs less than $1,500g$, he/she is classified as "very low birth weight" (VLBW). Newborns classified as VLBW are more likely to receive Intensive Medical Treatment (IMT), such as admission to neonatal intensive care, ventilation, incubation, etc.

2. Let Z be a dummy variable equal to 1 if a newborn is classified as VLBW, and 0 otherwise. Let w_i be the birth weight of a newborn. Hence, $Z = 1$ if $w_i < 1,500$ and $Z = 0$ otherwise. Would it be a good idea to use Z as an instrument for T to estimate the causal effect of intensive medical treatment on health outcomes? Which IV identification assumption might be invalid?

   Solution. This would not be a good idea. Z would satisfy one IV condition, because it is correlated with IMT. However, newborns with $T = 1$ would have lower birth weight than newborns with $T = 0$. Since birth weight might have an independent effect on health outcomes, the exclusion restriction would not be satisfied.

3. Instead, the authors use the discontinuity at $1,500g$ to estimate the causal effect of interest.

   (a) Explain the rationale of this strategy. What are the identification assumptions?

   (b) Write the appropriate RD estimator using causal notation.

   Solution. (a) The rationale of this strategy is that the only difference between newborns just below and just above the cutoff is that the one classified as VLBW are more likely to receive IMT. The key assumptions are (1) no manipulation of the running variable (weight), (2) the outcome variable evolves smoothly in the absence of treatment, i.e.

$$\lim_{\varepsilon \downarrow 0} E\left[Y_0 | w = 1500 + \varepsilon\right] = \lim_{\varepsilon \uparrow 0} E\left[Y_0 | w = 1500 + \varepsilon\right],$$

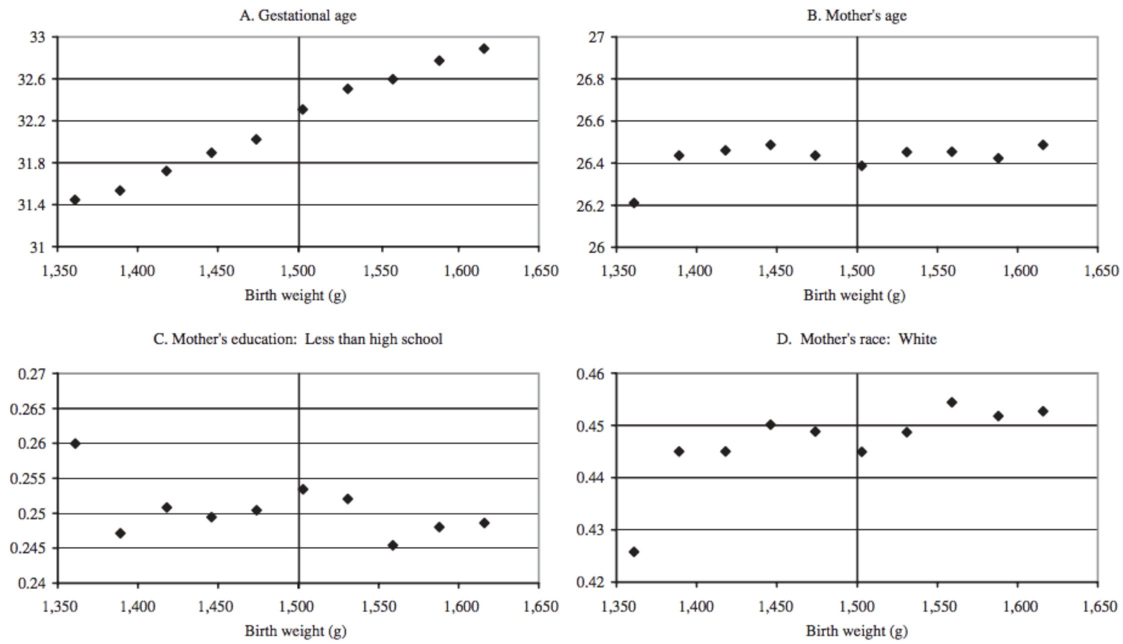
$$\lim_{\varepsilon \downarrow 0} E\left[Y_1 | w = 1500 + \varepsilon\right] = \lim_{\varepsilon \uparrow 0} E\left[Y_1 | w = 1500 + \varepsilon\right].$$

   and (3) intensive medical treatment is the only thing that changes when a baby crosses the $1500g$ threshold.

   (b) The RD estimator would be:

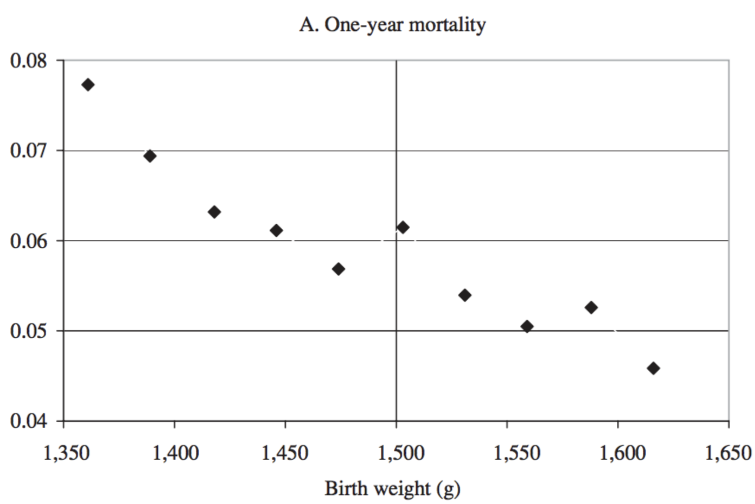
$$\hat{T}_{RD} = \lim_{\varepsilon \downarrow 0} E\left[Y | w = 1500 + \varepsilon\right] - \lim_{\varepsilon \uparrow 0} E\left[Y | w = 1500 + \varepsilon\right] = E\left[Y_1 - Y_0 | w = 1500\right].$$

4. The graphs in Figure V plot the relationship between several covariates and birth weight. Are these graphs consistent with the identification assumptions? Explain. What pattern(s) would tell you that the graphs are *not* consistent with the identification assumptions?



Solution. These graphs show that there are not any discontinuous change in these covariates around the threshold, which is consistent with the identification assumption. If there were a discontinuous change in some covariates around the threshold, we should worry that there are factors that change around the threshold other than IMT.

5. The graph in Figure II.A plots the relationship between one-year mortality and birth weight. Based on this graph and on the graph from III.A, what would you conclude about the causal effect of Intensive Medical Care on one-year mortality of newborns?



Solution. This graph shows that newborns just below the cutoff have a lower mortality rate. Under the *assumptions that the only difference between newborns just below and just above the cutoff is that the ones just below were more likely to receive IMT, this suggests that IMT reduces mortality.*

Signaling

In a world where skills or ability are private information, agents may engage in costly signaling to "prove" their skills. Below, we will go through two ways of modeling this. In one model, there are two types: high and low ability. In the other model, there is a continuum of types. In both setups, the broad strategy for solving for an equilibrium is the same. First, you assume one type of equilibrium. Second, you calculate the wages (and cutoff, if applicable). Finally, you check that the equilibrium type you assumed indeed holds.

Example - Signaling with Two Types

Setup:

- There are two types of ability. Half of agents are type H with $a_H = 1$ and half of agents are type L with $a_L = 0$.
- Productivity is given by $Y(a) = a$
- Before entering the workforce, agents can choose to obtain schooling $S \in \{0, 1\}$. Schooling is costly, but less costly if you are high ability. Let the cost be given by

$$c(a) = k(1 - a)$$

  where $k \geq 0$.
- The labor market is competitive, so firms pay employees their expected productivity. Wages are given by $W(S = 0)$ and $W(S = 1)$.
- The agents' utility is given by

$$u(W, c) = W - c.$$

Separating Equilibrium

Let $k = \frac{3}{2}$. We are looking for an equilibrium that consists of:

- Agents with a_H get schooling and agents with a_L don't get schooling
- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by assuming that we have a separating equilibrium. Then,

$$W(S = 0) = E[a|S = 0] = 0$$
$$W(S = 1) = E[a|S = 1] = 1.$$

Is this a truly a separating equilibrium, or will one type want to deviate? Compare the payoffs from school and no school for both types of agents. For the H type:

$$\text{no school: } u(W(S = 0), 0) = 0 - 0 = 0$$
$$\text{school: } u(W(S = 1), c(a_H)) = 1 - \frac{3}{2}(1 - a_H) = 1$$

For the L type:

$$\text{no school: } u(W(S=0),0) = 0 - 0 = 0$$
$$\text{school: } u(W(S=1),c(a_L)) = 1 - \frac{3}{2}(1-a_L) = -\frac{1}{2}.$$

So nobody will want to deviate. H prefers school, L prefers no school.

Pooling Equilibrium

Let $k = \frac{1}{4}$. We are looking for an equilibrium that consists of:

- All agents want to get schooling
- A set of wages for agents without and with schooling $W(S=0)$ and $W(S=1)$.

Start by assuming that we have a pooling equilibrium. Then,[2]

$$W(S=0) = 0$$
$$W(S=1) = E[a|S=1] = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$$

Is this a truly a pooling equilibrium, or will one type want to deviate? Compare the payoffs from school and no school for both types of agents. For the H type:

$$\text{no school: } u(W(S=0),0) = 0 - 0 = 0$$
$$\text{school: } u(W(S=1),c(a_H)) = \frac{1}{2} - \frac{1}{4}(1-a_H) = \frac{1}{2}$$

For the L type:

$$\text{no school: } u(W(S=0),0) = 0 - 0 = 0$$
$$\text{school: } u(W(S=1),c(a_L)) = \frac{1}{2} - \frac{1}{4}(1-a_L) = \frac{1}{4}.$$

So nobody will want to deviate. H and L both prefer school.

Example - Signaling with a Continuum of Types

Setup:

- There is a continuum of ability: $a \sim U[0,1]$ where $U[\cdot]$ is the uniform distribution.
- Productivity is given by $Y(a) = a$

[2]Why is $W(S=0)=0$ in a pooling equilibrium when in a pooling equilibrium everyone selects $S=1$? We have to assume that the employer will make some assumption about an agent's type if they go "off-equilibrium." It seems more likely that somebody who goes off equilibrium in this case is the L type. This is formalized by a notion called the "Intuitive Criterion" which is beyond the scope of this course.

- Before entering the workforce, agents can choose to obtain schooling $S \in \{0, 1\}$. Schooling is costly, but less costly if you are high ability. Let the cost be given by

$$c(a) = k(1 - a)$$

  where $k \geq 0$.

- The labor market is competitive, so firms pay employees their expected productivity. Wages are given by $W(S = 0)$ and $W(S = 1)$.

- The agents' utility is given by

$$u(W, c) = W - c.$$

Separating Equilibrium

Let $k = \frac{3}{4}$. We are looking for an equilibrium that consists of:

- A cutoff value a^*, above which agents will choose $S = 1$.
- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by just assuming that we have a separating equilibrium, with a cutoff of a^*. Then,

$$\begin{aligned} W(S = 0) &= E[a|S = 0] = E[a|a \leq a^*] = \frac{a^*}{2} \\ W(S = 1) &= E[a|S = 1] = E[a|a > a^*] = \frac{1 + a^*}{2}. \end{aligned}$$

Next, solve for the cutoff a^*. We want to find the agent who is indifferent between schooling and no schooling:

$$\begin{aligned} \underbrace{W(S = 0) - 0}_{\text{utility from no school}} &= \underbrace{W(S = 1) - c(a^*)}_{\text{utility from school}} \\ \frac{a^*}{2} - 0 &= \frac{1 + a^*}{2} - \frac{3}{4}(1 - a^*) \\ 0 &= \frac{1}{2} - \frac{3}{4} + \frac{3}{4}a^* \\ \frac{3}{4}a^* &= \frac{1}{4} \\ a^* &= \frac{1}{3}. \end{aligned}$$

Plugging in a^* to $W(S = 0)$ and $W(S = 1)$ gives us

$$\begin{aligned} W(S = 0) &= \frac{1}{6} \\ W(S = 1) &= \frac{2}{3}. \end{aligned}$$

Finally, we need to check whether this is a separating equilibrium. Will anyone want to deviate? Compare the payoffs from school and no school:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= \frac{1}{6} - 0 = \frac{1}{6} \\ \text{school: } u(W(S = 1), c(a)) &= \frac{2}{3} - \frac{3}{4}(1 - a) \end{aligned}$$

If $a = \frac{1}{3} - \epsilon$, then the utility from school is

$$\begin{aligned} \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} + \epsilon) &= \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} + \epsilon) \\ &= \frac{2}{3} - \frac{3}{4}(\frac{2}{3} + \epsilon) \\ &= \frac{1}{6} - \frac{3}{4}\epsilon < \frac{1}{6} \end{aligned}$$

so this person will prefer $S = 0$. If $a = \frac{1}{3} + \epsilon$, then the utility from school is

$$\begin{aligned} \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} - \epsilon) &= \frac{2}{3} - \frac{3}{4}(1 - \frac{1}{3} - \epsilon) \\ &= \frac{2}{3} - \frac{3}{4}(\frac{2}{3} - \epsilon) \\ &= \frac{1}{6} + \frac{3}{4}\epsilon > \frac{1}{6} \end{aligned}$$

so this person will prefer $S = 1$. So nobody will want to deviate. H prefers school, L prefers no school.

Pooling Equilibrium

Let $k = \frac{1}{4}$. We are looking for an equilibrium that consists of:

- All agents want to get schooling.
- A set of wages for agents without and with schooling $W(S = 0)$ and $W(S = 1)$.

Start by just assuming that we have a pooling equilibrium[3]

$$\begin{aligned} W(S = 0) &= E[a|S = 0] = 0 \\ W(S = 1) &= E[a|S = 1] = E[a] = \frac{1}{2}. \end{aligned}$$

Finally, we need to check whether this is a separating equilibrium. Will anyone want to deviate? Compare the payoffs from school and no school:

$$\begin{aligned} \text{no school: } u(W(S = 0), 0) &= 0 - 0 = 0 \\ \text{school: } u(W(S = 1), c(a)) &= \frac{1}{2} - \frac{1}{4}(1 - a) \end{aligned}$$

The utility of school is positive for all values of $a \in [0, 1]$, so nobody will want to deviate. H and L both prefer school.

Example - Signaling Spontaneity (2014 Problem Set #6)

Robin would like a spontaneous boyfriend. Men have a level of spontaneity (known to them) s_i distributed uniformly on interval $[0, 1]$. Robin doesn't observe spontaneity directly, but knows its distribution in the population. You can imagine that $s_i = 1$ corresponds to the type of person who would take you to Paris

[3]Again, we are assuming that $E[a|S = 0] = 0$ even though choosing $s = 0$ is off the equilibrium.

out of the blue, whereas a person with $s_i = 0$ wouldn't even take you to Somerville with less than 3 weeks' notice.

Robin goes on dates to a restaurant "Carmichaels" with randomly chosen men from the zero-one interval. Unfortunately, men cannot reveal that they are spontaneous since talk is cheap. That said, these potential partners *can* take a costly action that may signal that they are spontaneous: they can choose whether or not to steal the blue French horn that hangs on the restaurant wall, and give it to Robin. Stealing the French horn costs $1 - s_i^2$ for a person with level of spontaneity s_i (i.e. the action is costless for those with $s_i = 1$ who would go to Paris on a whim).

Robin decides which dates to accept as boyfriends [she can have more than one]. She gets utility $4s_i - 3$ from accepting a person with spontaneity s_i, and zero from not accepting. All candidates want to be Robin's boyfriend, which gives them utility $\frac{2}{3}$.

1. Find a separating equilibrium, i.e. an equilibrium where some dates steal the blue French horn and others don't bother. You should take the following steps to solve the problem:

   (a) Find which potential partners will steal the blue French horn if only those that steal it become Robin's boyfriends.

   (b) Determine which people Robin will accept as a function of her beliefs about their level of spontaneity (assuming again that only those who steal the French horn are eligible).

   (c) Confirm that if potential partners act according to (a), then Robin will indeed accept them if and only if they steal the blue French horn.

   Solution. (a) The benefit of stealing the blue French horn is $\frac{2}{3}$ (utility of being Robin's boyfriend) while the cost is heterogeneous, and equal to $1 - s_i^2$, so a person should steal if

   $$\frac{2}{3} \geq 1 - s_i^2 \Rightarrow s_i^2 \geq \frac{1}{3} \Rightarrow s_i \geq \sqrt{\frac{1}{3}} \approx 0.58$$

   so the cut-off $\tilde{s} = \sqrt{\frac{1}{3}}$.

   (b) Robin only observes whether a date steals the horn or not. She will accept those who steal only if her beliefs are that the average spontaneity among that groups is greater or equal to the one that solves for indifference in $4\hat{s} - 3 = 0$. Therefore she accepts all who steal iff

   $$E\left[s \mid steal\right] \geq \frac{3}{4}$$

   (c) In equilibrium if individuals steal iff $s \geq \tilde{s} = \sqrt{\frac{1}{3}}$, Robin is acting optimally in accepting all potential partners that steal since $E\left[s \mid steal\right] = \frac{1+\tilde{s}}{2} \approx 0.79 > \frac{3}{4}$. Similarly we have $E\left[s \mid not\, steal\right] = \frac{\tilde{s}}{2} \approx 0.29 < \frac{3}{4}$ so it is optimal not to accept these as boyfriends. This is a separating equilibrium.

2. Suppose Robin can only go on one date, with a randomly chosen man from the zero-one interval. What is Robin's expected utility of the date given this separating equilibrium?
   Solution. A date in which the horn is not stolen gives zero utility (since Robin doesn't accept), whereas a date where the horn is stolen gives Robin expected utility

   $$4E\left[s \mid steal\right] - 3 = 4\left(0.79\right) - 3 = 0.16$$

It follows that expected utility of the entire date is

$$P\,[not\ steal] \cdot 0 + P\,[steal] \cdot 0.16$$
$$= P\,[s < \tilde{s}] \cdot 0 + P\,[s \geq \tilde{s}] \cdot 0.16$$
$$= (1 - \tilde{s}) \cdot 0.16 \approx 0.067$$

3. The company Spontaneous Combustion, Inc. has developed a clever app that reveals a potential partner's exact level of spontaneity by scanning the iris of the person. The company sells use of the app for price p per person tested [assume that all utilities are measured in money, so buying the test also costs p units of utility]. What is the maximum price Robin would be willing to pay for use of the app? [You can again consider that she goes on only one date.]
Solution. Without the app, Robin gets expected utility 0.067 *(per date) in the separating equilibrium. With the app, Robin can use the efficient cutoff rule, only accepting as boyfriends those with* $s \geq \frac{3}{4}$*. This gives her per-date expected utility of*

$$P\left[s < \frac{3}{4}\right] \cdot 0 + P\left[s \geq \frac{3}{4}\right]\left(4E\left[s \mid s \geq \frac{3}{4}\right] - 3\right) - p$$
$$= \frac{1}{4}\left(4\left(\frac{7}{8}\right) - 3\right) - p = 0.125 - p$$

The maximum Robin is willing to pay is then $\bar{p}$ *such that* $0.125 - \bar{p} = 0.067$ *(making her indifferent between using the app or not). This gives us* $\bar{p} = 0.058$*.*