14.03/14.003 Recitation 8
Expected Utility

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Agenda

- Derivation of the Expected Utility Theorem
- Interesting Issues

Oskar Morgenstern on John von Neumann

Whenever Johnny saw a funny hat, he'd put [it] on...He had one with a little contraption that made a noise when you blew into it, and a light would flash at the same time. It was one of those children's things, and he loved it.

The hat picture

A gamble and a poll

0 1m 5m

(0, 1, 0)

(.01, .89, .10)

Experiment 1		**Experiment 2**	
Gamble 1.A	Gamble 1.B	Gamble 2.A	Gamble 2.B
{\$1m w.p.1	\$1m w.p. .89 \$0 w.p. .01 \$5m w.p. .10	\$0 w.p. .89 \$1m w.p. .11	\$0 w.p. .90 \$5m w.p. .10

Defining a Lottery

Definition

A simple lottery L is a list $L = (p_1, \dots p_N)$ with $p_n \geq 0$ for all n and $\Sigma_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Example:

$$\begin{cases} x_1 = \$100 \;\; w.p. \;\; p_1 = .5 \\ x_2 = \$200 \;\; w.p. \;\; p_2 = .5 \end{cases}$$

Define the set of alternatives/gambles the decision maker faces, denoted by $\mathscr{L}$, as the set of all simple lotteries over possible outcomes N. How do we represent preferences on this set? In this class, **vNM**, which assumes that the agents have a rational preference relation $\succsim$ on $\mathscr{L}$, that for all $L \in \mathscr{L}$ can be represented as:

$$U(L) = \sum_{j=1}^{N} p_j u(x_j).$$

$U(\cdot)$ s.t. $L' \succ L$

iff $U(L') > U(L)$

These representation requires **two axioms** on preferences (we also assume completeness and transitivity).

Axioms

Axiom 1. Continuity. *Small changes in probabilities do not change the nature of the ordering of two lotteries.*

$L \succ L'$ $\quad$ $p_1 x_1 + (1-p) x_2 \succ L'$ $\quad$ $\varepsilon > 0$ $\quad$ $(p_1+\varepsilon)\cdot x_1 + (1-(p_1+\varepsilon))x_2$

L'':

Axiom 2. Independence. *The preference relation* $\succsim$ *on the space of simple lotteries* $\mathscr{L}$ *satisfies the independence axiom if for all* $L, L', L'' \in \mathscr{L}$ *and* $\alpha \in (0,1)$, *we have*

$L'' \succ L'$

$$L \succsim L' \text{ if and only if } \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''.$$

$.5\,[p_1 x_1 + (1-p) x_2] + .5\,[\$1m] \succ .5 L' + .5\,[\$1m]$

Meaning:

1. Continuity: same as consumer theory. We can find a neighborhood in the space of lotteries where you prefer one lottery to another.
2. Independence (from irrelevant alternatives) if you prefer a lottery to another, and I mix both of them up with another in the same proportion, you still satisfy the same ordering. The "irrelevant alternative" is the additional lottery L'' that I am mixing in

Are they reasonable?

The vNM representation

The utility function $U : \mathscr{L} \rightarrow \mathbb{R}$ *has an expected utility form if there is an assignment of numbers* $(u_1, \dots u_N)$ *to the* N *outcomes such that for every simple lottery* $L = (p_1, \dots, p_N) \in \mathscr{L}$ *we have that*

$$U(L) = u_1 p_1 + \dots + u_N p_N. \quad = \sum p_n u(x_n)$$

In other words, a utility function has the expected utility form if and only if:

$$U\left(\sum\nolimits_{k=1}^{K} \alpha_k L_k\right) = \sum\nolimits_{k=1}^{K} \alpha_k \underline{U(L_k)} \qquad \alpha_k \in (0,1)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \dots, K$, and probabilities $(\alpha_1, \dots, \alpha_K) \geq 0$, $\Sigma_k \alpha_k = 1$. That is, the representation is linear in probabilities.

A utility function has the expected utility property if the utility of a lottery is simply the (probability-) weighted average of the utility of each of the outcomes.

Showing that the axioms imply vNM

Theorem

(Expected utility theory) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots p'_N)$, we have $L \succsim L'$ if and only if

$U(L) \geq U(L')$

$$\underbrace{\sum_{n=1}^{N} u_n p_n}_{} \geq \underbrace{\sum_{n=1}^{N} u_n p'_n}_{}$$

$L = (p_1, \ldots, p_N)$

.

We will show that, under the two axioms, and for any two lotteries L and L', and $p \in (0,1)$, there exists a utility function U representing preferences **over lotteries**, such that

$$\boxed{U(pL + (1-p)L') = pU(L) + (1-p)U(L').}$$

Does it satisfy the axioms?

$$U(pL + (1-p)L'') = \underbrace{p}_{-\varepsilon} U(L) + \underbrace{(1-p)}_{-\varepsilon} U(L') > p' U(L) + (1-p') U(L')$$

- The definition of continuity would be something like: if there is a sequence of lotteries $\{L_n\}$, such that $L_n \succsim L''$ for all n, and the limit of the sequence is L, then $L \succsim L''$.
- Since the vNM function is linear, it is also a continuous function, and weak inequalities are preserved under the limit operator.
- Independence is very trivial: if you add something on both sides of the equal sign and multiply both sides by some number the inequality is preserved.

Back to the Proof

We will show that, under the two axioms, and for any two lotteries L and L', and $p \in (0,1)$, we can build a utility function U representing preferences **over lotteries**, such that $U(pL + (1-p)L') = pU(L) + (1-p)U(L')$.

$x_1, \ldots, x_n$ $\qquad$ $x_n > \ldots > x_1 = \underline{L}$

$L_0 = \overline{L}$

- Step 1: Consider a fixed set of outcomes, and define $\overline{L}$ as the most-valued outcome and $\underline{L}$ as the least-valued. These outcomes are also *degenerate lotteries*: we get an outcome with certain probability.
- Step 2: continuity implies that, for each outcome $L_i \in [\underline{L}, \overline{L}]$, there exists a probability $p_i \in (0,1)$ such that $\boxed{p_i \overline{L} + (1-p_i)\underline{L}} \sim L_i$.

$L_i = \{ x_i \text{ w.p. } 1$

Proof cont.

- Step 3: define the utility function $\boxed{U(L_i) = p_i}$ (the utility *is the probability of the best outcome*, such that the mixture is indifferent to the lottery). Considering a lottery, L_i, we then have:

$$\boxed{L_i \sim p_i \overline{L} + (1-p_i)\underline{L}} = \boxed{U(L_i)\cdot \overline{L} + (1-U(L_i))\cdot \underline{L}}$$

$L_i \succ L_j$ iff $U(L_i) > U(L_j)$ increasing in U

- The above represents the preferences, as a rational agent will prefer the lottery that gives the higher payoff with larger probability, therefore:

$$L_1 \succ L_2 \;\; iff \;\; U(L_1) = p_1 \succ p_2 = U(p_2).$$

$L_1 \sim p_1 \overline{L} + (1-p_1)\underline{L}$

$L_2 \sim p_2 \overline{L} + (1-p_2)\underline{L}$

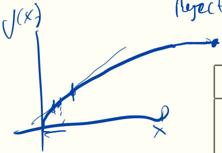
- Step 4: **by independence**, for any $p \in (0,1)$, the mixture:

$$\boxed{pL_1 + (1-p)L_2} \sim p\left[p_1\overline{L} + (1-p_1)\underline{L}\right] + (1-p)\left[p_2\overline{L} + (1-p_2)\underline{L}\right]$$

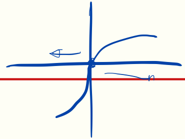
That is:

$$U(pL_1 + (1-p)L_2) = pU(L_1) + (1-p)U(L_2).$$

Allais "Paradox"



Reject -110 w.p. $.5$, 100 w.p. $.5$ $\quad$ -600 w.p. $.5$, ∞ w.p. $.5$



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If you choose 1.A and 2.B, your preferences are not vNM! They do not satisfy independence, as the above is the same as:

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Experiment 1 is just experiment 2, where in addition I give you a lottery where you win $\$1m$ in **both** scenarios. This should not flip your preferences according to vNM.