Recitation 8: Expected Utility

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1 Willingness to pay for insurance

Suppose that a risk-averse agent:

- has von Neumann-Morgenstern utility function given by $u(w) = \ln(w)$,
- faces a 50 percent chance of getting $w = \$200$ and a complementary 50 percent chance of having $w = \$100$.

Question: How much is the agent willing to pay, in expected value, to avoid this uncertainty?

- In absence of insurance, the agent's expected utility is:

$$E[u(w)] = 0.5 \ln 200 + 0.5 \ln 100 = 4.952.$$

- Expected wealth is:

$$E[w] = 0.5 \times 200 + 0.5 \times 100 = \$150.$$

- The certainty equivalent (i.e., the fixed amount of money the gives the same expected utility as the uncertain realization of the agent's wealth) is:

$$CE = \exp[4.952] = \$141.5,$$

because \$141.5 given to the agent with probability = 1 yield the same expected utility as the lottery described above.

*I thank Alex He for sharing his notes. Remaining errors are my own.

- Thus, to avoid uncertainty, the agent is willing to pay:

$$WTP = E[w] - CE = \$150 - \$141.5 = \$8.50.$$

The hard point to conceptualize in this example (and in the relevant question on the problem set) is that, given the uncertainty, ex-ante we don't know the exact realization of the agent's wealth (i.e., whether she'll have \$200 or \$100). Thus, the agent's willingness to pay for insurance has to be "benchmarked" to her expected wealth. Specifically, her willingness to pay is given by the difference between her expected wealth absent insurance and her certainty equivalent.

2 Deriving Expected Utility Theorem

Definition 1. A simple lottery L is a list $L = (p_1, ...p_N)$ with $p_n \geq 0$ for all n and $\Sigma_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Take the set of alternatives the decision maker faces, denoted by $\pounds$ to be the set of all simple lotteries over possible outcomes N. We assume the consumer has a rational preference relation $\succsim$ on $\pounds$, a *complete* and *transitive* relation allowing comparison among any pair of simple lotteries.

Axiom 1. Continuity. *Small changes in probabilities do not change the nature of the ordering of two lotteries. This can be made concrete here (I won't use formal notation b/c it's a mess). If a "bowl of miso soup" is preferable to a "cup of Kenyan coffee," then a mixture of the outcome "bowl of miso soup" and a sufficiently small but positive probability of "death by sushi knife" is still preferred to "cup of Kenyan coffee."*

Axiom 2. Independence. *The preference relation $\succsim$ on the space of simple lotteries $\pounds$ satisfies the independence axiom if for all $L, L', L'' \in \pounds$ and $\alpha \in (0, 1)$, we have*

$$L \succsim L' \text{ if and only if } \alpha L + (1 - \alpha) L'' \succsim \alpha L' + (1 - \alpha) L''.$$

In words, when we mix each of two lotteries with a third one, then the preference ordering of the two resulting mixtures does not depend on (is *independent of*) the particular third lottery used. Example: If a bowl of miso soup is preferred to cup of Peets coffee, then the lottery (bowl of miso soup with 50% probability, steak dinner with 50% probability) is preferred to the lottery (cup of Peets coffee with 50% probability, steak dinner with 50% probability).

2.1 Expected utility theory

- We now want to define a class of utility functions over risky choices that have the "expected utility form." We will then prove that if a utility function satisfies the definitions above for *continuity* and *independence* in preferences over lotteries, then the utility function has the expected utility form.

  Definition 2. *The utility function $U : \pounds \to \mathbb{R}$ has an expected utility form if there is an assignment of numbers $(u_1, ...u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, ..., p_N) \in \pounds$ we have that*

  $$U(L) = u_1 p_1 + ... + u_N p_N.$$

- A utility function with the expected utility form is called a Von Neumann-Morgenstern (VNM) expected utility function.

- In other words, a utility function has the expected utility form if and only if:

  $$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

  for any K lotteries $L_k \in \pounds$, $k = 1, ..., K$, and probabilities $(\alpha_1, ..., \alpha_K) \geq 0$, $\Sigma_k \alpha_k = 1$.

- Intuitively, a utility function has the expected utility property if the utility of a lottery is simply the (probability) weighted average of the utility of each of the outcomes.

2.2 Proof of expected utility property

Proposition. *(Expected utility theory) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\pounds$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, ..., N$ in such a manner that for any two lotteries $L = (p_1, ..., p_N)$ and $L' = (p'_1, ...p'_N)$, we have $L \succsim L'$ if and only if*

$$\sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n$$

.

Proof. We will show that for any two lotteries L and L', and $\beta \in (0, 1)$, there is a utility function U representing preferences over lotteries, such that $U(\beta L + (1 - \beta)L') = \beta U(L) + (1 - \beta)U(L')$. This is equivalent to showing the Expected Utility Property stated above because if we take L_n to be a lottery that results in outcome n with certainty, then $U(L) = U(\sum_n p_n L_n) = \sum_n p_n U(L_n) = \sum_n p_n u_n$

Expected Utility Property (in five steps) $\square$

Assume that there are best and worst lotteries in $\pounds$, $\bar{L}$ and $\underline{L}$.

1. If $L \succ L'$ and $\alpha \in (0,1)$, then $L \succ \alpha L + (1-\alpha) L' \succ L'$. This follows immediately from the independence axiom.

2. Let $\alpha, \beta \in [0,1]$. Then $\beta \bar{L} + (1-\beta)\underline{L} \succ \alpha \bar{L} + (1-\alpha)\underline{L}$ if and only if $\beta > \alpha$. This follows from the prior step.

3. For any $L \in \pounds$, there is a unique α_L such that $\left[\alpha_L \bar{L} + (1-\alpha_L)\underline{L}\right] \sim L$. Existence follows from continuity. Uniqueness follows from the prior step.

4. We now need to define a utility function that satisfies the expected utility property. Though there may be many choices, our proposition only requires us to pick one that represents the preferences over lotteries and satisfies the expected utility property. Condsider the function $U : \pounds \to \mathbb{R}$ that assigns $U(L) = \alpha_L$ for all $L \in \pounds$. It represents the preference relation $\succsim$ because from Step 3, we know that for any two lotteries $L, L' \in \pounds$, we have

$$L \succsim L' \text{ if and only if } \left[\alpha_L \bar{L} + (1-\alpha_L)\underline{L}\right] \succsim \left[\alpha_{L'} \bar{L} + (1-\alpha_{L'})\underline{L}\right].$$

   Thus $L \succsim L'$ if and only if $\alpha_L \geq \alpha_{L'}$.

5. The utility function $U(\cdot)$ that assigns $U(L) = \alpha_L$ for all $L \in \pounds$ is linear and therefore has the expected utility form.
   We want to show that for any $L, L' \in \pounds$, **and** $\beta[0,1]$, **we have** $U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta) U(L')$.
   By step (3) above, we have

$$\begin{aligned} L &\sim U(L)\bar{L} + (1-U(L))\underline{L} = \alpha_L \bar{L} + (1-\alpha_L)\underline{L} \\ L' &\sim U(L')\bar{L} + (1-U(L'))\underline{L} = \alpha_{L'} \bar{L} + (1-\alpha_{L'})\underline{L}. \end{aligned}$$

   By the Independence Axiom,

$$\begin{aligned} \beta L + (1-\beta)L' &\sim \beta\left[U(L)\bar{L} + (1-U(L))\underline{L}\right] + (1-\beta)\left[U(L')\bar{L} + (1-U(L'))\underline{L}\right] \\ &= \left[\beta U(L) + (1-\beta)U(L')\right]\bar{L} + \left[\beta(1-U(L)) + (1-\beta)\left(1-U(L)'\right)\right]\underline{L} \\ &= \left[\beta U(L) + (1-\beta)U(L')\right]\bar{L} + \left[1 - \beta U(L) + (\beta-1)U(L')\right]\underline{L}. \end{aligned}$$

By step (4), this expression can be written as

$$
\begin{aligned}
& \left[\beta \alpha_L + (1-\beta)\,\alpha_{L'}\right] \bar{L} + \left[1 - \beta \alpha_L + (\beta - 1)\,\alpha_{L'}\right] \underline{L} \\
= \; & \beta U(L) + (1-\beta)\, U(L').
\end{aligned}
$$

This establishes that a utility function that satisfies continuity and the Independence Axiom, has the expected utility property: $U(\beta L + (1-\beta) L') = \beta U(L) + (1-\beta) U(L')$