14.03 Recitation 4: Potential Outcomes and Consumer Choice

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I. Review of the Potential Outcome Framework

A. Objects of Interest

- X_i: treatment actually administered to an individual i. $X_i = 1$ usually refers to treatment being administered, while $X_i = 0$ to no treatment. Therefore all individuals with $X_i = 1$ are part of the *treatment group*, while those with $X_i = 0$ constitute the *control group*;

- Y_{ij} the outcomes of some individual i, after receiving treatment j

- $T_i = (Y_i|X_i = 1) - (Y_i|X_i = 0)$: treatment effect for individual i

Denote the treatment group by $\mathbf{T}$ and the control by $\mathbf{C}$. In the notation above, what we observe in the data is:

- $Y_{j1}|X_j = 1$, i.e. the outcomes of individuals j after receiving treatment, for the group $j \in \mathbf{T}$ that has been treated ($X_j = 1$);

- $Y_{k0}|X_k = 0$, $k \in \mathbf{C}$ i.e. the outcomes of individuals k after NOT receiving treatment, for the group k that has NOT been treated ($X_k = 0$);

This is where selection issues come in. If you worry that receiving the treatment (the value of variable X) is not random, you will also worry that outcomes will be systematically different across treatment and control *groups*, *independently of the nature of the treatment!*

This is the selection bias that we have seen in class:[1]

$$E_{\mathbf{T}}(Y_{j1}|X_j = 1) - E_{\mathbf{C}}(Y_{k,0}|X_k = 0) = \left[E_{\mathbf{T}}(Y_{j1}|X_j = 1) - E_{\mathbf{T}}(Y_{j,0}|X_j = 1)\right] + \left[E_{\mathbf{T}}(Y_{j,0}|X_j = 1) - E_{\mathbf{C}}(Y_{k,0}|X_k = 0)\right] \tag{1}$$

The last term in brackets is the selection effect (or bias), the systematic difference in outcomes between the two groups absent treatment. Therefore, if we want to obtain the average treatment-on-the-treated effect ATT:

$$E(Y_1|X = 1) - E(Y_0|X = 1)$$

we have to ensure that the bias term is 0 if we want to use the above difference in means (1).
Instead, if we want to use $E_{\mathbf{T}}(Y_{j1}|X_j = 1) - E_{\mathbf{C}}(Y_{k,0}|X_k = 0)$ to compute the average treatment effect ATE:

$$E(Y_1) - E(Y_0)$$

IN addition to a 0 selection effect, we would need to also assume

$$E_{\mathbf{T}}(Y_{j1}|X_j = 1) = E_{\mathbf{C}}(Y_{k1}|X_k = 0)$$

i.e. that the outcomes of the two groups would be the same if administered treatment. Randomization, if successfully carried out, ensures both approximately apply, since the treatment and control groups are very similar to each other.

B. Examples

Miguel & Kramer 2004 found deworming medicine increased probability of attending school for a sample of Kenyan school children by 7pp. Let's think through with RCM.

1. What are potential outcomes here?
    - Y_{i1} is school attendance for individual i if they take the deworming medicine
    - Y_{i0} is school attendance for individual i if they do not take the deworming medicine
2. What is the ATT?
    - $E[Y_{i1} - Y_{i0}|X_i = 1]$ is the average effect on school attendance for individuals who will choose to take the drug.

[1] Here I use superscripts $\mathbf{C}, \mathbf{T}$ to stress that averages are taken over control and treatment group. This is exactly the same as the notation in class:

$$E(Y_1|X = 1) - E(Y_0|X = 0) = [E(Y_1|X = 1) - E(Y_0|X = 0)] + [E(Y_1|X = 0) - E(Y_0|X = 0)]$$

3. What is the ATE?

   - $E[Y_{i1} - Y_{i0}]$ is the average effect on school attendance for all individuals

4. Can we use the difference in means $E[Y_{i1}|X_i = 1] - E[Y_{i0}|X_i = 0]$ to measure the ATT? i.e. is there selection bias $E[Y_{i0}|X_i = 1] - E[Y_{i0}|X_i = 0]$?

   - Here we might expect a selection bias. In particular, individuals who get the deworming drug are sick, which might be an indication that they come from poorer backgrounds, and are therefore likely to have lower attendance than non-takers in case they do not take the drug.

5. Can we use the ATT to measure the ATE? i.e. would outcome differ in case they both got the medicine? $E[Y_{i1}|X_i = 1] - E[Y_{i1}|X_i = 0]$?

   - The ATT is almost certainly biased upwards wrt ATE as individuals who have worms are more likely to take the medicine. While individuals without worms likely wouldn't benefit from the medicine and are unlikely to take it. (Very much like the cholesterol drug)

Another example: the good ol' Card and Krueger (1994). Denote by $\Delta Y \equiv Y_t - Y_{t-1}$ - the change in over a time period in some variable Y. Here:

- What are potential outcomes?
  - ΔY_{i1} is the change in employment for store i if they experience an increase in the minimum wage
  - ΔY_{i0} is the change in employment for store i if the minimum wage is unchanged.
- What is the ATT?
  - $E[\Delta Y_{i1} - \Delta Y_{i0}|X_i = 1]$ is the average effect on employment for stores that see an increase in the minimum wage (New Jersey stores).
- What is the ATE?
  - $E[\Delta Y_{i1} - \Delta Y_{i0}]$ is the average effect on stores of introducing the minimum wage.
- Can we use the difference in means $E[\Delta Y_{i1}|X_i = 1] - E[\Delta Y_{i0}|X_i = 0]$ to measure the ATT? Is there bias $E[\Delta Y_{i0}|X_i = 1] - E[\Delta Y_{i0}|X_i = 0]$?
  - As usual, the bias is given by the counterfactual evolution of outcomes across the two groups in the absence of treatment. Here the treatment is the introduction of the minimum wage, the two groups are NJ and PA. When does bias occur? When there are no *parallel trends*. Indeed, the parallel trends assumption precisely tells us that treatment and control are comparable by hypothesizing that they would have evolved in the same way absent treatment.

- Can we use the ATT to measure the ATE? i.e. would outcomes differ in case they both got the minimum wage? $E[Y_{i1}|X_i = 1] - E[Y_{i1}|X_i = 0]$?
  - Unclear. There might be arguments for both sides, and I am not sure this issue is tackled in general.

II. Inference/ Statistical Significance

Often we want to know whether a value we estimated is statistically different from zero. For means (and regression coefficients), we typically do this using a "t-test" on a sample mean. The basic logic is as follows:

- Suppose I assume that true mean of our data is zero.
- I can compute the t-statistic, that has a known distribution conditional on some value of the mean. This means that I can tell what is the probability that I observe certain values of t.[2]
- If I see a value that is unlikely if the data has indeed a zero mean, this suggests the original assumption that the mean is zero is wrong, and I can therefore reject my original hypothesis.

The above procedure is an application of the central limit theorem. The CLT tell us that if we have a sequence of independent and identically distributed or $i.i.d.$ random variables, where $E[X_i] = \mu$ and $Var[X_i] = \sigma^2 < \infty$ then

$$\sqrt{n}\left(\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right) \to^{d} N(0,\sigma^2)$$

Where $N(\cdot)$ is the normal distribution. That is, we know the distribution of sample averages converges to the normal distribution. And if that's true, we know that $T = \frac{\bar{x}-\mu}{\sqrt{s^2/n}}$ is distributed approximately as Student-t distribution $t(n-1)$ where s^2 is the sample variance.

So in order to test my hypotheses that the mean/coefficient is zero, I assume its true value in the population is zero, $\mu = 0$. Then compute $t^* = \frac{\bar{x}}{\sqrt{s^2/n}}$. If t^* takes on an extreme value given the distribution, it weighs against my hypotheses. In particular, the p-value $p = P(|T| \le t^*|\mu = 0)$ measures how likely it is to observe values that are more extreme than t^* if the hypothesis $\mu = 0$ is true.[3] $p \le 0.05$, which occurs when $t^* \approx 2$, is the most common rejection rule.

A useful rule of thumb to test significance is $|\bar{x} \pm 2 * \sqrt{s^2/n}| > 0$. This means that the (approximate) 95% *confidence interval*:

$$[\bar{x} - 2 * \sqrt{s^2/n}, \bar{x} + 2 * \sqrt{s^2/n}]$$

does not overlap with 0.

[2] This distributions of the statistic encodes the sampling uncertainty surrounding it. In particular they tell us the frequency with which we will observe a value of the statistic if we sample infinitely many times from the population and repeat the computation of the statistic.

[3] The p-value therefore tells us that if we extracted infinite samples from a population with mean 0, and computed a t-stat for each of those, we would have that only 5% of the values we computed would fall above the value t^*. This is clearly a frequentist definition.

Concretely: example for regression tables

I estimate a quantity for a coefficient/mean $\hat{\beta}$ and an associated standard error $\sigma(\hat{\beta})$. Usually in regression tables the standard error is reported below the value of the coefficient or estimated mean. Then:

- If $t = \dfrac{\hat{\beta}}{\sigma(\hat{\beta})} > 2$ (or $t < -2$), the estimated coefficient is statistically significant (at 5% confidence level);
- Otherwise, it is not.

An even quicker way to see whether a coefficient/mean is significant is to check whether: $|\hat{\beta}| > 2\sigma(\hat{\beta})$ (our brain is better at multiplying than dividing).

III. Theory of Consumer Choice

A. Primal Problem, Marshallian Demand, Indirect Utility

The consumer solves:

$$\max_{x,y} U(x, y)$$
$$\text{s.t.}\, p_x x + p_y y \leq I$$

Here, **prices** and **Income** are given. Therefore, solving the primal, we will obtain optimal quantities of goods for given income I and prices p_x, p_y. Under axiom 5 (strictly diminishing MRS), we will always have an interior solution, defined by:

$$MRS_{x,y} = \frac{\partial U(x, y)/\partial x}{\partial U(x, y)/\partial y} = \frac{p_x}{p_y} \tag{2}$$

The interpretation is that the "bang for the buck" in terms of util per dollar for good x -MU_x/p_x- equals the utils per dollar that I can get for good y - MU_y/p_y. Alternatively, I can think of the condition as stating "the psychological trade-off between goods (MRS) is the same as the market trade-off (price ratio)".
To obtain the above condition set up the Lagrangian:

$$L(\lambda, x, y) = U(x, y) + \lambda(I - p_x x - p_y y)$$

And combine FOC's for x and y. To fully solve, use (2) to replace x for y into the budget constraint. Something I like not to get tangled up in algebra is expressing $p_x x$ in terms of $p_y y$ and plug into the budget constraint.

Solving the primal problem, gives the **Marshallian demands**:

$$d_x(p_x, p_y, I)$$
$$d_y(p_x, p_y, I)$$

We can use them to answer in particular:

- **Fixing income** how does the consumption of y change as the price of x or y changes? (Taking the partial derivative wrt some price).

The optimal value (value function) of the problem is given by the **indirect utility function**, defined as:

$$V(p_x, p_y, I) = U(d_x(p_x, p_y, I), d_y(p_x, p_y, I))$$

This is the level of utility that the consumer optimally achieves given **prices** p_x, p_y and **income** I. We can therefore understand how utility changes with prices/income using the derivatives of this object.

B. Dual Problem, Hicksian Demand, Expenditure Function

The consumer solves:

$$\min_{x,y} p_x x + p_y y$$
$$\text{s.t.} U(x, y) \geq \bar{U}$$

Here, it is **utility** and **prices** that are given (note that income does not appear). The consumer is choosing a bundle (x, y) to minimize the expenditure to achieve a given level of utility. The FOC's for x and y will be the same, namely MRS equals price ratio, with the same interpretation. This is reassuring, as the dual and the primal should have the same solution, as long as we parametrize them to be equivalent. [4] The solution gives **Hicksian demands**, giving consumption as a function of utility and prices:

$$h_x(p_x, p_y, \bar{U})$$
$$h_y(p_x, p_y, \bar{U})$$

We can use them to answer in particular:

- **Fixing utility** how does the consumption of y change as the price of x or y changes? (Taking the partial derivative wrt some price).

The value function for this problem (its optimal value) is given by the **Expenditure function**:

$$E(p_x, p_y, \bar{U}) = p_x h_x(p_x, p_y, \bar{U}) + p_y h_y(p_x, p_y, \bar{U})$$

[4] Here, in particular, given some I in the primal problem, we need to set $\bar{U} = V(p_x, p_y, I)$ in order to obtain the exact same solution in the two problems. More in this further on...

It answers to the question: if prices are p_x, p_y and I want to achieve $\bar{U}$ what is the minimum I have to spend? Note that this also implies that $E(p_x, p_y, \bar{U})$ is the minimum **income** I need to enjoy utility $\bar{U}$ at the given prices.
Suppose we start with income I and obtain utility $\bar{U} = V(p_x, p_y, I)$. Then the price of x increases to p'_x. Then E allows us to answer the question:

- How much **additional** income I would need to be as happy as before the change? The answer is in particular:

$$E(p'_x, p_y, \bar{U}) - E(p_x, p_y, \bar{U})$$

  The first term is how much I need to get utility $\bar{U}$ after the price change, the second term is the amount needed *before* the change.

There is another wonderful thing about the expenditure function, which involves the envelope theorem. In particular, E is a minimized value, and h_x, h_y are minimizers. As a result, they will not change up to a first order approximation. This means in particular that:

$$\frac{dE(p_x, p_y, \bar{U})}{dp_x} = \frac{\partial E(p_x, p_y, \bar{U})}{\partial p_x}$$

Which can be applied to give:

$$\frac{\partial E(p_x, p_y, \bar{U})}{\partial p_x} = h_x(p_x, p_y, \bar{U})$$

This is **Shephard's Lemma**.

1. Awesomeness of the Expenditure Function

Why is this useful? Because thanks to some useful equivalences it can spare us a lot of pain computing the Hicksian demand if we already know the Marshallian demand and indirect utility. In particular, suppose we hace income I_0, know Marshallian demands (or we have solved for them) and we can write the indirect utility:

$$V(p_x, p_y, I_0) = U(d_x(p_x, p_y, I_0), d_y(p_x, p_y, I_0))$$

Suppose that we obtain a utility level U_0. Then what is the minimum income we need to get U_0? The answer is given by:

$$E(p_x, p_y, U_0) = I_0$$

Why is it equal to I_0?

- Short Answer: This grants the equivalence between primal and dual;
- Longer, economic, answer: if $E(p_x, p_y, U_0) < I_0$ it means that an optimizing consumer could obtain more than U_0 at income I_0, so it cannot be $V(p_x, p_y, I_0)$ (the expenditure function is

telling us that the consumer can get U_0 without using all her income, so she must be wasting something if U_0 is the achieved utility). Conversely $E(p_x, p_y, U_0) > I_0$ would mean that U_0 would be not achievable at I_0 (the needed bundle would be outside the budget set). The only way to avoid a contradiction is to have $E(p_x, p_y, U_0) = I_0$.

Given the above, we have:

$$V(p_x, p_y, I_0) = V(p_x, p_y, E(p_x, p_y, U_0))$$

This means that we can **invert** the indirect utility to get the expenditure function. And then, using Shephard's lemma, get Hicksian demand with very few computations!
Another equivalence that we get is:

$$d_x(p_x, p_y, I_0) = d_x(p_x, p_y, E(p_x, p_y, U_0))$$

which can be derived to get the Slutsky equation.

C. The Slutsky equation

Start from:

$$d_x(p_x, p_y, I_0) = d_x(p_x, p_y, E(p_x, p_y, U_0)) = h_x(p_x, p_y, U_0)$$

Derive in p_x (or p_y), using the chain rule and $d_x(p_x, p_y, I_0) = h_x(p_x, p_y, U_0)$:

$$\frac{\partial h_x(p_x, p_y, U_0)}{\partial p_x} = \frac{d d_x(p_x, p_y, E(p_x, p_y, U_0))}{d p_x} = \frac{\partial d_x(p_x, p_y, E(p_x, p_y, U_0))}{\partial p_x} + \frac{\partial d_x(p_x, p_y, E(p_x, p_y, U_0))}{\partial I} \frac{\partial E(p_x, p_y, U_0)}{\partial p_x}$$

Use Shephard's Lemma on the last term and rearrange:

$$\frac{d[d_x(p_x, p_y, E(p_x, p_y, U_0))]}{d p_x} = \frac{\partial h_x(p_x, p_y, U_0)}{\partial p_x} - \frac{\partial d_x(p_x, p_y, E(p_x, p_y, U_0))}{\partial I} h_x$$

See **slides** for a graphical intuition of what we are doing.

Worked-out example: getting everything as quickly as possible

Let us work with the function:

$$U(x, y) = x^{.5} y^{.5}$$

to be maximized subject to:

$$p_x x + p_y y = I_0$$

Step 1. Solve the primal problem for Marshallian demands. Set up the Lagrangian:

$$L(\lambda, x, y) = x^{.5} y^{.5} + \lambda(I_0 - p_x - p_y)$$

We could get FOC's, but in this case the solution is clearly interior so know that MRS will equal the price ratio. We have:

$$MU_x = .5\left(\frac{y}{x}\right)^{.5} \qquad MU_y = .5\left(\frac{x}{y}\right)^{.5}$$

Thus:

$$\frac{MU_x}{MU_y} = \frac{y}{x} = \frac{p_x}{p_y} \Rightarrow p_y y = p_x x$$

Plugging into the budget constraint:

$$2p_x d_x = I_0 \Rightarrow d_x(p_x, p_y, I_0) = \frac{I_0}{2p_x}$$

Similarly for y:

$$d_y(p_x, p_y, I_0) = \frac{I_0}{2p_y}$$

Step 2. Get the indirect utility function. Plug Marhsallian demands into the utility function:

$$V(p_x, p_y, I_0) = U(d_x, d_y) = \left(\frac{I_0}{2p_x}\right)^{.5}\left(\frac{I_0}{2p_y}\right)^{.5}$$

Simplifying:

$$V(p_x, p_y, I_0) = \frac{I_0}{2}(p_x p_y)^{-.5}$$

Step 3. Get the expenditure function inverting the above:

$$U_0 = V(p_x, p_y, I_0) = V(p_x, p_y, E(p_x, p_y, U_0)) = \frac{E(p_x, p_y, U_0)}{2}(p_x p_y)^{-.5} = U_0$$

Thus:

$$E(p_x, p_y, U_0) = 2U_0(p_x p_y)^{.5}$$

Step 4. Use Shephard's Lemma to get Hicksian demands:

$$h_x(p_x, p_y, U_0) = \frac{\partial E(p_x, p_y, U_0)}{\partial p_x} = .5 \times 2U_0\left(\frac{p_y}{p_x}\right)^{.5} = U_0\left(\frac{p_y}{p_x}\right)^{.5}$$

By $p_x x = p_y y$ we also have:

$$h_y(p_x, p_y, U_0) = U_0\left(\frac{p_x}{p_y}\right)^{.5}$$

Step 5. Decompose price changes into income and substitution effects for p_x (you should do p_y as an exercise), using Slutsky.

1. The substitution effect comes simply from deriving the Hicksian demand:

$$\frac{\partial h_x}{\partial p_x} = -\frac{U_0}{2}\left(\frac{p_y}{p_x}\right)^{.5}\frac{1}{p_x}$$

This can also be rewritten using $U_0 = V(p_x, p_y, I_0)$:

$$\frac{\partial h_x}{\partial p_x} = -\frac{I_0}{4}\frac{1}{p_x^2}$$

2. The total effect on Marshallian demand can be obtained deriving d_x in p_x:

$$\frac{\partial d_x}{\partial p_x} = -\frac{I_0}{2p_x^2}$$

3. By Slutsky, the income effect is then given by:

$$\frac{\partial d_x}{\partial p_x} - \frac{\partial h_x}{\partial p_x} = -\frac{I_0}{2p_x^2} + \frac{I_0}{4}\frac{1}{p_x^2} = -\frac{I_0}{4p_x^2}$$

Let us verify Slutsky. The income effect should equivalently be given by:

$$-\frac{\partial d_x}{\partial I} h_x = -\frac{1}{2p_x} U_0 \left(\frac{p_y}{p_x}\right)^{.5} = -\frac{1}{2p_x}\frac{I_0}{2}(p_x p_y)^{-.5}\left(\frac{p_y}{p_x}\right)^{.5} = -\frac{I_0}{4p_x^2}$$