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The Generalized Sprague–Grundy Function and Its Invariance under Certain Mappings

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The equivalence of three previously given definitions of the generalized Sprague–Grundy function G is established. A polynomial algorithm for the computation of G is given, and an optimal strategy for the sum of a wide class of two-player games which may contain cycles or loops is formulated, which is one of the main applications of G. Finally, the invariance of G under a mapping of digraphs which may contain cycles or loops is established.

1. INTRODUCTION

Our purpose is to unify the treatment of—and shed some light on the nature of—the generalized Sprague–Grundy function G (G-function in the sequel). This function determines an optimal strategy for the sum (disjunctive compound) of a wide class of combinatorial games represented by digraphs which may contain cycles or loops. Though the classical Sprague–Grundy function, which only permits treating games without cycles and without loops, is well known (see, e.g. [3, Chap. 14; 4, 5]), the more powerful G-function is little known to date.

In Section 2 we present some background and notation. Three distinct definitions of G given in [5, 6, 7, 8, 10] are listed in Section 3, which also contains an algorithm for computing G. In Section 4, the equivalence of the three definitions is established, as well as the uniqueness of G. The use of G for formulating a winning strategy for sums of games is given in Section 5.

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By selecting the proper definition of G, this can be done in a simple way. The main content of Section 6 is a new G-preserving mapping of digraphs which may contain cycles or loops. Applications of this mapping to the equivalence of annihilation games will be given elsewhere.

2. Background and Notation

Throughout, $R=(V, E)$ denotes a finite digraph with vertex-set $V=V(R)$ and edge-set $E=E(R)$, which may contain cycles or loops.

We consider two-person games without chance moves (no dice), with perfect information (i.e., each player has full knowledge of the present and possible past game positions), which are impartial (i.e., the possible moves from any position do not depend on which player is about to play). Any such game Γ can be represented by a digraph called a *game-graph* $R=(V, E)$, whose vertices are the game's positions, and $(u, v) \in E$ (directed from u to v) if and only if there is a move from position u to position v. Conversely, given any digraph R, we can define on it a game Γ by placing a token on one of its vertices. Each player at his turn moves the token to a neighboring vertex along a directed edge. Because of this duality, we often identify Γ with R, and the positions and moves of Γ with the vertices and edges of its corresponding game-graph, using them interchangeably.

We shall restrict attention to those games in which the player first unable to move is the loser, the opponent the winner. If there is no last move, the outcome is declared a draw.

For every $u \in V$, define its set of *followers* $F(u)$ by $F(u)=\{v \in V\colon (u, v) \in E\}$. If $F(u)=\varnothing$, u is called a *sink*. For every $v \in V$, define its set of *ancestors* by $F^{-1}(v)=\{u \in V\colon v \in F(u)\}$. In particular, in a loop (u, u), u is both a follower and an ancestor of itself.

The G-function, which is useful for game strategies as will be seen in the sequel, is a mapping $G\colon V \rightarrow Z^0 \cup \{\infty\}$, where Z^0 is the set of nonnegative integers, and where the symbol ∞ denotes a value larger than any natural number. If η is any mapping $\eta\colon V \rightarrow Z^0 \cup \{\infty\}$, we let $\eta(F(u))=\{\eta(v)<\infty\colon v \in F(u)\}$. If $G(u)=\infty$, $G(F(u))=\mathscr{K}$, we also write $G(u)=\infty(\mathscr{K})$. If $G(u)=k$, $G(v)=l$, then $G(u)=G(v)$ if and only if one of the following holds: (a) $k=l<\infty$; (b) $k=\infty(\mathscr{K})$, $l=\infty(\mathscr{L})$, and $\mathscr{K}=\mathscr{L}$. If L is any finite set of nonnegative integers, we denote by $\operatorname{mex} L$ the smallest nonnegative integer not in L. We use the notation $V^f=\{u \in V\colon G(u)<\infty\}$, $V^\infty=V \setminus V^f$.

3. Three Definitions and an Algorithm

Given a digraph $R = (V, E)$:

Definition 1. A function $G: V \to Z^0 \cup \{\infty\}$ is a *G-function* with *counterfunction* c: $V^f \to J$, where J is any well-ordered set, if the following three conditions hold:

A. If $G(u) < \infty$, then $G(u) = \operatorname{mex} G(F(u))$.

B. If $v \in F(u)$ and $G(v) > G(u)$, then there exists $w \in F(v)$ satisfying $G(w) = G(u)$ and $c(w) < c(u)$.

C. If $G(u) = \infty$, then there is $v \in F(u)$ with $G(v) = \infty(\mathscr{K})$ such that $\operatorname{mex} G(F(u)) \notin \mathscr{K}$.

The conceptually more difficult of these conditions is **C**. To make it more accessible, note that a stronger form of the opposite direction follows from **A** and **B**: If $G(u) < \infty$ and $v \in F(u)$ with $G(v) = \infty(\mathscr{K})$, then by **B** there is $w \in F(v)$ with $G(w) = G(u)$, which means, by **A**, $G(u) = \operatorname{mex} G(F(u)) \in \mathscr{K}$. Thus **C** could be reformulated in the stronger from: $G(u) = \infty$ if and only if there is $v \in F(u)$ with $G(v) = \infty(\mathscr{K})$ such that $\operatorname{mex} G(F(u)) \notin \mathscr{K}$. But the present weaker form of **C** is preferable for a definition.

The motivation for the counterfunction is that the G-value of a follower does not always determine the optimal next move completely. For example, for any followers with the same G-value, one which is not on a cycle may be the only choice for realizing a win. It will have smaller countervalue than the others, as will be seen from Algorithm G below. In order to win, a move to a vertex with given G-value and *minimal* countervalue may have to be selected.

Definition 2. A function $G_1: V \to Z^0 \cup \{\infty\}$ is a G-function with counterfunction c: $V_1^f \to J$ (where $V_1^f = \{u \in V: G_1(u) < \infty\}$), if the following conditions hold:

A′. If $G_1(u) < \infty$ and $i \in [0, G_1(u) - 1]$, then there exists $v \in F(u)$ satisfying $G_1(v) = i$, $c(v) < c(u)$.

A″. If $G_1(u) < \infty$ and $v \in F(u)$ satisfies $c(v) < c(u)$, then $G_1(v) \neq G_1(u)$.

B′. If $G_1(u) < \infty$ and $v \in F(u)$ satisfies either $G_1(v) = \infty$ or $c(v) \geq c(u)$, then there exists $w \in F(v)$ such that $G_1(w) = G_1(u)$ and $c(w) < c(u)$.

C. If $G_1(u) = \infty$, then there is $v \in F(u)$ with $G_1(v) = \infty(\mathscr{K})$ such that $\operatorname{mex} G_1(F(u)) \notin \mathscr{K}$.

Definition 3. A function $G_2: V \to Z^0 \cup \{\infty\}$ is a G-function, if it satisfies the following conditions:

A. If $G_2(u) < \infty$, then $G_2(u) = \operatorname{mex} G_2(F(u))$.

C. If $G_2(u) = \infty$, then there is $v \in F(u)$ with $G_2(v) = \infty(\mathscr{K})$ such that $\operatorname{mex} G_2(F(u)) \notin \mathscr{K}$.

C′. Among all the functions satisfying **A** and **C**, G_2 is a function with a maximal number of infinity values.

Definition 1 was used in [7] and in [8] without validity proof. This is also the definition used in the sequel. Definition 2 (here slightly simplified) was given in [10] and applied in [5, p. 133], and Definition 3 (here slightly weakened), was used in [6]. Note that no counterfunction is explicit in Definition 3.

The existence of a G-function for every finite digraph $R(V, E)$ which may contain cycles or loops, follows from Algorithm G below. Initially a special symbol v is attached to the label l of every vertex u, where $l(u) = v$ means that u has no label. We also let $V_v = \{u \in V: l(u) = v\}$, where $l(u)$ designates the label of u.

ALGORITHM G (for computing the generalized Sprague–Grundy function for a given finite digraph $R = (V, E)$).

1. (Initialize labels and counter) Put $i \leftarrow 0$, $m \leftarrow 0$, $l(u) \leftarrow v$ for all $u \in V$.

2. (Label and counter) If there exists $u \in V_v$ such that no follower of u is labeled i and every follower of u which is either unlabeled or labeled ∞ has a follower labeled i, then put $l(u) \leftarrow i$, $c(u) \leftarrow m$, $m \leftarrow m + 1$. Repeat 2.

3. (∞-label) For every $u \in V_v$ which has no follower labeled i, put $l(u) \leftarrow \infty$.

4. (Increase label) If $V_v \neq \varnothing$, put $i \leftarrow i + 1$ and return to 2; otherwise end.

If the digraph is stored as an *adjacency list* (see, e.g. [1, Sect. 2.3]), linked by both rows and columns in the fashion of a sparse matrix (see [9, Sect. 2.2.6]), then the number of steps of each iteration is $O(|V| + |E|)$. Letting $l\,\mathrm{MAX} = \max_{u \in L^f} G(u) \leqslant \max_{u \in L^f} |F(u)| < |V|$, the number of steps of the algorithm is bounded by $O((|V| + |E|)\, l\mathrm{MAX})$ or $O(|E|\, l\mathrm{MAX})$ for a connected digraph. Here L^f denotes the subset of V on which l is finite.

DEFINITION 4. A counterfunction c such that $G(u) < G(v) < \infty \Rightarrow c(u) < c(v)$ is called a *monotonic* counterfunction.

THEOREM 1. *The label l computed by Algorithm* G *satisfies Definitions* 1 *and* 2 *with monotonic counterfunction c for every finite digraph $R = (V, E)$.*

Proof. (i) Every $u \in V$ gets exactly one label. For if in the ith iteration no u gets labeled i, then all unlabeled vertices get assigned ∞ in step 3. It is also clear that c is monotonic.

(ii) For every u such that $l(u) = i < \infty$ and every $j \in [0, i-1]$, there exists $v \in F(u)$ such that $l(v) = j$. For suppose that there exists u such that $l(u) = i$ for which the claim does not hold. Then there exists $j \in [0, i-1]$ such that $l(v) = j$ for no $v \in F(u)$. Hence u is either labeled in some iteration prior to iteration j, or it is labeled ∞ in iteration j, a contradiction either way. This demonstrates **A**$'$, since $c(v) < c(u)$ by the monotonicity of c.

(iii) Suppose $l(u) = i < \infty$ and $v \in F(u)$. Then $l(v) \neq l(u)$. This follows immediately from step 2 of the algorithm. This demonstrates **A** and also **A**$''$.

(iv) Suppose $l(u) = i$, $v \in F(u)$, and $(l(v) > l(u)$, or $c(v) \geqslant c(u))$. If $c(v) \geqslant c(u)$, then $l(v) \geqslant l(u)$ by construction. By (iii), $l(v) > l(u)$, which thus holds in any case. Therefore in the ith iteration, v is either unlabeled or $l(v) = \infty$. Since u gets labeled in the ith iteration, step 2 implies existence of some $w \in F(v)$ which was already labeled i, and so $c(w) < c(u)$. This demonstrates **B** and **B**$'$.

(v) If $l(u) = \infty$, then there exists $v \in F(u)$ such that $l(v) = \infty(\mathscr{K})$, $\operatorname{mex} l(F(u)) \notin \mathscr{K}$. For let $\operatorname{mex} l(F(u)) = i$. For every $j \in [0, i-1]$, there exists $v \in F(u)$ such that $l(v) = j$. Hence u is not labeled ∞ in the jth iteration. Since $v \in F(u) \Rightarrow l(v) \neq i$, u is labeled ∞ in step 3 of the ith iteration. Moreover, since u was not labeled i in step 2 of the ith iteration, there exists $v \in F(u)$ with v unlabeled or $l(v) = \infty$, such that $w \in F(v) \Rightarrow l(w) \neq i$. If such a v was unlabeled, it got labeled ∞ in step 3 of the ith iteration. This demonstrates **C**. ∎

4. Equivalence and Uniqueness of G

For proving equivalence of the three definitions, we begin with an auxiliary result.

LEMMA 1. *If G satisfies Definition* 2, *then G satisfies Definition* 1.

Proof. Let $u \in V^f$, $v \in F(u)$, where V^f is the subset of V on which G satisfying Definition 2 is finite. Then $G(v) \neq G(u)$. This follows from **A**$''$ if $c(v) < c(u)$. So suppose $c(v) \geqslant c(u)$. By **B**$'$ there exists $w \in F(v)$ such that $G(w) = G(u)$, $c(w) < c(u) \leqslant c(v)$. If $G(v) < \infty$, then by **A**$''$, $c(w) < c(v) \Rightarrow G(v) \neq G(w) = G(u)$. Hence G satisfies **A**.

Let $u \in V^f$, $v \in F(u)$. If $G(v) = \infty$, then **B**$'$ implies **B**. If $G(u) < G(v) < \infty$,there exists $z \in F(v)$ such that $G(z) = G(u)$, $c(z) < c(v)$ by **A**$'$. If

$c(z) < c(u)$, we let $w = z$ and **B** is established. Otherwise $c(u) \leqslant c(z) < c(v)$, and so by **B**$'$ there exists $w \in F(v)$ such that $G(w) = G(u)$, $c(w) < c(u)$, establishing **B**. ∎

Though a counterfunction can obviously be selected in various ways, the above three definitions define the same *unique* function G.

THEOREM 2. *Definitions* 1,2, *and* 3 *define the same function* G, *which exists uniquely for every finite digraph* $R = (V, E)$.

Proof. By Theorem 1, there exists a function G satisfying Definitions 1 and 2. In any case it satisfies Definition 1, by Lemma 1. Since such G satisfies **A** and **C**, there also exists a function G satisfying Definition 3 for every finite digraph R. Let G be any function satisfying Definition 1 or 2 with counter function c, G' any function satisfying Definition 3. Let

$$K = \{u \in V: G(u) < \infty, \qquad G(u) \neq G'(u)\}, \qquad k = \min_{u \in K}(G(u), G'(u)).$$

If there exists $v \in K$ such that $G'(v) = k$, then $k < G(v) < \infty$, and so by **A** there exists $u \in F(v)$ such that $G(u) = k$. Again by **A**, $G'(u) > k$, so $u \in K$. Let $U = \{u \in K\colon\ G(u) = k\}$. Then $K \neq \emptyset \Rightarrow U \neq \emptyset$. Pick $u \in U$ with $c(u)$ minimal. We have $G'(u) > k$. Suppose first there exists $v \in F(u)$ such that $G'(v) = k$. Now $G(v) \neq k$ by **A**, hence $v \in K$. By the minimality of k we then have actually $G(v) > k$ (see Fig. 1). By **B**, there exists $w \in F(v)$ such that $G(w) = k$, $c(w) < c(u)$. By **A**, $G'(w) > k$, hence $w \in U$, contradicting the minimality of $c(u)$.

If, on the other hand, $G'(v) \neq k$ for all followers v of u, then, by **A**, $G'(u) = \infty$. By the minimality of k, if $i \in [0, k-1]$, there exists $z \in F(u)$ such that $G(z) = G'(z) = i$. This shows that mex $G'(F(u)) = k$. By **C** there exists $v \in F(u)$ such that $G'(v) = \infty(\mathscr{L})$, $k \notin \mathscr{L}$. Now $G(v) > k$ by **A**, and so by **B**, there exists $w \in F(v)$ such that $G(w) = k$, $c(w) < c(u)$. But $G'(w) \neq k$, hence $G'(\omega) > k$, so $w \in U$, a contradiction to the minimality of c. Hence $K = \emptyset$.

Since G satisfies **A** and **C**, and since G' is a function with a maximal number of infinity values among all the functions satisfying **A** and **C**, we

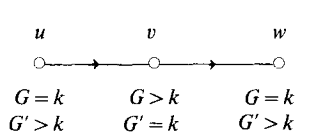

FIG. 1. An impossible situation.

have $G(u)=\infty \Rightarrow G'(u)=\infty$. Hence $G'(u)=G(u)$ for all $u \in V$. Thus Definitions 1, 2, and 3 define the same function.

Suppose that G_1, G_2 satisfy Definition 1 or 2, and G' satisfies Definition 3. By the first part of this proof, $G_1=G'$, $G_2=G'$, hence $G_1=G_2$, proving uniqueness. ∎

If $R=(V, E)$ is any finite digraph, then the classical Sprague–Grundy function $g: V \to Z^0$ is defined by $g(u)=\operatorname{mex} g(F(u))$ for all $u \in V$. We observe that G is indeed a generalization of g: If $V^\infty=\varnothing$, then **A** of Definition 1 implies $G=g$. In particular, if R is finite, acyclic, and loopless, then $G=g$, because then $V^\infty=\varnothing$ by **C**. Incidentally, g exists uniquely if R is finite, acyclic, and loopless [3, Chap. 14]. But if R contains cycles, g may exist uniquely with $g=G$; it may exist uniquely but $g \neq G$; it may exist non-uniquely; or it may exist not at all [6]. Note that if there is a loop at u, then **A** implies $G(u)=\infty$.

5. Games with Draws and the G-Function

Given a finite digraph $R=(V, E)$ with possible cycles and loops. Recall the duality between games and digraphs pointed out in Section 2.

A *P-position* of R is any position $u \in V$ from which the *Previous* player can force a win, that is, the player not called upon to move from u. An *N-position* is any position v from which the *Next* player can force a win, that is, the player whose turn it is to move from v. A *D-position* is any position from which neither player can force a win in a finite number of moves, but both can continue to move indefinitely, that is, both can *Draw*.

The set of all P-positions, N-positions, and D-positions is denoted by $\mathscr{P}$, $\mathscr{N}$, and $\mathscr{D}$, respectively.

LEMMA 2. *Let $R=(V, E)$ be a finite digraph. Then $u \in \mathscr{P}$ if and only if $F(u) \subseteq \mathscr{N}$; $u \in \mathscr{N}$ if and only if $F(u) \cap \mathscr{P} \neq \varnothing$; and $u \in \mathscr{D}$ if and only if $F(u) \cap \mathscr{P}=\varnothing$ and $F(u) \cap \mathscr{D} \neq \varnothing$.*

Proof. The three conditions are a direct consequence of the definitions of the subsets $\mathscr{P}$, $\mathscr{N}$, and $\mathscr{D}$. ∎

We now show that every position of any game of the type we are considering, has a unique P, N, or D-label.

THEOREM 3. *Let Γ be a two-person game with perfect information and no chance moves, either terminating in a finite number of moves by one of the players winning and the other losing, or in a draw. Then there are two possibilities:* (i) *There exists a winning move for precisely one of the two* *players.* (ii) *There is a winning move for neither player, but a drawing move for both.*

Proof. It suffices to show that the set of positions of Γ can be partitioned into three subsets $\mathscr{P}$, $\mathscr{N}$, and $\mathscr{D}$.

Suppose there exists a position u_0 which is neither in $\mathscr{P}$ nor in $\mathscr{N}$. Then $F(u_0) \cap \mathscr{P} = \emptyset$, otherwise u_0 would be an N-position; and $F(u_0) \not\subseteq \mathscr{N}$, otherwise u_0 would be a P-position. Hence there exists $u_1 \in F(u_0)$ such that also u_1 is neither in $\mathscr{P}$ nor in $\mathscr{N}$. Moreover, the only nonlosing move for the player moving from u_0 is to move to such u_1. Thus there is an infinite sequence $u_0, u_1, \ldots$ with $u_{i+1} \in F(u_i)$ such that $u_i \notin \mathscr{P} \cup \mathscr{N}$, and a best move is to go from u_i to u_{i+1} $(i \geqslant 0)$. Thus $u_0 \in \mathscr{D}$. Hence every position has a P, N, or D-label.

Suppose $w \in \mathscr{P} \cap \mathscr{N}$. Starting to play from w, both the Next and the Previous player can win—in contradiction to the possible outcomes. Thus $\mathscr{P} \cap \mathscr{N} = \emptyset$. In a similar way it is shown that $\mathscr{P} \cap \mathscr{D} = \mathscr{N} \cap \mathscr{D} = \emptyset$. ∎

The importance of the G-function is that it determines the P, N, and D-labels. This is shown in Theorem 4 for games and in Theorem 5 for sums of games. Given finitely many games, their *sum* is the game in which each player at his turn selects a game and makes a move in it.

THEOREM 4. *Let $R = (V, E)$ be a finite digraph. Then V can be partitioned as follows:*

$$\mathscr{P} = \{u \in V: G(u) = 0\}, \qquad \mathscr{D} = \{u \in V: G(u) = \infty(\mathscr{K}), 0 \notin \mathscr{K}\},$$

$$\mathscr{N} = \{u \in V: 0 < G(u) < \infty\} \cup \{u \in V: G(u) = \infty(\mathscr{K}), 0 \in \mathscr{K}\}.$$

Proof. By Theorem 3, every vertex of R has a unique P, N, or D-label and by Theorem 2 it has a unique G-function. Denote the three sets on the right-hand sides by $\mathscr{P}'$, $\mathscr{D}'$, and $\mathscr{N}'$, respectively. We use Lemma 2 for the proof. Note that

$$\mathscr{P}' \cup \mathscr{N}' \cup \mathscr{D}' = V, \qquad \mathscr{P}' \cap \mathscr{N}' = \mathscr{P}' \cap \mathscr{D}' = \mathscr{N}' \cap \mathscr{D}' = \emptyset.$$

Let $u \in \mathscr{P}'$, $v \in F(u)$. Then $G(v) > 0$ by **A** of Definition 1. If $G(v) = \infty(K)$, then $0 \in K$ by **B**. Hence $F(u) \subseteq \mathscr{N}'$. Now let $u \in \mathscr{N}'$. Then clearly $F(u) \cap \mathscr{P}' \neq \emptyset$. Finally, let $u \in \mathscr{D}'$. Then $F(u) \cap \mathscr{P}' = \emptyset$. Furthermore, $\operatorname{mex} G(F(u)) = 0$. Hence by **C** there is $v \in F(u)$ satisfying $G(v) = \infty(\mathscr{L})$, $0 \notin \mathscr{L}$, so $v \in \mathscr{D}'$. ∎

COROLLARY 1. *At the end of the first iteration $(i = 0)$ of Algorithm* G, *we have*

$$P = \{u: l(u) = 0\}, \qquad N = V_v, \qquad D = V^\infty.$$

Thus the N, P, D-labels can be computed in only $O(|V|+|E|)$ steps, or $O(|E|)$ for a connected digraph.

Proof. Follows immediately from Theorem 4 and the complexity argument following Algorithm G. ∎

The main use of the G-function is to provide a strategy for a finite sum of finite games. For doing this, we define the *Nim-sum* as follows.

For any nonnegative integer, write $H=\sum_{i\geqslant 0} h^i 2^i$ for the binary encoding of H ($h^i \in \{0, 1\}$). If A and B are nonnegative integers, then their Nim-sum

$$A \oplus B = C$$

is defined by $c^i \equiv a^i + b^i \pmod 2$, $c^i \in \{0, 1\}$ ($i \geqslant 0$). The generalized Nim-sum of a nonnegative integer A and $\infty(K)$ is defined by

$$A \oplus \infty(\mathscr{K}) = \infty(\mathscr{K}) \oplus A = \infty(\mathscr{K} \oplus A),$$

where $\infty(\mathscr{K} \oplus A) = \{k \oplus A : k \in \mathscr{K}\}$. The generalized Nim-sum of $\infty(\mathscr{K}_1)$ and $\infty(\mathscr{K}_2)$ is defined by

$$\infty(\mathscr{K}_1) \oplus \infty(\mathscr{K}_2) = \infty(\mathscr{K}_2) \oplus \infty(\mathscr{K}_1) = \infty(\varnothing).$$

Finally, the generalized Nim-sum of $m \geqslant 2$ summands is $\sum'^{m}_{i=1} A_i = A_1 \oplus \cdots \oplus A_m$ which is well defined since the generalized Nim-sum is clearly associative.

THEOREM 5. *Let $R=(V, E)$ be the game-graph of the sum of the finite digraphs $R_1, \ldots, R_m$, and let $\sigma(\mathbf{u}) = \sum'^{m}_{i=1} G(u_i)$ for every $\mathbf{u}=(u_1, \ldots, u_m) \in V$. Then σ is the unique G-function of R with counterfunction $c(\mathbf{u}) = \sum_{i=1}^{m} c_i(u_i)$, where the c_i are monotonic counterfunctions of G in R_i $(1 \leqslant i \leqslant m)$.* ∎

For the proof see [6, 10]. We point out, however, that the proof using Definition 1 is somewhat simpler than the proof using Definitions 2 and 3 which had been employed in [10] and [6], respectively.

COROLLARY 2. *Let $R=(V, E)$ be the game-graph of the sum of the finite digraphs $R_1, \ldots, R_m$ $(m \geqslant 1)$. Then the P, N, and D-labels of R are determined by*

$$\mathscr{P} = \{\mathbf{u} \in V : \sigma(\mathbf{u}) = 0\}, \qquad \mathscr{D} = \{\mathbf{u} \in V : \sigma(\mathbf{u}) = \infty(\mathscr{K}), 0 \notin \mathscr{K}\}$$

$$\mathscr{N} = \{\mathbf{u} \in V : 0 < \sigma(\mathbf{u}) < \infty\} \cup \{\mathbf{u} \in V : \sigma(\mathbf{u}) = \infty(\mathscr{K}), 0 \in \mathscr{K}].$$

Let $n = \max(|V_1|, \ldots, |V_m|)$. If the input size of the games is at least some positive power of mn and during the play no position grows larger than a *fixed power of mn, then the computation of the P, N, and D-labels is polynomial.*

Proof. Follows from Theorems 4, 5, and Algorithm G. ∎

In addition to giving the nature of the game positions (P, N, or D), the proof of Theorem 5 also indicates an optimal next move (for N and D-positions).

The gist of the argument is that if $\mathbf{u} = (u_1, ..., u_m)$ is a P-position, that is, a position with $\sigma(\mathbf{u}) = 0$, then our opponent moves to $\mathbf{v} \in F(\mathbf{u})$ with $\sigma(\mathbf{v}) > 0$, and we can find some $\mathbf{w} \in F(\mathbf{v})$ with $\sigma(\mathbf{w}) = 0$ and $c(\mathbf{w}) < c(\mathbf{u})$. The latter inequality is realized by moving in a component game with minimal c_i-value from among all components having the specified G-value. Since the counterfunction is well ordered, it follows that we can realize a win in a finite number of moves. Also **C** of Definition 1 implies that if $\mathbf{u} \in \mathscr{D}$, then we can find a follower $\mathbf{v} \in \mathscr{D}$, so a D-position can be preserved.

6. G-PRESERVING MAPS

Throughout $R = (V, E)$ is a finite digraph which may have cycles or loops. We introduce the notation $F'(u) = F(u) - \{u\}$. For a loopless digraph, of course, $F'(u) = F(u)$ for all $u \in V$.

DEFINITION 5. Let R, $\hat{R}$ be finite digraphs. A mapping λ: $V(R) \to V(\hat{R})$ is called a *D-morphism* if for every $u \in V(R)$,

$$F_{\hat{R}}(\lambda(u)) \subseteq \lambda(F_R(u)) \tag{1}$$

$$\lambda(F'_R(u)) \subseteq F_{\hat{R}}(\lambda(u)) \cup F_{\hat{R}}^{-1}(\lambda(u)), \tag{2}$$

where for any set S, $\lambda(S) = \{\lambda(s): s \in S\}$, hence $\lambda(S_1 \cup S_2) = \lambda(S_1) \cup \lambda(S_2)$, and the subscripts R, $\hat{R}$ of F indicate the digraph to which the corresponding followers belong.

If R has no loops, then Definition 5 coincides with Banerji's definition of a D-morphism. For acyclic and loopless digraphs, Banerji [2, Sect. 3.6] proved that if $\hat{R}$ has a classical Sprague–Grundy function g, then the function $g'(u) = g(\lambda(u))$ is a g-function on R.

If R and $\hat{R}$ are acyclic and loopless, then $\hat{R}$ has a g-function, and the g-function on the game-graph R determines a winning strategy there. Thus λ relates the winning strategy of $\hat{R}$ to that of R. If either R or $\hat{R}$ has cycles or loops, R or $\hat{R}$ may not have a g-function. In the cyclic case, even if R or $\hat{R}$ has a g-function, it does not necessarily determine a winning strategy. Since, on the other hand, G exists uniquely on every finite digraph and always determines a strategy, it is of interest to investigate G-preserving maps. In [8] we proved and used the following result:

THEOREM 6. *Let R and $\hat{R}$ be finite digraphs, and λ: $V(R) \to V(\hat{R})$ a D-morphism. Then $G(u) = G(\lambda(u))$ for every $u \in V^f(R)$.*

It was also shown in [8] that if $V^\infty(R) \neq \varnothing$, then a D-morphism does not preserve G in general. But under somewhat stronger hypotheses, G is preserved as we will see in Theorem 7 below.

If f_i and g_i are in well-ordered sets ($i \in \{1, 2\}$), then the usual lexicographic ordering of pairs is defined by $(f_1, g_1) < (f_2, g_2)$ if (i) $f_1 \leqslant f_2$ and (ii) $f_1 = f_2 \Rightarrow g_1 < g_2$.

THEOREM 7. *Let R and $\hat{R}$ be finite digraphs, λ: $V(R) \to V(\hat{R})$ and d: $V(R) \to J$ mappings, where J is any well-ordered set, with the following properties:*

λ satisfies (1).

For every $u \in V(R)$ and every $v \in F_R(u)$, either

(a) *$\lambda(v) \in F_{\hat{R}}(\lambda(u))$, or*

(b) *there exists $w \in F_R(v)$ such that $\lambda(w) = \lambda(u)$ and $d(w) < d(u)$.*

Let $G'(u) = G(\lambda(u))$ for all $u \in V(R)$; let $c(u) = (\hat{c}(\lambda(u)), d(u))$, ordered lexicographically, whenever $\hat{c}(\lambda(u))$ is defined, where $\hat{c}$ is a monotonic counterfunction on $\hat{R}$. Then G' is a G-function on R with monotonic counterfunction c.

EXAMPLE. For R and $\hat{R}$ given in Fig. 2, let $\lambda(u_i) = v_i$ $(1 \leqslant i \leqslant 4)$, $\lambda(u_5) = v_2$, $\lambda(u_6) = v_1$. Then (1) is clearly satisfied. Also (a) and (b) are satisfied. For example, the pair (u_5, u_6) satisfies (a); (u_5, u_3) does not satisfy (a), but it satisfies (b) with $w = u_2$, and $d(u_2)$ and $d(u_5)$, say, the same as any counterfunction values of R at u_2 and u_5.

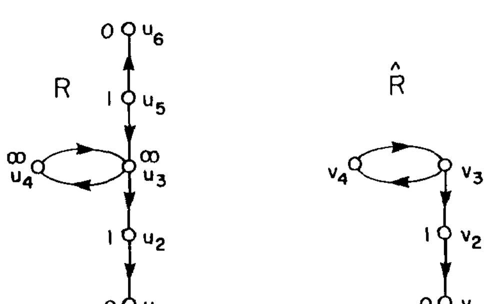

FIG. 2. Two digraphs and a G-preserving map.

Proof. We first show that λ satisfies

$$\lambda(F_R(u)) \subseteq F_{\hat{R}}(\lambda(u)) \cup F_{\hat{R}}^{-1}(\lambda(u)) \tag{3}$$

for every $u \in V(R)$. So in particular, λ is a D-morphism. Let

$$U = \{(u, v) \in V(R) \times V(R) \colon v \in F_R(u),\ \lambda(v) \notin F_{\hat{R}}(\lambda(u)) \cup F_{\hat{R}}^{-1}(\lambda(u))\}.$$

For every $(u, v) \in V(R) \times V(R)$, let $K(u, v) = d(u) + d(v)$. If $U \neq \emptyset$, let $K_0 = \min\{K(u, v) \colon (u, v) \in U\}$, and let $(u_0, v_0) \in U$ satisfy $K(u_0, v_0) = K_0$. Then u_0, v_0 do not satisfy (a). Thus by (b) there exists $w \in F(v_0)$ such that $\lambda(w) = \lambda(u_0)$, $d(w) < d(u_0)$. Hence

$$K(v_0, w) = d(v_0) + d(w) < d(v_0) + d(u_0) = K_0.$$

By the minimality of K_0, $\lambda(w) \in F_{\hat{R}}(\lambda(v_0)) \cup F_{\hat{R}}^{-1}(\lambda(v_0))$. Equivalently,

$$\lambda(v_0) \in F_{\hat{R}}(\lambda(w)) \cup F_{\hat{R}}^{-1}(\lambda(w)) = F_{\hat{R}}(\lambda(u_0)) \cup F_{\hat{R}}^{-1}(\lambda(u_0)),$$

contradicting $(u_0, v_0) \in U$. Hence $U = \emptyset$, and (3) is established.

Since λ is a D-morphism Theorem 6 applies, and so $u \in V^f(R) \Rightarrow G(u) = G'(u)$, where, throughout this proof, $G(u)$ is the G-function on R ($\hat{R}$) for all $u \in R$ ($\hat{u} \in \hat{R}$). Let $K = \{u \in V(R) \colon G(u) = \infty,\ G'(u) < \infty\}$. If $K \neq \emptyset$, let $k = \min_{u \in K}(G'(u))$, $c_0 = \min\{c(u) \colon u \in K,\ G'(u) = k\}$. Pick $u \in K$ such that $G'(u) = k$, $c(u) = c_0$.

For every $j \in [0, k-1]$, there exists $\hat{v} \in F_{\hat{R}}(\lambda(u))$ such that $G(\hat{v}) = j$, and by monotonicity, $\hat{c}(\hat{v}) < \hat{c}(\lambda(u))$. By (1), there exists $v \in F_R(u)$ such that $\lambda(v) = \hat{v}$. Since $c(v) = (\hat{c}(\hat{v}), d(v)) < (\hat{c}(\lambda(u), d(u)) = c_0$, we have $G(v) = j$. Let $w \in F_R(u)$. By (3), $\lambda(w) \in F_{\hat{R}}(\lambda(u)) \cup F_{\hat{R}}^{-1}(\lambda(u))$. Since $G(\lambda(u)) = k$, we have $G(\lambda(w)) \neq k$. But then also $G(w) \neq k$, since $G(w) = k \Rightarrow G(\lambda(w)) = k$. Hence mex $G(F_R(u)) = k$.

By **C** of Definition 1, there exists $v \in F_R(u)$ such that $G(v) = \infty$, $k \notin G(F_R(v))$. By hypothesis, either $\lambda(v) \in F_{\hat{R}}(\lambda(u))$, or there exists $w \in F_R(v)$ such that $\lambda(w) = \lambda(u)$, $d(w) < d(u)$. In the former case $G(\lambda(v)) > k$ by the minimality of k. Hence there exists $\hat{w} \in F_{\hat{R}}(\lambda(v))$ such that $G(\hat{w}) = k$, $\hat{c}(\hat{w}) < \hat{c}(\lambda(u))$. By (1), there exists $w \in F_R(v)$ such that $\lambda(w) = \hat{w}$. Since $c(w) < c_0$, we have $G(w) = k$, a contradiction. In the latter case, $c(w) < c_0$ by the lexicographic ordering, since $d(w) < d(u)$, $\bar{c}(\lambda(w)) = \bar{c}(\lambda(u))$. By the minimality of c_0, $G(w) = G(\lambda(w)) = G(\lambda(u)) = k$, again a contradiction. Thus $K = \emptyset$.

Suppose that for some $u \in V(R)$, $G(u) = \infty(\mathscr{K})$, $G'(u) = \infty(\mathscr{M})$. By (1), $G(F_{\hat{R}}(\lambda(u)) \subseteq G(\lambda(F_R(u)))\ =\ G(F_R(u))$. Hence $\mathscr{M} \subseteq \mathscr{K}$. Let $k \in \mathscr{K}$. Then there exists $v \in F_R(u)$ such that $G(v) = k$, and so $G(\lambda(v)) = k$. By (3), $\lambda(v) \in F_{\hat{R}}(\lambda(u)) \cup F_{\hat{R}}^{-1}(\lambda(u))$. If $\lambda(v) \in F_{\hat{R}}(\lambda(u))$ then $k \in \mathscr{M}$. If $\lambda(u) \in F_{\hat{R}}(\lambda(v))$, then **B** implies $k \in \mathscr{M}$, and so $\mathscr{K} = \mathscr{M}$.

It remains only to show that c is a monotonic counterfunction. We first show that c is a counterfunction. Suppose that $v \in F_R(u)$, $G(v) > G(u)$. Then $G(\lambda(v)) = G(v) > G(u) = G(\lambda(u))$. If $\lambda(v) \in F_{\hat{R}}(\lambda(u))$, then by **B** there exists $\hat{w} \in F_{\hat{R}}(\lambda(v))$ such that $G(\hat{w}) = G(\lambda(u))$, $\hat{c}(\hat{w}) < \hat{c}(\lambda(u))$. By (1), there exists $w \in F_R(v)$ such that $\lambda(w) = \hat{w}$. Thus $G(w) = G(\lambda(w)) = G(\lambda(u)) = G(u)$ and $c(w) < c(u)$. If, on the other hand, there exists $w \in F(v)$ such that $\lambda(w) = \lambda(u)$ and $d(w) < d(u)$, then again $G(w) = G(\lambda(w)) = G(\lambda(u)) = G(u)$ and $c(w) < c(u)$. Hence c is a counterfunction. It is monotonic, because

$$G(u) < G(v) < \infty \Rightarrow$$

$$G(\lambda(u)) < G(\lambda(v)) < \infty \Rightarrow \hat{c}(\lambda(u)) < \hat{c}(\lambda(v)) \Rightarrow c(u) < c(v). \quad \blacksquare$$

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