$e ::=$	$x \mid (e\ e) \mid \lambda x^{\tau}.e \mid (\mathbf{if}\ e\ e\ e) \mid c \mid \#\mathrm{t} \mid \#\mathrm{f} \mid (\mathit{cons}\ e\ e) \mid (\mathit{vec}\ \overrightarrow{e}) \mid n$	Expressions
$c ::=$	$\mathit{add1} \mid =_n \mid \leq_n \mid \mathit{num?} \mid \mathit{bool?} \mid \mathit{proc?} \mid \mathit{cons?} \mid \mathit{vec?} \mid \mathit{car} \mid \mathit{cdr} \mid \mathit{len} \mid \mathit{ref}_n$	Primitive Operations
$pe ::=$	$\mathbf{car} \mid \mathbf{cdr} \mid \mathbf{ref}_n \mid \mathbf{len}$	Path Elements
$\pi ::=$	$\overrightarrow{pe}$	Paths
$o ::=$	$\emptyset \mid \pi(x)$	Objects
$\phi ::=$	$a_1 o_1 + ... + a_{(i-1)} \leq a_i o_i + ... + a_j$	Linear Inequalities
$\Phi ::=$	$\overrightarrow{\phi}$	System of Linear Inequalities
$\sigma, \tau ::=$	$\top \mid \mathbf{N} \mid \{\diamond : \mathbf{N} \mid \Phi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid [[\overrightarrow{\tau}]] \mid x{:}\sigma \xrightarrow[o]{\psi} \tau$	Types
$\psi ::=$	$\tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbb{T} \mid \mathbb{F} \mid \mathbb{U}$	Propositions
$\Gamma ::=$	$\overrightarrow{\psi}$	Environments

Figure 1: Syntax of Types, Propositions, Terms, etc...

T-Num
$$\Gamma \vdash n : \{\diamond : \mathbf{N} \mid \diamond = n\} \,;\, \mathbb{T} \,;\, \emptyset$$

T-Const
$$\Gamma \vdash c : \delta_\tau(c) \,;\, \mathbb{T} \,;\, \emptyset$$

T-True
$$\Gamma \vdash \#\mathrm{t} : \mathbf{T} \,;\, \mathbb{T} \,;\, \emptyset$$

T-False
$$\Gamma \vdash \#\mathrm{f} : \mathbf{F} \,;\, \mathbb{F} \,;\, \emptyset$$

T-Var
$$\frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau \,;\, \overline{\mathbf{F}}_x \,;\, x}$$

T-Abs
$$\frac{\Gamma, \sigma_x \vdash e : \tau \,;\, \psi \,;\, o}{\Gamma \vdash \lambda x^\sigma.e : x{:}\sigma \xrightarrow[o]{\psi} \tau \,;\, \mathbb{T} \,;\, \emptyset}$$

T-App
$$\frac{\begin{array}{c}\Gamma \vdash e : x{:}\sigma \xrightarrow[o_f]{\psi_f} \tau \,;\, \psi \,;\, o \\ \Gamma \vdash e' : \sigma \,;\, \psi' \,;\, o'\end{array}}{\Gamma \vdash (e\ e') : \tau[o'/x] \,;\, \psi_f[o'/x] \,;\, o_f[o'/x]}$$

T-If
$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \,;\, \psi_1 \,;\, o_1 \\ \Gamma \vdash e_2 : \tau \,;\, \psi_2 \,;\, o \\ \Gamma \vdash e_3 : \tau \,;\, \psi_3 \,;\, o\end{array}}{\Gamma \vdash (\mathbf{if}\ e_1\ e_2\ e_3) : \tau \,;\, \psi_2 \vee \psi_3 \,;\, o}$$

T-Let
$$\frac{\begin{array}{c}\Gamma \vdash e_0 : \tau \,;\, \psi_0 \,;\, o_0 \\ \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_0, \mathbf{F}_x \supset \neg\psi_0 \vdash e_1 : \sigma \,;\, \psi_1 \,;\, o_1\end{array}}{\Gamma \vdash (\mathbf{let}\ (x\ e_0)\ e_1) : \sigma[o_0/x] \,;\, \psi_1[o_0/x] \,;\, o_1[o_0/x]}$$

T-Cons
$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \,;\, \psi_1 \,;\, o_1 \\ \Gamma \vdash e_2 : \tau_2 \,;\, \psi_2 \,;\, o_2\end{array}}{\Gamma \vdash (\mathit{cons}\ e_1\ e_2) : \langle \tau_1, \tau_2 \rangle \,;\, \mathbb{T} \,;\, \emptyset}$$

T-Car
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \,;\, \psi \,;\, o}{\Gamma \vdash (\mathit{car}\ e) : \tau_1 \,;\, \overline{\mathbf{F}}_{\mathbf{car}(x)}[o/x] \,;\, \mathbf{car}(x)[o/x]}$$

T-Vec
$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \,;\, \psi_1 \,;\, o_1 \\ \cdots \\ \Gamma \vdash e_n : \tau_n \,;\, \psi_n \,;\, o_n\end{array}}{\Gamma \vdash (\mathit{vec}\ e_1\ \ldots\ e_n) : [[\tau_1\ \ldots\ \tau_n]] \,;\, \mathbb{T} \,;\, \emptyset}$$

T-Cdr
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \,;\, \psi \,;\, o}{\Gamma \vdash (\mathit{cdr}\ e) : \tau_2 \,;\, \overline{\mathbf{F}}_{\mathbf{cdr}(x)}[o/x] \,;\, \mathbf{cdr}(x)[o/x]}$$

T-Subsume
$$\frac{\begin{array}{c}\Gamma \vdash e : \tau \,;\, \psi \,;\, o \\ \Gamma, \psi \vdash \psi' \qquad \tau <: \tau' \qquad o <: o'\end{array}}{\Gamma \vdash e : \tau' \,;\, \psi' \,;\, o'}$$

T-Ref
$$\frac{\Gamma \vdash e : [[\ldots \tau_i \ldots]] \,;\, \psi_i \,;\, o_i}{\Gamma \vdash (\mathit{ref}_i\ \ e) : \tau_i \,;\, \overline{\mathbf{F}}_{\mathbf{ref_i}(x)}[o/x] \,;\, \mathbf{ref_i}(x)[o/x]}$$

Figure 2: Typing Rules

S-Refl
$$\vdash \tau <: \tau$$

S-Top
$$\vdash \tau <: \top$$

S-UnionSuper
$$\frac{\exists \sigma \in \overrightarrow{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \overrightarrow{\sigma})}$$

S-UnionSub
$$\frac{\forall \tau \in \overrightarrow{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \overrightarrow{\tau}) <: \sigma}$$

S-Pair
$$\frac{\vdash \sigma_1 <: \tau_1 \qquad \vdash \sigma_2 <: \tau_2}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}$$

S-Fun
$$\frac{\vdash \sigma' <: \sigma \qquad \vdash \tau <: \tau' \qquad \psi \vdash \psi' \qquad \vdash o <: o'}{\vdash x{:}\sigma \xrightarrow[o]{\psi} \tau <: x{:}\sigma' \xrightarrow[o']{\psi'} \tau'}$$

S-Vec
$$\frac{\forall i, \vdash \sigma_i <: \tau_i}{\vdash [[\sigma_o \ldots \sigma_i]] <: [[\tau_o \ldots \tau_i]]}$$

S-Ineq
$$\frac{\text{unsatisfiable}(\Phi \bowtie \overline{\Phi'})}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \{\diamond : \mathbf{N} \mid \Phi'\}}$$

S-IneqNum
$$\frac{\vdash \mathbf{N} <: \tau}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \tau}$$

Figure 3: Subtyping Rules

L-Atom
$$\frac{\nu_{\pi(x)} \in \Gamma}{\Gamma \vdash \nu_{\pi(x)}}$$

L-True
$$\Gamma \vdash \mathbb{T}$$

L-False
$$\frac{\Gamma \vdash \mathbb{F}}{\Gamma \vdash \psi}$$

L-AndI
$$\frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}$$

L-AndE
$$\frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$$

L-ImpI
$$\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$$

L-ImpE
$$\frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}$$

L-OrI
$$\frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}$$

L-OrE
$$\frac{\Gamma, \psi_1 \vdash \psi \qquad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi}$$

L-Sub
$$\frac{\Gamma \vdash \tau_x \qquad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_x}$$

L-SubNot
$$\frac{\Gamma \vdash \overline{\sigma}_x \qquad \vdash \tau <: \sigma}{\Gamma \vdash \overline{\tau}_x}$$

L-Bot
$$\frac{\Gamma \vdash \bot_x}{\Gamma \vdash \psi}$$

L-Update
$$\frac{\Gamma \vdash \tau_x \qquad \Gamma \vdash \nu_x}{\Gamma \vdash \text{update}(\tau, \nu)_x}$$

L-Ineq
$$\frac{\Gamma \vdash \mathbf{N}_x \qquad \Gamma \vdash \Phi' \qquad \text{unsatisfiable}(\Phi' \bowtie (\Phi[x/\diamond]))}{\Gamma \vdash \{\diamond : \mathbf{N} \mid \Phi\}_x}$$

L-IneqSimple
$$\frac{\Gamma, \mathbf{N}_x, \Phi[x/\diamond] \vdash \psi}{\Gamma, \{\diamond : \mathbf{N} \mid \Phi\}_x \vdash \psi}$$

L-IneqJoin
$$\frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi}$$

L-IneqUnsat
$$\frac{\Gamma \vdash \Phi \qquad \text{unsatisfiable}(\Phi)}{\Gamma \vdash \psi}$$

Figure 4: Logic Rules

$$\begin{array}{rcl}
\neg\mathbb{T} & = & \mathbb{F} \\
\neg\mathbb{F} & = & \mathbb{T} \\
\neg\mathbb{U} & = & \mathbb{U} \\
\neg\Phi & = & \overline{\Phi} \\
\neg\tau_{\pi(x)} & = & \overline{\tau}_{\pi(x)} \\
\neg\overline{\tau}_{\pi(x)} & = & \tau_{\pi(x)} \\
\neg(P \vee Q) & = & \neg P \wedge \neg Q \\
\neg(P \wedge Q) & = & \neg P \vee \neg Q \\
\neg(P \supset Q) & = & \neg P \wedge Q
\end{array}$$

Figure 5: Proposition Negation

$$
\begin{array}{lll}
\nu_{\pi(x)}[\pi'(y)/x] & = (\nu[\pi'(y)/x])_{\pi(\pi'(y))} & \\
\nu_{\pi(x)}[\emptyset/x]^{+} & = \mathbb{T} & \\
\nu_{\pi(x)}[\emptyset/x]^{-} & = \mathbb{F} & \\
\nu_{\pi(x)}[o/z] & = \nu_{\pi(x)} & x \neq z \text{ and } z \notin \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^{+} & = \mathbb{T} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^{-} & = \mathbb{F} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\mathbb{T}[o/x] & = \mathbb{T} & \\
\mathbb{F}[o/x] & = \mathbb{F} & \\
\mathbb{U}[o/x] & = \mathbb{U} & \\
(\psi_1 \supset \psi_2)[o/x]^{+} & = \psi_1[o/x]^{-} \supset \psi_2[o/x]^{+} & \\
(\psi_1 \supset \psi_2)[o/x]^{-} & = \psi_1[o/x]^{+} \supset \psi_2[o/x]^{-} & \\
(\psi_1 \vee \psi_2)[o/x] & = \psi_1[o/x] \vee \psi_2[o/x] & \\
(\psi_1 \wedge \psi_2)[o/x] & = \psi_1[o/x] \wedge \psi_2[o/x] & \\
\Phi[o/x] & = \Phi & x \notin \Phi \\
\Phi[\pi'(y)/x] & = \forall \phi \in \Phi,\ \phi[x \mapsto \pi'(y)] & x \in \Phi \\
\Phi[\emptyset/x]^{\pm} & = \mathrm{FMelim}(\Phi,\ x,\ \pm) & x \in \Phi \\
 & & \\
\pi(x)[\pi'(y)/x] & = \pi(\pi(y)) & \\
\pi(x)[\emptyset/x] & = \emptyset & \\
\pi(x)[o/z] & = \pi(x) & x \neq z \\
\emptyset[o/x] & = \emptyset &
\end{array}
$$

Figure 6: Substitution