$$
\begin{array}{rll}
e ::= & x \mid (e\, e) \mid \lambda x^{\tau}.e \mid (\textbf{if}\ e\ e\ e) \mid c \mid \#\mathrm{t} \mid \#\mathrm{f} \mid (\mathit{cons}\ e\ e) \mid (\mathit{vec}\ \overrightarrow{e}) \mid n & \text{Expressions} \\
c ::= & + \mid * \mid \leq \mid \mathit{num?} \mid \mathit{bool?} \mid \mathit{proc?} \mid \mathit{cons?} \mid \mathit{vec?} \mid \mathit{car} \mid \mathit{cdr} \mid \mathit{len} \mid \mathit{ref} & \text{Primitive Operations} \\
pe ::= & \textbf{car} \mid \textbf{cdr} \mid \textbf{len} & \text{Path Elements} \\
\pi ::= & \overrightarrow{pe} & \text{Paths} \\
\mathfrak{L} ::= & n \mid \pi(x) \mid n\mathfrak{L} \mid \mathfrak{L} + \mathfrak{L} & \text{Linear Expressions} \\
o ::= & \emptyset \mid \pi(x) \mid \mathfrak{L} & \text{Objects} \\
\phi ::= & \mathfrak{L} \leq \mathfrak{L} & \text{Linear Inequalities} \\
\Phi ::= & \overrightarrow{\phi} & \text{System of Linear Inequalities} \\
\sigma, \tau ::= & \top \mid \mathbf{N} \mid \{x : \tau \mid \psi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid [[\tau]] \mid x{:}\sigma \xrightarrow[o]{\psi} \tau & \text{Types} \\
\psi ::= & \tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbb{T} \mid \mathbb{F} \mid \mathbb{U} & \text{Propositions} \\
\Gamma ::= & \overrightarrow{\psi} & \text{Environments}
\end{array}
$$

$\perp$ is defined as $(\bigcup)$, $\mathbf{B}$ is defined as $(\bigcup\ \mathbf{T}\ \mathbf{F})$.

Figure 1: Syntax of Types, Propositions, Terms, etc...

T-Num
$$\Gamma \vdash n : \{x : \mathbf{N} \mid x = n\} \ ; \ \mathbb{T} \ ; \ n$$

T-Const
$$\Gamma \vdash c : \delta_{\tau}(c) \ ; \ \mathbb{T} \ ; \ \emptyset$$

T-True
$$\Gamma \vdash \#\mathrm{t} : \mathbf{T} \ ; \ \mathbb{T} \ ; \ \emptyset$$

T-False
$$\Gamma \vdash \#\mathrm{f} : \mathbf{F} \ ; \ \mathbb{F} \ ; \ \emptyset$$

T-Var
$$\frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau \ ; \ \overline{\mathbf{F}}_x \ ; \ x}$$

T-Abs
$$\frac{\Gamma, \sigma_x \vdash e : \tau \ ; \ \psi \ ; \ o}{\Gamma \vdash \lambda x^{\sigma}.e : x{:}\sigma \xrightarrow[o]{\psi} \tau \ ; \ \mathbb{T} \ ; \ \emptyset}$$

T-App
$$\frac{\Gamma \vdash e : x{:}\sigma \xrightarrow[o_f]{\psi_f} \tau \ ; \ \psi \ ; \ o \qquad \Gamma \vdash e' : \sigma \ ; \ \psi' \ ; \ o'}{\Gamma \vdash (e\, e') : \tau[o'/x] \ ; \ \psi_f[o'/x] \ ; \ o_f[o'/x]}$$

T-If
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_1 \ ; \ o_1 \qquad \Gamma \vdash e_2 : \tau \ ; \ \psi \ ; \ o \qquad \Gamma \vdash e_3 : \tau \ ; \ \psi \ ; \ o}{\Gamma \vdash (\textbf{if}\ e_1\ e_2\ e_3) : \tau \ ; \ \psi \ ; \ o}$$

T-Let
$$\frac{\Gamma \vdash e_0 : \tau \ ; \ \psi_0 \ ; \ o_0 \qquad \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_0, \mathbf{F}_x \supset \neg\psi_0 \vdash e_1 : \sigma \ ; \ \psi_1 \ ; \ o_1}{\Gamma \vdash (\textbf{let}\ (x\ e_0)\ e_1) : \sigma[o_0/x] \ ; \ \psi_1[o_0/x] \ ; \ o_1[o_0/x]}$$

T-Cons
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_1 \ ; \ o_1 \qquad \Gamma \vdash e_2 : \tau_2 \ ; \ \psi_2 \ ; \ o_2}{\Gamma \vdash (\mathit{cons}\ e_1\ e_2) : \langle \tau_1, \tau_2 \rangle \ ; \ \mathbb{T} \ ; \ \emptyset}$$

T-Car
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \ ; \ \psi \ ; \ o \qquad \psi_r = \overline{\mathbf{F}}_{\mathbf{car}(x)}[o/x]}{\Gamma \vdash (\mathit{car}\ e) : \tau_1 \ ; \ \psi_r \ ; \ \mathbf{car}(x)[o/x]}$$

T-Cdr
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \ ; \ \psi \ ; \ o \qquad \psi_r = \overline{\mathbf{F}}_{\mathbf{cdr}(x)}[o/x]}{\Gamma \vdash (\mathit{cdr}\ e) : \tau_2 \ ; \ \psi_r \ ; \ \mathbf{cdr}(x)[o/x]}$$

T-Vec
$$\frac{\Gamma \vdash e_1 : \tau \ ; \ \psi_1 \ ; \ o_1 \qquad \ldots \qquad \Gamma \vdash e_n : \tau \ ; \ \psi_n \ ; \ o_n}{\Gamma \vdash (\mathit{vec}\ e_1\ \ldots\ e_n) : [[\tau]] \ ; \ \mathbb{T} \ ; \ \emptyset}$$

T-Ref
$$\frac{\Gamma \vdash e_1 : [[\tau]] \ ; \ \psi_1 \ ; \ o_1 \qquad \Gamma \vdash e_2 : \{x : \mathbf{N} \mid 0 \leq x \leq \mathbf{len}(o_1)\} \ ; \ \psi_2 \ ; \ o_2}{\Gamma \vdash (\mathit{ref}\ e_1\ e_2) : \tau \ ; \ \mathrm{prop}(\tau) \ ; \ \emptyset}$$

T-Subsume
$$\frac{\Gamma \vdash e : \tau \ ; \ \psi \ ; \ o \qquad \Gamma \vdash \psi <: \psi' \qquad \vdash \tau <: \tau' \qquad \vdash o <: o'}{\Gamma \vdash e : \tau' \ ; \ \psi' \ ; \ o'}$$

Figure 2: Typing Rules

SP-Impl
$$\frac{\Gamma, \psi \vdash \psi'}{\Gamma \vdash \psi <: \psi'}$$

SP-Refl
$$\Gamma \vdash \psi <: \psi$$

SP-Top
$$\Gamma \vdash \psi <: \mathbb{U}$$

SO-Refl
$$\vdash o <: o$$

SO-Top
$$\vdash o <: \emptyset$$

S-Refl
$$\vdash \tau <: \tau$$

S-Top
$$\vdash \tau <: \top$$

S-UnionSuper
$$\frac{\exists \sigma \in \overrightarrow{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \overrightarrow{\sigma})}$$

S-UnionSub
$$\frac{\forall \tau \in \overrightarrow{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \overrightarrow{\tau}) <: \sigma}$$

S-Pair
$$\frac{\vdash \sigma_1 <: \tau_1 \quad \vdash \sigma_2 <: \tau_2}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}$$

S-Fun
$$\frac{\vdash \sigma' <: \sigma \qquad \vdash \tau <: \tau' \qquad \vdash \psi <: \psi' \qquad \vdash o <: o'}{\vdash x{:}\sigma \xrightarrow[o]{\psi} \tau <: x{:}\sigma' \xrightarrow[o']{\psi'} \tau'}$$

S-Vec
$$\frac{\vdash \sigma <: \tau}{\vdash [[\sigma]] <: [[\tau]]}$$

S-Dep
$$\frac{\vdash \tau <: \sigma \qquad \psi \vdash \psi'}{\vdash \{x : \tau \mid \psi\} <: \{x : \sigma \mid \psi'\}}$$

S-DepWeak
$$\frac{\vdash \tau <: \sigma}{\vdash \{x : \tau \mid \psi\} <: \sigma}$$

Figure 3: Subtyping Rules

L-Atom
$$\frac{\chi \in \Gamma}{\Gamma \vdash \chi}$$

L-True
$$\Gamma \vdash \mathbb{T}$$

L-False
$$\frac{\Gamma \vdash \mathbb{F}}{\Gamma \vdash \psi}$$

L-AndI
$$\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}$$

L-AndE
$$\frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$$

L-ImpI
$$\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$$

L-ImpE
$$\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}$$

L-OrI
$$\frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}$$

L-OrE
$$\frac{\Gamma, \psi_1 \vdash \psi \quad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi}$$

L-Sub
$$\frac{\Gamma \vdash \tau_{\pi(x)} \qquad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_{\pi(x)}}$$

L-SubNot
$$\frac{\Gamma \vdash \overline{\sigma}_{\pi(x)} \qquad \vdash \tau <: \sigma}{\Gamma \vdash \overline{\tau}_{\pi(x)}}$$

L-Bot
$$\frac{\Gamma \vdash \bot_{\pi(x)}}{\Gamma \vdash \psi}$$

L-Update
$$\frac{\Gamma \vdash \tau_{\pi'(x)} \qquad \Gamma \vdash \nu_{\pi(\pi'(x))}}{\Gamma \vdash \text{update}(\tau, \nu, \pi)_{\pi'(x)}}$$

L-IneqImpl
$$\frac{\Gamma \vdash \Phi_1 \quad \text{unsatisfiable}(\Phi_1 \bowtie \overline{\Phi}_2)}{\Gamma \vdash \Phi_2}$$

L-IneqJoin
$$\frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi}$$

L-IneqUnsat
$$\frac{\Gamma \vdash \Phi \quad \text{unsatisfiable}(\Phi)}{\Gamma \vdash \psi}$$

Where χ ranges over propositions of the form $\tau_{\pi(x)}$, $\overline{\tau}_{\pi(x)}$, and Φ.

Figure 4: Logic Rules

$$
\begin{array}{lll}
\psi_1 \mid \psi_2[o/x] & = \psi_1[o/x] \mid \psi_2[o/x] & \\
\\
\nu_{\pi(x)}[\pi'(y)/x] & = (\nu[\pi'(y)/x])_{\pi(\pi'(y))} & \\
\nu_{\pi(x)}[\emptyset/x]^+ & = \mathbb{T} & \\
\nu_{\pi(x)}[\emptyset/x]^- & = \mathbb{F} & \\
\nu_{\pi(x)}[o/z] & = \nu_{\pi(x)} & x \neq z \text{ and } z \notin \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^+ & = \mathbb{T} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^- & = \mathbb{F} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\mathbb{T}[o/x] & = \mathbb{T} & \\
\mathbb{F}[o/x] & = \mathbb{F} & \\
(\psi_1 \supset \psi_2)[o/x]^+ & = \psi_1[o/x]^- \supset \psi_2[o/x]^+ & \\
(\psi_1 \supset \psi_2)[o/x]^- & = \psi_1[o/x]^+ \supset \psi_2[o/x]^- & \\
(\psi_1 \vee \psi_2)[o/x] & = \psi_1[o/x] \vee \psi_2[o/x] & \\
(\psi_1 \wedge \psi_2)[o/x] & = \psi_1[o/x] \wedge \psi_2[o/x] & \\
\Phi[o/x] & = \Phi & x \notin \Phi \\
\Phi[\pi'(y)/x] & = \forall \phi \in \Phi,\ \phi[x \mapsto \pi'(y)] & x \in \Phi \\
\Phi[\emptyset/x]^{\pm} & = \mathrm{FMelim}(\Phi,\ x,\ \pm) & x \in \Phi \\
\\
\pi(x)[\pi'(y)/x] & = \pi[\pi'(y)/x](\pi'(y)) & \\
\pi(x)[\emptyset/x] & = \emptyset & \\
\pi(x)[o/z] & = \pi[o/z](x) & x \neq z \\
\emptyset[o/x] & = \emptyset &
\end{array}
$$

Figure 5: Substitution

$$
\begin{array}{ll}
\mathrm{update}(\langle \tau, \sigma \rangle, \nu, \pi :: \mathbf{car}) & = \langle \mathrm{update}(\tau, \nu, \pi), \sigma \rangle \\
\mathrm{update}(\langle \tau, \sigma \rangle, \nu, \pi :: \mathbf{cdr}) & = \langle \tau, \mathrm{update}(\sigma, \nu, \pi) \rangle
\end{array}
$$