$$
\begin{array}{rcll}
e & ::= & x \mid (e\;e) \mid \lambda x^{\tau}.e \mid (\textbf{if}\;e\;e\;e) \mid c \mid \#\mathrm{t} \mid \#\mathrm{f} \mid (\mathit{cons}\;e\;e) \mid (\mathit{vec}\;\overrightarrow{e}) \mid n & \text{Expressions} \\
c & ::= & \mathit{add1} \mid = \mid \leq \mid \mathit{num?} \mid \mathit{bool?} \mid \mathit{proc?} \mid \mathit{cons?} \mid \mathit{vec?} \mid \mathit{car} \mid \mathit{cdr} \mid \mathit{len} \mid \mathit{ref}[n] & \text{Primitive Operations} \\
\mathit{pe} & ::= & \textbf{car} \mid \textbf{cdr} \mid \textbf{ref}_n \mid \textbf{len} & \text{Path Elements} \\
\pi & ::= & \overrightarrow{\mathit{pe}} & \text{Paths} \\
o & ::= & \emptyset \mid \pi(x) & \text{Objects} \\
\phi & ::= & a_0 o_0 + \ldots + a_n o_n \leq a_{n+1} & \text{Linear Inequalities} \\
\Phi & ::= & \overrightarrow{\phi} & \text{System of Linear Inequalities} \\
\sigma, \tau & ::= & \top \mid \mathbf{N} \mid \{\diamond : \mathbf{N} \mid \Phi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid [[\overrightarrow{\tau}]] \mid x{:}\sigma \xrightarrow[o]{\psi \mid \psi} \tau & \text{Types} \\
\psi & ::= & \tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbb{T} \mid \mathbb{F} & \text{Propositions} \\
\Gamma & ::= & \overrightarrow{\psi} & \text{Environments}
\end{array}
$$

Note: $\mathbf{B}$ is equivalent to $(\bigcup \mathbf{T}\;\mathbf{F})$, and $\bot$ is equivalent to $(\bigcup)$.

T-Num
$$\Gamma \vdash n : \{\diamond : \mathbf{N} \mid \diamond = n\}\;;\;\mathbb{T} \mid \mathbb{F}\;;\;\emptyset$$

T-Const
$$\Gamma \vdash c : \delta_\tau(c)\;;\;\mathbb{T} \mid \mathbb{F}\;;\;\emptyset$$

T-True
$$\Gamma \vdash \#\mathrm{t} : \mathbf{T}\;;\;\mathbb{T} \mid \mathbb{F}\;;\;\emptyset$$

T-False
$$\Gamma \vdash \#\mathrm{f} : \mathbf{F}\;;\;\mathbb{F} \mid \mathbb{T}\;;\;\emptyset$$

T-Var
$$\frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau\;;\;\overline{\mathbf{F}}_x \mid \mathbf{F}_x\;;\;x}$$

T-Abs
$$\frac{\Gamma, \sigma_x \vdash e : \tau\;;\;\psi_+ \mid \psi_-\;;\;o}{\Gamma \vdash \lambda x^{\sigma}.e : x{:}\sigma \xrightarrow[o]{\psi_+ \mid \psi_-} \tau\;;\;\mathbb{T} \mid \mathbb{F}\;;\;\emptyset}$$

T-App
$$\frac{\Gamma \vdash e : x{:}\sigma \xrightarrow[o_f]{\psi_{f+} \mid \psi_{f-}} \tau\;;\;\psi_+ \mid \psi_-\;;\;o \qquad \Gamma \vdash e' : \sigma\;;\;\psi'_+ \mid ; \mid \psi'_-\;;\;o'}{\Gamma \vdash (e\;e') : \tau[o'/x]\;;\;\psi_{f+} \mid \psi_{f-}[o'/x]\;;\;o_f[o'/x]}$$

T-If
$$\frac{\Gamma \vdash e_1 : \tau_1\;;\;\psi_{1+} \mid \psi_{1-}\;;\;o_1 \qquad \Gamma \vdash e_2 : \tau\;;\;\psi_{2+} \mid \psi_{2-}\;;\;o \qquad \Gamma \vdash e_3 : \tau\;;\;\psi_{3+} \mid \psi_{3-}\;;\;o}{\Gamma \vdash (\textbf{if}\;e_1\;e_2\;e_3) : \tau\;;\;\psi_{2+} \vee \psi_{3+} \mid \psi_{2-} \vee \psi_{3-}\;;\;o}$$

T-Let
$$\frac{\Gamma \vdash e_0 : \tau\;;\;\psi_{0+} \mid \psi_{0-}\;;\;o_0 \qquad \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_{0+}, \mathbf{F}_x \supset \psi_{0-} \vdash e_1 : \sigma\;;\;\psi_{1+} \mid \psi_{1-}\;;\;o_1}{\Gamma \vdash (\textbf{let}\;(x\;e_0)\;e_1) : \sigma[o_0/x]\;;\;\psi_{1+} \mid \psi_{1-}[o_0/x]\;;\;o_1[o_0/x]}$$

T-Cons
$$\frac{\Gamma \vdash e_1 : \tau_1\;;\;\psi_{1+} \mid \psi_{1-}\;;\;o_1 \qquad \Gamma \vdash e_2 : \tau_2\;;\;\psi_{2+} \mid \psi_{2-}\;;\;o_2}{\Gamma \vdash (\mathit{cons}\;e_1\;e_2) : \langle \tau_1, \tau_2 \rangle\;;\;\mathbb{T} \mid \mathbb{F}\;;\;\emptyset}$$

T-Car
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle\;;\;\psi_+ \mid \psi_-\;;\;o \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{car}(x)} \mid \mathbf{F}_{\mathbf{car}(x)}[o/x]}{\Gamma \vdash (\mathit{car}\;e) : \tau_1\;;\;\psi_{r+} \mid \psi_{r-}\;;\;\mathbf{car}(x)[o/x]}$$

T-Cdr
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle\;;\;\psi_+ \mid \psi_-\;;\;o \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{cdr}(x)} \mid \mathbf{F}_{\mathbf{cdr}(x)}[o/x]}{\Gamma \vdash (\mathit{cdr}\;e) : \tau_2\;;\;\psi_{r+} \mid \psi_{r-}\;;\;\mathbf{cdr}(x)[o/x]}$$

T-Vec
$$\frac{\Gamma \vdash e_1 : \tau_1\;;\;\psi_{1+} \mid \psi_{1-}\;;\;o_1 \qquad \ldots \qquad \Gamma \vdash e_n : \tau_n\;;\;\psi_{n+} \mid \psi_{n-}\;;\;o_n}{\Gamma \vdash (\mathit{vec}\;e_1\;\ldots\;e_n) : [[\tau_1\;\ldots\;\tau_n]]\;;\;\mathbb{T} \mid \mathbb{F}\;;\;\emptyset}$$

T-Ref
$$\frac{\Gamma \vdash e : [[\ldots \tau_i \ldots]]\;;\;\psi_{i+} \mid \psi_{i-}\;;\;o_i \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{ref_i}(x)} \mid \mathbf{F}_{\mathbf{ref_i}(x)}[o/x]}{\Gamma \vdash (\mathit{ref}[i]\;\;e) : \tau_i\;;\;\psi_{r+} \mid \psi_{r-}\;;\;\mathbf{ref_i}(x)[o/x]}$$

T-Subsume
$$\frac{\Gamma \vdash e : \tau\;;\;\psi_+ \mid \psi_-\;;\;o \qquad \Gamma, \psi_+ \vdash \psi'_+ \qquad \Gamma, \psi_- \vdash \psi'_- \qquad \tau <: \tau' \qquad o <: o'}{\Gamma \vdash e : \tau'\;;\;\psi'_+ \mid \psi'_-\;;\;o'}$$

S-Refl
$$\vdash \tau <: \tau$$

S-Top
$$\vdash \tau <: \top$$

S-UnionSuper
$$\frac{\exists \sigma \in \overrightarrow{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \overrightarrow{\sigma})}$$

S-UnionSub
$$\frac{\forall \tau \in \overrightarrow{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \overrightarrow{\tau}) <: \sigma}$$

S-Pair
$$\frac{\vdash \sigma_1 <: \tau_1 \qquad \vdash \sigma_2 <: \tau_2}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}$$

S-Fun
$$\frac{\vdash \sigma' <: \sigma \qquad \vdash \tau <: \tau' \qquad \psi_+ \vdash \psi'_+ \qquad \psi_- \vdash \psi'_- \qquad \vdash o <: o'}{\vdash x{:}\sigma \xrightarrow[o]{\psi_+|\psi_-} \tau <: x{:}\sigma' \xrightarrow[o']{\psi'_+|\psi'_-} \tau'}$$

S-Vec
$$\frac{\forall i, \vdash \sigma_i <: \tau_i}{\vdash [[\sigma_o \ldots \sigma_i]] <: [[\tau_o \ldots \tau_i]]}$$

S-Ineq
$$\frac{\text{overconstrained}(\Phi \bowtie \overline{\Phi'})}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \{\diamond : \mathbf{N} \mid \Phi'\}}$$

S-IneqNum
$$\frac{\vdash \mathbf{N} <: \tau}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \tau}$$

L-Atom
$$\frac{\psi \in \Gamma}{\Gamma \vdash \psi}$$

L-True
$$\Gamma \vdash \mathbb{T}$$

L-False
$$\frac{\Gamma \vdash \mathbb{F}}{\Gamma \vdash \psi}$$

L-AndI
$$\frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}$$

L-AndE
$$\frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$$

L-ImpI
$$\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$$

L-ImpE
$$\frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}$$

L-OrI
$$\frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}$$

L-OrE
$$\frac{\Gamma, \psi_1 \vdash \psi \qquad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi}$$

L-Sub
$$\frac{\Gamma \vdash \tau_x \qquad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_x}$$

L-SubNot
$$\frac{\Gamma \vdash \overline{\sigma}_x \qquad \vdash \tau <: \sigma}{\Gamma \vdash \overline{\tau}_x}$$

L-Bot
$$\frac{\Gamma \vdash \bot_x}{\Gamma \vdash \psi}$$

L-Update
$$\frac{\Gamma \vdash \tau_x \qquad \Gamma \vdash \nu_x}{\Gamma \vdash \text{update}(\tau, \nu)_x}$$

L-Ineq
$$\frac{\Gamma \vdash \mathbf{N}_x \qquad \Gamma \vdash \Phi' \qquad \text{overconstrained}(\Phi' \bowtie (\Phi[x/\diamond]))}{\Gamma \vdash \{\diamond : \mathbf{N} \mid \Phi\}_x}$$

L-IneqSimple
$$\frac{\Gamma, \mathbf{N}_x, \Phi[x/\diamond] \vdash \psi}{\Gamma, \{\diamond : \mathbf{N} \mid \Phi\}_x \vdash \psi}$$

L-IneqJoin
$$\frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi}$$