$$
\begin{array}{rll}
e ::= & x \mid (e\ e) \mid \lambda x^{\tau}.e \mid (\textbf{if}\ e\ e\ e) \mid c \mid \#\text{t} \mid \#\text{f} \mid (\mathit{cons}\ e\ e) \mid (\mathit{vec}\ \overrightarrow{e}) \mid n & \text{Expressions} \\
c ::= & + \mid * \mid \leq \mid \mathit{num?} \mid \mathit{bool?} \mid \mathit{proc?} \mid \mathit{cons?} \mid \mathit{vec?} \mid \mathit{car} \mid \mathit{cdr} \mid \mathit{len} \mid \mathit{ref} & \text{Primitive Operations} \\
pe ::= & \textbf{car} \mid \textbf{cdr} \mid \textbf{len} & \text{Path Elements} \\
\pi ::= & \overrightarrow{pe} & \text{Paths} \\
\mathfrak{L} ::= & n \mid \pi(x) \mid n\mathfrak{L} \mid \mathfrak{L} + \mathfrak{L} & \text{Linear Expressions} \\
o ::= & \emptyset \mid \pi(x) \mid \mathfrak{L} & \text{Objects} \\
\phi ::= & \mathfrak{L} \leq \mathfrak{L} & \text{Linear Inequalities} \\
\Phi ::= & \overrightarrow{\phi} & \text{System of Linear Inequalities} \\
\sigma, \tau ::= & \top \mid \mathbf{N} \mid \{x : \tau \mid \psi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid [[\tau]] \mid x{:}\sigma \xrightarrow[o]{\psi} \tau & \text{Types} \\
\psi ::= & \tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbb{T} \mid \mathbb{F} \mid \mathbb{U} & \text{Propositions} \\
\Gamma ::= & \overrightarrow{\psi} & \text{Environments}
\end{array}
$$

$\bot$ is defined as $(\bigcup)$, $\mathbf{B}$ is defined as $(\bigcup\ \mathbf{T}\ \mathbf{F})$.

Figure 1: Syntax of Types, Propositions, Terms, etc...

$$\textsc{T-Num}\quad \Gamma \vdash n : \{x : \mathbf{N} \mid x = n\}\ ;\ \mathbb{T}\ ;\ n$$

$$\textsc{T-Const}\quad \Gamma \vdash c : \delta_{\tau}(c)\ ;\ \mathbb{T}\ ;\ \emptyset$$

$$\textsc{T-True}\quad \Gamma \vdash \#\text{t} : \mathbf{T}\ ;\ \mathbb{T}\ ;\ \emptyset$$

$$\textsc{T-False}\quad \Gamma \vdash \#\text{f} : \mathbf{F}\ ;\ \mathbb{F}\ ;\ \emptyset$$

$$\textsc{T-Var}\quad \frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau\ ;\ \overline{\mathbf{F}}_x\ ;\ x}$$

$$\textsc{T-Abs}\quad \frac{\Gamma, \sigma_x \vdash e : \tau\ ;\ \psi\ ;\ o}{\Gamma \vdash \lambda x^{\sigma}.e : x{:}\sigma \xrightarrow[o]{\psi} \tau\ ;\ \mathbb{T}\ ;\ \emptyset}$$

$$\textsc{T-App}\quad \frac{\begin{array}{c}\Gamma \vdash e : x{:}\sigma \xrightarrow[o_f]{\psi_f} \tau\ ;\ \psi\ ;\ o \\ \Gamma \vdash e' : \sigma\ ;\ \psi'\ ;\ o'\end{array}}{\Gamma \vdash (e\ e') : \tau[o'/x]\ ;\ \psi_f[o'/x]\ ;\ o_f[o'/x]}$$

$$\textsc{T-If}\quad \frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1\ ;\ \psi_1\ ;\ o_1 \\ \Gamma \vdash e_2 : \tau\ ;\ \psi\ ;\ o \\ \Gamma \vdash e_3 : \tau\ ;\ \psi\ ;\ o\end{array}}{\Gamma \vdash (\textbf{if}\ e_1\ e_2\ e_3) : \tau\ ;\ \psi\ ;\ o}$$

$$\textsc{T-Let}\quad \frac{\begin{array}{c}\Gamma \vdash e_0 : \tau\ ;\ \psi_0\ ;\ o_0 \\ \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_0, \mathbf{F}_x \supset \neg\psi_0 \vdash e_1 : \sigma\ ;\ \psi_1\ ;\ o_1\end{array}}{\Gamma \vdash (\textbf{let}\ (x\ e_0)\ e_1) : \sigma[o_0/x]\ ;\ \psi_1[o_0/x]\ ;\ o_1[o_0/x]}$$

$$\textsc{T-Cons}\quad \frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1\ ;\ \psi_1\ ;\ o_1 \\ \Gamma \vdash e_2 : \tau_2\ ;\ \psi_2\ ;\ o_2\end{array}}{\Gamma \vdash (\mathit{cons}\ e_1\ e_2) : \langle \tau_1, \tau_2 \rangle\ ;\ \mathbb{T}\ ;\ \emptyset}$$

$$\textsc{T-Car}\quad \frac{\begin{array}{c}\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle\ ;\ \psi\ ;\ o \\ \psi_r = \overline{\mathbf{F}}_{\mathbf{car}(x)}[o/x]\end{array}}{\Gamma \vdash (\mathit{car}\ e) : \tau_1\ ;\ \psi_r\ ;\ \mathbf{car}(x)[o/x]}$$

$$\textsc{T-Cdr}\quad \frac{\begin{array}{c}\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle\ ;\ \psi\ ;\ o \\ \psi_r = \overline{\mathbf{F}}_{\mathbf{cdr}(x)}[o/x]\end{array}}{\Gamma \vdash (\mathit{cdr}\ e) : \tau_2\ ;\ \psi_r\ ;\ \mathbf{cdr}(x)[o/x]}$$

$$\textsc{T-Vec}\quad \frac{\begin{array}{c}\Gamma \vdash e_1 : \tau\ ;\ \psi_1\ ;\ o_1 \\ \ldots \\ \Gamma \vdash e_n : \tau\ ;\ \psi_n\ ;\ o_n\end{array}}{\Gamma \vdash (\mathit{vec}\ e_1\ \ldots\ e_n) : [[\tau]]\ ;\ \mathbb{T}\ ;\ \emptyset}$$

$$\textsc{T-Ref}\quad \frac{\begin{array}{c}\Gamma \vdash e_1 : [[\tau]]\ ;\ \psi_1\ ;\ o_1 \\ \Gamma \vdash e_2 : \{x : \mathbf{N} \mid 0 \leq x \leq \mathbf{len}(o_1)\}\ ;\ \psi_2\ ;\ o_2\end{array}}{\Gamma \vdash (\mathit{ref}\ e_1\ e_2) : \tau\ ;\ \text{prop}(\tau)\ ;\ \emptyset}$$

$$\textsc{T-Subsume}\quad \frac{\begin{array}{c}\Gamma \vdash e : \tau\ ;\ \psi\ ;\ o \\ \Gamma \vdash \psi <: \psi' \qquad \vdash \tau <: \tau' \qquad \vdash o <: o'\end{array}}{\Gamma \vdash e : \tau'\ ;\ \psi'\ ;\ o'}$$

Figure 2: Typing Rules

$$\textsc{SP-Impl}\ \frac{\Gamma, \psi \vdash \psi'}{\Gamma \vdash \psi <: \psi'} \qquad \textsc{SP-Refl}\ \Gamma \vdash \psi <: \psi \qquad \textsc{SP-Top}\ \Gamma \vdash \psi <: \mathbb{U} \qquad \textsc{SO-Refl}\ \vdash o <: o \qquad \textsc{SO-Top}\ \vdash o <: \emptyset \qquad \textsc{S-Refl}\ \vdash \tau <: \tau$$

$$\textsc{S-Top}\ \vdash \tau <: \top \qquad \textsc{S-UnionSuper}\ \frac{\exists \sigma \in \overrightarrow{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \overrightarrow{\sigma})} \qquad \textsc{S-UnionSub}\ \frac{\forall \tau \in \overrightarrow{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \overrightarrow{\tau}) <: \sigma} \qquad \textsc{S-Pair}\ \frac{\begin{array}{c}\vdash \sigma_1 <: \tau_1 \\ \vdash \sigma_2 <: \tau_2\end{array}}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}$$

$$\textsc{S-Fun}\ \frac{\begin{array}{cc}\vdash \sigma' <: \sigma & \vdash \tau <: \tau' \\ \vdash \psi <: \psi' & \vdash o <: o'\end{array}}{\vdash x{:}\sigma \xrightarrow[o]{\psi} \tau <: x{:}\sigma' \xrightarrow[o']{\psi'} \tau'} \qquad \textsc{S-Vec}\ \frac{\vdash \sigma <: \tau}{\vdash [[\sigma]] <: [[\tau]]} \qquad \textsc{S-Dep}\ \frac{\vdash \tau <: \sigma \qquad \psi \vdash \psi'}{\vdash \{x : \tau \mid \psi\} <: \{x : \sigma \mid \psi'\}} \qquad \textsc{S-DepWeak}\ \frac{\vdash \tau <: \sigma}{\vdash \{x : \tau \mid \psi\} <: \sigma}$$

Figure 3: Subtyping Rules

$$\textsc{L-Atom}\ \frac{\chi \in \Gamma}{\Gamma \vdash \chi} \qquad \textsc{L-True}\ \Gamma \vdash \mathbb{T} \qquad \textsc{L-False}\ \frac{\Gamma \vdash \mathbb{F}}{\Gamma \vdash \psi} \qquad \textsc{L-AndI}\ \frac{\begin{array}{c}\Gamma \vdash \psi_1 \\ \Gamma \vdash \psi_2\end{array}}{\Gamma \vdash \psi_1 \wedge \psi_2} \qquad \textsc{L-AndE}\ \frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi} \qquad \textsc{L-ImpI}\ \frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$$

$$\textsc{L-ImpE}\ \frac{\begin{array}{c}\Gamma \vdash \psi_1 \\ \Gamma \vdash \psi_1 \supset \psi_2\end{array}}{\Gamma \vdash \psi_2} \qquad \textsc{L-OrI}\ \frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2} \qquad \textsc{L-OrE}\ \frac{\begin{array}{c}\Gamma, \psi_1 \vdash \psi \\ \Gamma, \psi_2 \vdash \psi\end{array}}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi} \qquad \textsc{L-Sub}\ \frac{\Gamma \vdash \tau_{\pi(x)} \qquad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_{\pi(x)}}$$

$$\textsc{L-SubNot}\ \frac{\Gamma \vdash \overline{\sigma}_{\pi(x)} \qquad \vdash \tau <: \sigma}{\Gamma \vdash \overline{\tau}_{\pi(x)}} \qquad \textsc{L-Bot}\ \frac{\Gamma \vdash \bot_{\pi(x)}}{\Gamma \vdash \psi} \qquad \textsc{L-Update}\ \frac{\Gamma \vdash \tau_{\pi'(x)} \qquad \Gamma \vdash \nu_{\pi(\pi'(x))}}{\Gamma \vdash \text{update}(\tau, \nu, \pi)_{\pi'(x)}} \qquad \textsc{L-Dep}\ \frac{\Gamma \vdash \tau_{\pi(x)} \qquad \Gamma \vdash \psi}{\Gamma \vdash \{y : \tau \mid \psi\}_{\pi(x)}}$$

$$\textsc{L-DepWeak}\ \frac{\Gamma \vdash \{y : \tau \mid \psi\}_{\pi(x)}}{\Gamma \vdash \psi[\pi(x)/y]} \qquad \textsc{L-IneqImpl}\ \frac{\begin{array}{c}\Gamma \vdash \Phi_1 \\ \text{unsatisfiable}(\Phi_1 \bowtie \overline{\Phi}_2)\end{array}}{\Gamma \vdash \Phi_2} \qquad \textsc{L-IneqJoin}\ \frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi} \qquad \textsc{L-IneqUnsat}\ \frac{\begin{array}{c}\Gamma \vdash \Phi \\ \text{unsatisfiable}(\Phi)\end{array}}{\Gamma \vdash \psi}$$

Where χ ranges over propositions of the form $\tau_{\pi(x)}$, $\overline{\tau}_{\pi(x)}$, and Φ.

Figure 4: Logic Rules

$$
\begin{array}{lll}
\psi_1 \mid \psi_2[o/x] & = \psi_1[o/x] \mid \psi_2[o/x] & \\
 & & \\
\nu_{\pi(x)}[\pi'(y)/x] & = (\nu[\pi'(y)/x])_{\pi(\pi'(y))} & \\
\nu_{\pi(x)}[\emptyset/x]^+ & = \mathbb{T} & \\
\nu_{\pi(x)}[\emptyset/x]^- & = \mathbb{F} & \\
\nu_{\pi(x)}[o/z] & = \nu_{\pi(x)} & x \neq z \text{ and } z \notin \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^+ & = \mathbb{T} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^- & = \mathbb{F} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\mathbb{T}[o/x] & = \mathbb{T} & \\
\mathbb{F}[o/x] & = \mathbb{F} & \\
(\psi_1 \supset \psi_2)[o/x]^+ & = \psi_1[o/x]^- \supset \psi_2[o/x]^+ & \\
(\psi_1 \supset \psi_2)[o/x]^- & = \psi_1[o/x]^+ \supset \psi_2[o/x]^- & \\
(\psi_1 \vee \psi_2)[o/x] & = \psi_1[o/x] \vee \psi_2[o/x] & \\
(\psi_1 \wedge \psi_2)[o/x] & = \psi_1[o/x] \wedge \psi_2[o/x] & \\
\Phi[o/x] & = \Phi & x \notin \Phi \\
\Phi[\pi'(y)/x] & = \forall \phi \in \Phi,\ \phi[x \mapsto \pi'(y)] & x \in \Phi \\
\Phi[\emptyset/x]^\pm & = \mathrm{FMelim}(\Phi,\ x,\ \pm) & x \in \Phi \\
 & & \\
\pi(x)[\pi'(y)/x] & = \pi[\pi'(y)/x](\pi'(y)) & \\
\pi(x)[\emptyset/x] & = \emptyset & \\
\pi(x)[o/z] & = \pi[o/z](x) & x \neq z \\
\emptyset[o/x] & = \emptyset &
\end{array}
$$

Figure 5: Substitution

$$
\begin{array}{ll}
\mathrm{update}(\langle \tau, \sigma \rangle, \nu, \pi :: \mathbf{car}) & = \langle \mathrm{update}(\tau, \nu, \pi), \sigma \rangle \\
\mathrm{update}(\langle \tau, \sigma \rangle, \nu, \pi :: \mathbf{cdr}) & = \langle \tau, \mathrm{update}(\sigma, \nu, \pi) \rangle
\end{array}
$$