$$
\begin{array}{rll}
e ::= & x \mid (e\ e) \mid \lambda x^{\tau}.e \mid (\textbf{if}\ e\ e\ e) \mid c \mid \#\mathrm{t} \mid \#\mathrm{f} \mid (cons\ e\ e) \mid (vec\ \overrightarrow{e}) \mid n & \text{Expressions} \\
c ::= & add1 \mid = \mid \leq \mid num? \mid bool? \mid proc? \mid cons? \mid vec? \mid car \mid cdr \mid len \mid ref[n] & \text{Primitive Operations} \\
pe ::= & \textbf{car} \mid \textbf{cdr} \mid \textbf{ref}_n \mid \textbf{len} & \text{Path Elements} \\
\pi ::= & \overrightarrow{pe} & \text{Paths} \\
o ::= & \emptyset \mid \pi(x) & \text{Objects} \\
\phi ::= & a_0 o_0 + ... + a_n o_n \leq a_{n+1} & \text{Linear Inequalities} \\
\Phi ::= & \overrightarrow{\phi} & \text{System of Linear Inequalities} \\
\sigma, \tau ::= & \top \mid \mathbf{N} \mid \{\diamond : \mathbf{N} \mid \Phi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid [[\overrightarrow{\tau}]] \mid x{:}\sigma \xrightarrow[o]{\psi \mid \psi} \tau & \text{Types} \\
\psi ::= & \tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbb{T} \mid \mathbb{F} & \text{Propositions} \\
\Gamma ::= & \overrightarrow{\psi} & \text{Environments}
\end{array}
$$

Figure 1: Syntax of Types, Propositions, Terms, etc...

$$
\textsc{T-Num}\ \ \Gamma \vdash n : \{\diamond : \mathbf{N} \mid \diamond = n\}\ ;\ \mathbb{T} \mid \mathbb{F}\ ;\ \emptyset
\qquad
\textsc{T-Const}\ \ \Gamma \vdash c : \delta_\tau(c)\ ;\ \mathbb{T} \mid \mathbb{F}\ ;\ \emptyset
\qquad
\textsc{T-True}\ \ \Gamma \vdash \#\mathrm{t} : \mathbf{T}\ ;\ \mathbb{T} \mid \mathbb{F}\ ;\ \emptyset
\qquad
\textsc{T-False}\ \ \Gamma \vdash \#\mathrm{f} : \mathbf{F}\ ;\ \mathbb{F} \mid \mathbb{T}\ ;\ \emptyset
$$

$$
\textsc{T-Var}\ \ \frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau\ ;\ \overline{\mathbf{F}}_x \mid \mathbf{F}_x\ ;\ x}
\qquad
\textsc{T-Abs}\ \ \frac{\Gamma, \sigma_x \vdash e : \tau\ ;\ \psi_+ \mid \psi_-\ ;\ o}{\Gamma \vdash \lambda x^{\sigma}.e : x{:}\sigma \xrightarrow[o]{\psi_+ \mid \psi_-} \tau\ ;\ \mathbb{T} \mid \mathbb{F}\ ;\ \emptyset}
$$

$$
\textsc{T-App}\ \ \frac{\Gamma \vdash e : x{:}\sigma \xrightarrow[o_f]{\psi_{f+} \mid \psi_{f-}} \tau\ ;\ \psi_+ \mid \psi_-\ ;\ o \qquad \Gamma \vdash e' : \sigma\ ;\ \psi'_+ \mid\ ;\ \mid \psi'_-\ ;\ o'}{\Gamma \vdash (e\ e') : \tau[o'/x]\ ;\ \psi_{f+} \mid \psi_{f-}[o'/x]\ ;\ o_f[o'/x]}
$$

$$
\textsc{T-If}\ \ \frac{\Gamma \vdash e_1 : \tau_1\ ;\ \psi_{1+} \mid \psi_{1-}\ ;\ o_1 \qquad \Gamma \vdash e_2 : \tau\ ;\ \psi_{2+} \mid \psi_{2-}\ ;\ o \qquad \Gamma \vdash e_3 : \tau\ ;\ \psi_{3+} \mid \psi_{3-}\ ;\ o}{\Gamma \vdash (\textbf{if}\ e_1\ e_2\ e_3) : \tau\ ;\ \psi_{2+} \vee \psi_{3+} \mid \psi_{2-} \vee \psi_{3-}\ ;\ o}
$$

$$
\textsc{T-Let}\ \ \frac{\Gamma \vdash e_0 : \tau\ ;\ \psi_{0+} \mid \psi_{0-}\ ;\ o_0 \qquad \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_{0+}, \mathbf{F}_x \supset \psi_{0-} \vdash e_1 : \sigma\ ;\ \psi_{1+} \mid \psi_{1-}\ ;\ o_1}{\Gamma \vdash (\textbf{let}\ (x\ e_0)\ e_1) : \sigma[o_0/x]\ ;\ \psi_{1+} \mid \psi_{1-}[o_0/x]\ ;\ o_1[o_0/x]}
$$

$$
\textsc{T-Cons}\ \ \frac{\Gamma \vdash e_1 : \tau_1\ ;\ \psi_{1+} \mid \psi_{1-}\ ;\ o_1 \qquad \Gamma \vdash e_2 : \tau_2\ ;\ \psi_{2+} \mid \psi_{2-}\ ;\ o_2}{\Gamma \vdash (cons\ e_1\ e_2) : \langle \tau_1, \tau_2 \rangle\ ;\ \mathbb{T} \mid \mathbb{F}\ ;\ \emptyset}
$$

$$
\textsc{T-Car}\ \ \frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle\ ;\ \psi_+ \mid \psi_-\ ;\ o \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{car}(x)} \mid \mathbf{F}_{\mathbf{car}(x)}[o/x]}{\Gamma \vdash (car\ e) : \tau_1\ ;\ \psi_{r+} \mid \psi_{r-}\ ;\ \mathbf{car}(x)[o/x]}
$$

$$
\textsc{T-Cdr}\ \ \frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle\ ;\ \psi_+ \mid \psi_-\ ;\ o \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{cdr}(x)} \mid \mathbf{F}_{\mathbf{cdr}(x)}[o/x]}{\Gamma \vdash (cdr\ e) : \tau_2\ ;\ \psi_{r+} \mid \psi_{r-}\ ;\ \mathbf{cdr}(x)[o/x]}
$$

$$
\textsc{T-Vec}\ \ \frac{\Gamma \vdash e_1 : \tau_1\ ;\ \psi_{1+} \mid \psi_{1-}\ ;\ o_1 \qquad \ldots \qquad \Gamma \vdash e_n : \tau_n\ ;\ \psi_{n+} \mid \psi_{n-}\ ;\ o_n}{\Gamma \vdash (vec\ e_1\ \ldots\ e_n) : [[\tau_1\ \ldots\ \tau_n]]\ ;\ \mathbb{T} \mid \mathbb{F}\ ;\ \emptyset}
$$

$$
\textsc{T-Ref}\ \ \frac{\Gamma \vdash e : [[\ldots \tau_i \ldots]]\ ;\ \psi_{i+} \mid \psi_{i-}\ ;\ o_i \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{ref_i}(x)} \mid \mathbf{F}_{\mathbf{ref_i}(x)}[o/x]}{\Gamma \vdash (ref[i]\ \ e) : \tau_i\ ;\ \psi_{r+} \mid \psi_{r-}\ ;\ \mathbf{ref_i}(x)[o/x]}
$$

$$
\textsc{T-Subsume}\ \ \frac{\Gamma \vdash e : \tau\ ;\ \psi_+ \mid \psi_-\ ;\ o \qquad \Gamma, \psi_+ \vdash \psi'_+ \qquad \Gamma, \psi_- \vdash \psi'_- \qquad \tau <: \tau' \qquad o <: o'}{\Gamma \vdash e : \tau'\ ;\ \psi'_+ \mid \psi'_-\ ;\ o'}
$$

Figure 2: Typing Rules

S-Refl
$$\vdash \tau <: \tau$$

S-Top
$$\vdash \tau <: \top$$

S-UnionSuper
$$\frac{\exists \sigma \in \overrightarrow{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \overrightarrow{\sigma})}$$

S-UnionSub
$$\frac{\forall \tau \in \overrightarrow{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \overrightarrow{\tau}) <: \sigma}$$

S-Pair
$$\frac{\vdash \sigma_1 <: \tau_1 \quad \vdash \sigma_2 <: \tau_2}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}$$

S-Fun
$$\frac{\vdash \sigma' <: \sigma \quad \vdash \tau <: \tau' \quad \psi_+ \vdash \psi'_+ \quad \psi_- \vdash \psi'_- \quad \vdash o <: o'}{\vdash x{:}\sigma \xrightarrow[o]{\psi_+ | \psi_-} \tau <: x{:}\sigma' \xrightarrow[o']{\psi'_+ | \psi'_-} \tau'}$$

S-Vec
$$\frac{\forall i, \vdash \sigma_i <: \tau_i}{\vdash [[\sigma_o \dots \sigma_i]] <: [[\tau_o \dots \tau_i]]}$$

S-Ineq
$$\frac{\text{overconstrained}(\Phi \bowtie \overline{\Phi'})}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \{\diamond : \mathbf{N} \mid \Phi'\}}$$

S-IneqNum
$$\frac{\vdash \mathbf{N} <: \tau}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \tau}$$

Figure 3: Subtyping Rules

L-Atom
$$\frac{\psi \in \Gamma}{\Gamma \vdash \psi}$$

L-True
$$\Gamma \vdash \mathbb{T}$$

L-False
$$\frac{\Gamma \vdash \mathbb{F}}{\Gamma \vdash \psi}$$

L-AndI
$$\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}$$

L-AndE
$$\frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$$

L-ImpI
$$\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$$

L-ImpE
$$\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}$$

L-OrI
$$\frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}$$

L-OrE
$$\frac{\Gamma, \psi_1 \vdash \psi \quad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi}$$

L-Sub
$$\frac{\Gamma \vdash \tau_x \quad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_x}$$

L-SubNot
$$\frac{\Gamma \vdash \overline{\sigma}_x \quad \vdash \tau <: \sigma}{\Gamma \vdash \overline{\tau}_x}$$

L-Bot
$$\frac{\Gamma \vdash \bot_x}{\Gamma \vdash \psi}$$

L-Update
$$\frac{\Gamma \vdash \tau_x \quad \Gamma \vdash \nu_x}{\Gamma \vdash \text{update}(\tau, \nu)_x}$$

L-Ineq
$$\frac{\Gamma \vdash \mathbf{N}_x \quad \Gamma \vdash \Phi' \quad \text{overconstrained}(\Phi' \bowtie (\Phi[x/\diamond]))}{\Gamma \vdash \{\diamond : \mathbf{N} \mid \Phi\}_x}$$

L-IneqSimple
$$\frac{\Gamma, \mathbf{N}_x, \Phi[x/\diamond] \vdash \psi}{\Gamma, \{\diamond : \mathbf{N} \mid \Phi\}_x \vdash \psi}$$

L-IneqJoin
$$\frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi}$$

Figure 4: Logic Rules

$$\begin{array}{lll}
\psi_1 \mid \psi_2[o/x] & = \psi_1[o/x] \mid \psi_2[o/x] & \\
\\
\nu_{\pi(x)}[\pi'(y)/x] & = (\nu[\pi'(y)/x])_{\pi(\pi'(y))} & \\
\nu_{\pi(x)}[\emptyset/x]^+ & = \mathbb{T} & \\
\nu_{\pi(x)}[\emptyset/x]^- & = \mathbb{F} & \\
\nu_{\pi(x)}[o/z] & = \nu_{\pi(x)} & x \neq z \text{ and } z \notin \text{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^+ & = \mathbb{T} & x \neq z \text{ and } z \in \text{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^- & = \mathbb{F} & x \neq z \text{ and } z \in \text{fv}(\nu) \\
\mathbb{T}[o/x] & = \mathbb{T} & \\
\mathbb{F}[o/x] & = \mathbb{F} & \\
(\psi_1 \supset \psi_2)[o/x]^+ & = \psi_1[o/x]^- \supset \psi_2[o/x]^+ & \\
(\psi_1 \supset \psi_2)[o/x]^- & = \psi_1[o/x]^+ \supset \psi_2[o/x]^- & \\
(\psi_1 \vee \psi_2)[o/x] & = \psi_1[o/x] \vee \psi_2[o/x] & \\
(\psi_1 \wedge \psi_2)[o/x] & = \psi_1[o/x] \wedge \psi_2[o/x] & \\
\Phi[o/x] & = \Phi & x \notin \Phi \\
\Phi[\pi'(y)/x] & = \forall \phi \in \Phi,\ \phi[x \mapsto \pi'(y)] & x \in \Phi \\
\Phi[\emptyset/x]^{\pm} & = \text{FME}(\Phi,\ x,\ \pm) & x \in \Phi \\
\\
\pi(x)[\pi'(y)/x] & = \pi(\pi(y)) & \\
\pi(x)[\emptyset/x] & = \emptyset & \\
\pi(x)[o/z] & = \pi(x) & x \neq z \\
\emptyset[o/x] & = \emptyset &
\end{array}$$

Figure 5: Substitution