

$e ::=$	$x \mid (e \ e) \mid \lambda x^\tau. e \mid (\mathbf{if} \ e \ e \ e) \mid c \mid \#t \mid \#f \mid (cons \ e \ e) \mid (vec \ \vec{e}) \mid n$	Expressions
$c ::=$	$add1 \mid =_n \mid \leq_n \mid num? \mid bool? \mid proc? \mid cons? \mid vec? \mid car \mid cdr \mid len \mid ref_n$	Primitive Operations
$pe ::=$	$\mathbf{car} \mid \mathbf{cdr} \mid \mathbf{ref}_n \mid \mathbf{len}$	Path Elements
$\pi ::=$	$\vec{p}\vec{e}$	Paths
$o ::=$	$\emptyset \mid \pi(x)$	Objects
$\phi ::=$	$a_1 o_1 + \dots + a_{(i-1)} \leq a_i o_i + \dots + a_j$	Linear Inequalities
$\Phi ::=$	$\vec{\phi}$	System of Linear Inequalities
$\sigma, \tau ::=$	$\top \mid \mathbf{N} \mid \{\diamond : \mathbf{N} \mid \Phi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \vec{\tau}) \mid \langle \tau, \tau \rangle \mid [[\vec{\tau}]] \mid x : \sigma \xrightarrow[o]{\psi \mid \psi} \tau$	Types
$\psi ::=$	$\tau_{\pi(x)} \mid \bar{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbf{T} \mid \mathbf{F}$	Propositions
$\Gamma ::=$	$\vec{\psi}$	Environments

Figure 1: Syntax of Types, Propositions, Terms, etc...

<b>T-NUM</b>	$\Gamma \vdash n : \{\diamond : \mathbf{N} \mid \diamond = n\} ; \mathbb{T} \mid \mathbb{F} ; \emptyset$	<b>T-CONST</b>	$\Gamma \vdash c : \delta_\tau(c) ; \mathbb{T} \mid \mathbb{F} ; \emptyset$	<b>T-TRUE</b>	$\Gamma \vdash \#t : \mathbf{T} ; \mathbb{T} \mid \mathbb{F} ; \emptyset$	<b>T-FALSE</b>	$\Gamma \vdash \#f : \mathbf{F} ; \mathbb{F} \mid \mathbb{T} ; \emptyset$
<b>T-VAR</b>	$\frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau ; \bar{\mathbf{F}}_x \mid \mathbf{F}_x ; x}$	<b>T-ABS</b>	$\frac{\Gamma, \sigma_x \vdash e : \tau ; \psi_+ \mid \psi_- ; o}{\Gamma \vdash \lambda x^\sigma. e : x : \sigma \xrightarrow[o]{\psi_+ \mid \psi_-} \tau ; \mathbb{T} \mid \mathbb{F} ; \emptyset}$	<b>T-APP</b>	$\frac{\Gamma \vdash e : x : \sigma \xrightarrow[o_f]{\psi_{f+} \mid \psi_{f-}} \tau ; \psi_+ \mid \psi_- ; o \quad \Gamma \vdash e' : \sigma ; \psi'_+ \mid ; \mid \psi'_- ; o'}{\Gamma \vdash (e \ e') : \tau[o'/x] ; \psi_{f+} \mid \psi_{f-}[o'/x] ; o_f[o'/x]}$		
<b>T-IF</b>	$\frac{\Gamma \vdash e_1 : \tau_1 ; \psi_{1+} \mid \psi_{1-} ; o_1 \quad \Gamma \vdash e_2 : \tau ; \psi_{2+} \mid \psi_{2-} ; o \quad \Gamma \vdash e_3 : \tau ; \psi_{3+} \mid \psi_{3-} ; o}{\Gamma \vdash (\mathbf{if} \ e_1 \ e_2 \ e_3) : \tau ; \psi_{2+} \vee \psi_{3+} \mid \psi_{2-} \vee \psi_{3-} ; o}$	<b>T-LET</b>	$\frac{\Gamma \vdash e_0 : \tau ; \psi_{0+} \mid \psi_{0-} ; o_0 \quad \Gamma, \tau_x, \bar{\mathbf{F}}_x \supset \psi_{0+}, \mathbf{F}_x \supset \psi_{0-} \vdash e_1 : \sigma ; \psi_{1+} \mid \psi_{1-} ; o_1}{\Gamma \vdash (\mathbf{let} \ (x \ e_0) \ e_1) : \sigma[o_0/x] ; \psi_{1+} \mid \psi_{1-}[o_0/x] ; o_1[o_0/x]}$	<b>T-CAR</b>	$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle ; \psi_+ \mid \psi_- ; o \quad \psi_{r+} \mid \psi_{r-} = \bar{\mathbf{F}}_{\mathbf{car}(x)} \mid \mathbf{F}_{\mathbf{car}(x)}[o/x]}{\Gamma \vdash (car \ e) : \tau_1 ; \psi_{r+} \mid \psi_{r-} ; \mathbf{car}(x)[o/x]}$		
<b>T-CONS</b>	$\frac{\Gamma \vdash e_1 : \tau_1 ; \psi_{1+} \mid \psi_{1-} ; o_1 \quad \Gamma \vdash e_2 : \tau_2 ; \psi_{2+} \mid \psi_{2-} ; o_2}{\Gamma \vdash (cons \ e_1 \ e_2) : \langle \tau_1, \tau_2 \rangle ; \mathbb{T} \mid \mathbb{F} ; \emptyset}$	<b>T-VEC</b>	$\frac{\Gamma \vdash e_1 : \tau_1 ; \psi_{1+} \mid \psi_{1-} ; o_1 \quad \dots \quad \Gamma \vdash e_n : \tau_n ; \psi_{n+} \mid \psi_{n-} ; o_n}{\Gamma \vdash (vec \ e_1 \ \dots \ e_n) : [[\tau_1 \ \dots \ \tau_n]] ; \mathbb{T} \mid \mathbb{F} ; \emptyset}$	<b>T-SUBSUME</b>	$\frac{\Gamma \vdash e : \tau ; \psi_+ \mid \psi_- ; o \quad \Gamma, \psi_+ \vdash \psi'_+ \quad \Gamma, \psi_- \vdash \psi'_- \quad \tau <: \tau' \quad o <: o'}{\Gamma \vdash e : \tau' ; \psi'_+ \mid \psi'_- ; o'}$		
<b>T-CDR</b>	$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle ; \psi_+ \mid \psi_- ; o \quad \psi_{r+} \mid \psi_{r-} = \bar{\mathbf{F}}_{\mathbf{cdr}(x)} \mid \mathbf{F}_{\mathbf{cdr}(x)}[o/x]}{\Gamma \vdash (cdr \ e) : \tau_2 ; \psi_{r+} \mid \psi_{r-} ; \mathbf{cdr}(x)[o/x]}$	<b>T-REF</b>	$\frac{\Gamma \vdash e : [[\dots \tau_i \dots]] ; \psi_{i+} \mid \psi_{i-} ; o_i \quad \psi_{r+} \mid \psi_{r-} = \bar{\mathbf{F}}_{\mathbf{ref}_i(x)} \mid \mathbf{F}_{\mathbf{ref}_i(x)}[o/x]}{\Gamma \vdash (ref_i \ e) : \tau_i ; \psi_{r+} \mid \psi_{r-} ; \mathbf{ref}_i(x)[o/x]}$				

Figure 2: Typing Rules

$$\begin{array}{c}
\text{S-REFL} \\
\vdash \tau <: \tau
\end{array}
\quad
\begin{array}{c}
\text{S-TOP} \\
\vdash \tau <: \top
\end{array}
\quad
\begin{array}{c}
\text{S-UNIONSUPER} \\
\frac{\exists \sigma \in \vec{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \vec{\sigma})}
\end{array}
\quad
\begin{array}{c}
\text{S-UNIONSUB} \\
\frac{\forall \tau \in \vec{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \vec{\tau}) <: \sigma}
\end{array}
\quad
\begin{array}{c}
\text{S-PAIR} \\
\frac{\vdash \sigma_1 <: \tau_1 \quad \vdash \sigma_2 <: \tau_2}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}
\end{array}$$
  

$$\begin{array}{c}
\text{S-FUN} \\
\frac{\vdash \sigma' <: \sigma \quad \vdash \tau <: \tau' \quad \psi_+ \vdash \psi'_+ \quad \psi_- \vdash \psi'_- \quad \vdash o <: o'}{\vdash x:\sigma \xrightarrow[o]{\psi_+|\psi_-} \tau <: x:\sigma' \xrightarrow[o']{\psi'_+|\psi'_-} \tau'}
\end{array}
\quad
\begin{array}{c}
\text{S-VEC} \\
\frac{\forall i, \vdash \sigma_i <: \tau_i}{\vdash [[\sigma_o \dots \sigma_i]] <: [[\tau_o \dots \tau_i]]}
\end{array}
\quad
\begin{array}{c}
\text{S-INEQ} \\
\frac{\text{unsatisfiable}(\Phi \bowtie \overline{\Phi'})}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \{\diamond : \mathbf{N} \mid \Phi'\}}
\end{array}$$
  

$$\begin{array}{c}
\text{S-INEQNUM} \\
\frac{\vdash \mathbf{N} <: \tau}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \tau}
\end{array}$$

Figure 3: Subtyping Rules

$$\begin{array}{c}
\text{L-ATOM} \\
\frac{\psi \in \Gamma}{\Gamma \vdash \psi}
\end{array}
\quad
\begin{array}{c}
\text{L-TRUE} \\
\Gamma \vdash \mathbb{T}
\end{array}
\quad
\begin{array}{c}
\text{L-FALSE} \\
\Gamma \vdash \mathbb{F}
\end{array}
\quad
\begin{array}{c}
\text{L-ANDI} \\
\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}
\end{array}
\quad
\begin{array}{c}
\text{L-ANDE} \\
\frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}
\end{array}
\quad
\begin{array}{c}
\text{L-IMPI} \\
\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}
\end{array}$$
  

$$\begin{array}{c}
\text{L-IMPE} \\
\frac{\Gamma \vdash \psi_1 \quad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}
\end{array}
\quad
\begin{array}{c}
\text{L-ORI} \\
\frac{\Gamma \vdash \psi_1 \text{ or } \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}
\end{array}
\quad
\begin{array}{c}
\text{L-ORE} \\
\frac{\Gamma, \psi_1 \vdash \psi \quad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi}
\end{array}
\quad
\begin{array}{c}
\text{L-SUB} \\
\frac{\Gamma \vdash \tau_{\pi(x)} \quad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_{\pi(x)}}
\end{array}$$
  

$$\begin{array}{c}
\text{L-SUBNOT} \\
\frac{\Gamma \vdash \bar{\sigma}_{\pi(x)} \quad \vdash \tau <: \sigma}{\Gamma \vdash \bar{\tau}_{\pi(x)}}
\end{array}
\quad
\begin{array}{c}
\text{L-BOT} \\
\frac{\Gamma \vdash \perp_{\pi(x)}}{\Gamma \vdash \psi}
\end{array}
\quad
\begin{array}{c}
\text{L-UPDATE} \\
\frac{\Gamma \vdash \tau_{\pi'(x)} \quad \Gamma \vdash \nu_{\pi(\pi'(x))}}{\Gamma \vdash \mathbf{update}((, \tau, , )\nu, \pi)_{\pi(x)}}
\end{array}
\quad
\begin{array}{c}
\text{L-INEQ} \\
\frac{\Gamma \vdash \mathbf{N}_{\pi(x)} \quad \Gamma \vdash \Phi' \quad \text{unsatisfiable}(\Phi' \bowtie (\Phi[x/\diamond]))}{\Gamma \vdash \{\diamond : \mathbf{N} \mid \Phi\}_{\pi(x)}}
\end{array}$$
  

$$\begin{array}{c}
\text{L-INEQSIMPLE} \\
\frac{\Gamma, \mathbf{N}_{\pi(x)}, \Phi[x/\diamond] \vdash \psi}{\Gamma, \{\diamond : \mathbf{N} \mid \Phi\}_{\pi(x)} \vdash \psi}
\end{array}
\quad
\begin{array}{c}
\text{L-INEQJOIN} \\
\frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi}
\end{array}
\quad
\begin{array}{c}
\text{L-INEQUNSAT} \\
\frac{\Gamma \vdash \Phi \quad \text{unsatisfiable}(\Phi)}{\Gamma \vdash \psi}
\end{array}$$

Figure 4: Logic Rules

$$\begin{array}{ll}
\psi_1 \mid \psi_2[o/x] & = \psi_1[o/x] \mid \psi_2[o/x] \\
\\
\nu_{\pi(x)}[\pi'(y)/x] & = (\nu[\pi'(y)/x])_{\pi(\pi'(y))} \\
\nu_{\pi(x)}[\emptyset/x]^+ & = \mathbb{T} \\
\nu_{\pi(x)}[\emptyset/x]^- & = \mathbb{F} \\
\nu_{\pi(x)}[o/z] & = \nu_{\pi(x)} & x \neq z \text{ and } z \notin \text{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^+ & = \mathbb{T} & x \neq z \text{ and } z \in \text{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^- & = \mathbb{F} & x \neq z \text{ and } z \in \text{fv}(\nu) \\
\mathbb{T}[o/x] & = \mathbb{T} \\
\mathbb{F}[o/x] & = \mathbb{F} \\
(\psi_1 \supset \psi_2)[o/x]^+ & = \psi_1[o/x]^- \supset \psi_2[o/x]^+ \\
(\psi_1 \supset \psi_2)[o/x]^- & = \psi_1[o/x]^+ \supset \psi_2[o/x]^- \\
(\psi_1 \vee \psi_2)[o/x] & = \psi_1[o/x] \vee \psi_2[o/x] \\
(\psi_1 \wedge \psi_2)[o/x] & = \psi_1[o/x] \wedge \psi_2[o/x] \\
\Phi[o/x] & = \Phi & x \notin \Phi \\
\Phi[\pi'(y)/x] & = \forall \phi \in \Phi, \phi[x \mapsto \pi'(y)] & x \in \Phi \\
\Phi[\emptyset/x]^\pm & = \text{FMelim}(\Phi, x, \pm) & x \in \Phi \\
\\
\pi(x)[\pi'(y)/x] & = \pi(\pi(y)) \\
\pi(x)[\emptyset/x] & = \emptyset \\
\pi(x)[o/z] & = \pi(x) & x \neq z \\
\emptyset[o/x] & = \emptyset
\end{array}$$

Figure 5: Substitution

$$\begin{array}{ll}
\text{update}(\langle \tau, \sigma \rangle, \nu, \pi :: \text{car}) & = \langle \text{update}(\tau, \nu, \pi), \sigma \rangle \\
\text{update}(\langle \tau, \sigma \rangle, \nu, \pi :: \text{cdr}) & = \langle \tau, \text{update}(\sigma, \nu, \pi) \rangle
\end{array}$$