$$
\begin{array}{rll}
e ::= & x \mid (e\, e) \mid \lambda x^{\tau}.e \mid (\textbf{if}\ e\ e\ e) \mid c \mid \#\mathrm{t} \mid \#\mathrm{f} \mid (cons\ e\ e) \mid (vec\ \overrightarrow{e}) \mid n & \text{Expressions} \\
c ::= & add1 \mid = \mid \leq \mid num? \mid bool? \mid proc? \mid cons? \mid vec? \mid car \mid cdr \mid len \mid ref[n] & \text{Primitive Operations} \\
pe ::= & \textbf{car} \mid \textbf{cdr} \mid \textbf{ref}_n \mid \textbf{len} & \text{Path Elements} \\
\pi ::= & \overrightarrow{pe} & \text{Paths} \\
o ::= & \emptyset \mid \pi(x) & \text{Objects} \\
\phi ::= & a_0 o_0 + ... + a_n o_n \leq a_{n+1} & \text{Linear Inequalities} \\
\sigma, \tau ::= & \top \mid \mathbf{N} \mid \{x : \mathbf{N} \mid \overrightarrow{\phi}\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid \lfloor \overrightarrow{\tau} \rfloor \mid x : \sigma \xrightarrow[o]{\psi \mid \psi} \tau & \text{Types} \\
\psi ::= & \tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \mathbb{T} \mid \mathbb{F} & \text{Propositions} \\
\Gamma ::= & (\overrightarrow{\psi}, \overrightarrow{\phi}) & \text{Environments}
\end{array}
$$

T-Num
$$\Gamma \vdash n : \{x : \mathbf{N} \mid x = n\} \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset$$

T-Const
$$\Gamma \vdash c : \delta_{\tau}(c) \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset$$

T-True
$$\Gamma \vdash \#\mathrm{t} : \mathbf{T} \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset$$

T-False
$$\Gamma \vdash \#\mathrm{f} : \mathbf{F} \ ; \ \mathbb{F} \mid \mathbb{T} \ ; \ \emptyset$$

T-Var
$$\frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau \ ; \ \overline{\mathbf{F}}_x \mid \mathbf{F}_x \ ; \ x}$$

T-Abs
$$\frac{\Gamma, \sigma_x \vdash e : \tau \ ; \ \psi_+ \mid \psi_- \ ; \ o}{\Gamma \vdash \lambda x^{\sigma}.e : x : \sigma \xrightarrow[o]{\psi_+ \mid \psi_-} \tau \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset}$$

T-App
$$\frac{\Gamma \vdash e : x : \sigma \xrightarrow[o_f]{\psi_{f+} \mid \psi_{f-}} \tau \ ; \ \psi_+ \mid \psi_- \ ; \ o \qquad \Gamma \vdash e' : \sigma \ ; \ \psi'_+ \mid \psi'_- \ ; \ o'}{\Gamma \vdash (e\, e') : \tau[o'/x] \ ; \ \psi_{f+} | \psi_{f-}[o'/x] \ ; \ o_f[o'/x]}$$

T-If
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1 \qquad \Gamma \vdash e_2 : \tau \ ; \ \psi_{2+} \mid \psi_{2-} \ ; \ o \qquad \Gamma \vdash e_3 : \tau \ ; \ \psi_{3+} \mid \psi_{3-} \ ; \ o}{\Gamma \vdash (\textbf{if}\ e_1\ e_2\ e_3) : \tau \ ; \ \psi_{2+} \vee \psi_{3+} \mid \psi_{2-} \vee \psi_{3-} \ ; \ o}$$

T-Let
$$\frac{\Gamma \vdash e_0 : \tau \ ; \ \psi_{0+} \mid \psi_{0-} \ ; \ o_0 \qquad \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_{0+}, \mathbf{F}_x \supset \psi_{0-} \vdash e_1 : \sigma \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1}{\Gamma \vdash (\textbf{let}\ (x\ e_0)\ e_1) : \sigma[o_0/x] \ ; \ \psi_{1+} \mid \psi_{1-}[o_0/x] \ ; \ o_1[o_0/x]}$$

T-Cons
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1 \qquad \Gamma \vdash e_2 : \tau_2 \ ; \ \psi_{2+} \mid \psi_{2-} \ ; \ o_2}{\Gamma \vdash (cons\ e_1\ e_2) : \langle \tau_1, \tau_2 \rangle \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset}$$

T-Car
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \ ; \ \psi_+ \mid \psi_- \ ; \ o}{\Gamma \vdash (car\ e) : \tau_1 \ ; \ \overline{\mathbf{F}}_{\mathbf{car}(x)} \mid \mathbf{F}_{\mathbf{car}(x)}[o/x] \ ; \ \mathbf{car}(x)[o/x]}$$

T-Cdr
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \ ; \ \psi_+ \mid \psi_- \ ; \ o}{\Gamma \vdash (cdr\ e) : \tau_2 \ ; \ \overline{\mathbf{F}}_{\mathbf{cdr}(x)} \mid \mathbf{F}_{\mathbf{cdr}(x)}[o/x] \ ; \ \mathbf{cdr}(x)[o/x]}$$

T-Vec
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1 \qquad \ldots \qquad \Gamma \vdash e_n : \tau_n \ ; \ \psi_{n+} \mid \psi_{n-} \ ; \ o_n}{\Gamma \vdash (vec\ e_1\ \ldots\ e_n) : \lfloor \tau_1\ \ldots\ \tau_n \rfloor \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset}$$

T-Ref
$$\frac{\Gamma \vdash e : \lfloor \ldots \tau_i \ldots \rfloor \ ; \ \psi_{i+} \mid \psi_{i-} \ ; \ o_i}{\Gamma \vdash (ref[i]\ \ e) : \tau_i \ ; \ \overline{\mathbf{F}}_{\mathbf{ref}[i](x)} \mid \mathbf{F}_{\mathbf{ref}[i](x)}[o/x] \ ; \ \mathbf{ref}[i](x)[o/x]}$$

T-Subsume
$$\frac{\Gamma \vdash e : \tau \ ; \ \psi_+ \mid \psi_- \ ; \ o \qquad \Gamma, \psi_+ \vdash \psi'_+ \qquad \Gamma, \psi_- \vdash \psi'_- \qquad \vdash \tau <: \tau' \qquad \vdash o <: o'}{\Gamma \vdash e : \tau' \ ; \ \psi'_+ \mid \psi'_- \ ; \ o'}$$

$$\frac{(x_1{:}A_1, \ldots, x_n{:}A_n)\ \mathsf{ctx}}{x_1{:}A_1, \ldots, x_n{:}A_n \vdash x_i : A_i}\ \mathsf{Vble}$$

As with ctx-ext, the hypothesis and conclusion of the rule Vble are judgments of different forms, only now they are reversed: we start with a well-formed context and derive a typing judgment.

The following important principles, called **substitution** and **weakening**, need not be explicitly assumed. Rather, it is possible to show, by induction on the structure of all possible derivations, that whenever the hypotheses of these rules are derivable, their conclusion is also derivable.[1] For the typing judgments these principles are manifested as

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x{:}A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] : B[a/x]}\ \mathsf{Subst}_1 \qquad\qquad \frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma, \Delta \vdash b : B}{\Gamma, x{:}A, \Delta \vdash b : B}\ \mathsf{Wkg}_1$$

and for judgmental equalities they become

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x{:}A, \Delta \vdash b \equiv c : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv c[a/x] : B[a/x]}\ \mathsf{Subst}_2 \qquad\qquad \frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma, \Delta \vdash b \equiv c : B}{\Gamma, x{:}A, \Delta \vdash b \equiv c : B}\ \mathsf{Wkg}_2$$

In addition to the judgmental equality rules given for each type former, we also assume that judgmental equality is an equivalence relation respected by typing.

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B : \mathcal{U}_i}{\Gamma \vdash a : B}$$

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash A \equiv B : \mathcal{U}_i}{\Gamma \vdash a \equiv b : B}$$

Additionally, for all the type formers below, we assume rules stating that each constructor preserves definitional equality in each of its arguments; for instance, along with the Π-INTRO rule, we assume the rule

$$\frac{\Gamma \vdash A \equiv A' : \mathcal{U}_i \qquad \Gamma, x{:}A \vdash B \equiv B' : \mathcal{U}_i \qquad \Gamma, x{:}A \vdash b \equiv b' : B}{\Gamma \vdash \lambda x.\, b \equiv \lambda x.\, b' : \prod_{(x:A)} B}\ \Pi\text{-INTRO-EQ}$$

However, we omit these rules for brevity.

[1] Such rules are called **admissible**.