$$
\begin{array}{rll}
e ::= & x \mid (e\ e) \mid \lambda x^{\tau}.e \mid (\textbf{if}\ e\ e\ e) \mid c \mid \#\mathrm{t} \mid \#\mathrm{f} \mid (\mathit{cons}\ e\ e) \mid (\mathit{vec}\ \overrightarrow{e}) \mid n & \text{Expressions} \\
c ::= & \mathit{add1} \mid =_n \mid \leq_n \mid \mathit{num?} \mid \mathit{bool?} \mid \mathit{proc?} \mid \mathit{cons?} \mid \mathit{vec?} \mid \mathit{car} \mid \mathit{cdr} \mid \mathit{len} \mid \mathit{ref}_n & \text{Primitive Operations} \\
pe ::= & \textbf{car} \mid \textbf{cdr} \mid \textbf{ref}_n \mid \textbf{len} & \text{Path Elements} \\
\pi ::= & \overrightarrow{pe} & \text{Paths} \\
o ::= & \emptyset \mid \pi(x) & \text{Objects} \\
\phi ::= & a_1 o_1 + ... + a_{(i-1)} \leq a_i o_i + ... + a_j & \text{Linear Inequalities} \\
\Phi ::= & \overrightarrow{\phi} & \text{System of Linear Inequalities} \\
\sigma, \tau ::= & \top \mid \mathbf{N} \mid \{\diamond : \mathbf{N} \mid \Phi\} \mid \mathbf{T} \mid \mathbf{F} \mid (\bigcup \overrightarrow{\tau}) \mid \langle \tau, \tau \rangle \mid [[\overrightarrow{\tau}]] \mid x{:}\sigma \xrightarrow[o]{\psi|\psi} \tau & \text{Types} \\
\psi ::= & \tau_{\pi(x)} \mid \overline{\tau}_{\pi(x)} \mid \psi \supset \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Phi \mid \mathbb{T} \mid \mathbb{F} & \text{Propositions} \\
\Gamma ::= & \overrightarrow{\psi} & \text{Environments}
\end{array}
$$

Figure 1: Syntax of Types, Propositions, Terms, etc...

T-Num
$$\Gamma \vdash n : \{\diamond : \mathbf{N} \mid \diamond = n\} \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset$$

T-Const
$$\Gamma \vdash c : \delta_\tau(c) \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset$$

T-True
$$\Gamma \vdash \#\mathrm{t} : \mathbf{T} \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset$$

T-False
$$\Gamma \vdash \#\mathrm{f} : \mathbf{F} \ ; \ \mathbb{F} \mid \mathbb{T} \ ; \ \emptyset$$

T-Var
$$\frac{\Gamma \vdash \tau_x}{\Gamma \vdash x : \tau \ ; \ \overline{\mathbf{F}}_x \mid \mathbf{F}_x \ ; \ x}$$

T-Abs
$$\frac{\Gamma, \sigma_x \vdash e : \tau \ ; \ \psi_+ \mid \psi_- \ ; \ o}{\Gamma \vdash \lambda x^{\sigma}.e : x{:}\sigma \xrightarrow[o]{\psi_+|\psi_-} \tau \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset}$$

T-App
$$\frac{\Gamma \vdash e : x{:}\sigma \xrightarrow[o_f]{\psi_{f+}|\psi_{f-}} \tau \ ; \ \psi_+ \mid \psi_- \ ; \ o \qquad \Gamma \vdash e' : \sigma \ ; \ \psi'_+ \mid ; \mid \psi'_- \ ; \ o'}{\Gamma \vdash (e\ e') : \tau[o'/x] \ ; \ \psi_{f+} \mid \psi_{f-}[o'/x] \ ; \ o_f[o'/x]}$$

T-If
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1 \qquad \Gamma \vdash e_2 : \tau \ ; \ \psi_{2+} \mid \psi_{2-} \ ; \ o \qquad \Gamma \vdash e_3 : \tau \ ; \ \psi_{3+} \mid \psi_{3-} \ ; \ o}{\Gamma \vdash (\textbf{if}\ e_1\ e_2\ e_3) : \tau \ ; \ \psi_{2+} \vee \psi_{3+} \mid \psi_{2-} \vee \psi_{3-} \ ; \ o}$$

T-Let
$$\frac{\Gamma \vdash e_0 : \tau \ ; \ \psi_{0+} \mid \psi_{0-} \ ; \ o_0 \qquad \Gamma, \tau_x, \overline{\mathbf{F}}_x \supset \psi_{0+}, \mathbf{F}_x \supset \psi_{0-} \vdash e_1 : \sigma \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1}{\Gamma \vdash (\textbf{let}\ (x\ e_0)\ e_1) : \sigma[o_0/x] \ ; \ \psi_{1+} \mid \psi_{1-}[o_0/x] \ ; \ o_1[o_0/x]}$$

T-Cons
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1 \qquad \Gamma \vdash e_2 : \tau_2 \ ; \ \psi_{2+} \mid \psi_{2-} \ ; \ o_2}{\Gamma \vdash (\mathit{cons}\ e_1\ e_2) : \langle \tau_1, \tau_2 \rangle \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset}$$

T-Car
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \ ; \ \psi_+ \mid \psi_- \ ; \ o \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{car}(x)} \mid \mathbf{F}_{\mathbf{car}(x)}[o/x]}{\Gamma \vdash (\mathit{car}\ e) : \tau_1 \ ; \ \psi_{r+} \mid \psi_{r-} \ ; \ \mathbf{car}(x)[o/x]}$$

T-Cdr
$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle \ ; \ \psi_+ \mid \psi_- \ ; \ o \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{cdr}(x)} \mid \mathbf{F}_{\mathbf{cdr}(x)}[o/x]}{\Gamma \vdash (\mathit{cdr}\ e) : \tau_2 \ ; \ \psi_{r+} \mid \psi_{r-} \ ; \ \mathbf{cdr}(x)[o/x]}$$

T-Vec
$$\frac{\Gamma \vdash e_1 : \tau_1 \ ; \ \psi_{1+} \mid \psi_{1-} \ ; \ o_1 \qquad \dots \qquad \Gamma \vdash e_n : \tau_n \ ; \ \psi_{n+} \mid \psi_{n-} \ ; \ o_n}{\Gamma \vdash (\mathit{vec}\ e_1\ \dots\ e_n) : [[\tau_1\ \dots\ \tau_n]] \ ; \ \mathbb{T} \mid \mathbb{F} \ ; \ \emptyset}$$

T-Ref
$$\frac{\Gamma \vdash e : [[\dots \tau_i \dots]] \ ; \ \psi_{i+} \mid \psi_{i-} \ ; \ o_i \qquad \psi_{r+} \mid \psi_{r-} = \overline{\mathbf{F}}_{\mathbf{ref_i}(x)} \mid \mathbf{F}_{\mathbf{ref_i}(x)}[o/x]}{\Gamma \vdash (\mathit{ref}_i\ \ e) : \tau_i \ ; \ \psi_{r+} \mid \psi_{r-} \ ; \ \mathbf{ref_i}(x)[o/x]}$$

T-Subsume
$$\frac{\Gamma \vdash e : \tau \ ; \ \psi_+ \mid \psi_- \ ; \ o \qquad \Gamma, \psi_+ \vdash \psi'_+ \quad \Gamma, \psi_- \vdash \psi'_- \qquad \tau <: \tau' \quad o <: o'}{\Gamma \vdash e : \tau' \ ; \ \psi'_+ \mid \psi'_- \ ; \ o'}$$

Figure 2: Typing Rules

$$\textsc{S-Refl}\quad \vdash \tau <: \tau$$

$$\textsc{S-Top}\quad \vdash \tau <: \top$$

$$\textsc{S-UnionSuper}\quad \frac{\exists \sigma \in \overrightarrow{\sigma}, \vdash \tau <: \sigma}{\vdash \tau <: (\bigcup \overrightarrow{\sigma})}$$

$$\textsc{S-UnionSub}\quad \frac{\forall \tau \in \overrightarrow{\tau}, \vdash \tau <: \sigma}{\vdash (\bigcup \overrightarrow{\tau}) <: \sigma}$$

$$\textsc{S-Pair}\quad \frac{\vdash \sigma_1 <: \tau_1 \qquad \vdash \sigma_2 <: \tau_2}{\vdash \langle \sigma_1, \sigma_2 \rangle <: \langle \tau_1, \tau_2 \rangle}$$

$$\textsc{S-Fun}\quad \frac{\begin{array}{c}\vdash \sigma' <: \sigma \qquad \vdash \tau <: \tau' \\ \psi_+ \vdash \psi'_+ \qquad \psi_- \vdash \psi'_- \qquad \vdash o <: o'\end{array}}{\vdash x{:}\sigma \xrightarrow[o]{\psi_+|\psi_-} \tau <: x{:}\sigma' \xrightarrow[o']{\psi'_+|\psi'_-} \tau'}$$

$$\textsc{S-Vec}\quad \frac{\forall i, \vdash \sigma_i <: \tau_i}{\vdash [[\sigma_o \ldots \sigma_i]] <: [[\tau_o \ldots \tau_i]]}$$

$$\textsc{S-Ineq}\quad \frac{\text{unsatisfiable}(\Phi \bowtie \overline{\Phi'})}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \{\diamond : \mathbf{N} \mid \Phi'\}}$$

$$\textsc{S-IneqNum}\quad \frac{\vdash \mathbf{N} <: \tau}{\vdash \{\diamond : \mathbf{N} \mid \Phi\} <: \tau}$$

Figure 3: Subtyping Rules

$$\textsc{L-Atom}\quad \frac{\psi \in \Gamma}{\Gamma \vdash \psi}$$

$$\textsc{L-True}\quad \Gamma \vdash \mathbb{T}$$

$$\textsc{L-False}\quad \frac{\Gamma \vdash \mathbb{F}}{\Gamma \vdash \psi}$$

$$\textsc{L-AndI}\quad \frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}$$

$$\textsc{L-AndE}\quad \frac{\Gamma, \psi_1, \psi_2 \vdash \psi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \psi}$$

$$\textsc{L-ImpI}\quad \frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \supset \psi_2}$$

$$\textsc{L-ImpE}\quad \frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_1 \supset \psi_2}{\Gamma \vdash \psi_2}$$

$$\textsc{L-OrI}\quad \frac{\Gamma \vdash \psi_1 \ \text{ or } \ \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \vee \psi_2}$$

$$\textsc{L-OrE}\quad \frac{\Gamma, \psi_1 \vdash \psi \qquad \Gamma, \psi_2 \vdash \psi}{\Gamma, \psi_1 \vee \psi_2 \vdash \psi}$$

$$\textsc{L-Sub}\quad \frac{\Gamma \vdash \tau_x \qquad \vdash \tau <: \sigma}{\Gamma \vdash \sigma_x}$$

$$\textsc{L-SubNot}\quad \frac{\Gamma \vdash \overline{\sigma}_x \qquad \vdash \tau <: \sigma}{\Gamma \vdash \overline{\tau}_x}$$

$$\textsc{L-Bot}\quad \frac{\Gamma \vdash \bot_x}{\Gamma \vdash \psi}$$

$$\textsc{L-Update}\quad \frac{\Gamma \vdash \tau_x \qquad \Gamma \vdash \nu_x}{\Gamma \vdash \text{update}(\tau, \nu)_x}$$

$$\textsc{L-Ineq}\quad \frac{\begin{array}{c}\Gamma \vdash \mathbf{N}_x \qquad \Gamma \vdash \Phi' \\ \text{unsatisfiable}(\Phi' \bowtie (\Phi[x/\diamond]))\end{array}}{\Gamma \vdash \{\diamond : \mathbf{N} \mid \Phi\}_x}$$

$$\textsc{L-IneqSimple}\quad \frac{\Gamma, \mathbf{N}_x, \Phi[x/\diamond] \vdash \psi}{\Gamma, \{\diamond : \mathbf{N} \mid \Phi\}_x \vdash \psi}$$

$$\textsc{L-IneqJoin}\quad \frac{\Gamma, (\Phi_1 \bowtie \Phi_2) \vdash \psi}{\Gamma, \Phi_1, \Phi_2 \vdash \psi}$$

$$\textsc{L-IneqUnsat}\quad \frac{\Gamma \vdash \Phi \qquad \text{unsatisfiable}(\Phi)}{\Gamma \vdash \psi}$$

Figure 4: Logic Rules

$$
\begin{array}{lll}
\psi_1 \mid \psi_2[o/x] & = \psi_1[o/x] \mid \psi_2[o/x] & \\
 & & \\
\nu_{\pi(x)}[\pi'(y)/x] & = (\nu[\pi'(y)/x])_{\pi(\pi'(y))} & \\
\nu_{\pi(x)}[\emptyset/x]^+ & = \mathbb{T} & \\
\nu_{\pi(x)}[\emptyset/x]^- & = \mathbb{F} & \\
\nu_{\pi(x)}[o/z] & = \nu_{\pi(x)} & x \neq z \text{ and } z \notin \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^+ & = \mathbb{T} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\nu_{\pi(x)}[o/z]^- & = \mathbb{F} & x \neq z \text{ and } z \in \mathrm{fv}(\nu) \\
\mathbb{T}[o/x] & = \mathbb{T} & \\
\mathbb{F}[o/x] & = \mathbb{F} & \\
(\psi_1 \supset \psi_2)[o/x]^+ & = \psi_1[o/x]^- \supset \psi_2[o/x]^+ & \\
(\psi_1 \supset \psi_2)[o/x]^- & = \psi_1[o/x]^+ \supset \psi_2[o/x]^- & \\
(\psi_1 \vee \psi_2)[o/x] & = \psi_1[o/x] \vee \psi_2[o/x] & \\
(\psi_1 \wedge \psi_2)[o/x] & = \psi_1[o/x] \wedge \psi_2[o/x] & \\
\Phi[o/x] & = \Phi & x \notin \Phi \\
\Phi[\pi'(y)/x] & = \forall \phi \in \Phi,\ \phi[x \mapsto \pi'(y)] & x \in \Phi \\
\Phi[\emptyset/x]^{\pm} & = \mathrm{FMelim}(\Phi,\ x,\ \pm) & x \in \Phi \\
 & & \\
\pi(x)[\pi'(y)/x] & = \pi(\pi(y)) & \\
\pi(x)[\emptyset/x] & = \emptyset & \\
\pi(x)[o/z] & = \pi(x) & x \neq z \\
\emptyset[o/x] & = \emptyset &
\end{array}
$$

Figure 5: Substitution