Suppose f is a real-valued function defined on $\mathbb{R}^n$. We say that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L,$$

and call L the **limit** of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$, if $f(\mathbf{x})$ becomes as close as we wish to L provided $\mathbf{x}$ is sufficiently close to, but not equal to, $\mathbf{a}$. More formally, the statement $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for any positive number ϵ there is a positive number δ so that

$$(1.6) \qquad |f(\mathbf{x}) - L| < \epsilon \text{ whenever } 0 < |\mathbf{x} - \mathbf{a}| < \delta.$$

This condition can be rephrased in terms of the individual components $x_j - a_j$ of $\mathbf{x} - \mathbf{a}$, as follows: $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if and only if for every positive number ϵ there is a positive number δ' so that

$$(1.7) \qquad |f(\mathbf{x}) - L| < \epsilon \text{ whenever } 0 < \max\big(|x_1 - a_1|, \ldots, |x_n - a_n|\big) < \delta'.$$

The equivalence of (1.6) and (1.7) follows from (1.3): If (1.6) is satisfied, then (1.7) is satisfied with $\delta' = \delta/\sqrt{n}$; and if (1.7) is satisfied, then (1.6) is satisfied with $\delta = \delta'$.

More generally, we can consider functions f that are only defined on a subset S of $\mathbb{R}^n$ and points $\mathbf{a}$ that lie in the closure of S. The definition of $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ is the same as before except that $\mathbf{x}$ is restricted to lie in the set S. It may be necessary, for the sake of clarity, to specify this restriction explicitly; for this purpose we use the notation

$$\lim_{\mathbf{x}\to\mathbf{a},\ \mathbf{x}\in S} f(\mathbf{x}).$$

In particular, for a function f on the real line we often need to consider the **one-sided limits**

$$\lim_{x\to a+} f(x) = \lim_{x\to a,\ x>a} f(x) \quad \text{and} \quad \lim_{x\to a-} f(x) = \lim_{x\to a,\ x<a} f(x).$$

For example, let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x + 1$ for $|x| \le 1$ and $f(x) = 0$ for $|x| > 1$. Then $\lim_{x\to 1} f(x)$ does not exist, but $\lim_{x\to 1-} f(x) = 2$ and $\lim_{x\to 1+} f(x) = 0$.

Notice that the definition of $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ does not involve the value $f(\mathbf{a})$ at all; only the values of f at points near $\mathbf{a}$ but unequal to $\mathbf{a}$ are relevant. Indeed, f need not even be defined at $\mathbf{a}$ — a situation that arises, for example, in the limits that define derivatives. On the other hand, if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ and $f(\mathbf{a})$ both exist and are equal, that is, if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}),$$

then f is said to be **continuous at** $\mathbf{a}$.

If f is continuous at every point of a set $U \subset \mathbb{R}^n$, f is said to be **continuous on** U. Going back to the condition (1.6) that defines limits, we see that the continuity of f on U is equivalent to the following condition: For every positive number ϵ and every $\mathbf{a} \in U$ there is a positive number δ so that

$$|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon \text{ whenever } |\mathbf{x} - \mathbf{a}| < \delta. \tag{1.8}$$

Informally speaking, f is continuous if changing the input values by a small amount changes the output values by only a small amount.

The same definitions apply equally well to vector-valued functions, that is, functions $\mathbf{f}$ with values in $\mathbb{R}^k$ for some $k > 1$. In this case the limit $\mathbf{L}$ is an element of $\mathbb{R}^k$, and $|\mathbf{f}(\mathbf{x}) - \mathbf{L}|$ is the norm of the vector $\mathbf{f}(\mathbf{x}) - \mathbf{L}$. In view of (1.3), it is clear that

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \quad \iff \quad \lim_{\mathbf{x}\to\mathbf{a}} f_j(\mathbf{x}) = L_j \text{ for } j = 1, \dots, k.$$

Thus the study of limits and continuity of vector-valued functions is easily reduced to the scalar case, to which we now return out attention.

We often express the relation $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ informally by saying that $f(\mathbf{x})$ approaches L as $\mathbf{x}$ approaches $\mathbf{a}$. In one dimension this works quite well; we can envision $\mathbf{x}$ as the location of a particle that moves toward $\mathbf{a}$ from the right or the left. But in higher dimensions there are infinitely many different paths along which a particle might move toward $\mathbf{a}$, and for the limit to exist one must get the same result no matter which path is chosen. It is safer to abandon the "dynamic" picture of a particle moving toward $\mathbf{a}$; we should simply think in terms of $f(\mathbf{x})$ being close to L provided that $\mathbf{x}$ is close to $\mathbf{a}$, without reference to any motion.

EXAMPLE 1. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$, and let $f(0, 0) = 0$. Show that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist — and, in particular, f is discontinuous at $(0, 0)$.

Solution. First, note that $f(x, 0) = f(0, y) = 0$ for all x and y, so $f(x, y) \to 0$ as (x, y) approaches $(0, 0)$ along the x-axis or the y-axis. But if we consider other straight lines passing through the origin, say $y = cx$, we have $f(x, cx) = cx^2/(x^2 + c^2x^2) = c/(1 + c^2)$, so the limit as (x, y) approaches $(0, 0)$ along the line $y = cx$ is $c/(1 + c^2)$. Depending on the value of c, this can be anything between $-\frac{1}{2}$ and $\frac{1}{2}$ (these two extreme values being achieved when $c = -1$ or $c = 1$). So there is no limit as (x, y) approaches $(0, 0)$ unrestrictedly.

The argument just given suggests the following line of thought. We wish to know if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists. We look at all the straight lines passing through $\mathbf{a}$ and evaluate the limit of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ along each of those lines by one-variable techniques; if we always get the same answer L, then we should have $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, right? Unfortunately, this doesn't work:

EXAMPLE 2. Let $g(x,y) = \dfrac{x^2y}{x^4+y^2}$ if $(x,y) \neq (0,0)$ and $g(0,0) = 0$. Again we have $g(x,0) = g(0,y) = 0$, so the limit as $(x,y) \to (0,0)$ along the coordinate axes is 0. Moreover, if $c \neq 0$,

$$g(x,cx) = \frac{cx^4}{x^4+c^2x^2} = \frac{cx}{c^2+x^2} \to 0 \text{ as } x \to 0,$$

so the limit as $(x,y) \to (0,0)$ along any other straight line is also 0. But if we approach along a *parabola* $y = cx^2$, we get

$$g(x,cx^2) = \frac{cx^3}{x^4+c^2x^4} = \frac{c}{1+c^2},$$

which can be anything between $-\frac{1}{2}$ and $\frac{1}{2}$ as before, so the limit does not exist. (The similarity with Example 1 is not accidental: If f is the function in Example 1 we have $g(x,y) = f(x^2,y)$.)

After looking at examples like this one, one might become discouraged about the possibility of ever proving that limits *do* exist! But things are not so bad. If f is a *continuous* function, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ is simply $f(\mathbf{a})$. Moreover, most of the functions of several variables that one can easily write down are built up from continuous functions of one variable by using the arithmetic operations plus composition, and these operations all preserve continuity (except for division when the denominator vanishes).

Here are the precise statements and proofs of the fundamental results. (The reader may wish to skip the proofs; they are of some value as illustrations of the sort of formal arguments involving limits that are important in more advanced analysis, but they contribute little to an intuitive understanding of the results.)

1.9 Theorem. *Suppose* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous on* $U \subset \mathbb{R}^n$ *and* $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^k$ *is continuous on* $\mathbf{f}(U) \subset \mathbb{R}^m$. *Then the composite function* $\mathbf{g} \circ \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ *is continuous on* U.

Proof. Let $\epsilon > 0$ and $\mathbf{a} \in U$ be given, and let $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Since $\mathbf{g}$ is continuous on $\mathbf{f}(U)$, we can choose $\eta > 0$ so that $|\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{b})| < \epsilon$ whenever $|\mathbf{y} - \mathbf{b}| < \eta$. Having chosen this η, since $\mathbf{f}$ is continuous on U we can find $\delta > 0$ so that $|\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \eta$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. Thus,

$$|\mathbf{x} - \mathbf{a}| < \delta \quad \Longrightarrow \quad |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \eta \quad \Longrightarrow \quad |\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{g}(\mathbf{f}(\mathbf{a}))| < \epsilon,$$

which says that $\mathbf{g} \circ \mathbf{f}$ is continuous on U. □

1.10 Theorem. *Let $f_1(x, y) = x + y$, $f_2(x, y) = xy$, and $g(x) = 1/x$. Then f_1 and f_2 are continuous on $\mathbb{R}^2$ and g is continuous on $\mathbb{R} \setminus \{0\}$.*

Proof. To prove continuity of f_1 and f_2, we need to show that $\lim_{(x,y)\to(a,b)} x+y = a + b$ and $\lim_{(x,y)\to(a,b)} xy = ab$ for every $a, b \in \mathbb{R}$. That is, given $\epsilon > 0$ and $a, b \in \mathbb{R}$, we need to find $\delta > 0$ so that if $|x - a| < \delta$ and $|y - b| < \delta$, then (i) $|(x + y) - (a + b)| < \epsilon$ or (ii) $|xy - ab| < \epsilon$. For (i) we can simply take $\delta = \frac{1}{2}\epsilon$, for if $|x - a| < \frac{1}{2}\epsilon$ and $|y - b| < \frac{1}{2}\epsilon$, then

$$|(x + y) - (a + b)| = |(x - a) + (y - b)| \le |x - a| + |y - b| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

For (ii) we observe that $xy - ab = (x - a)y + a(y - b)$, so we can make $xy - ab$ small by making the two terms on the right small. Indeed, let

$$\delta = \min\left(1,\ \frac{\epsilon}{2(|a| + 1)},\ \frac{\epsilon}{2(|b| + 1)}\right).$$

If $|x - a| < \delta$ and $|y - b| < \delta$, then $|y| < |b| + \delta \le |b| + 1$, so

$$\begin{aligned} |xy - ab| &\le |x - a||y| + |a||y - b| \\ &\le \frac{\epsilon}{2(|b| + 1)}(|b| + 1) + |a|\frac{\epsilon}{2(|a| + 1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves the continuity of f_1 and f_2. As for g, to show that $\lim_{x\to a} 1/x = 1/a$ for $a \neq 0$, we observe that

$$\frac{1}{x} - \frac{1}{a} = \frac{a - x}{ax}.$$

Given $\epsilon > 0$, let δ be the smaller of the numbers $\frac{1}{2}|a|$ and $\frac{1}{2}\epsilon a^2$. If $|x - a| < \delta$, then $|a| \le |a - x| + |x| < \frac{1}{2}|a| + |x|$ and hence $|x| > \frac{1}{2}|a|$, so

$$\left|\frac{x - a}{ax}\right| < \left|\frac{\epsilon a^2}{2ax}\right| = \epsilon \left|\frac{a}{2x}\right| < \epsilon,$$

as desired. □

1.11 Corollary. *The function $f_3(x, y) = x - y$ is continuous on $\mathbb{R}^2$, and the function $f_4(x, y) = x/y$ is continuous on $\{(x, y) : y \neq 0\}$.*

Proof. With notation as in Theorem 1.10, we have $f_4(x,y) = f_2(x, g(y))$, so f_4 is the composition of continuous mappings and hence is continuous on the set where $y \neq 0$. Likewise, $f_3(x,y) = f_1(x, f_2(-1,y))$, so f_3 is continuous. (Alternatively, continuity for f_3 may be proved in exactly the same way as for f_1.) $\square$

1.12 Corollary. *The sum, product, or difference of two continuous functions is continuous; the quotient of two continuous functions is continuous on the set where the denominator is nonzero.*

Proof. Combine Theorem 1.10 and Corollary 1.11 with Theorem 1.9. For example, if f and g are continuous functions on $U \subset \mathbb{R}^n$, then $f+g$ is continuous because it is the composition of the continuous map (f,g) from U to $\mathbb{R}^2$ and the continuous map $(x,y) \mapsto x+y$ from $\mathbb{R}^2$ to $\mathbb{R}$. Likewise for the other arithmetic operations. $\square$

The elementary functions of a single variable (polynomials, trig functions, exponential functions, etc.) are all continuous on their domains of definition, and elementary functions of several variables are generally built up out of functions of one variable by the arithmetic operations and composition. The preceding results therefore allow the continuity of such functions to be established almost immediately in most cases. For example, the function $\varphi(x,y) = \dfrac{\sin(3x+2y)}{x^2 - y}$ is continuous everywhere except along the parabola $y = x^2$, because it is built up from the continuous functions of one variable $3x$, $2y$, x^2, and $-y$ by taking sums ($3x + 2y$ and $x^2 - y$), composing with the sine function ($\sin(3x+2y)$), and then taking a quotient. For another example, the function $\psi(x,y) = x^y$, defined on the region where $x > 0$, is continuous there, because it can be rewritten as $\psi(x,y) = e^{y \log x}$, which is assembled from the (continuous) exponential and logarithmic functions and the operation of multiplication ($y \cdot \log x$). Similarly, the functions in Examples 1 and 2 are continuous everywhere except at the origin.

Let us look at one more example:

EXAMPLE 3. Let $h(x,y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ for $(x,y) \neq (0,0)$ and $h(0,0) = 0$. Evaluate $\lim_{(x,y)\to(2,3)} h(x,y)$ and $\lim_{(x,y)\to(0,0)} h(x,y)$. Is h continuous at $(0,0)$?

Solution. The first limit is easy: Clearly h is continuous everywhere except at the origin, so $\lim_{(x,y)\to(2,3)} h(x,y) = h(2,3) = 6(4-9)/(4+9) = -\frac{30}{13}$. The behavior of h at the origin requires a closer examination. Since $h(x,0) = 0$ for all x, if the limit exists it must equal 0. Experimentation with lines and parabolas as in Examples 1 and 2 fails to yield any evidence to the contrary.

In fact, the limit *is* 0, and this can be established with a little ad hoc estimating. Clearly $|x^2 - y^2| \le x^2 + y^2$, so $|h(x,y)| \le |xy|$. But $xy \to 0$ as $(x,y) \to (0,0)$, so $h(x,y)$, being even smaller in absolute value than xy, must also approach 0. Thus $\lim_{(x,y)\to(0,0)} h(x,y) = 0$ and h is continuous at $(0,0)$.

We now establish the relation between inequalities on continuous functions and open and closed sets that was mentioned at the end of the preceding section.

1.13 Theorem. *Suppose* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ *is continuous and* U *is a subset of* $\mathbb{R}^k$*, and let* $S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}) \in U\}$*. Then* S *is open if* U *is open, and* S *is closed if* U *is closed.*

Proof. Suppose U is open. We shall show that S is open by showing that every point $\mathbf{a}$ in S is an interior point of S. If $\mathbf{a} \in S$, then $f(\mathbf{a}) \in U$. Since U is open, some ball centered at $f(\mathbf{a})$ is contained in U; that is, there is a positive number ϵ such that every $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{f}(\mathbf{a})| < \epsilon$ is in U. Since f is continuous, there is a positive number δ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. But this means that $f(\mathbf{x}) \in U$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$, that is, $\mathbf{x} \in S$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. Thus $\mathbf{a}$ is an interior point of S.

On the other hand, suppose U is closed. Then the complement of U in $\mathbb{R}$ is open by Proposition 1.4b, so the set $S' = \{\mathbf{x} : f(\mathbf{x}) \in U^c\}$ is open by the argument just given. But S' is just the complement of S in $\mathbb{R}^n$, so S is closed by Proposition 1.4b again. □

The result about the openness or closedness of sets defined by inequalities or equations at the end of §1.2 is a corollary of Theorem 1.13. For example, if $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, the set $\{\mathbf{x} : f(\mathbf{x}) > 0\}$ (resp.[2] $\{\mathbf{x} : f(\mathbf{x}) = 0\}$) is of the form $\{\mathbf{x} : f(\mathbf{x}) \in U\}$ where $U = (0, \infty)$ (resp. $U = \{0\}$), and this U is open (resp. closed).

Theorem 1.13 can be generalized to functions that are only defined on subsets of $\mathbb{R}^n$; with notation as above, the correct statement is that if U is open (resp. closed) then S is the intersection of the domain of $\mathbf{f}$ with an open (resp. closed) set. (For example, the set $\{x \in \mathbb{R} : \log x \le 0\}$, namely $(0, 1]$, is the intersection of the domain of log, namely $(0, \infty)$, with the closed set $[0, 1]$. On the other hand, the set $\{x \in \mathbb{R} : \sqrt{x} < 1\}$, namely $[0, 1)$, is the intersection of the domain of the square root function, namely $[0, \infty)$, with the open set $(-1, 1)$.) In particular, if U and the domain of $\mathbf{f}$ are both open (resp. closed), then so is S.

The converse of Theorem 1.13 is also true; see Exercise 8.

[2]"resp." is an abbreviation for "respectively."