Data Structures

Gabriel Istrate

May 14, 2025

Where we are

Why did we want dynamic sets to start with ?

- To improve algorithms.
- First part of today: complete Computational geometry

Second part: Augmenting Data Structures (e.g. Red-Black Trees)

- What happens if a data structure doesn't support all the operations you want ?
- Augment it: modify it to support the new operations.
- Might need to add additional fields. These need to be maintained.

- Convex hull of a set of points: smallest convex polygon that contains the set of points.
- place elastic rubber band around set of points and let it shrink.
- Two algorithms: Graham's Scan $O(n \log n)$.
- Jarvis's March $O(n \cdot h)$, h the number of points on the convex hull.
- Other algorithms:
- Incremental: points sorted from left to right forming sequence $p_1, \ldots, p_n$. At stage i add p_i to convex hull $CH(p_1, \ldots, p_{i-1})$, forming $CH(p_1, \ldots, p_i)$.
- Divide-and-conquer: divide into leftmost n/2 points and rightmost n/2 points. Compute convex hulls and combine them.
- Prune-and-search method.

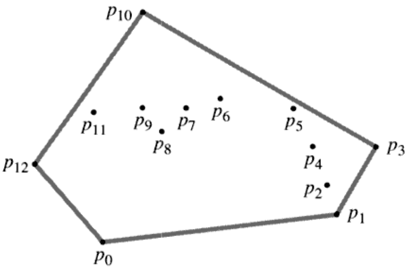


Figure 33.6 A set of points $Q = \{p_0, p_1, \ldots, p_{12}\}$ with its convex hull $\mathrm{CH}(Q)$ in gray.

Graham's scan

- Maintains a stack S of candidate points.
- Each point of Q is pushed onto the stack.
- Points not in $CH(Q)$ eventually popped from the stack.
- $TOP(S)$, $NEXT-TO-TOP(S)$: stack functions, do not change its contents.
- Stack returned by the algorithm: points of $CH(Q)$ in counterclockwise order.

Convex hull algorithm

GRAHAM-SCAN(Q)

let p_0 be the point in Q with the minimum y-coordinate, or the leftmost such point in case of a tie
let $\langle p_1, p_2, \ldots, p_m \rangle$ be the remaining points in Q, sorted by polar angle in counterclockwise order around p_0 (if more than one point has the same angle, remove all but the one that is farthest from p_0)
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i \leftarrow 3$ **to** m
  do while the angle formed by points NEXT-TO-TOP(S), TOP(S), and p_i makes a nonleft turn
    do POP(S)
  PUSH(p_i, S)
return S

Graham's Scan:Example

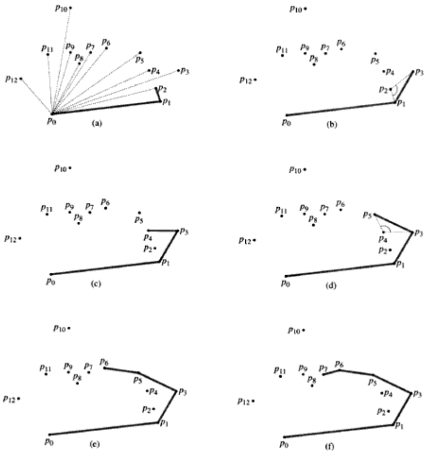
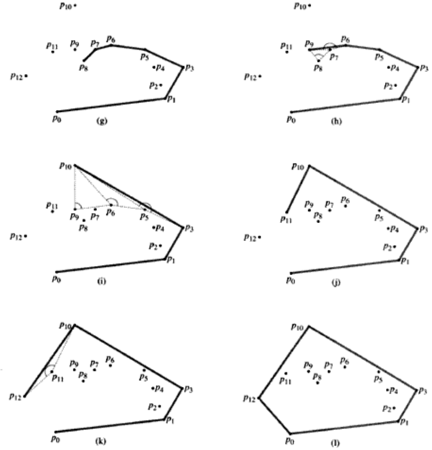


Figure 33.7 The execution of GRAHAM-SCAN on the set Q of Figure 33.6. The current convex hull contained in stack S is shown in gray at each step. **(a)** The sequence $\langle p_1, p_2, \ldots, p_{12}\rangle$ of points numbered in order of increasing polar angle relative to p_0, and the initial stack S containing p_0, p_1, and p_2. **(b)–(k)** Stack S after each iteration of the **for** loop of lines 6–9. Dashed lines show nonleft turns, which cause points to be popped from the stack. In part (h), for example, the right turn at angle $\angle p_7 p_8 p_9$ causes p_8 to be popped, and then the right turn at angle $\angle p_6 p_7 p_9$ causes p_7 to be popped. **(l)** The convex hull returned by the procedure, which matches that of Figure 33.6.

Graham’s Scan:Example



Graham's Scan: Correctness and Performance

- Invariant: at the beginning of each iteration of the for loop stack S contains (from bottom to top) exactly the vertices of $CH(Q_{i-1})$ in counterclockwise order.
- Line 1: $\theta(n)$ time.
- Sorting $\theta(n \log n)$ time.
- Testing for left/right turn: vector product $\theta(1)$ time.
- The rest of the algorithm $O(n)$ time.

Graham's Scan: Correctness

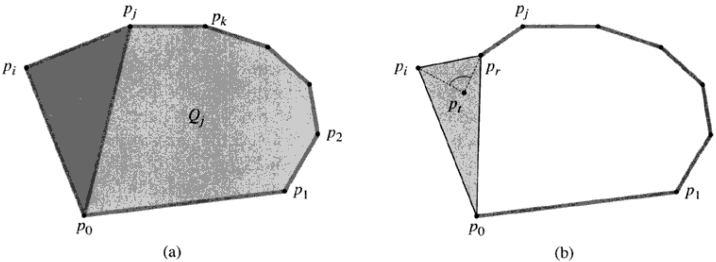


Figure 33.8 The proof of correctness of GRAHAM-SCAN. **(a)** Because p_i's polar angle relative to p_0 is greater than p_j's polar angle, and because the angle $\angle p_k p_j p_i$ makes a left turn, adding p_i to $\text{CH}(Q_j)$ gives exactly the vertices of $\text{CH}(Q_j \cup \{p_i\})$. **(b)** If the angle $\angle p_r p_t p_i$ makes a nonleft turn, then p_t is either in the interior of the triangle formed by p_0, p_r, and p_i or on a side of the triangle, and it cannot be a vertex of $\text{CH}(Q_i)$.

Jarvis's March

- uses a technique known as gift wrapping.
- Simulates wrapping a piece of paper around set Q.
- Start at the same point p_0 as in Graham's scan.
- Pull the paper to the right, then higher until it touches a point. This point is a vertex in the convex hull. Continue this way until we come back to p_0.
- Formally: start at p_0. Choose p_1 as the point with the smallest polar angle from p_0. Choose p_2 as the point with the smallest polar angle from $p_1 \ldots$
- $\ldots$ until we reached the highest point p_k.
- We have constructed the right chain.
- Construct the left chain by starting from p_k and measuring polar angles with respect to the negative x-axis.

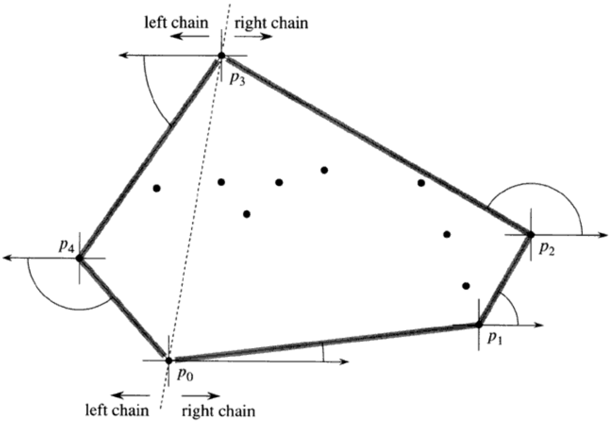


Figure 33.9 The operation of Jarvis's march. The first vertex chosen is the lowest point p_0. The next vertex, p_1, has the smallest polar angle of any point with respect to p_0. Then, p_2 has the smallest polar angle with respect to p_1. The right chain goes as high as the highest point p_3. Then, the left chain is constructed by finding smallest polar angles with respect to the negative x-axis.

Augmenting Data Structures

- What if no existing data structure fits your needs ?
- Invent a new one, or ...
- More realistic (in practice): slightly modify a "standard" data structure to support more operations.
- Done by storing extra information in it
- Not always straightforward: new information **must be updated and maintained** by D.S. operations.

Augmenting Data Structures

Example: two data structures obtained by modifying red-black trees

- First data structure: supports order statistics queries on a dynamic set.
  - $\boxed{\text{Find } i\text{'th number in a set or the rank of an element}}$.
- Second data structure: maintain a set of intervals (e.g. time intervals).
- Plus: a general result about augmenting Data Structures.

Dynamic order statistics

- Order statistic tree: red-black tree with one extra field per node: size of the subtree rooted at that node.
- Thus fields: *key*, *color*, *p*, *left*, *right*, *size*.
- $size[nil[T]] = 0$.
- $size[x] = size[left[x]] + size[right[x]] + 1$.
- Supports $OS-SELECT(x, i)$: return i'th smallest element in the tree rooted at x. $O(\log n)$ time.
- Supports $OS-RANK(T, x)$: return the rank of x in the tree T. $O(\log n)$ time.

Order statistics tree

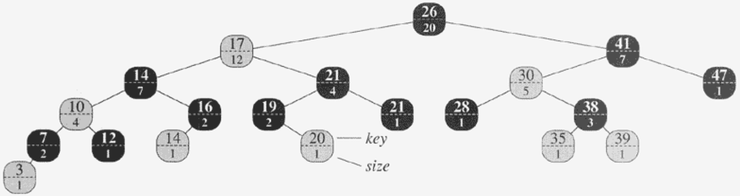


Figure 14.1 An order-statistic tree, which is an augmented red-black tree. Shaded nodes are red, and darkened nodes are black. In addition to its usual fields, each node x has a field $size[x]$, which is the number of nodes in the subtree rooted at x.

Selecting i'th element

- If $i = size[left(x)] + 1$ then (by BST property) node x is the i'th element. Return x.
- If $i \leq size[left(x)]$ then node is in $left[x]$. i'th element. Call procedure recursively.
- If $i > size[left(x)] + 1$ then node is in $right[x]$. $i - size[left(x)]$'th element. Call procedure recursively.
- Running time: proportional to the height of the tree: $O(\log n)$.

Selecting *i*'th element

```
OS-SELECT(x, i)
r ← size[left[x]]+1
if i = r
    then return x
elseif i < r
    then return OS-SELECT(left[x], i)
else return OS-SELECT(right[x], i − r)
```

Rank of an element

```
OS-RANK(T, x)
r ← size[left[x]] + 1
y ← x
while y ≠ root[T]
      do if y = right[p[y]]
            then r ← r + size[left[p[y]]] + 1
         y ← p[y]
return r
```

Rank of an element

- Perform inorder traversal.
- Return rank of node x in this traversal.
- Move pointer y from x up towards $root(T)$.
- Maintains the following invariant: at the start of each iteration of the while loop, r is the rank of $key[x]$ in the subtree rooted at y.
- If y is a right child, add the size of its left child to the count.
- Each iteration: $O(1)$ time. y goes up the tree, time complexity $O(\log n)$.

Maintaining subtree sizes: Insertion.

- During LEFT/RIGHT rotations.
- INSERTION. First phase: go from the root to the frontier, inserting the new node as the child of an existing node. new node gets size of 1. Each node from x to the path: size increases by 1. $O(\log n)$.
- Second phase: go up the tree, changing colors, and maintaining the red-black property by rotations.
- Second phase: changes via LEFT/RIGHT rotations.
- LEFT-ROTATE: add lines

- $size[y] \leftarrow size[x]$.
- $size[x] \leftarrow size[left[x]] + size[right[x]] + 1$.

- to rotation pseudocode.
- RIGHT-ROTATE: symmetric.

Maintaining *size* during rotations.

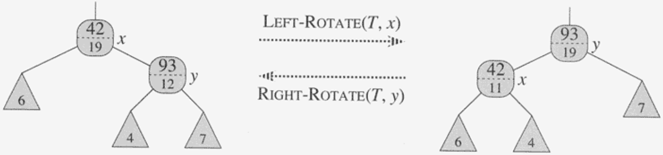


Figure 14.2 Updating subtree sizes during rotations. The link around which the rotation is performed is incident on the two nodes whose *size* fields need to be updated. The updates are local, requiring only the *size* information stored in x, y, and the roots of the subtrees shown as triangles.

Maintaining subtree sizes: Deletion.

- DELETION: two phases.
- First phase: delete node. Update tree size on the path from the node to the top. Decrement by 1 for each node.
- Rotations: as for insertion.

How to augment a data structure

- Four steps:
- 1. Choose underlying data structure.
- 2. Determine additional information to be maintained.
- 3. Verify that additional information can be maintained in the D.S. operations.
- 4. develop new operations required by new fields.

How to augment a data structure (II)

1. Choose red-black trees. Clue: supports other dynamic set operations on total order: MINIMUM, MAXIMUM, SUCCESSOR, PREDECESSOR.
2. We didn't need field size to implement OS-SELECT, OS-RANK, but then operations wouldn't run in $O(\log n)$ time. Additional information to be maintained: sometimes pointer rather than data.
3. Ideally only a few elements need to be updated to maintain D.S. E.g. if we simply stored in each node it rank in the tree then OS-SELECT and OS-UPDATE would be efficient but inserting a smallest node causes changes in the whole tree.
4. Developed OS-SELECT, OS-RANK. Occasionally, instead of new operations, speed-up old ones.

Augmenting red-black trees

Theorem

Let f be a field that augments a RB tree of n nodes, and suppose the contents of f for node x can be computed in $O(1)$ using only information in node x, $left[x]$ and $right[x]$, including $f[left[x]]$ and $f[right[x]]$. Then we can maintain the values of f in all nodes in T during insertion and deletion without asymptotically affecting $O(\log n)$ performance.

Proof idea: change in field f at a node x propagates only to ancestors of x in the tree.

Interval trees

- closed interval: $[t_1, t_2]$. Also open, half-open intervals.
- $i = [t_1, t_2]$. $low[i] = t_1$, $high[i] = t_2$.
- i and i' overlap if $i \bigcap i' \neq \emptyset$. That is $low[i] \leq high[i']$ and $low[i'] \leq high[i]$.
- Want: Data structure representing a dynamic set of intervals.
- Must support the following operations:
- $INTERVAL - INSERT(T, x)$: adds element x, whose *int* field contains an interval.
- $INTERVAL - DELETE(T, x)$: removes element x from T.
- $INTERVAL - SEARCH(T, i)$: return pointer to an element x such that $int[x]$ overlaps i, or *nil* if no such element found.

Intervals

- Any two intervals satisfy interval trichotomy: three alternatives:
  1. i and i' overlap.
  2. i is to the left of i' ($high[i] < low[i']$).
  3. i is to the right of i'.($low[i] > high[i']$).

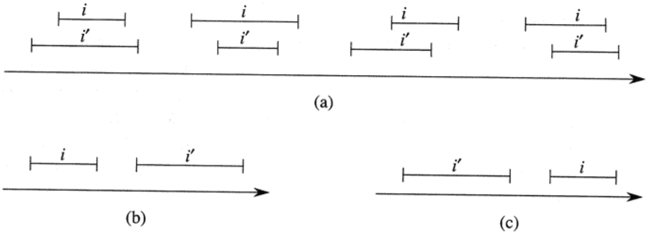


Figure 14.3 The interval trichotomy for two closed intervals i and i'. **(a)** If i and i' overlap, there are four situations; in each, $low[i] \le high[i']$ and $low[i'] \le high[i]$. **(b)** The intervals do not overlap, and $high[i] < low[i']$. **(c)** The intervals do not overlap, and $high[i'] < low[i]$.

Interval trees: Implementation

1. Possible clue: intervals (partial) ordering. Might try to modify a total order. Then red-black tree. Each node x stores an interval $int[x]$.

 - $key[x] = low[int[x]]$.

2. Additional info: $max[x]$, the maximum value of any endpoint of an interval stored in the subtree rooted at x.
3. Maintain info: $max[x] = \max(high[int[x]], max[left[x]], max[right[x]])$.
4. By applying previous theorem: insertion/deletion $O(\log n)$ while maintaining $max[x]$.

Interval tree

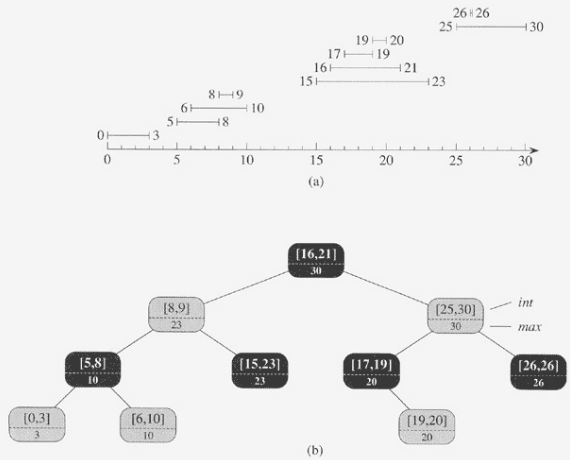


Figure 14.4 An interval tree. **(a)** A set of 10 intervals, shown sorted bottom to top by left endpoint. **(b)** The interval tree that represents them. An inorder tree walk of the tree lists the nodes in sorted order by left endpoint.

INTERVAL-SEARCH

- finds a node in tree T whose interval overlaps interval i, returns sentinel node $nil[T]$ if no overlapping interval found.
- Search starts at the root and proceeds downwards.
- Chooses *left* or *right* subtree based on the maximum element in the left subtree of x.
- If $max[left[x]]$ is $\geq low[i]$ (of course, $left[x] \neq nil[T]$) go left.
- otherwise go right.
- takes $O(\log n)$ time since each basic loop takes $O(1)$ time and the height of the RB tree is $O(\log n)$.

INTERVAL-SEARCH

```
INTERVAL-SEARCH(T, i)
x ← root[T]
while x ≠ nil[T] and i does not overlap int[x]
      do if left[x] ≠ nil[T] and max[left[x]] ≥ low[i]
            then x ← left[x]
            else x ← right[x]
return x
```

Correctness of INTERVAL-SEARCH

- Why is it enough to examine a single path ?
- Idea: search proceeds in a "safe direction".
- INVARIANT: If tree T contains an interval that overlaps i then there is such an interval in the subtree rooted at x.
- Initialization: clearly satisfied, $x = root[T]$.
- Either line 4 or line 5 executed.
- Line 5 executed: because $left[x] = nil[T]$ or $max[left[x]] < low[i]$. The subtree rooted at left[x] does not contain any interval that overlaps i.
- If such an interval is found in T, it must be in $right[x]$.

Correctness of INTERVAL-SEARCH

- Line 4 executed: contrapositive of loop invariant holds.
- If there is no such an interval in the subtree rooted at $left[x]$ then there is no such interval in tree T.
- Since line 4 executed $max[left[x]] \geq low[i]$. There exists i' with $high[i'] = max[left[x]] \geq low[i]$.
- i and i' do not overlap, by assumption. By trichotomy $high[i] < low[i']$.
- i'' interval in $right[x]$. Intervals keyed on the low endpoints.
- $high[i] < low[i'] \leq low[i'']$.
- Conclusion: no interval in $right[x]$ (and thus in T) overlaps i.

Next: B-Trees

Outline:

- Search in secondary storage
- B-Trees
  - properties
  - search
  - insertion

Complexity Model

Complexity Model

- Basic assumption so far: *data structures fit completely in main memory (RAM)*
  - all basic operations have the same cost
    - even this is a rough approximation, since the main-memory system is not at all "flat"

Complexity Model

- Basic assumption so far: *data structures fit completely in main memory (RAM)*
  - all basic operations have the same cost
    - even this is a rough approximation, since the main-memory system is not at all "flat"
- However, some applications require more storage than what fits in main memory
  - we must use data structures that reside in *secondary storage* (i.e., disk)

Complexity Model

- Basic assumption so far: *data structures fit completely in main memory (RAM)*
  - all basic operations have the same cost
    - even this is a rough approximation, since the main-memory system is not at all "flat"
- However, some applications require more storage than what fits in main memory
  - we must use data structures that reside in *secondary storage* (i.e., disk)

Disk is 10,000–100,000 times slower than RAM

Idea

Idea

- In a balanced *binary* tree, n keys require a tree of height $h = \lfloor \log_2 n \rfloor$
  - all the important operations require access to $O(h)$ nodes
  - each one accounting for *one or very few* basic operations

Idea

- In a balanced *binary* tree, n keys require a tree of height $h = \lfloor \log_2 n \rfloor$
  - all the important operations require access to $O(h)$ nodes
  - each one accounting for *one or very few* basic operations

- **Idea:** store several keys and pointers to children nodes in a single node

Idea

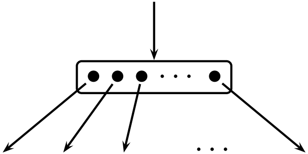
- In a balanced *binary* tree, n keys require a tree of height $h = \lfloor \log_2 n \rfloor$
  - all the important operations require access to $O(h)$ nodes
  - each one accounting for *one or very few* basic operations

- **Idea:** store several keys and pointers to children nodes in a single node
  - in practice we ***increase the degree*** (or *branching factor*) of each node up to $d > 2$, so $h = \lfloor \log_d n \rfloor$
    - in practice d can be as high as a few thousands

Idea

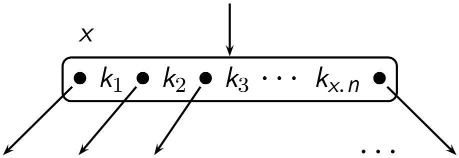
- In a balanced *binary* tree, n keys require a tree of height $h = \lfloor \log_2 n \rfloor$
  - all the important operations require access to $O(h)$ nodes
  - each one accounting for *one or very few* basic operations

- **Idea:** store several keys and pointers to children nodes in a single node
  - in practice we ***increase the degree*** (or *branching factor*) of each node up to $d > 2$, so $h = \lfloor \log_d n \rfloor$
    - in practice d can be as high as a few thousands

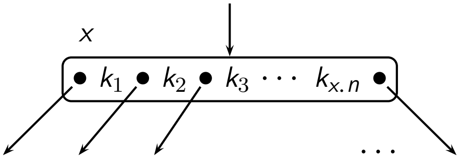


E.g., if $d = 1000$, then
only three accesses $(h = 2)$
cover ***up to one billion keys***

Definition of a B-Tree

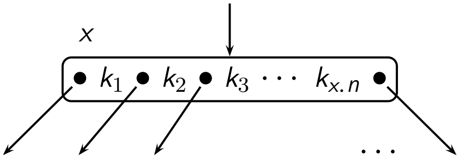


Definition of a B-Tree



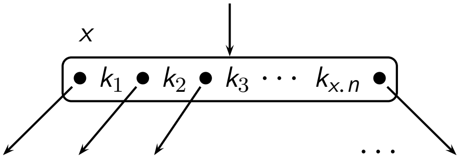
- Every node x has the following fields
  - $x.n$ is the number of keys stored at each node

Definition of a B-Tree



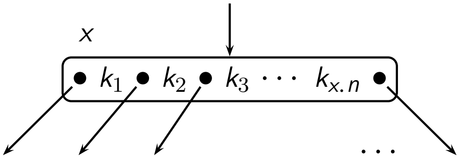
- Every node x has the following fields
  - $x.n$ is the number of keys stored at each node
  - $x.key[1] \le x.key[2] \le \ldots x.key[x.n]$ are the $x.n$ keys stored in nondecreasing order

Definition of a B-Tree



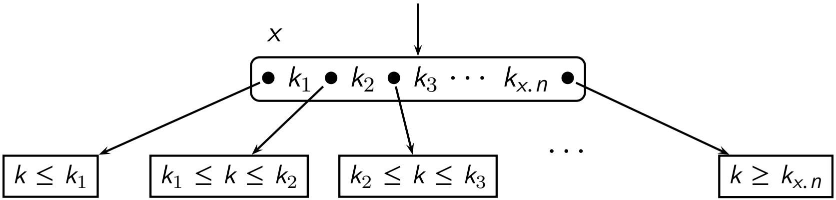
- Every node x has the following fields
  - $x.n$ is the number of keys stored at each node
  - $x.key[1] \leq x.key[2] \leq \dots x.key[x.n]$ are the $x.n$ keys stored in nondecreasing order
  - $x.leaf$ is a Boolean flag that is TRUE if x is a *leaf node* or FALSE if x is an *internal node*

Definition of a B-Tree

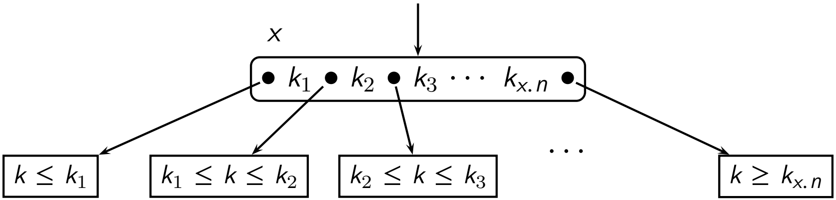


- Every node x has the following fields
  - $x.n$ is the number of keys stored at each node
  - $x.key[1] \leq x.key[2] \leq \ldots x.key[x.n]$ are the $x.n$ keys stored in nondecreasing order
  - $x.leaf$ is a Boolean flag that is TRUE if x is a *leaf node* or FALSE if x is an *internal node*
  - $x.c[1], x.c[2], \ldots, x.c[x.n+1]$ are the $x.n+1$ pointers to its children, if x is an *internal node*

Definition of a B-Tree (2)

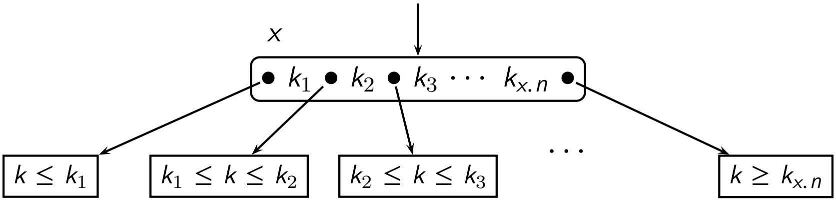


Definition of a B-Tree (2)



- The keys $x.key[i]$ delimit the ranges of keys stored in each subtree

Definition of a B-Tree (2)



- The keys $x.key[i]$ delimit the ranges of keys stored in each subtree

  $x.c[1] \longrightarrow$ subtree containing keys $k \leq x.key[1]$

  $x.c[2] \longrightarrow$ subtree containing keys k, $x.key[1] \leq k \leq x.key[2]$

  $x.c[3] \longrightarrow$ subtree containing keys k, $x.key[2] \leq k \leq x.key[3]$

  $\ldots$

  $x.c[x.n+1] \longrightarrow$ subtree containing keys k, $k \geq x.key[x.n]$

Definition of a B-Tree (3)

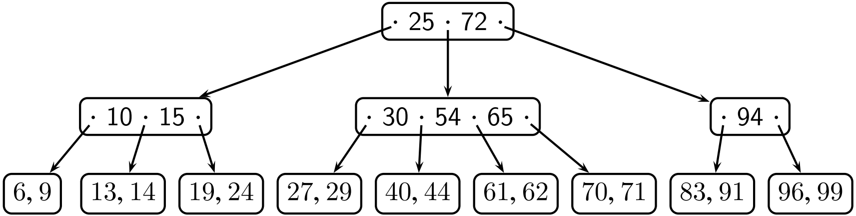
Definition of a B-Tree (3)

- ***All leaves have the same depth***

Definition of a B-Tree (3)

- ***All leaves have the same depth***
- Let $t \geq 2$ be the ***minimum degree*** of the B-tree
  - every node other than the root must have ***at least*** $t-1$ ***keys***
  - every node must contain ***at most*** $2t-1$ ***keys***
    - a node is *full* when it contains exactly $2t-1$ keys
    - a full node has $2t$ children

Example



· 25 · 72 ·
· 10 · 15 ·
· 30 · 54 · 65 ·
· 94 ·
6, 9
13, 14
19, 24
27, 29
40, 44
61, 62
70, 71
83, 91
96, 99

Search in B-Trees

Search in B-Trees

```
B-Tree-Search(x, k)
  i = 1
  while i ≤ x.n and k > x.key[i]
        i = i + 1
  if i ≤ x.n and k == x.key[i]
        return (x, i)
  if x.leaf
        return NIL
  else Disk-Read(x.c[i])
        return B-Tree-Search(x.c[i], k)
```

Height of a B-Tree

- **Theorem:** the height of a B-tree containing $n \geq 1$ keys and with a minimum degree $t \geq 2$ is

$$h \leq \log_t \frac{n+1}{2}$$

- **Theorem:** the height of a B-tree containing $n \geq 1$ keys and with a minimum degree $t \geq 2$ is

$$h \leq \log_t \frac{n+1}{2}$$

Proof:

  - $n \geq 1$, so the root has at least one key (and therefore two children)

Height of a B-Tree

- **Theorem:** the height of a B-tree containing $n \geq 1$ keys and with a minimum degree $t \geq 2$ is

$$h \leq \log_t \frac{n+1}{2}$$

Proof:

  - $n \geq 1$, so the root has at least one key (and therefore two children)
  - every other node has at least t children

- **Theorem:** the height of a B-tree containing $n \geq 1$ keys and with a minimum degree $t \geq 2$ is

$$h \leq \log_t \frac{n+1}{2}$$

Proof:

  - $n \geq 1$, so the root has at least one key (and therefore two children)
  - every other node has at least t children
  - in the worst case, there are two subtrees (of the root) each one containing a total of $(n-1)/2$ keys, and each one consisting of t-degree nodes, with each node containing $t-1$ keys

Height of a B-Tree

- **Theorem:** the height of a B-tree containing $n \geq 1$ keys and with a minimum degree $t \geq 2$ is

$$h \leq \log_t \frac{n+1}{2}$$

Proof:

- $n \geq 1$, so the root has at least one key (and therefore two children)
- every other node has at least t children
- in the worst case, there are two subtrees (of the root) each one containing a total of $(n-1)/2$ keys, and each one consisting of t-degree nodes, with each node containing $t-1$ keys
- each subtree contains $1 + t + t^2 \cdots + t^{h-1}$ nodes, each one containing $t-1$ keys

Height of a B-Tree

- **Theorem:** the height of a B-tree containing $n \geq 1$ keys and with a minimum degree $t \geq 2$ is

$$h \leq \log_t \frac{n+1}{2}$$

Proof:

  - $n \geq 1$, so the root has at least one key (and therefore two children)
  - every other node has at least t children
  - in the worst case, there are two subtrees (of the root) each one containing a total of $(n-1)/2$ keys, and each one consisting of t-degree nodes, with each node containing $t-1$ keys
  - each subtree contains $1 + t + t^2 \cdots + t^{h-1}$ nodes, each one containing $t-1$ keys, so

$$n \geq 1 + 2(t^h - 1)$$