Tema 1 - geometrie

$E^3 = (\mathbb{R}^3/\mathbb{R}, \langle , \rangle)$

$B = \{ f_1 = (1,1,1), f_2 = (1,1,-1), f_3 = (1,-1,-1)\}$

- $e_1 = \frac{f_1}{\|f_1\|}$, $\|f_1\| = \sqrt{\langle f_1, f_1 \rangle} = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$e_1 = \frac{1}{\sqrt{3}} \cdot (1,1,1)$

- $e_2 = \frac{e_2'}{\|e_2'\|}$, unde $e_2' = f_2 - \langle f_2, e_1 \rangle e_1 =$

$= (1,1,-1) - \frac{1}{\sqrt{3}} \cdot \langle f_2, f_1 \rangle \cdot e_1 =$

$= (1,1,-1) - \frac{1}{\sqrt{3}} \cdot [1\cdot 1 + 1/1 + 1/(-1)] \cdot e_1 =$

$= (1,1,-1) - \frac{1}{\sqrt{3}} \cdot e_1 =$

$= (1,1,-1) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot (1,1,1) =$

$= (1,1,-1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-4}{3}\right) =$

$= \frac{2}{3}(1,1,-2)$

$\|e_2'\|^2 = \frac{4}{9} \cdot (1^2+1^2+(-2)^2) = \frac{4}{\cancel{9}_3} \cdot \cancel{6}^2 = \frac{8}{3}$

$\Rightarrow \|e_2'\| = \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{6}}{3}$

$$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{\cancel{2}}{\cancel{3}}(1,1,-2)\cdot\frac{\cancel{3}}{\cancel{2}\sqrt{6}} = \frac{1}{\sqrt{6}}(1,1,-2)$$

$$\bullet\; e_3 = \frac{e_3'}{\|e_3'\|}$$

unde $e_3' = f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2 =$

$$= (1,-1,-1) - \frac{1}{\sqrt{3}}\cdot\langle f_3, f_1\rangle\cdot e_1 - \frac{1}{\sqrt{6}}\langle f_3, (1,1,-2)\rangle\cdot e_2 =$$

$$= (1,-1,-1) - \frac{1}{\sqrt{3}}\cdot\left(1\cdot 1 + 1\cdot(-1) + 1\cdot(-1)\right)e_1 - \frac{1}{\sqrt{6}}\left(1\cdot 1 + (-1)\cdot 1 + (-1)\cdot(-2)\right)e_2$$

$$= (1,-1,-1) - \frac{1}{\sqrt{3}}\cdot(-1)\cdot e_1 - \frac{1}{\sqrt{6}}\cdot 2\cdot e_2 =$$

$$= (1,-1,-1) + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\cdot(1,1,1) - \frac{2}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}(1,1,-2) =$$

$$= (1,-1,-1) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \frac{2}{6}(1,1,-2) =$$

$$= \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right) - \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) = (1,-1,0)$$

$$\|e_3'\|^2 = 1^2 + (-1)^2 + 0 = 1+1 = 2$$

$$\Rightarrow \|e_3'\| = \sqrt{2}$$

$$e_3 = \frac{e_3'}{\|e_3'\|} = (1,-1,0)\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot(1,-1,0)$$

Deci: $\tilde{B} = \left\{e_1 = \frac{1}{\sqrt{3}}(1,1,1),\ e_2 = \frac{1}{\sqrt{6}}(1,1,-2),\ e_3 = \frac{1}{\sqrt{2}}(1,-1,0)\right\}$

baza ortonormată

Tema 2 - geometrie

Considerând spațiul vectorial euclidian $E_3 = (\mathbb{R}^3/_{\mathbb{R}}, <,>)$ și baza ortonormată B' obținută în aplicația anterioară, să se determine coordonatele vectorului în această bază:

a) $w = (-1, 1, 2)$

Considerăm scrierea vectorului w în baza B' data:

$w = w_1 e_1 + w_2 e_2 + w_3$, unde

$$\tilde{B} = \{ e_1 = \frac{1}{\sqrt{3}}(1,1,1),\ e_2 = \frac{1}{\sqrt{6}}(1,1,-2),\ e_3 = \frac{1}{\sqrt{2}}(1,-1,0) \}$$

baza ortonormată

$$[w]_{\tilde{B}} = (w_1, w_2, w_3)$$

$$w_1 = \langle w, e_1 \rangle = w_1 \langle e_1, \overset{1}{e_1} \rangle + w_2 \langle e_2, \overset{0}{e_1} \rangle + w_3 \langle e_3, \overset{0}{e_1} \rangle =$$

$$= \langle w, e_1 \rangle = (-1, 1, 2) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) =$$

$$= -\cancel{\frac{1}{\sqrt{3}}} + \cancel{\frac{1}{\sqrt{3}}} + \frac{2}{\sqrt{3}} = \frac{2^{\sqrt{3}}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$w_2 = \langle w, e_2 \rangle = (-1, 1, 2) \cdot \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) =$$

$$= -\cancel{\frac{1}{\sqrt{6}}} + \cancel{\frac{1}{\sqrt{6}}} - \frac{4}{\sqrt{6}} = -\frac{4^{\sqrt{6}}}{\sqrt{6}} = -\frac{\cancel{4}^{2}\sqrt{6}}{\cancel{6}_{3}} = -\frac{2\sqrt{6}}{3}$$

$$w_3 = \langle w, e_3 \rangle = (-1,1,2)\cdot\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) =$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 = -\frac{\overset{\sqrt{2}}{\cancel{2}}}{\sqrt{2}} = -\frac{\cancel{2}\sqrt{2}}{\cancel{2}_1} = -\sqrt{2}$$

$$\Rightarrow w = \langle w, e_1 \rangle e_1 + \langle w, e_2 \rangle e_2 + \langle w, e_3 \rangle e_3 =$$

$$= \frac{2\sqrt{3}}{3} e_1 + \left(-\frac{2\sqrt{6}}{3}\right) e_2 + (-\sqrt{2})\, e_3$$

$$[w]_{\check{B}} = \left(\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{6}}{3}, -\sqrt{2}\right)$$

Tema 3 - geometrie

Fie spațiul vectorial euclidian $E_3 = (\mathbb{R}^3/_{\mathbb{R}}, \underset{p.s.}{<,>})$.

$B_0 = \{e_1, e_2, e_3\} \subset E_3$

baza canonică

c) $T : E_3 \to E_3$, $\begin{cases} T(e_1) = \frac{2}{3} e_1 + \frac{2}{3} e_2 - \frac{1}{3} e_3 \\ T(e_2) = \frac{2}{3} e_1 - \frac{1}{3} e_2 + \frac{2}{3} e_3 \\ T(e_3) = -\frac{1}{3} e_1 + \frac{2}{3} e_2 + \frac{2}{3} e_3 \end{cases}$

$B_0 = \{e_1, e_2, e_3\} \subset E_3$

baza ortonormată $(\langle e_i, e_j \rangle = \delta_{ij}, (\forall) i, j = \overline{1,3})$

A = $\begin{array}{c} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\ \begin{array}{ccc} T(e_1) & T(e_2) & T(e_3) \end{array} \end{array}$

matricea asociată endomorfismului T în raport cu baza canonică B_0

$$^tA \cdot A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{I}_3$$

Avem: ${}^tA \cdot A = \mathcal{I}_3 \Rightarrow A$ matrice ortogonală

Deci T este transformare ortogonală (mai exact, o rotație în jurul axei Ox)

Tema 4 | - geometrie

Fie $F: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$F(x, y) = 2x_1y_1 + x_2y_2 - 2x_1y_2 - 2x_2y_1 - 2x_2y_3 - 2x_3y_2,$$

$$(\forall)\ x = (x_1, x_2, x_3) \in \mathbb{R}^3$$
$$y = (y_1, y_2, y_3)$$

d) Determinați forma pătratică Q corespunzătoare lui F și să se aducă la o formă canonică utilizând metodele Gauss, respectiv Jacobi.

$Q: \mathbb{R}^3 \longrightarrow \mathbb{R}$, $Q(x) = Q(x_1, x_2, x_3) = F(x, x)$, $\forall x \in \mathbb{R}^3$

forma pătratică a lui F

unde $x = (x_1, x_2, x_3)$

1) Metoda Gauss

$$Q(x) = 2x_1^2 + x_2^2 - 2x_1x_2 - 2x_2x_1 - 2x_2x_3 - 2x_3x_2 =$$

$$= \underline{2x_1^2} + \underline{x_2^2} \underline{- 4x_1x_2} - 4x_2x_3 =$$

$$= (\sqrt{2}x_1 - \sqrt{2}x_2)^2 \underline{- x_2^2} \underline{- 4x_2x_3} =$$

$$= (\sqrt{2}x_1 - \sqrt{2}x_2)^2 - (x_2 + 2x_3)^2 + 4x_3^2$$

schimbare de coordonate $\begin{cases} y_1 = \sqrt{2}(x_1 - x_2) \\ y_2 = x_2 + 2x_3 \\ y_3 = x_3 \end{cases}$

$$Q(y) = y_1^2 - y_2^2 + 4y_3^2$$

$$\begin{cases} r = 3 \\ p = 2 \\ q = 1 \\ s = 2 - 1 = 1 \end{cases}$$

2) Metoda Jacobi

Matricea simetrică asociată:

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\Delta_1 = 2 \neq 0$$

$$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2 + 4 = 6 \neq 0$$

$$\Delta_3 = \begin{vmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{vmatrix} = 0 + 0 - 0 - 0 - 8 - 0 = -8 \neq 0$$

Conform Th Jacobi: $(\exists)\ B' \subset \mathbb{R}^3$ a.î. $Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2$

unde $x = x_1' e_1' + x_2' e_2' + x_3' e_3'$

$$Q(x) = \frac{1}{2} x_1'^2 + \frac{1}{3} x_2'^2 + \frac{-3}{4} x_3'^2, \quad x = (x_1', x_2', x_3')_{B'}$$

$$\begin{cases} r = 3 \\ p = 2 \\ q = 1 \\ s = 2 - 1 = 1 \end{cases}$$

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Tema 5 |

Fie $E^3 = (\mathbb{R}^3/\mathbb{R}, \langle , \rangle)$ şi baza:

$$B = \{\beta_1 = (1,2,0),\ \beta_2 = (2,1,1),\ \beta_3 = (1,0,3)\}$$

• $e_1 = \frac{\beta_1}{\|\beta_1\|}$, $\|\beta_1\| = \sqrt{1^2+2^2+0^2} = \sqrt{5}$

$\Rightarrow e_1 = \frac{1}{\sqrt{5}}(1,2,0)$

• $e_2 = \frac{e_2'}{\|e_2'\|}$, unde $e_2' = \beta_2 - \langle \beta_2, e_1 \rangle e_1 =$

$= (2,1,1)\cdot e_1\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) = \frac{2+2}{\sqrt{5}} e_1 = \frac{4}{\sqrt{5}} \cdot e_1 =$

$= \left(\frac{4}{5}, \frac{8}{5}, 0\right)$

$\|e_2'\|^2 = \left(\frac{6}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 + 1^2 = \frac{36+9+25}{25} = \frac{70}{25}$

$\Rightarrow \|e_2'\| = \sqrt{\frac{70}{25}} = \frac{\sqrt{70}}{5}$

$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\frac{\sqrt{70}}{5}} \cdot \left(\frac{6}{5}, -\frac{3}{5}, 1\right) = \frac{5}{\sqrt{70}}\left(\frac{6}{5}, -\frac{3}{5}, 1\right)$

• $e_3 = \frac{e_3'}{\|e_3'\|}$

unde $e_3' = \beta_3 - \langle \beta_3, e_1 \rangle e_1 - \langle \beta_3, e_2 \rangle e_2 =$

$= (1,0,3) - \frac{1}{\sqrt{5}} \cdot (1+0+0) \cdot e_1 - \frac{5}{\sqrt{70}}\left(\frac{6}{5}\cdot 1 + 0 + 3 \cdot 1\right) \cdot e_2 =$

$= (1,0,3) - \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}(1,2,0) - \frac{\cancel{5}}{\sqrt{70}} \cdot \frac{21}{\cancel{5}} \cdot \frac{5}{\sqrt{70}}\left(\frac{6}{5}, -\frac{3}{5}, 1\right) =$

$$= (1,0,3) - \left(\frac{1}{5}, \frac{2}{5}, 0\right) - \left(\frac{18}{10}, -\frac{9}{10}, \frac{3}{2}\right) =$$

$$= \left(-1, \frac{7}{10}, \frac{3}{2}\right)$$

$$\|e_3'\|^2 = (-1)^2 + \left(\frac{7}{10}\right)^2 + \left(\frac{3}{2}\right)^2 = 1 + \frac{49}{100} + \frac{9}{4} = \frac{374}{100}$$

$$\Rightarrow \|e_3'\| = \sqrt{\frac{374}{100}} = \frac{\sqrt{374}}{10}$$

$$e_3 = \frac{e_3'}{\|e_3'\|} = \frac{10}{\sqrt{374}} \cdot \left(-1, \frac{7}{10}, \frac{3}{2}\right)$$

Baza ortonormata:

$$\tilde{B} = \left\{ e_1 = \frac{1}{\sqrt{5}}(1,2,0),\ e_2 = \frac{5}{\sqrt{70}}\left(\frac{6}{5}, -\frac{3}{5}, 1\right), \right.$$
$$\left. e_3 = \frac{10}{\sqrt{374}} \cdot \left(-1, \frac{7}{10}, \frac{3}{2}\right) \right\}$$

Tema 5

Fie conica $x_1^2+2x_1x_2-6x_2^2+4x_1-8x_2-17=0$.

Să se aducă la o formă canonică Γ prin izometrii.

Forma pătratică: $Q(x_1,x_2)=x_1^2+2x_1x_2-6x_2^2$

Matricea asociată: $A=\begin{pmatrix}1 & 1\\ 1 & -6\end{pmatrix}$

$\delta=\det A=1\cdot(-6)-1\cdot 1=-6-1=-7<0$

$\delta<0 \Rightarrow$ conica este hiperbolă

$$\begin{cases}\dfrac{\partial f}{\partial x_1}=2x_1+2x_2+4=0 & (1)\\[2mm] \dfrac{\partial f}{\partial x_2}=2x_1-12x_2-8=0 & (2)\end{cases}$$

$(1): x_1+x_2=-2 \Rightarrow x_1=-2-x_2$

substituim $(2): 2(-2-x_2)-12x_2-8=0 \Rightarrow -4-2x_2-12x_2-8=0 \Rightarrow$

$\Rightarrow -14x_2=12 \Rightarrow x_2=-\frac{6}{7}$

$x_1=-2-\left(-\frac{6}{7}\right)=-\frac{8}{7}$

$\Rightarrow$ centrul conicii este: $C\left(-\frac{8}{7},-\frac{6}{7}\right)$

Translația: $t \begin{cases} x_1' = x_1 - x_1^0 \\ x_2' = x_2 - x_2^0 \end{cases} \Leftrightarrow \begin{cases} x_1' = x_1 + \frac{8}{7} \\ x_2' = x_2 + \frac{6}{7} \end{cases}$

$$\Leftrightarrow \begin{cases} x_1 = x_1' - \frac{8}{7} \\ x_2 = x_2' - \frac{6}{7} \end{cases}$$

$$t(\Gamma): x_1^2 + 2x_1x_2 - 6x_2^2 + 4x_1 - 8x_2 - 17 = 0$$

$$\xrightarrow{ec\ tr} t(\Gamma): x_1'^2 + 2x_1'x_2' - 6x_2'^2 - \frac{111}{7} = 0$$

Forma pătratică: $Q(x_1', x_2') = x_1'^2 + 2x_1'x_2' - 6x_2'^2$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -6 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 5\lambda - 7 = 0 \Rightarrow \lambda_{1,2} = \frac{-5 \pm \sqrt{53}}{2}$$

$$\begin{cases} v_1 = \begin{pmatrix} \frac{7}{2} + \frac{\sqrt{53}}{2} \\ 1 \end{pmatrix} & \text{- vector propriu pentru } \lambda_1 \\ v_2 = \begin{pmatrix} \frac{7}{2} - \frac{\sqrt{53}}{2} \\ 1 \end{pmatrix} & \text{- vector propriu pentru } \lambda_2 \end{cases}$$

Vectorii ortonormați:

$$e_1 = \frac{1}{\sqrt{1 + \left(\frac{7}{2} + \frac{\sqrt{53}}{2}\right)^2}} \begin{pmatrix} \frac{7}{2} + \frac{\sqrt{53}}{2} \\ 1 \end{pmatrix}$$

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$$e_2 = \frac{1}{\sqrt{1+\left(\frac{7}{2}-\frac{\sqrt{53}}{2}\right)^2}}\begin{pmatrix}\frac{7}{2}-\frac{\sqrt{53}}{2}\\ 1\end{pmatrix}$$

Matricea de rotație $R = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$

Schimbarea de var prin rotație:

$$\begin{cases} x_1' = e_{11} x_1'' + e_{12} x_2'' \\ x_2' = e_{21} x_1'' + e_{22} x_2'' \end{cases}$$

$$\Rightarrow \lambda_1 (x_1'')^2 + \lambda_2 (x_2'')^2 = \frac{111}{7} \Rightarrow \frac{(x_1'')^2}{\frac{111}{7\lambda_1}} - \frac{(x_2'')^2}{\frac{111}{7|\lambda_2|}} = 1$$

Forma canonică a hiperbolei:

$$\frac{(x_1'')^2}{a^2} - \frac{(x_2'')^2}{b^2} = 1$$

unde $a^2 = \frac{111}{7\lambda_1}$

$b^2 = \frac{111}{7\lambda_2}$