Algorithms and Data Structures (II)

Gabriel Istrate

June 2, 2025

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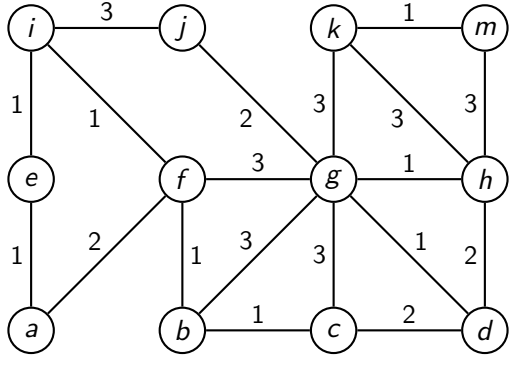
- T's total weight of the tree is minimal

$$w(T) = \sum_{(u,v) \in T} w(u, v)$$

  - a ***minimum-weight spanning tree***, or "minimum spanning tree"

Example



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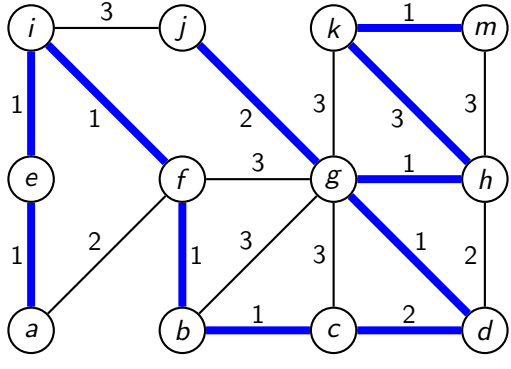

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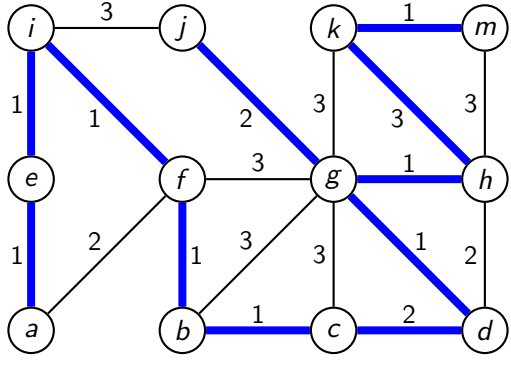

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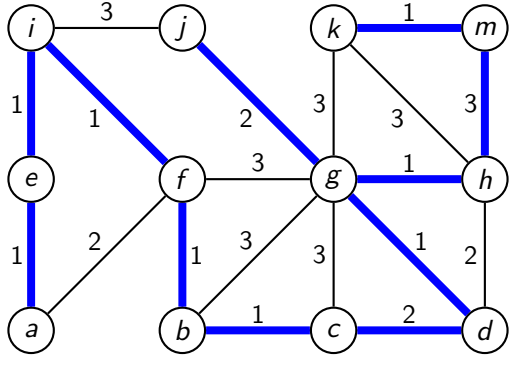

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- This is a *greedy algorithm*
- Does it work?

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  - more or less the *definition* of a greedy algorithm

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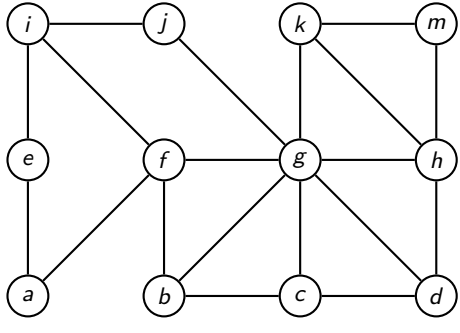

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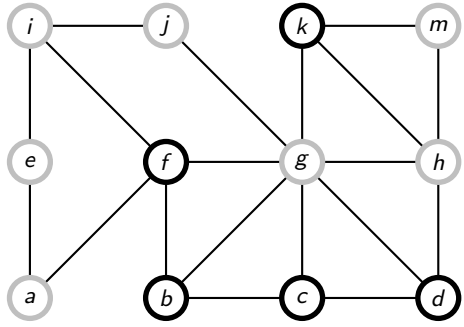

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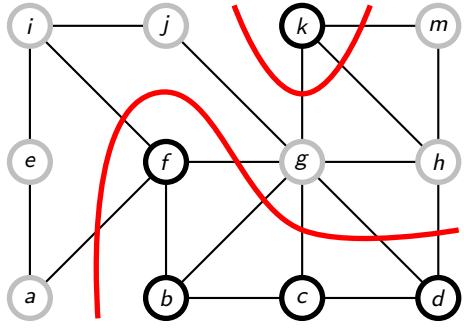

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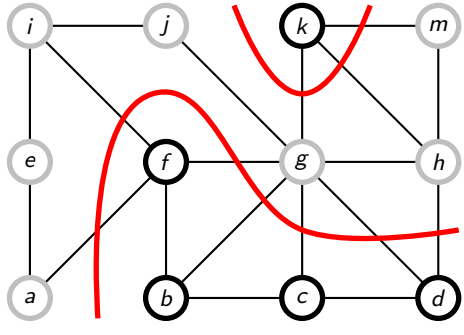

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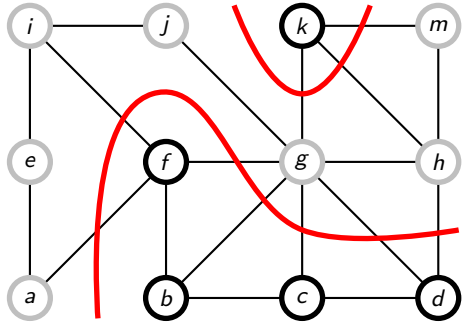

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- A cut $(S, V - S)$ *respects* a set of edges A if no edge in A crosses the cut

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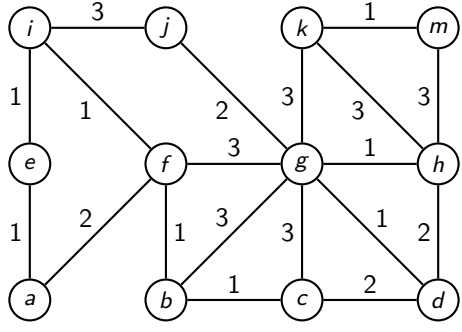

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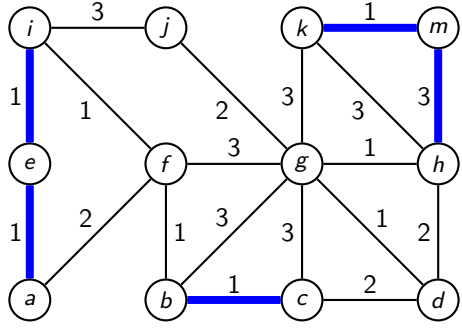

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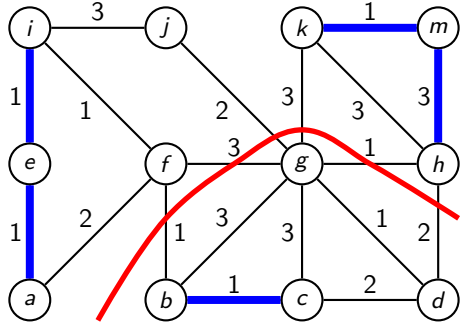

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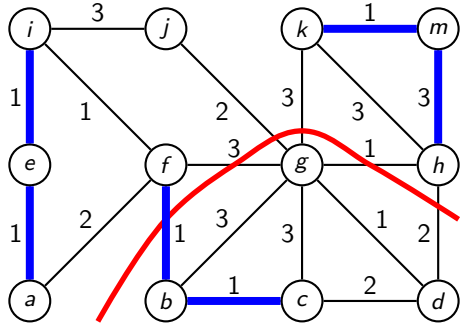

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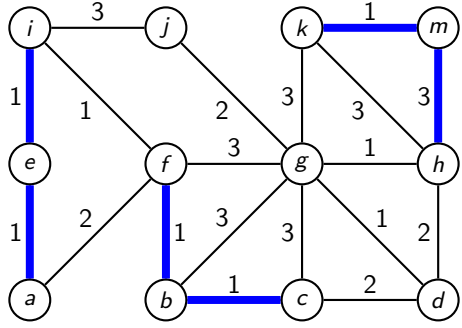

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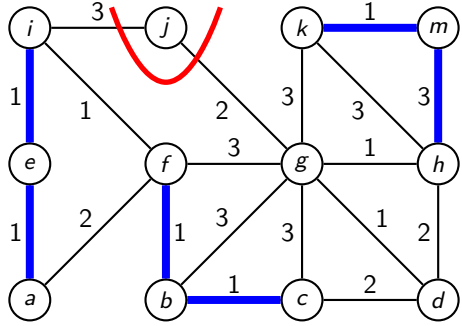

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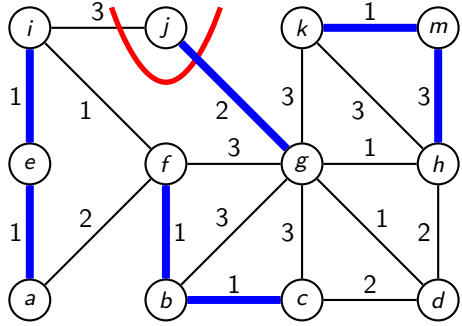

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Concrete Algorithms

- Kruskal's algorithm (1956)
  - based on the generic minimum-spanning-tree algorithm
  - incrementally builds a ***forest*** A
- Prim's algorithm (1957)
  - based on the generic minimum-spanning-tree algorithm
  - incrementally builds a ***single tree*** A

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- *Union*(x, y) joins the sets containing x and y

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2  for each vertex v ∈ V(G)
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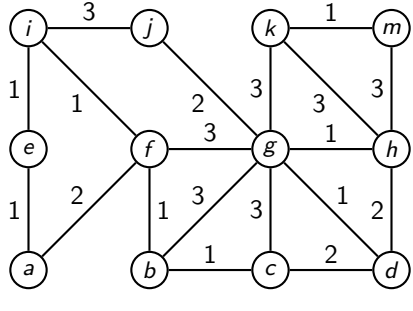

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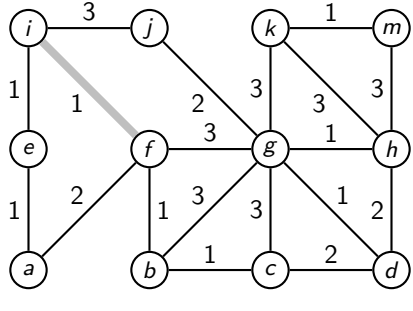

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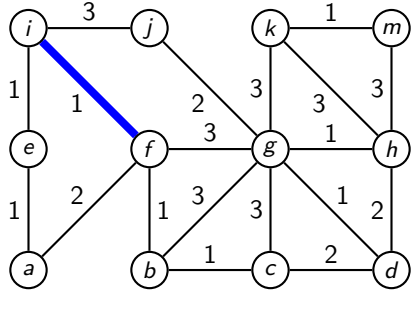

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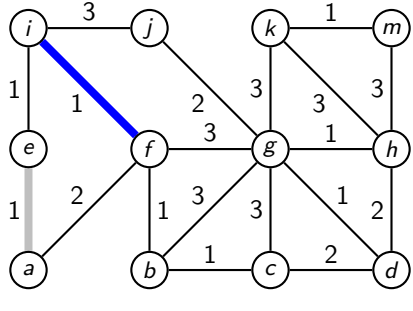

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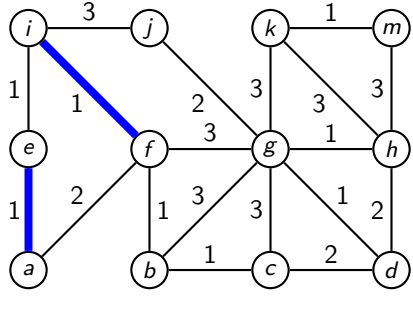

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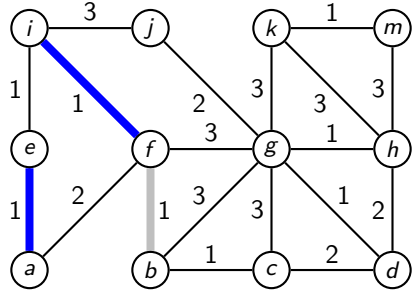

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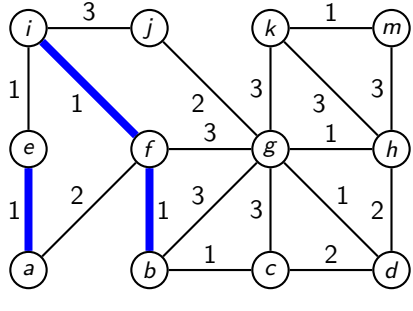

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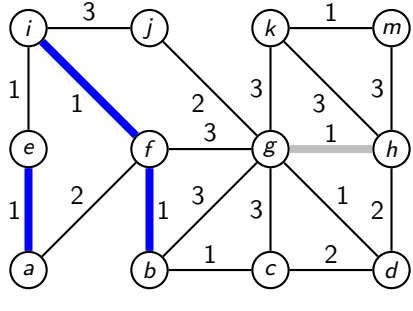

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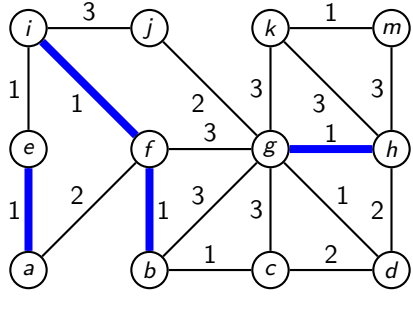

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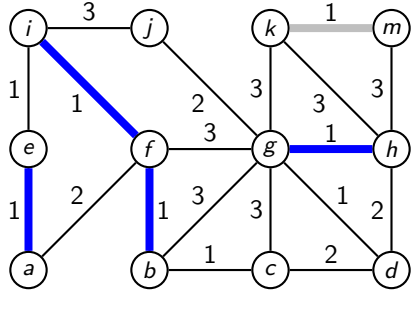

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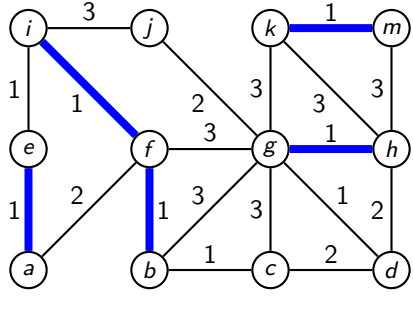

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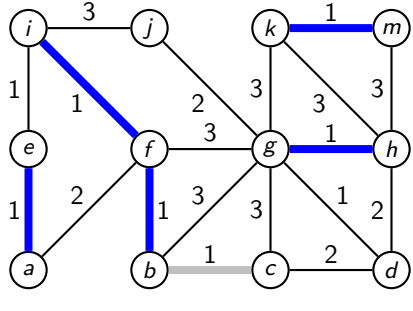

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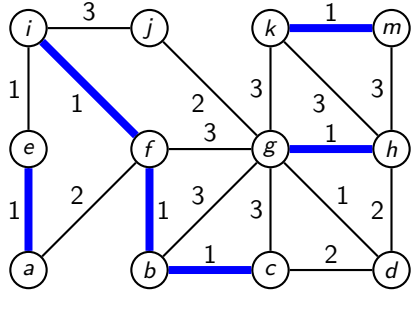

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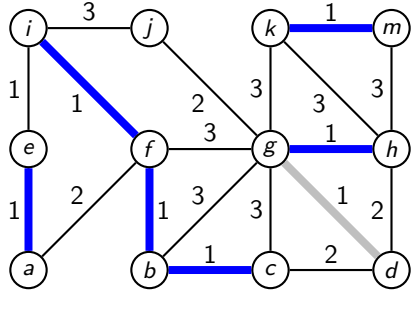

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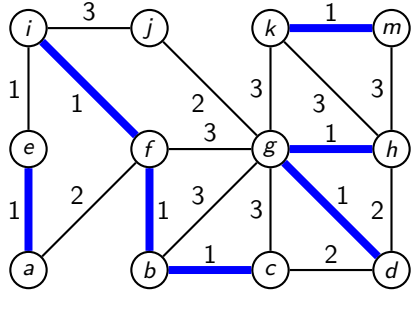

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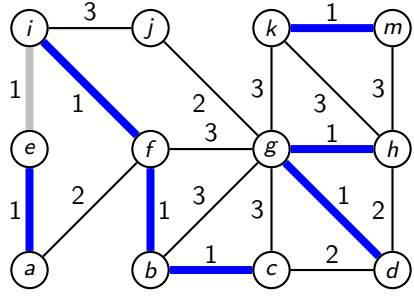

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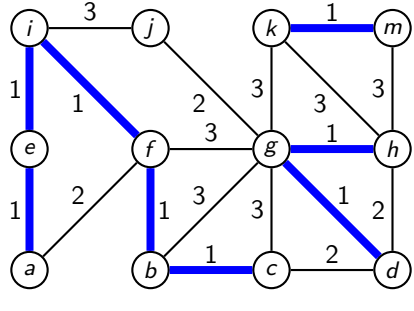

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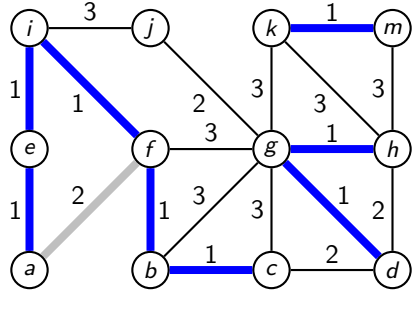

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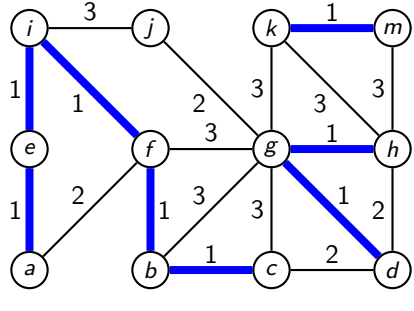

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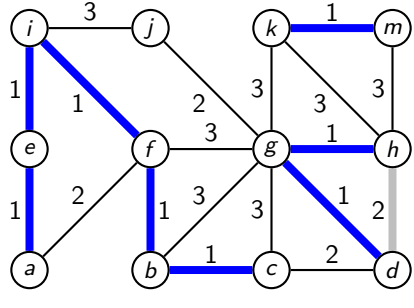

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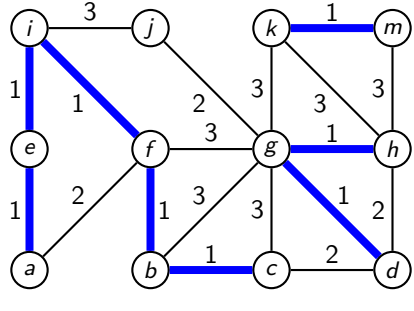

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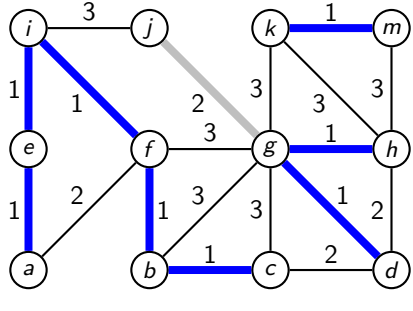

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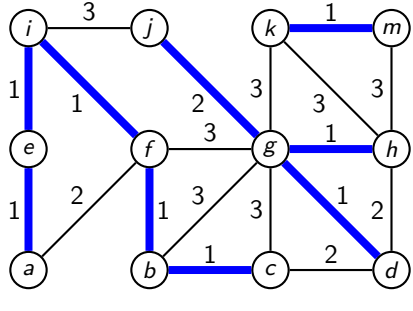

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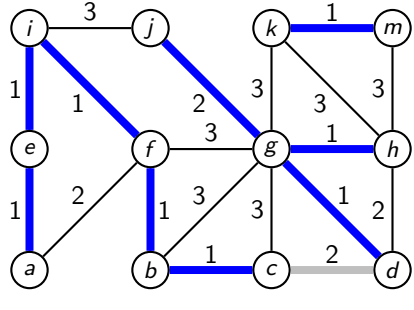

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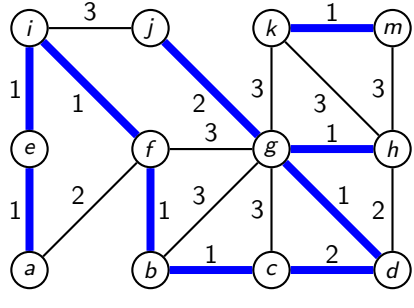

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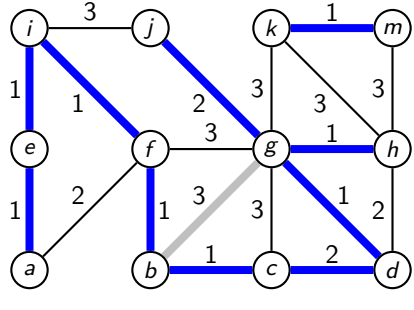

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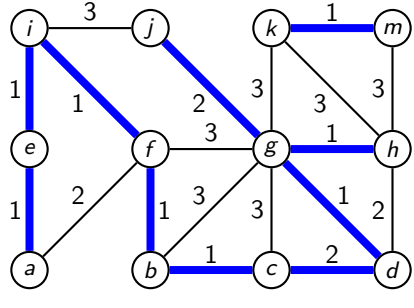

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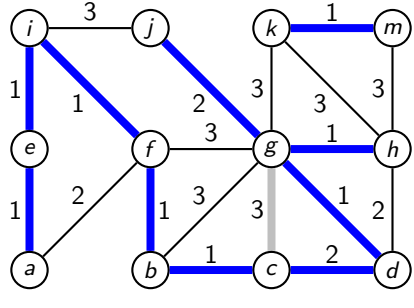

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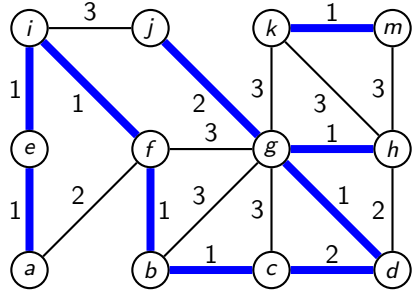

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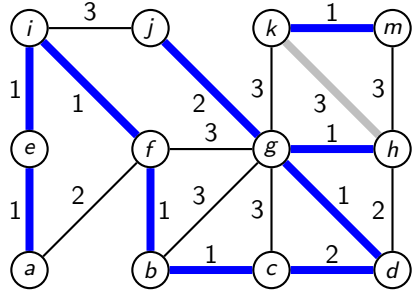

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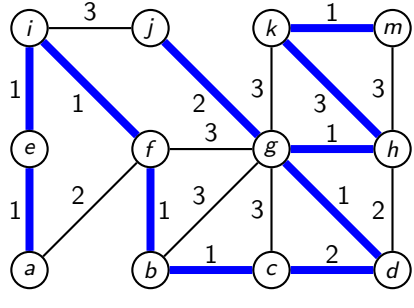

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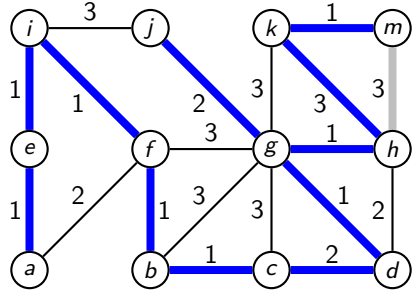

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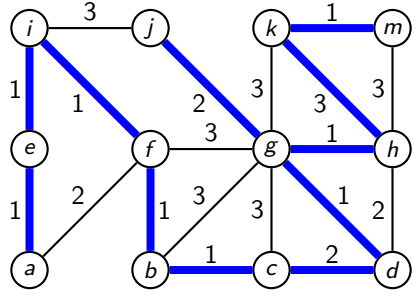

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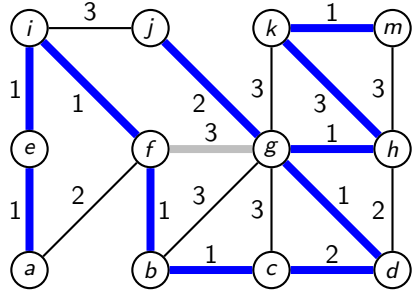

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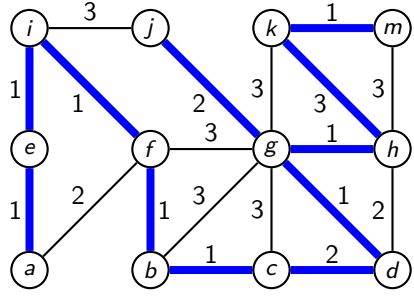

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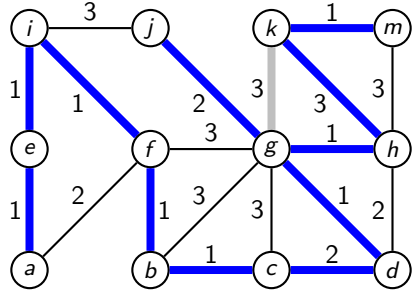

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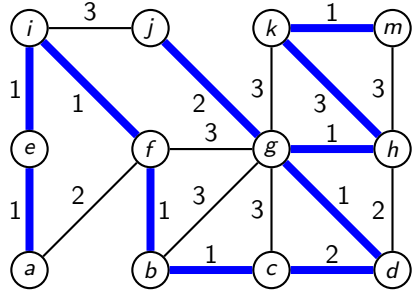

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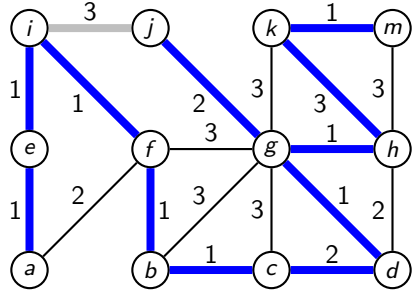
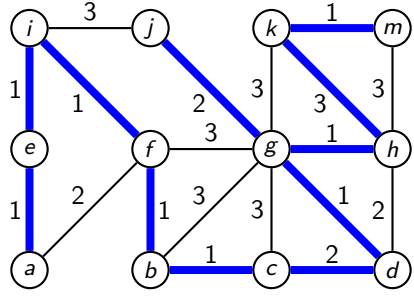


Kruskal's Algorithm

```
MST-Kruskal(G, w)
1  A = ∅
2  for each vertex v ∈ V(G)
3      Make-Set(v)                    // disjoint-set data structure
4  sort E in non-decreasing order by weight w
5  for each edge (u, v) ∈ E, taken in non-decreasing order by w
6      if Find-Set(u) ≠ Find-Set(v)
7          A = A ∪ {(u, v)}
8          Union(u, v)
```



Complexity of MST-Kruskal

```
MST-Kruskal(G, w)
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- $|V|$ times **Make-Set** (loop of line 2–3)

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Complexity of MST-Kruskal

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- $O(|E| \log |E|)$ for sorting E (line 4)
- $2|E|$ times **Find-Set**
- $O(|E|)$ times **Union**

Prim's Algorithm

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MST-Prim(G, w, r)
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 5  Q = V(G)
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 7      u = Extract-Min(Q) // min by key[u]
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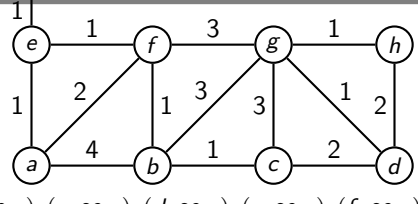

$Q = \{(a, 0, \cdot), (b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

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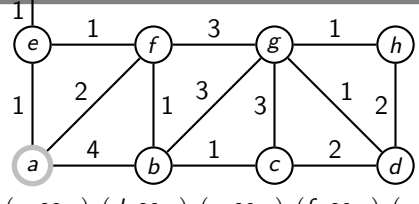

$Q = \{(b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

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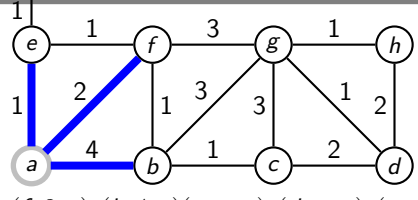

$Q = \{(e, 1, a), (f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

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MST-Prim(G, w, r)
 1  for each vertex u ∈ V(G)
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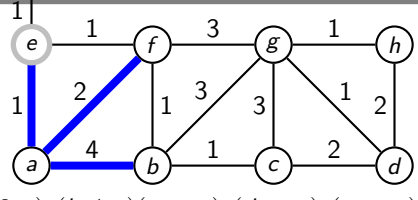

$Q = \{(f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

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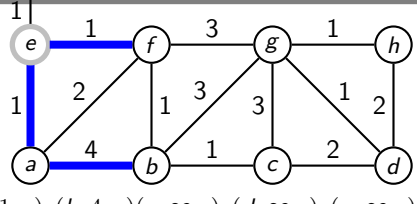

$Q = \{(f, 1, e), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

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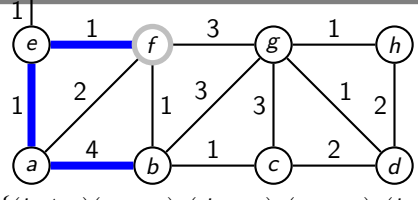

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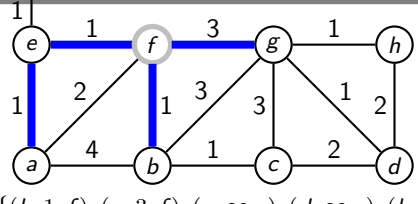

$Q = \{(b, 1, f), (g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

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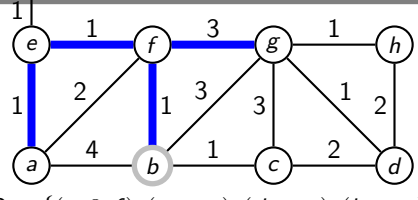

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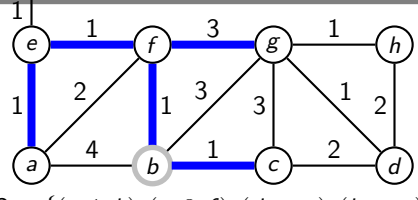

$Q = \{(c, 1, b), (g, 3, f), (d, \infty, \cdot), (h, \infty, \cdot)\}$

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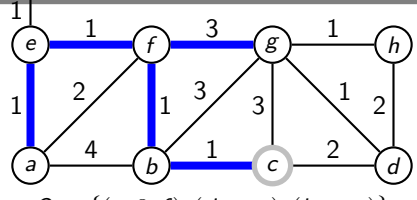

$Q = \{(g, 3, f), (d, \infty, \cdot), (h, \infty, \cdot)\}$

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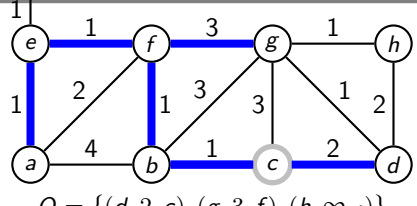

$Q = \{(d, 2, c), (g, 3, f), (h, \infty, \cdot)\}$

Prim's Algorithm

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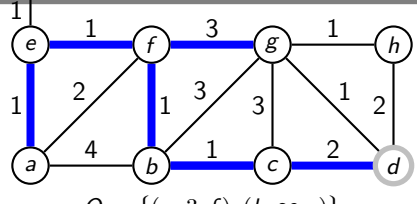

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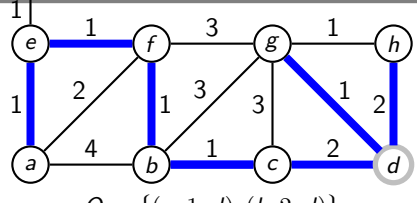

$Q = \{(g, 1, d), (h, 2, d)\}$

Prim's Algorithm

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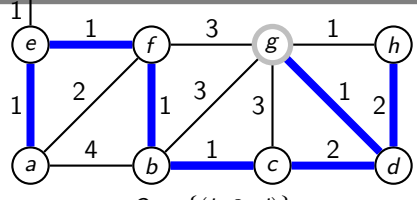

$Q = \{(h, 2, d)\}$

Prim's Algorithm

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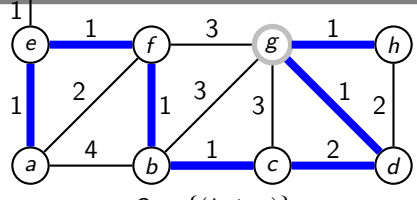

$Q = \{(h, 1, g)\}$

Prim's Algorithm

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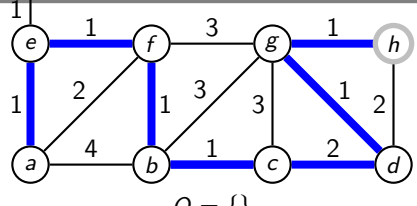

$Q = \{\}$

Prim's Algorithm

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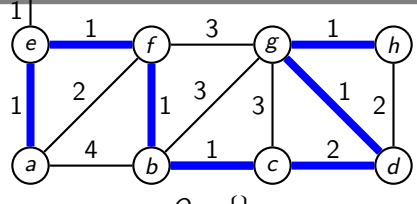

$Q = \{\}$