Temă 1

a) $(S_2)\begin{cases} 2x+y-z+t=1 \\ x-y+2z+t=3 \\ x+2y-3z-2t=6 \end{cases}$ G-J

$$\bar{A}=\left(\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 1 \\ 1 & -1 & 2 & 1 & 3 \\ 1 & 2 & -3 & -2 & 6 \end{array}\right) \xrightarrow{L_1=L_1-L_2}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -2 \\ 1 & -1 & 2 & 1 & 3 \\ 1 & 2 & -3 & -2 & 6 \end{array}\right) \xrightarrow{L_3=L_3-L_1}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -2 \\ 1 & -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & -2 & 8 \end{array}\right) \xrightarrow[L_2=L_2-L_1]{L_3=\frac{L_3}{-2}}$$

$$\longrightarrow \left(\begin{array}{cccc|c} \boxed{1} & 2 & -3 & 0 & -2 \\ 0 & \boxed{-3} & 5 & 1 & 5 \\ 0 & 0 & 0 & \boxed{1} & -4 \end{array}\right) \xrightarrow{L_2=L_2-L_3}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -2 \\ 0 & -3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{L_2=\frac{L_2}{-3}} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -2 \\ 0 & 1 & -\frac{5}{3} & 0 & -3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right)$$

$$\xrightarrow{L_1=L_1-2L_2} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 1/3 & 0 & 4 \\ 0 & \boxed{1} & -5/3 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & -4 \end{array}\right)$$

$$\begin{cases} x + \frac{z}{3} = 4 \\ y - \frac{5z}{3} = -3 \\ t = -4 \end{cases} \rightarrow \begin{cases} 3x + z = 12 \\ 3y - 5z = -9 \end{cases}$$

$$z = s \Rightarrow \begin{cases} 3x + s = 12 \Rightarrow 3x = 12 - s \Rightarrow x = \frac{12-s}{3} \\ 3y - 5s = -9 \Rightarrow 3y = -9 + 5s \Rightarrow y = \frac{-9+5s}{3} \end{cases}$$

$$\Rightarrow \mathcal{S} = \left\{ \begin{pmatrix} \frac{12-s}{3} \\ \frac{-9+5s}{3} \\ s \\ -4 \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}$$

Tema 2

1.

$$\begin{cases} \alpha x + y + z = 1 \\ x + \alpha y + z = 1 \\ x + y + \alpha z = 1 \end{cases}, \quad \alpha \in \mathbb{R}$$

$$\bar{A} = \left(\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \end{array}\right) \xrightarrow[L_2 = L_2 - L_3]{L_1 = L_1 - L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \alpha - 1 & 0 & 1-\alpha & 0 \\ 0 & \alpha - 1 & 1-\alpha & 0 \\ 1 & 1 & \alpha & 1 \end{array}\right) \xrightarrow[L_2 = L_2 \cdot \frac{1}{\alpha-1}]{L_1 = L_1 \cdot \frac{1}{\alpha-1}}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \overset{= -(\alpha-1)}{\frac{1-\alpha}{\alpha-1}} & 0 \\ 0 & 1 & \frac{1-\alpha}{\alpha-1} & 0 \\ 1 & 1 & \alpha & 1 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 1 & 1 & \alpha & 1 \end{array}\right) \xrightarrow{L_3 = L_3 - L_1 - L_2}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{\alpha + 2} & 1 \end{array}\right) \xrightarrow{L_3 = L_3 \cdot \frac{1}{\alpha+2}}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{1} & \frac{1}{\alpha+2} \end{array}\right) \xrightarrow[L_2 = L_2 + L_3]{L_1 = L_1 + L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & \frac{1}{\alpha+2} \\ 0 & \boxed{1} & 0 & \frac{1}{\alpha+2} \\ 0 & 0 & \boxed{1} & \frac{1}{\alpha+2} \end{array}\right)$$

forma eșalon redusă

$$S = \left\{ \left(\frac{1}{\alpha+2} ; \frac{1}{\alpha+2} , \frac{1}{\alpha+2} \right) \middle| \alpha \in \mathbb{R} \right\} \Rightarrow \begin{cases} x = \frac{1}{\alpha+2} \\ y = \frac{1}{\alpha+2} \\ z = \frac{1}{\alpha+2} \end{cases}$$

$\Rightarrow$ sistemul este compatibil s.nedeterminat

2.

$$\begin{cases} \alpha x + y + z = 1 \\ x + \alpha y + z = \alpha \\ x + y + \alpha z = \alpha^2 \end{cases}, \quad \alpha \in \mathbb{R}$$

$$\bar{A} = \left(\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & \alpha \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right) \xrightarrow[L_2 = L_2 - L_3]{L_1 = L_1 - L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \alpha - 1 & 0 & 1 - \alpha & 1 - \alpha^2 \\ 0 & \alpha - 1 & 1 - \alpha & \alpha - \alpha^2 \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right) \xrightarrow[L_2 = L_2 \cdot \frac{1}{\alpha - 1}]{L_1 = L_1 \cdot \frac{1}{\alpha - 1}}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{\overset{-1}{\cancel{(1-\alpha)}}(1+\alpha)}{\cancel{\alpha - 1}} \\ 0 & 1 & -1 & \frac{\alpha \overset{-1}{\cancel{(1-\alpha)}}}{\cancel{\alpha - 1}} \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 - \alpha \\ 0 & 1 & -1 & -\alpha \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right) \xrightarrow{L_3 = L_3 - L_1}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 - \alpha \\ 0 & 1 & -1 & -\alpha \\ 0 & 1 & \alpha + 1 & \alpha^2 + \alpha + 1 \end{array}\right) \xrightarrow{L_3 = L_3 - L_2}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1-\alpha \\ 0 & 1 & -1 & -\alpha \\ 0 & 0 & \alpha+2 & \alpha^2+2\alpha+1 \end{array}\right) \xrightarrow{L_3 = L_3 \cdot \frac{1}{\alpha+2}}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -(\alpha+1) \\ 0 & \boxed{1} & -1 & -\alpha \\ 0 & 0 & \boxed{1} & \frac{(\alpha+1)^2}{\alpha+2} \end{array}\right) \xrightarrow[L_2 = L_2 + L_3]{L_1 = L_1 + L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -\frac{\alpha+1}{\alpha+2} \\ 0 & \boxed{1} & 0 & \frac{1}{\alpha+2} \\ 0 & 0 & \boxed{1} & \frac{(\alpha+1)^2}{\alpha+2} \end{array}\right)$$

forma eșalon redusă

• $$-{}^{\alpha+2)}(\alpha+1) + \frac{(\alpha+1)^2}{\alpha+2} = \frac{-(\alpha+2)(\alpha+1) + (\alpha+1)^2}{\alpha+2} =$$

$$= \frac{(\alpha+1)(-\cancel{\alpha}-2+\cancel{\alpha}+1)}{\alpha+2} = \frac{-(\alpha+1)}{\alpha+2} = \frac{-\alpha-1}{\alpha+2}$$

• $$-{}^{\alpha+2)}\alpha + \frac{(\alpha+1)^2}{\alpha+2} = \frac{-\cancel{\alpha^2}-\cancel{2\alpha}+\cancel{\alpha^2}+\cancel{2\alpha}+1}{\alpha+2} = \frac{1}{\alpha+2}$$

$$S = \left\{ \left(-\frac{\alpha+1}{\alpha+2}, \frac{1}{\alpha+2}, \frac{(\alpha+1)^2}{\alpha+2} \right) \middle| \alpha \in \mathbb{R} \right\}$$

$$x = -\frac{\alpha+1}{\alpha+2}, \quad y = \frac{1}{\alpha+2}, \quad z = \frac{(\alpha+1)^2}{\alpha+2}$$

$\Rightarrow$ sistemul este compatibil s. nedeterminat

3. Inversa

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$(B|I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_1 = L_1 + L_2}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_3 = L_3 - L_1}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & -1 & 1 \end{array}\right) \xrightarrow{L_1 = L_1 \cdot \frac{1}{2}}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & -1 & 1 \end{array}\right) \xrightarrow{L_2 = L_2 - L_1}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & -1 & -1 & 1 \end{array}\right) \xrightarrow{L_3 = L_3 + L_2}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & -\frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{L_3 = L_3 \cdot \frac{1}{3}}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \end{array}\right) \xrightarrow{L_2 = L_2 - L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \boxed{1} & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \end{array}\right) \xrightarrow{L_2 = L_2 \cdot (-1)}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \end{array}\right)$$

$$\underbrace{\hphantom{1 \quad 0 \quad 0}}_{y_3}$$

$\Rightarrow$ B este inversabila, $B^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$

Tema 3

1. a) Folosind dezvoltarea după linia 3, calculați det A.

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} \xlongequal[\substack{C_2 = C_2 - 5 \cdot C_3 \\ C_4 = C_4 + C_3}]{C_1 = C_1 - 2 \cdot C_3} \begin{vmatrix} -3 & -9 & 2 & 5 \\ -5 & -14 & 3 & 7 \\ 0 & 0 & 1 & 0 \\ -5 & -12 & 2 & 6 \end{vmatrix} =$$

pivot

nr linie + nr coloana

$$= 1 \cdot (-1)^{3+3} \cdot \begin{vmatrix} -3 & -9 & 5 \\ -5 & -14 & 7 \\ -5 & -12 & 6 \end{vmatrix} = (-1)^6 \cdot \begin{vmatrix} -3 & -9 & 5 \\ -5 & -14 & 7 \\ -5 & -12 & 6 \end{vmatrix} =$$

$$= (-3)\cdot(-14)\cdot 6 + (-9)\cdot 7 \cdot(-5) + (-5)\cdot(-12)\cdot 5 - 5\cdot(-14)\cdot(-5) - 7\cdot(-12)\cdot(-3) - (-9)\cdot(-5)\cdot 6 =$$

$$= 18 \cdot 14 + 63 \cdot 5 + 60 \cdot 5 - 70 \cdot 5 - 84 \cdot 3 - 45 \cdot 6 =$$

$$= 252 + 315 + 300 - 350 - 252 - 270 = -5$$

b) Calculați det A, folosind regula lui Laplace prin dezvoltare după ultimele 2 linii:

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} \overset{\text{Laplace}}{=} \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} \cdot (-1)^{3+4+1+2} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \cdot (-1)^{3+4+1+3} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} \cdot (-1)^{3+4+1+4} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} +$$

$$+\begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} \cdot (-1)^{3+4+2+3} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} \cdot (-1)^{3+4+2+4} \cdot$$

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} \cdot (-1)^{3+4+3+4} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} =$$

$$= (-4+5) \cdot (-1) \cdot 2 + 5 \cdot (-1) \cdot 1 + (8-1) \cdot 1 + 12 \cdot 1 + (20-2) \cdot (-1) \cdot 1$$

$$+ 6 \cdot 0 =$$

$$= -2 - 5 + 7 + 12 - 18 + 0 = 19 - 25 = -6$$

2.c) Determinați dacă sunt sau nu subspații vectoriale:

- $\mathcal{S} = \{A \in \mathcal{M}_2(\mathbb{R}) \mid {}^tA = A\} \subset \mathcal{M}_2(\mathbb{R})/\mathbb{R}$

  m. simetrice

- $\mathcal{A} = \{B \in \mathcal{M}_2(\mathbb{R}) \mid {}^tB = -B\} \subset \mathcal{M}_2(\mathbb{R})/\mathbb{R}$

  m. antisimetrice

- $\mathcal{U} = \{C \in \mathcal{M}_2(\mathbb{R}) \mid \underset{\overset{\shortparallel}{a+d}}{\mathrm{Tr}\, C} = 0\}$

  $\overset{\shortparallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

  m. de urmă nulă

- $\mathcal{D} = \{D \in \mathcal{M}_2(\mathbb{R}) \mid (\exists)\, \lambda \in \mathbb{R}$ a. î. $D = \lambda I_2\}$

  m. diagonală

Rezolvare:

- $\mathcal{S} = \{A \in \mathcal{M}_2(\mathbb{R}) \mid {}^tA = A\} \subset \mathcal{M}_2(\mathbb{R})/\mathbb{R}$

Verificarea cond. de subspațiu vectorial:

1. Conține matricea nulă

${}^tO_2 = O_2 \Rightarrow O_2 \in \mathcal{S}$

2. Închiderea la adunare

$+ : \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2(\mathbb{R})$ op. internă

$A, B \in \mathcal{S} \Rightarrow {}^tA = A, \; {}^tB = B$

$\underline{{}^t(A+B)} = {}^tA + {}^tB = \underline{A + B} \Rightarrow A + B \in \mathcal{S}$

3. Închiderea la înmulțirea cu scalari

$\cdot : \mathcal{M}_2(\mathbb{R}) \times \mathbb{R} \rightarrow \mathcal{M}_2(\mathbb{R})$ op. externă

$\begin{cases} A \in \mathcal{S} \Rightarrow {}^tA = A \\ \lambda \in \mathbb{R} \end{cases}$

${}^t(\lambda A) = \lambda\, {}^tA = \lambda A \Rightarrow \lambda A \in \mathcal{S}$

$\Rightarrow \mathcal{S}$ este subspațiu vectorial

- $\mathcal{A} = \{B \in \mathcal{M}_2(\mathbb{R}) \mid {}^tB = -B\} \subset \mathcal{M}_2(\mathbb{R})/\mathbb{R}$

1. ${}^tO_2 = -O_2 \Rightarrow O_2 \in \mathcal{A}$

2. $+ : \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R}) \rightarrow \mathcal{M}_2(\mathbb{R})$ op. internă

$$A, B \in \mathcal{A} \Rightarrow \begin{cases} {}^tA = -A \\ {}^tB = -B \end{cases}$$

$${}^t(A+B) = {}^tA + {}^tB = -A - B = -(A+B)$$

$$\Rightarrow A+B \in \mathcal{A}$$

3. $\cdot : \mathcal{M}_2(\mathbb{R}) \times \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$ op. externă

$A \in \mathcal{A} \Rightarrow {}^tA = -A$

$\alpha \in \mathbb{R}$

$${}^t(\alpha A) = \alpha \cdot {}^tA = \alpha \cdot (-A) = -\alpha A \Rightarrow \alpha A \in \mathcal{A}$$

$\Rightarrow \mathcal{A}$ este un subspațiu vectorial

- $\mathcal{U} = \{ C \in \mathcal{M}_2(\mathbb{R}) \mid \mathrm{Tr}\, C = 0 \}$

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathrm{Tr}\, C = a + d$$

$a + d = 0$

1. $\mathrm{Tr}(O_2) = 0 + 0 = 0 \Rightarrow O_2 \in \mathcal{U}$

2. $+ : \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ op. internă

$$A, B \in \mathcal{U} \Rightarrow \begin{cases} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, & a + d = 0 \\ B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, & e + h = 0 \end{cases}$$

$$A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \Rightarrow \mathrm{Tr}(A+B) = \overbrace{a + \underbrace{e + d}_{} + h}^{0} = 0$$

$$\Rightarrow A + B \in \mathcal{U}$$

3. $\cdot : \mathcal{M}_2(\mathbb{R}) \times \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$ op. externă

$A \in \mathcal{U} \Rightarrow a + d = 0$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\alpha \in \mathbb{R}$

$\alpha \cdot A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \Rightarrow Tr(\alpha A) = \alpha \underbrace{(a + d)}_{0} = 0$

$\Rightarrow \alpha A \in \mathcal{U}$

$\Rightarrow \mathcal{U}$ este subspațiu vectorial

$\cdot\ \mathcal{D} = \{ D \in \mathcal{M}_2(\mathbb{R}) \mid (\exists)\ \lambda \in \mathbb{R}$ a.î. $D = \lambda I_2 \}$

$D = \lambda I_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

1. $\lambda = 0 \to 0 \cdot I_2 = O_2 \Rightarrow O_2 \in \mathcal{D}$

2. $+ : \mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ op. internă

$A, B \in \mathcal{D} \Rightarrow A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

$B = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$A + B = \begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{pmatrix} = (\lambda_1 + \lambda_2) \cdot I_2$

$\Rightarrow A + B \in \mathcal{D}$

3. $\cdot : \mathcal{M}_2(\mathbb{R}) \times \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$ op. externă

$$\begin{cases} A \in \eta \\ \alpha \in \mathbb{R} \end{cases}, \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\alpha A = \begin{pmatrix} \alpha\lambda & 0 \\ 0 & \alpha\lambda \end{pmatrix} = (\alpha\lambda) \cdot I_2 \Rightarrow \alpha A \in \eta$$

$\Rightarrow \eta$ subspațiu vectorial

Tema 4

1. Fie $u_1=(1,3,5)$, $u_2=(2,-1,4)$, $u_3=(8,3,22)$, vectori din $\mathbb{R}^3$. Să se arate că $\{u_1,u_2,u_3\}$ este liniar dependentă.

$$\begin{cases}\alpha_1+2\alpha_2+8\alpha_3=0\\ 3\alpha_1-\alpha_2+3\alpha_3=0\\ 5\alpha_1+4\alpha_2+22\alpha_3=0\end{cases} \Rightarrow A=\begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{pmatrix}1&2&8\\3&-1&3\\5&4&22\end{pmatrix}\end{array}$$

$$\det A=\begin{vmatrix}1&2&8\\3&-1&3\\5&4&22\end{vmatrix}=-22+30+96+40-12-132=$$

$= 166-166=0 \Rightarrow$ sistemul este liniar dependent

$$\Delta_p=\begin{vmatrix}1&2\\3&-1\end{vmatrix}=-1-6=-7\neq 0 \Rightarrow rgA=2$$

α_1,α_2 - necunoscute principale

$\alpha_3=\lambda\in\mathbb{R}$ - necunoscută secundară

$$\begin{cases}\alpha_1+2\alpha_2=-8\lambda\\ 3\alpha_1-\alpha_2=-3\lambda \;|\cdot 2\end{cases} \Rightarrow \begin{cases}\alpha_1+2\alpha_2=-8\lambda\\ 6\alpha_1-2\alpha_2=-6\lambda\end{cases}\;(+)$$

$$7\alpha_1 \;/ = -14\lambda$$

$$\alpha_1=\frac{-14\lambda}{7}=-2\lambda$$

$\alpha_1+2\alpha_2=-8\lambda \Leftrightarrow -2\lambda+2\alpha_2=-8\lambda \Leftrightarrow 2\alpha_2=-6\lambda \Leftrightarrow$

$$\Leftrightarrow \alpha_2=\frac{-6\lambda}{2}=-3\lambda$$

$\alpha_1 + 2\alpha_2 + 8\alpha_3 = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3} \Leftrightarrow$

$\Leftrightarrow -2\lambda v_1 - 3\lambda v_2 + \lambda v_3 = 0_{\mathbb{R}^3} \,|: (\lambda) \Leftrightarrow$

$\Leftrightarrow -v_1 - 3v_2 + v_3 = 0_{\mathbb{R}^3}$ (relația de dependență liniară)

2. Să se determine $a \in \mathbb{R}$ pentru care vectorii $v_1 = (1, 3, a)$, $v_2 = (2, -1, 1)$ și $v_3 = (0, 4, a)$ din $\mathbb{R}^3$ sunt liniar independenți

$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ A = & \left(\begin{array}{c} 1 \\ 3 \\ 2 \end{array}\right. & \begin{array}{c} 2 \\ -1 \\ 1 \end{array} & \left.\begin{array}{c} 0 \\ 4 \\ a \end{array}\right) \end{array} \Rightarrow \det A = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 2 & 1 & a \end{vmatrix} =$$

$$= -a + 16 + 0 - 0 - 4 - 6a =$$

$$= 12 - 7a$$

$\det A = 0 \Leftrightarrow 12 - 7a = 0 \Leftrightarrow 12 = 7a \Leftrightarrow a = \frac{12}{7}$

sistemul este liniar independent $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow$

$\Leftrightarrow a \in \mathbb{R} \setminus \{\frac{12}{7}\}$

Tema 5

(11) 1. Fie $m, n \in \mathbb{N}^*$. Să se arate că dimensiunea K-spațiului vectorial $M_{m,n}(K)$ este mn (K este subcorp al lui $\mathbb{C}$).

Fie $A \in \mathcal{M}_{m,n}(K)$ de forma:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{unde } a_{ij} \in K$$

Coeficienți $a_{ij} \in K$ iau orice val din K, spațiul este generat de matricele unitare (matrici care au un singur element 1 și restul 0).

Baza este formată din matricele:

$$E_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \cdots 0 & \cdots 1 & \cdots 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

unde 1 este pe poz (i,j)

$\exists$ $m \times n$ astfel de matrice $\Rightarrow \dim M_{m,n}(K) = m \cdot n$

$\downarrow$

baza

(18) 2. Fie K un subcorp al. lui $\mathbb{C}$ * și $n \in \mathbb{N}^*$. Atunci $\dim K[x]_n = n+1$.

Un polinom în acest spațiu:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n,$$

unde $a_0, a_1, \dots a_n \in K$ (corp)

orice polinom de grad cel mult n poate fi scris ca o combinație liniară a polinoamelor

$$\{1, x, x^2, \dots, x^n\}$$

$\Rightarrow$ aceste polinoame sunt liniar independente, deoarece dacă avem o relație de forma $c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n = 0, \forall x$ atunci coeficienții trebuie să fie toți 0, adică $c_0 = c_1 = \dots = c_n = 0$ $\Rightarrow$ această mulțime formează o bază a spațiului $K[x]_{\leq n}$

Numărul de elemente din această bază este $n+1$, deci:

$$\dim K[x]_{\leq n} = n+1$$

(20) 3. Să se arate că $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$.

Dimensiunea unui spațiu vectorial peste un corp este numărul de vectori dintr-o bază a acelui spațiu.

• $\dim_{\mathbb{C}} \mathbb{C} = 1$

$\left.\begin{array}{l} \text{Baza } \{1\} \\ \forall z \in \mathbb{C} \text{ poate fi scris ca } z \cdot 1 \end{array}\right\} \Rightarrow \dim_{\mathbb{C}} \mathbb{C} = 1$

• $\dim_{\mathbb{R}} \mathbb{C} = 2$

$\forall z \in \mathbb{C}$, $z = a + bi$, $a, b \in \mathbb{R}$

Baza $\{1, i\}$: $v = 1 \cdot v_1 + i \cdot v_2$

$\Rightarrow \dim_{\mathbb{R}} \mathbb{C} = 2$

(12) 4. Fie $v_1 = (1,1,1)$, $v_2 = (2,3,-1)$ și $v_3 = (4,5,1)$ din $\mathbb{R}^3$. Dacă $U = \langle v_1, v_2, v_3 \rangle$, să se determine dimensiunea lui U și o bază a sa.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & -1 & 1 \end{vmatrix} = 3 + 10 - 4 - 12 + 5 - 2 = 18 - 18 = 0$$

$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

$\Rightarrow \text{rang} A = 2$ $\Rightarrow$ 3 vectori nu sunt liniar independenți

$\dim(A) = 2$

GAUSS-JORDAN

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow[C_3 = C_3 - 4C_1]{C_2 = C_2 - 2C_1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -3 & -3 \end{pmatrix} \xrightarrow{C_3 = C_3 - C_2}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & \boxed{0} \\ 1 & 1 & \boxed{0} \\ 1 & -3 & \boxed{0} \end{pmatrix}$$

baza este formată din coloanele nenule rămase

$a_1 = (1,1,1)$, $a_2 = (0,1,-3)$

(4) 5. Fie $U = \{(x+3y+5z, x-y+z, x+2y, x+9z) \mid x, y \in \mathbb{R}\}$. Să se arate că U este subspațiu al lui $\mathbb{R}^4$, să se determine dimensiunea lui U și o bază a sa.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 9 \end{pmatrix} \xrightarrow[\substack{L_3 = L_3 - L_1 \\ L_4 = L_4 - L_1}]{L_2 = L_2 - L_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -4 & -4 \\ 0 & -1 & -5 \\ 0 & -3 & 4 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{L_2 = \frac{L_2}{-4}} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & -3 & 4 \end{pmatrix} \xrightarrow[L_4 = L_4 + 3L_2]{L_3 = L_3 + L_2} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 7 \end{pmatrix} \xrightarrow{L_3 = \frac{L_3}{-4}}$$

$$\longrightarrow \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{pmatrix} \xrightarrow{L_4 = L_4 - 7L_3} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3 rânduri nenule $\Rightarrow$ rang$(A) = 3 \Rightarrow$ dim $(V) = 3$

subspațiu

Pentru a extrage o bază, luăm coloanele corespunzătoare pivoților.

$\langle (1,1,1,1),\ (3,-1,2,0),\ (5,1,0,9) \rangle$

Tema 6

7. Fie $u_1 = (1,2,1,0)$, $u_2 = (-1,1,1,1)$, $v_1 = (2,-1,0,1)$ și $v_2 = (1,-1,3,7)$ vectori din $\mathbb{R}^4$. Fie subspațiile $U = \langle u_1, u_2 \rangle$ și $V = \langle v_1, v_2 \rangle$. Să se determine dimensiunea și o bază pentru fiecare dintre subspațiile U, V, $U+V$, $U \cap V$.

• $U = \langle u_1, u_2 \rangle = \langle \{u_1, u_2\} \rangle = \langle \{(1,2,1,0), (-1,1,1,1)\} \rangle =$

$= \langle S_u \rangle$

$$\left.\begin{array}{l} S_u \subset U \\ \text{sistem de generatori} \end{array}\right\} ①$$

$$\left.\begin{array}{l} A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{L_2 = L_2 + L_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{3} & 2 & 0 \end{pmatrix} \\ \text{avem pivot pe fiecare coloană} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ S.L.I (sistem liniar independent) ②

$$①, ② \Rightarrow \left.\begin{array}{l} S_u \subset U \\ \text{bază } (|S_u| = 2) \end{array}\right\} \Rightarrow \dim_{\mathbb{R}} U = 2$$

$\Downarrow$

U este bază

$\cdot V = \langle v_1, v_2 \rangle = \langle \{v_1, v_2\} \rangle = \langle \{(2,-1,0,1), (1,-1,3,7)\} \rangle =$

$= \langle S_V \rangle$

$V = \langle S_V \rangle \Rightarrow S_V \subset V$ sistem de generatori $\Big\}$ ①

$$B = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 7 \end{pmatrix} \xrightarrow{L_2 = L_2 - \frac{1}{2}L_1}$$

$$\longrightarrow \begin{pmatrix} \boxed{2} & -1 & 0 & 1 \\ 0 & \boxed{-\frac{1}{2}} & 3 & \frac{13}{2} \end{pmatrix}$$

avem pivot pe fiecare coloană $\Big\}$.

$\Rightarrow$ S.L.I ②

①, ② $\Rightarrow$ $S_V \subset V$, $\text{baza}(|S_V|) = 2$ $\Big\} \Rightarrow \dim_{\mathbb{R}} V = 2$

$\Downarrow$

V este bază

$\cdot\ U+V = \mathbb{R}^4$

$U+V = \langle U \cup V \rangle = \{u+v \mid u \in U \text{ și } v \in V\}$

$U = \langle S_u \rangle$, $V = \langle S_v \rangle$ $\Rightarrow U+V = \langle S_u \cup S_v \rangle \Rightarrow S_u \cup S_v \subset U+V$ sistem de generatori.

Forma Gauss (eșalon redusă)

$$C = \begin{array}{c} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & -1 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 7 \end{pmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \end{array} \longrightarrow$$

$$\xrightarrow[L_3 = L_3 - L_1]{L_2 = L_2 - 2L_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & -3 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & 1 & 7 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{L_2 = \frac{L_2}{3}} \begin{pmatrix} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{1} & -\frac{5}{3} & -1 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & 1 & 7 \end{pmatrix} \xrightarrow[\substack{L_3 = L_3 - 2L_2 \\ L_4 = L_4 - L_2}]{L_1 = L_1 + L_2}$$

$$\longrightarrow \begin{pmatrix} \boxed{1} & 0 & 2-\frac{5}{3} & 0 \\ 0 & \boxed{1} & -\frac{5}{3} & -1 \\ 0 & 0 & 4/3 & 4 \\ 0 & 0 & 8/3 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 1/3 & 0 \\ 0 & \boxed{1} & -5/3 & -1 \\ 0 & 0 & 4/3 & 4 \\ 0 & 0 & 8/3 & 8 \end{pmatrix} \xrightarrow{L_3 = \frac{3}{4}L_3}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 & \frac{1}{3} & 0 \\ 0 & \boxed{1} & -\frac{5}{3} & -1 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & \frac{8}{3} & 8 \end{pmatrix} \xrightarrow[\substack{L_2 = L_2 + \frac{5}{3}L_3 \\ L_4 = L_4 - \frac{8}{3}L_3}]{L_1 = L_1 - \frac{1}{3}L_3} \begin{pmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

sunt pivoți pe 3 coloane

$\Rightarrow \dim(U + V) = 3$

• $U \cap V$

T. Grassman : $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V) \Leftrightarrow$

$\Leftrightarrow 3 = 2 + 2 - \dim(U \cap V) \Leftrightarrow$

$\Leftrightarrow \dim(U \cap V) = 1$

$U = \langle S_U \rangle = \{ a u_1 + b u_2 \mid a, b \in \mathbb{R} \}$

$V = \langle S_V \rangle = \{ c v_1 + d v_2 \mid c, d \in \mathbb{R} \}$

$U \cap V \Rightarrow a u_1 + b u_2 = c v_1 + d v_2 \Rightarrow$

$\Rightarrow a u_1 + b u_2 - c v_1 - d v_2 = 0 \Rightarrow$

$\Rightarrow a(1, 2, 1, 0) + b(-1, 1, 1, 1) - c(2, -1, 0, 1) - d(1, -1, 3, 7) = 0 \Rightarrow$

$$\Rightarrow \begin{cases} a - b - 2c - d = 0 & (1) \\ 2a + b + c + d = 0 & (2) \\ a + b - 3d = 0 & (3) \\ b - c - 7d = 0 & (4) \end{cases}$$

$(4): b - c - 7d = 0 \Rightarrow c = b - 7d$

înlocuim

$$\begin{cases} (1): a - b - 2b + 14d - d = 0 \\ (2): 2a + b + b - 7d + d = 0 \end{cases} \Rightarrow \begin{cases} a - 3b + 13d = 0 \\ 2a + 2b - 6d = 0 \end{cases} \Rightarrow ,$$

$$\Rightarrow a - 3b + 13d = 2a + 2b - 6d \Rightarrow$$

$$\Rightarrow a + 5b - 19d = 0$$

$$\begin{cases} (1): a = 3b - 13d \\ (2): a = -b + 3d \end{cases} \Rightarrow 3b - 13d = -a + 3d \Rightarrow$$

$$\Rightarrow 4b = 16d \Rightarrow b = 4d$$

$$c = b - 7d = 4d - 7d = -3d$$

$$(3): a + b - 3d = 0 \Leftrightarrow a + 4d - 3d = 0 \Leftrightarrow a + d = 0 \Leftrightarrow$$

$$\Leftrightarrow a = -d$$

$$d = \alpha, \quad \alpha \in \mathbb{R}$$

$$a u_1 + b u_2 = c v_1 + d v_2 \Leftrightarrow$$

$$\Leftrightarrow -\alpha u_1 + 4\alpha u_2 = -3\alpha v_1 + \alpha v_2 \mid : \alpha \Leftrightarrow$$

$$\Leftrightarrow -u_1 + 4u_2 = -3v_1 + v_2$$

$$U \cap V = \{ \{-1u_1 + 4u_2\} \} = < \{w\} >$$

$$B_n = \{w\} < U \cap V$$

bază

Tema 7

1. Fie $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x-3y, 2x+y, -3x+2y) \in \mathbb{R}^3$

a) f este aplicație liniară

1. Aditivitatea $\quad f(u+v) = f(u) + f(v)$

$u = (x_1, y_1)$, $v = (x_2, y_2)$, $u+v = (x_1+x_2, y_1+y_2)$

$$f(u+v) = (x_1+x_2 - 3(y_1+y_2), 2(x_1+x_2) + y_1+y_2, -3(x_1+x_2) + 2(y_1+y_2)) =$$
$$= (x_1+x_2-3y_1-3y_2, 2x_1+2x_2+y_1+y_2, -3x_1-3x_2+2y_1+2y_2)$$

$$f(u) + f(v) = (x_1-3y_1, 2x_1+y_1, -3x_1+2y_1) + (x_2-3y_2, 2x_2+y_2, -3x_2+2y_2) =$$
$$= (x_1+x_2-3y_1-3y_2, 2x_1+2x_2+y_1+y_2, -3x_1-3x_2+2y_1+2y_2)$$

$\Rightarrow f(u+v) = f(u) + f(v) \quad (1)$

2. Omogenitatea: $f(\lambda \cdot u) = \lambda \cdot f(u)$

$u = (x,y)$, $\lambda \in \mathbb{R}$

$\lambda \cdot u = (\lambda x, \lambda y)$

$f(\lambda u) = f(\lambda x, \lambda y) = (\lambda x - 3\lambda y, 2\lambda x + \lambda y, -3\lambda x + 2\lambda y)$

$= \lambda \cdot f(u)$ (2)

3. $f(0,0) = (0,0,0)$ (3)

(1), (2), (3) $\Rightarrow$ $f(x,y)$ este aplicație liniară

2. Fie $f(x,y,z) = (2x - y + 2z, -x + 2y - z, x + y + z)$.

$$f(x,y,z) = \begin{pmatrix} 2x - y + 2z \\ -x + 2y - z \\ x + y + z \end{pmatrix}$$

Toate componentele sunt combinații liniare de x, y, z (nu apare niciun termen constant sau o putere) $\Rightarrow$ f este aplicație liniară

a) $\mathrm{Ker}\, f$, $\mathrm{Im}\, f$

• $\mathrm{Ker}\, f = \{ \underset{(x,y,z)}{\overset{\shortparallel}{v}} \in \mathbb{R}^3 \mid f(u) = 0_{\mathbb{R}^3} \subseteq \mathbb{R}^3 \}$

$$f(x,y,z) = (0,0,0) \underset{SLO}{\Leftrightarrow} \begin{cases} 2x - y + 2z = 0 \\ -x + 2y - z = 0 \\ x + y + z = 0 \end{cases}$$

$$\underset{\text{mat. asociata}}{A} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\det A = \cancel{4} + \cancel{1} - \cancel{2} - \cancel{4} + \cancel{2} - \cancel{1} = 0 \Rightarrow \mathrm{Ker}(f)$ are soluții nenule

$A \cdot X = O_3$

$$\begin{cases} 2x - y + 2z = 0 \\ -x + 2y - z = 0 \\ x + y + z = 0 \end{cases} \Rightarrow x = -y - z \quad \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2(-y-z) - y + 2z = 0 \\ -(-y-z) + 2y - z = 0 \\ \cancel{-y} - \cancel{z} + \cancel{y} + \cancel{z} = 0 \end{cases} \Leftrightarrow \begin{cases} -2y - \cancel{2z} - y + \cancel{2z} = 0 \\ y + \cancel{z} + 2y - \cancel{z} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -3y = 0 \\ 3y = 0 \end{cases} \Leftrightarrow y = 0$$

$$x = -y - z = 0 - z = -z$$

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \middle| z \in \mathbb{R} \right\}$$

$\Rightarrow \dim(\mathrm{Ker} f) = 1$

- $\mathrm{Im}\, f$

T. rang-defect : $\dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f) = \dim_{\mathbb{R}} \mathbb{R}^3$

$\Leftrightarrow 1 + \dim(\mathrm{Im} f) = 3 \Leftrightarrow$

$\Leftrightarrow \dim(\mathrm{Im} f) = 2$

b) f este bijectivă

$\dim(\mathrm{Ker} f) = 1 \Rightarrow f$ nu este injectivă

$\dim(\mathrm{Im} f) = 2 \neq \dim(\mathbb{R}^3) \Rightarrow f$ nu este surjectivă $\Big] \Rightarrow$

$\Rightarrow f$ nu este bijectivă

c) Spec (f)

Rezolvare ec. caracteristica

$$\det(A - \lambda I_3) = 0$$

$$[f]_{B_{0_3}} = A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} =$$

$$= (2-\lambda)^2(1-\lambda) + 1 - 2 - 2(2-\lambda) + (2-\lambda) - (1-\lambda) =$$

$$= (2-\lambda)^2(1-\lambda) - \cancel{1} - 4 + 2\lambda + \cancel{2} - \cancel{\lambda} - \cancel{1} + \cancel{\lambda} =$$

$$= (2-\lambda)^2(1-\lambda) + 2(\lambda - 2) = (2-\lambda)^2(1-\lambda) - 2(2-\lambda) =$$

$$= (2-\lambda)[(2-\lambda)(1-\lambda) - 2]$$

$\det(A - \lambda I_3) = 0 \Leftrightarrow (2-\lambda)[(2-\lambda)(1-\lambda) - 2] = 0$

I $2 - \lambda = 0 \Rightarrow \lambda = 2$

II $(2-\lambda)(1-\lambda) - 2 = 0 \Leftrightarrow$

$\Leftrightarrow \cancel{2} - 2\lambda - \lambda + \lambda^2 - \cancel{2} = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 - 3\lambda = 0 \Leftrightarrow \lambda(\lambda - 3) = 0 \begin{cases} \lambda = 0 \\ \lambda = 3 \end{cases}$

Spec $(f) = \{0, 2, 3\}$ (valorile proprii ale lui $f(A)$)

$\cdot\ V_{\lambda_1} = \{ \underset{\overset{\shortparallel}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}}{v} \in \mathbb{R}^3 \mid f(v) = \lambda_1 v \}$

$(A - \lambda_1 I_3) v = O_3 \Leftrightarrow (A - 0 \cdot I_3) v = O_3 \Leftrightarrow$

$\Leftrightarrow A \cdot v = O_3 \overset{a)}{\Longrightarrow} v = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$V_{\lambda_1} = \{ \alpha \underbrace{(-1 \ 0 \ 1)}_{v_1} \mid \alpha \in \mathbb{R} \} = \langle v_1 \rangle$

$\underset{\text{baza}}{B_1} = \{ v_1 \} \subset V_{\lambda_1} \Rightarrow \dim V_{\lambda_1} = 1 \Rightarrow mg(\lambda_1) = 1$

$\cdot\ V_{\lambda_2} = \{ \underset{\overset{\shortparallel}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}}{v} \in \mathbb{R}^3 \mid f(v) = \lambda_2 v \}$

$(A - \lambda_2 I_3) v = O_3 \Leftrightarrow (A - 2 I_3) v = O_3$

$A - 2 I_3 = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$

$= \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

$$(A-2\mathcal{Y}_3)v = 0_3 \Leftrightarrow \begin{array}{c} \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} \\ \begin{matrix} x & \quad y & \quad z \end{matrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0_3$$

$$\Delta_p = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

$\Rightarrow rg(A - 2\mathcal{Y}_3) = 2$

$$\begin{cases} x, y - \text{necunoscute principale} \\ z = \beta \quad - \text{necunoscută secundară} \end{cases}$$

$$\begin{cases} -y = -2\beta \\ -x = \beta \end{cases} \Rightarrow \begin{cases} y = 2\beta \\ x = -\beta \end{cases}, \ \beta \in \mathbb{R}$$

$$V_{\lambda_2} = \{ \beta \underbrace{(-1, 2, 1)}_{v_2} \mid \beta \in \mathbb{R} \} = \langle v_2 \rangle$$

$\underset{\text{bază}}{B_2} = \{v_2\} \subset V_{\lambda_2} \Rightarrow \dim V_{\lambda_2} = 1 \Rightarrow mg(\lambda_2) = 1$

$\cdot \; V_{\lambda_3} = \{ \underset{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\overset{}{v}} \in \mathbb{R}^3 \mid f(v) = \lambda_3 v \}$

$$(A - \lambda_3 \mathcal{Y}_3)v = 0_3 \Leftrightarrow (A - 3\mathcal{Y}_3)v = 0_3$$

$$A - 3\mathcal{Y}_3 = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$(A-3I_3)v = O_3 \iff \begin{array}{c} \begin{array}{ccc} x & \ \ y & \ \ z \end{array} \\ \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O_3$$

$$\Delta_p = \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = 1+2 = 3 \neq 0$$

$\Rightarrow rg(A-3I_3) = 2$

$\begin{cases} y, z & \text{- necunoscute principale} \\ x = \gamma & \text{- necunoscută secundară} \end{cases}$

$\begin{cases} -y+2z = \gamma \\ -y-z = \gamma \mid \cdot(-1) \end{cases} \Rightarrow \begin{cases} -y+2z = \gamma \\ +y+z = -\gamma \end{cases}$ (+)

$3z = 0 \Rightarrow z = 0$

$-y+2z = \gamma \iff -y+0 = \gamma \iff y = -\gamma, \ \gamma \in \mathbb{R}$

$V_{\lambda_3} = \{\gamma(\underbrace{1,-1,0}_{v_3}) \mid \gamma \in \mathbb{R}\} = \langle v_3 \rangle$

$B_3 = \{v_3\} \subset V_{\lambda_3} \Rightarrow \dim V_{\lambda_3} = 1 \Rightarrow mg(\lambda_3) = 1$
bază

d) f este endomorfism diagonalizabil

Polinomul caracteristic: $\lambda(\lambda-2)(\lambda-3)$

$\Rightarrow m_a(0) = 1, \ ma(2) = 1, \ ma(3) = 1$

L multiplicitatea algebrică

Conform teoremei de echivalență

$$\begin{cases} 1)\ ma(\lambda_1) + ma(\lambda_2) + ma(\lambda_3) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \\ 2)\ ma(\lambda_i) = mg(\lambda_i),\ \forall i = \overline{1,3} \end{cases}$$

$\Rightarrow f$ este endomorfism diagonalizabil

$\Rightarrow (\exists)\ \tilde{B} \subset \mathbb{R}^3$ a.î. $[f]_{\tilde{B}} = D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\tilde{B} = B_1 \cup B_2 \cup B_3 = \{v_1, v_2, v_3\}$

e) $A^n,\ n \in \mathbb{N}^*$

$$A = [f]_{B_{0_3}} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} B_{0_3} & \xrightarrow{C} & \tilde{B} \\ \downarrow & & \downarrow \\ A & & D = C^{-1}AC \end{array} \Rightarrow A = CDC^{-1}$$

$$A^n = CD\underbrace{C^{-1} \cdot C}_{I_3}D\underbrace{C^{-1} \cdot \ldots \cdot C}_{I_3}\underbrace{\ }_{I_3}DC^{-1} = CD^nC^{-1}$$

$$D^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \qquad C = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$\boxed{C^{-1}}$ → Gauss-Jordan

$$C = \left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_1 = L_1 \cdot (-1)}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_3 = L_3 - L_1}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{L_2 = L_2 + L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{L_2 = \frac{L_2}{2}}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{L_1 = L_1 - L_2 + L_3}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cancel{-1} - \frac{1}{2} + \cancel{1} & -\frac{1}{2} & -\frac{1}{2} + 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \rightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\cdot) \quad C^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^n = C D^n C^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix}$$

$\forall n \in \mathbb{N}^*$

Tema 8

A1. Considerăm aplicația liniară:

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (2x-2y, -2x+y-2z, -2y)$, $(\forall)(x,y,z) \in \mathbb{R}^3$

a) Ker f, Im f

• $\text{Ker} f = \{ v \in \mathbb{R}^3 \mid f(v) = 0_{\mathbb{R}^3} \subseteq \mathbb{R}^3 \}$

$v = (x,y,z)$

$$f(x,y,z) = (0,0,0) \overset{}{\underset{S.L.O}{\Leftrightarrow}} \begin{cases} 2x-2y=0 \\ -2x+y-2z=0 \\ -2y=0 \Rightarrow y=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x-0=0 \\ -2x+0-2z=0 \end{cases} \begin{cases} 2x=0 \\ -2x=2z \end{cases} \Leftrightarrow \begin{cases} x=0 \\ z=0 \end{cases}$$

$x=y=z=0 \Rightarrow \text{Ker} f = \{0_{\mathbb{R}^3}\}$

$\Rightarrow \dim_{\mathbb{R}} \text{Ker} f = 0$

• Im f

Th. rang-defect $\Rightarrow \underbrace{\dim \text{Ker} f}_{0} + \dim \text{Im} f = \underbrace{\dim_{\mathbb{R}} \mathbb{R}^3}_{3} \Rightarrow$

$\Rightarrow \dim \text{Im} f = 3$

$\Rightarrow \text{Im} f = \mathbb{R}^3$

a) f injectivă, surjectivă, bijectivă

$$\left.\begin{aligned} Ker\, f = \{0_{\mathbb{R}^3}\} &\Rightarrow f \text{ inj} \\ Im\, f = \mathbb{R}^3 &\Rightarrow f \text{ surj} \end{aligned}\right\} \Rightarrow f \text{ bijectivă}$$

c) $Spec(f)$

Ecuația caracteristică : $det(A - \lambda I_3) = 0$

$$[f]_{B_0 3} = A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 1-\lambda & -2 \\ 0 & -2 & -\lambda \end{vmatrix} =$$

$$= (2-\lambda)(1-\lambda)\cdot(-\lambda) + 0 + 0 - 0 - 4(2-\lambda) + 4\lambda =$$

$$= (2 - \underbrace{2\lambda - \lambda}_{-3\lambda} + \lambda^2)\cdot(-\lambda) - 8 + 4\lambda + 4\lambda =$$

$$= -2\lambda + 3\lambda^2 - \lambda^3 - 8 + 8\lambda =$$

$$= -\lambda^3 + 3\lambda^2 + 6\lambda - 8 = -(\lambda - 1)(\lambda - 4)(\lambda + 2)$$

$$det(A - \lambda I_3) = 0 \Leftrightarrow -(\lambda - 1)(\lambda - 4)(\lambda + 2) = 0$$

I $\lambda = 1$

II $\lambda = 4$

III $\lambda = -2$

$\mathrm{Spec}(f) = \{-2, 1, 4\}$

• $V_{\lambda_1} = \{ u \in \mathbb{R}^3 \mid f(u) = \lambda_1 u \}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(A - \lambda_1 I_3) u = 0_3 \Leftrightarrow (A + 2 I_3) u = 0_3$

$$A + 2I_3 = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$(A + 2I_3) u = 0_3 \Leftrightarrow \begin{array}{c} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \\ \begin{matrix} x & \;\; y & \;\; z \end{matrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0_3$$

$$\Delta_p = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12 - 4 = 8 \neq 0$$

$\rightarrow rg(A + 2I_3) = 2$

$\begin{cases} x, y - \text{necunoscute principale} \\ z = \alpha - \text{necunoscută secundară} \end{cases}$

$$\begin{cases} 4x - 2y = 0 \mid :2 \\ -2x + 3y = 2\alpha \end{cases} \Rightarrow \begin{cases} 2x - y = 0 \\ -2x + 3y = 2\alpha \end{cases} \oplus$$

$$/ \; 2y = 2\alpha \Rightarrow y = \alpha$$

$$2x - \alpha = 0 \Rightarrow 2x = \alpha \Rightarrow x = \frac{\alpha}{2}$$

$$V_{\lambda_1} = \{ \alpha \underbrace{\left(\tfrac{1}{2}, 1, 1 \right)}_{v_1} \mid \alpha \in \mathbb{R} \} = \langle v_1 \rangle$$

$$B_1 = \{ v_1 \} \subset V_{\lambda_1} \Rightarrow \dim V_{\lambda_1} = 1 \Rightarrow mg(\lambda_1) = 1$$

bază

• $V_{\lambda_2} = \{ \underset{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}{v} \in \mathbb{R}^3 \mid f(v) = \lambda_2 v \}$

$$(A - \lambda_2 I_3) v = O_3 \Leftrightarrow (A - 4 I_3) v = O_3$$

$$A - 4I_3 = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix}$$

$$(A - 4I_3) v = O_3 \Leftrightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O_3$$

$$\Delta p = \begin{vmatrix} -2 & -2 \\ -2 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

$\Rightarrow rg(A - 4 I_3) = 2$

$\begin{cases} x, y - \text{necunoscute principale} \\ z = \beta - \text{necunoscută secundară} \end{cases}$

$\begin{cases} -2x - 2y = 0 \quad | \cdot (-1) \\ -2x - 3y = 2\beta \end{cases} \Leftrightarrow \begin{cases} 2x + 2y = 0 \\ -2x - 3y = 2\beta \end{cases} \oplus$

$1 - y = 2\beta \Rightarrow y = -2\beta$

$-2x - 4\beta = 0 \Rightarrow -2x = 4\beta \Rightarrow x = -2\beta$

$V_{\lambda_2} = \{\beta(\underbrace{-2, -2, 1)}_{v_2} \mid \beta \in \mathbb{R}\} = \langle v_2 \rangle$

$B_2 = \{v_2\} \subset V_{\lambda_2} \Rightarrow \dim V_{\lambda_2} = 1 \Rightarrow m_g(\lambda_2) = 1$

bază

$\cdot\ V_{\lambda_3} = \{ \underset{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\overset{}{v}} \in \mathbb{R}^3 \mid f(v) = \lambda_3 v \}$

$(A - \lambda_3 I_3) v = 0 \Leftrightarrow (A - I_3) v = 0$

$$A - \lambda_3 = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}$$

$$(A - \lambda_3)u = 0_3 \iff \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \iff$$

$$\iff \begin{cases} x - 2y = 0 \Rightarrow x = 2y & (1) \\ -2x - 2z = 0 & (2) \\ -2y - z = 0 & (3) \end{cases}$$

$$\begin{cases} (2): -2 \cdot 2y - 2z = 0 \Rightarrow -4y - 2z = 0 \mid \cdot(-2) \iff \\ (3): -2y - z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2y + z = 0 \\ -2y - z = 0 \end{cases} \quad (+)$$

$$/$$

$$(3): z = -2y$$

$$\begin{cases} (2): -2x - 2 \cdot (-2y) = 0 \Rightarrow -2x + 4y = 0 \\ (1): x - 2y = 0 \end{cases}$$

$$\Rightarrow (x, y, z) = (0, 0, 0)$$

$$V_{\lambda_3} = \{\gamma(0,0,0) \mid \gamma \in \mathbb{R}\} = \{\underbrace{(0,0,0)}_{v_3}\} = \langle v_3 \rangle$$

$$B_{3_{bază}} = \{v_3\} \subset V_{\lambda_3} \Rightarrow \dim V_{\lambda_3} = 0 \Rightarrow mg(\lambda_3) = 0$$

d) f este endomorfism diagonalizabil

Conform teoremei de echivalență

$$\begin{cases} 1) \ ma(\lambda_1) + ma(\lambda_2) + ma(\lambda_3) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \\ 2) \ ma(\lambda_i) = mg(\lambda_i), \ \forall i = \overline{1,3} \end{cases}$$

$\xRightarrow{(2) \text{ fals}}$ f nu este endomorfism diagonalizabil