Lista 2

2.a

Temos que:

t_j: instante com alguma falha observada

n_j: número de indivíduos em risco em t_j

d_j: número de falhas em t_j

$n_1 = n$

O estimador de Kaplan-Meier é:

$$\hat{S}(t) = \begin{cases} 1, & \text{se } t < t_1 \\ \prod_{t_j \le t} \left(1 - \frac{d_j}{n_j}\right), & \text{se } t \geq t_1 \end{cases}$$

Se não existem censuras:

$$n_{j+1} = n_j - d_j = n_{j-1} - d_{j-1} - d_j = n - \sum_{k=1}^{j} d_k$$

Assim, se $t \geq t_1$

$$\hat{S}(t) = \prod_{t_j \le t} \left(1 - \frac{d_j}{n_j}\right)$$

$$\hat{S}(t) = \prod_{t_j \le t} \left(\frac{n_j - d_j}{n_j}\right)$$

Sendo t_i o t_j máximo que respeita $t_j \le t$:

$$\hat{S}(t) = \left(\frac{n_i - d_i}{n_i}\right) \cdot \left(\frac{n_{i-1}\ d_{i-1}}{n_{i-1}}\right) \cdots \left(\frac{n - d_1}{n}\right)$$

Como $\quad n_i = n_{i-1} - d_{i-1}$

$$n_{i-1} = n_{i-2} - d_{i-2}$$

$$\vdots$$

$$n_2 = n - d_1$$

Então:

$$\hat{S}(t) = \left(\frac{n_i - d_i}{\cancel{n_i}}\right)\left(\frac{\cancel{n_i}}{n_{i-1}}\right)\left(\frac{\cancel{n_{i-1}}}{n_{i-2}}\right)\cdots\left(\frac{\cancel{n_2}}{n}\right)$$

$$\hat{S}(t) = \frac{n_i - d_i}{n} = \frac{n_{i+1}}{n} = \frac{n^{\circ}\text{ obs} > t}{n} \quad \text{cqd}$$

2.b

Temos que:

$$\hat{Var}(\hat{S}(t)) = [\hat{S}(t)]^2 \cdot \sum_{t_j \leq t}\left[\left(\frac{d_j}{n_j - d_j}\right)n_j\right]$$

Lembremos que, sem censura:

$$d_j = n_{j+1} - n_j$$

Assim:

$$\frac{d_j}{n_j - d_j} = \frac{n_j - n_{j+1}}{n_j - n_j + n_{j+1}} = \frac{n_j - n_{j+1}}{n_{j+1}}$$

Assim:

$$\left(\frac{d_j}{n_j - d_j}\right)n_j = \frac{n_j - n_{j+1}}{n_{j+1}\, n_j} = \frac{1}{n_{j+1}} - \frac{1}{n_j}$$

Assim

$$\sum_{j:t_j\le t}\left(\frac{d_j}{n_j-d_j}\right)_n=\left(\frac{1}{n_{i+1}}-\frac{1}{n_i}\right)+\left(\frac{1}{n_i}-\frac{1}{n_{i-1}}\right)+\dots+\left(\frac{1}{n_2}-\frac{1}{n_1}\right)$$

$n_1=n$

$$\hat{V}ar(\hat{S}(t))=[\hat{S}(t)]^2\cdot\left[\frac{1}{n_{i+1}}-\frac{1}{n}\right]$$

$\hat{S}(t)=\dfrac{n_{i+1}}{n}$, portanto:

$$\hat{V}ar(\hat{S}(t))=\frac{n_{i+1}^2}{n^2}\left[\frac{1}{n_{i+1}}-\frac{1}{n}\right]$$

$$\hat{V}ar(\hat{S}(t))=\hat{S}(t)\cdot\left[\frac{n_{i+1}}{n}\right]\cdot\left[\frac{1}{n_{i+1}}-\frac{1}{n}\right]$$

$$=\hat{S}(t)\cdot\left[\frac{1}{n}-\frac{n_{i+1}}{n^2}\right]$$

$$=\hat{S}(t)\cdot n\cdot\left[1-\frac{n_{i+1}}{n}\right]$$

$\therefore\ \hat{V}ar(\hat{S}(t))=\hat{S}(t)\cdot[1-\hat{S}(t)]\cdot n$ cqd