2.

Temos que: $X \sim exp(\lambda)$

$Y \sim exp(\theta)$

X e Y independentes

$\delta = I(X \leq Y) \qquad Z = \min(X, Y)$

2.a

Sem perda de generalidade, no lugar de $P(X \in [t, t+\Delta t])$ com $\Delta t \to 0$, vamos escrever simplesmente $P(X = t)$.

Assim:

$$P(\delta = 1) = \int_0^\infty P(X = t, Y \geq t)\,dt \overset{ind}{=} \int_0^\infty P(X = t)\,P(Y \geq t)\,dt$$

$$P(X = t) = \lambda e^{-\lambda t}$$

$$P(Y \geq t) = e^{-\theta t}$$

$$P(\delta = 1) = \int_0^\infty \lambda e^{-(\lambda+\theta)t}\,dt$$

$$P(\delta = 1) = \lambda \cdot \left[- \frac{e^{-(\lambda+\theta)t}}{(\lambda+\theta)} \right]_{t=0}^{t \to \infty}$$

Lembremos que: $\lim_{t\to\infty} \frac{e^{-at}}{a} = 0$, se $a>0$

Assim:

$$P(\delta=1) = \frac{\lambda}{\lambda+\theta}$$

2.b

$$P(Z>z) = P(X>z\,;\,Y>z) \overset{ind}{=} P(X>z)P(Y>z)$$

$$P(Z>z) = e^{-(\lambda+\theta)z}, \quad z>0$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\left[1 - e^{-(\lambda+\theta)z}\right]$$

Assim:

$$F_Z(z) = 1 - e^{-(\lambda+\theta)z}$$

$$f_Z(z) = (\lambda+\theta)\,e^{-(\lambda+\theta)z}, \quad z>0$$

$$\therefore\ Z \sim \exp(\lambda+\theta)$$

2.c

Se forem independentes:

$$P(\delta = k \mid Z = z) = P(\delta = k) \quad \forall \ k \in \{0,1\}$$
$$z > 0$$

Para $k = 1$:

$$P(\delta = 1 \mid Z = z) = \frac{P(X \leq Y \cap \min(X,Y) = z)}{P(Z = z)}$$

$$P(\delta = 1 \mid Z = z) = \frac{P(X = z)\, P(Y \geq z)}{P(Z = z)}$$

$$= \frac{\lambda e^{-\lambda t} e^{-\theta t}}{(\lambda + \theta)\, e^{-(\lambda+\theta)t}}$$

$$\therefore \ P(\delta = 1 \mid Z = z) = \frac{\lambda}{\lambda + \theta} = P(\delta = 1) \quad \forall z > 0$$

Analogamente:

$$P(\delta = 0 \mid Z = z) = \frac{\theta}{\lambda + \theta} = P(\delta = 0) \quad \forall z > 0$$

Portanto

δ e Z são independentes //

2.d

Lembremos que:

$$P(\delta=1)=\frac{\lambda}{\lambda+\theta} \qquad P(\delta=0)=\frac{\theta}{\lambda+\theta}=1-\frac{\lambda}{\lambda+\theta}$$

Assim, escrevendo $p=\frac{\lambda}{\lambda+\theta}$, temos que:

$$P(\delta=1)=p \qquad f_\delta(x)=P(\delta=x)=\left[\frac{\lambda}{\lambda+\theta}\right]^{x}\left[\frac{\theta}{\lambda+\theta}\right]^{1-x}\mathbb{1}_{\{0,1\}}(x)$$

$$P(\delta=0)=1-p$$

Logo, torna-se evidente que

$$\delta \sim \text{Bernoulli}\left(\frac{\lambda}{\lambda+\theta}\right)$$

De forma então que a distribuição da soma de n Bernoullis iid é:

$$D \sim \text{Binomial}\left(n,\frac{\lambda}{\lambda+\theta}\right)$$

2.e

$$E(\hat{\lambda})=E\left[\frac{\sum\delta_i}{\sum Z_i}\right]=E\left[\frac{\delta_1}{\sum Z_i}+\frac{\delta_2}{\sum Z_i}+\dots+\frac{\delta_n}{\sum Z_i}\right]$$

Onde $\sum Z_i \sim \text{Gama}(n,\lambda+\theta)$, pois $Z_i \sim \exp(\lambda+\theta)$

Além disso:

$$E\left[\frac{\delta_i}{\sum z_i}\right] = E\left[E\left(\frac{\delta_i}{\sum z_i} \,\middle|\, \delta_i\right)\right]$$

$$= P(\delta_i = 0) \cdot E\left[\frac{0}{\sum z_i}\right] + P(\delta_i = 1)\, E\left[\frac{1}{\sum z_i}\right]$$

$$= 0 + \frac{\lambda}{\lambda+\theta} \cdot \int_0^\infty \frac{1}{x} \cdot \frac{[\lambda+\theta]^n}{\Gamma(n)} \cdot e^{-(\lambda+\theta)x} \cdot x^{n-1}\, dx$$

$$= \lambda \cdot \int_0^\infty \frac{1}{\Gamma(n)} \cdot [\lambda+\theta]^{n-1} \cdot e^{-(\lambda+\theta)x} \cdot x^{n-2}\, dx$$

Sabemos que, sendo n um inteiro positivo:

$$\Gamma(n) = (n+1)! = (n+1) \cdot n! = (n+1)\,\Gamma(n-1)$$

Assim:

$$E\left[\frac{\delta_i}{\sum z_i}\right] = \frac{\lambda}{(n+1)} \cdot \underbrace{\int_0^\infty \frac{[\lambda+\theta]^{n-1}}{\Gamma(n-1)} \cdot e^{-(\lambda+\theta)x} \cdot x^{n-2}\, dx}_{= \int_0^\infty \text{Gama}(n-1,\, \lambda+\theta)\, dx = 1}$$

Assim:

$$E\left[\frac{\delta_i}{\sum z_i}\right] = \frac{\lambda}{n+1} \qquad \therefore \quad E[\hat{\lambda}] = \frac{n}{n+1}\lambda$$