Seminar 7 - group 812

16 November 2022 10:11

$\lim_{n\to\infty} x_n = 3 \neq 0 \Rightarrow \sum x_n$ D.

$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = \frac{1}{\infty} = 0 \Rightarrow ?$

Ideea: $\lim_{n\to\infty} x_n \neq 0 \Rightarrow \sum x_n$ D.

$n! < n^n \quad \forall n \geq 2$

$\frac{1}{\sqrt[n]{n}} < \frac{1}{\sqrt[n]{n!}} \Leftrightarrow \frac{1}{n} < x_n$; $\sum \frac{1}{n}$ D. $\xRightarrow{C1C}$ $\sum x_n$ D.

$$a) \sum_{n\geq1} \frac{3n^4+7}{\sqrt{n^8+7n^3}}, \quad b) \sum_{n\geq1} \frac{1}{\sqrt[n]{n}}, \quad c) \sum_{n\geq1} \frac{1}{\sqrt[n]{n!}}, \quad d) \sum_{n\geq1} \left(1+\frac{1}{3n}\right)^{-n}.$$

$\to$ D. (b)

$e^{-1/3} \neq 0 \Rightarrow$ D.

Exercise 2: Determine the nature (convergence or divergence) of the following series of real numbers:

Ideea C1C

it is a sum of 2 geo. series

$\frac{2^n+3^n}{5^n} < \frac{2^n \cdot 3^n}{5^n}$ ✓

$$a) \sum_{n\geq1} \frac{2^n+3^n}{5^n}, \quad b) \sum_{n\geq1} \frac{2^n}{3^n+5^n}.$$

$\lim_{n\to\infty} x_n = 0 \Rightarrow ?$ $\quad 3^n+5^n > 2\cdot 3^n \Rightarrow \frac{2^n}{3^n+5^n} < \frac{1}{2}\cdot\left(\frac{2}{3}\right)^n$ C. $\quad \xRightarrow{C1C}$ C.

Exercise 3: Determine the nature (convergence or divergence) of the following series of real numbers: ideea C2C

$\sim \frac{1}{n}$ D. $\quad \sim \frac{1}{n^3}$ $\quad \to$ D $\quad y_n = \frac{1}{n} \quad \lim \frac{x_n}{y_n} = \frac{1}{2}$

$$a) \sum_{n\geq2} \frac{1}{3n-4}, \quad b) \sum_{n\geq1} \frac{1}{(4n-1)^3}, \quad c) \sum_{n\geq1} \frac{1}{\sqrt{4n^2-1}}, \quad d) \sum_{n\geq1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}.$$

$\lim_{n\to\infty} \frac{x_n}{y_n} \in \boxed{(0,\infty)}$

$\lim_{n\to\infty} \frac{x_n}{\frac{1}{n^\alpha}} = \lim_{n\to\infty} n^\alpha \cdot x_n \in (0,\infty)$

α - at our choice

Exercise 4: Determine the nature (convergence or divergence) of the following series of real numbers:

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$ C

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{2}{e} < 1$ C.

$\frac{3}{e} > 1$ D $\quad \to$ C.

$$a) \sum_{n\geq1} \frac{100^n}{n!}, \quad b) \sum_{n\geq1} \frac{2^n n!}{n^n}, \quad c) \sum_{n\geq1} \frac{3^n n!}{n^n}, \quad d) \sum_{n\geq1} \frac{(n!)^2}{2^{n^2}}, \quad e) \sum_{n\geq1} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}.$$

Exercise 5: Determine the nature (convergence or divergence), by discussing the value of the parameter $a > 0$, of the following series of real numbers:

$$a) \sum_{n\geq1} \frac{a^n}{n^n}, \quad b) \sum_{n\geq1} \left(\frac{n^2+n+1}{n^2}a\right)^n, \quad c) \sum_{n\geq1} \frac{3^n}{2^n+a^n}.$$

$$) \sum_{n\geq1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}.$$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n}}}{n^{5/3}\sqrt{1-\frac{1}{n^4}}} = 0 \Rightarrow ?$

Because in the general term x_n we encounter n at a constant power first we try a comparison to $y_n = \frac{1}{n^\alpha}$

$\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} \cdot n^\alpha = \lim_{n\to\infty} \frac{n^\alpha \cdot n\sqrt{1+\frac{1}{n}}}{n^{5/3}\sqrt{1-\frac{1}{n^4}}} = \lim_{n\to\infty} \frac{n^{1+\alpha}}{n^{5/3}} = 1 \in (0,\infty)$

$1+\alpha = \frac{5}{3}$

$\boxed{\alpha = \frac{2}{3}}$

$\lim_{n\to\infty} \frac{x_n}{y_n} \in (0,\infty)$

$\sum y_n = \sum \frac{1}{n^{2/3}}$ D. $\left(\frac{2}{3} < 1\right)$

$\xRightarrow{C2C} \sum x_n \sim \sum \frac{1}{n^{2/3}}$ D.

d) $\sum \frac{(n!)^2}{\ldots}$ $\quad \lim \frac{x_{n+1}}{x_n}$ $\quad \lim \frac{[(n+1)!]^2}{\ldots} \cdot \frac{2^{n^2}}{\ldots} =$

4d) $\sum \frac{(n!)^2}{2^{n^2}}$

$$\boxed{\lim_{n\to\infty} \frac{x_{n+1}}{x_n}} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{\cancel{[n!]^2}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n^2+2n+1}} \cdot 2^{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}} =$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} \left(\frac{n+1}{2^n}\right)^2 = \frac{1}{2}\left[\lim_{n\to\infty}\left(\frac{n+1}{2^n}\right)\right]^2 = \frac{1}{2}\cdot 0 = 0$$

$\boxed{\lim_{n\to\infty} \frac{n+1}{2^n} = ?}$ A nice proof for $\boxed{\lim_{n\to\infty} \frac{n+1}{2^n} = 0}$

- Let us $t_n = \frac{n+1}{2^n}$ and consider $\sum t_n = \sum \frac{n+1}{2^n}$

D'Alembert $\Rightarrow \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \lim_{n\to\infty} \frac{n+2}{2^{n+1}} \cdot \frac{2^n}{n+1} = \frac{1}{2} < 1$

$\Downarrow$

$\sum t_n$ C.

$\Downarrow$

$\lim_{n\to\infty} x_n = 0$

- If we consider $\lim_{n\to\infty} \sqrt[n]{t_n} = \lim_{n\to\infty} \frac{\sqrt[n]{(n+1)}}{2} = \frac{1}{2} < 1$ C

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0 < 1 \Rightarrow \sum x_n \text{ is C.}$$

e) $\sum_{n\geq 1} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$.

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{\left(\overset{1}{\sqrt[n]{n}}\right)^2}{\left(2+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1 \quad \text{C.}$$

$\boxed{a>0}$

5a) $\sum_{n\geq 1} \frac{a^n}{n^n}$

! whenever we have $\boxed{n^n}$ or... similar things the first choice should be $\lim_{n\to\infty} \sqrt[n]{x_n}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{a}{n}\right)^n} = \lim_{n\to\infty} \frac{a}{n} = 0 < 1 \Rightarrow \sum x_n \text{ is C (no matter how } a \text{ looks like)}$$

5b) $\sum_{n\geq 1} \left(\frac{n^2+n+1}{n^2} a\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{n^2+n+1}{n^2} \cdot a = a \begin{cases} <1 & \text{C.} \\ >1 & \text{D.} \\ =1 & ? \end{cases}$$

$\boxed{a=1}$ $x_n = \left(\frac{n^2+n+1}{n^2} \cdot 1\right)^n \Rightarrow \lim_{n\to\infty} \left(\frac{n^2+n+1}{n^2}\right)^n \overset{1^\infty}{=} \left(1+\frac{n+1}{n^2}\right)^n = e \neq 0$

$\Downarrow$

$\boxed{\sum x_n \text{ D.}}$

Conclusion $\sum x_n \begin{cases} \text{C} & a<1 \\ \text{D} & a\geq 1 \end{cases}$

5c) $\sum_{n\geq 1} \frac{3^n}{2^n+a^n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3^n}{2^n+a^n} = \begin{cases} \frac{3^n}{2^n\left(1+\left(\frac{a}{2}\right)^n\right)} & : 2>a \\ \frac{1}{2}\cdot\left(\frac{3}{2}\right)^n & : 2=a \end{cases} =$$

5c) $\sum_{n\geq 1} \frac{3^n}{2^n+a^n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3^n}{2^n+a^n} = \begin{cases} 2^n\left(1+\left(\frac{a}{2}\right)^n\right) \\ \frac{1}{2}\cdot\left(\frac{3}{2}\right)^n & : 2=a \\ \left(\frac{3}{a}\right)^n \cdot \frac{1}{\left(\frac{2}{a}\right)^n+1} & : a>2 \end{cases} =$$

$$= \begin{cases} \lim_{n\to\infty}\left(\frac{3}{2}\right)^n & : 2>a \\ \frac{1}{2}\lim\left(\frac{3}{2}\right)^n & : 2=a \\ \lim_{n\to\infty}\left(\frac{3}{a}\right)^n & : a>2 \end{cases} = \begin{cases} \infty \neq 0 & : a\leq 2 \\ \infty \neq 0 & : 3>a \\ 1 \neq 0 & : 3=a \\ 0 & : \boxed{3<a,\ a>2} \end{cases}$$

$$\boxed{\lim_{n\to\infty} x_n \neq 0 \quad \forall\, a\leq 3 \Rightarrow \sum x_n \text{ este } D.}$$

II $a>3$ $\quad \lim_{n\to\infty} x_n = 0 \Rightarrow ? \longrightarrow \frac{3^n}{2^n+a^n} \leq \frac{3^n}{a^n} = \left(\frac{3}{a}\right)^n = q^n$

$|q| = \left|\frac{3}{a}\right| < 1 \Rightarrow \sum q^n$ C $\quad$ $x_n \leq q^n$ $\Big\} \Rightarrow \sum x_n$ C.

sau $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} \dots$

$$\sum_{n\geq 1} \frac{5^n}{7^n+a^n} \leq \left(\frac{5}{7}\right)^n \text{ C.}$$

Determine the nature of the series $\sum_{n\geq 1} \frac{e^n\cdot n!}{n^n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{e^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{e^n\cdot n!} = \lim_{n\to\infty} e\cdot\left(\frac{n}{n+1}\right)^n = e\cdot\frac{1}{e} = 1 \Rightarrow ?$$

Raabe-Duhamel

$$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left(\frac{1}{e}\cdot\frac{(n+1)^n}{n^n}-1\right) = \lim_{n\to\infty} \overset{\infty}{n}\cdot\overset{0}{\left(\frac{1}{e}\cdot\left(1+\frac{1}{n}\right)^n-1\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{e}\left(1+\frac{1}{n}\right)^n-1}{\frac{1}{n}} = -\frac{e}{2} < 1 \quad \boxed{\Rightarrow D.}$$

We define $f:(0,\infty)\to\mathbb{R}$ $\quad f(x) = \frac{1}{e}\cdot\left(1+\frac{1}{x}\right)^x - 1$

$g:(0,\infty)\to\mathbb{R}$ $\quad g(x) = \frac{1}{x}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)}$ $\quad$ $\left.\begin{matrix}\frac{1}{x}=t \\ x\to\infty\end{matrix}\right\} \Rightarrow t\to 0$

$$\| \quad \lim_{t\to 0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to 0} \frac{\frac{1}{e}(1+t)^{\frac{1}{t}}-1}{t} \overset{\frac{0}{0}}{\underset{LH}{=}} \lim_{t\to 0} \frac{1}{e}\left[(1+t)^{\frac{1}{t}}\right]'$$

$$\left(\ \ \frac{1}{t}\right)' \quad \left(\ln(1+t)^{\frac{1}{t}}\right)' \quad \left(e^{\frac{1}{t}\ln(1+t)}\right)' = e^{\frac{1}{t}\ln(1+t)}\cdot\left(\frac{1}{t}\cdot\ln(1+t)\right)' =$$

$g(t)$

$$\left((1+t)^{\frac1t}\right)' = \left(e^{\ln(1+t)^{\frac1t}}\right)' = \left(e^{\frac1t \ln(1+t)}\right)' = e^{\frac1t\ln(1+t)}\cdot\left(\frac1t\cdot\ln(1+t)\right)' =$$

$$= (1+t)^{\frac1t}\cdot\left(-\frac{1}{t^2}\ln(1+t)+\frac1t\cdot\frac{1}{t+1}\right) =$$

$$= \lim_{t\to0}\frac{(1+t)^{\frac1t}}{t^2(t+1)}\cdot\left(t-(1+t)\cdot\ln(1+t)\right) =$$

$$= \underbrace{\lim_{t\to0}\left[(1+t)^{\frac1t-1}\right]}_{1^\infty}\xrightarrow{} e \cdot\lim_{t\to0}\frac{t-(1+t)\ln(1+t)}{t^2(t+1)} = \boxed{-\frac e2}$$

$$e^{\lim_{t\to0} t\cdot\left(\frac1t-1\right)} \overset{\lim_{t\to0}(1-t)}{=} e = e$$

$$\lim_{t\to0}\frac{t-(1+t)\ln(1+t)}{t^3+t^2} \underset{L'H}{\overset{\frac00}{=}} \lim_{t\to0}\frac{1-\ln(1+t)-1}{3t^2+2t} =$$

$$= \lim_{t\to0}\frac{-\ln(1+t)}{3t^2+2t} \overset{\frac00}{=} \lim_{t\to0}\frac{-\frac{1}{1+t}}{6t+2} =$$

$$= \lim_{t\to0} -\frac{1}{(1+t)(6t+2)} = -\frac12$$