812 Seminar 12

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Power series

$\sum_{n\geq 0} a_n x^n$ $(a_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$

- $\mathcal{C} = \{ x \in \mathbb{R} : \sum_{n\geq 0} a_n \cdot x^n \text{ is C} \}$ → convergence set

$0 \in \mathcal{C}$ $a_0 + a_1 \cdot x^1 + \dots a_n x^n + \dots$

$x = 0 \Rightarrow a_0$

$\mathcal{C} \neq \phi$

- Abel - Dirichlet th. $(-R, R) \subseteq \mathcal{C} \subseteq [R, R]$
- R - the convergence radius $R = \frac{1}{\lambda}$, $\lambda = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

↳ CAUCHY HADAMARD Th.

Algorithm for handling power series:

(1) Determine $\lambda \Rightarrow R = \frac{1}{\lambda} \Rightarrow (-R, R) \subseteq \mathcal{C} \subseteq [-R, R]$

(2) Analyse separately the nature of the series:

a) $x = -R$

b) $x = R$

(3) Use (1) & (2) $\Rightarrow \mathcal{C}$.

Example 1: Determine the convergence set for the following power series:

$$\sum_{n\geq 0} x^n \qquad a_n = 1 \quad \forall n \in \mathbb{N}$$

(1) $\lambda = \lim_{n\to\infty} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1 \Rightarrow R = 1 \Rightarrow (-1, 1) \subseteq \mathcal{C} \subseteq [-1, 1]$

(2) a) $\boxed{x = -1} \Rightarrow \sum_{n\geq 0} (-1)^n$ D. (without a sum) $\Rightarrow -1 \notin \mathcal{C}$

b) $\boxed{x = 1} \Rightarrow \sum_{n\geq 0} 1^n = \sum_{n\geq 0} 1$

$s_1 = 1$

$s_2 = 1 + 1 = 2$

D. (with the sum ∞)

$s_n = n \Rightarrow \lim_{n\to\infty} s_n = \infty \Rightarrow 1 \notin \mathcal{C}$

(3) $\mathcal{C} = (-1, 1) \Rightarrow \forall x \in (-1, 1)$ $\sum_{n\geq 0} x^n$ is C.

$$\underbrace{\qquad\qquad\qquad\qquad}$$

$|x|<1$ $\sim \sum q^{n-1}$ the geometric series

<u>Example 2</u> $\sum_{n\geq 1} \frac{(-1)^n}{n(2n+1)} \cdot x^n$ $\qquad a_n = \frac{(-1)^n}{n(2n+1)}$

① $\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)(2n+2)}{n(2n+1)} = 1 \Rightarrow R = \frac{1}{1} = 1 \Rightarrow (-1,1) \subseteq \mathcal{C} \subseteq [-1,1]$

② $\boxed{x=-1}$ $\sum_{n\geq 0} \frac{(-1)^n}{n(2n+1)} \cdot (-1)^n = \sum_{n\geq 0} \frac{(-1)^{2n}}{n(2n+1)} = \sum_{n\geq 0} \frac{1}{n(2n+1)} \sim \sum \frac{1}{n^2}$

$\leq \frac{1}{n^2}$

$\alpha = 2 \Rightarrow C. \Rightarrow -1 \in \mathcal{C}$

$\boxed{x=1}$ $\underbrace{\sum_{n\geq 1} \frac{(-1)^n}{n(2n+1)}}_{u_n}$

$\sum |u_n| = \sum \frac{1}{n(2n+1)}$ C. $\Rightarrow \sum u_n$ A.C. $\Rightarrow \sum u_n$ C.

or

with Leibniz. $\qquad \Rightarrow 1 \in \mathcal{C}$

③ $\mathcal{C} = [-1,1]$

<u>Example 3:</u> $\sum_{n\geq 0} (n+1)^n \cdot x^n$ $\qquad a_n = (n+1)^n$

① $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (n+1) = \infty \Rightarrow R = \frac{1}{\infty} = 0 \Rightarrow \left.\begin{matrix}\mathcal{C} \subseteq \{0\} \\ 0 \in \mathcal{C}\end{matrix}\right\} \Rightarrow \mathcal{C} = \{0\}$

$\forall x \in \mathbb{R}^*$ the series $\sum (n+1)^n \cdot x^n$ is D.

$\sum (n+1)^n \cdot 2^n$ is D

$3^n \quad \ldots$

$(-2)^n \ldots$

<u>Example 4:</u> $\sum_{n\geq 1} \frac{1}{n!} x^n$ $\qquad a_n = \frac{1}{n!}$

Example 4: $\sum_{n\geq 1} \frac{1}{n!} x^n$ $\qquad a_n = \frac{1}{n!}$

① $\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0 \Rightarrow R = \infty \Rightarrow$

$(-\infty, \infty) \subseteq \mathcal{C} \subseteq [-\infty, \infty]$

$$\left.\begin{array}{l} (!)\ \mathcal{C} \subseteq \mathbb{R} \\ \mathbb{R} \subseteq \mathcal{C} \subseteq \overline{\mathbb{R}} \end{array}\right\} \Rightarrow \mathcal{C} = \mathbb{R}$$

$\forall x \in \mathbb{R}$, $\sum \frac{1}{n!} x^n$ is C.

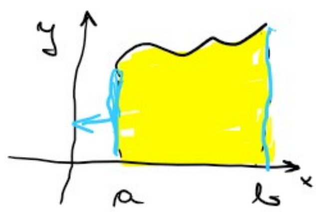
$$\left.\begin{array}{l} \downarrow \\ 2^n \\ 3^n \\ \\ (-5)^n \ \ldots \end{array}\right\} C.$$

Consequence: $\sum \frac{x^n}{n!}$ is C $\Rightarrow \boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$

Improper integrals

- $(a, b]$
- $[a, b)$
- $[a, \infty)$

$$\underbrace{\int_{a^+}^{b} \quad \int_a^{b^-} \quad \int_a^{\infty}}$$



$(a, b]$

$\lim_{x \to \text{to point}} F(x) \longrightarrow$ antiderivative of f

↳ primitiva

$\int_a^b f(x)\,dx = F(b) - F(a)$

Examples: $f : (0,1) \to \mathbb{R}$ $\qquad f(x) = \frac{1}{x(x+1)}$. Study its ii on $(0,1)$.

① $\int f(x)dx = \int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \ln x - \ln(x+1) + \mathcal{C} =$

$= \ln \frac{x}{x+1} + \mathcal{C}$

We choose $F(x) = \ln \frac{x}{x+1}$ as an antiderivative for f

② problems: $\boxed{0 \text{ and } 1}$ $\rightarrow \boxed{0}$ $\lim_{x\to 0} F(x) = \lim_{x\to 0} \ln \frac{x}{x+1} = \ln 0 = -\infty$ $\nexists$

② problems: $\boxed{0 \text{ and } 1}$ $\rightarrow \boxed{0}$ $\lim\limits_{\substack{x\to 0\\ x>0}} F(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \ln \frac{x}{x+1} = \ln 0 = -\infty \; \nexists$

$(0,1)$

1 $\lim\limits_{\substack{x\to 1\\ x<1}} F(x) = \lim\limits_{\substack{x\to 1\\ x<1}} \ln \frac{x}{x+1} = \ln \frac{1}{2} \; \exists$

$$\int_{0^+}^{1^-} f(x)dx = \lim\limits_{\substack{x\to 1\\ x<1}} F(x) - \lim\limits_{\substack{x\to 0\\ x>0}} F(x) = \ln\frac{1}{2} - (-\infty) = \infty \; \exists \notin \mathbb{R}$$

$\boxed{\text{the ii exists and is D. (with the value } \infty)}$

Example 2: $f: (1,\infty) \to \mathbb{R}$ $\quad f(x) = \frac{1}{x(x+1)}$

① $F: (1,\infty) \to \mathbb{R}$ $\quad F(x) = \ln \frac{x}{x+1}$

② problem points: 1 and ∞

$\lim\limits_{\substack{x\to 1\\ x>1}} F(x) = \ln \frac{1}{2}$ (Ex. 1) $\quad \exists \in \mathbb{R}$

$\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty} \ln \frac{x}{x+1} = \ln 1 = 0 \quad \exists \in \mathbb{R}$

$$\Rightarrow \exists \int_{1^+}^{\infty} f(x)dx = \lim\limits_{x\to\infty} F(x) - \lim\limits_{\substack{x\to 1\\ x>1}} F(x) = 0 - \ln\frac{1}{2} = \ln\left(\frac{1}{2}\right)^{-1} = \ln 2 \in \mathbb{R}$$

$\Rightarrow \boxed{\text{the ii is C.}}$

Example 3: $f: (-1,1) \to \mathbb{R}$ $\quad f(x) = \frac{1}{\sqrt{1-x^2}}$

① $\int f(x)dx = \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + \mathcal{C}$

② problem points: (-1) and 1

⊙ $\lim\limits_{\substack{x\to -1\\ x>-1}} F(x) = \arcsin(-1) = -\arcsin 1 = -\frac{\pi}{2} \quad \exists \in \mathbb{R}$

⊙ $\lim\limits_{\substack{x\to 1\\ x>1}} F(x) = \arcsin 1 = \frac{\pi}{2} \quad \exists \in \mathbb{R}$

$$\Rightarrow \exists \int_{-1^+}^{1^-} f(x)\,dx = \lim_{\substack{x\to 1\\ x<1}} F(x) - \lim_{\substack{x\to -1\\ x>-1}} F(x) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \quad \checkmark\ C.$$

Example 4: $f:(0,\infty)\to\mathbb{R}$ $\quad f(x)=\ln x$

① $\int f(x)dx = \int \ln x\,dx = \int (x)'\ln x\,dx = x\ln x - \int x\cdot \ln' x\,dx = x\ln x - \int x\cdot\frac{1}{x}dx =$

$= x\ln x - \int dx = x\ln x - x + \mathcal{C} = x(\ln x - 1) + \mathcal{C}$

We choose $F:(0,\infty)\to\mathbb{R}$ $\quad F(x) = x(\ln x-1)$ as an antiderivative of f

② problem points: 0 and ∞

- $\lim_{\substack{x\to 0\\ x>0}} F(x) = \lim_{\substack{x\to 0\\ x>0}} x(\ln x - 1) \overset{0\cdot(-\infty)}{=\!=} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln x - 1}{\frac{1}{x}} \overset{\frac{-\infty}{\infty}}{\underset{L'H}{=}}$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{\substack{x\to 0\\ x>0}} -x = 0 \quad \exists \in \mathbb{R}$$

- $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} x\cdot(\ln x - 1) = \infty\cdot(-\infty) = -\infty \quad \exists \in \mathbb{R}$

$$\text{so } \int_{0^+}^{\infty} f(x)\,dx = \lim_{x\to\infty} x(\ln x-1) - \lim_{x\to 0} F(x) = -\infty - 0 = -\infty$$

$\exists \notin \mathbb{R}$

$\Rightarrow$ it is $\boxed{D}$

Example 5: $f:\left(\frac{1+\sqrt{3}}{2}, 2\right] \to \mathbb{R}$ $\quad \boxed{f(x) = \frac{1}{x\sqrt{2x^2-2x-1}}}$

① $\int f(x)dx = \int \frac{1}{x\sqrt{2x^2-2x-1}}dx = \int \frac{1}{x^2\sqrt{2-\frac{2}{x}-\frac{1}{x^2}}}dx =$

$$= \int \frac{\left(-\frac{1}{x}\right)'\,dx}{\sqrt{2-\frac{2}{x}-\frac{1}{x^2}}} \longrightarrow = \int \frac{-dt}{\sqrt{2-2t-t^2}} =$$

$t = \frac{1}{x}$ $\qquad dt = -\frac{1}{x^2}dx \Rightarrow -dt = \frac{1}{x^2}dx$

$$\boxed{\begin{array}{ll} x = \frac{1+\sqrt{3}}{2} & \Rightarrow t = \frac{2}{1+\sqrt{3}} \\ x = 2 & \Rightarrow t = \frac{1}{2} \end{array}}$$

$$= \int \frac{-dt}{\sqrt{3-(t+1)^2}} =$$

$$= \frac{-1}{\sqrt{3}} \int \frac{dt}{\sqrt{1-\left(\frac{t+1}{\sqrt{3}}\right)^2}} =$$

$$\frac{t+1}{\sqrt{3}} = u \qquad \frac{dt}{\sqrt{3}} = du \Rightarrow dt = \sqrt{3}\,du$$

$$= -\frac{1}{\sqrt{3}} \cdot \int \frac{\sqrt{3}\,du}{\sqrt{1-u^2}} = -\int \frac{du}{\sqrt{1-u^2}} =$$

$$= -\arcsin u = -\arcsin\left(\frac{t+1}{\sqrt{3}}\right) + \mathcal{C}$$

$$= -\arcsin\left(\frac{\frac{1}{x}+1}{\sqrt{3}}\right) + \mathcal{C}$$

$$= -\arcsin \frac{1+x}{x\sqrt{3}} + \mathcal{C}$$

We choose $F(x) = -\arcsin \frac{1+x}{x\sqrt{3}}$

• pb. $\frac{1+\sqrt{3}}{2}$ $\lim\limits_{\substack{x \to \frac{1+\sqrt{3}}{2} \\ >}} F(x) = -\arcsin \frac{1+\frac{1+\sqrt{3}}{2}}{\frac{1+\sqrt{3}}{2}\cdot\sqrt{3}} =$

$$= -\arcsin \frac{3+\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{3+\sqrt{3}} =$$

$$= -\arcsin 1 = -\frac{\pi}{2} \quad \exists \in \mathbb{R}$$

$$\int_{\frac{1+\sqrt{3}}{2}^+}^{2} f(x)\,dx = F(2) - \lim_{\substack{x \to \frac{1+\sqrt{3}}{2} \\ x >}} F(x) =$$

$$= -\arcsin \frac{1+2}{2\sqrt{3}} - \left(-\frac{\pi}{2}\right) = -\arcsin \boxed{\frac{3}{2\sqrt{3}}} + \frac{\pi}{2} =$$

$$\to \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{2} - \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{3\pi - 2\pi}{6} = \boxed{\frac{\pi}{6}}$$