Lecture 2

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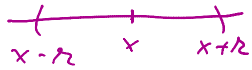
The extension of the topology from $\mathbb{R}$, to $\overline{\mathbb{R}}$

Def: For $r>0$, • $B(\infty, r) = (r, \infty]$
the ball centered at ∞, of radius r

• $B(-\infty, r) = [-\infty, -r)$
the ball centered at $-\infty$, of radius r

• if $x \in \mathbb{R}$, $B(x, r) = (x-r, x+r)$



Def 2: A set $V \subseteq \overline{\mathbb{R}}$ is a neighbourhood of an element $\boxed{a \in \overline{\mathbb{R}}}$ if $\exists r>0$ s.t. $B(a, r) \subseteq V$.

Notation: $\mathcal{V}(a) = \{ V \subseteq \overline{\mathbb{R}},\ V$ is a neighbourhood of $a \}$.

Examples of neighbourhoods in $\overline{\mathbb{R}}$:

• $a = \infty$, $\underbrace{(2, \infty],\ [2, \infty],\ \mathbb{Z} \cup [2, \infty],\ \mathbb{Q} \cup [2, \infty]}_{\in \mathcal{V}(\infty)}$

$\underbrace{\overline{\mathbb{R}} \setminus \mathbb{Q},\ \mathbb{R},\ \mathbb{N} \cup \{\infty\},\ \mathbb{Q} \cup \{\infty\}}_{\notin \mathcal{V}(\infty)}$

• $a = -\infty$ $\underbrace{\overline{\mathbb{R}},\ [-\infty, c),\ [-\infty, 0],\ [-\infty, 0] \cup \mathbb{N}}_{\in \mathcal{V}(-\infty)}$

$\underbrace{\mathbb{R},\ \mathbb{Q} \cup \{-\infty\},\ (0, \infty],\ \dots}_{\notin \mathcal{V}(-\infty)}$

Remark: T_1 & T_2 are the same for $\mathbb{R} \to \overline{\mathbb{R}}$.

II Sequences of real numbers

Consider $k \in \mathbb{N} \cup \{0\}$ $\quad \mathbb{N}_k = \{ n : n \geq k,\ \in \mathbb{N} \cup \{0\} \}$

Def: Each function $f : \mathbb{N}_k \to \mathbb{R}$, is a sequence of real numbers.

Notation: • $\forall n \in \mathbb{N}_k$ $\quad f(n) := x_n$ - the general term of the sequence of rank n

• $(x_n)_{n \geq 1} = (x_n)_{n \in \mathbb{N}} = (x_n)$ is the notation for a sequence of real numbers

$\boxed{\text{Remark:}}$ Sequences may be introduced:

• explicitly : $\forall n \in \mathbb{N}$ $\quad x_n = n^2 + 7n + 8$

• implicitly (when the expression of x_n depending on n is sometimes unachivable)

$\to$ recurrent sequences

$$\begin{cases} x_1 = 7 \\ x_2 = 20 \\ x_{n+2} = x_n^2 + 7x_{n-1} \end{cases}$$

II.1. Limits of sequences

Def: Consider $(a_n) \subseteq \mathbb{R}$, a sequence of real numbers
$l \in \overline{\mathbb{R}}$, an element.

l is said to be a LIMIT of the sequence (a_n) if
$\forall V \subseteq \overline{\mathbb{R}} \in \mathcal{V}(l),\ \exists m_V \in \mathbb{N}$, s.t. $\forall m \geq m_V,\ x_m \in V$.

T (characterizes the limit of sequences in terms of balls) $\quad V = B(l, \varepsilon)$

Consider $(a_n) \subseteq \mathbb{R}$ a sequence of real numbers
$l \in \overline{\mathbb{R}}$, an element.

$l = \lim\limits_{m \to \infty} a_m \quad \boxed{\Leftrightarrow}$

a) $(l \in \mathbb{R})$ $\forall \varepsilon > 0,\ \exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon,\ |x_m - l| < \varepsilon$

b) $(l = \infty)$ $\forall \varepsilon > 0,\ \exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon,\ x_m > \varepsilon$

c) $(l = -\infty)$ $\forall \varepsilon > 0,\ \exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon,\ x_m < -\varepsilon$.

Proof:

a) $l \in \mathbb{R}$

$l = \lim\limits_{m \to \infty} a_m \Leftrightarrow \forall \varepsilon > 0,\ \exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon,\ |a_n - l| < \varepsilon \quad \boxed{2}$

$\boxed{\Rightarrow}$ We know: $l = \lim\limits_{m \to \infty} a_m \overset{def}{\Leftrightarrow} \forall V \in \mathcal{V}(l),\ \exists m_V \in \mathbb{N}$ s.t. $\forall m \geq m_V,\ a_m \in V \quad (1)$

Choose $\varepsilon > 0$ randomly ... ? $|a_m - l| < \varepsilon \Leftrightarrow a_m \in B(l, \varepsilon)$

$B(l, \varepsilon) \in \mathcal{V}(l) \xrightarrow{(1)} \exists m' \in \mathbb{N}$ s.t. $\forall m \geq m',\ a_m \in B(l, \varepsilon)$

$\Updownarrow$

$|a_m - l| < \varepsilon$

ε - random $\Rightarrow (2)$ ✓

$\boxed{\Leftarrow}$ We know: (2)
We want: (1)

Choose $V \in \mathcal{V}(l)$ randomly $\quad$? $a_n \in V$

$\Updownarrow$ def.

$\exists r > 0$, s.t. $B(l, r) \subseteq V$

$r \to \varepsilon \xrightarrow{(2)} \exists m_r \in \mathbb{N}$ s.t. $\forall m \geq m_r,\ |a_m - l| < r$

$\Updownarrow$

$a_m \in B(l, r) \subseteq V \Rightarrow a_m \in V$

V - random $\Rightarrow (1)$ is true ✓

that is why $\Leftrightarrow$ ✓

b) $l = \lim\limits_{m \to \infty} a_m = \infty \Leftrightarrow \forall \varepsilon > 0,\ \exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon,\ x_m > \varepsilon \quad (3)$

$(1) \Leftrightarrow (3)$

$\boxed{\Rightarrow}$ $(1) \Rightarrow (3)$

Choose $\varepsilon > 0$ randomly $\quad$? $x_m > \varepsilon$

$B(\infty, \varepsilon) \in \mathcal{V}(\infty)$

$\xrightarrow{(1)} \exists m'' \in \mathbb{N}$ s.t. $\forall m \geq m'',\ x_m \in B(\infty, \varepsilon)$

$\Uparrow$

$\xrightarrow{(1)} \exists n'' \in \mathbb{N}$ s.t. $\forall n \geq n''$, $x_n \in B(\infty, \varepsilon)$

$\Updownarrow$

$x_n > \varepsilon$

ε-random $\Rightarrow$ (3) ✓

$\boxed{\Leftarrow}$ (3) $\Rightarrow$ (1)

Choose $\underbrace{V \in \mathcal{V}(\ell)}_{=\mathcal{V}(\infty)}$ randomly

$\Updownarrow$ def

$\exists r > 0$ s.t. $B(\infty, r) \subseteq V$

$\xrightarrow[r=\varepsilon]{(3)} \exists \overline{n} \in \mathbb{N}$ s.t. $\forall n \geq \overline{n}$, $x_n > r$

$\Updownarrow$

$x_n \in B(\infty, r) \subseteq V$

$x_n \in V$

V-random $\Rightarrow$ (1) ✓

Classical examples:

a) $a_n := \frac{1}{n}$, $\forall n \in \mathbb{N}$ $\qquad \lim\limits_{n\to\infty} a_n = 0$

Prove that $\lim\limits_{n\to\infty} a_n = 0 \overset{T_1}{\underset{a)}{\Longleftrightarrow}} \underbrace{\forall \varepsilon > 0}_{1}, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $\underbrace{|a_n - 0| < \varepsilon}_{2}$ $\quad (*)$

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We prove $(*)$. Choose $\varepsilon > 0$ randomly

$|a_n| < \varepsilon \Leftrightarrow \left|\frac{1}{n}\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon$

Archimedes axiom:

$\forall x > 0$, $\exists n_x \in \mathbb{N}$ s.t. $x < n_x$ $\qquad n_x := [x] + 1$

$\Leftrightarrow \forall r > 0$ $\exists n_r \in \mathbb{N}$ s.t. $0 < \boxed{\frac{1}{n_r} < r}$

$\frac{1}{n_r} < r \Leftrightarrow \boxed{\frac{1}{r}} < n_r$ ✓ $\quad \to x = \frac{1}{r}$

$\xRightarrow{\text{Arch. Ax}} \exists n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{n_\varepsilon} < \varepsilon$

if $n \geq n_\varepsilon \Rightarrow \frac{1}{n} \leq \frac{1}{n_\varepsilon}$ $\qquad \frac{1}{n} < \varepsilon$

For $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $\frac{1}{n} < \varepsilon \Leftrightarrow \left|\frac{1}{n}\right| < \varepsilon \Leftrightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$ $\qquad (*)$ ✓ for $\varepsilon > 0$ random $\Rightarrow (*)$ ✓

$(*) \Leftrightarrow \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

... Prove that $\lim b_n = \infty$.

$(*) \iff \lim_{n\to\infty} \frac{1}{n} = 0.$

b) Consider $b_n := n \quad \forall n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} b_n = \infty$.

$$\lim_{n\to\infty} b_n = \infty \overset{T2b)}{\iff} \underbrace{\forall \varepsilon > 0}_{1}, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \underbrace{\forall n \ge n_\varepsilon}_{3}, \underbrace{b_n > \varepsilon}_{2}. \quad (**)$$

We prove $(**)$. Choose $\varepsilon > 0$ randomly

$? \{ \ b_n > \varepsilon \iff n > \varepsilon$

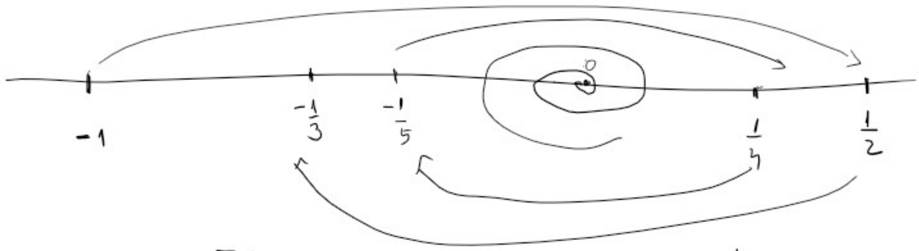
$\downarrow$ Arh.

$\left.\begin{array}{l} \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \varepsilon < n_\varepsilon \\ \text{for } n \ge n_\varepsilon \end{array}\right\} \Rightarrow \varepsilon < n = b_n$

For $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \ge n_\varepsilon$, $b_n > \varepsilon \implies \left.\begin{array}{l} (**) \text{ true for } \varepsilon > 0 \\ \varepsilon \text{ random} \end{array}\right\} \Rightarrow (**)$ ✓

$\overset{T2b}{\Rightarrow} \lim_{n\to\infty} n = \infty.$

c) Consider $c_n = \frac{(-1)^n}{n}$. $\quad ? \lim_{n\to\infty} c_n = 0$



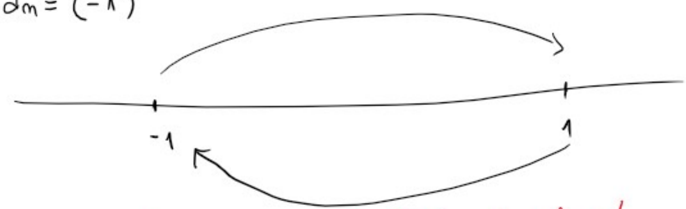
Prove that $\lim_{n\to\infty} c_n = 0 \overset{T1a)}{\iff} \underbrace{\forall \varepsilon > 0}_{1}, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \ge n_\varepsilon$, $\underbrace{|c_n - 0| < \varepsilon}_{2}$.

Choose $\varepsilon > 0$

$? \{ \ |c_n - 0| < \varepsilon \iff \left| \frac{(-1)^n}{n} - 0 \right| < \varepsilon \iff \frac{1}{n} < \varepsilon$ (true a))

from a) $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \ge n_\varepsilon$ $\frac{1}{n} < \varepsilon \implies \left| \frac{(-1)^n}{n} - 0 \right| < \varepsilon \iff \lim_{n\to\infty} c_n = 0$

d) $\forall n \in \mathbb{N}$, $d_n = (-1)^n$



$\nexists \lim_{n\to\infty} d_n$ (Prove it) $\iff \forall l \in \overline{\mathbb{R}}, \ l \neq \lim_{n\to\infty} d_n$

$\iff \forall l \in \overline{\mathbb{R}}, \exists V \in \mathcal{V}(l), \forall n_V \in \mathbb{N}, \exists n \ge n_V$ s.t. $d_n \notin V$

Case 1: $\boxed{l \in \mathbb{R}} \iff \exists \varepsilon > 0, \forall n_\varepsilon \in \mathbb{N}, \exists \boxed{n' \ge n_\varepsilon}$ with $|(-1)^n - l| \not< \varepsilon$

$n = $ even $\quad (-1)^n = 1$

$|1 - l| = \begin{cases} 1 - l \ ; & 1 \ge l \\ l - 1 \ ; & 1 < l \end{cases}$

$n =$ even $\quad (-1) = 1$

$$|1-l| = \begin{cases} 1-l \ ; \ 1 \geq l \\ l-1 \ ; \ 1 < l \end{cases}$$

$t \geq 0$

Subcase: $\boxed{t > 0}$ $\quad t \geq \varepsilon$

for $\boxed{\varepsilon := t}$, $\forall n_\varepsilon \in \mathbb{N}, \exists n' = 2n_\varepsilon$ with

$|(-1)^{2n_\varepsilon} - l| = t \geq \varepsilon = t$ ~~sd.~~

$\not< \varepsilon$

$\Rightarrow l \neq \lim_{n \to \infty} d_n$

Subcase: $t = 0$

$\Rightarrow \boxed{l = 1 \text{ or } l = -1}$

$\hookrightarrow$ HW

Case: $\boxed{l \neq \infty} \Leftrightarrow \exists \varepsilon > 0, \forall n_\varepsilon \in \mathbb{N}, \exists n \geq n_\varepsilon$ with $d_n \not> \varepsilon$

$\varepsilon = 2, \quad n = 2n_\varepsilon \quad (-1)^{2n} = 1 \not> 2$

Case 3: $\boxed{l = -\infty}$ HW

$\boxed{\text{Def:}}$ We say that a sequence $(a_n) \subseteq \mathbb{R}$ is:

- INCREASING if $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$ (strict crescător)
- NONDECREASING if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$ (crescător)
- DECREASING if $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$ (strict descrescător)
- NONINCREASING if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$ (descrescător)
- MONOTONIC if either non incr. or non decr.
- STRICTLY MONOTONIC if either incr. or decr.
- NON-MONOTONIC if none of the above

Consider a sequence $(a_n) \subseteq \mathbb{R}$, denote by $A_n := \{a_n : n \in \mathbb{N}\} = f(\mathbb{N})$

- the set of all the values of the sequence
- the image of the function

Def: A sequence $(a_n) \subseteq \mathbb{R}$ is said to be

- LOWER BOUNDED if $LB(A_n) \neq \emptyset$
- UPPER BOUNDED if $UB(A_n) \neq \emptyset$
- BOUNDED if it is both LB and UB
- UNBOUNDED if both LB and UB are empty

T (Weirstrass theorem on monotonic sequences)

Consider $(a_n) \subseteq \mathbb{R}$ a sequence of real numbers.

T (Weirstrass theorem on monotonic sequences")

Consider $(a_n) \subseteq \mathbb{R}$ a sequence of real numbers.

a) if (a_n) is non decreasing $\Rightarrow \exists \lim\limits_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$

b) if (a_n) is non increasing $\Rightarrow \exists \lim\limits_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$

Proof: → next time

Remark: • A sequence is said to be CONVERGENT if $\exists\, l = \lim\limits_{n\to\infty} a_n$ and $l \in \mathbb{R}$.

• A sequence is said to be DIVERGENT if it is not convergent

So either → $\exists \lim\limits_{n\to\infty} a_n \in \{-\infty, \infty\}$

or ↘ $\nexists \lim\limits_{n\to\infty} a_n$.

Basically speaking $b_n := n,\ \forall n \in \mathbb{N}$ is divergent, and it has $\lim\limits_{n\to\infty} b_n = n$.

$d_n = (-1)^n,\ \forall n \in \mathbb{N}$ is divergent, but it does not have a limit.