Mathematical Logic
Lecture 5
04.11.2022

Theorem 1 (characterization of injective functions).

Let $f : A \to B$ be a function. The following statements are equivalent:

(i) f is injective.

(ii) f is <u>left-cancellable</u>:

$$A' \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \xrightarrow{f} B$$

i.e. $\forall\, \alpha, \beta : A' \to A$,

$$f \circ \alpha = f \circ \beta \implies \alpha = \beta$$

(iii) (Assume $A \neq \emptyset$) f has a <u>left inverse</u> (<u>retraction</u>):

$$A \underset{r}{\overset{f}{\rightleftarrows}} B$$

i.e. $\exists\, r : B \to A$ s.t. $r \circ f = 1_A$

$$\boxed{\begin{array}{l}(i) \Longleftrightarrow (ii)\\ \Updownarrow \\ (iii)\end{array}}$$

$$\begin{tikzcd} & (i) \arrow[dl, Rightarrow] & \\ (iii) \arrow[rr, Rightarrow, "?"'] & & (ii) \arrow[ul, Rightarrow] \end{tikzcd}$$

H.W. (cyclic proof)

Proof

<u>(i) $\Rightarrow$ (ii)</u> We assume that f is injective.

$\boxed{p \to (q \to r) \Longleftrightarrow p \wedge q \to r}$

Let $\alpha, \beta : A' \to A$ such that $f \circ \alpha = f \circ \beta$.

We prove that $\alpha = \beta$.

Let $a' \in A'$. We know $(f \circ \alpha)(a') = (f \circ \beta)(a')$

hence $f(\alpha(a')) = f(\beta(a')) \overset{f\ \text{inj}}{\Longrightarrow} \underline{\alpha(a') = \beta(a')}$.

So $\alpha = \beta$.

$\boxed{p \to q \Longleftrightarrow \neg q \to \neg p}$

contraposition.

proof by contradiction.

<u>(ii) $\Rightarrow$ (i)</u>

<u>$\neg$(i) $\Rightarrow$ $\neg$(ii)</u> We assume that f is not inj.

So $\exists\, x_1, x_2 \in A$, $x_1 \neq x_2$ and $f(x_1) = f(x_2)$

$\neg(p \to q) \Longleftrightarrow \neg(\neg p \vee q)$

$\Longleftrightarrow p \wedge \neg q$

We will prove that $\exists\, \alpha, \beta : A' \to A$ s.t.

$f \circ \alpha = f \circ \beta$ and $\alpha \neq \beta$.

Let $A' = \{x_1, x_2\}$. Let $\alpha, \beta : A' \longrightarrow A$

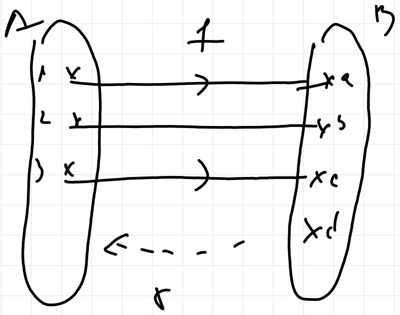
	x_1	x_2
$\alpha(x)$	x_1	x_2
$\beta(x)$	x_1	x_1

so $\alpha \neq \beta$.

$$\left.\begin{array}{l} \begin{cases} (f\circ\alpha)(x_1) = f(\alpha(x_1)) = f(x_1) \\ (f\circ\alpha)(x_2) = f(\alpha(x_2)) = f(x_2) \end{cases} \\ \begin{cases} (f\circ\beta)(x_1) = f(\beta(x_1)) = f(x_1) \\ (f\circ\beta)(x_2) = f(\beta(x_2)) = f(x_1) \underset{hyp}{=} f(x_2) \end{cases} \end{array}\right\} \Longrightarrow f\circ\alpha = f\circ\beta.$$

(i) $\Longrightarrow$ (iii) We assume that f is injective as $A \neq \emptyset$.

Example



$r(a) = 1$
$r(b) = 2$
$r(c) = 3$
$r(d) \in A$ so there are 3 possibly.
so f has 3 different retractions

We define $r : B \longrightarrow A$. Let $b \in B$.

- If $b \in \operatorname{Im} f$ then $\exists!\, a \in A$ s.t. $f(a) = b$. Define $r(b) = a$
- If $b \notin \operatorname{Im} f$ then $r(b)$ can be any element of A

We have $(r\circ f)(a) = r(f(a)) = r(b) = \underset{\mathbb{1}_A(a)}{\underset{\shortparallel}{a}}$ $\forall a \in A$

hence $r\circ f = \mathbb{1}_A$

(iii) $\Rightarrow$ (i). Let r be a left inverse of f.

We show that f is injective.

Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$.

We get $r(f(x_1)) = r(f(x_2)) \Longrightarrow (r \circ f)(x_1) = (r \circ f)(x_2)$

By assumption we get $1(x_1) = 1(x_2)$, i.e. $x_1 = x_2$. $\blacksquare$

Theorem 2 (characterization of surjective functions)

Let $f : A \to B$ be a function. The foll. statements are equiv:

(i) f is surjective.

(ii) f is right-cancellable:

$$A \xrightarrow{f} B \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B' \qquad \text{i.e. } \forall \alpha, \beta : B \to B'$$

$$\alpha \circ f = \beta \circ f \Longrightarrow \alpha = \beta$$

(iii) f has a right inverse (section):

$$A \underset{s}{\overset{f}{\rightleftarrows}} B \qquad \text{i.e. } \exists s : B \to A \text{ st } f \circ s = 1_B$$

Proof (i) $\Rightarrow$ (ii) Assume that f is surjective.

Let $\alpha, \beta : B \to B'$ st $\alpha \circ f = \beta \circ f$.

We will prove that $\alpha = \beta$.

Let $b \in B$. Then $\exists a \in A$ st $f(a) = b$.

We have $\alpha(b) = \alpha(f(a)) = (\alpha \circ f)(a) \overset{hyp}{=} (\beta \circ f)(a)$

$= \beta(f(a)) = \beta(b)$

Hence $\alpha = \beta$.

(ii) $\Longrightarrow$ (i)

$\Updownarrow$

$\neg$(i) $\Rightarrow \neg$(ii)

We assume that f is not surjective.

hence $\exists\, b_0 \in B \setminus \operatorname{Im} f$, i.e.

$$f(a) \neq b_0 \quad \forall\, a \in A.$$

We want to find two functions $\alpha, \beta : B \to B'$ st

$$\alpha \circ f = \beta \circ f \quad \text{and} \quad \alpha \neq \beta.$$

Let $B' := B$. Let $\alpha : B \to B$, $\alpha(b) = b \quad \forall\, b \in B$.

Let $\beta : B \to B$, $\beta(b) = \begin{cases} b, & \text{if } b \neq b_0 \\ b_1 \neq b_0, & \text{if } b = b_0 \end{cases}$

hence $\alpha \neq \beta$.

Now let $a \in A$: we have

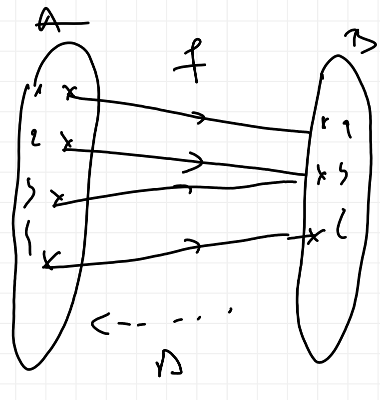
$$(\alpha \circ f)(a) = \alpha(f(a)) = f(a).$$

$$(\beta \circ f)(a) = \beta(f(a)) = f(a) \text{ because } f(a) \neq b_0$$

Hence $\alpha \circ f = \beta \circ f$

(i) $\Longrightarrow$ (iii)

Exple:



$s(a) = 1$

$s(b)$ can be 2 or 3

$s(c) = 4$.

Assume that f is surjective. Let $b \in B$. We chose an element a st $f(a) = b$ (we have $f^{-1}(b) \neq \emptyset$)

Define $s(b) = a$. We have:

$$(f \circ s)(b) = f(s(b)) = f(a) = b = 1_B(b)$$

Hence $f \circ s = 1_B$.

(iii) $\Rightarrow$ (i) Let $s : B \to A$ s.t. $f \circ s = \mathbb{1}_B$.

We prove that f is surjective.

Let $b \in B$. Let $a := s(b)$

We have $f(a) = f(s(b)) = (f \circ s)(b) \overset{\text{hypothesis}}{=} \mathbb{1}_B(b) = b$.

Hence $b \in \operatorname{Im} f$, hence f is surj. ■

Theorem 3 (characterization of bijective functions)

Let $f : A \to B$ be a function. Then:

f is bijective $\Longleftrightarrow$ f is invertible (i.e. $\exists\, g : B \to A$ s.t. $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$.)

Proof. "$\Rightarrow$". Assume that f is bijective. We want to find an inverse of f.

Let $b \in B$. Then $\exists!\, a \in A$ s.t. $f(a) = b$

We define $g(b) = a$.

We have: $(g \circ f)(a) = g(f(a)) = g(b) = a = \mathbb{1}_A(a)$

$(f \circ g)(b) = f(g(b)) = f(a) = b = \mathbb{1}_B(b)$

"$\Leftarrow$". • g is a left inverse for f $\overset{\text{Thm 1}}{\Longrightarrow}$ f is injective

• g is a right inverse for f $\overset{\text{Thm}}{\Longrightarrow}$ f is surjective.

Remarks 1) A bijective function has a unique inverse.

Indeed: if g_1, g_2 are inverses, then

$\mathbb{1}_A = g_1 \circ f = g_2 \circ f \Longrightarrow g_1 = g_2$

We denote the unique inverse of f by f^{-1}.

We have:
$$\boxed{f(x) = y \iff f^{-1}(y) = x}$$

2) We know that any function f has the inverse relation $f^{-1} = (B, A, F^{-1})$, which is not a function in general. The above arguments show that the rel f^{-1} is a function $\iff$ f is bijective

Homework ex 42 – 46, 51, 52

Equivalence relations

Def 1. A homogeneous relation $\rho = (A, A, R)$ is called an equivalence relation if:

preorder:

(R) ρ is reflexive: $\forall x \in A \quad x \rho x$
($\text{i.e. } 1_A \subseteq \rho$)

not reflexive

ex $d_1 \perp d_2$
because $\cancel{d \perp d}$

(T) ρ is transitive: $\forall x, y, z \in A \quad x \rho y$ and $y \rho z \Longrightarrow x \rho z$
(i.e. $x\, \rho \circ \rho\, z \Longrightarrow x \rho z$
i.e. $\rho \circ \rho \subseteq \rho$)

(S) ρ is symmetric: $\forall x, y \in A \quad x \rho y \Longrightarrow y \rho x$
(ρ is not sym: $\exists\, x, y \in A$ st $x \rho y$ and $y \not\rho x$) (i.e. $\rho = \rho^{-1}$)

Examples 1). the equality relation on A: $1_A = (A, A, \Delta_A)$
is equivalence

2). Divisibility on $\mathbb{Z}$: $a \mid b \overset{\text{def}}{\Longleftrightarrow} \exists\, x \in \mathbb{Z}$ st $b = ax$.

Rem. • $a \mid 0 \ \forall a \in \mathbb{Z}$ bc $0 = 0a$
• in part $0 \mid 0$ ($0 : 0$ is not defined)
• $0 \mid a \Longleftrightarrow a = 0$

(R) $a \mid a$ is true because $a = 1 \cdot a$

(T) assume $a \mid b$, $b \mid c$ so $\exists\, x, y \in \mathbb{Z}$ st $b = ax$, $c = by$.
then $c = axy$ so $a \mid c$

~~(S)~~ $a \mid b \overset{?}{\Longrightarrow} b \mid a$ not in gen. e.g. $2 \mid 4$ and $4 \nmid 2$

3) The relation of <u>congruence modulo n</u> on $\mathbb{Z}$

Let $n \in \mathbb{N}$. We define the relation $\equiv (\text{mod } n)$ on $\mathbb{Z}$:

If $a, b \in \mathbb{Z}$, then $a \equiv b (\text{mod } n) \overset{\text{def}}{\Longleftrightarrow} n \mid b-a$

e.g. $22 \equiv 57 (\text{mod } 5)$

(R) $a \equiv a (\text{mod } n) \Longleftrightarrow n \mid a-a$ (true)

(T) $a \equiv b (\text{mod } n), b \equiv c (\text{mod } n) \Longrightarrow$

$\Longrightarrow n \mid b-a, \; n \mid c-b \Longrightarrow n \mid (b-a)+(c-b)$

$\Longrightarrow n \mid c-a \Longrightarrow a \equiv c (\text{mod } n)$

(S) $a \equiv b (\text{mod } n) \Longrightarrow n \mid b-a \Longrightarrow n \mid a-b \Longrightarrow b \equiv a (\text{mod } n)$

Hence $\equiv (\text{mod } n)$ is an equivalence relation on $\mathbb{Z}$.

<u>Def 2</u> Let A be a set. A subset $\pi \subseteq \mathcal{P}(A) \setminus \{\emptyset\}$

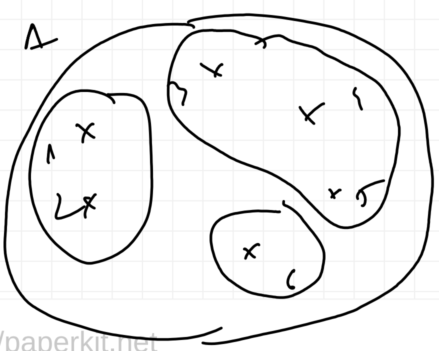
(i.e. π is a set of nonempty subsets of A)

is called a <u>partition of A</u> if

$\forall x \in A \quad \exists! B \in \pi$ such that $x \in B$

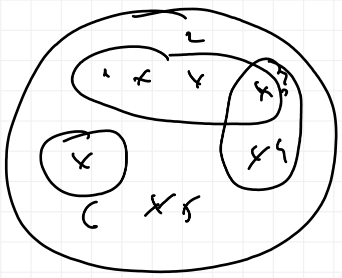
(i.e. any element of A belongs to exactly one class of the partition π)

Equivalently, π satisfies:
(1) $\bigcup_{B \in \pi} B = A$
(2) $\forall B, B' \in \pi, B \neq B' \Longrightarrow B \cap B' = \emptyset$



$A = \{1, 2, 3, 4, 5, 6\}$

$\pi = \{\{1, 2\}, \{3, \cancel{4}, 5\}, \{6\}\}$.



not a partition

$$\pi = \{\{1,2,3\}, \{3,4\}, \{6\}\}$$

Theorem 1 Let π be a partition of the set A.

We define on A the relation $\rho_\pi = (A, A, R_\pi)$ as follows:

$$\boxed{\forall\, x,y \in A : \quad x \,\rho_\pi\, y \overset{def}{\Longleftrightarrow} \exists\, B \in \pi \text{ s.t. } x,y \in B.}$$

Then: ρ_π is an equivalence relation on A.

Example: in the previous example:

$$R_\pi = \{(1,1),(2,2),(1,2),(2,1),(3,4),(3,3),(3,5),(4,4),(4,3),(4,5),$$
$$(5,5),(5,3),(5,4),(6,6)\}$$

Homework: ex. 55 – 61

http://paperkit.net