Mathematical logic

Lecture 9

09.12 2022

Chapter 6. Lattices and Boole algebras

Lattices as algebraic structures

Def 1. An ordered set $(A, \leq)$ is called a lattice if for any $x, y \in A$ $\exists \inf\{x,y\}$, $\exists \sup\{x,y\}$

Def 2 The algebraic structure $(A, \vee, \wedge)$ is called a lattice if the following axioms are satisfied:

(1) commutativity: $\forall x, y \in A$ $\quad x \vee y = y \vee x$, $\quad x \wedge y = y \wedge x$

(2) associativity $\forall x, y, z \in A$ $\quad (x \vee y) \vee z = x \vee (y \vee z)$, $\quad (x \wedge y) \wedge z = x \wedge (y \wedge z)$

(3) absorbtion: $\forall x, y \in A$ $\quad x \vee (x \wedge y) = x$, $\quad x \wedge (x \vee y) = x$.

We'll show that the two notions are equivalent

Theorem 1 1). Let $(A, \leq)$ be a lattice (as an ordered set).
We define on A the following operations:

$$\forall x, y \in A: \quad x \vee y \overset{def}{=} \sup\{x, y\}$$
$$x \wedge y \overset{def}{=} \inf\{x, y\}.$$

Then $(A, \vee, \wedge)$ is a lattice (as an alg. structure)

2). Let $(A, \vee, \wedge)$ be a lattice (as an alg. structure).
We define on A the relation „$\leq$": $\forall x, y \in A$ $\quad x \leq y \overset{def}{\Longleftrightarrow} x \vee y = y$.
Then „$\leq$" is an order relation on A, and moreover, $(A, \leq)$ is a lattice, where $\sup\{x,y\} = x \vee y$ and $\inf\{x,y\} = x \wedge y$.

<u>Proof</u> 1). Homework! see Th. 6.1.2 / 39.

2) First, we prove that $\forall x, y \in A$.

$$x \vee y = y \iff x \wedge y = x$$ (so we may also use the oper. $\wedge$ to define "$\leq$")

"$\Rightarrow$" $x \vee y = y \xrightarrow{x\wedge\cdot} x \wedge (x \vee y) = x \wedge y \xrightarrow{abs} x = x \wedge y$

"$\Leftarrow$" $x \wedge y = x \xrightarrow{\cdot\vee y} (x \wedge y) \vee y \Rightarrow x \vee y \xrightarrow{abs} y = x \vee y$

We check that "$\leq$" is an order relation:

(R). $\forall a \in A$: $a \overset{?}{\leq} a \overset{def}{\iff} a \vee a \overset{?}{=} a$

$\iff a \wedge a = a.$

(so we need to prove that both operations are idempotent)

Indeed, by absorption, we have,

• $a = a \vee (a \wedge a) \xrightarrow{a\wedge\cdot} a \wedge a = a \wedge (a \vee \underbrace{(a \wedge a)}_{b})$

$\xRightarrow{abs} a \wedge a = a$

• similarly with $a = a \wedge (a \vee a) \xrightarrow{a\vee} \underline{a \vee a} = a \vee (a \wedge \underline{(a \vee a)}) = \underline{a}$

(T) $\forall a, b, c \in A$ if $a \leq b$, $b \leq c \Rightarrow$

$\Rightarrow a \vee b = b$, $b \vee c = c \Rightarrow \underline{a \vee c} = a \vee (b \vee c) \overset{ass.}{=}$

$= \underline{(a \vee b)} \vee c = b \vee c = \underline{c}$, hence $a \leq c$

(A) let $a, b \in A$ st $a \leq b$ and $b \leq a$.

Then: $a \vee b = b$ and $b \vee a = a \xrightarrow{comm} a = b$

Next, we prove that $\sup\{x, y\} = x \vee y$.

• we check that $x \vee y$ is an upper bound for $\{x, y\}$.

$x \vee (x \vee y) \overset{ass}{=} (x \vee x) \vee y \overset{idemp}{=} x \vee y. \Rightarrow x \leq x \vee y$

$$y \vee (x \vee y) = x \vee (y \vee y) = x \vee y \Rightarrow y \le x \vee y.$$

Let z be another upper bound for $\{x, y\}$

$$\underline{z \vee (x \vee y)} = (z \vee x) \vee y = z \vee y = \underline{z}$$

then $x \vee y \le z$

hence $x \vee y$ is the least upper bound.

Similarly: $\inf\{x, y\} = x \wedge y$ (Homework!!)

Example. 1). Let $(A, \le)$ be a totally ordered set. so

$$\inf\{x, y\} = \min\{x, y\} \; ; \; \sup\{x, y\} = \max\{x, y\}.$$

2). Let M be a set and consider $(\mathcal{P}(M), \subseteq)$

Let $X, Y \subseteq M$. Then: $\inf\{X, Y\} = X \cap Y$

$\sup\{X, Y\} = X \cup Y.$

3). Let $(\mathbb{N}, |)$. Let $x, y \in \mathbb{N}$.

$\inf\{x, y\} = (x, y)$ g.c.d.

$\sup\{x, y\} = [x, y]$ l.c.m.

Def 3 A function $f : (A, \vee, \wedge) \to (A', \vee', \wedge')$ is called a morphism of lattices if $\forall x, y \in A$:

$$f(x \vee y) = f(x) \vee' f(y) \text{ and } f(x \wedge y) = f(x) \wedge' f(y).$$

Rem (see the exercises!)

f morph of lattice $\underset{\not\Leftarrow}{\Longrightarrow}$ f is monotony.

Boole algebras and Boole rings

Def. A lattice $(A, \vee, \wedge)$ is called a Boole lattice (Boole algebra) if the following additional axioms hold:

(1) distributivity: $\forall\, x, y, z \in A$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(one is enough to state! see ex)

(2). $\exists \min A =: 0$; $\exists \max A =: 1$

(i.e.: $0 \wedge x = 0$, $0 \vee x = x$; $1 \wedge x = x$, $1 \vee x = 1$.)

(3). $\forall x \in A$ has a complement x' satisfying:

$$x \vee x' = 1 \quad ; \quad x \wedge x' = 0$$

Proposition 1 Let $(A, \vee, \wedge, 0, 1, ')$ be a Boole algebra. Then:

1). $\forall\, x \in A$, the complement of x is unique

2). $0' = 1$, $1' = 0$, $(x')' = x$

3). $\forall\, x, y \in A$ we have:
$$\left.\begin{array}{l}(x \vee y)' = x' \wedge y' \\ (x \wedge y)' = x' \vee y'\end{array}\right\}\ \text{De Morgan}$$

Proof 1). We assume that x', $\bar{x}$ are both complements of $x \in A$.
We have:

- $x' = x' \wedge 1 = x' \wedge (x \vee \bar{x}) \overset{\text{distr}}{=} (x' \wedge x) \vee (x' \wedge \bar{x}) =$
  $= 0 \vee (x' \wedge \bar{x}) = x' \wedge \bar{x}$
- $\bar{x} = \bar{x} \wedge 1 = \ .\ \text{HW}\ .\ = x' \wedge \bar{x}$

hence $x' = \bar{x}$

2). • $0\vee 1 = 1$, $0\wedge 1 = 0 \Rightarrow 0' = 1$, $1' = 0$

• $x\vee x' = 1$, $x\wedge x' = 0 \Rightarrow (x')' = x$

3) we calculate

• $(x\vee y)\wedge(x'\wedge y') \overset{dist}{=} (\underbrace{x\wedge x'}_{0}\wedge y')\vee(\underbrace{y\wedge x'\wedge y'}_{0})$

$= (0\wedge y')\vee(0\wedge x') = 0\vee 0 = 0$

• $(x\vee y)\vee(x'\wedge y') \overset{dist}{=} (\underbrace{x\vee y\vee x'})\wedge(x\vee \underbrace{y\vee y'}) =$

$= (1\vee y)\wedge(x\vee 1) = 1\wedge 1 = 1$

Thus $(x\vee y)' = x'\wedge y'$

Similarly, we prove the other equality! HW!

Def 5. Any $(B,+,\cdot)$ (assoc. with 1) is called a Boole ring if $\forall x\in B$ we have $x^2 = x$ (every elem. is idempotent)

$(B,+)$ ab. group
$(B,\cdot)$ monoid
$\cdot$ distrib. w.r.t. $+$.

Prop. Let $(B,+,\cdot,0,1)$ be a Boole ring. Then:

1). $1+1=0$ (hence $x+x=0$ $\forall x\in B$)

(we say that B has characteristic 2)

2) B is commutative, i.e. $xy = yx$ $\forall x,y\in B$.

Proof 1). $\cancel{1+1} = (1+1)^2 = \cancel{1+1}+1+1$; but $(B,+)$ is a group, hence $0=1+1$

2). $\cancel{x+y} = (x+y)^2 = x^2+xy+yx+y^2 = \cancel{x+y}+xy+yx$

$\Rightarrow 0 = xy + yx \Rightarrow yx = -xy = xy$
because $1 = -1$

<u>Theorem 2</u> (Stone).

1) Let $(A, \vee, \wedge, 0, 1, ')$ be a Boole algebra. We define on A the following operations:

- $x + y \stackrel{def}{=} (x \wedge y') \vee (x' \wedge y)$
- $xy \stackrel{def}{=} x \wedge y$

Then: $(A, +, \cdot, 0, 1)$ is a Boole ring.

2). Let $(B, +, \cdot, 0, 1)$ be a Boole ring.
We define on B the foll. operations:

- $x \vee y \stackrel{def}{=} x + y + xy$
- $x \wedge y \stackrel{def}{=} xy$
- $x' \stackrel{def}{=} 1 + x$

Then $(B, \vee, \wedge, 0, 1, ')$ is a Boole algebra.

3). The constructions from 1) and 2) are inverses of each other:

- $\underbrace{(A, \vee, \wedge, 0, 1, ')}_{\text{Boole algebra}} \xrightarrow{1)} \underbrace{(A, +, \cdot, 0, 1)}_{\text{Boole ring}} \xrightarrow{2)} \underbrace{(A, \cup, \cap, 0, 1, ^-)}_{\text{Boole algebra}}$

Then: $\cup = \vee$, $\cap = \wedge$, $^- = '$

- $\underbrace{(B, +, \cdot, 0, 1)}_{\text{Boole ring}} \xrightarrow{2)} \underbrace{(B, \vee, \wedge, 0, 1, ')}_{\text{Boole algebra}} \xrightarrow{1)} \underbrace{(B, \oplus, \odot, 0, 1)}_{\text{Boole ring}}$

Then: $\oplus = +$, $\odot = \cdot$

<u>Proof</u> see Thm. 6.2.6 / 41. HW!

Examples

1) Let $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$ be the ring of residues modulo 2.

$$\hat{0} = 2\mathbb{Z} \ (\text{even}), \qquad \hat{1} = 2\mathbb{Z}+1 \ (\text{odd})$$

So $(\mathbb{Z}_2, +, \cdot)$ is a Boole ring: $\hat{0}^2 = \hat{0}$, $\hat{1}^2 = \hat{1}$

We determine the associated Boole algebra.

$$\begin{cases} x \wedge y = xy \\ x \vee y = x+y+xy \end{cases} \qquad x' = 1+x$$

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

x	x'
0	1
1	0

$0 \vee 0 = 0+0+0\cdot 0 = 0$

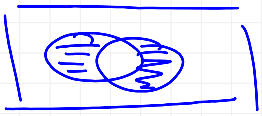
$0 \vee 1 = 0+1+0\cdot 1 = 1$

$1 \vee 1 = 1+1+1\cdot 1 = 1$

2). Consider the structure $(\mathcal{P}(M), \cup, \cap, \underset{\hat{0}}{\emptyset}, \underset{\hat{1}}{M}, \complement)$.

From set theory we know that this is a Boole algebra. We determine the corresponding Boole ring.

Let $X, Y \in \mathcal{P}(M)$. Then,



$$X \cdot Y = X \cap Y$$

$$\begin{aligned} X + Y &= (X \cap \complement Y) \cup (Y \cap \complement X) \\ &= (X \setminus Y) \cup (Y \setminus X) \\ &= X \,\Delta\, Y \quad (\text{symmetric difference}) \end{aligned}$$

We get the Boole ring $(\mathcal{P}(M), \Delta, \cap, \emptyset, M)$.

3) The <u>Lindenbaum – Tarski algebra</u>

Let $X = \{x_1, \dots, x_n\}$ be a set of atomic formulas in propositional logic.
Let $\mathcal{F}$ be the set of formulas containing these atoms.
On $\mathcal{F}$ we consider the equivalence relation "$\Longleftrightarrow$"

$A \Longleftrightarrow B$ if and only if the formula: $(A \Longleftrightarrow B)$ is a tautology.

We consider the quotient set $\mathcal{F}/_{\Longleftrightarrow} = \{\hat{A} \mid A \in \mathcal{F}\}$

where $\hat{A} = \{B \in \mathcal{F} \mid A \Longleftrightarrow B\}$

On $\mathcal{F}/_{\Longleftrightarrow}$ we define the following operations:

$$\underset{\substack{\uparrow \\ \text{operation}}}{\hat{A} \vee \hat{B}} \stackrel{def}{=} \underset{\substack{\uparrow \\ \text{connective}}}{\widehat{A \vee B}}$$

$\hat{0}$ = the set of all contradictions

$$\hat{A} \wedge \hat{B} \stackrel{def}{=} \widehat{A \wedge B}$$

$\hat{1}$ = the set of all tautologies.

$$\hat{A}' \stackrel{def}{=} \widehat{\neg A}$$

From propositional logic it follows that $(\mathcal{F}/_{\Longleftrightarrow}, \vee, \wedge, \hat{0}, \hat{1}, ')$ is a Boole algebra.

<u>Rem</u> Examples 2) and 3) give the possibility to use algebraic methods in set theory and propositional logic.

Homework Fill in the proofs of the above theorems.
ex 76 – 83.