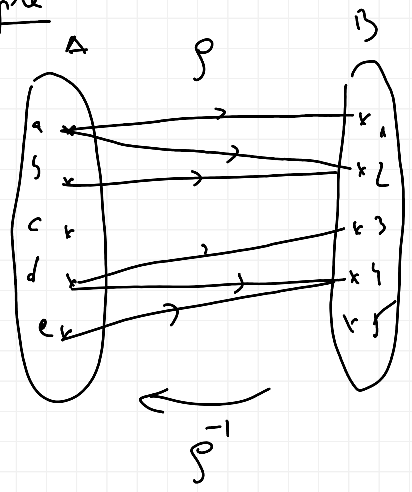
Mathematical Logic
Lecture 4
28.10.2022

The section of a relation w.r.t. a subset
(the image of a subset under a relation)

Example



$\rho(\{a\}) = \{1,2\}$

$\rho(\{a,c,d\}) = \{1,2,3,4\}$

$\rho(\{c\}) = \emptyset$

$\rho^{-1}(\{2\}) = \{a,b\}$

$\rho^{-1}(B) = \{a,b,d,e\}$

$\rho^{-1}(\{5\}) = \emptyset$

Def Let $\rho = (A,B,R)$ be a relation, and let $X \subseteq A$.
The section of ρ w.r.t. X is:

$$\rho(X) = \{b \in B \mid \exists x \; x \in X \text{ and } x\rho b\} \subseteq B$$

that is, for $b \in B$, we have

$$b \in \rho(X) \overset{def}{\Longleftrightarrow} \exists x \; x \in X \text{ and } x \rho b$$

(informally $\exists x \in X$ s.t. $x \rho b$)

Particular case: if $X = \{x\}$, we denote $\rho(\{x\}) = \rho\langle x\rangle = \{b \in B \mid x\rho b\}$

Proposition (the behaviour of the section w.r.t composition).
Let $\rho = (A,B,R)$, $\sigma = (C,D,S)$, and let $X \subseteq A$.
Then:

$$\boxed{(\sigma\circ\rho)(X) = \sigma\big(C \cap \rho(X)\big)}$$

In particular, if $\rho(X) \subseteq C$, then $(\sigma\circ\rho)(X) = \sigma(\rho(X))$

Proof. Both sets are subsets of D. Let $d \in D$. We have:

$$d \in (\sigma \circ \rho)(X) \overset{\text{by def}}{\iff} \exists x \;\; x \in X \text{ and } x(\sigma\circ\rho)d$$

$$\overset{\text{def of } \circ}{\iff} \exists x \;\; x \in X \text{ and } \exists y \;\; y \in B\cap C \text{ and } x\rho y \text{ and } y\sigma d$$

$$\overset{(*)}{\iff} \exists y \;\; y \in B\cap C \text{ and } \exists x \;\; x \in X \text{ and } x\rho y \text{ and } y \sigma d$$

$$\overset{\text{def of sect}}{\iff} \exists y \;\; y \in B\cap C \text{ and } y \in \rho(X) \text{ and } y\sigma d$$

$$\overset{\text{def } \cap}{\iff} \exists y \;\; y \in B \cap \underbrace{C \cap \rho(X)} \text{ and } y \sigma d$$

$$\overset{\text{def sect}}{\iff} d \in \sigma\big(C \cap \rho(X)\big)$$

(*) We have used the foll tautologies:

$$(A\wedge B)\wedge C \iff A\wedge(B\wedge C) \qquad (\text{assoc})$$

$$A\wedge B \iff B \wedge A \qquad (\text{comm})$$

$$\exists x \exists y \;\; A(x,y) \iff \exists y \exists x \;\; A(x,y) \qquad (2.3.5\ (1)\ p.17)$$

$$\exists x\,(A \wedge C(x)) \iff A \wedge \exists x\, C(x) \qquad (2.3.5\ (2)\quad)$$

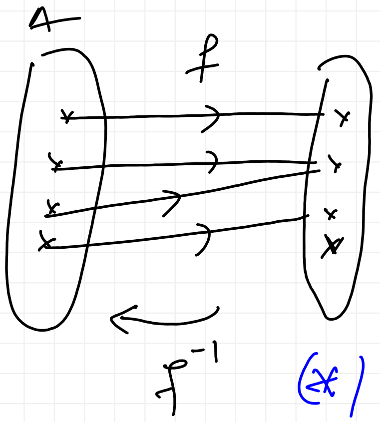
Functions (as particular cases of relations)

Def Let $f = (A, B, F)$ be a relation, where $F \subseteq A\times B$.
We say that f is a function if $\forall x \in A$ the section $f\langle x\rangle = \{ b \in B \mid x f b\}$ contains exactly one element, i.e. $|f\langle x \rangle| = 1$.

We denote: $f\langle x\rangle = \{f(x)\}$ Euler's notation

$f : A \to B$, $\begin{cases} A \xrightarrow{f} B \\ x \longmapsto f(x) \end{cases}$ maps to

Example. In the previous example, ρ is not a function because $|\rho\langle a\rangle| = 2$, and also because $|\rho\langle c\rangle| = \emptyset$.
i.e. $|\rho\langle c\rangle| = 0$



f is a function but the inverse relation f^{-1} is <u>not</u> a function.

Remarks and examples

1) the equality relation $1_A = (A, A, \Delta_A)$ is a function because $1_A\langle a\rangle = \{a\}$ $\forall a \in A$.

We call $1_A : A \longrightarrow A$, $1_A(a) = a$ the <u>identity function of A</u>.

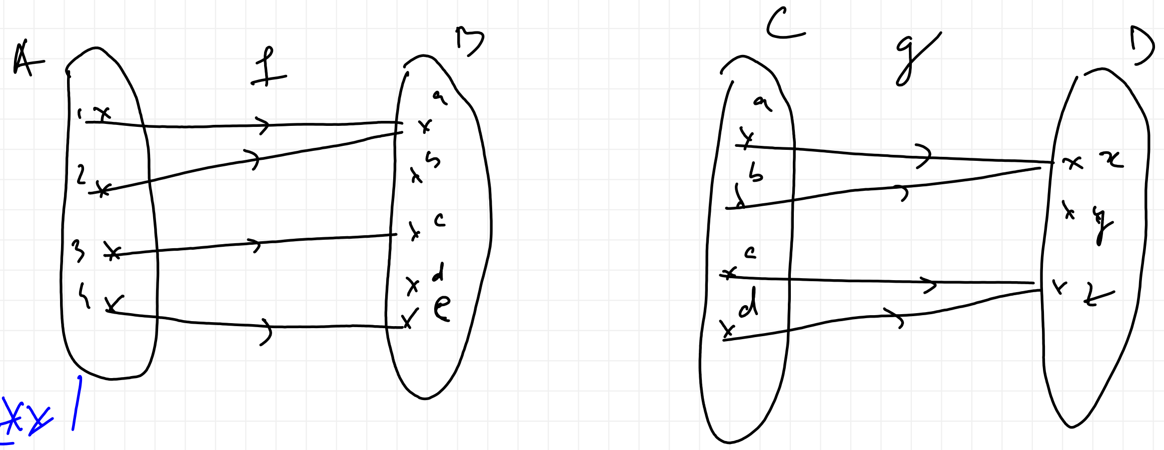
2) • Assume that $A = \emptyset$. Then the relation $(\emptyset, B, \emptyset)$ is a function.

• Assume that $A \neq \emptyset$ and $B = \emptyset$. Then the relation $(A, \emptyset, \emptyset)$ is <u>not</u> a function.

3). <u>Equality of functions</u>: the functions $f : A \to B$ and $g : C \to D$ are equal $(f = g)$ iff

$$\underbrace{f = (A, B, F)}_{} = \underbrace{g = (C, D, G)}_{} \iff \begin{cases} A = C & \text{(same domain)} \\ B = D & \text{(same codomain)} \\ \underset{\{(a, f(a)) \mid a \in A\}}{F} = G \iff f(a) = g(a) \ \forall a \in A \end{cases}$$

4) Composition of functions



(**)

In this example, the composed relation

$$g\circ f = (A, D, G\circ F) \text{ is } \underline{\text{not}} \text{ a function}$$

because $(g\circ f)\langle 4\rangle = \emptyset$

It is clear that, in general, $g\circ f$ is a function $\Longleftarrow$ $f(A) \subseteq C$

5) Properties of the composition of function

(a) the identity function is a neutral element

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[rr, bend right, "f\circ 1_A"'] & A \arrow[r, "f"] \arrow[rr, bend right, "1_B\circ f"'] & B \arrow[r, "1_B"] & B \end{tikzcd}$$

$$f\circ 1_A = 1_B\circ f = f$$

(b) the comp. of functions is associative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend right=50, "h\circ(g\circ f)"'] \arrow[rrr, bend right=70, "(h\circ g)\circ f"'] & B \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

We have: $h\circ(g\circ f) = (h\circ g)\circ f$

Proof: $\forall a \in A$ we have

$$(h\circ(g\circ f))(a) = h((g\circ f)(a)) = h(g(f(a)))$$
$$= (h\circ g)(f(a)) = ((h\circ g)\circ f)(a)$$

c). Let $f: A \to B$ be a function. We may consider the inverse relation $f^{-1} = (B, A, \bar{F}^{-1})$.

The f^{-1} is not a function in general! (see ex. (∗1))

Image and inverse image (pre image)

(particular case of the notion of a relation.)

Let $f: A \to B$ be a function.

- If $X \subseteq A$, then $f(X) \overset{def}{=} \{ f(x) \mid x \in X \}$

(the image of X under f)

i.e. if $b \in B$, then

$b \in f(X) \overset{def}{\Longleftrightarrow} \exists x \in X$ s.t. $b = f(x)$

$(\exists x \;\; x \in X$ and $b = f(x))$

In example (∗1) $f(\{1,2,3\}) = \{a, c\}$

particular case. $\operatorname{Im} f = f(A) = \{ f(a) \mid a \in A \}$.

the image of f

In ex (∗1) $\operatorname{Im} f = \{a, c, e\}$

- If $Y \subseteq B$, then $f^{-1}(Y) \overset{def}{=} \{ a \in A \mid f(a) \in Y \}$.

In ex (∗1). $f^{-1}(a) = \{1, 2\}$

$f^{-1}(\{b, c\}) = \{3\}$

$f^{-1}(b) = \emptyset$

$f^{-1}(B) = A$ always!

Commutative diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[ur, "h"'] & \end{tikzcd}$$

this diagram is comm $\iff f = h \circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

is comm $\iff k \circ f = h \circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[r, "h"'] & D \arrow[u, "k"'] \end{tikzcd}$$

is comm $\iff k \circ h \circ g = f$

Families of elements and sets

example: consider the sequence (şir)

string = coordin

$(a_n)_{n \in \mathbb{N}}$, $a_n = (-1)^n$

$1, -1, 1, -1, \ldots$

the set of elements of the sequence is $\{1, -1\}$

Def. A family of elements of a set A indexed by the index set I is a function $f : I \to A$

Not $f(i) = a_i$, $f = (a_i)_{i \in I}$.

Def A <u>family of sets</u> (subsets of U) <u>indexed</u> by I is a function $f : I \to \mathcal{P}(U)$

Not $f(i) = A_i$, $f = (A_i)_{i \in I}$.

<u>Operations</u> with families of sets:

- $\bigcup_{i \in I} A_i \overset{def}{=} \{ x \in U \mid \exists i \ i \in I \text{ and } x \in A_i \}$
- $\bigcap_{i \in I} A_i \overset{def}{=} \{ x \in U \mid \forall i \ i \in I \text{ and } x \in A_i \}$

Particular case: assume $I = \emptyset$.

$$\bigcup_{i \in \emptyset} A_i = \emptyset \qquad \bigcap_{i \in \emptyset} A_i = U$$

Injective, surjective and bijective functions

one-to-one　　　　onto.

$\boxed{\begin{array}{c} p \to q \iff \neg q \to \neg p \\ \text{contraposition} \end{array}}$

Def 1 A function $f : A \to B$ is <u>injective</u> if

$\forall x_1, x_2 \in A \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

(or equivalently) $\forall x_1, x_2 \in A \quad f(x_1) = f(x_2) \implies x_1 = x_2$

Rem f is <u>not</u> inj $\iff \exists x_1, x_2 \in A$ s.t. $x_1 \neq x_2$ and $f(x_1) = f(x_2)$

$\boxed{\begin{array}{l} \neg(p \to q) \iff \neg(\neg p \vee q) \\ \qquad \iff p \wedge \neg q \end{array}}$

Def 2 A function $f: A \to B$ is **surjective** if $\forall y \in B \ \exists x \in A$ s.t. $y = f(x)$

or equivalently, $\operatorname{Im} f = B$.

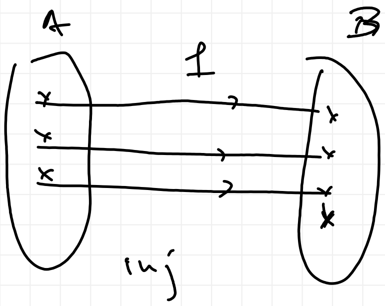
$$\operatorname{Im} f = f(A) = \{ f(x) \mid x \in A \}$$

Rem f is **not** surjective $\iff \exists y \in B \ \forall x \in A \quad y \neq f(x)$

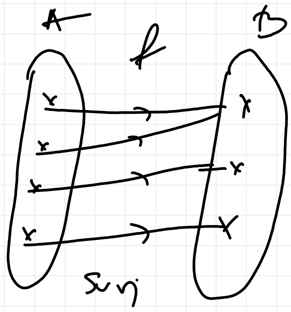
Def 3 A function $f: A \to B$ is **bijective** if f is injective and surjective,

i.e. $\forall y \in B \ \exists! \, x \in A$ s.t. $y = f(x)$
(unique)

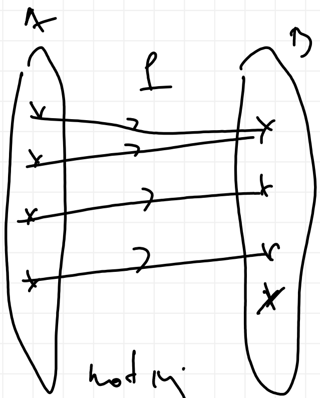
Example



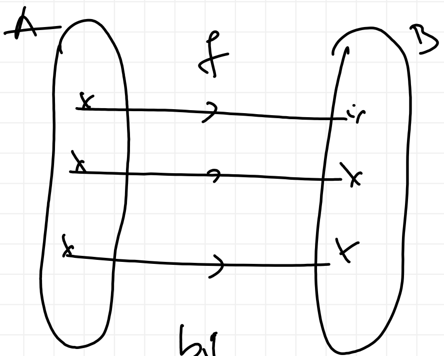
inj
not surj



surj
not inj



not inj
not surj



bij

Georg Cantor

Homework: ex. 31 – 41