Mathematical Logic
lecture 8
18.11.2022

# Chapter 5. Ordered sets

## Order relations

Def. A homogeneous relation $\rho = (A, A, R)$ is called an order relation if $\rho$ is:

pre-order $\begin{cases} (R) \text{ reflexive } (\forall x \in A \quad x\rho x) \text{ i.e. } 1_A \subseteq \rho \\ (T) \text{ transitive } (\forall x,y,z \in A \quad x\rho y, y\rho z \Rightarrow x\rho z) \ (\rho\circ\rho \subseteq \rho) \end{cases}$

(A) antisymmetric $(\forall x,y \in A \quad x\rho y, y\rho x \Longrightarrow x=y)$

i.e. $\rho \cap \rho^{-1} \subseteq 1_A$, in fact $\rho \cap \rho^{-1} = 1_A$

Rem a rel. which is R, T, S, A $\Longleftrightarrow \rho = 1_A$

- If, in addition, $\rho$ satisfies:

$\forall x,y \in A$ we have $x\rho y$ or $y \rho x$

(i.e. any two elements are comparable, ie $\rho \cup \rho^{-1} = A \times A$)

then we say that $\rho$ is a total order

Terminology the pair $(A, \rho)$ is called an ordered set (a totally ordered set)

(in many books: – order ⟶ partial order
– total order ⟶ order)

poset

Notations: for an order relation $\rho$: $\leq$, $\subseteq$, $\preccurlyeq$, ...

$< = \lneq = \rho \setminus 1_A$

strict order

Examples 1). $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$

these are totally ordered sets.

2). $(\mathbb{Z}, |)$. $\quad$ ( $a|b \overset{def}{\iff} \exists x \in \mathbb{Z}$ s.t $b = ax$ ).

we know that „|" on $\mathbb{Z}$ is R, T, $\not S$

$\boxed{\neg(p \to q) \iff p \wedge \neg q}$

(A) we study antisymmetry:

$\forall a, b \in \mathbb{Z}$, $a|b$, $b|a \overset{?}{\implies} a = b$

not in general! e.g. $3|-3$, $-3|3$, $3 \neq -3$

hence $(\mathbb{Z}, |)$ is a preordered set

2') $(\mathbb{N}, |)$. it is an ordered set

is it totally ordered?

$\forall x, y \in \mathbb{N}$ do we have $x|y$ or $y|x$ ?

no! e.g. $2 \nmid 3$, $3 \nmid 2$

3). Let $M$ be any set and consider $(\mathcal{P}(M), \subseteq)$

we know that this is an ordered set.

To discuss total order, we consider cases

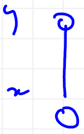
- $\mathcal{P}(\emptyset) = \{\emptyset\}$ totally ordered
- Let $M = \{1\}$, $\mathcal{P}(M) = \{\emptyset, M\}$ is totally ordered.
- Assume $|M| \geq 2$ say, $x, y \in M$, $x \neq y$.

$\{x\}$, $\{y\}$ are not comparable

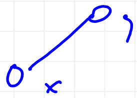
hence $(\mathcal{P}(M), \subseteq)$ is not totally ordered

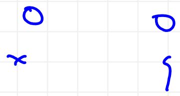
# Hasse diagrams

(are used to visualize ordered sets with few elements)

Conventions: 


means $x < y$, and there are no elements between $x$ and $y$

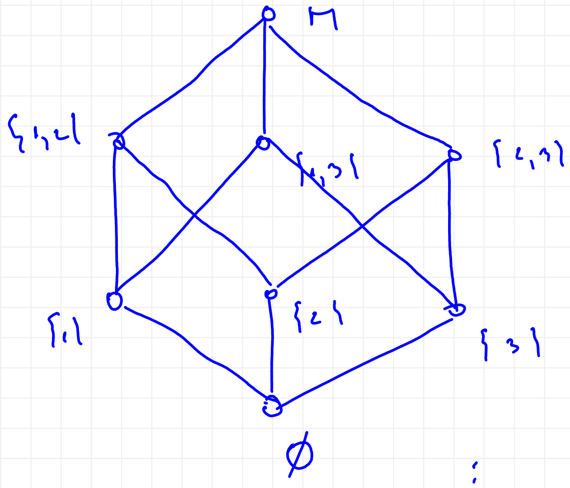
variant: 


means that $x, y$ are not comparable.

makes no sense.

Example. 1) Let $M = \{1,2,3\}$. Draw the Hasse diagram of $(\mathcal{P}(M), \subseteq)$. We have $\mathcal{P}(M) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, M\}$




2). $(\mathbb{N}, \leq)$


totally ordered (also called chain)

# Morphisms between ordered sets

Def Let $(A, \rho)$, $(B, \sigma)$ be two ordered sets, and let $f: A \to B$ be a function:

a). $f$ is increasing (morphism of ordered sets). if
$$\forall\, x, x' \in A \quad x \rho x' \Longrightarrow f(x)\, \sigma\, f(x'),$$

a') $f$ is strictly increasing if
$$\forall x, x' \in A \quad x \rho x',\ x \neq x' \Longrightarrow f(x) \sigma f(x'),\ f(x) \neq f(x')$$
$(x < x')$ $\qquad f(x) < f(x')$

(ie $f$ is increasing and injective)

b) $f$ is decreasing (antimorphism). if
$$\forall x, x' \in A \quad x \rho x' \Longrightarrow f(x')\, \sigma\, f(x)$$

b') $f$ is strictly decreasing if
$$\forall x, x' \in A,\ x \rho x',\ x \neq x' \Longrightarrow f(x') \sigma f(x),\ f(x) \neq f(x')$$

ie. $f$ is decreasing and injective

c). $f$ is an isomorphism of ordered sets if
$f$ is increasing, bijective, and $f^{-1}$ is increasing

c') $f$ is an anti-isomorphism of ordered sets if
$f$ is decreasing, bijective and $f^{-1}$ is decreasing

Example. Consider $1_{\mathbb{N}^*}: (\mathbb{N}^*, |) \longrightarrow (\mathbb{N}^*, \leq)$, $1_{\mathbb{N}^*}(x) = x$ $\forall x \in \mathbb{N}^*$

- this is bijective, $1^{-1}_{\mathbb{N}^*} = 1_{\mathbb{N}^*}$
- increasing? $x \mid y \overset{?}{\Longrightarrow} x \leq y$ true! (morphism)
- $1_{\mathbb{N}^*}: (\mathbb{N}^*, \leq) \to (\mathbb{N}^*, |)$ is increasing?

$x \leq y \overset{?}{\Longrightarrow} x \mid y$ no! in general. e.g. $2 \leq 3$ and $2 \nmid 3$.

# Special elements in ordered sets

<u>Def</u> Let $(A, \leq)$ be an ordered set.

a) The elem $a \in A$ is called <u>the least</u> (minimum) elem. of $A$ (not $a = \min A$) if $\forall x \in A \quad a \leq x$.

a') $a \in A$ is the <u>largest elem</u> (maximum) of $A$

if $\forall x \in A \quad x \leq a$

<u>Rem</u> $\min A$, $\max A$ don't always exist, but when they exist, they are unique!

b). the elem $a \in A$ is a <u>minimal element</u> if there are no strictly smaller elements:

i.e. $\forall x \in A \quad x \leq a \implies x = a$

b') the elem. $a \in A$ is a <u>maximal element</u> if there are no strictly larger elements:

i.e. $\forall x \in A \quad a \leq x \implies x = a$.

<u>Examples</u> 1). In $(\mathbb{N}, \leq)$ $\quad \min(\mathbb{N}, \leq) = 0$

$\nexists \max(\mathbb{N}, \leq)$

2). $(\mathcal{P}(M), \subseteq)$ $\quad \min \mathcal{P}(M) = \emptyset$, $\max \mathcal{P}(M) = M$.

Now, let $(\mathcal{P}(M) \setminus \{\emptyset, M\}, \subseteq)$. Then $\nexists \min$, $\nexists \max$

- every subset with 1 elem, $\{x\}$, is minimal
- sets of the form $M \setminus \{x\}$ are maximal

3). $(\mathbb{N}, |)$ $\quad \min(\mathbb{N}, |) = 1$, because $1 | a \ \forall a \in \mathbb{N}$

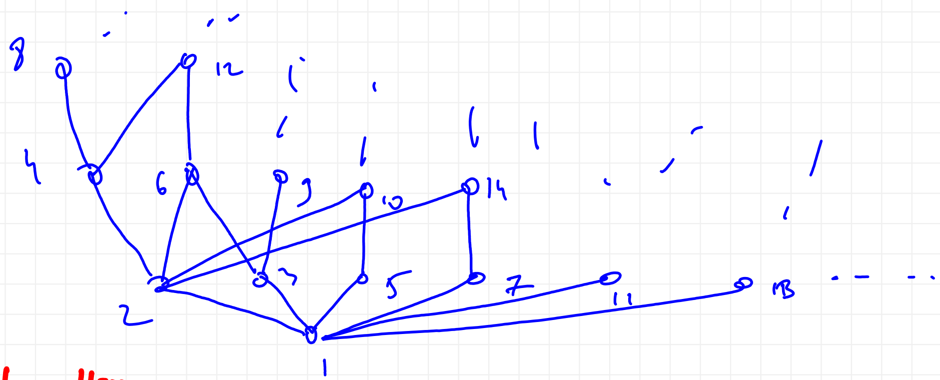
$\max(\mathbb{N}, |) = 0$, because $a | 0 \ \forall a \in \mathbb{N}$

$0 \mid 0$ is true
$0 : 0$ makes no sense

Consider $(\mathbb{N} \setminus \{0,1\}, \mid)$.

- $\nexists$ min, $\nexists$ max.
- minimal elements are the prime numbers
- $x \mid 2x \ \forall x \in \mathbb{N}$, so $\nexists$ maximal element

Hasse diagram:




## Lattices

**Def** Let $(A, \leq)$ be an ordered set and $B \subseteq A$.

a) the elem $a \in A$ is a minorant (lower bound) for $B$ if $\forall x \in B \quad a \leq x$

a') $a \in A$ is a majorant (upper bound) for $B$ if $\forall x \in B \quad x \leq a$.

b) $a \in A$ is the infimum of $B$ ($\inf B$) if $a$ is the largest minorant for $B$.

ie. $\begin{cases} a \leq x \quad \forall x \in B \\ \forall a' \in A \ a' \leq x \ \forall x \in B \Rightarrow a' \leq a \end{cases}$

b') $a \in A$ is the supremum of $B$ ($\sup. B$) if $a$ is the least majorant for $B$.

ie. $\begin{cases} x \leq a \quad \forall x \in B \\ \forall a' \in A \ x \leq a' \ \forall x \in B \Rightarrow a \leq a' \end{cases}$

Rm inf B, sup B do not always exist, but when they exist, they are unique

Def c). the ordered set $(A, \leq)$ is called a lattice if $\forall x, y \in A$ $\exists \inf\{x,y\}$ and $\exists \sup\{x,y\}$

Rm by induction, every finite subset will have inf, sup

d). $(A, \leq)$ is called a complete lattice if $\forall B \subseteq A$ has $\inf B$, $\sup B$

Examples 1) Let $(A, \leq)$ be totally ordered.

then $\inf\{x,y\} = \min\{x,y\}$

$\sup\{x,y\} = \max\{x,y\}$

Hence any totally ordered set is a lattice.

$(\mathbb{R}, \leq)$ is totally ordered, but it is not a lattice

because $\nexists \inf(-\infty, a)$, $\nexists \sup(a, +\infty)$

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a complete lattice.

Rm every finite lattice is complete

2). Let $(\mathbb{N}, |)$. Let $a, b \in \mathbb{N}$.

- $\inf\{a,b\} = \gcd(a,b) = d \overset{not}{=} (a,b)$

by def $\begin{cases} d \mid a, \ d \mid b \\ d' \mid a, \ d' \mid b \Longrightarrow d' \mid d \end{cases}$

- $\sup\{a,b\} = \operatorname{lcm}(a,b) = m \overset{not}{=} [a,b]$

by def: $\begin{cases} a \mid m, \ b \mid m \\ a \mid m', \ b \mid m' \Longrightarrow m \mid m' \end{cases}$

3). $(\mathcal{P}(M), \subseteq)$.

e.g. let $X = \{1,2\}$, $Y = \{2,3\}$

$\inf\{X,Y\} = \{2\} = X \cap Y$

$\sup\{X,Y\} = M = X \cup Y$

In general, let $\mathcal{B} \subseteq \mathcal{P}(M)$. Then,

$$\inf \mathcal{B} = \bigcap_{X \in \mathcal{B}} X = \{x \in M \mid \forall X \in \mathcal{B}, x \in X\}$$

$$\sup \mathcal{B} = \bigcup_{X \in \mathcal{B}} X = \{x \in M \mid \exists X \in \mathcal{B}, x \in X\}.$$

conclusion: $(\mathcal{P}(M), \subseteq)$ is a complete lattice.

4). let $(A, \leq)$ be an ordered set.

$\inf \emptyset = \max A$ (because $\forall a \in A$ is minorant for $\emptyset$)
(if it exists)

$\sup \emptyset = \min A$ (if it exists; because $\forall a \in A$ is majorant for $\emptyset$)

$\boxed{\text{Homework: } 65 - 72}$