Mathematical Logic
Lecture 3
21.10.2022

Chapter 3. Sets

"Naively", a set is a collection of uniquely determined abstract objects.

This theory was introduced by Georg Cantor (~1870)
Later, contradictions have been found in this theory
(i.e. Cantor's set theory is not consistent!)

Around 1920 → axiomatic set theory has been developed.

- The concepts of – set
  – element
  – belongs ∈ – binary predicate
  } symbols of the language

  are primary, i.e. they are not defined.

A set can be given by:
- enumerating the elements e.g. $A = \{a, b, c\}$
- a property e.g. $A = \{x \mid P(x)\}$
  where P is a predicate
  e.g. $\{x \mid x \neq x\} = \emptyset$ the empty set
  $\neg(x \overset{\text{!!}}{=} x)$ (it is unique)

- two sets are equal if they have the same elements:
  $A = B \iff \forall x\,(x \in A \leftrightarrow x \in B)$.

- A is a subset of B
  $A \subseteq B \iff \forall x\,(x \in A \rightarrow x \in B)$

• the power set of A is the set of all subsets of A:

$$\mathcal{P}(A) \overset{\text{def}}{=} \{X \mid X \subseteq A\}.$$

rem $\quad X \subseteq A \iff X \in \mathcal{P}(A)$.

Example $\quad \mathcal{P}(\emptyset) = \{\emptyset\}.$

$$\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

Rem. we will consider that all our sets are subsets of a "big" set U (universe)

Operations with sets

($\vee$ (connective))

1) union $\quad A \cup B \overset{\text{def}}{=} \{x \in U \mid x \in A \text{ or } x \in B\}$

2) intersection $\quad A \cap B \overset{\text{def}}{=} \{x \mid x \in A \text{ and } x \in B\}$

3) difference $\quad A \setminus B \overset{\text{def}}{=} \{x \mid x \in A \text{ and } x \notin B\}$

particular: $\complement A = U \setminus A = \{x \mid x \notin A\}$ $\quad \neg(x \in B)$

complement of A

hence $A \setminus B = A \cap \complement B$

4) symmetric difference $\quad \begin{cases} \Delta - \text{Delta} \\ \delta - \text{delta} \end{cases}$

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cap \complement B) \cup (B \cap \complement A)$$

$$= \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

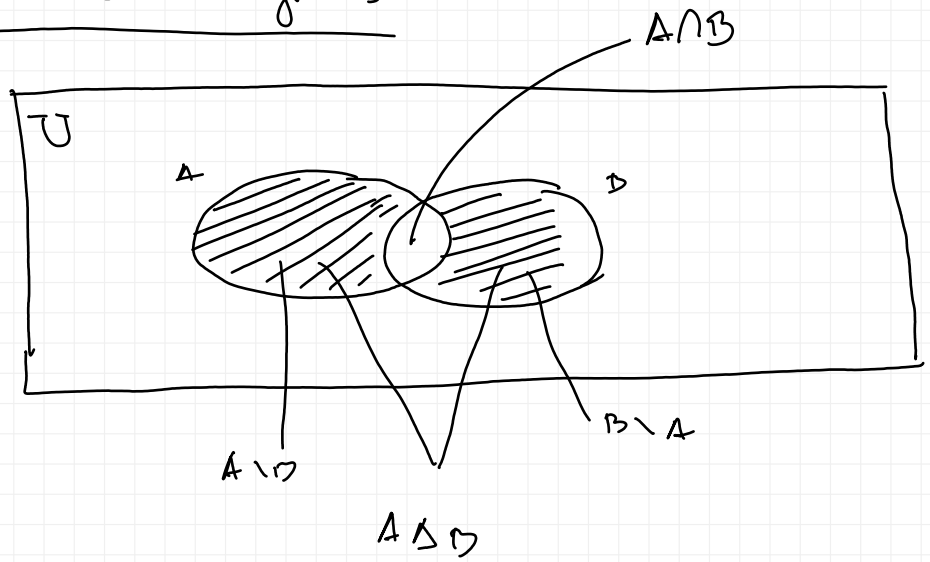
$$= \{x \mid x \in A \text{ xor } x \in B\}$$

$$= (A \cup B) \setminus (A \cap B) = (A \cup B) \cap \complement(A \cap B)$$

truth table

P	Q	P xor Q
0	0	0
0	1	1
1	0	1
1	1	0

Euler-Venn diagrams



Ordered pair of elements

Consider the element $a, b \in U$.

recall: $\{a, b\} = \{b, a\}$

if $a = b$ $\{a, b\} = \{a\}$

We want to define the ordered pair (a, b) such that the following property is satisfied:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d$$

Def (Kuratowski, 1921).

$$(a, b) \overset{\text{def}}{=} \{\{a\}, \{a, b\}\}$$

Variant: $(a, b) = \{a, \{a, b\}\}$

Def by recursion we define ordered n-tuples:

$$(a, b, c) \overset{\text{def}}{=} ((a, b), c)$$

$$(a_1, a_2, \dots, a_n) \overset{\text{def}}{=} ((a_1, \dots, a_{n-1}), a_n)$$

5). Cartesian product (René Descartes $\approx$ 1630 / Renatus Cartesius)

$$A \times B \overset{\text{def}}{=} \{(a,b) \mid a \in A \text{ and } b \in B\}$$

$$A_1 \times \dots \times A_n \overset{\text{def}}{=} \{(a_1, \dots, a_n) \mid \forall i \in \{1, \dots, n\} \; a_i \in A_i\}$$

Russell's paradox (Bertrand Russell).

We may consider "the set of all sets" $S \in S$!?

Let $R = \{X \mid X \text{ a set}, X \notin X\}$. – this is exact according to Cantor

Question: does $R \in R$?

Case 1 Assume $R \in R$. Then R does not satisfy the condition in the def. of R, hence $R \notin R$, contrad.

Case 2 Assume $R \notin R$. Then R satisfies the condition in the definition of R, hence $R \in R$, contrad.

This lead to axiomatic set theory

– $X \in X$ is not allowed

Homework ex 19–26

Chapter 4. Relations (Correspondences)

Def A relation is a triple $\rho = (A, B, R)$ where $R \subseteq A \times B$ and A, B are sets.

- A is called the domain of ρ : $\mathrm{dom}\,\rho$
- B is called the codomain of ρ : $\mathrm{codom}\,\rho$
- $R \subseteq A \times B$ is the graph of ρ

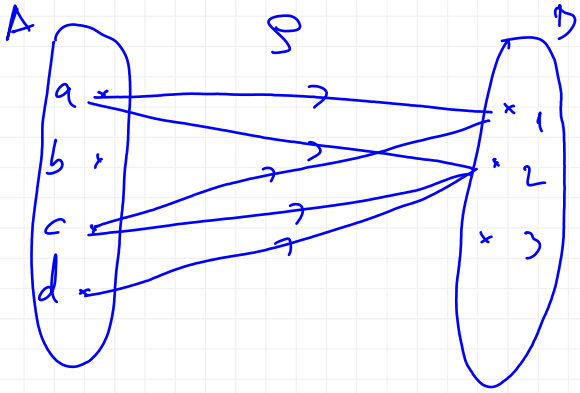
Rem a) equality of relations $\rho = (A, B, R)$, $\sigma = (C, D, S)$.

$$\rho = \sigma \iff \begin{cases} A = C & \text{(have the same domain)} \\ B = D & \text{(have the same codomain)} \\ R = S & \text{(have the same graph)} \end{cases}$$

b). We write: $(a,b) \in R \overset{\text{not.}}{\iff} a \rho b$ e.g. $a < b$, $a \mid b$, $d_1 \perp d_2$

$$a \rho b \iff a \sigma b$$

c). We may represent relations by using oriented graphs (quiver)



$$R = \{(a,1), (a,2), (c,1), (c,2), (d,2)\}$$

$$R^{-1} = \{(1,a), (2,a), (1,c), (2,c), (2,d)\}$$

(we invert the arrows)

Examples 1) the universal relation $\rho = (A, B, A \times B)$.

i.e. $\forall a \in A \ \forall b \in B \quad a \rho b$

the empty relation $(A, B, \emptyset)$

2). The identity (equality) relation on the set A.

$1_A = (A, A, \Delta_A)$ where

$\Delta_A = \{(a, a) \mid a \in A\}$.

i.e. $a \, 1_A \, b \overset{df}{\iff} (a, b) \in \Delta_A$

$\iff a = b$.

Operations with relations

1) union Let $\rho = (A, B, R)$, $\rho' = (A, B, R')$

$\rho \cup \rho' \overset{def}{=} (A, B, R \cup R')$.

i.e. $\forall (a, b) \in A \times B$: $a \, \rho \cup \rho' \, b \overset{def}{\iff} a \rho b$ or $a \rho' b$.

2) intersection $\rho \cap \rho' \overset{def}{=} (A, B, R \cap R')$

i.e. $\forall (a, b) \in A \times B$: $a \, \rho \cap \rho' \, b \overset{def}{\iff} a \rho b$ and $a \rho' b$

3) the inverse of a relation Let $\rho = (A, B, R)$.

then $\rho^{-1} \overset{df}{=} (B, A, R^{-1})$, where $R^{-1} \subseteq B \times A$

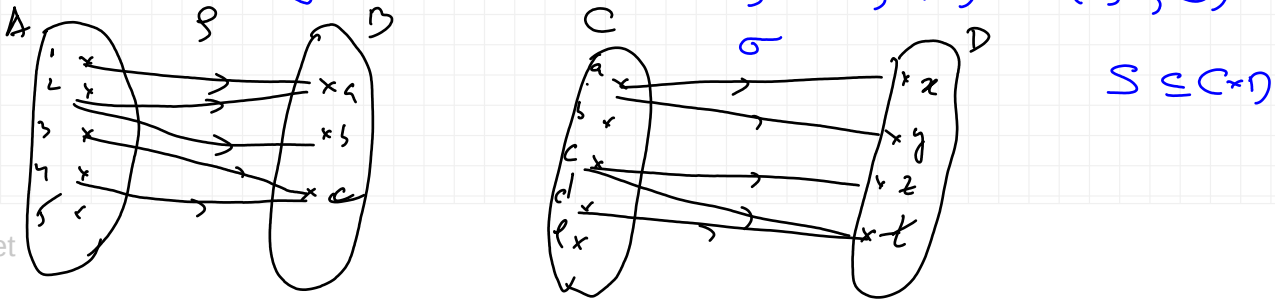
$R^{-1} \overset{df}{=} \{(b, a) \mid (a, b) \in R\}$

i.e. $\forall (a, b) \in A \times B$: $a \rho b \iff b \rho^{-1} a$

(ρ^{-1} is not the inverse w.r.t. an operation)

4) The composition of relations Let $\rho = (A, B, R)$, $\sigma = (C, D, S)$

Example



$S \subseteq C \times D$

$$\sigma \circ \rho = (A, D, S \circ R)$$

σ: 2nd, ρ: 1st, A: dom ρ, D: codom σ, where $S \circ R \subseteq A \times D$

In our example: $S \circ R = \{(1,x), (1,y), (2,x), (2,y), (3,z), (3,t), (4,z), (4,t)\}$

In general:

$$S \circ R \overset{\text{def}}{=} \{(a,d) \in A \times D \mid (\exists x)\ x \in B \cap C \text{ s.t. } (a,x) \in R \text{ and } (x,d) \in S\}$$

(B: codom ρ, C: dom σ)

equivalently.

For $(a,d) \in A \times D$ we have.

$$a\ \sigma \circ \rho\ d \overset{\text{def}}{\iff} (\exists x)\ x \in B \cap C \text{ and } a \rho x \text{ and } x \sigma d$$

Theorem (properties of the composition),

1). the identity relation is neutral element w.r.t. "$\circ$".

i.e. $\rho \circ 1_A = 1_B \circ \rho = \rho$, where $\rho = (A, B, R)$

2). "$\circ$" is associative: if $\rho = (A,B,R)$, $\sigma = (C,D,S)$, $\tau = (E,F,T)$

then $\tau \circ (\sigma \circ \rho) = (\tau \circ \sigma) \circ \rho$

Proof 1). $\rho \circ 1_A = (A, B, R \circ \Delta_A)$.

$1_B \circ \rho = (A, B, \Delta_B \circ R)$.

so when the same domain and the same codomain

Let $(a,b) \in A \times B$. We have:

- $a\ \rho \circ 1_A\ b \iff (\exists x)\ x \in A \cap A$ and $a 1_A x$ and $x \rho b \iff a \rho b$
- $a\ 1_B \circ \rho\ b \iff (\exists x)\ x \in B \cap B$ and $a \rho x$ and $x 1_B b \iff a \rho b$.

b). $\sigma \circ \rho = (A, D, S \circ R)$, $\tau \circ (\sigma \circ \rho) = (A, F, T \circ (S \circ R))$

$\tau \circ \sigma = (C, F, T \circ S)$; $(\tau \circ \sigma) \circ \rho = (A, F, (T \circ S) \circ R)$

So both relations have the same domain and the same codomain

Let $(a, f) \in A \times F$. We have

$a \; (\tau \circ \sigma) \circ \rho \; f \overset{\text{def.}\circ}{\iff} (\exists x) \; x \in B \cap C$ and $a \rho x$ and $x \tau \circ \sigma f \overset{\text{def.}\circ}{\iff}$

$\iff (\exists x) \; x \in B \cap C$ and $a \rho x$ and $(\exists y) \; y \in D \cap E$ and $x \sigma y$ and $y \tau f$

$\iff (\exists x)(\exists y) \; x \in B \cap C$ and $a \rho x$ and $y \in D \cap E$ and $x \sigma y$ and $y \tau f$

$\iff (\exists y)(\exists x) \; y \in D \cap E$ and $x \in B \cap C$ and $a \rho x$ and $x \sigma y$ and $y \tau f$

$\iff (\exists y) \; y \in B \cap E$ and $(\exists x) \; x \in B \cap C$ and $a \rho x$ and $x \sigma y$ and $y \tau f$

$\overset{\text{def.}\circ}{\iff} (\exists y) \; y \in B \cap E$ and $a \; \sigma \circ \rho \; y$ and $y \tau f$

$\overset{\text{def.}\circ}{\iff} a \; \tau \circ (\sigma \circ \rho) \; f$

We have used the following tautologies:

- $A \wedge (B \wedge C) \iff (A \wedge B) \wedge B$ (assoc. of $\wedge$)
- $A \wedge B \iff B \wedge A$ (comm. of $\wedge$)
- $\exists x \exists y \; A \iff \exists y \exists x \; A$
- $\exists x \; (A \wedge C(x)) \iff A \wedge \exists x \; C(x)$ (A does not depend on the var x)

remind 2.3.3 (1) and (8) pg. 17

Homework: ex. 27 – 30