Chapter 2. First order logic (Predicate logic)

In addition to what we did in Propositional Logic, we will formalize other expressions from the natural language

- we introduce quantifiers $\exists$ existential (there is) $\forall$ universal (for all)

Remark 1) "First order" means that we only quantify elements of sets: $\forall x$, $\exists x$, $x \in M$

"Second order" means that we also quantify sets

$\exists M$, $\forall M$ M set

2) "Naively", a predicate is a kind of "open sentence" e.g. $x+y=1$, such that if we replace the variables with elements of a set, we get a proposition (i.e. true or false)

First order language

Def A first order language consists of the following data:

a) symbols:
1) parentheses: (,)
2) connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$
3) quantifiers: $\exists$, $\forall$
4) symbol of equality: $=$
5) constants: $a, b, c, \dots, a_1, a_2, \dots$
6) variables: $x, y, z, \dots, x_1, x_2, \dots$

We are also given a natural number $n \in \mathbb{N}$, called the arity

7). functions: $f, g, h, \dots, f_1, f_2, \dots$

8) predicates: $P, Q, R, \dots, P_1, Q_2, \dots$

arity (nullary, unary, binary, ..., n-ary ...)
→ shows the number of variables

b) terms (expressions) - these are defined recursively (by induction)

1) constants are terms
2) variables are terms
} atomic terms

3) If f is a symbol of an n-ary function, and $t_1, \dots, t_n$ are terms, then $f(t_1, \dots, t_n)$ is also a term.

c) formulae: defined recursively

atomic formulae
1) If P is a symbol of an n-ary predicate, and $t_1, \dots, t_n$ are terms, then $P(t_1, \dots, t_n)$ is a formula.
2) If t_1, t_2 are terms, then $t_1 = t_2$ is a formula.

3) If A and B are formulae, then

$(\neg A)$, $(A \vee B)$, $(A \wedge B)$, $(A \to B)$, $(A \leftrightarrow B)$ are also formulae.

4) If $A(x)$ is a formula which depends on the variable x, then $(\exists x A(x))$, $(\forall x A(x))$ are also formulae.

Remarks 1) In the formula $\exists x\, A(x, y)$, x is called a bound variable and y is called a free variable

2) If a formula like $\forall x \exists y\, A(x,y)$ has no free variables, then this formula is called a closed formula.

A closed formula may be regarded as a proposition. (it will have a truth value) e.g. $\forall y \exists x\, (x + y = 1)$ closed formula

The structure of a first order Language. Interpretations

We will give values to expressions, and truth values to formulas.

Def A structure of a first order language consists of the following data:

1) a set M
2) to every constant symbol a we associate a fixed element $\tilde{a} \in M$
3) to every n-ary function symbol f we associate a

$$\tilde{f} : M^n \longrightarrow M$$

$$M^n = \underbrace{M \times \cdots \times M}_{n \text{ times}} = \{(x_1, \dots, x_n) \mid x_i \in M, i = 1, \dots, n\}$$

4) to any n-ary predicate symbol P we associate a subset $\tilde{P} \subseteq M^n$
5) To the equality symbol '$=$' we associate the equality relation on M, i.e. the subset

$$\Delta(M) = \{(x, x) \mid x \in M\} \subseteq M^2 = M \times M$$

Def An interpretation of a first order language (w.r.t a given structure) is a function

$$s : \mathcal{V} \longrightarrow M.$$

$\mathcal{V}$ = the set of variables

Def. the value of a term (w.r.t a given structure and interpretation) is defined recursively:

1). the value of a constant a is the element $\tilde{a} \in M$
2). The value of a variable x is the element $\tilde{x} = s(x) \in M$
3) If f is an n-ary function symbol, and $t_1, \dots, t_n$ are terms, then the value of the expression $f(t_1, \dots, t_n)$ is $\tilde{f}(\tilde{t}_1, \dots, \tilde{t}_n) \in M$

Def the <u>truth value of a formula</u> (w.r.t. a given structure and interpretation) is defined recursively:

1) the truth value of the formula $P(t_1, \dots, t_n)$ is 1 if and only if $(\tilde{t}_1, \dots, \tilde{t}_n) \in \tilde{P}$, and is 0 otherwise.

2) the truth value of the formula $t_1 = t_2$ is 1 if and only if $\tilde{t}_1 = \tilde{t}_2$ in M, and is 0 otherwise.

3) the truth values of the formulas $(\neg A)$, $(A \vee B)$, $(A \wedge B)$, $(A \to B)$, $(A \leftrightarrow B)$ are defined as in the propositional logic

4) - The truth value of the formula $\exists x A$ is 1 if and only if there is an element $x \in M$ such that $A(x)$ is true.
- The truth value of the formula $\forall x A$ is 1 if and only if for any $x \in M$, $A(x)$ is true.

<u>Example</u> consider the formula $x + y = 1$

equality

here: x, y are variables

1 constant

$+$ is a symbol of a binary function; alternate notation

$$x+y = +(x,y)$$

$x+y$, 1 are terms.

structure: let $M = \mathbb{R}$, $\tilde{1} = 1 \in \mathbb{R}$

$\tilde{+} = + : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$; $V = \{x, y\}$

interpretation: $s : V \longrightarrow M$, $s(x) = 2$, $s(y) = 3$

so in this case $\tilde{x} = 2$, $\tilde{y} = 3$; the value of the term $x+y$ is $5 \in \mathbb{R}$

so the truth value of our formula is 0 (false)

- consider the closed formulas:
    - $\forall y \exists x \, (x+y=1)$ is true $(x = 1-y)$
    - $\exists x \forall y \, (x+y=1)$ is false

Def a) A formula is called a <u>tautology</u> if it takes the truth value 1 for any structure M and any interpretation ρ.

b) a formula is called a <u>contradiction</u> if it takes the truth value 0 for any structure M and any interpretation ρ.

c) a formula is <u>satisfiable</u> if it is not a contradiction

<u>Examples</u>

De Morgan
$$\begin{cases} \neg(\forall x \, A(x)) \Longleftrightarrow \exists x \, \neg A(x) \\ \neg(\exists x \, A(x)) \Longleftrightarrow \forall x \, \neg A(x) \end{cases}$$

(i.e. the formula $(\neg(\forall x \, A(x)) \longleftrightarrow \exists x \neg A(x))$ is a tautology

- $\exists x \forall y \, A(x,y) \Longrightarrow \forall y \exists x \, A(x,y)$

<u>Remark</u> the previous example shows that the converse implication is not true <u>in general</u>.

i.e. $\underbrace{\forall y \exists x \, A(x,y)}_{\text{can be } 1} \longrightarrow \underbrace{\exists x \forall y \, A(x,y)}_{\text{can be } 0}$

List of fundamental tautologies: 2.3.9. p. 17.

$\boxed{\text{Homework:} \longrightarrow \text{ex. 19.}}$