Mathematical Logic
Lecture 8
25.11.2022

Complete lattices (continued)

Def. 1). The ordered set $(A,\le)$ is called a lattice if $\forall x,y\in A$ $\exists \inf\{x,y\}$, $\exists \sup\{x,y\}$.

2). $(A,\le)$ is called a complete lattice if $\forall B\subseteq A$ $\exists \inf B$ and $\exists \sup B$.

Theorem (characterization of complete lattices)

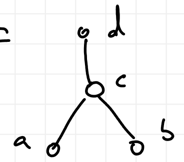
Let $(A,\le)$ be an ordered set. The following statements are equivalent:

(i) A is a complete lattice

(ii) $\forall B\subseteq A$ $\exists \inf B$.

(iii) $\forall B\subseteq A$ $\exists \sup B$.

Example



$\sup\{c,d\}=d$.

$\sup\{a,b\}=c$

$\sup\{a,c\}=c$

$\nexists \inf\{a,b\}$

$\nexists \sup\emptyset$ hence $\nexists \min A$

Proof. (i) $\Rightarrow$ (ii), (iii) by def.

(ii) $\Rightarrow$ (i) Let $B\subseteq A$. We have to show that $\exists \sup B$.

Let $C=\{y\in A \mid \forall x\in B\ x\le y\}$ the set of all upper bounds for B.

By the assumption (ii) $\exists \inf C =: a$.

We prove that $a=\sup B$.

- we show that a is a majorant of B, i.e. $x\le a$ $\forall x\in B$.

Indeed, let $x\in B$; then $\forall y\in C$ we have $x\le y$. This means that x is a minorant for C, but $a=\inf C$, hence $x\le a$.

- we show that a is the least majorant for B.

Let $a'\in A$ be another majorant for B, i.e. $a'\in C$.

But $a=\inf C$, hence $a\le a'$.

(iii) $\Rightarrow$ (i) Homework!

<u>Conclusion</u>: A is a complete lattice.

$$\sup B = \inf \{ y \in A \mid y \text{ is a majorant of } B \}$$
$$\inf B = \sup \{ y \in A \mid y \text{ is a lower bound for } B \}$$
(minorant)

<u>Example</u> 1). $(\mathcal{P}(M), \subseteq)$ is a complete lattice,

where $\inf (X_i)_{i \in I} = \bigcap\limits_{i \in I} X_i$

$\sup (X_i)_{i \in I} = \bigcup\limits_{i \in I} X_i$ $\quad \forall$ subsets $X_i \subseteq M$, $i \in I$.

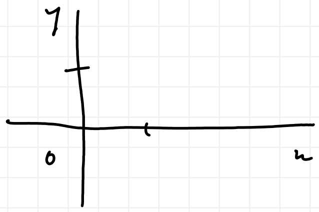
2). Let $(G, \cdot)$ be a group.

Denote $\mathcal{S}(G) = \{ H \mid H \text{ subgroup of } G \}$

Let $(H_i)_{i \in I}$ be a family of subgroups of G

then $\inf (H_i)_{i \in I} = \bigcap\limits_{i \in I} H_i$ because the intersection of subgroups is a subgroup

<u>Remark</u> the union of subgroups is, in general, not a subgroup



$(\mathbb{R} \times \mathbb{R}, +)$ grp

$H_1 = Ox = \{ (x, 0) \mid x \in \mathbb{R} \}$ subgrp of $\mathbb{R} \times \mathbb{R}$

$H_2 = Oy = \{ (0, y) \mid y \in \mathbb{R} \}$ subgrp of $\mathbb{R} \times \mathbb{R}$

but $H_1 \cup H_2$ is not a subgrp. Let $\underset{\overset{\cap}{H_1}}{a} = (1, 0)$, $\underset{\overset{\cap}{H_2}}{b} = (0, 1)$

$a, b \in H_1 \cup H_2$. but $a + b = (1, 1) \notin H_1 \cup H_2$

According to the theorem:

$$\operatorname{Sp}(H_i)_{i\in I} = \bigcap_{\substack{H\le G\\ H\supseteq H_i\,\forall i\in I}} H$$

Well-ordered sets

Def. Let $(A,\le)$ be an ordered set. We say that A is well-ordered, if $\forall\, \emptyset\ne B\subseteq A\ \exists \min B$.

Examples. 1) $(\mathbb{N},\le)$ is well-ordered (we'll see!)

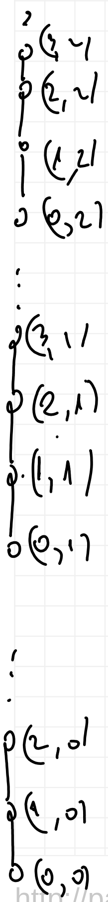
(Note that $\nexists \inf\emptyset = \max(\mathbb{N},\le)$)

2). If A is well-ordered, then A is totally ordered, because $\forall\, x,y\in A\ \exists \min\{x,y\}$.

The converse is not true.

e.g. $(\mathbb{R},\le)$ is totally ordered, but not well-ordered because, for instance, $\nexists \min(a,b)$, or $\nexists \min\{\frac{1}{n} \mid n\in\mathbb{N}^*\}$ ($\frac{1}{n} > \frac{1}{n+1}$, but it has inf equal to 0. ($\forall a>0\ \exists n$ s.t. $\frac{1}{n}<a$))

3) any finite totally ordered set is well-ordered (by induction)



4). $(\mathbb{N},|)$, $(\mathcal{P}(M),\subseteq)$ where $|M|\ge 2$ are not totally ordered, hence they are not well-ordered.

5) $\emptyset$ is well-ordered, by def

6) Let $A=\bigcup_{i\in\mathbb{N}}\mathbb{N}\times\{i\}$ ← see the diagram. it is well-ordered

Theorem (Characterization of well-ordered sets)

Let $(A, \leq)$ be a non-empty ordered set.

The following statements are equivalent:

(i) A is well-ordered (A satisfies the minimum condition)

(ii) (inductive condition) A is totally ordered, $\exists \min A =: a_0$,

and for any subset $B \subseteq A$ which satisfies:

1° $a_0 \in B$,

2° $\forall y \in A$, if $\{x \in A \mid x < y\} \subseteq B$,

then $y \in B$.

we have $B = A$.

Corollary (The principle of (strong) complete induction).

Let $(A, \leq)$ be a non-empty well-ordered set, and let P be a predicate on A. Assume that:

1° $P(a_0)$ is true, where $a_0 = \min A$

2° $\forall y \in A$, if $P(x)$ is true $\forall x < y$, then $P(y)$ is true

Then $P(a)$ is true for any $a \in A$

Proof (cor) Let $B = \{a \in A \mid P(a) \text{ is true}\}$

By assumption, B satisfies conditions 1°, 2° of the theorem

it follows that $B = A$

Proof (thm) $\underline{(ii) \Rightarrow (i)}$ Let $\emptyset \neq B \subseteq A$ be a non-empty subset of A; we have to show that $\exists \min B$.

Assume, by contradiction, that $\nexists \min B$.

We consider the subset $A \setminus B = C_A(B)$

- We have that $a_0 \in A \setminus B$, because otherwise, if $a_0 \in B$ then a_0 would be the smallest of B.

  Hence $A \setminus B$ satisfies $1°$

- Let $y \in A$ s.t. $\forall x \in A$, $x < y$ we have $x \in A \setminus B$

  If $y \in B$, then we would get that $y = \min B$, impossible

  It follows that $y \in A \setminus B$.

  This means that $A \setminus B$ satisfies cond $2°$.

By assumption (ii), we get $A \setminus B = A$. Then $B = \emptyset$, contrad.

<u>(i) $\Rightarrow$ (ii)</u> Assume that $\underset{\neq \emptyset}{A}$ is well-ordered. Then $\exists\, a_0 = \min A$ and is totally ordered.

Let $B \subseteq A$ satisfying conditions $1°$, $2°$.

We have to show that $B = A$.

Assume, by contradiction, that $B \neq A$, ie $A \setminus B = \complement_A B \neq \emptyset$ (1)

By (1), $\exists\; a := \min(A \setminus B)$, hence $\underline{\underline{a \notin B}}$.

Let $x \in A$ s.t. $x < a$, hence $x \notin A \setminus B$, hence $x \in B$

By cond $2°$ on B, we deduce that $\underline{\underline{a \in B}}$ contrad.

It follows that $B = A$.

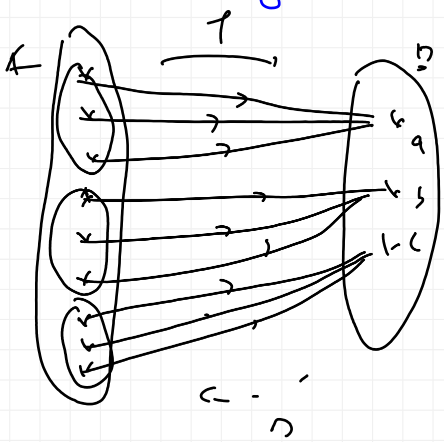
The Axiom of Choice

(AC) Let $I \neq \emptyset$ be a index set, let $(A_i)_{i \in I}$ be a family of non-empty sets such that $A_i \cap A_j = \emptyset$ for $i \neq j$

Then there is $\underset{\text{a set}}{X}$ s.t. $\forall i \in I$ $|X \cap A_i| = 1$

Statements equivalent to the axiom of choice

1). Every surjective function has a right inverse (section)



$f \circ s = 1_B$.

$s(c) = ?$ we choose an element from $f^{-1}(c) = \{x \in A \mid f(x) = c\}$

2). Zermelo's well-ordering principle — Ernst Zermelo

On any set A there is an order relation "$\leq$" such that $(A, \leq)$ is well-ordered.

3) (Zorn's Lemma) Max Zorn

Let A be a non-empty ordered set. Assume that any totally ordered subset B of A has a majorant in A.
Then in A there exist maximal elements.

Application: any vector space has a basis
"
a maximal linearly independent subset

Homework: ex. 73^*, 74^*, 75^*