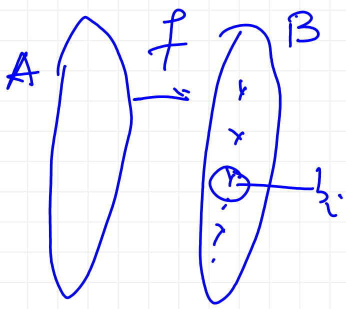
Exercises

Ex. 106 Let $|A| = k$, $|B| = n$.
Find the number of surjective functions $f: A \to B$.

Solution Observe that if $f: A \to B$, then $|A| \geq |B|$,
Because f has a right inverse (a section) $s: B \to A$
$f \circ s = 1_B$, But then s has a left inverse,
so s is injective, hence $|B| \leq |A|$

- If $k < n$, then $|Hom_{surj}(A,B)| = 0$

So we assume $k \geq n$, $k \neq 0$, $n \neq 0$
(for $k = n = 0$, $Hom(A,B) = \{\emptyset\}$
and $\emptyset$ is surjective.)



We know: $|Hom(A,B)| = n^k$

Let $B = \{b_1, \dots, b_n\}$.

f is not surjective $\iff \exists i \in \{1,\dots,n\}$ s.t. $b_i \notin Im f$
$\iff \exists i$ s.t. $f \in A_i$

Not. $A_i = \{ f: A \to B \mid b_i \notin Im f \}$

hence f is not surj $\iff f \in \bigcup_{i=1}^{n} A_i$

We get: $|Hom_{surj}(A,B)| = n^k - \left|\bigcup_{i=1}^{n} A_i\right|$

We use the inclusion - exclusion principle :

$$\left|\bigcup_{i=1}^{n} A_i\right| = \underbrace{\sum_{i=1}^{n}}_{\binom{n}{1}=n \text{ terms}} |A_i| - \underbrace{\sum_{i<j}}_{\binom{n}{2}=C_n^2 \text{ terms}} |A_i \cap A_j| + \underbrace{\sum_{i_1<i_2<i_3}}_{\binom{n}{3}=C_n^3 \text{ terms}} |A_{i_1} \cap A_{i_2} \cap A_{i_3}|$$

$$+ \dots + (-1)^{m+1} \underbrace{\sum_{1\le i_1 < \dots < i_m \le n}}_{\binom{n}{m}=C_n^m \text{ terms}} |A_{i_1} \cap \dots \cap A_{i_m}| +$$

$$+ \dots + (-1)^{n+1} \left|\bigcap_{i=1}^{n} A_i\right|$$

$$\binom{n}{n} = C_n^n = 1 \text{ term}$$

We have :

- $|A_i| = |\{f : A \to B \mid b_i \notin \mathrm{Im} f\}| = |\{f : A \to B \setminus \{b_i\}\}| = (n-1)^k$
- $|A_i \cap A_j| = |\{f \mid b_i, b_j \notin \mathrm{Im} f\}| = |\{f : A \to B \setminus \{b_i, b_j\}\}| = (n-2)^k$

...

- $|A_{i_1} \cap \dots \cap A_{i_m}| = (n-m)^k$
- $\left|\bigcap_{i=1}^{n} A_i\right| = 0 = |\{f : A \to \emptyset\}|$
- $|A_{i_1} \cap \dots \cap A_{i_{n-1}}| = (n-(n-1))^k = 1^k$

We get :

$$\left|\bigcup A_i\right| = \binom{n}{1}(n-1)^k - \binom{n}{2}(n-2)^k + \binom{n}{3}(n-3)^k - \dots$$

$$\dots + (-1)^{m+1}\binom{n}{m}(n-m)^k + \dots$$

$$+ \dots (-1)^n \binom{n}{n-1} 1^k + 0$$

Finally :

$$|\mathrm{Hom}_{surj}(A,B)| = n^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \dots + (-1)^m\binom{n}{m}(n-m)^k +$$

$$+ \dots + (-1)^{n-1}\binom{n}{n-1} 1^k$$

97 a) Prove that $\aleph_0 + \aleph_0 = \aleph_0$

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

Sol: we know $\aleph_0 := |\mathbb{N}|$

- We write $\mathbb{N} = \underbrace{2\mathbb{N}}_{\text{even}} \cup \underbrace{(2\mathbb{N}+1)}_{\text{odd}}$

We have $f : \mathbb{N} \longrightarrow 2\mathbb{N}$, $f(x) = 2x$ bij

$g : \mathbb{N} \longrightarrow 2\mathbb{N}+1$, $g(x) = 2x+1$ bij

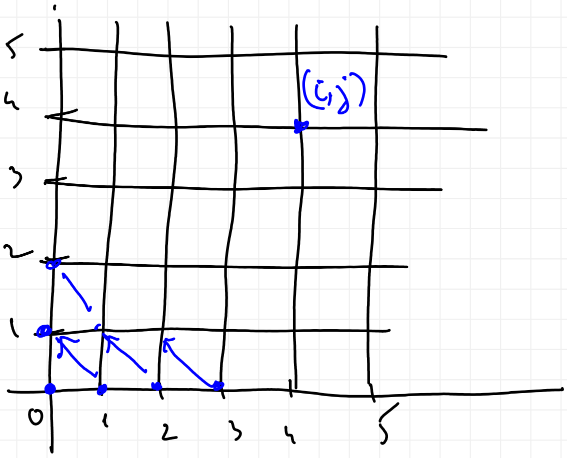
Hence $\aleph_0 = |\mathbb{N}| = |2\mathbb{N}| = |2\mathbb{N}+1|$

hence $\aleph_0 = \aleph_0 + \aleph_0$

- 1st sol: we know that $\aleph_0 \cdot \aleph_0 = |\mathbb{N} \times \mathbb{N}|$

We need to find a bijective function

$$f : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$$



We define f as follows:

$f(0) = (0,0)$

$f(1) = (1,0)$

$f(2) = (0,1)$

$f(3) = (2,0)$

$f(4) = (1,1)$

$f(5) = (0,2)$

$f(6) = (3,0)$

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it is clear that $\forall (i,j) \in \mathbb{N} \times \mathbb{N}$ $\exists!\, n \in \mathbb{N}$ st $f(n) = (i,j)$

hence we obtain in this way a bij. funct. $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

2nd sol We will use the fact that $\forall n \in \mathbb{N}^*$ can be written uniquely as $n = 2^{m-1}(2k-1)$, with $m, k \in \mathbb{N}^*$

e.g. $n = 18 = 2 \cdot 9$, $m = 2$, $k = 5$

$n = 52 = 4 \cdot 13$, $m = 3$, $k = 7$

It follows that the function : $\begin{cases} f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{N}^* \\ f(m,k) = 2^{m-1}(2k-1) \end{cases}$ is bijective.

Then $\aleph_0 \cdot \aleph_0 = \aleph_0$.

96 Let A be an infinite set. Prove that

a) $|A| + n = |A|$

b) $|A| + \aleph_0 = |A|$

Proof We know that $n \leq \aleph_0$

hence $|A| \leq |A| + n \leq |A| + \aleph_0 \overset{?}{=} |A|$

This means that it is enough to prove b)

We know that $\exists\, f : \mathbb{N} \longrightarrow A$ injective function

we get that $\aleph_0 = |\mathbb{N}| = |\operatorname{Im} f|$

We have $A = \operatorname{Im} f \cup (A \setminus \operatorname{Im} f)$,

hence $|A| = |\operatorname{Im} f| + |A \setminus \operatorname{Im} f|$

$= \aleph_0 + |A \setminus \operatorname{Im} f|$

$+ (\aleph_0 + \aleph_0) + |A \setminus \operatorname{Im} f|$

$$= \aleph_0 + (\aleph_0 + |A \setminus \operatorname{Im} f|)$$
$$= \underline{\aleph_0 + |A|}.$$

99 a) $c^2 = c^{\aleph_0} = c$ $\quad$ (where $c := |\mathbb{R}| = 2^{\aleph_0}$)

b) $c + c = c \cdot \aleph_0 = \aleph_0^{\aleph_0} = c$

<u>Sol</u> a) $c^2 = (2^{\aleph_0})^2 = 2^{2\aleph_0} = 2^{\aleph_0} = c$

$$c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$$

b) • $c \leq c + c = 2c \leq c^2 = c$

we get (by antisymmetry) $c + c = c$

• $c \leq c \cdot \aleph_0 \leq c \cdot c = c \implies c \cdot \aleph_0 = c$

• $c = 2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} = c \implies \aleph_0^{\aleph_0} = c.$

97 d) Prove that $\mathbb{Q}$ is countable (i.e. $\mathbb{Q} \approx \mathbb{N}$, i.e. $|\mathbb{Q}| = \aleph_0$)

<u>Sol</u> We will simplify the problem a bit.

We write $\mathbb{Q} = \mathbb{Q}^*_- \cup \{0\} \cup \mathbb{Q}^*_+$

Clearly $|\mathbb{Q}^*_-| = |\mathbb{Q}^*_+|$. So it is enough to

prove that $|\mathbb{Q}^*_+| = \aleph_0$

We have that $|\mathbb{Q}_+^*| = \{ \frac{m}{n} \mid m,n \in \mathbb{N}^* \}$

Let $\begin{cases} f : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{Q}_+^* \\ f(m,n) = \frac{m}{n} \end{cases}$ f is surjective, f is not injective

so $\aleph_0 = \aleph_0 \cdot \aleph_0 = |\mathbb{N}^* \times \mathbb{N}^*| \geq |\mathbb{Q}_+^*|$

but obviously $|\mathbb{N}| \leq |\mathbb{Q}_+^*|$ (because the function $\mathbb{N} \to \mathbb{Q}_+^*$, $n \mapsto \frac{n}{1}$ is injective)

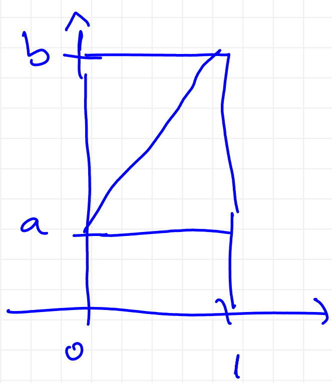
We get $|\mathbb{Q}_+^*| = \aleph_0$.

98 a) Prove that any interval has the power of the continuum. i.e.

$$\mathbb{R} \sim (a,b) \sim [a,b) \sim (a,b] \sim [a,b]$$

b) the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable (i.e. $\aleph_0 < |\mathbb{R} \setminus \mathbb{Q}|$)

<u>Sol</u> a) We prove that $[0,1] \sim [a,b]$

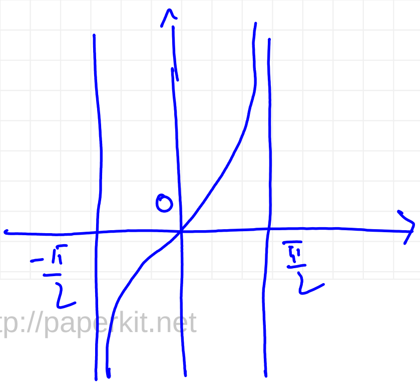


We need a bijective function $f : [0,1] \to [a,b]$

Let $f(x) = (b-a)x + a$

Because (a,b) is infinite, we have

$$(a,b) \sim [a,b) \sim [a,b] \sim (a,b]$$



The map $\begin{cases} f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \\ f(x) = tg\, x \end{cases}$ is bijective.

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b) We write $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$

hence $c = \aleph_0 + |\mathbb{R} \setminus \mathbb{Q}|$

if, by contradiction, we assume that $\mathbb{R} \setminus \mathbb{Q}$ is countable, then we get $c = \aleph_0 + \aleph_0 = \aleph_0$

contradiction

(we know $c > \aleph_0$)