Mathematical Logic
Lecture 12
13.01.2023

Finite, infinite, and countable sets

Recall: the set of natural numbers:

$\mathbb{N} = \{0, 1, 2, \ldots\}$, where $0 = \emptyset$, $1 = \emptyset^+ = \{\emptyset\}$, $2 = 1^+$

where the successor of the set X is $X^+ = X \cup \{X\}$

Not. 1) If $n \in \mathbb{N}$, then we denote by the same n the cardinal number of n:

$|n| = n$ so nat. numbers are also regarded as cardinal numbers.

(recall: $|A| = \{B \text{ set} \mid A \sim B \text{ equipotent}\}$)

2). $|\mathbb{N}| \overset{\text{not}}{=\!:} \aleph_0$ aleph zero

3) $|\mathbb{R}| \overset{\text{not}}{=\!:} c$ — the power of the continuum.

Def Let A be a set.

1) We say that A is finite if A is equipotent to a nat. number (i.e. $\exists n \in \mathbb{N}$ s.t. $A \sim n$)

2). A is infinite if A is not finite (i.e. $\forall n \in \mathbb{N}$ $A \nsim n$.)

3) A is countable if A is equipotent to a subset of $\mathbb{N}$.

Rem A is countable means that either A is finite or A is infinite countable i.e. $A \sim \mathbb{N}$

Theorem (characterization of infinite sets). Let A be a set. The foll. statements are equivalent:

(i) A is infinite

(ii) A is equipotent to some proper subset of A

(i.e. $\exists\, B \subsetneq A$ s.t. $A \sim B$)

(iii) $\exists\, f : \mathbb{N} \to A$ injective function.

Rem: cond (iii) says that $\aleph_0$ is the smallest transfinite cardinal number.

Theorem (characterization of finite sets)

Let A be a set. The foll. statements are equivalent:

(i) A is finite

(ii) $\forall\, B \subsetneq A$ $\quad A \not\sim B$.

(iii) $\forall\, f : \mathbb{N} \to A$ is not injective

Example: Consider $2\mathbb{N} = \{2k \mid k \in \mathbb{N}\} \subsetneq \mathbb{N}$

Let $f : \mathbb{N} \to 2\mathbb{N}$, $f(k) = 2k$. Then f is bijective.

so $|\mathbb{N}| = |2\mathbb{N}|$ $\qquad$ simil. $|\mathbb{N}| \sim |\mathbb{N}^*|$

Theorem (Cantor). 1). $\mathbb{R}$ is not countable (i.e. $\aleph_0 < c$)

2). More precisely: $c = 2^{\aleph_0}$

Proof 1) we use the decimal representation of real numbers, and "Cantor's diagonal argument"

By an (ex) $\mathbb{R} \sim (a,b) \sim [a,b] \sim [a,b) \sim (a,b]$

So we will prove that the interval $[0,1)$ is not countable

If $a \in [0,1)$, then we may write:

$$a = 0.a_1 a_2 a_3 \cdots \ (10)$$
$$= \sum_{k=1}^{\infty} \frac{a_k}{10^k}, \quad \text{where } a_k \in \{0, \dots, 9\}.$$

 Recall also that this representation is unique if we exclude period 9:

$$0.(9) = 0.999\ldots = 1.000\ldots = 1.(0)$$

So some rational numbers have two representations:

$$0.a_1 a_2 \ldots \underset{\substack{\neq \\ 9}}{a_n}(9) = 0.a_1 a_2 \ldots a_{n-1}(a_n+1)\underset{(0)}{(0)}$$

$$= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n} \in \mathbb{Q}.$$

Assume by contradiction that $\exists\ f: \mathbb{N}^* \to [0,1)$ bijective.

Let:

$$f(1) = 0.a_{11}\,a_{12}\,a_{13}\,a_{14}\,a_{15}\ldots$$

$$f(2) = 0.a_{21}\,a_{22}\,a_{23}\,a_{24}\,a_{25}\ldots \quad \text{(without period 9)}$$

$$f(3) = 0.a_{31}\,a_{32}\,a_{33}\,a_{34}\,a_{35}\ldots$$

$$f(4) = 0.a_{41}\,a_{42}\,a_{43}\,a_{44}\,a_{45}\ldots$$

$$f(5) = 0.a_{51}\,a_{52}\,a_{53}\,a_{54}\,a_{55}\ldots$$

$\vdots$

Let $a = 0.a_1 a_2 a_3 a_4 a_5 \ldots$, where we choose

$$a_1 \neq a_{11},\ 0,\ 9$$

$$a_2 \neq a_{22},\ 0,\ 9$$

$$\ldots$$

$$a_n \neq a_{nn},\ 0,\ 9.$$

We have $a \in [0,1)$, and by the uniqueness of representation, we have that $a \neq f(k)$ $\forall k \in \mathbb{N}^*$.

But this contradicts the surjectivity of f.

2) We know that $2^{\aleph_0} = |\mathrm{Hom}(\mathbb{N}^*, \{0,1\})|$

and that a function $f: \mathbb{N}^* \to \{0,1\}$ is just a sequence $(a_k)_{k\in\mathbb{N}^*}$, $a_k \in \{0,1\}$

$$\overset{\shortparallel}{(a_1, a_2, \dots)}$$

Anyway we can get $\mathbb{R} \sim [0,1)$, i.e. $|[0,1)| = c$.

We use the representation of numbers from $[0,1)$ in base 2.

recall: $a = 0.a_1 a_2 a_3 \dots_{(2)}$

$$= \sum_{k=1}^{\infty} \frac{a_k}{2^k}, \text{ where } a_k \in \{0,1\}$$

we have to avoid period 1 because

$$0.(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}}$$

So $0.a_1 a_2 \dots a_k 0(1) = 0.a_1 a_2 \dots a_k 1(0) \in \mathbb{Q}$

$$= \frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_k}{2^k} + \frac{1}{2^{k+1}}$$

So some rational numbers have exactly two representations, and if we avoid period 1 the repres. is unique.

Let $f : [0,1) \longrightarrow \mathrm{Hom}(\mathbb{N}^*, \{0,1\}) = \{0,1\}^{\mathbb{N}^*}$

where if $a = 0.a_1 a_2 a_3 \dots_{(2)}$

then we define $f(a) = (a_k)_{k\in\mathbb{N}^*} \in \{0,1\}^{\mathbb{N}^*}$.

By the uniqueness of repr. of a, we get that f is a well defined function and f is injective.

It follows that $c = |[0,1)| = |\mathrm{Im} f|$

Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be the set of sequences having period 1. Then we have the disjoint union:

$$\{0,1\}^{\mathbb{N}} = \mathrm{Inf} \cup A \text{, hence}$$

$$2^{\aleph_0} = |\{0,1\}^{\mathbb{N}}| = c + |A|$$

But sequences with period 1 represent rational numbers, and by (ex) we have $|A| = \aleph_0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad |\mathbb{Q}|$

we get $2^{\aleph_0} = c + \aleph_0$

$$\overset{\text{ex}}{=} c \quad .$$

Rem $c \overset{?}{=} \aleph_1$, the continuum hypothesis is indep from other axioms of set theory.

(i.e. between $\aleph_0$ and c there are no other cardinals)

Combinatorics

We will compute the number of elements of certain finite sets.

Let $k, n \in \mathbb{N}$. Consider the totally ordered finite sets

$$A = \{a_1 < a_2 < \dots < a_k\}$$

$$B = \{b_1 < b_2 < \dots < b_n\}.$$

1). Arrangements with repetition

Def A k-arrangement with repetition of n elements is a sequence of length k of elements of B

Example Let $k=2$, $n=4$. We write down all the 2-arrangements with rep. of 4 elem:

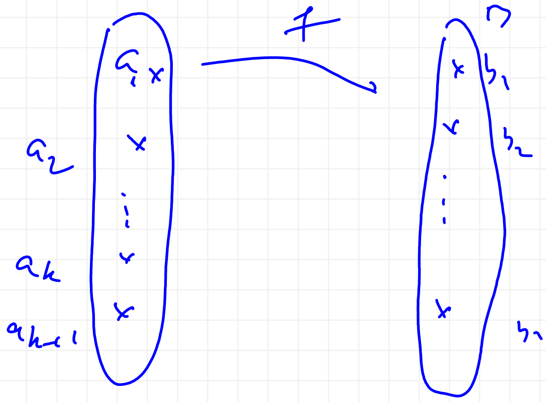
$$\begin{array}{cccc} b_1b_1 & b_2b_1 & b_3b_1 & b_4b_1 \\ b_1b_2 & b_2b_2 & b_3b_2 & b_4b_2 \\ b_1b_3 & b_2b_3 & b_3b_3 & b_4b_3 \\ b_1b_4 & b_2b_4 & b_3b_1 & b_4b_4 \end{array}$$

$\bar{A}_4^2 = 16$

Not. $\bar{A}_n^k$ = the number of k-arrangements with rep. of n elem.

Problem Calculate $\bar{A}_n^k = ?$

Observe that a k-arrangement with rep of n elem is identified with a function $f: A \to B$.



Let $k=1$.
$|B^A| = n$
Let $k=2$
$|B^A| = n^2$

induction on k: $P(k)$: $\bar{A}_n^k = n^k$

Assume $P(k)$ is true. Then for any choice of $f(a_1), \dots, f(a_k)$ we have n choices for $f(a_{k+1})$.

We get $\bar{A}_n^{k+1} = \bar{A}_n^k \cdot n \overset{P(k)}{=} n^{k+1}$.

Conclusion. $\boxed{\bar{A}_n^k = |\mathrm{Hom}(A,B)| = n^k}$

e). Arrangements

Def A *k-arrangement of n elements* is a sequence of length n of elements of B, such that every element occurs at most once.

Exple $n=4$, $k=2$ we write down all the 2-arrangements of 4 elements:

$$\begin{matrix} b_1 b_2 & b_2 b_1 & b_3 b_1 & b_4 b_1 \\ b_1 b_3 & b_2 b_3 & b_3 b_2 & b_4 b_2 \\ b_1 b_4 & b_2 b_4 & b_3 b_4 & b_4 b_3 \end{matrix} \qquad A_4^2 = 12$$

Not $A_n^k :=$ the number of k-arrangements of n elements.

Problem: Calculate $A_n^k = ?$

Observe that a k-arrangement of n elements can be identified with an injective function $f: A \to B$

hence $A_n^k = |\mathrm{Hom}_{inj}(A, B)|$

We proceed by induction on k.

- for $k=1$, any funct $f: A \to B$ is injective
- for $k=2$, for any values of $f(e_1)$ we have $n-1$ possibilities for $f(e_2)$
- from k to $k+1$: for any values of $f(e_1), \dots, f(e_k)$ we have $n-k$ possibilities for $f(e_k)$

We get $A_n^{k+1} = A_n^k \cdot (n-k)$.

Hence $$\boxed{A_n^k = n(n-1)(n-2)\cdots(n-k+1)}$$

Rem 1) if $k > n$, then $A_n^k = 0$

2) if $k=0$, i.e. $A=\emptyset$, then $\mathrm{Hom}(\emptyset, B) = \{\emptyset\}$

hence $\bar{A}_n^0 = A_n^0 = 1.$ in part $0^0 = 1$ in this context!!

3) Permutations

Def A perm. of n elmts is a sequence of length n of elements of B s.t. each elem occur exactly once.

Rem a perm. of n elmts is identified with a bijective function $f: A \to B$, where $k = n$.

Not P_n = no. of perm. of n elmts.

We have: $\boxed{P_n = A_n^n = n!}$

Rem $P_0 = 1$, so $0! = 1$ by convention.

$$P_0 = \left|\mathrm{Hom}_{bij.}(\emptyset, \emptyset)\right|$$

4) Combinations

Def A k-combination of n elements is a strictly increasing sequence of length k of elem of B.

Ex $n = 5$, $k = 3$. We write down all the 3-combination of 5 elements:

$b_1 b_2 b_3$ $b_1 b_3 b_4$ $b_2 b_3 b_4$ $b_3 b_4 b_5$.

$b_1 b_2 b_4$ $b_1 b_3 b_5$ $b_2 b_3 b_5$

$b_1 b_2 b_5$ $b_1 b_4 b_5$ $b_2 b_4 b_5$

Not $\binom{n}{k} = C_n^k$ = no. of k-combination of n elmts

Problem calculate $\binom{n}{k} = ?$

Observe that from any k-comb of n elements we get P_k k-arrangem of n elements.

Hence $A_n^k = C_n^k \cdot P_k$.

Then
$$\binom{n}{k} = \frac{A_n^k}{P_k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

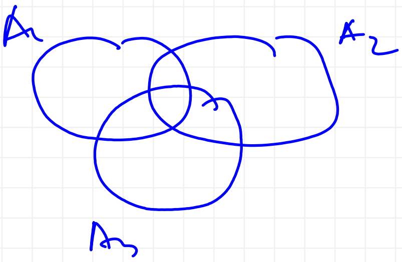
(for $k \leq n$)

(for $k > n$ $\binom{n}{k} = 0$)

Rem any k-combination of n elements is determined uniquely by a subset with k-elements of B.

hence $\binom{n}{k}$ = number of subsets with k-elements of a set with n elements.

5). The Inclusion exclusion principle



- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

Theorem let $A_1, \dots, A_n$ be finite sets.

Then
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}| + \dots + (-1)^{n+1} \left|\bigcap_{i=1}^{n} A_i\right|$$

Proof HW

Pag. 8.4.3 / 55

Homework : ex. 95 – 106.