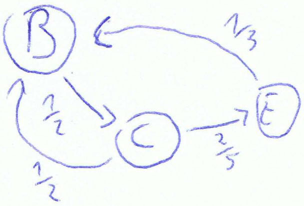
B) 1) $S = \{A, B, C, D, E\}$ COMM. CLASSES $= \{\{A, B, C, E\}, \{D\}\}$

$|$ COMM. CLASS. $| > 1 \Rightarrow$ REDUCIBLE CHAIN

$\{D\} \to$ PERIOD $= 1 \to$ APERIODIC, POS. RECURRENT



PERIOD$(B) = GCD(2,3) = 1 \Rightarrow$ PERIOD $\{A, B, C, E\} = 1$ - APERIODIC

$\hookrightarrow$ TRANSIENT

$\left(P(T_A < \infty) < \frac{4}{5} \neq 1\right)$

STATIONARY DISTRIBUTION:

$\pi K = \pi \iff [a\ b\ c\ d\ e]K = \left[\frac{b}{4} + \frac{e}{5}, \frac{4a}{5} + \frac{c}{2}, \frac{b}{2} + \frac{e}{3}, \frac{a}{5} + \frac{b}{4} + \frac{c}{10} + d + \frac{e}{3}, \frac{2e}{5}\right] = [a\ b\ c\ d\ e]$

CONDITION: $a + b + c + d + e = 1$

SOLVE SYSTEM OF LINEAR EQUATIONS: ONLY SOLUTION: $\pi = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = 0$

$\sum_{i=1}^{5} \pi_i \neq 1 \Rightarrow$ THERE IS NO STATIONARY DISTRIBUTION

2) $S = \{A, B, C, D\}$ COMM. CLASSES $= \{\{A\}, \{B\}, \{C, D\}\} \to$ REDUCIBLE

$K(x,x) > 0 \ \forall x \in S \Rightarrow$ APERIODIC

$\{A\}, \{B\}$: transient; $\{C, D\}$: $\hookrightarrow$ POS. RECURRENT

STATIONARY DISTRIBUTION:

$E[T_D] = \frac{9}{10} + 2 \cdot \frac{1}{10} \cdot \frac{7}{10} + 3 \cdot \frac{1}{10} \cdot \frac{3}{10} \cdot \frac{7}{10} + \dots =$

$[a\ b\ c\ d] = \left[\frac{a}{10}, \frac{2a}{10} + \frac{5b}{10}, \frac{3a}{10} + \frac{2b}{10} + \frac{3c}{10} + \frac{2d}{10}, \frac{4a}{10} + \frac{3b}{10} + \frac{7c}{10} + \frac{9d}{10}\right]$ $= \frac{9}{10} + \frac{7}{100}\left(\sum_{i=2}^{\infty} i \left(\frac{3}{10}\right)^{i-2}\right) < \infty$

$a = \frac{a}{10} \Rightarrow a = 0$

$b = \frac{5b}{10} \Rightarrow b = 0$ $\Rightarrow$ THERE IS NO STATIONARY DISTRIBUTION

$c = \frac{3}{7}d$, $d = 7c = 3d \Rightarrow c = 0, d = 0$

3) $S = \{A, B, C, D\}$ COMM. CLASSES $= \{\{A\}, \{D\}, \{B, C\}\} \to$ REDUCIBLE

$K(x,x) > 0 \ \forall x \in S \Rightarrow$ APERIODIC

$\{A\}$: POS. RECURRENT; $\{D\}$: TRANSIENT; $\{B, C\}$: $\hookrightarrow$ POS. RECURRENT

STATIONARY DISTRIBUTION:

$[a\ b\ c\ d] = \left[a + \frac{d}{2}, \frac{b}{2} + \frac{c}{2}, \frac{b}{2} + \frac{c}{2}, \frac{d}{2}\right]$ $\Rightarrow \pi = [1-2b, b, b, 0]$, $b \in [0, \frac{1}{2}]$

$d = 0$, $a = a$ $\qquad a + 2b = 1 \Rightarrow a = 1 - 2b$ $\qquad$ STAT. DISTR.

$b = \frac{b}{2} + \frac{c}{2} = c \Rightarrow b = c$

4) $S = \mathbb{Z}$ COMM. CLASS $= S \to$ IRREDUCIBLE

PERIOD $= 2$

$K = \begin{bmatrix} \ddots & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \ddots \\ & & \ddots & \end{bmatrix}$

$\pi K = \pi \iff \pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1} \ \forall i \in \mathbb{Z}$

$\pi_i = c \ \forall \mathbb{R}$ IS A SOLUTION, BUT

$\sum \pi_i = \infty$

(4) CONTINUED...

SUPPOSE $\exists j \in \mathbb{Z}: \pi_{j-1} < \pi_j \Rightarrow \pi_{j+1} = 2\pi_j - \pi_{j-1} > \pi_j$

$\Rightarrow \pi_i$ is increasing from j onwards $\Rightarrow \sum_{i=-\infty}^{\infty} \pi_i = \infty$

$\exists j \in \mathbb{Z}: \pi_{j-1} > \pi_j \Rightarrow \pi_{j-2} = 2\pi_{j-1} - \pi_j > \pi_{j-1} \Rightarrow$

$\Rightarrow \pi_i$ is increasing from $j \to -\infty \Rightarrow \sum_{i=-\infty}^{\infty} \pi_i = \infty$

$\Rightarrow$ THERE IS NO STATIONARY DISTRIBUTION

INTUITION: S is ~~transient~~ null recurrent, don't know how to prove it.

$\Downarrow$

$P(T_x < \infty) = 1, \quad E[T_x] = \infty$

$P(T_x < \infty) = 1 \to$ not transient (there is a non-zero probability that we will eventually return in each state)

(5) $S = \mathbb{Z}$ IRREDUCIBLE (COMM. CLASS $= \mathbb{Z}$), PERIOD $= 2$

$$K = \begin{bmatrix} \ddots & & & & & \\ \frac{1}{5} & 0 & \frac{4}{5} & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \frac{4}{5} & 0 & \frac{1}{5} & \\ & & & & & \ddots \end{bmatrix}$$

(IF AN IRREDUCIBLE MC HAS A UNIQUE STAT. DISTR. $\Rightarrow$ POS. RECURRENT.)

$\pi K = \pi \Rightarrow$

$$\pi_j = \frac{4}{5}\pi_{j-1} + \frac{1}{5}\pi_{j+1} \qquad j \le -2$$

$$\pi_{-1} = \frac{4}{5}\pi_{-2} + \frac{1}{2}\pi_0$$

$$\pi_0 = \frac{1}{2}(\pi_{-1} + \pi_1)$$

$$\pi_1 = \frac{1}{2}\pi_0 + \frac{4}{5}\pi_2$$

$$\pi_i = \frac{1}{5}\pi_{i-1} + \frac{4}{5}\pi_{i+1} \qquad i \ge 2$$

IF $\pi_i \neq \pi_{-i}$ WE GET INCREASING SEQUENCES AS IN (4) $(\pi_1 = \pi_{-1} \Rightarrow \pi_i = \pi_{-i} \; \forall i)$

$\Rightarrow \pi_0 = \cancel{\frac{1}{2}(\dots)} = \frac{4}{5}(\pi_{-1} + \pi_1) = \frac{16}{5}(\pi_2 + \pi_{-2}) \Rightarrow (\pi_1 + \pi_{-1}) = 4(\pi_2 + \pi_{-2}) \dots$

$(\pi_i + \pi_{-i}) = 4^{-(i-1)} \cdot (\pi_1 + \pi_{-1}) \Rightarrow \sum \pi_i = \pi_0 + \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^{i-1} (\pi_1 + \pi_{-1}) =$

$= \pi_0 + \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^{i-1} \cdot \frac{5}{4}\pi_0 = \pi_0 + 5\pi_0 \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i =$

$= \pi_0 \left(1 + 5\frac{1}{1-\frac{1}{4}}\right) = \pi_0 \left(\frac{23}{3}\right) = 1 \Rightarrow \pi_0 = \frac{3}{23}$

$\Rightarrow 2\pi_1 = \frac{5}{4} \cdot \frac{3}{23} \Rightarrow \pi_1 = \frac{15}{8 \cdot 23} = \pi_{-1}$

so: $\pi_i = \pi_{-i} = \left(\frac{1}{4}\right)^{i-1} \cdot \frac{15}{4 \cdot 23} = \left(\frac{1}{4}\right)^i \cdot \frac{15}{23}$

UNIQUE STAT. DISTR. $\pi \Rightarrow$ POS RECURRENT //