

# **Modelling and Numerical Simulation of the Sterile Insect Technique for Mosquito Population Control**

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# Abstract

This project concerns feedback control strategies for applying the Sterile Insect Technique (SIT) to eliminate mosquito populations. This strategy entails releasing sterilized males into the environment to compete with wild males for mating with females, thereby reducing reproduction. Several control strategies are analyzed for two types of population models: one incorporating an Allee effect and one without. Numerical simulations that illustrate the effectiveness of these strategies are presented. Additionally, a cost minimization problem is formulated as an optimal control problem, with the goal of reducing the total number of sterile males released while ensuring mosquito population extinction.

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# 1

## **Introduction**

Mosquitoes serve as vectors for numerous diseases, including malaria, dengue, yellow fever and Zika virus, which, according to the World Health Organization, contribute to a substantial number of fatalities worldwide [10, 11]. Despite the availability of several effective vector control strategies for malaria and other arboviral diseases, some methods can be environmentally harmful and lead to unintended ecological consequences. For instance, the widespread application of insecticides can negatively impact non-target species. Moreover, prolonged use of chemical insecticides often drives the development of resistance in mosquito populations, reducing the long-term effectiveness of such measures.

The sterile insect technique (SIT) has emerged as a promising alternative for controlling mosquito populations. This method entails sterilizing male mosquitoes—typically through ionizing radiation—before releasing them into the environment to compete for mates with wild females.

This first cycle integrated project mainly aims to study [5] and [4]. In these articles, mathematical control theory is used through the construction of feedback laws to determine the appropriate release rates of sterile males to drive the population of tiger mosquitoes toward extinction equilibrium.

We undergo an analysis of the wild mosquito population model from [5] and its global stability properties, and study its evolution numerically, according to different initial conditions and parameter choices. We perform the same work regarding the SIT model built upon it with the three presented feedback laws, and aim to reproduce the numerical results from that article. Additionally, we build a new feedback law that still allows for the extinction of the mosquito population, with relevant differences.

Moreover, we carry out similar effort considering an altered version of the original model that now takes into consideration an Allee Effect, both for the wild dynamics and the SIT adaptation, regarding two out of the three original feedback laws, as well as the one we previously built.

We test the robustness of these strategies against perturbations of both the system dynamics parameters and the initial conditions.

Finally, to address the practical concern of minimizing intervention costs, we formulate and attempt to solve an optimal control problem aimed at minimizing a cost functional that balances population suppression with the operational costs associated with releasing sterile males.

# 2

## Models for mosquito population dynamics

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## 2.1 Model for wild mosquito population dynamics

The mosquito life cycle involves several stages. In order to reduce complexity, this model considers only two main phases: the aquatic phase and the adult phase, the former including the (fecund) egg, larva, and pupa stages. After the pupa stage, mosquitoes become adults. It is in this phase that they reproduce, and only female mosquitoes bite. To lay fecund eggs, female mosquitoes need to be fertilized and must consume a blood meal. Adult mosquitoes typically live for 2 to 4 weeks. The model represents the aquatic phase as  $E$ , and divides the adult phase into two groups:  $M$  for males and  $F$  for (fertilized) females.

We now define the mathematical model representing the two-phase mosquito life cycle:

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \quad (2.1)$$

$$\dot{M} = (1 - \nu) \nu_E E - \delta_M M, \quad (2.2)$$

$$\dot{F} = \nu \nu_E E - \delta_F F, \quad (2.3)$$

in which,  $E(t), M(t), F(t) \geq 0$  correspond, respectively, to the aquatic phase, adult male and fertilized female densities;  $\beta_E > 0$  corresponds to the oviposition rate, i.e., the average number of eggs that a female lays over a time unit (day);  $\delta_E, \delta_M, \delta_F > 0$  are, respectively, the death rates for the aquatic phase, adult males and fertilized females;  $\nu_E > 0$  corresponds to the hatching rate for eggs;  $\nu \in (0, 1)$  is the probability that a female emerges from a pupa (for simplicity, it is assumed that a female is immediately fertilized after this process);  $K > 0$  is the environmental capacity for eggs. It represents the maximum density of eggs females can lay in breeding sites. Since larval and pupal stages are not modeled separately,  $E$  includes all aquatic stages. Thus,  $K$  acts as the carrying capacity in a logistic growth model, accounting for larval competition, and has units of spatial density.

From these parameters we construct the basic offspring number

$$\mathcal{R}_0 = \frac{\beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E)} \quad (2.4)$$

which has the biological meaning of the average number of (fertilized) female mosquitoes produced by a single female in its lifetime.

Let us set  $x = (E, M, F)^T$ ,  $\mathcal{D} = \{x \in \mathbb{R}^3 : x \geq 0\}$ , in order to write the model as  $\dot{x} = f(x)$ , where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is clearly continuously differentiable on  $\mathbb{R}^3$ . For any initial condition  $x(0) \in \mathcal{D}$  we observe that  $x(t) \in \mathcal{D}$  for  $\forall t \geq 0$ .

Solving  $\dot{x} = 0$  gives the two following equilibria:

$$x = (0, 0, 0)^T = \mathbf{0} \quad (2.5)$$



$$x = (K(1 - \frac{1}{\mathcal{R}_0}), \frac{(1 - \nu)}{\delta_M} K(1 - \frac{1}{\mathcal{R}_0}), \frac{\nu \nu_E}{\delta_F} K(1 - \frac{1}{\mathcal{R}_0}))^T \quad (2.6)$$

Heed that the latter belongs to  $\mathcal{D}$  iff  $1 - \frac{1}{\mathcal{R}_0} \geq 0 \implies \mathcal{R}_0 \geq 1$ .

We now present an important theorem that will be used in the proof of the theorem that classifies equilibria for (2.1) - (2.3).

**Theorem 1.** (*Barbashin-Krasovskii*)

Consider the autonomous system

$$\dot{x} = f(x), \quad (2.7)$$

where  $f : D \rightarrow \mathbb{R}^n$  is a locally Lipschitz map from a domain  $D \subset \mathbb{R}^n$  into  $\mathbb{R}^n$ .

Let  $x$  be an equilibrium point for (4.1), and  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  a continuously differentiable function.

If the following hold:

$$V(\mathbf{0}) = 0 \text{ and } V(x) > 0, \forall x \neq \mathbf{0}, \quad (2.8)$$

$$\|x\| \rightarrow +\infty \implies V(x) \rightarrow +\infty, \quad (2.9)$$

$$\dot{V}(x) = \nabla V \cdot f(x) < 0, \forall x \neq \mathbf{0}. \quad (2.10)$$

then  $x = \mathbf{0}$  is globally asymptotically stable.

The proof may be consulted in [8].

**Theorem 2.** *The following properties hold.*

1. If  $\mathcal{R}_0 \leq 1$ , then  $\mathbf{0} \in \mathbb{R}^3$  is a globally asymptotically stable equilibrium in  $\mathcal{D}$ .
2. If  $\mathcal{R}_0 > 1$ , then the system has two equilibria,  $\mathbf{0}$  and (2.6). The latter is stable with basin of attraction  $\mathcal{D} = \{x = (E, M, F)^T \in \mathbb{R}_+^3 : E = F = 0\}$  whereas the former is unstable in  $\mathcal{D}$  with the non negative  $M$  - axis as a stable manifold.

*Proof.* We prove part one. The proof of the second part is presented in [5].

Let  $t \mapsto x(t) = (E(t), M(t), F(t))^T$  be a solution of (2.1) - (2.3) such that  $(E(0), M(0), F(0))^T \in \mathcal{D}$ .

Since (2.2) is linear and non-homogeneous, it follows directly that

$$M(t) = e^{-\delta_M t} M(0) + (1 - \nu) \nu_E \int_0^t e^{-\delta_M(t-s)} E(s) ds. \quad (2.11)$$

Then, we find the following upper bounds for  $M$ , where  $A = \sup\{E(s) : s \geq 0\}$ ,  $B(t) = \max\{E(s) : 0 \leq$

$s \leq \frac{t}{2}$  and  $C(t) = \sup\{E(s) : s \geq \frac{t}{2}\}$ :

$$M \leq M(0) + (1 - \nu)\nu_E \int_0^t e^{-\delta_M(t-s)} E(s) ds \quad (2.12)$$

$$\leq M(0) + (1 - \nu)\nu_E \int_0^t e^{-\delta_M(t-s)} A ds \quad (2.13)$$

$$= M(0) + (1 - \nu)\nu_E A \int_0^t e^{-\delta_M(t-s)} ds \quad (2.14)$$

$$= M(0) + \frac{(1 - \nu)\nu_E}{\delta_M} A, \quad (2.15)$$

$$M \leq e^{-\delta_M t} M(0) + (1 - \nu)\nu_E \int_0^{\frac{t}{2}} e^{-\delta_M(t-s)} E(s) ds + (1 - \nu)\nu_E \int_{\frac{t}{2}}^t e^{-\delta_M(t-s)} E(s) ds \quad (2.16)$$

$$\leq e^{-\delta_M t} M(0) + (1 - \nu)\nu_E \int_0^{\frac{t}{2}} e^{-\delta_M(t-s)} B(t) ds + (1 - \nu)\nu_E \int_{\frac{t}{2}}^t e^{-\delta_M(t-s)} C(t) ds \quad (2.17)$$

$$= e^{-\delta_M t} M(0) + (1 - \nu)\nu_E B(t) \int_0^{\frac{t}{2}} e^{-\delta_M(t-s)} ds + (1 - \nu)\nu_E C(t) \int_{\frac{t}{2}}^t e^{-\delta_M(t-s)} ds \quad (2.18)$$

$$= e^{-\delta_M t} M(0) + \frac{(1 - \nu)\nu_E}{\delta_M} (e^{-\delta_M \frac{t}{2}} B(t) + C(t)). \quad (2.19)$$

Now, let us set the subsystem

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \quad (2.20)$$

$$\dot{F} = \nu \nu_E E - \delta_F F, \quad (2.21)$$

in  $[0, +\infty)^2$ .

If  $\mathbf{0} \in \mathbb{R}^2$  is a stable equilibrium in  $[0, +\infty)^2$ , then  $\forall \epsilon > 0, \exists \delta_\epsilon > 0 : E(0)^2 + F(0)^2 < \delta_\epsilon^2 \implies E(t)^2 + F(t)^2 < \epsilon^2 \forall t > 0$ .

If  $E(t)^2 + M(t)^2 + F(t)^2 \leq E(t)^2 + F(t)^2 + (M(0) + \frac{(1-\nu)\nu_E}{\delta_M} A)^2 < \epsilon'^2 \Leftrightarrow E(t)^2 + F(t)^2 < \epsilon'^2 - (M(0) + \frac{(1-\nu)\nu_E}{\delta_M} A)^2$ , we can set  $\epsilon = \sqrt{\epsilon'^2 - (M(0) + \frac{(1-\nu)\nu_E}{\delta_M} A)^2}$  wlog.

Therefore, since  $E(0)^2 + F(0)^2 < \delta_\epsilon^2$  we may take  $\delta'_{\epsilon'} = \sqrt{M(0)^2 + \delta_\epsilon^2}$  so that  $E(0)^2 + M(0)^2 + F(0)^2 < \delta_{\epsilon'}^2 \implies E(t)^2 + M(t)^2 + F(t)^2 < \epsilon'^2 \forall t > 0$ , hence  $\mathbf{0} \in \mathbb{R}^3$  is a stable equilibrium in  $\mathcal{D}$ .

Moreover, if  $\mathbf{0} \in \mathbb{R}^2$  is a global attractor in  $[0, +\infty)^2$ , since by (2.19)  $E(t) \rightarrow 0$  implies that  $M(t) \rightarrow 0$ , we conclude that  $\mathbf{0} \in \mathbb{R}^3$  is a global attractor in  $\mathcal{D}$ .

Thus, we need only assert that  $\mathbf{0} \in \mathbb{R}^2$  is globally asymptotically stable in  $[0, +\infty)^2$  for the subsystem. To do so, we consider  $V : [0, +\infty)^2 \rightarrow \mathbb{R}, y = (E, F)^T \mapsto V(y)$  defined by

$$V(y) = \delta_F E + \beta_E F \quad (2.22)$$

Notice that  $V$  is clearly continuously differentiable and holds for (2.8) - (2.9). It remains to check (2.10):

$$\dot{V} = \delta_F(\beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E) + \beta_E(\nu\nu_E E - \delta_F F) \quad (2.23)$$

$$= -(\delta_F(\nu_E + \delta_E) - \beta_E\nu\nu_E)E - \frac{\delta_F\beta_E}{K}EF \quad (2.24)$$

Since  $\mathcal{R}_0 \leq 1 \implies \delta_F(\nu_E + \delta_E) - \beta_E\nu\nu_E \geq 0$  and  $(E, F)^T \in [0, +\infty)^2$ , we can set the following inequality

$$\dot{V} \leq -(\delta_F(\nu_E + \delta_E) - \beta_E\nu\nu_E)E - \frac{\delta_F\beta_E}{K}EF \leq -\frac{\delta_F\beta_E}{K}EF \leq 0 \quad (2.25)$$

Let  $t \in \mathbb{R} \mapsto y(t) = (E(t), F(t))^T \in [0, \infty)^2$  be a trajectory that verifies  $\dot{V}(y(t)) = 0 \forall t \in \mathbb{R}$ . Then,

$$0 = \dot{V} \leq \frac{\delta_F\beta_E}{K}EF \leq 0 \implies EF = 0 \forall t \in \mathbb{R} \quad (2.26)$$

If for some  $t_0$  we have  $E(t_0) \neq 0$ , by continuity we have  $t \in B_\epsilon(t_0) : E(t) \neq 0$  for some  $\epsilon > 0$ . This gives  $F(t) = 0, \forall t \in B_\epsilon(t_0)$ , which when differentiated w.r.t.  $t$  gives  $\dot{F} = 0 \Leftrightarrow \nu\nu_E E - \delta_F F = 0 \Leftrightarrow E = 0 \forall t \in B_\epsilon(t_0)$ , a contradiction. Thus, there is no  $t_0$  such that  $E(t_0) \neq 0$ , i.e.,  $E(t) = 0 \forall t \in \mathbb{R}$ . Differentiating w.r.t.  $t$ , one gets  $\dot{E} = 0 \Leftrightarrow \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E = 0 \Leftrightarrow F = 0 \forall t \in \mathbb{R}$ .

Combining,  $\dot{V}((E(t), F(t))^T) = 0 \implies (E(t), F(t))^T = \mathbf{0}$ , with (2.25) gives  $\dot{V}((E(t), F(t))) < 0 \forall (E(t), F(t)) \neq \mathbf{0}$ . Hence, we are in the conditions of Theorem 1, which concludes the proof that  $\mathbf{0}$  is globally asymptotically stable for  $\mathcal{R}_0 \leq 1$ .  $\square$

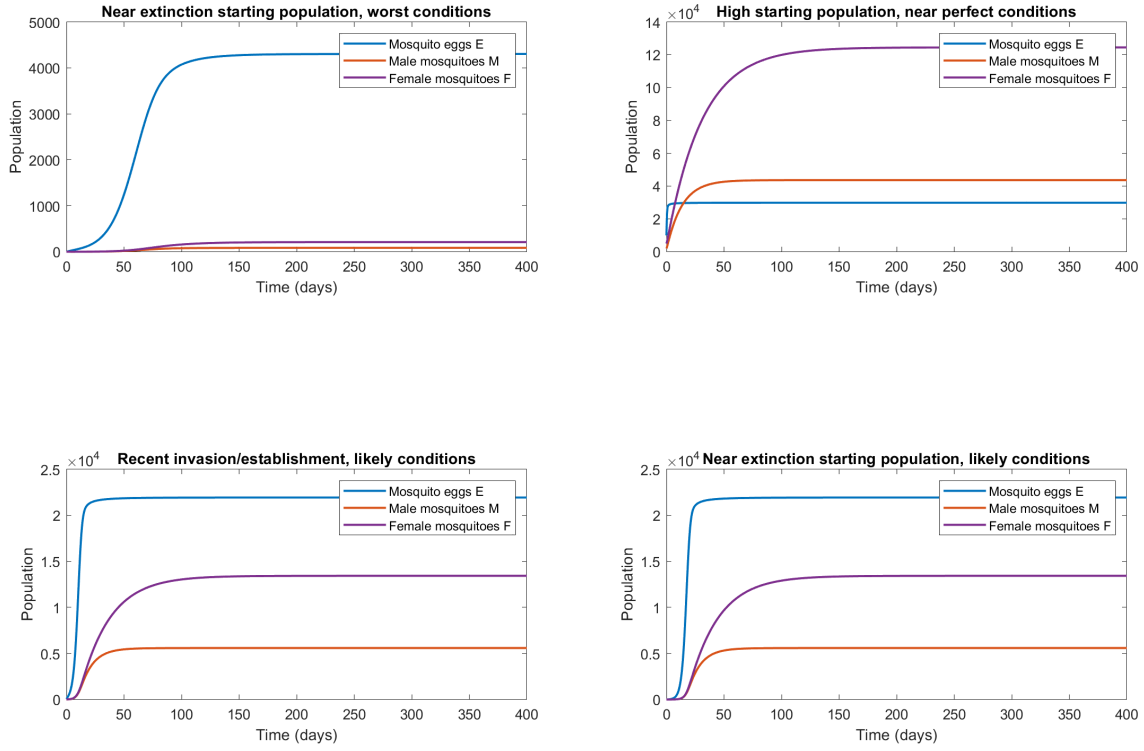
## 2.1.1 Numerical simulations

**Remark 1.** The numerical simulations of this work were obtained through a Fourth Order Runge-Kutta method implementation in Matlab. We opted to use this method since due to the small step choice ( $h = 0.001$ ) stability is assured. Moreover, the problems we consider are not stiff regarding the taken initial conditions. [1]

The parameter intervals for every numerical simulation in all sections of this project were taken from [9]. Figure 4.1 illustrates the evolution of (2.1) - (2.3) for the following scenarios, respectively:

- Near extinction starting population ( $E_0 = 0, M_0 = 1, F_0 = 1$ ), minimal  $\mathcal{R}_0$  (7.15), low  $K$  (5000), low  $\nu$  (0.45);
- High starting population ( $E_0 = 10000, M_0 = 2000, F_0 = 5000$ ), maximal  $\mathcal{R}_0$  (226.65), high  $K$  (30000), high  $\nu$  (0.55);
- Modest starting population ( $E_0 = 50, M_0 = 10, F_0 = 20$ ), moderate  $\mathcal{R}_0$  (76.56), moderate  $K$  (22200), likely  $\nu$  (0.49);

- Near extinction starting population ( $E_0 = 0, M_0 = 1, F_0 = 1$ ), moderate  $\mathcal{R}_0$  (76.56), moderate  $K$  (22200), likely  $\nu$  (0.49);



**Figure 2.1:** Plot of  $E$ ,  $M$  and  $F$  when taking several initial conditions and parameter choices.

As one would expect from Theorem 2, given that for any biologically reasonable parameter choice we get  $\mathcal{R}_0 > 1$ , the population is persistent, i.e., there is a non-zero equilibrium for any initial condition with  $E_0 \neq 0$  or  $F_0 \neq 0$ , which corresponds, of course, to what is observed in nature.

## 2.2 SIT model in mosquito population dynamics

SIT models are based on the premise of rendering fertile females useless from a reproductive standpoint through their mating with sterile males. When these matings happen, females are effectively bound to lay infecund eggs (that do not hatch) for the rest of their lives. This trends the fertile population, and consequently the whole population, to extinction. The SIT model built upon (2.1) - (2.3) is:

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \quad (2.27)$$

$$\dot{M} = (1 - \nu)\nu_E E - \delta_M M, \quad (2.28)$$

$$\dot{F} = \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F, \quad (2.29)$$

$$\dot{M}_s = u - \delta_s M_s, \quad (2.30)$$

in which,  $M_s(t) \geq 0$  corresponds to the sterilized adult male density,  $\delta_s \geq \delta_M$  is the death rate for sterilized adult males,  $u \geq 0$  is the control representing the density of sterile males released at time  $t$ , and  $\gamma_s \in (0, 1]$  takes into account the possibility of females having a preference for fertile males ( $\gamma_s = 1$  meaning that they do not differentiate between fertile and sterilized males). Much like in (2.1) - (2.3),  $F$  only denotes fertilized females. Then, the factor

$$\frac{M}{M + \gamma_s M_s} \quad (2.31)$$

in (2.29) stands for the probability that a female mates with a fertile male.

We now set  $x = (E, M, F, M_s)^T$  and  $\mathcal{D}' = \{x \in \mathbb{R}^4 : x \geq 0\}$ , so that when applying a feedback law  $u : \mathcal{D}' \rightarrow \mathbb{R}_0^+$ , the closed-loop system is

$$\dot{x} = F(x, u(x)) \quad (2.32)$$

where

$$F(x, u) = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ u - \delta_s M_s \end{pmatrix} \quad (2.33)$$

and  $u \in L_{\text{loc}}^\infty(\mathcal{D}')$ .

# 3

## Feedback laws

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In this chapter we address different types of feedback laws for the closed-loop system (2.32) presented in the previous section.

### 3.1 Backstepping feedback law

This feedback law is regarded solely from a computational and practical standpoint, due to the highly technical nature of the theory concerning it, which lies beyond the scope of this work. We refer the reader to [3] for a formal definition along classical theorems for backstepping feedback laws.

Let us define

$$\mathcal{R}(\theta) = \frac{\mathcal{R}_0}{1 + \gamma_s \theta} \quad (3.1)$$

$$\psi = \frac{2\beta_E \nu \nu_E}{\delta_F(1 - \mathcal{R}(\theta))(1 + \gamma_s \theta)} \quad (3.2)$$

$$G : \mathcal{D}' \rightarrow \mathbb{R} \quad (3.3)$$

such that

$$G(x) = \begin{cases} \frac{\gamma_s \psi E(\theta M + M_s)^2}{\alpha(M + \gamma_s M_s)(3\theta M + M_s)} + \frac{((1 - \nu)\nu_E \theta E - \theta \delta_M M)(\theta M + 3M_s)}{3\theta M + M_s} \\ \quad + \delta_s M_s + \frac{\beta_s}{\alpha}(\theta M - M_s), & \text{if } M + M_s \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

where  $\alpha, \beta_s \in (0, +\infty)$ .

Our feedback law  $u$  is then

$$u(x) = \max(0, G(x)) \quad (3.4)$$

Hence, we arrive at the closed-loop system

$$\dot{z} = F(z, u(z)), \quad z = (E, M, F, M_s)^T \in \mathcal{D}' \quad (3.5)$$

where

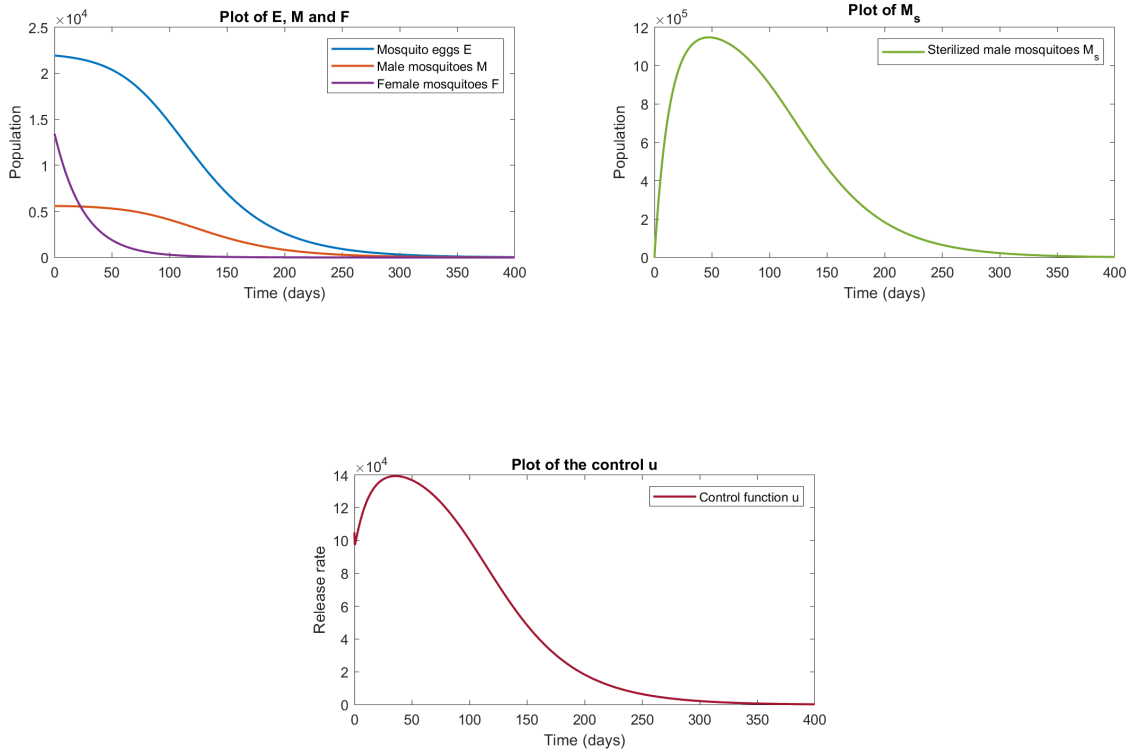
$$F(z, u) = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu \nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ u - \delta_s M_s \end{pmatrix} \quad (3.6)$$

**Theorem 3.** *If  $\mathcal{R}(\theta) < 1$ , then  $0 \in \mathbb{R}^4$  is globally asymptotically stable in  $\mathcal{D}'$  for (2.27) - (2.30) with control (3.4).*

The proof along with all the calculations needed to derive the previous expressions are omitted. Nonetheless, they can be consulted in [5].

### 3.1.1 Numerical simulations

For Figure 3.1 we take  $\beta_E = 10, \gamma_s = 1, \nu_E = 0.05, \delta_E = 0.03, \delta_F = 0.04, \delta_M = 0.1, \delta_s = 0.12, \nu = 0.49$  and the persistence equilibrium for these parameter choices (which may be observed in the last two graphs in Figure 4.1) as the initial condition - ( $E_0 = 21910, M_0 = 5587, F_0 = 13419, M_{s_0} = 0$ ). From (3.1) we obtain  $\theta > 75.56$ , and set  $\theta = 220, \alpha = 13, \beta_s = 1$ .



**Figure 3.1:** Plot of  $E, M$  and  $F, M_s$  and the control  $u$ .

The amount of sterile males released in the first 360 days is

$$\int_0^{360} u(t) dt \approx 18\,722\,253 \text{ mosquitoes} \quad (3.7)$$

These results appear to coincide with the ones presented in [5].

**Remark 2.** Numerical integration results were obtained through the trapezoidal rule Matlab command,



since due to the small step, higher order Gauss quadratures are not required, for they produce very similar values.

### 3.2 Total male dependent feedback law

Several kinds of traps are able to track and/or distinguish genders of these insects. Such devices make a feedback law depending only on the total male population ( $M + M_s$ ) a viable option. Additionally, the traps may be placed in various locations in the field to better their assessment. For this reason, we present a feedback linearly depending on total males.

Consider

$$\dot{z} = F(z, u(z)), \quad z = (E, M, F, M_s)^T \in \mathcal{D}' \quad (3.8)$$

where

$$u(z) = k(M + M_s), \quad (3.9)$$

$$F(z, u) = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ u - \delta_s M_s \end{pmatrix}, \quad k \in \mathbb{R}. \quad (3.10)$$

- Let  $k \geq \delta_s$ , then:

$$\dot{M}_s = kM + kM_s - \delta_s M_s \geq 0, \text{ therefore the sterilized male population never decreases;}$$

- Let  $k < 0$ , then:

$$\dot{M}_s = kM + kM_s - \delta_s M_s \leq 0, \text{ therefore the sterilized male population never increases;}$$

Hence, for the control to show usefulness,

$$k \in [0, \delta_s). \quad (3.11)$$

We know that the probability that a given female is fertilized is (2.31). Thus,  $\mathcal{R}_1(k)$ , the amount of new fertile females per each fertile female that dies under the SIT variant, must be  $\mathcal{R}_0 \cdot \frac{M}{M + \gamma_s M_s}$ .

Since  $M$  and  $M_s$  are time dependent, we must first assume equilibrium:

$$\dot{M}_s = 0 \implies k(M + M_s) = \delta_s M_s \implies M_s = \frac{k}{\delta_s - k} M \quad (3.12)$$

Then,

$$\mathcal{R}_1(k) := \mathcal{R}_0 \cdot \frac{M}{M + \gamma_s M_s} = \mathcal{R}_0 \cdot \frac{M}{M + \gamma_s \frac{k}{\delta_s - k} M} = \frac{(\delta_s - k)\beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E)(\delta_s - (1 - \gamma_s)k)} \quad (3.13)$$

Equilibria of (3.8) may be obtained by solving the system

$$\begin{cases} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E = 0 \\ (1 - \nu)\nu_E E - \delta_M M = 0 \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F = 0 \\ kM - (\delta_s - k)M_s = 0 \end{cases} \quad (3.14)$$

which gives both the extinction equilibrium,  $(E, M, F, M_s)^T = (0, 0, 0, 0)^T$ , and a biologically meaningless result ( $E < 0$ ) when  $\mathcal{R}_1(k) < 1$ . Hence, we shall not consider the second equilibria.

Now, from

$$\begin{cases} \mathcal{R}_0 > 1 \\ \mathcal{R}_1(k) = \mathcal{R}_0 \cdot \frac{\delta_s - k}{\delta_s - (1 - \gamma_s)k} < 1 \\ 0 \leq k < \delta_s \end{cases} \quad (3.15)$$

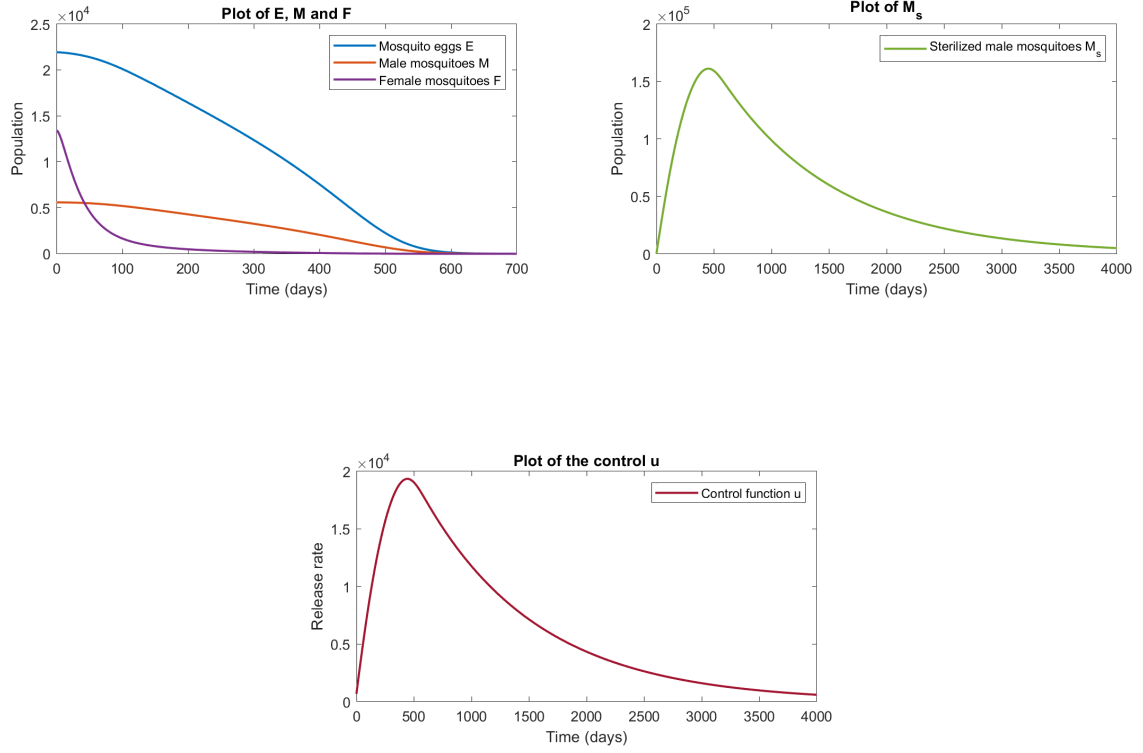
it follows that

$$\begin{aligned} \mathcal{R}_0 \delta_s - \mathcal{R}_0 k < \delta_s - (1 - \gamma_s)k &\implies \delta_s(\mathcal{R}_0 - 1) < k(\mathcal{R}_0 - (1 - \gamma_s)) \implies \\ &\frac{\beta_E \nu \nu_E - (\nu_E + \delta_E)\delta_F}{\beta_E \nu \nu_E - (1 - \gamma_s)(\nu_E + \delta_E)\delta_F} \delta_s < k < \delta_s. \end{aligned} \quad (3.16)$$

To conclude this section, note that it has not been proved that  $\mathbf{0}$  is globally asymptotically stable for (3.8). Nonetheless, there is robust numerical evidence for such a claim, which is presented in the following section.

### 3.2.1 Numerical simulations

For Figure 3.2 we take the same parameters and initial condition from the last section. Computing (3.16) we get  $0.11843 < k < 0.12$ , and choose  $k = 0.119$ .



**Figure 3.2:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$  and the control  $u$ .

The amount of sterile males released in the first 700 days is

$$\int_0^{700} u(t) dt \approx 10\,354\,540 \text{ mosquitoes} \quad (3.17)$$

The latter result does not match with the one presented in [5], although it seems correct upon closer inspection.

### 3.3 Wild male dependent feedback law

In some scenarios, sterile males may be picked out by the presence of a marker applied before their release. Furthermore, PCR analysis of the captured mosquitoes is currently used due to genetic bar-coding. For these reasons, we consider in this section a feedback that depends only on  $M$ .

Let

$$\dot{z} = F(z, u(z)), \quad z = (E, M, F, M_s)^T \in \mathcal{D}' \quad (3.18)$$

with

$$u(z) = \lambda M, \quad (3.19)$$

$$F(z, u) = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ \lambda M - \delta_s M_s \end{pmatrix}, \quad \lambda \in \mathbb{R}. \quad (3.20)$$

Now, to obtain the new offspring number,  $\mathcal{R}_2(\lambda)$ , we set

$$\dot{M}_s = 0 \implies \lambda M = \delta_s M_s \implies M_s = \frac{\lambda}{\delta_s} M \quad (3.21)$$

Then,

$$\mathcal{R}_2(k) := \mathcal{R}_0 \cdot \frac{M}{M + \gamma_s M_s} = \mathcal{R}_0 \cdot \frac{M}{M + \gamma_s \frac{\lambda}{\delta_s} M} = \frac{\delta_s \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\delta_s + \gamma_s \lambda)} \quad (3.22)$$

Assuming

$$\mathcal{R}_2(k) < 1 \quad (3.23)$$

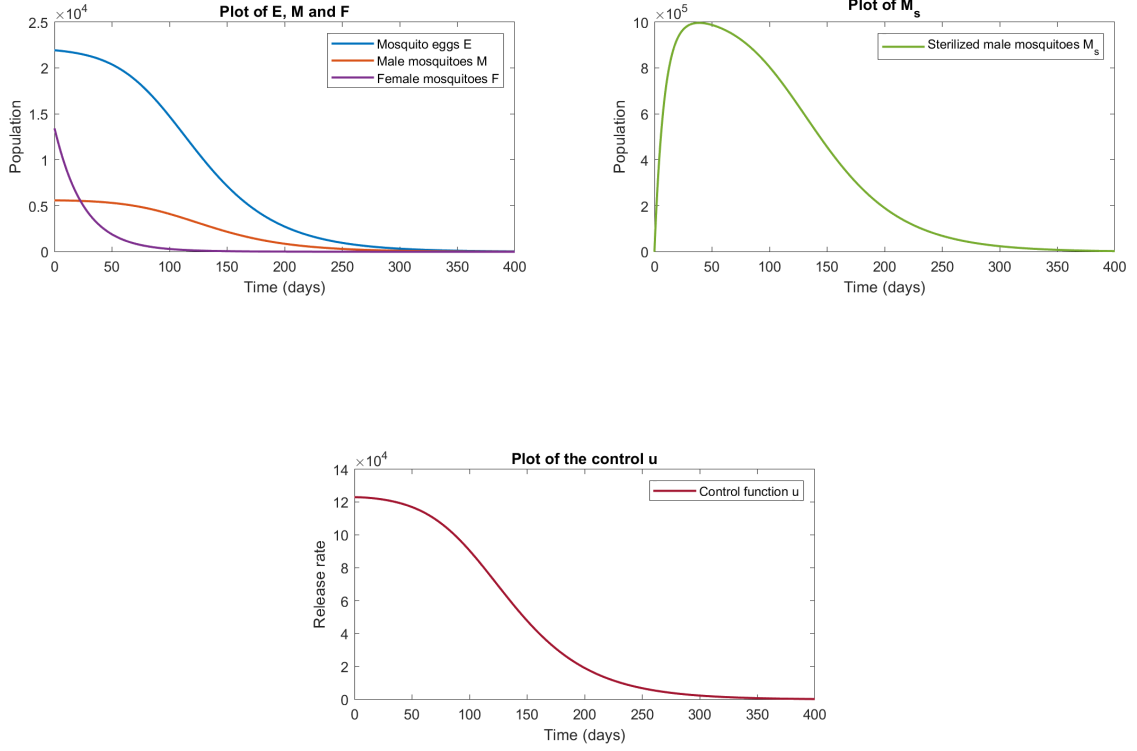
we get

$$\frac{\delta_s \beta_E \nu \nu_E}{\delta_F (\nu_E + \delta_E) (\delta_s + \gamma_s \lambda)} < 1 \implies \lambda > \frac{(\beta_E \nu \nu_E - (\nu_E + \delta_E) \delta_F) \delta_s}{\gamma_s (\nu_E + \delta_E) \delta_F} \quad (3.24)$$

Similarly to the previous section, it has not been proved that  $\mathbf{0}$  is globally asymptotically stable for (3.18), although numerical evidence suggests this is in fact true.

### 3.3.1 Numerical simulations

For Figure 3.3 we take the same parameters and initial condition from the previous sections. Computing (3.24) we get  $\lambda > 9.06$ , and choose  $\lambda = 22$ .



**Figure 3.3:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$  and the control  $u$ .

The amount of sterile males released in the first 400 days is

$$\int_0^{400} u(t) dt \approx 17\,311\,090 \text{ mosquitoes} \quad (3.25)$$

which is close to the result presented in the original article.

One issue that arises from this feedback (and the previous one, for that matter) is the disproportionately high amount of sterile males that must be released in order to extinguish a population at persistence equilibrium. In [5],  $\lambda$  may have been set to 22 to drive the population down rather quickly, since it might not be realistic to dispose sterilized males at a rate of twenty two times the amount of wild males.

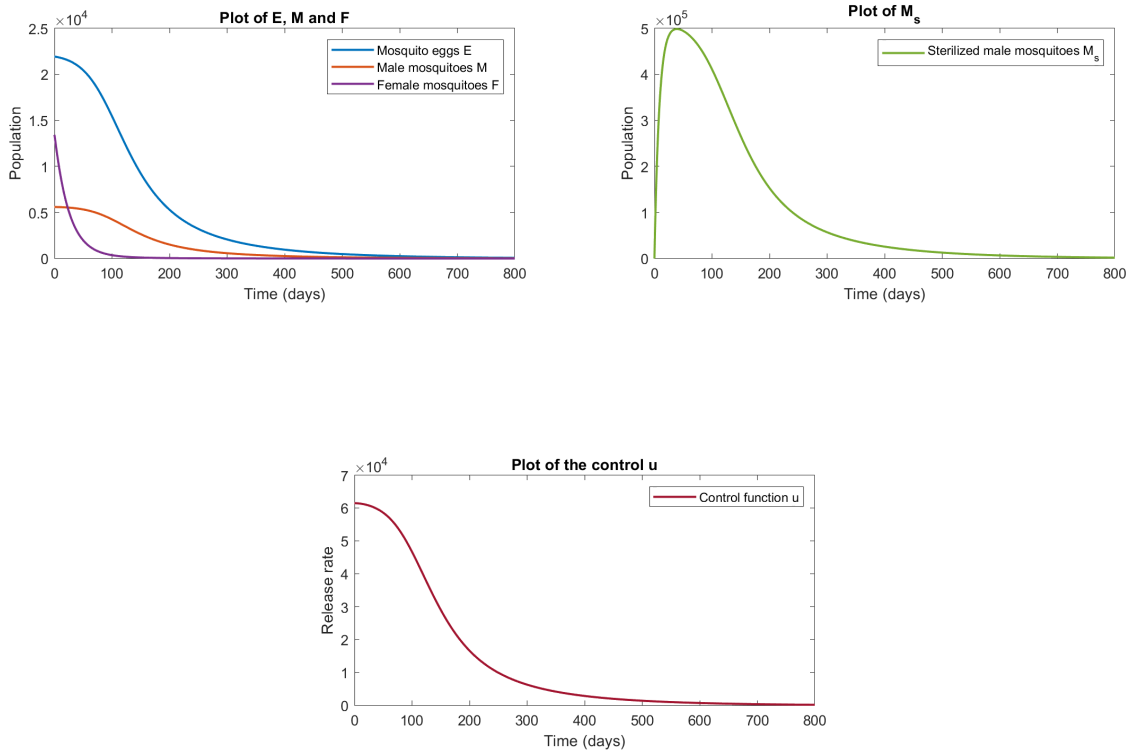
Table 3.1 shows the amount of time required to drive  $E$  down to  $K/100$ , and the total amount of

sterile males released during that time, for  $\lambda = 9.06, 10, \dots, 22$ .

$\lambda$	Intervention time (days)	Total $M_s$ released
9.06	3185	12 876 440
10	874	10 442 640
11	621	10 519 067
12	517	10 916 355
13	459	11 432 280
14	422	12 007 560
15	396	12 617 424
16	377	13 249 422
17	362	13 896 536
18	350	14 554 470
19	340	15 220 439
20	332	15 892 547
21	325	16 569 460
22	319	17 250 207

**Table 3.1:** Intervention time and control cost for several  $\lambda$ .

For  $\lambda > 9.06$ , it appears that the relationship with the total  $M_s$  released is monotonic. Hence, by halving  $\lambda$  to 11, one still achieves extinction while almost halving the total amount of released sterile males. For Figure 3.4 we keep the same parameter and initial conditions from Figure 3.3, except for  $\lambda$  which is now set to 11.



**Figure 3.4:** Plot of  $E, M$  and  $F, M_s$  and the control  $u$  for  $\lambda = 11$ .

### 3.4 Aquatic phase dependent feedback law

An inexpensive and low-labor way to estimate the aquatic phase is through ovitraps – a container filled with water (sometimes with a paddle or strip inside) that mimics a breeding site. Females lay eggs on the strip or container wall above the waterline. This, along several other methods, turns a feedback law linearly dependent on  $E$  a feasible alternative. Thus, consider the closed-loop system

$$\dot{z} = F(z, u(z)), \quad z = (E, M, F, M_s)^T \in \mathcal{D}' \quad (3.26)$$

where now we set

$$u(z) = \omega E, \quad (3.27)$$

$$F(z, u) = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_s M_s} - \delta_F F \\ \omega E - \delta_s M_s \end{pmatrix}, \quad \omega \in \mathbb{R}. \quad (3.28)$$

In order to find the new offspring number,  $\mathcal{R}_3(\omega)$ , we perform similar calculations to those in the last two sections

$$\dot{M}_s = 0 \implies \omega E = \delta_s M_s \implies M_s = \frac{\omega}{\delta_s} E \quad (3.29)$$

$$\dot{M} = 0 \implies (1 - \nu)\nu_E E = \delta_M M \implies M = \frac{(1 - \nu)\nu_E E}{\delta_M} \quad (3.30)$$

so that,

$$\frac{M}{M + \gamma_s M_s} = \frac{\frac{(1 - \nu)\nu_E E}{\delta_M}}{\frac{(1 - \nu)\nu_E E}{\delta_M} + \gamma_s \frac{\omega}{\delta_s} E} = \frac{\delta_s(1 - \nu)\nu_E}{\delta_s(1 - \nu)\nu_E + \delta_M \gamma_s \omega} \quad (3.31)$$

Assuming

$$\mathcal{R}_3(\omega) = \mathcal{R}_0 \frac{\delta_s(1 - \nu)\nu_E}{\delta_s(1 - \nu)\nu_E + \delta_M \gamma_s \omega} < 1 \quad (3.32)$$

we then get

$$\omega > \frac{(\beta_E \nu \nu_E - (\nu_E + \delta_E)\delta_F)\delta_s}{\gamma_s(\nu_E + \delta_E)\delta_F} \quad (3.33)$$

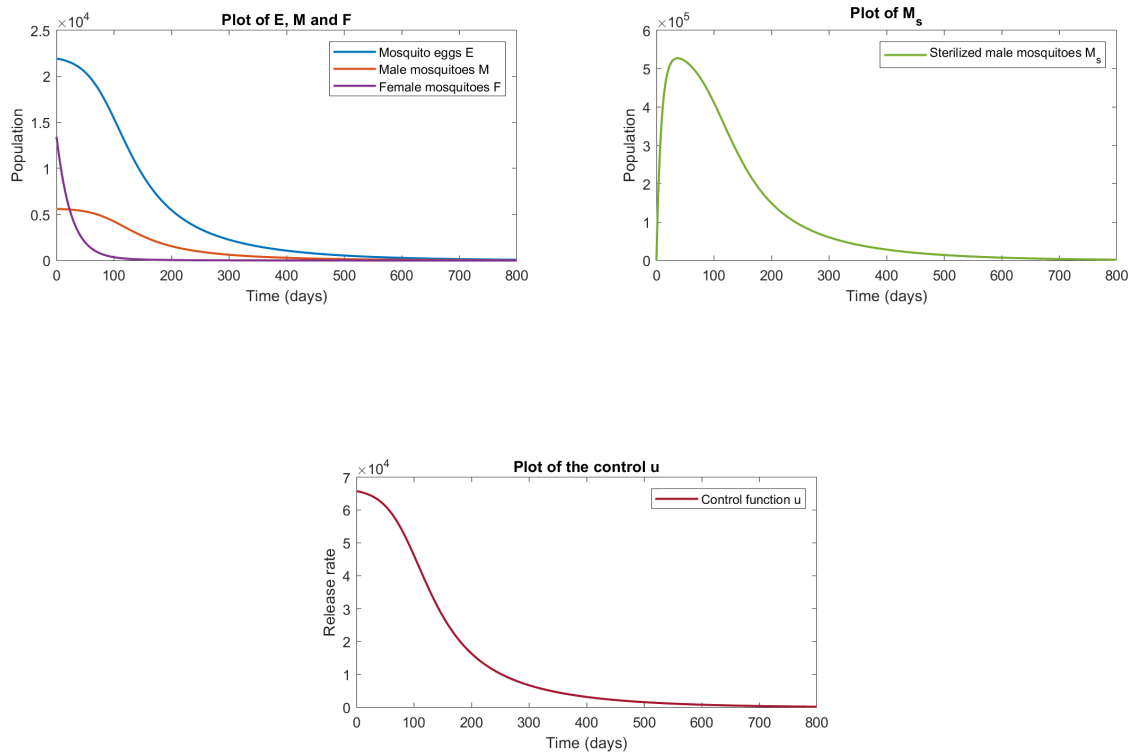
#### 3.4.1 Numerical simulations

Let the initial condition be the persistence equilibrium. Taking the same parameters from the previous sections, (3.33) gives  $\omega > 2.31$ . The amount of time required to drive  $E$  down to  $K/100$ , and the total amount of sterile males released during that time, for  $\omega = 2.31, 3, 3.5, \dots, 6$ , is shown in Table 3.2.

$\omega$	Intervention time (days)	Total $M_s$ released
2.31	4224	14 396 720
3	638	10 765 054
3.5	497	11 571 111
4	431	12 603 669
4.5	393	13 728 613
5	368	14 901 147
5.5	349	16 101 637
6	336	17 320 016

**Table 3.2:** Intervention time and control cost for several  $\omega$ .

Similarly to the what was observed with the previous control, the relationship between  $\omega > 2.31$  and the total  $M_s$  released seems monotonic. Thus, opting for  $\omega = 3$  to reduce costs, produces the following plots:



**Figure 3.5:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$  and the control  $u$ .



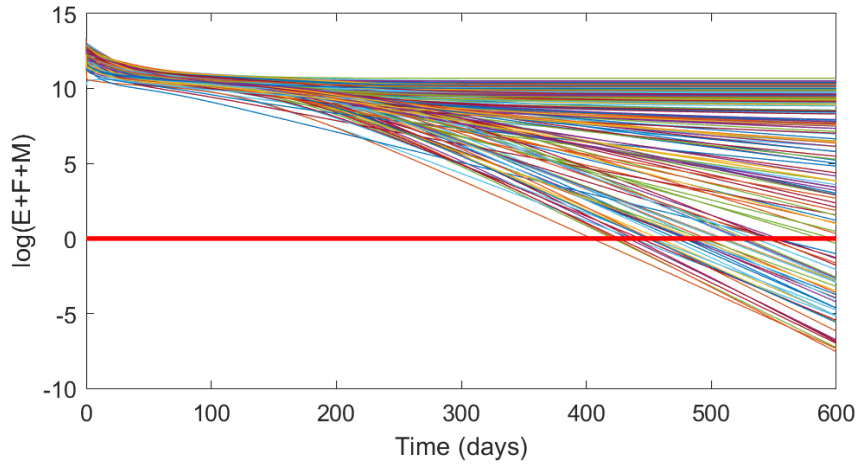
## 3.5 Robustness Tests

### 3.5.1 Backstepping feedback law

For the control (3.4) we fix the same parameters as in 3.1.1. Then, for the true dynamics of (2.32) we consider 200 random perturbations of the system parameters such that

$$\begin{aligned}
 \hat{\beta}_E &\sim \mathcal{U}(7.46, 14.85), \\
 \hat{\nu}_E &\sim \mathcal{U}(0.005, 0.25), \\
 \hat{\delta}_E &\sim \mathcal{U}(0.023, 0.046), \\
 \hat{\delta}_F &\sim \mathcal{U}(0.033, 0.046), \\
 \hat{\delta}_M &\sim \mathcal{U}(0.077, 0.139), \\
 \hat{\delta}_s &\sim \mathcal{U}(0.077, 0.139), \\
 \hat{\gamma}_s &\sim \mathcal{U}(0.5, 1).
 \end{aligned} \tag{3.34}$$

Taking 200 random initial conditions uniformly distributed in  $[0, 10K]^4$  gives

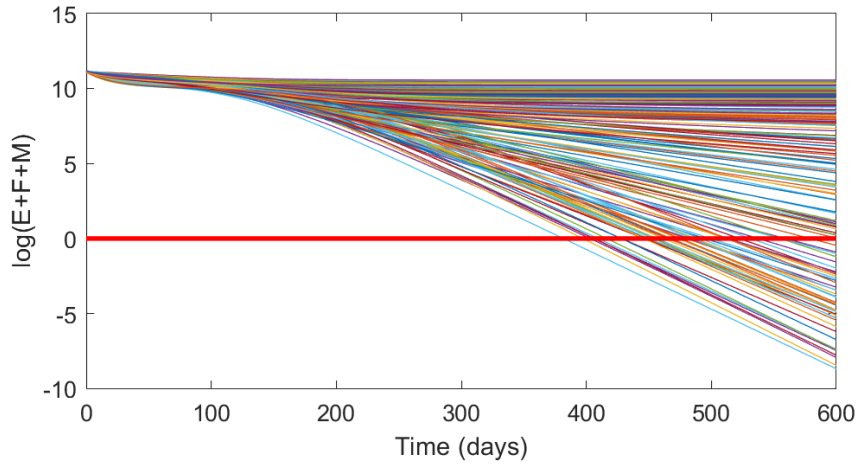


**Figure 3.6:** Evolution of  $\log(E + F + M)$  up to  $t = 600$  days when  $x_0$  is uniformly distributed in  $[0, 10K]^4$ .

which clearly does not match with what is presented in [5]. Seeking results closer to those, we tried generating values closer to  $K$  through the empirical rule by setting  $\mu = K = 22200$  and finding  $\sigma$  such

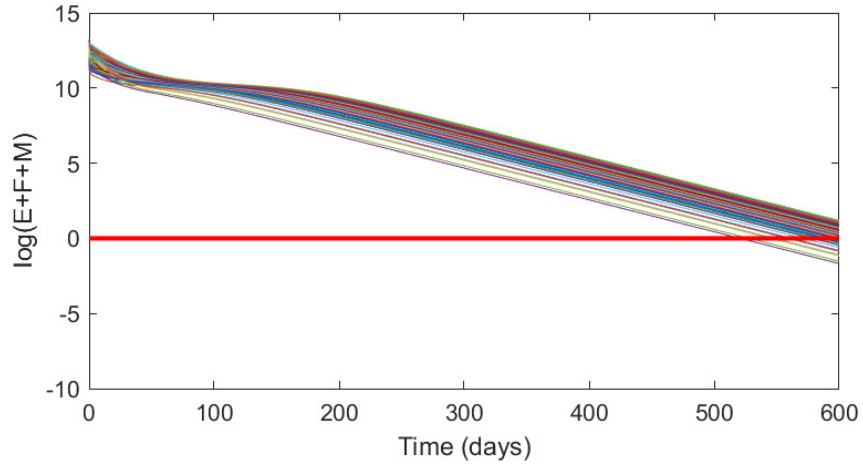
that 99.7% of the values from  $\mathcal{N}(\mu, \sigma)$  deviate only 10% from  $K$ .

$$K - 3\sigma = \mu - 0.1K \implies K - 3\sigma = K - 0.1K \implies \sigma = \frac{0.1K}{3} = 740 \quad (3.35)$$



**Figure 3.7:** Evolution of  $\log(E + F + M)$  up to  $t = 600$  days when  $x_0$  comes from  $\mathcal{N}(K, \frac{0.1K}{3})$ .

Even then, the feedback (3.4) does not seem robust: it doesn't stabilize the dynamics at extinction equilibrium if its parameters differ significantly from those used for simulating the mosquito dynamics. For this reason, we speculate that this protocol, originally presented in [5], was not correctly carried forward. It seems that there was only a perturbation of the initial conditions, not the system parameters, which was likely overlooked. Further testing that hypothesis granted the following result, which is much closer to the one in the original article.



**Figure 3.8:** Evolution of  $\log(E + F + M)$  up to  $t = 600$  days when  $x_0$  is uniformly distributed in  $[0, 10K]^4$ .

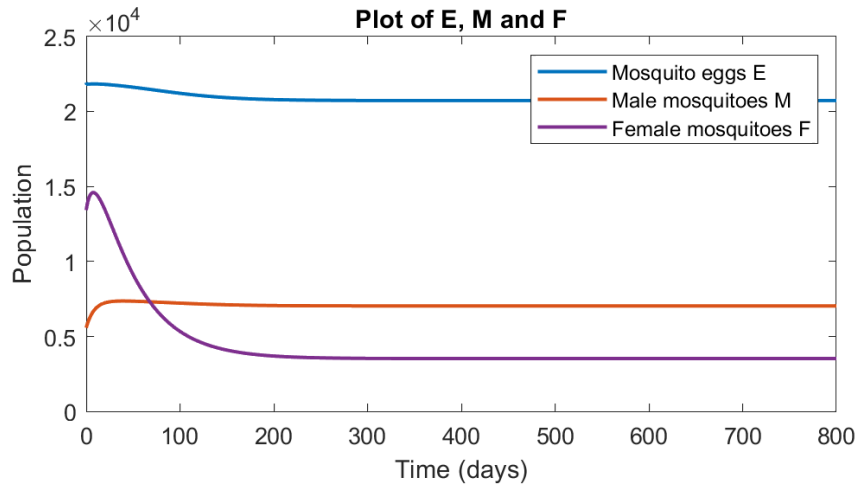
Additionally, another disadvantage of this method is the dependence on all state variables, which may not be feasible from a realistic standpoint in the field.

### 3.5.2 Total male dependent feedback law

Let the initial condition remain the persistence equilibrium. Carrying out the following parameter variation while still taking  $k \in (0.11843, 0.12)$  (opting again for 0.119)

$$\begin{aligned}
 \beta_E &= 11, \\
 \nu_E &= 0.08, \\
 \delta_E &= 0.046, \\
 \delta_F &= 0.033, \\
 \delta_M &= 0.12, \\
 \delta_s &= 0.139.
 \end{aligned} \tag{3.36}$$

shows that feedback (3.9) is not robust for rather small perturbations of the system parameters.

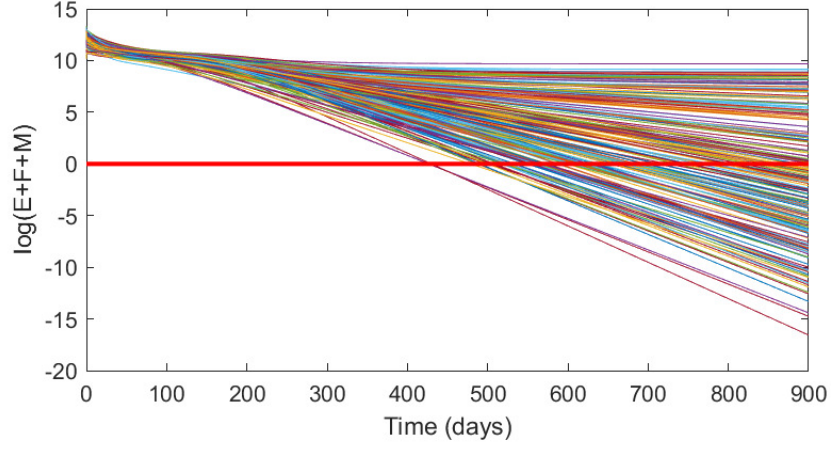


**Figure 3.9:** Plot of  $E$ ,  $M$  and  $F$ .

This is to be expected, since now (3.16) gives  $0.13765 < k < 0.139$  which clearly does not hold anymore. The narrow interval allowed for  $k$  is an important drawback from this feedback.

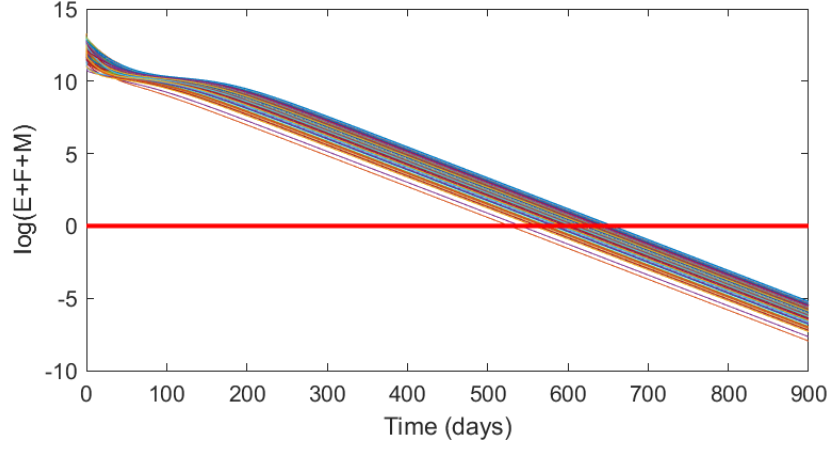
### 3.5.3 Wild males dependent feedback law

Taking  $\lambda = 22$  and following the same protocol as in 3.5.1, one gets



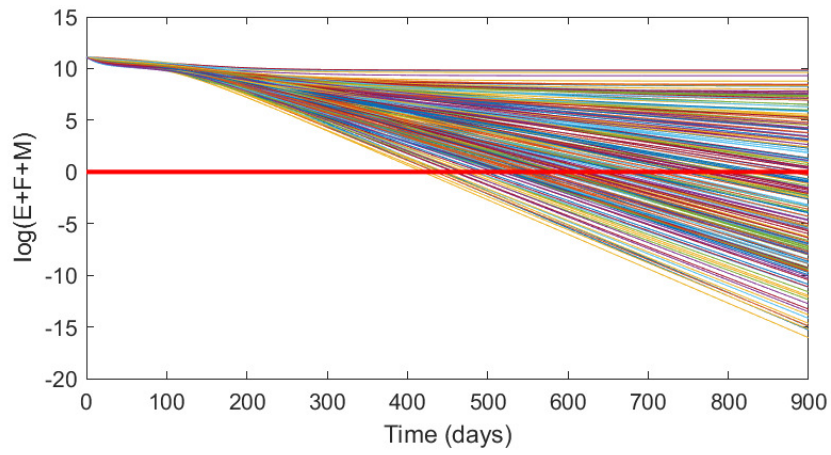
**Figure 3.10:** Evolution of  $\log(E + F + M)$  up to  $t = 900$  days when  $x_0$  is uniformly distributed in  $[0, 10K]^4$ .

which again does not match with the original article. This may be once more due to what was mentioned in 3.5.1 regarding the supposed perturbation of the system parameters. For instance, if only the initial conditions are perturbed, results are not far from those in [5]:



**Figure 3.11:** Evolution of  $\log(E + F + M)$  up to  $t = 900$  days when  $x_0$  is uniformly distributed in  $[0, 10K]^4$ .

When the 200 initial conditions come from  $\mathcal{N}(K, \frac{0.1K}{3})$ , one gets

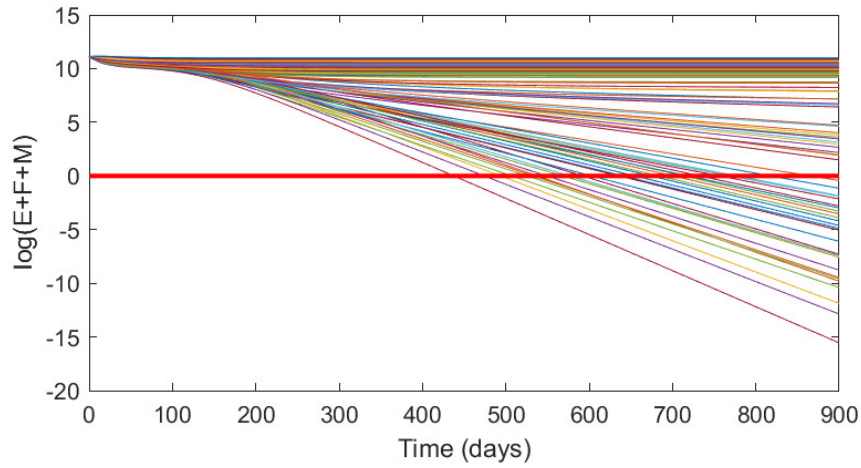


**Figure 3.12:** Evolution of  $\log(E + F + M)$  up to  $t = 900$  days when  $x_0$  comes from  $\mathcal{N}(K, \frac{0.1K}{3})$ .

Since  $\lambda = 22$  was chosen due to (3.24) giving  $\lambda > 9.06$  for the fixed parameters, it is not surprising that the feedback does not stabilize the dynamics at 0 if the mosquito dynamics parameters demand  $\lambda > 22$ . Therefore, we come to the fact that feedback (3.19) is also not robust against variations of the mosquito dynamics parameters.

### 3.5.4 Aquatic phase dependent feedback law

On account of the extinction condition for feedback (3.27) giving  $\omega > 2.31$ , inferring from the previous linearly dependent feedback laws it is clear that this feedback should not be robust against variations of the mosquito dynamics parameters. This is shown in Figure 3.13, produced by following the same protocol as in 3.5.1 when fixing  $\omega = 3$  and taking 200  $x_0$  from  $\mathcal{N}(K, \frac{0.1K}{3})$ .



**Figure 3.13:** Evolution of  $\log(E + F + M)$  up to  $t = 900$  days when  $x_0$  comes from  $\mathcal{N}(K, \frac{0.1K}{3})$ .

# 4

## Models for mosquito population dynamics with Allee Effect

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## 4.1 Model for wild mosquito population dynamics with Allee Effect

If one seeks to better entail the mosquito population dynamics natural behavior when the population density is rather low, an adequate procedure is to introduce an *Allee Effect*. This may be achieved through the inclusion of a Michaelis-Menter type factor, such as

$$\frac{\eta M}{1 + \eta M} \quad (4.1)$$

in the aquatic phase equation, where  $\eta$  is a search efficiency parameter. This considers the fact that in the aforementioned conditions, females may not easily find a fertile male with whom to mate, in order to lay fecund eggs. The constant male density for which the mating probability of a female would be 0.5 is  $\frac{1}{\eta}$ . We now present the resulting system:

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1 + \eta M} - (\nu_E + \delta_E) E, \quad (4.2)$$

$$\dot{M} = (1 - \nu) \nu_E E - \delta_M M, \quad (4.3)$$

$$\dot{F} = \nu \nu_E E - \delta_F F, \quad (4.4)$$

An important nuance arises from this version –  $F$  now represents the whole of females, not simply the ones that are fertilized. Similarly to the previous model, the eggs encapsulated by  $E$  are strictly fecund.

Solving (4.2) - (4.4) for equilibria gives  $\mathbf{0} \in \mathbb{R}^3$  and a non-trivial equilibrium. When plugged back into the system, the latter results in a quadratic equation in  $E$ . One of the roots of it's discriminant is

$$r = 1 + \frac{2\delta_M}{\eta K (1 - \nu) \nu_E} \left( 1 + \sqrt{1 + \frac{\eta K (1 - \nu) \nu_E}{\delta_M}} \right). \quad (4.5)$$

**Theorem 4.** *If  $\mathcal{R}_0 < r$ , then  $\mathbf{0} \in \mathbb{R}^3$  is a globally asymptotically stable equilibrium in  $\mathcal{D}$ .*

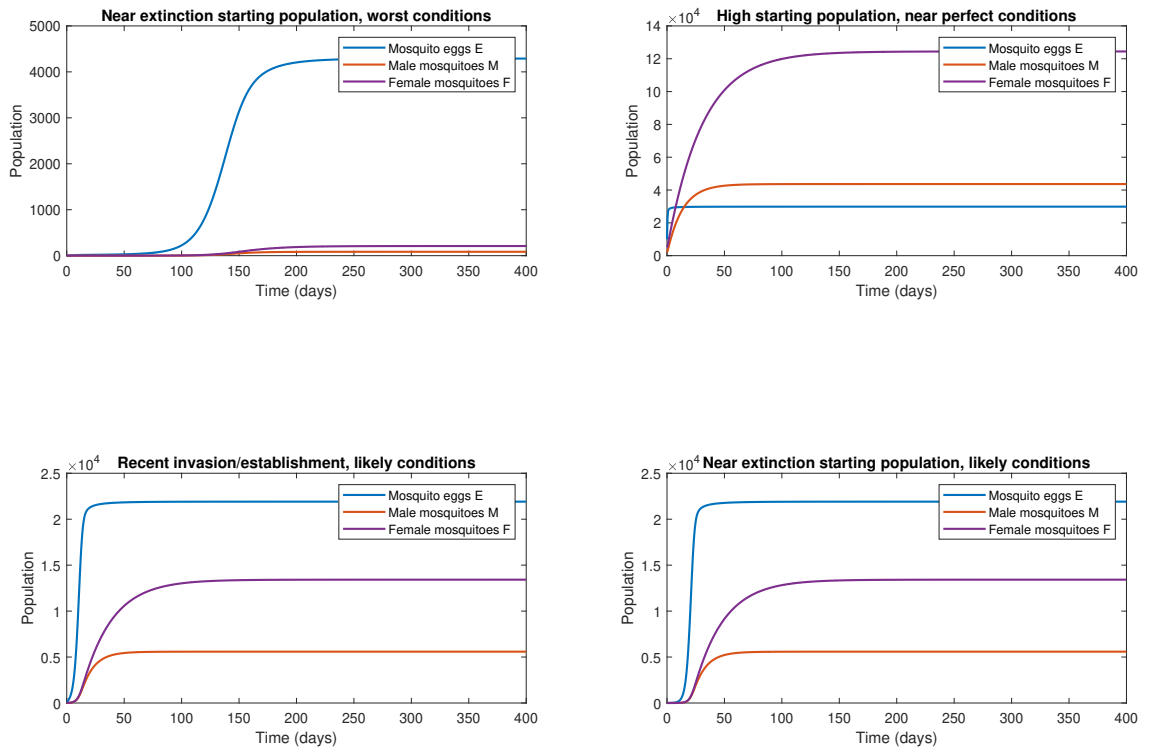
The proof is presented in [4].

### 4.1.1 Numerical simulations

Figure 4.1 illustrates the evolution of (4.2) - (4.4) for the following scenarios, respectively:

- Near extinction starting population ( $E_0 = 0, M_0 = 1, F_0 = 1$ ), minimal  $\mathcal{R}_0$  (7.15), low  $K$  (5000), low  $\nu$  (0.45);

- High starting population ( $E_0 = 10000, M_0 = 2000, F_0 = 5000$ ), maximal  $\mathcal{R}_0$  (226.65), high  $K$  (30000), high  $\nu$  (0.55);
- Modest starting population ( $E_0 = 50, M_0 = 10, F_0 = 20$ ), moderate  $\mathcal{R}_0$  (76.56), moderate  $K$  (22200), likely  $\nu$  (0.49);
- Near extinction starting population ( $E_0 = 0, M_0 = 1, F_0 = 1$ ), moderate  $\mathcal{R}_0$  (76.56), moderate  $K$  (22200), likely  $\nu$  (0.49);



**Figure 4.1:** Plot of  $E$ ,  $M$  and  $F$  when taking several initial conditions and parameter choices.

## 4.2 SIT model in mosquito population dynamics with Allee Effect

By modifying model (2.27) - (2.30) accordingly, one gets

$$\dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1 + \eta(M + \gamma_s M_s)} - (\nu_E + \delta_E) E, \quad (4.6)$$

$$\dot{M} = (1 - \nu) \nu_E E - \delta_M M, \quad (4.7)$$

$$\dot{F} = \nu\nu_E E - \delta_F F, \quad (4.8)$$

$$\dot{M}_s = u - \delta_s M_s. \quad (4.9)$$

We again set  $x = (E, M, F, M_s)^T$  and  $\mathcal{D}' = \{x \in \mathbb{R}^4 : x \geq 0\}$ , so that when applying a feedback law  $u : \mathcal{D}' \rightarrow \mathbb{R}_0^+$ , the closed-loop system is

$$\dot{x} = F(x, u(x)) \quad (4.10)$$

where

$$F(x, u) = \begin{pmatrix} \beta_E F(1 - \frac{E}{K}) \frac{\eta M}{1 + \eta(M + \gamma_s M_s)} - (\nu_E + \delta_E)E \\ (1 - \nu)\nu_E E - \delta_M M \\ \nu\nu_E E - \delta_F F \\ u - \delta_s M_s \end{pmatrix} \quad (4.11)$$

and  $u \in L_{\text{loc}}^\infty(\mathcal{D}')$ .

# 5

## Feedback laws with Allee Effect

### Contents

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## 5.1 Feedback law depending only on the total number of male mosquitoes

In this section we analyze the possible application of feedback (3.9) for (4.10). Observe that (3.11) still holds. To find a lower bound for  $k$  such that the feedback drives the population to extinction, we follow this approach:

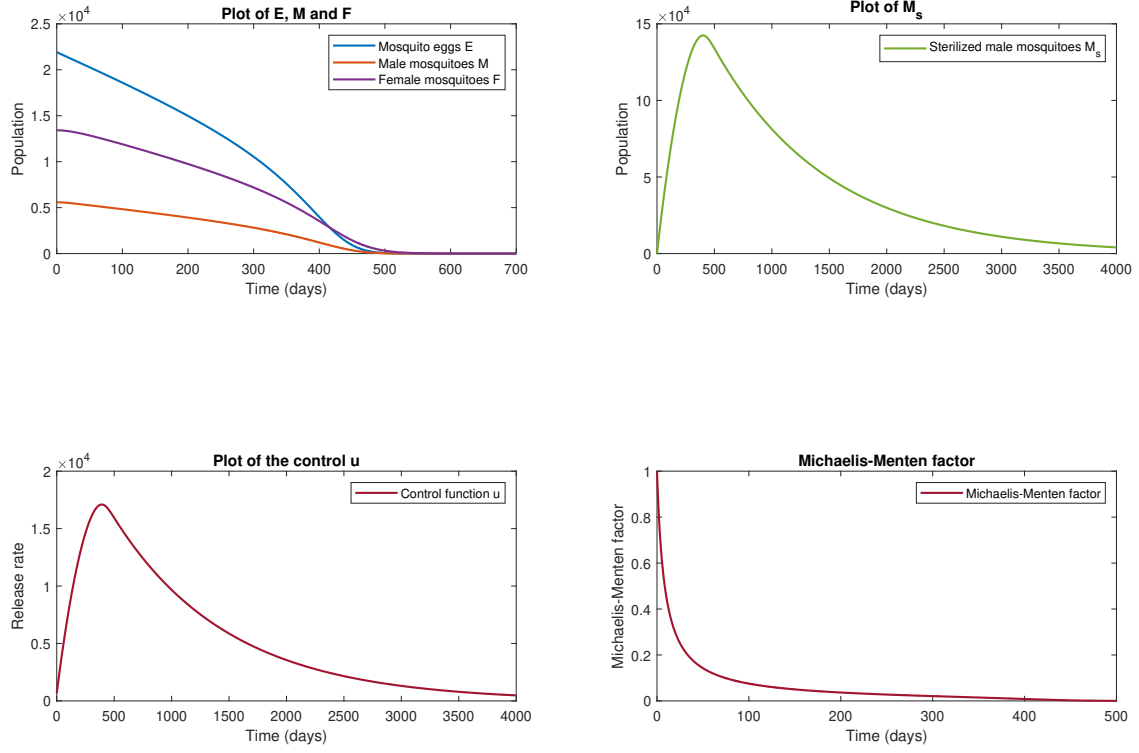
- Let  $\mathcal{R}_1(k) := \mathcal{R}_0 \frac{\eta M}{1 + \eta(M + \gamma_s M_s)}$ .
- At equilibria,  $\dot{M}_s = 0 \implies k(M + M_s) = \delta_s M_s \implies M_s = \frac{k}{\delta_s - k} M$ . Setting  $\mathcal{R}_1(k) = r$ , i.e., to the threshold for extinction, implies  $M = \frac{r}{\eta R(1 - \frac{1}{R} r(1 + \frac{\gamma_s k}{\delta_s - k}))}$ .
- For an equilibrium in  $\mathcal{D}'$ , we must have  $M > 0 \implies \frac{1}{R} r(1 + \frac{\gamma_s k}{\delta_s - k}) < 1$ . Hence, we seek  $\frac{1}{R} r(1 + \frac{\gamma_s k}{\delta_s - k}) \geq 1$ , which gives

$$k \geq \frac{\delta_s(\mathcal{R}_0 - r)}{\mathcal{R}_0 + r(\gamma_s - 1)} \quad (5.1)$$

This coincides with (3.16) when  $r = 1$ .

### 5.1.1 Numerical simulations

For Figure 5.5 we take the same parameters and initial condition as in section 3.2.1, then set  $\eta = 0.7 \text{ km}^2$ . This gives  $r = 1.03227$  and  $k \geq 0.11838$ . With  $k = 0.119$ ,



**Figure 5.1:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$ , the control  $u$  and the Michaelis-Menten factor from (4.6).

The amount of sterile males released in the first 700 days is

$$\int_0^{700} u(t) dt \approx 9\,146\,233 \text{ mosquitoes} \quad (5.2)$$

## 5.2 Feedback law depending only on wild male mosquitoes

In this section we analyze the possible application of feedback (3.19) for (4.10). To find a lower bound for  $\lambda$  such that the feedback drives the population to extinction, we follow this approach:

- Let  $\mathcal{R}_2(\lambda) := \mathcal{R}_0 \frac{\eta M}{1 + \eta(M + \gamma_s M_s)}$ .
- At equilibria,  $\dot{M}_s = 0 \implies \lambda M = \delta_s M_s \implies M_s = \frac{\lambda}{\delta_s} M$ . Setting  $\mathcal{R}_2(\lambda) = r$ , i.e., to the threshold for extinction, implies  $M = \frac{r}{\eta R(1 - \frac{1}{R} r(1 + \frac{\gamma_s \lambda}{\delta_s}))}$ .
- For an equilibrium in  $\mathcal{D}'$ , we must have  $M > 0 \implies \frac{1}{R} r(1 + \frac{\gamma_s \lambda}{\delta_s}) < 1$ . Hence, we seek  $\frac{1}{R} r(1 + \frac{\gamma_s \lambda}{\delta_s}) \geq 1$ , which gives

$$\lambda \geq \frac{\delta_s(\mathcal{R}_0 - r)}{r\gamma_s} \quad (5.3)$$

This coincides with (3.24) when  $r = 1$ .

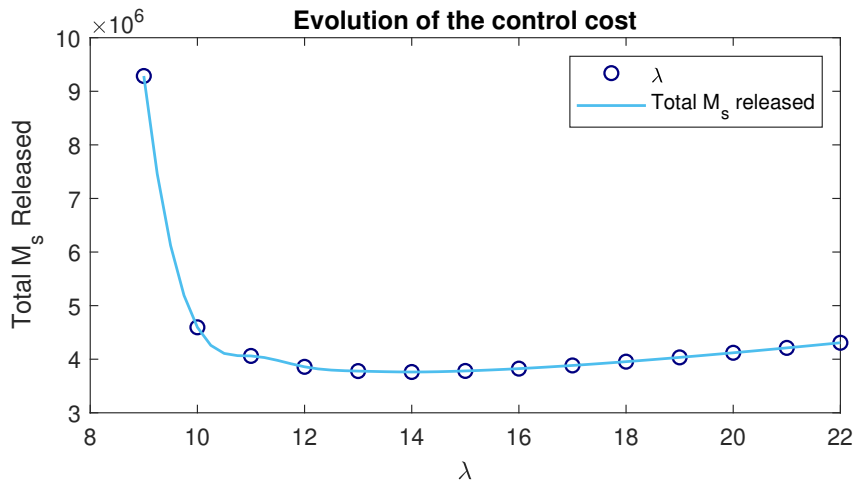
### 5.2.1 Numerical simulations

Let the initial condition be the persistence equilibrium. Taking the same parameters from the previous sections, (5.3) gives  $\lambda > 8.78$ . The amount of time required to drive  $E$  down to  $K/100$ , and the total amount of sterile males released during that time, for  $\lambda = 9, 10, \dots, 22$ , is shown in Table 5.1.

$\lambda$	Intervention time (days)	Total $M_s$ released
9	5106	9 285 484
10	750	4 595 287
11	492	4 061 116
12	386	3 857 979
13	326	3 778 576
14	287	3 761 570
15	259	3 781 149
16	238	3 824 269
17	222	3 883 536
18	209	3 954 414
19	198	4 033 943
20	189	4 120 107
21	182	4 211 478
22	175	4 307 018

**Table 5.1:** Intervention time and control cost for several  $\lambda$ .

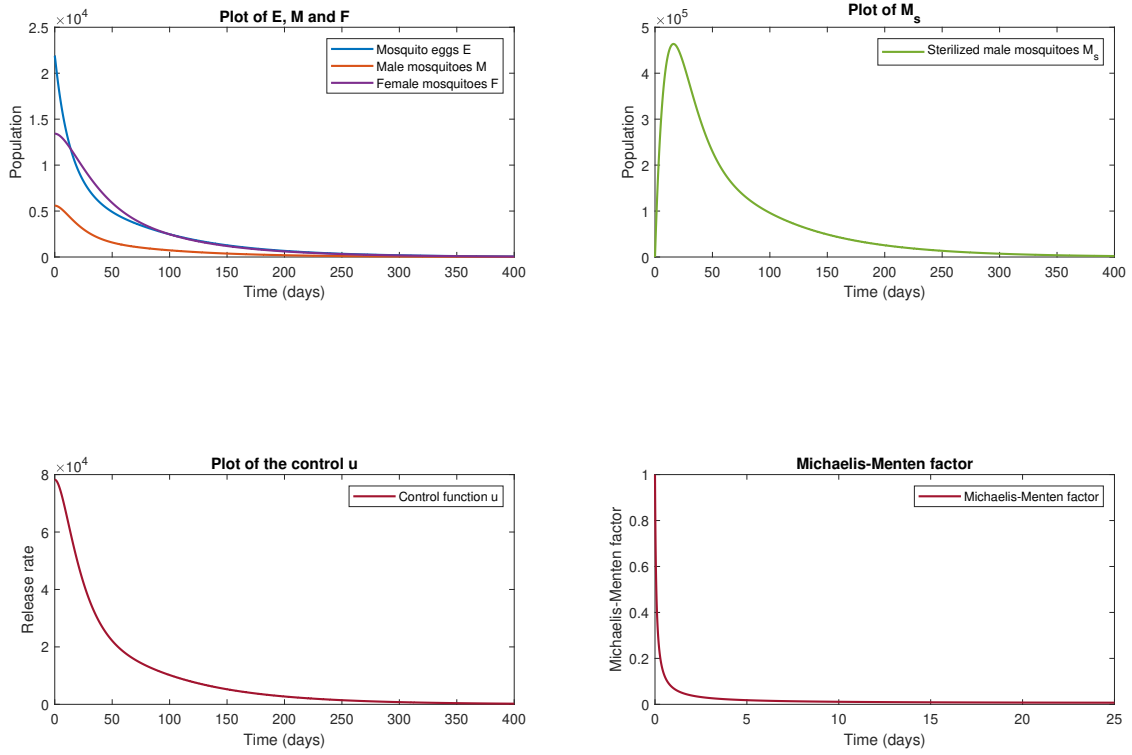
Figure 5.2 comprises this data in a graphic manner:



**Figure 5.2:** Evolution of the control cost for each  $\lambda$ .

The relationship between  $\lambda$  and the control cost is clearly not monotonic, with a minimum at  $\lambda \approx 14$ . We take this value for the following simulation.





**Figure 5.3:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$ , the control  $u$  and the Michaelis-Menten factor with  $\lambda = 14$ .

The amount of sterile males released in the first 400 days is

$$\int_0^{400} u(t) dt \approx 3\,816\,396 \text{ mosquitoes} \quad (5.4)$$

### 5.3 Feedback law depending only on aquatic phase mosquitoes

In this section we analyze the possible application of feedback (3.27) for (4.10). To find a lower bound for  $\omega$  such that the feedback drives the population to extinction, we follow this approach:

- Let  $\mathcal{R}_3(\omega) := \mathcal{R}_0 \frac{\eta M}{1 + \eta(M + \gamma_s M_s)}$ .
- At equilibria,  $\dot{M}_s = 0 \implies \omega E = \delta_s M_s \implies M_s = \frac{\omega}{\delta_s} E$ , and  $\dot{M} = 0 \implies (1 - \nu)\nu_E E = \delta_M M \implies M = \frac{(1 - \nu)\nu_E E}{\delta_M}$ . Setting  $\mathcal{R}_3(\omega) = r$ , i.e., to the threshold for extinction, implies  $E = \frac{r}{\mathcal{R}_0 a - r(a + b)}$ , with  $a = \frac{\eta(1 - \nu)\nu_E}{\delta_M}$  and  $b = \frac{\eta\gamma_s \omega}{\delta_s}$ .
- For an equilibrium in  $\mathcal{D}'$ , we must have  $E > 0 \implies r(a + b) < \mathcal{R}_0 a$ . Hence, we seek  $r(a + b) \geq \mathcal{R}_0 a$ , which gives

$$\omega \geq \frac{\delta_s(\mathcal{R}_0 - r)(1 - \nu)\nu_E}{r\gamma_s\delta_M} \quad (5.5)$$

This coincides with (3.33) when  $r = 1$ .

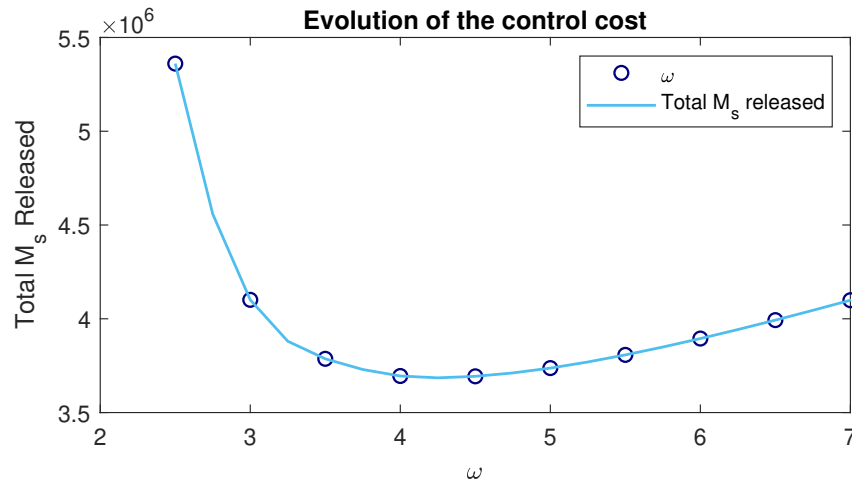
### 5.3.1 Numerical simulations

Let the initial condition be the persistence equilibrium. Taking the same parameters from the previous sections, (5.5) gives  $\omega > 2.24$ . The amount of time required to drive  $E$  down to  $K/100$ , and the total amount of sterile males released during that time, for  $\omega = 2.5, 3, 3.5, \dots, 7$ , is shown in Table 3.2.

$\omega$	Intervention time (days)	Total $M_s$ released
2.5	1000	5 360 656
3	512	4 101 006
3.5	370	3 786 560
4	303	3 694 915
4.5	263	3 693 145
5	236	3 737 084
5.5	217	3 807 494
6	202	3 894 591
6.5	190	3 992 860
7	181	4 098 951

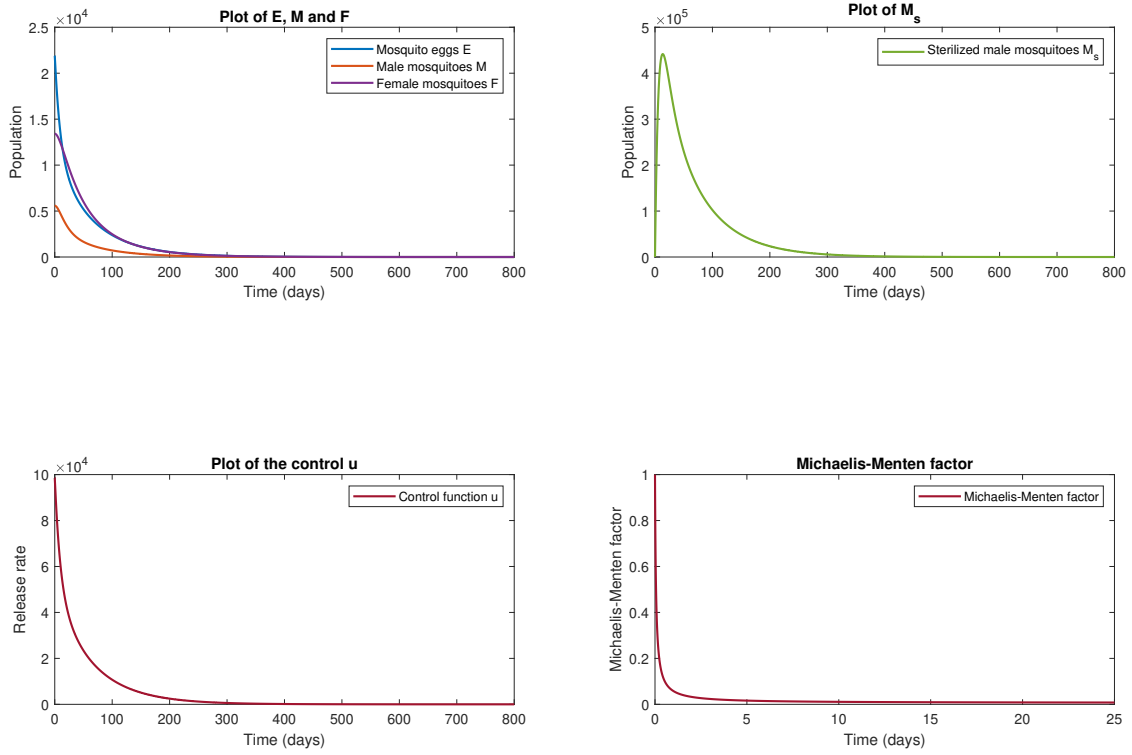
**Table 5.2:** Intervention time and control cost for several  $\omega$ .

Figure 5.4 comprises this data in a graphic manner:



**Figure 5.4:** Evolution of the control cost for each  $\omega$ .

The relationship between  $\omega$  and the control cost is clearly not monotonic. We take  $\omega = 4.5$  for the following simulation.



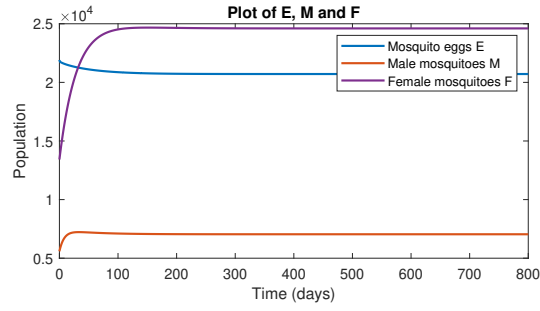
**Figure 5.5:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$ , the control  $u$  and the Michaelis-Menten factor with  $\omega = 4.5$ .

The amount of sterile males released in the first 800 days is

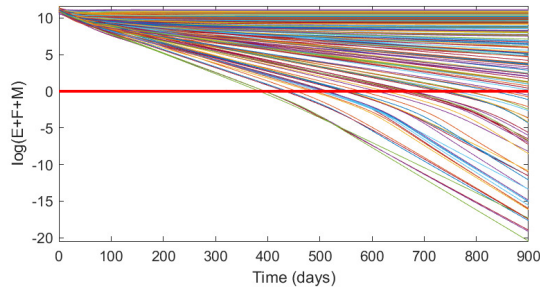
$$\int_0^{800} u(t) dt \approx 3\,763\,238 \text{ mosquitoes} \quad (5.6)$$

## 5.4 Robustness Tests

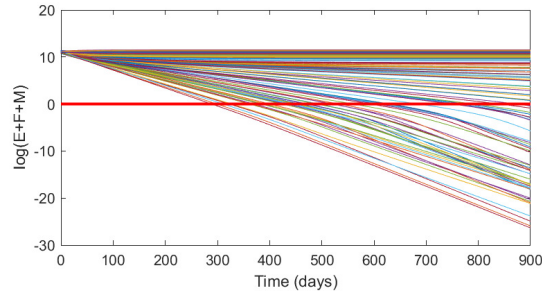
Robustness was tested for the adapted feedback laws by following the same protocol as in section 3.5, although in this instance, initial conditions for the two latter controls were generated as shown in the end of 3.5.1. The following figures correspond to the total male, wild male, and aquatic phase dependent feedback laws, respectively.



**Figure 5.6:** Plot of  $E$ ,  $M$  and  $F$ .



**Figure 5.7:** Evolution of  $\log(E + F + M)$  up to  $t = 900$  days when  $x_0$  comes from  $\mathcal{N}(K, \frac{0.1K}{3})$ .



**Figure 5.8:** Evolution of  $\log(E + F + M)$  up to  $t = 900$  days when  $x_0$  comes from  $\mathcal{N}(K, \frac{0.1K}{3})$ .

Similarly to what was observed in section 3.5, these feedback laws remain not robust against variations of the mosquito dynamics parameters.

# 6

## **Finding an optimal control**

This chapter concerns the problem of finding a control strategy that minimizes the cost, i.e, the total amount of released sterile mosquitoes, of the intervention while still assuring the extinction of the mosquito population.

Consider the optimal control problem

$$\min_{(x,u) \in C^1([0,T];[0,\infty)^4) \times C^1([0,T];[0,\infty))} J(x,u) = \int_0^T \alpha_1 E(t)^2 + \alpha_2 M(t)^2 + \alpha_3 F(t)^2 + \beta u(t)^2 dt \quad (6.1)$$

subject to

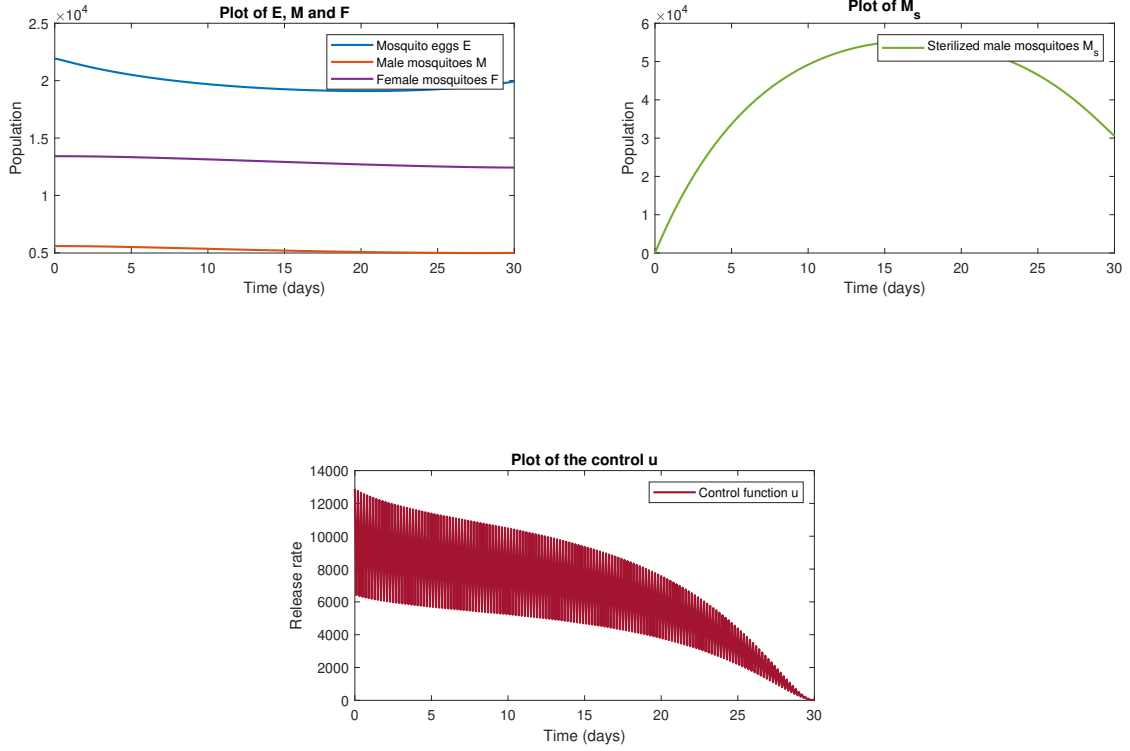
$$\begin{cases} \dot{E} = \beta_E F \left(1 - \frac{E}{K}\right) \frac{\eta M}{1 + \eta(M + \gamma_s M_s)} - (\nu_E + \delta_E) E \\ \dot{M} = (1 - \nu) \nu_E E - \delta_M M \\ \dot{F} = \nu \nu_E E - \delta_F F \\ \dot{M}_s = u - \delta_s M_s \end{cases} \quad (6.2)$$

To find a numerical solution, we first approximate  $J$  through composite Simpson's 1/3 rule. Then, formulate the dynamic constraints by discretizing the system of ODEs through the fourth-order Runge-Kutta method, ensuring that at each time step, the state variables satisfy the method's update equations.

This allows us to recast the optimal control problem as a nonlinear programming problem, which we solve using Matlab's `fmincon` function with the SQP algorithm, as it consistently yields better results than other available methods such as interior-point or active-set. No lower bounds are initially imposed on the control variable, and the model parameters are kept consistent with previous sections. It is worth noting that this formulation involves thousands of optimization variables, which poses significant computational challenges.

We observe that regardless of the weights choice, minimizing (6.1) produces a highly oscillatory (discrete) distribution of  $u$ , without an apparent smooth or piece-wise continuous counterpart. Moreover, the resulting control fails to trend the population toward extinction.





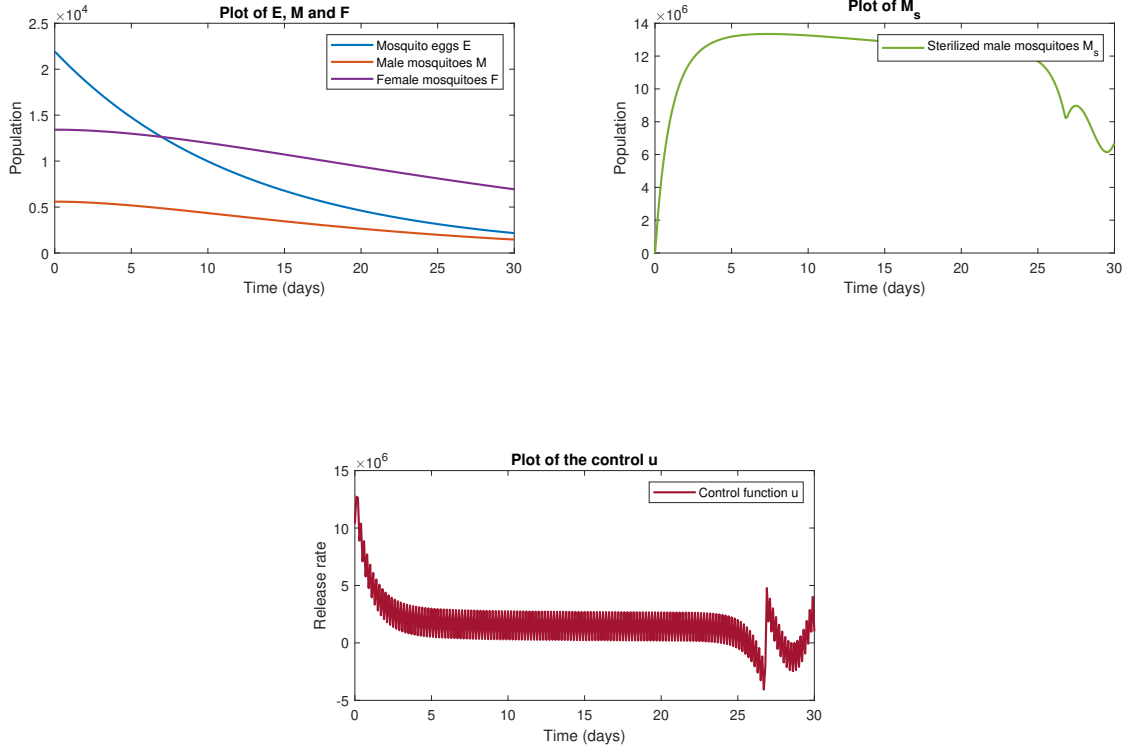
**Figure 6.1:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$ , and the control  $u$  when  $\alpha_2 = \alpha_3 = 0$  and  $\alpha_1 = \beta = 1$ , for  $T = 30$ .

Further refining the step did not seem to improve this behavior. This suggests that minimizing the norm of the control's derivative instead could be of better interest.

Alternatively, we regard a different formulation

$$\min_{(x,u) \in C^1([0,T];[0,\infty)^4) \times C^1([0,T];[0,\infty))} J(x,u) = \int_0^T \alpha_1 E(t)^2 + \alpha_2 M(t)^2 + \alpha_3 F(t)^2 + \beta |u(t)| dt. \quad (6.3)$$

where we replace  $u^2$  by  $|u|$ . Similarly to the previous strategy, there is a high oscillatory behavior of  $u$ .

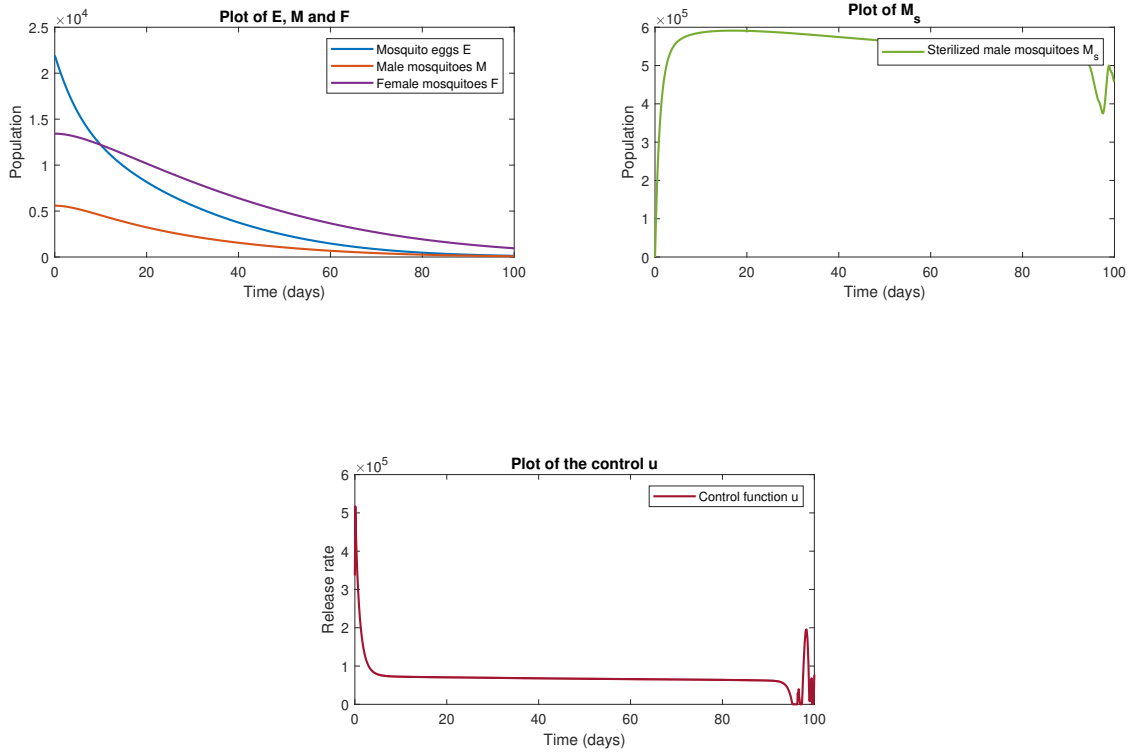


**Figure 6.2:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$ , and the control  $u$  when  $\alpha_2 = \alpha_3 = 0$  and  $\alpha_1 = \beta = 1$ , for  $T = 30$ .

Consequently, we settle with the final control problem

$$\min_{(x,u) \in C^1([0,T];[0,\infty)^4) \times C^1([0,T];[0,\infty))} J(x,u) = \int_0^T \alpha_1 E(t)^2 + \alpha_2 M(t)^2 + \alpha_3 F(t)^2 + \beta u(t) dt, \quad (6.4)$$

where we now impose a lower bound of zero on the control. This produced less fluctuating control profiles, although we were still unable to obtain a solution that both minimized the number of released sterile males and effectively drove the population to extinction. We show the result given by `fmincon` when  $\alpha_2 = \alpha_3 = 0$  and  $\alpha_1 = \beta = 1$ , for  $T = 100$ . Setting different  $\alpha_i$  to 0 did not seem to give improved results.



**Figure 6.3:** Plot of  $E$ ,  $M$  and  $F$ ,  $M_s$  and the control  $u$ .

The amount of sterile males released in the first 100 days is

$$\int_0^{100} u(t) dt \approx 7\,064\,478 \text{ mosquitoes,} \quad (6.5)$$

a figure considerably higher than those achieved with the previous feedback control strategies. Nevertheless, we take notice of the fact that this control allows for a seemingly slightly faster decay of the mosquito population.

# 7

## **Conclusion**

In this project, we studied mathematical models for mosquito population dynamics, focusing on the sterile insect technique (SIT) as a biological control strategy. We analyzed both the classical model and its variant incorporating an Allee effect, exploring their behavior under various feedback control laws. Several feedback strategies were investigated, including controls based on a backstepping approach, and on a direct dependence from the total male population, the wild male population, and the aquatic phase.

Through extensive numerical simulations, we verified the effectiveness of the proposed feedback laws in driving the mosquito population to extinction under ideal conditions. Although these feedback laws were originally developed for the model without an Allee effect, they remained effective when applied to the extended model incorporating such an effect. Furthermore, for both the wild male-dependent and aquatic phase-dependent feedbacks, we observed a non-monotonic relationship between the control parameter and the associated cost.

When comparing the feedback strategies within the Allee effect model, we conclude that if the goal is to minimize the total number of sterilized males to be released, the aquatic phase-dependent feedback yields good results within approximately 260 days when using  $\omega = 4.5$ . These results are comparable to those obtained with the wild male-dependent feedback when  $\lambda = 14$ , so the choice between them should depend on which population component is more practical to monitor in the field. On the other hand, if a faster drive to extinction is prioritized, it can be achieved by increasing the number of sterile males released—though at a higher operational cost. The total male-dependent feedback emerged as the least efficient in terms of time and cost.

Additionally, robustness tests revealed that these strategies are highly sensitive to parameter variations, which limits their practical reliability in real-world scenarios where biological and environmental parameters are often uncertain or variable. Another drawback concerns the significant gap between the size of the wild male population and the number of sterile males that would need to be released, which may pose logistical challenges and question the feasibility of the method in practice.

We formulated and attempted to solve an optimal control problem aimed at minimizing a cost functional. Despite correctly implementing the problem formulation, the numerical optimization (via `fmincon` in Matlab) proved computationally challenging, often converging prematurely or returning suboptimal, irregular control profiles. Future work can be done in further methodological refinement, such as better initialization strategies, alternative optimization algorithms, or improved constraint handling to achieve practical and cost-effective solutions.

Overall, this project deepened our understanding of SIT dynamics and control, while highlighting both the potential and the challenges of applying mathematical optimization to ecological interventions.

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