The Galerkin and Collocation Methods for Two-Dimensional Integral Equations

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***Abstract*—The Galerkin and Collocation methods are regularization strategies for inverse problems. They are based on projection operators, which are formal strategies to discretize the equation. In "*An Introduction to Mathematical Theory of Inverse Problems*", written by Andreas Kirsch, the two methods are applied for unidimensional integral equations, i.e., with only one independent variable. But there are problems in which the integral equation is bidimensional. This paper presents the Galerkin and Collocation Methods for integral equations with two independent variables.**

I. Introduction

The project methods are regularizing strategies for operator equations of the form $Kx = y$, where x and y are elements of certain function spaces. These methods raised from the fact that numerical treatment of integral equations normally require the discretization of the continuous problem and the reduction to a finite system of linear or nonlinear equations. Any discretization scheme is a regularization strategy, i.e., a linear and bounded operator such that it converges pointwise to the identity.

Andreas Kirsch presents in his book [1] some projection methods such as the Collocation and Galerkin methods. The first is based on orthogonal projection, while the second is based on collocation points for interpolation. These methods are applied in his book on integral equations with only one independent variable. However, many inverse problems in the form of integral equations have two or three independent variables – problems with two or three dimensions in space, for example.

This report intents to present the formulation of the Collocation and Galerkin methods for bidimensional problems, hoping that it may be useful for someone who is reading the Kirsch's book and is looking for help for applying these methods in their own problems. In a worst-case scenario, this report will be useful for remembering these formulations in possibly new aplications.

In the next section, the general formulation of the integral equation will be presented. Sections III and IV introduce the Galerkin and Collocation formulations. The section V shows some results for an example problem. Conclusions are drawn in section VI.

II. Integral Equations

We are interested in the following kind of equation:

$$\int_a^b \int_c^d k(x,y,u,v) f(u,v) dv du = g(x,y) \tag{1}$$

where $x \in [a,b]$, $y \in [c,d]$, F and G are (real or complex) Hilbert spaces; $f(x,y) \in F$ and $g(x,y) \in G$; and $k(x,y,u,v)$ is a continuous or weakly singular kernel in $L^2(a,b) \times (c,d)$ or $C[a,b] \times [c,d]$.

We also denote the operator $K : F \rightarrow G$ as

$$K \coloneqq \int_a^b \int_c^d k(x,y,u,v) dv du,$$

and we assume that K is a linear, bounded and one-to-one operator.

When $g(x,y)$ is known and $f(x,y)$ is unknown, the integral equation is an inverse problem. Such inverse problems are normally ill-posed [2] because either the solution does not exist or it is not unique or it does not depend continuously on the data. Regularization strategies are operators which are applied on these problems for recovering the solution function with less error and even a discretization scheme may be considered as a regularization strategy.

III. Galerkin Method

Based on the assumptions, we also define $F_{mn} \in X$ and $G_{mn} \in Y$ as finite-dimensional subspaces with dim F_{mn} = dim $G_{mn} = mn$; and $Q_{mn} : G \rightarrow G_{mn}$ as a orthogonal projection operator onto G_{mn}. Then the equation $Q_{mn} K f_{mn} = Q_{mn} g$ reduces to the Galerkin equations

$$(K f_{mn}, z_{mn}) = (g, z_{mn}), \ \forall z_{mn} \in G_{mn},$$

where $f_{mn} \in F_{mn}$.

If F_{mn} = span$\{\hat{f}_1, \cdots, \hat{f}_{mn}\}$ and G_{mn} = span$\{\hat{g}_1, \cdots, \hat{g}_{mn}\}$, then we are looking for a solution $f_{mn} = \sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} \hat{f} jk$ in which α is the solution of the following linear system:

$$\sum_{j=1}^m \sum_{k=1}^n \alpha_{jk} (K \hat{f}_{jk}, \hat{g}_{pq}) = (g, \hat{g}_{pq}), \tag{2}$$

where $p = 1, \cdots, m$, and $q = 1, \cdots, n$; and $\hat{g}_{mn} = K\hat{f}_{mn}$. The linear system may be rewritten as $A\alpha = \beta$ where

$$A_{(pq),(jk)} = \int_a^b \int_c^d \int_a^b \int_c^d k(x,y,u,v)\hat{f}_{jk}(u,v) \hat{g}_{pq}(x,y)dvdudydx, \quad (3)$$

$$\beta_{pq} = \int_a^b \int_c^d g(x,y)\hat{g}_{pq}(x,y)dydx. \quad (4)$$

IV. Collocation Method

The Collocation Method is based on collocations points. Therefore, let $a \leq x_1, \cdots, x_m \leq b$ and $c \leq y_1, \cdots, y_n \leq d$ be the collocation points. Then, with the same assumptions for the Galerkin Method, we have the collocation equations

$$Kf_{mn}(x_p, y_q) = g(x_p, y_q), \ \ p = 1, \cdots, m, \ \ q = 1, \cdots, n.$$

Then, being $F_{mn} = \text{span}\{\hat{f}_1, \cdots, \hat{f}_{mn}\}$ and $G_{mn} = \text{span}\{\hat{g}_1, \cdots, \hat{g}_{mn}\}$, then we are again looking for a solution $f_{mn} = \sum_{j=1}^m \sum_{k=1}^n \alpha_{jk}\hat{f}jk$ in which α is the solution of $A\alpha = \beta$ where $A_{(pq),(jk)} = K\hat{f}_{jk}(x_p, y_q)$ and $\beta_{pq} = g(x_p, y_q)$. Therefore, we may rewrite $A_{(pq),(jk)}$ as

$$A_{(pq),(jk)} = \int_a^b \int_c^d k(x_p, y_q, u, v)\hat{f}_{jk}(u,v)dvdu \quad (5)$$

V. Experimentation

In order to illustrate the application of the two methods, we choose the differential problem given by the following integral equation:

$$Kf(x,y) := \int_0^x \int_0^y f(u,v)dvdu = g(x,y), \quad (6)$$

where $x \in [0,1]$ and $y \in [0,1]$. We define a kernel $k(x,y,u,v)$ as

$$k(x,y,u,v) = \begin{cases} 1, & \text{if } u \leq x \text{ and } v \leq y \\ 0, & \text{otherwise,} \end{cases}$$

then eq.(6) may be rewritten as

$$Kf(x,y) = \int_0^1 \int_0^1 k(x,y,u,v)f(u,v)dvdu = g(x,y). \quad (7)$$

For both methods, we are assuming the basis function $\hat{f}_{jk}(u,v) = k(x_j, y_k, u, v)$.

A. Galerkin Method

According to eq.(3), the matrix coefficients are given by

$$A_{(pq),(jk)} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 k(x,y,u,v)k(x_j,y_k,u,v) \hat{g}_{pq}(x,y)dvdudydx \quad (8)$$

$$= \int_0^1 \int_0^1 \int_0^{x_j} \int_0^{y_k} k(x,y,u,v)\hat{g}_{pq}(x,y) dvdudydx \quad (9)$$

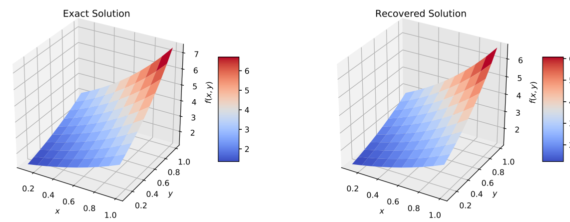

Fig. 1. Results for Galerkin Method.

where

$$\hat{g}_{pq}(x,y) = \int_0^1 \int_0^1 g(u,v)k(x_p, x_q, u, v)dvdu \quad (10)$$

$$= \int_0^{x_p} \int_0^{y_q} g(u,v)dvdu \quad (11)$$

B. Collocation Method

According to eq. (5), the coefficients matrix are given by

$$A_{(pq),(jk)} = \int_0^1 \int_0^1 k(x_p, y_q, u, v)k(x_j, y_k, u, v) dvdu \quad (12)$$

$$= \int_0^{x_{min}} \int_0^{y_{min}} dvdu \quad (13)$$

$$= x_{min}y_{min}, \quad (14)$$

where $x_{min} = \min\{x_p, x_j\}$ and $y_{min} = \min\{y_q, y_k\}$.

C. Results

Both methods are applied to solve eq. (6) considering $g(x,y) = (e^x - 1)(e^y - 1)$. In this case, the exact solution of the integral equation is $f(x,y) = e^{x+y}$, which will be considered to compare with the recovered function. The intervals $x, y \in [0,1]$ are divided in 10 samples points, equally spaced, which means 10 collocation points for the Collocation Method. Therefore, the coefficients matrix is 100×100. The integrals (3), (4) and (5) were approximated using the trapezoidal method[1]. The error is measured according to the following formula:

$$\text{error} = \frac{1}{mn} \sum_{p=1}^m \sum_{q=1}^n \frac{|e^{x_p+y_q} - f_{mn}(x_p, y_q)|}{e^{x_p+y_q}} \quad (15)$$

Figures 1 and 2 show the results for Galerkin and Collocation Method, respectively. The surfaces were recovered with the same error for both methods: 9.44×10^{-2}. The author believes that both methods reached the same error due to the simplicity of the problem and the basis function chosen.

[1]The implementation is available in https://github.com/andre-batista/Methods-for-Inverse-Problems

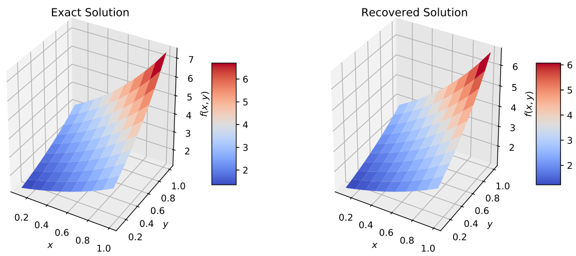

Fig. 2. Results for Collocation Method.

VI. Conclusion

This report was written with the purpose of helping readers (including me) of "An Introduction of Mathematical Theory of Inverse Problems" [1] to apply the projection methods described in the book on integral equations with two independent variables.

The Galerkin and Collocation Methods were introduced and the integrals for the linear system definition were presented. A simple example of the application of the methods was considered and the results validated the formulation.

Further works may include three dimensional integral equations, a study of different basis functions and the application to different problems.

If you find any mistake in this paper, please, e-mail me with your contribution.

Acknowledgment

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