Fast Gabor notes

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1 Introduction

motivation:
- continuous version: motivated by biology: cochlea: driven membrane
- discrete version: motivated technically/dsp: coupled filter-bank -¿ high-dimensional state space filter, where every sub-space is one desired filter output

1.1 Open Problems

- yet no closed solution for full discrete version
- no analytic approach for non equally spaced frequencies, especially for log-scaled frequency distribution
- no better understanding of borders, if introduced. Currently lead to instabilities if to 'unsmooth'
- stability: double is good, float is bad
- is implicit version more stable?
- faster implicit version?
- serial filtering has less adds then parallel version
- skip infinite tail, use just tails from neighbor mirror unit circle half
- thus this would be three filters fwd, bwd w/ nieghbor, fwd neighbor

2 Continuous Version

2.1 Diffusion

$$\partial_t u = (\gamma + ix + \varkappa\partial_x^2)u + f(t), \tag{1}$$

where $u(x,t)$ is the deviation of the membrane from it point of rest, γ is the damping coefficient, ωx is the inhomogeneous frequency term and $\varkappa$ the membrane's coupling strength (TODO correct term).

The solution of the unforced system ($f \equiv 0$) can be found analytically. We use the ansatz

$$u(x,t) = e^{at+bt^2+ct^3}e^{ixt} \tag{2}$$

Inserting (2) into (1) one obtains

$$a + 2bt + 3ct^2 + ix \ = \ \gamma + ix - \varkappa t^2,$$

and after collecting all terms the coefficients can be determined to be $a = \gamma$, $b = 0$, and $\varkappa = -3c$ which leaves us with the final solution

$$u(x,t) = e^{\gamma t - \varkappa t^3/3}e^{ixt} \tag{3}$$

Conclusion: γ has to be positive (unstable) for gaussian response shape, any positive $\varkappa$ immedietely stabilizes the system

2.2 Advection — true gaussian

$$\partial_t u = (\gamma + ix + \varkappa\partial_x)u + f(t), \tag{4}$$

Substituting the ansatz for the homogenous solution ($f \equiv 0$) in (4)

$$u(t) = \alpha(t)e^{ixt} \tag{5}$$

and its derivatives

$$\begin{aligned} \partial_t u &= (\dot\alpha + ix\alpha)e^{ixt} = (\dot\alpha + ix\alpha)\frac{u}{\alpha} \\ \partial_x u &= it\alpha e^{ixt} = itu \end{aligned}$$

we get

$$\begin{aligned} (\dot\alpha + ix\alpha)\frac{u}{\alpha} &= (\gamma + ix)u + \varkappa itu \\ \dot\alpha + ix\alpha &= (\gamma + ix + \varkappa it)\alpha \\ \dot\alpha &= (\gamma + \varkappa it)\alpha \end{aligned}$$

which can be solved by separation of variables

$$\int \frac{d\alpha}{\alpha} \ = \ \int dt\ (\gamma + \varkappa it). \tag{6}$$

The solution is

$$\log\alpha \ = \ c + \gamma t + \frac{i\varkappa}{2}t^2. \tag{7}$$

The integration constant c would be found from the initial conditions and can be factored out and will be set to 0. From the solution it's clear that in order to obtain a gaussian envelope $\varkappa$ must be pure imaginary. Thus by replacing $\varkappa \to -i\varkappa$ we get the differential equation

$$\partial_t u = (\gamma + ix - i\varkappa\partial_x)u + f(t), \tag{8}$$

with the homogenous solution

$$u(x,t) \;=\; e^{\gamma t - \varkappa t^2/2} e^{ixt} \tag{9}$$

2.3 scale invariant advection – Wavelet filterbank

We modify Eq. 4 to scale the frequencies exponentially ($x \to e^x$) and also make the damping scale accordingly ($\gamma \to \partial_x e^x = e^x$).

$$\partial_t u = e^x(\gamma + i + \varkappa\partial_x)u + f(t), \tag{10}$$

The ansatz condition is that the solution must be scale invariant, that is

$$u(x,t) = \phi(e^x t). \tag{11}$$

Substituting the ansatz and its derivatives

$$\begin{aligned}\partial_t u &= \phi' e^x\\ \partial_x u &= \phi' t e^x\end{aligned}$$

in Eq. 10 we get

$$\phi' e^x = e^x(\gamma + i)\phi + \varkappa\phi' t e^{2x}. \tag{12}$$

Substituting $\tau = e^x t$ yields the ordinary differential equation

$$\begin{aligned}\phi' &= (\gamma + i)\phi + \varkappa\tau\phi'\\ \phi'(1 - \varkappa\tau) &= (\gamma + i)\phi\end{aligned}$$

which can be solved by separation of variables

$$\int \frac{d\phi}{\phi} \;=\; \int d\tau\, \frac{\gamma + i}{1 - \varkappa\tau}. \tag{13}$$

The solution is

$$\ln\phi \;=\; c - (\gamma + i)\frac{1 - \varkappa\tau}{\varkappa} \tag{14}$$

Substituting back t and letting $c = 0$ we have the homogenous solution for (10)

$$u(x,t) \;=\; \exp\big(-\tfrac{\gamma+i}{\varkappa}\, ln(1 - \varkappa e^x t)\big) \tag{15}$$

2.4 Spatially discretized – Advection

Nabla is discretized by 2nd order finite difference scheme:

$$\partial_t u_n = (\gamma + in)u_n - i\varkappa(u_{n+1} - u_{n-1}) + f(t),$$
(16)

(TODO: there should be an ω factor ?)

Using the usual ansatz

$$u(t) = \alpha(t)e^{int} \tag{17}$$

with

$$\begin{aligned}\partial_t u_n &= (\dot\alpha + in\alpha)e^{int} = (\dot\alpha + in\alpha)\frac{u}{\alpha}\\ u_{n\pm1} &= e^{\pm it}u_n\end{aligned}$$

in Eq. (21) yields

$$\begin{aligned}(\dot\alpha + in\alpha)\frac{u_n}{\alpha} &= (\gamma + in)u_n - i\varkappa(e^{+it} - e^{-it})u_n\\ \dot\alpha + in\alpha &= \big((\gamma + in) + 2\varkappa\sin t\big)\alpha\\ \dot\alpha &= \big(\gamma + 2\varkappa\sin t\big)\alpha,\end{aligned}$$

an ordinary differential equation for the amplitude which can be solved by separation of variables

$$\int \frac{d\alpha}{\alpha} \;=\; \int dt\, (\gamma + 2\varkappa\sin t). \tag{18}$$

The solution is

$$\log\alpha \;=\; c + \gamma t - 2\varkappa\cos t \tag{19}$$

Thus we get the homogenous solution for Eq. (21) ($c = 0$)

$$u(x,t) \;=\; e^{\gamma t - 2\varkappa\cos t} e^{int} \tag{20}$$

2.5 Spatially discretized – Diffusion

This version has the spatial laplacian term discretized by finite difference scheme (classical discrete laplacian):

$$\partial_t u_n = (\gamma + in)u_n + \varkappa(u_{n-1} - 2u_n + u_{n+1}) + f(t),$$
(21)

(TODO: there should be an ω factor ?)

Using the usual ansatz

$$u(t) = \alpha(t)e^{int} \tag{22}$$

with

$$\begin{aligned}\partial_t u_n &= (\dot\alpha + in\alpha)e^{int} = (\dot\alpha + in\alpha)\frac{u}{\alpha}\\ u_{n\pm1} &= e^{\pm it}u_n\end{aligned}$$

in Eq. (21) yields

$$\begin{aligned}(\dot\alpha + in\alpha)\frac{u_n}{\alpha} &= (\gamma + in)u_n + \varkappa(e^{+it} + e^{-it} - 2)u_n \\ \dot\alpha + in\alpha &= \big((\gamma + in) + \varkappa 2(\cos t - 1)\big)\alpha \\ \dot\alpha &= \big(\gamma + 2\varkappa(\cos t - 1)\big)\alpha,\end{aligned}$$

an ordinary differential equation for the amplitude which can be solved by separation of variables

$$\int \frac{d\alpha}{\alpha} = \int dt\ \big(\gamma + 2\varkappa(\cos t - 1)\big). \tag{23}$$

The solution is

$$\log\alpha = c + (\gamma - 2\varkappa)t + 2\varkappa \sin t \tag{24}$$

Thus we get the homogenous solution for Eq. (21) ($c = 0$)

$$u(x,t) = e^{(\gamma - 2\varkappa)t + 2\varkappa \sin t} e^{int} \tag{25}$$

Conclusion: The adjecency version is better suited as the stability is merely controlled by γ whereas $\varkappa$ controls ...

2.6 discrete time, continuous space

$$u^{t+1} = re^{ix}(u^t + \varkappa \partial_x^2 u^t) \tag{26}$$

Ansatz:

$$u^t = \alpha^t e^{ixt} \tag{27}$$

with

$$\begin{aligned}u^{t+1} &= \alpha_{t+1} e^{ix} e^{ixt} = e^{ix}\frac{\alpha_{t+1}}{\alpha_t}u^t \\ \partial_x^2 u^t &= -t^2 \alpha_t e^{ixt} = -t^2 u^t\end{aligned}$$

inserting:

$$\begin{aligned}e^{ix}\frac{\alpha_{t+1}}{\alpha_t}u^t &= re^{ix}(1 - t^2\varkappa)u^t \\ \alpha_{t+1} &= \alpha_t r(1 - \varkappa t^2)\end{aligned}$$

This difference equation for α diverges for all non-zero values of $\varkappa$.

2.7 Full discrete version

The spatially discretized Eq. (21)

$$\partial_t u_n = \big[(\gamma + i\omega n) + \varkappa(\mathbb{S}_{-1} + \mathbb{S}_{+1})\big]u_n$$

(here in it's adjacency formulation) has the general solution

$$\mathbf{u}(t) = e^{t\left[(\gamma + i\omega n) + \varkappa(\mathbb{S}_{-1} + \mathbb{S}_{+1})\right]}\mathbf{u}(0) \tag{28}$$

which can be rewritten as a difference equation by iterating the solution step-wise

$$\mathbf{u}(t+1) = e^{\gamma + i\omega n} e^{\varkappa(\mathbb{S}_{-1} + \mathbb{S}_{+1})}\mathbf{u}(t)$$

The second exponential can be expanded which yields

$$\mathbf{u}(t+1) = e^{\gamma + i\omega n}\big(1 + \varkappa(\mathbb{S}_{-1} + \mathbb{S}_{+1}) + \dots\big)\mathbf{u}(t)$$

We cut off after the second term of the expansion in order to maintain the same algebraic complexity and the same coupling depth as the original differential equation. Introducing a shorter notation for time and space indices and applying the shift operator to the state u yields

$$u_n^{t+1} = re^{i\omega n}\big(u_n^t + \varkappa(u_{n-1}^t + u_{n+1}^t)\big) + f^t. \tag{29}$$

Inserting the ansatz

$$u_n^t = \alpha^t e^{i\omega nt} \tag{30}$$

in (29) ($f \equiv 0$) with corresponding shifted versions

$$\begin{aligned}u_n^{t+1} &= \alpha_{t+1} e^{i\omega n} e^{i\omega nt} \\ &= e^{i\omega n}\frac{\alpha_{t+1}}{\alpha_t}u_n^t \\ u_{n\pm1}^t &= e^{\pm i\omega t}u_n^t\end{aligned}$$

yields

$$\begin{aligned}e^{i\omega n}\frac{\alpha_{t+1}}{\alpha_t}u_n^t &= re^{i\omega n}\big(u_n^t + \varkappa(e^{-i\omega t} + e^{+i\omega t})u_n^t\big) \\ \frac{\alpha_{t+1}}{\alpha_t} &= r(1 + \varkappa(e^{-i\omega t} + e^{+i\omega t})) \\ \alpha_{t+1} &= r(1 + 2\varkappa\cos(\omega t))\alpha_t\end{aligned}$$

Thus, the solution for α_t will be

$$\begin{aligned}\alpha_t &= \alpha_0 \prod_{\tau=1}^{t} r(1 + 2\varkappa\cos(\omega\tau)) \\ \alpha_t &= \alpha_0 r^t \prod_{\tau=1}^{t}(1 + 2\varkappa\cos(\omega\tau))\end{aligned}$$

Side notes: By introducing $a_t = \log\alpha_t$ the above equation can be read

$$a_{t+1} = a_t + \log r + \log(1 + 2\varkappa\cos\omega t)$$

Open problem: find solution of $\prod(1+2\dots)$ or $\sum\log(1+2\dots)$, respectively.

$$\begin{aligned}&\prod_t(1+2\varkappa\cos t)\\&\quad=(1+2\varkappa\cos 0)(1+2\varkappa\cos 1)(1+2\varkappa\cos 2)\dots\\&\quad=1+2\varkappa(\cos 0+\cos 1+\cos 2+\dots)+\dots\end{aligned}\tag{31}$$

2.7.1 Scaling

Eq. (28) is of the form

$$\mathbf{u}(t)=e^{t\mathbb{P}}\mathbf{u}(0)$$

with $\mathbb{P}=(\gamma+i\omega n)+\varkappa(\triangleleft+\triangleright)$ being the continuous time operator, where $\triangleleft$ and $\triangleright$ denote the left-shift ($\mathbb{S}_{-1}$) and right-shift ($\mathbb{S}_{+1}$) operators, respectively. By choosing different time steps than unity, say τ, the discrete time version can be scaled arbitrary:

$$\mathbf{u}(t+\tau)=e^{\tau\mathbb{P}}\mathbf{u}(t)$$

Thus, the general version of Eq. (29) is

$$\begin{aligned}\mathbf{u}(t+1)&=e^{\tau(\gamma+i\omega n)}e^{\tau\varkappa(\triangleleft+\triangleright)}\mathbf{u}(t)\\&=e^{\tau(\gamma+i\omega n)}\big(1+\tau\varkappa(\triangleleft+\triangleright)+\dots\big)\mathbf{u}(t)\\&\approx e^{\tau(\gamma+i\omega n)}\big(1+\tau\varkappa(\triangleleft+\triangleright)\big)\mathbf{u}(t)\end{aligned}$$

So we arrive at the scaleable discrete time, discrete space filter bank:

$$\mathbf{u}_{t+1}=e^{\tau(\gamma+i\omega n)}(1+\tau\varkappa(\triangleleft+\triangleright))\mathbf{u}_t\tag{32}$$

2.7.2 Transfer function

The partial difference equation (29) in matrix form is written

$$\mathbf{u}_{t+1}=\mathbf{A}\big(\mathbf{1}+\varkappa\mathbf{K}\big)\mathbf{u}_t+\mathbf{B}f_t$$

where $\mathbf{A}=diag\{a_n=e^{\gamma+i\omega n}\}$. Applying the z-tranform with yields

$$z\mathbf{U}(z)=\mathbf{A}\big(\mathbf{1}+\varkappa\mathbf{K}\big)\mathbf{U}(z)+\mathbf{B}F(z)$$

Rearranging for the definition of the transfer function $\mathbf{H}(z)=\mathbf{U}(z)/F(z)$

$$\big[z\mathbf{1}-\mathbf{A}\big(\mathbf{1}+\varkappa\mathbf{K}\big)\big]\mathbf{H}(z)=\mathbf{B}$$

and thus

$$\mathbf{H}(z)=\big[z\mathbf{1}-\mathbf{A}\big(\mathbf{1}+\varkappa\mathbf{K}\big)\big]^{-1}\mathbf{B}$$

$$\begin{pmatrix}1&\varkappa&&&\varkappa\\\varkappa&1&\varkappa&&\\&&\dots&&\\&&\varkappa&1&\varkappa\\\varkappa&&&\varkappa&1\end{pmatrix}\tag{33}$$

3 Discretization

3.1 Approximation of Shift

The Shift operator $\mathbb{S}_h$ is defined as $\mathbb{S}_h u(t):u(t)\to u(t+h)$ The Shift operator can be written in terms of derivatives $\mathbb{S}_h=e^{h\frac{d}{dt}}$ (assuming smooth functions, definition?) Lets define the unit delay as $\mathbb{D}=\mathbb{S}_{-1}$. The Unit delay can be taylored in terms of derivatives:

$$\mathbb{D}=e^{-\frac{d}{dt}}=\frac{e^{-\frac{1}{2}\frac{d}{dt}}}{e^{\frac{1}{2}\frac{d}{dt}}}\approx\frac{1-\frac{1}{2}\frac{d}{dt}}{1+\frac{1}{2}\frac{d}{dt}}\tag{34}$$

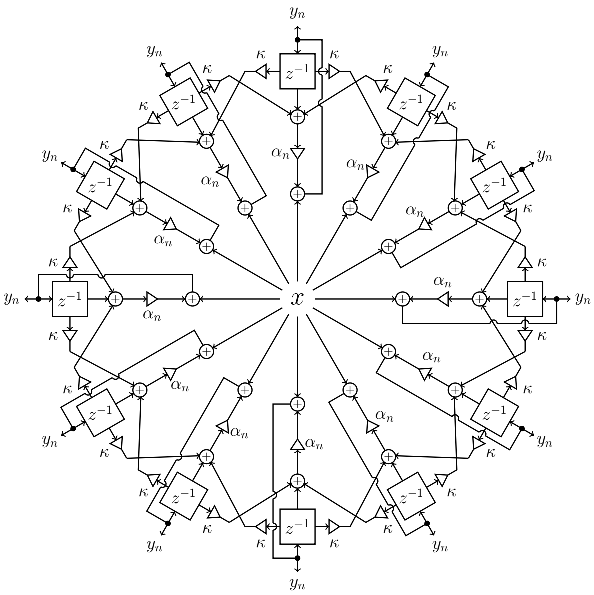
Rearranging and solving for $\frac{d}{dt}$ yields

$$\frac{d}{dt}\approx 2\frac{\mathbb{D}+1}{\mathbb{D}-1}\tag{35}$$

3.1.1 Application for simple ODE

Given $\frac{d}{dt}y=ay+x$m replacing the derivative with the shift approximation gives

$$\begin{aligned}2\tfrac{\mathbb{D}+1}{\mathbb{D}-1}y&=ay+x\\2(\mathbb{D}+1)y&=(\mathbb{D}-1)(ay+x)\end{aligned}$$



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