Linear Algebra

This is a summary of the linear algebra concepts used in the course:

Matrices and Vectors

- $A \in \mathbb{R}^{m \times n}$: **matrix** with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- $x \in \mathbb{R}^n$: **(column) vector** with n rows and 1 column:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- A **row vector** is a matrix with 1 row and n columns;
- A **scalar** is a matrix with 1 row and 1 column (i.e. a number).

Matrix Transpose and Matrix Product

- **Transpose** of a matrix $A \in \mathbb{R}^{m \times n}$ is the matrix $A^T \in \mathbb{R}^{n \times m}$ such that $A^T_{ij} = A_{ji}$;
- Matrix A is **symmetric** if $A = A^T$;
- The **product** of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is the matrix $C \in \mathbb{R}^{m \times p}$:

$$C = AB \quad \text{where} \quad C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

- The **inner product** (or **dot product**) of two vectors $x, y \in \mathbb{R}^n$ is the scalar $x^T y \in \mathbb{R}$:

$$< x, y >= x^T y = y^T x = \sum_{i=1}^{n} x_i y_i$$

- The **outer product** of two vectors $x, y \in \mathbb{R}^n$ is the matrix $xy^T \in \mathbb{R}^{n \times n}$:

$$xy^T = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \dots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \dots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \dots & x_n y_n \end{bmatrix}$$

- The **Hadamard/Schur product** of two vectors $x, y \in \mathbb{R}^n$ is the vector $x \circ y \in \mathbb{R}^n$ such that $(x \circ y)_i = x_i y_i$:

$$x \circ y = \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{bmatrix}$$

Properties

- **Associativity**: $(AB)C = A(BC)$;
- It is **not commutative**: $AB \neq BA$ - unless A or B is a scalar;
- **Transpose of a product**: $(AB)^T = B^T A^T$;
- **Transpose of sum**: $(A + B)^T = A^T + B^T$.

Norms

- The **norm** of a vector is its **length** or **magnitude**;
- The euclidean norm (or l_2 norm) of a vector $x \in \mathbb{R}^n$ is:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x} = \sqrt{< x, x >}$$

- More generally, the l_p norm of a vector $x \in \mathbb{R}^n$ is:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- The l_1 norm of a vector $x \in \mathbb{R}^n$ is: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$;
- The l_∞ norm of a vector $x \in \mathbb{R}^n$ is: $\|x\|_\infty = \max_i |x_i|$.

Special Kinds of Matrices and Vectors

- **Diagonal matrix**: a matrix $D \in \mathbb{R}^{n \times n}$ such that $D_{ij} = 0$ for $i \neq j$;
- **Identity matrix**: a diagonal matrix $I \in \mathbb{R}^{n \times n}$ such that $I_{ii} = 1$ for all i;
  - $AI = IA = A$ for any matrix $A \in \mathbb{R}^{m \times n}$ - **neutral element** of matrix multiplication;
- **Upper triangular matrix**: a matrix $U \in \mathbb{R}^{n \times n}$ such that $U_{ij} = 0$ for $i > j$;
- **Lower triangular matrix**: a matrix $L \in \mathbb{R}^{n \times n}$ such that $L_{ij} = 0$ for $i < j$.

Eigendecomposition

- **Eigenvector** of a matrix $A \in \mathbb{R}^{n \times n}$ is a nonzero vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar $\lambda \in \mathbb{R}$;

$$Ax = \lambda\vec{v} \quad \Leftrightarrow \quad (A - \lambda I)\vec{v} = 0$$

- The eigenvalues of a diagonal matrix are its diagonal elements;
- The matrix **trace** is the sum of its diagonal elements: $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i$;
- The matrix **determinant** is the product of its eigenvalues: $\det(A) = |A| = \prod_{i=1}^{n} \lambda_i$;
  - $|A| = |A^T|$;
  - $|AB| = |A||B|$;
  - $|\alpha A| = \alpha^n |A|$.

Matrix Inverse

- A matrix is **invertible** if there exists a matrix B such that $AB = BA = I$;
- The **inverse** of a matrix $A \in \mathbb{R}^{n\times n}$ is the matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that $AA^{-1} = A^{-1}A = I$;
- $det(A) \neq 0 \quad \Leftrightarrow \quad A$ is invertible;
- $det(A^{-1}) = \frac{1}{det(A)}$;
- If A is invertible, then $Ax = b$ has a unique solution $x = A^{-1}b$ for any b;
- Computational cost of inverting a matrix is $O(n^3)$.

Properties:

- $(A^{-1})^{-1} = A$;
- $(AB)^{-1} = B^{-1}A^{-1}$;
- $(A^T)^{-1} = (A^{-1})^T$.

Quadratic Form and Positive Semidefinite Matrices

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$, the **quadratic form** is:

$$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \quad \text{where} \quad A \in \mathbb{R}^{n\times n}$$

This can be written as:

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- A **positive semidefinite matrix** is a symmetric matrix $A \in \mathbb{R}^{n \times n}$ such that $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$;
  - A is positive semidefinite $\quad \Leftrightarrow \quad \lambda_i(A) \geq 0$ for all i;
- A **positive definite matrix** is a symmetric matrix $A \in \mathbb{R}^{n \times n}$ such that $x^T Ax > 0$ for all $x \in \mathbb{R}^n$;
  - A is positive definite $\quad \Leftrightarrow \quad \lambda_i(A) > 0$ for all i.