Project Thesis
Solving Cahn Hilliard Equation

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1 Introduction

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PLAN FOR REPORT

1) Introduction

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2 DG for Possion Problem

2.1 Possion Problem

Lets define the problem

$$\begin{aligned} -\varepsilon \nabla u &= f \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \Gamma_D \\ \partial_n u &= g \quad \text{on } \Gamma_N \\ \partial_n u + \beta u &= h \quad \text{on } \Gamma_R \end{aligned}$$

Here is $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$.

2.2 Classical DG

2.3 Hybrid DG Method

We want to write this on a weak form. Let the spaces we work on be

$$H^1(\mathcal{T}_h) = \left\{ u \in L^2(\Omega), u \in H^1(T)\, \forall T \in \mathcal{T}_h \right\}$$

.

For the problem to be discontinuous do we define the trial and test function to be $u \in H^1(\Omega)$ and $v \in H^1(\mathcal{T}_h)$. Thus,

$$-\sum_{T \in \mathcal{T}_h} \int_T \varepsilon \nabla^2 u \cdot v dx = \sum_{T \in \mathcal{T}_h} \left\{ \int_T \varepsilon \nabla u \nabla v dx - \int_{\partial T} \varepsilon \cdot \partial_n u \cdot v ds \right\} = \sum_{T \in \mathcal{T}_h} \int_T f \cdot v dx. \tag{1}$$

But we want to introduce the shorter notation equivalently such that

$$\sum_{T \in \mathcal{T}_h} \{ \varepsilon (\nabla u, \nabla v)_T - \varepsilon \langle \partial_n u, v \rangle_{\partial T} \} = \sum_{T \in \mathcal{T}_h} (f, v)\,. \tag{2}$$

Where $\langle \cdot, \cdot \rangle$ is the surface integral operator. Before we contitinue do we want to introduce a alternative method to integrate using edges. Let $v_F \in L^2(\mathcal{F}_h)$ for the set of all facets $\mathcal{F}_h$. Now the surface integral can be rewritten such that

$$\sum_{T\in\mathcal{T}_h} \varepsilon \langle \partial_n u, v_F\rangle = \sum_{E\in\mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F\rangle_E + \varepsilon \langle \partial_{n^-} u, v_F\rangle_E + \sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F\rangle. \tag{3}$$

Here are we using the definitions n^+ and n^- illustrated using figure 1. Lets define some crucial spaces for the DG method

$$V = \left\{(u, u_F) : u \in H^2(\mathcal{T}_h) \cap H^1(\Omega), u_F \in L^2(\mathcal{F}_h)\right\}$$
$$V_h = \left\{(u, u_F) : u \in \mathcal{P}^k(T)\, \forall T \in \mathcal{T}_h, \quad u_F \in \mathcal{P}^k(E)\, \forall E \in \mathcal{F}_h\right\}$$

and now including drichlet conditions using the previous definition

$$V_D = \{(u, u_F) \in V, u_F = u_D \quad \text{on } \Gamma_D\} \quad V_{h,D} = \{(u, u_F) \in V_h, u_F = u_D \quad \text{on } \Gamma_D\}$$
$$V_0 = \{(u, u_F) \in V, u_F = 0 \quad \text{on } \Gamma_D\} \quad V_{h,0} = \{(u, u_F) \in V_h, u_F = 0 \quad \text{on } \Gamma_D\}$$

Defining $(u, u_F) \in V_D$ and $(v, v_F) \in V_0$. Now adding (2) and (3) can we easily see that

$$\sum_{T\in\mathcal{T}_h} \{\varepsilon (\nabla u, \nabla v)_T\} = \sum_{T\in\mathcal{T}_h} (f, v)_T + \sum_{E\in\mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F\rangle_E + \varepsilon \langle \partial_{n^-} u, v_F\rangle_E + \sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F\rangle. \tag{4}$$

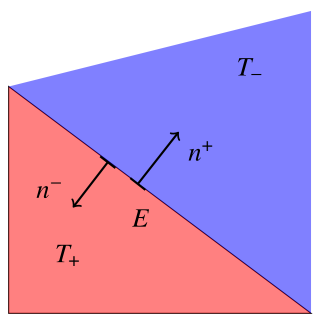

Figure 1: Edge E shared by the triangles T_- and T_+ and the normal unit vectors n^+ and n^-.

Applying the Neumann conditions on Γ_N and Γ_R, can the condition on the exterior facets be rewritten such that

$$\sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F\rangle = \varepsilon \langle g, v_F\rangle_{\Gamma_N} + \varepsilon \langle h - \beta u, v_F\rangle_{\Gamma_R}$$

Keep in mind that we on the exterior boundaries define the integral so $\langle f, v_F\rangle_\Gamma = \int_\Gamma f \cdot v_F \cdot n ds$ for a arbitary neumann boundary function f on some surface Γ. Hence (4) ends up being

$$\sum_{T\in\mathcal{T}_h} \varepsilon (\nabla u, \nabla v) - \sum_{E\in F^{int}} (\varepsilon \langle \partial_{n^+} u, v_F\rangle_E + \varepsilon \langle \partial_{n^-} u, v_F\rangle_E) + \beta \langle \varepsilon u, v_F\rangle_{\Gamma_R} = \sum_{T\in T_h} (f, v)_T + \langle g, v_F\rangle_{\Gamma_N} + \langle h, v_F\rangle_{\Gamma_R}. \tag{5}$$

According to Lehrenfeld 2010 [1] at page 13 on equation (1.2.7) is (5) equivalent to

$$\sum_{T\in\mathcal{T}_h} (\varepsilon \nabla u, \nabla v)_T - \sum_{T\in\mathcal{T}_h} \langle \varepsilon \partial_n u, [\![v]\!]\rangle_{\partial T} + \beta \langle \varepsilon u, v_F\rangle_{\Gamma_R} = \sum_{T\in\mathcal{T}_h} (f, v) + \langle \varepsilon g, v_F\rangle_{\Gamma_N} + \langle \varepsilon h, v_F\rangle_{\Gamma_R} \tag{6}$$

Where, $u, u_F \in V_D$ and $v, v_F \in V_h$ Here is the jump defined simply as $[\![v]\!] = v - v_F$. Remember that $v_F = tr_{\partial T}(v)$. What we see is for (5) and (6) to be equivalent must this be true.

$$\sum_{T\in\mathcal{T}_h} \langle \varepsilon \partial_n u, [\![v]\!]\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \langle \varepsilon \partial_n u, v\rangle_{\partial T} - \langle \varepsilon \partial_n u, v_F\rangle_{\partial T} = \sum_{E\in\mathcal{F}^{int}} \varepsilon \left(\langle \partial_{n^+} u, v_F\rangle_E + \langle \partial_{n^-} u, v_F\rangle_E\right). \tag{7}$$

Since $(u, u_F) \in V$ is has to be continious, hence the jump is $[\![u]\!] = 0$ for the correct solution. Hence, adding $-\langle \varepsilon \partial_n v, [\![u]\!]\rangle_{\partial T}$ for symmetry and $\tau_h \langle \varepsilon [\![u]\!], [\![v]\!]\rangle_{\partial T}$ for stability with some stabilization parameter τ_h for each $T \in \mathcal{T}_h$. This can be added to lhs on (6) such that,

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h} (\varepsilon\nabla u, \nabla v)_t - \sum_{T\in\mathcal{T}_h} \{&\langle \varepsilon\partial_n u, [\![v]\!]\rangle_{\partial T} - \langle \varepsilon\partial_n v, [\![u]\!]\rangle_{\partial T} + \tau_h \langle \varepsilon [\![u]\!], [\![v]\!]\rangle_{\partial T} \} \\
&+ \beta \left\langle \varepsilon u, v_f \right\rangle_{\Gamma_R} \\
&= \sum_{T\in\mathcal{T}_h} (f, v) + \langle \varepsilon g, v_F\rangle_{\Gamma_N} + \langle \varepsilon h, v_F\rangle_{\Gamma_R}
\end{aligned} \tag{8}
$$

Finally, we can now construct the discrete system. Let now $u, u_F \in V_{h,D}$ and $v, v_F \in V_{h,0}$ be the discretized spaces. Using what we have in (6) can we define

$$
\begin{aligned}
F(v, v_F) &= \sum_{T\in\mathcal{T}_h} (f, v) + \langle \varepsilon g, v_F\rangle_{\Gamma_N} + \langle \varepsilon h, v_F\rangle_{\Gamma_R} \\
B(u, u_F, v, v_F) &= \sum_{T\in\mathcal{T}_h} (\varepsilon\nabla u, \nabla v)_t - \sum_{T\in\mathcal{T}_h} \{\langle \varepsilon\partial_n u, [\![v]\!]\rangle_{\partial T} - \langle \varepsilon\partial_n [\![u]\!]\rangle_{\partial T} + \tau_h \langle \varepsilon [\![u]\!], [\![v]\!]\rangle_{\partial T} \} + \beta \langle \varepsilon u, v_F\rangle_{\Gamma_R}
\end{aligned}
$$

Hence, the numerical method must solve

$$
B(u, u_F, v, v_F) = F(v, v_F)\,. \tag{9}
$$

3 C^0 Interior Penalty Method for Biharmonic Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega$ be its corresponding boundary. Let the fourth order biharmonic equation have the form,

$$\begin{aligned} \nabla^4 u + \gamma u = f \quad &\text{in } \Omega \\ \partial_n u = g_1(x) \quad &\text{on } \partial\Omega\,. \\ \partial_n \nabla^2 u = g_2(x) \quad &\text{on } \partial\Omega. \end{aligned} \tag{10}$$

We will assume for now that $u \in H^4(\Omega)$, γ is nonnegative constants and $f \in L_2(\Omega)$. We may consider the functions g_1 and g_2 as time independent boundary conditions. Such problems as (10) are often associated with the Cahn-Hilliard model [2] for phase seperation. As a matter of fact, the major difference is that (10) has no time dependencie. However, depending on how Cahn-Hilliard model is time discretized numerically can (10) naturally arise. I refer to [3] for more informastion on this.

Before we start constructing a numerical method, we might want to introduce the basic weak formulation of (10). Now, let the solution space be on the form,

$$V = \left\{ v \in H^2(\Omega) : \partial_n v = g_1 \text{ on } \partial\Omega \right\}..$$

Consider the weak formulation to solve for a $u \in V$ such that

$$a(u, v)_\Omega = F(v). \quad \forall v \in V, \tag{11}$$

where the terms are computed as

$$\begin{aligned} a(u, v)_\Omega &= \int_\Omega \left(\nabla^2 u : \nabla^2 v + \gamma u v\right) dx, \\ F(v)_\Omega &= (f, v)_\Omega - \langle g_2, v\rangle_{\partial\Omega} + \langle \nabla g_1, \nabla v\rangle_{\partial\Omega}\,. \end{aligned}$$

In fact, the solution is unique for $\gamma > 0$. However, for $\gamma = 0$ must we assume the solvability condtion,

$$\int_\Omega f\, dx = \int_{\partial\Omega} g_2\, ds.$$

This condition easily arise when using the substitution $v = 1$ in (11). To handle this, can we extended the solution space

$$V^* = \begin{cases} V & \gamma > 0 \\ \{v \in V : v(p_*) = 0\}\,, & \gamma = 0 \end{cases}$$

where p_* is a corner of the polygonal domain Ω. Thus, the unique solution in $v \in V^*$ belongs to $H^3(\Omega)$ and we get the follwing elliptic regularity estimate [4],

Why is setting test function $v(p_*) = 0$ on corners interesting?

$$|u|_{H^3(\Omega)} \le C_\Omega \left(\|f\|_{L_2(\Omega)} + (1 + \gamma^{\frac{1}{2}}) \cdot \|w\|_{H^4(\Omega)} \right), \quad w \in H^4(\Omega)\,. \tag{12}$$

This regularity estimate may be important for further usecases in terms of error analysis.

To solve this numerically do we want to introduce the C^0 Interior Penalty Method (C0IP), which is a Discontinious Galerkin method (DG) using C^0 finite elements. There is several reasons why we want to apply C^0 instead of the often used C^1 finite elements for fourth order problems. First and foremost is the C^0 finite elements simpler than obtaining C^1 finite elements. Also, compared to other methods similar to the mixed finite element method for the problem (10), C0IP has in fact preserved the symmetric positive definiteness, which means the stability analysis is more straight forward. Finally and most importantly according to [3] can naive use mixed methods of splitting the boundary conditions of the problem (10) produce wrong solutions if Ω is nonconvex.

Let $w, v \in H^4(T)$ and $\mathcal{T}_h$ the simplicial triangulation of Ω. Using the same method as in [3, 4] can we deduce that for every triangle $T \in \mathcal{T}_h$ is

$$\begin{aligned} \left(\nabla^4 w, v\right)_T &= \left\langle \partial_n \nabla^2 w, v\right\rangle_{\partial T} - \left(\nabla\left(\nabla^2 w\right), \nabla v\right)_T \\ &= \left(D^2 w, D^2 v\right)_T + \left\langle \partial_n \nabla^2 w, v\right\rangle_{\partial T} - \langle \partial_n \nabla w, \nabla v\rangle_{\partial T} \\ &= \left(D^2 w, D^2 v\right)_T - \langle \partial_{nt} w, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} w, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 w, v\right\rangle_{\partial T} \end{aligned}$$

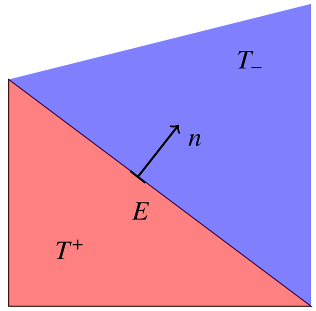

Figure 2: Edge $E \in \mathcal{F}_h$ shared by the triangles $T^+, T^- \in \mathcal{T}_h$ and the normal unit vector n.

Keep in mind that this result naturally arise when defining $\nabla = (\partial_n, \partial_t)$ such that

$$\langle \partial_n \nabla w, \nabla v \rangle_{\partial T} = \langle \partial_{nt} w, \partial_t v \rangle_{\partial T} + \langle \partial_{nn} w, \partial_n v \rangle_{\partial T} \,.$$

Thus, letting $u, v \in H^4(T)$ does this hold ,

$$\left(\nabla^4 u, v\right)_T = \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v \rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v \right\rangle_{\partial T} \,. \tag{13}$$

For global continuity, let $v \in V = \left\{ v \in H^1(\Omega) : v_T \in H^4(T),\ \forall T \in \mathcal{T}_h \right\} \cap C^0(\overline{\Omega})$ and $u \in H^4(\Omega)$ such that,

$$\left(\nabla^4 u, v\right)_\Omega = \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v \rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v \right\rangle_{\partial T} \,. \tag{14}$$

However, this can be simplified to

$$\begin{aligned}
\left(\nabla^4 u, v\right)_\Omega &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{E \in \mathcal{F}^{ext}} \left\langle \partial_n \nabla^2 u, v \right\rangle_E - \langle \partial_{nt} u, \partial_t v \rangle_E + \langle \partial_{nn} u, \partial_n v \rangle_E + \sum_{E \in \mathcal{F}^{int}} \langle \partial_{nn} u, [\![\partial_n v]\!] \rangle_E \\
&= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{E \in \mathcal{F}^{ext}} \langle g_2, v \rangle_E + \langle n g_2, \nabla_n v \rangle_E + \langle \partial_t g_1, \partial_t v \rangle_E + \sum_{E \in \mathcal{F}^{int}} \langle \partial_{nn} u, [\![\partial_n v]\!] \rangle_E
\end{aligned} \tag{15}$$

Where $\mathcal{F}_h^{int}, \mathcal{F}_h^{ext} \subset \mathcal{F}_h$ be the set of interior and exterior facets of the triangulation $\mathcal{T}_h$. Keep in mind that the jump over and edge E, visualized in figure 2, is defined as $[\![a]\!] = a^+ - a^-$ and similarly will the mean be defined as $\{\!\{a\}\!\} = \frac{1}{2}(a^+ + a^-)$. The equivalence of (14) and (15) comes from the following argumentation.

$$\begin{aligned}
\left(\nabla^4 u, v\right)_\Omega &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v \rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v \right\rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T \\
&\quad + \sum_{E \in \mathcal{F}_h^{ext}} \underbrace{\left\langle \partial_n \nabla^2 u, v \right\rangle_E}_{= \langle g_2, v \rangle_E} - \underbrace{\langle \partial_{nt} u, \partial_t v \rangle_E}_{= \langle \partial_t g_1, \partial_t v \rangle} - \underbrace{\langle \partial_{nn} u, \partial_n v \rangle}_{= \langle n g_2, \partial_n v \rangle_E} \\
&\quad + \sum_{E \in \mathcal{F}_h^{int}} \underbrace{\left(\left\langle \partial_{n^+} \nabla^2 u^+, v^+ \right\rangle_E + \left\langle \partial_{n^-} \nabla^2 u^+, v^- \right\rangle_E \right)}_{(I)} + \underbrace{\left(\left\langle \partial_{n^+ t} u^+, \partial_t v^+ \right\rangle_E + \langle \partial_{n^- t} u^-, \partial_t v^- \rangle_E \right)}_{(II)} + \underbrace{\left(\left\langle \partial_{n^+ n^+} u^+, v^+ \right\rangle_E + \langle \partial_{n^- n^-} u^-, v^- \rangle_E \right)}_{(III)}
\end{aligned} .$$

Where integration over all interior edges $\forall E \in \mathcal{F}_h^{int}$ is computed in this way:

$$\begin{aligned}
(I) &= \left\langle \partial_{n^+} \nabla^2 u^+, v^+ \right\rangle_E + \left\langle \partial_{n^-} \nabla^2 u^+, v^- \right\rangle_E = \int_E [\![\partial_n \nabla^2 u \cdot v]\!] = \int_E \{\!\{\partial_n \nabla^2 u\}\!\} \underbrace{[\![v]\!]}_{=0} + \underbrace{[\![\partial_n \nabla^2 u]\!]}_{=0} \{\!\{v\}\!\} = 0 \\
(II) &= \left\langle \partial_{n^+ t} u^+, \partial_t v^+ \right\rangle_E + \langle \partial_{n^- t} u^-, \partial_t v^- \rangle_E = \int_E [\![\partial_{nt} u \cdot \partial_t v]\!] = \int_E \{\!\{\partial_{nt} u\}\!\} \underbrace{[\![\partial_t v]\!]}_{=0} + \underbrace{[\![\partial_{nt} u]\!]}_{=0} \{\!\{\partial_t v\}\!\} = 0 \\
(III) &= \left\langle \partial_{n^+ n^+} u^+, \partial_{n^+} v^+ \right\rangle_E + \langle \partial_{n^- n^-} u^-, \partial_{n^-} v^- \rangle_E = \int_E [\![\partial_{nn} u \cdot \partial_n v]\!] = \int_E \{\!\{\partial_{nn} u\}\!\} \underbrace{[\![\partial_n v]\!]}_{\neq 0} + \underbrace{[\![\partial_{nn} u]\!]}_{=0} \{\!\{\partial_n v\}\!\} = \langle \partial_{nn} u, [\![\partial_n v]\!] \rangle_E
\end{aligned} .$$

Observe that the cancellations in the term (I) appears of the continuity of $v \in V$ and $u \in H^4(\Omega)$ which makes the jumps zero. For the second term (II) does the terms become zero cancelled because the tangential derivative at the edge has no jump. However, The third term (III) is fairly interesting since the discontinuity in normal vector for $v \in V$ is a jump, while the second term is still continious. It can also be raised that $\{\!\{\partial_{nn} u\}\!\} = \partial_{nn} u$ holds by the continuity of $H^4(\Omega)$. Anyhow, the definition of jump of should more interesting when we later weakend the continuity of u during discretization. Hence, (14) and (15) is equivalent.

We can finally start defining the fully discrete formulation. Let the basis be a $\mathcal{P}_2$ Lagrange finite element space so,

$$V_h = \left\{v \in C^0(\Omega) : v_T = v|_T \in P_2(T), \forall T \in \mathcal{T}_h\right\}$$

and

$$V_h^* = \begin{cases} V_h & \text{if } \gamma > 0 \\ \{v \in V_h : v(p_*) = 0\} & \text{if } \gamma = 0 \end{cases}$$

Now, if we choose $u \in V_h$ must we take account that the jump is discrete. We have now the final C0IP formulation. The discretized numerical problem is to solve $w_h \in V_h^*$ such that

$$\mathcal{A}(w_h, v_h) = F(v_h), \quad \forall v_h \in V_h^*.$$

where

$$\begin{aligned} \mathcal{A}(w_h, v_h) = \quad & (\gamma w_h, v_h)_\Omega \\ & + \sum_{T \in \mathcal{T}_h} \left(D^2 w_h, D^2 v_h\right)_T \\ & + \sum_{E \in \mathcal{F}_h^{int}} \langle \{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!]\rangle_E + \langle \{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w]\!]\rangle_E + \tau_E \langle [\![\partial_n w_h]\!], [\![\partial_n v_h]\!]\rangle_E \end{aligned} \tag{16}$$

and

$$F(v_h) = (f, v_h)_\Omega + \sum_{\mathcal{F}^{ext}} \langle g_2, v_h\rangle_E + \langle n g_2, \partial_n v_h\rangle_E + \langle \partial_t g_1, \partial_t v_h\rangle_E \tag{17}$$

Notice that the regulation term with $\tau_E = \sigma/|E|$ is to be determined by respectively a global tuning parameter $\sigma > 0$ and edge length $|E|$. Another key component to the formulation in (16) after introduction of $w_h, v_h \in V_h^*$ is that we expanded $\langle \partial_{nn} w, [\![\partial_n v]\!]\rangle_E \to \langle \{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!]\rangle_E$ since we can longer not guarantee a continious jump. For symmetric purposes we also added $\langle \{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w_h]\!]\rangle_E$.

3.0.1 HC0IP Method copied from NGSolve

We consider the Kirchhoff plate equation: Find $w \in H^2$, such that

$$\int \nabla^2 w : \nabla^2 v = \int f v$$

A conforming method requires C^1 continuous finite elements. But there is no good option available, and thus there is no H^2 conforming finite element space in NGSolve.

$$\sum_T \nabla^2 w : \nabla^2 v - \int_E \{\nabla^2 w\}_{nn} [\partial_n v] - \int_E \{\nabla^2 v\}_{nn} [\partial_n w] + \alpha \int_E [\partial_n w][\partial_n v]$$

[Baker 77, Brenner Gudi Sung, 2010]

We consider its hybrid DG version, where the normal derivative is a new, facet-based variable:

$$\sum_T \nabla^2 w : \nabla^2 v - \int_{\partial T} (\nabla^2 w)_{nn} (\partial_n v - \widehat{v_n}) - \int_{\partial T} (\nabla^2 v)_{nn} (\partial_n w - \widehat{w_n}) + \alpha \int_E (\partial_n v - \widehat{v_n})(\partial_n w - \widehat{w_n})$$

4 Cahn Hilliard Equation on a Closed Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a a 2 -phase system such that $c_0(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0, \infty] \to [0, 1]$ and similarly $c_1(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0, \infty] \to [0, 1]$, where $\mathbf{x}$ is a element of some surface $\boldsymbol{\Omega}$ and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$ at any $\mathbf{x}$ at time t. A property of the restriction is that we now can express c_0 using c_1, with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0, \infty] \to [0, 1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase seperation described by a evolutional differential equation [2] using a model based on chemical energy of the substances. However, further development has been done [5] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface Γ, so that $c(\mathbf{x}, t) : \Gamma \times [0, T] \to [0, 1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma \left(M \nabla_\Gamma \left(f_0' - \varepsilon^2 \nabla_\Gamma^2 c \right) \right) = 0 \quad \text{on } \Gamma. \tag{18}$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n} \nabla c)\, \mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2. The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divaded by the total surface area of Γ. Here is the mobility M often derived such that is is dependent on c and is crucial for the result during a possible coarsering event [5]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [5] be nonconvex and nonlinear.

A important observation is that equation (18) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

5 Appendix

5.1 The Space $L^2(\Omega)$

Using the definition from [6] and we let Ω be a an open set in $\mathbb{R}^d$ and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \to \mathbb{R}$. Then is $L^2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^p(\Omega)} = \int_\Omega uv.$$

If the integral is finite do we say that $u, v \in L^p(\Omega)$.

5.2 The Space $H^m(\Omega)$, $m > 1$

Again using the definition from [6]. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha \geq 0$, such that $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Now we define the space

$$H^m(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall\alpha : |\alpha| \leq m\right\}.$$

Suppose that u, v is measurable functions. We can now define $u \in H^m(\Omega)$ the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)} + \sum_{k=1}^{m} |u|^2_{H^k(\Omega)}\right), \quad |u|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|^2_{L^2(\Omega)}}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v$$

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