Project Thesis
Solving Cahn Hilliard Equation

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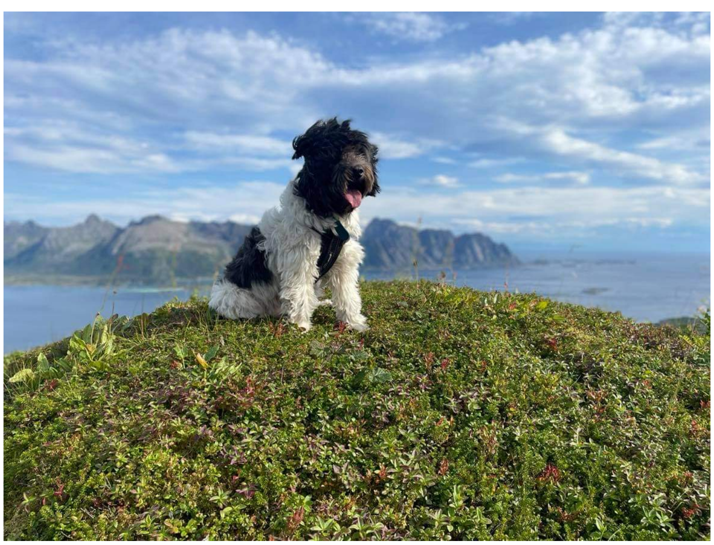

1 Introduction

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PLAN FOR REPORT

1) Introduction

2) DG for poission problem
- classical DG
- (Sability/ apriori error analysis)
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- HDG for Possion Equation

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2 DG for Possion Problem

2.1 Possion Problem

Lets define the problem

$$\begin{aligned} -\varepsilon \nabla u &= f \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \Gamma_D \\ \partial_n u &= g \quad \text{on } \Gamma_N \\ \partial_n u + \beta u &= h \quad \text{on } \Gamma_R \end{aligned}$$

Here is $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$.

2.2 Classical DG

2.3 Hybrid DG Method

We want to write this on a weak form. Let the spaces we work on be

$$H^1(\mathcal{T}_h) = \left\{ u \in L^2(\Omega), u \in H^1(T)\, \forall T \in \mathcal{T}_h \right\}$$

.

For the problem to be discontinuous do we define the trial and test function to be $u \in H^1(\Omega)$ and $v \in H^1(\mathcal{T}_h)$. Thus,

$$-\sum_{T \in \mathcal{T}_h} \int_T \varepsilon \nabla^2 u \cdot v dx = \sum_{T \in \mathcal{T}_h} \left\{ \int_T \varepsilon \nabla u \nabla v dx - \int_{\partial T} \varepsilon \cdot \partial_n u \cdot v ds \right\} = \sum_{T \in \mathcal{T}_h} \int_T f \cdot v dx. \tag{1}$$

But we want to introduce the shorter notation equivalently such that

$$\sum_{T \in \mathcal{T}_h} \left\{ \varepsilon (\nabla u, \nabla v)_T - \varepsilon \langle \partial_n u, v \rangle_{\partial T} \right\} = \sum_{T \in \mathcal{T}_h} (f, v). \tag{2}$$

Where $\langle \cdot, \cdot \rangle$ is the surface integral operator. Before we contitinue do we want to introduce a alternative method to integrate using edges. Let $v_F \in L^2(\mathcal{F}_h)$ for the set of all facets $\mathcal{F}_h$. Now the surface integral can be rewritten such that

$$\sum_{T\in\mathcal{T}_h} \varepsilon \langle \partial_n u, v_F \rangle = \sum_{E\in\mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E + \sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle. \tag{3}$$

Here are we using the definitions n^+ and n^- illustrated using figure 1. Lets define some crucial spaces for the DG method

$$V = \left\{ (u, u_F) : u \in H^2(\mathcal{T}_h) \cap H^1(\Omega), u_F \in L^2(\mathcal{F}_h) \right\}$$
$$V_h = \left\{ (u, u_F) : u \in \mathcal{P}^k(T)\, \forall T \in \mathcal{T}_h, \quad u_F \in \mathcal{P}^k(E)\, \forall E \in \mathcal{F}_h \right\}$$

and now including drichlet conditions using the previous definition

$$V_D = \{(u, u_F) \in V, u_F = u_D \quad \text{on } \Gamma_D\} \quad V_{h,D} = \{(u, u_F) \in V_h, u_F = u_D \quad \text{on } \Gamma_D\}$$
$$V_0 = \{(u, u_F) \in V, u_F = 0 \quad \text{on } \Gamma_D\} \quad V_{h,0} = \{(u, u_F) \in V_h, u_F = 0 \quad \text{on } \Gamma_D\}$$

Defining $(u, u_F) \in V_D$ and $(v, v_F) \in V_0$. Now adding (2) and (3) can we easily see that

$$\sum_{T\in\mathcal{T}_h} \{\varepsilon (\nabla u, \nabla v)_T\} = \sum_{T\in\mathcal{T}_h} (f, v)_T + \sum_{E\in\mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E + \sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle. \tag{4}$$

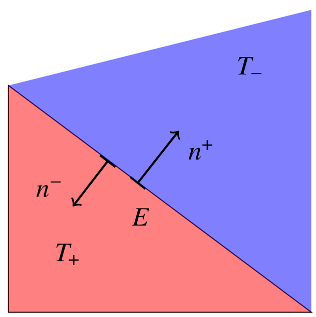

Figure 1: Edge E shared by the triangles T_- and T_+ and the normal unit vectors n^+ and n^-.

Applying the Neumann conditions on Γ_N and Γ_R, can the condition on the exterior facets be rewritten such that

$$\sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle = \varepsilon \langle g, v_F \rangle_{\Gamma_N} + \varepsilon \langle h - \beta u, v_F \rangle_{\Gamma_R}$$

Keep in mind that we on the exterior boundaries define the integral so $\langle f, v_F \rangle_\Gamma = \int_\Gamma f \cdot v_F \cdot n ds$ for a arbitary neumann boundary function f on some surface Γ. Hence (4) ends up being

$$\sum_{T\in\mathcal{T}_h} \varepsilon (\nabla u, \nabla v) - \sum_{E\in F^{int}} (\varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E) + \beta \langle \varepsilon u, v_F \rangle_{\Gamma_R} = \sum_{T\in T_h} (f, v)_T + \langle g, v_F \rangle_{\Gamma_N} + \langle h, v_F \rangle_{\Gamma_R}. \tag{5}$$

According to Lehrenfeld 2010 [1] at page 13 on equation (1.2.7) is (5) equivalent to

$$\sum_{T\in\mathcal{T}_h} (\varepsilon \nabla u, \nabla v)_T - \sum_{T\in\mathcal{T}_h} \langle \varepsilon \partial_n u, [\![v]\!] \rangle_{\partial T} + \beta \langle \varepsilon u, v_F \rangle_{\Gamma_R} = \sum_{T\in\mathcal{T}_h} (f, v) + \langle \varepsilon g, v_F \rangle_{\Gamma_N} + \langle \varepsilon h, v_F \rangle_{\Gamma_R} \tag{6}$$

Where, $u, u_F \in V_D$ and $v, v_F \in V_h$ Here is the jump defined simply as $[\![v]\!] = v - v_F$. Remember that $v_F = tr_{\partial T}(v)$. What we see is for (5) and (6) to be equivalent must this be true.

$$\sum_{T\in\mathcal{T}_h} \langle \varepsilon \partial_n u, [\![v]\!] \rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \langle \varepsilon \partial_n u, v \rangle_{\partial T} - \langle \varepsilon \partial_n u, v_F \rangle_{\partial T} = \sum_{E\in\mathcal{F}^{int}} \varepsilon \left(\langle \partial_{n^+} u, v_F \rangle_E + \langle \partial_{n^-} u, v_F \rangle_E \right). \tag{7}$$

Since $(u, u_F) \in V$ is has to be continious, hence the jump is $[\![u]\!] = 0$ for the correct solution. Hence, adding $-\langle \varepsilon \partial_n v, [\![u]\!] \rangle_{\partial T}$ for symmetry and $\tau_h \langle \varepsilon [\![u]\!], [\![v]\!] \rangle_{\partial T}$ for stability with some stabilization parameter τ_h for each $T \in \mathcal{T}_h$. This can be added to lhs on (6) such that,

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h}(\varepsilon\nabla u,\nabla v)_t - &\sum_{T\in\mathcal{T}_h}\{\langle\varepsilon\partial_n u,[\![v]\!]\rangle_{\partial T}-\langle\varepsilon\partial_n v,[\![u]\!]\rangle_{\partial T}+\tau_h\langle\varepsilon[\![u]\!],[\![v]\!]\rangle_{\partial T}\} \\
&+\beta\left\langle\varepsilon u,v_f\right\rangle_{\Gamma_R} \\
&=\sum_{T\in\mathcal{T}_h}(f,v)+\langle\varepsilon g,v_F\rangle_{\Gamma_N}+\langle\varepsilon h,v_F\rangle_{\Gamma_R}
\end{aligned} \tag{8}
$$

Finally, we can now construct the discrete system. Let now $u,u_F\in V_{h,D}$ and $v,v_F\in V_{h,0}$ be the discretized spaces. Using what we have in (6) can we define

$$
\begin{aligned}
F(v,v_F)&=\sum_{T\in\mathcal{T}_h}(f,v)+\langle\varepsilon g,v_F\rangle_{\Gamma_N}+\langle\varepsilon h,v_F\rangle_{\Gamma_R} \\
B(u,u_F,v,v_F)&=\sum_{T\in\mathcal{T}_h}(\varepsilon\nabla u,\nabla v)_t-\sum_{T\in\mathcal{T}_h}\{\langle\varepsilon\partial_n u,[\![v]\!]\rangle_{\partial T}-\langle\varepsilon\partial_n[\![u]\!]\rangle_{\partial T}+\tau_h\langle\varepsilon[\![u]\!],[\![v]\!]\rangle_{\partial T}\}+\beta\langle\varepsilon u,v_F\rangle_{\Gamma_R}
\end{aligned}
$$

Hence, the numerical method must solve

$$
B(u,u_F,v,v_F)=F(v,v_F). \tag{9}
$$

3 C^0 Interior Penalty Method for Biharmonic Equation

3.1 Introduction of the Boundary Value Problem

In this section do we want to establish a numerical method to fourth order equations. Instead of embarking on the special case of surface PDE described in (19) can we establish a general numerical theory on $\mathbb{R}^2$, which we later can generalize on closed surface later. Assume that we restrict ourself to a compact surface $\Omega\in\mathbb{R}^2$ and let $f\in L^2(\Omega)$ as defined in 6.1. Let say we want to solve the equation on the form.

$$
\begin{aligned}
\Delta^2 u-\beta\Delta u+\gamma u&=f\quad \beta,\gamma\geq 0 \\
\frac{\partial u}{\partial n}&=0\quad \text{on }\Omega \\
\frac{\partial\Delta u}{\partial n}&=q\quad \text{on }\partial\Omega
\end{aligned}. \tag{10}
$$

For convenience are the boundary condition q chosen to be defined via a $\phi\in H^4(\Omega)$ such that $q=\frac{\partial\Delta\phi}{\partial n}$ so $\frac{\partial\phi}{\partial n}=0.\ \partial\Omega$.

3.2 Weak Formulation

We want to rewrite (10) on weak formulation. Now define the Hilbert space

$$
V=\left\{v\in H^2(\Omega):\frac{\partial v}{\partial n}=0\quad\text{on }\partial\Omega\right\}.
$$

It can be shown [2] that a convinient form is to write it as

$$
\begin{aligned}
a(u,v)&=(f,v)_{L^2(\Omega)}-(q,v)_{L^2(\partial\Omega)} \\
&=\int_\Omega D^2w:D^2v\,dx+\int_\Omega\nabla w\nabla v\,dx+\int_\Omega\gamma w\cdot v\,dx.
\end{aligned} \tag{11}
$$

For all $\forall v\in V$, where

$$
D^2w:D^2v=\sum_{i,j=1}^{2}\frac{\partial^2 w}{\partial x_i\partial x_j}\cdot\frac{\partial^2 v}{\partial x_i\partial x_j}.
$$

In fact, according to [2] can it be shown that the problem has a unique solution if and only if $\gamma>0$. However, in the case where $\gamma=0$ can we provoke a unique solution by introducing the condition

$$
\int_\Omega f\,dx=\int_{\partial\Omega}q\,ds
$$

Taking this into account can we expand the solution space such that

$$V^* = \begin{cases} V, & \text{if } \gamma > 0 \\ \{v \in V : v(p^*) = 0\}, & \text{if } \gamma = 0 \end{cases}$$

Where p^* is a corner in Ω. In fact, now all solutions of (11) exists in V^*.

3.3 Construction of C^0 Interior Penalty Method

We want to construct a C^0 interior penalty method based on C^0 Lagrange elements. Assume $\mathcal{T}_h$ be a triangulation of Ω and V_h be the à $\mathcal{P}_2$ Lagrange finite element space associated with $\mathcal{T}_h$

$$V_h = \left\{v \in C\left(\overline{\Omega}\right) : v_T = v|_T \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h\right\}$$

So that we can earn a similar space for the approximated solution space ,

$$V_h^* = \begin{cases} V_h, & \text{for } \gamma > 0 \\ \{v \in V_h : v(p^*) = 0\} & \text{for } \gamma = 0. \end{cases}$$

Here is p^* again a corner in Ω. Let us now generalize the Hilbert space as well to the approximated solution space by defining

$$H^k(\Omega, \mathcal{T}_h) = \left\{H^1(\Omega) : v_T \in H^k(T) \quad \forall T \in \mathcal{T}_h\right\}.$$

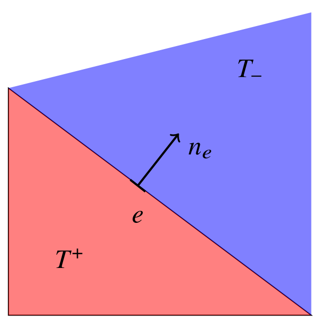

Figure 2: Edge e shared by the triangles T_- and T_+ and the normal unit vector n_e.

Now assume that that $e \in \mathcal{E}_h^i$ is shared between two triangles $T_-, T_+ \in \mathcal{T}_h$. Then we can assume that the unit normal from T_- to T_+ is described as n_e as illustrated in figure 2. Finally, we now want to define jumps internally,

$$\left[\!\left[\frac{\partial v_h}{\partial n_e}\right]\!\right] = \frac{\partial v_{T_+}}{\partial n_e}|_e - \frac{\partial v_{T_-}}{\partial n_e}|_e, \quad \forall v \in H^2(\Omega, \mathcal{T}_h)$$
$$\left[\!\left[\frac{\partial^2 v_h}{\partial n_e^2}\right]\!\right] = \frac{\partial^2 v_{T_+}}{\partial n_e^2}|_e - \frac{\partial^2 v_{T_-}}{\partial n_e^2}|_e \quad \forall v \in H^3(\Omega, \mathcal{T}_h).$$

And similarly for means internally,

$$\left\{\!\!\left\{\frac{\partial v_{T_-}}{\partial n_e}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial v_{T_+}}{\partial n_e}|_e + \frac{\partial v_{T_-}}{\partial n_e}|_e\right) \quad \forall v \in H^2(\Omega, \mathcal{T}_h)$$
$$\left\{\!\!\left\{\frac{\partial^2 v_h}{\partial n_e^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_{T_+}}{\partial n_e^2}|_e + \frac{\partial^2 v_{T_-}}{\partial n_e^2}|_e\right) \quad \forall v \in H^3(\Omega.\mathcal{T}_h),$$

Let the edges along the boundary be defined as $e \in \mathcal{E}_h^b$ along a some boundary triangle $\mathcal{T}_h$. We can then define the jump and mean as

$$\left[\!\left[\frac{\partial v_h}{\partial n_e}\right]\!\right] = -\frac{\partial v_T}{\partial n_e}|_e \quad \forall v \in H^2(\Omega, \mathcal{T}_h)$$
$$\left\{\!\!\left\{\frac{\partial^2 v_h}{\partial n_e^2}\right\}\!\!\right\} = \frac{\partial v_T}{\partial n_e}|_e \quad \forall v \in H^3(\Omega, \mathcal{T}_h)$$

Using the results from [2] can we formulate the discrete formulation the boundary value problem (10) using C^0 interior penalty method. Our goals is to find a $u_h \in V_h^*$ such that this is true,

$$\mathcal{A}\left(u_h, v_h\right) = \left(f, v_h\right)_{L^2(\Omega)} - \left(q, v_h\right)_{L^2(\partial\Omega)} \quad \forall v_h \in V_h^*. \tag{12}$$

Where $w_h, v_h \in V_h$ and

$$\begin{aligned} \mathcal{A}\left(w_h, v_h\right) = & \sum_{T \in \mathcal{T}_h} \int_T D^2 w_h : D^2 v_h \\ & + \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w_h}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\!\right] ds \\ & + \sum_{e \in \mathcal{E}_h} \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w_h}{\partial n_e} \right]\!\!\right] ds \\ & + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\!\left[\frac{\partial w_h}{\partial n_e} \right]\!\!\right] \left[\!\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\!\right] ds \\ & + \int_\Omega \beta \nabla w_h \cdot \nabla v_h dx + \int_\Omega \gamma w_h v_h dx. \end{aligned} \tag{13}$$

The notation $|e|$ is to describe the length of the edge e and $\sigma \geq 1$ is a penalty parameter.

4 C^0 Interior Penalty Method for Biharmonic Equation

We will now try to formulate and analyze a similar discontinious Galerkin approach for the biharmonic equation

4.1 Hybrid DG Biharmonic Equation

Let us again define the problem

$$\begin{aligned} \nabla^4 u &= f \quad \text{in } \Omega \\ \partial_n u &= \partial_n \nabla^2 u = 0 \quad \text{on } \partial\Omega \end{aligned}. \tag{14}$$

In fact, we must also assume the solvability condtion $\int_\Omega f dx = 0$ to obtain a unique solution according to equation (2.4) in Brenner [3], which also can be related to equation (3.4) in Gu [2].

Anyhow, let us first define the Hilbert space of the discrete solution

$$\begin{aligned} H^1(\mathcal{T}_h) &= \left\{ v \in L_2(\Omega) : v \mid_T \in H^1(T) \, \forall T \in \mathcal{T}_h \right\} \\ V &= \left\{ v \in H^2(\Omega) : \partial_n v = 0 \text{ and } \partial_n \nabla^2 v = 0 \text{ on } \partial\Omega \right\} \end{aligned}$$

Probably need to work more on this argumentation

Let us now define $u, v \in V$.

4.1.1 HDG Method

Let us now define our workspace using the spaces

$$\begin{aligned} V &= \left\{ (u, u_F) : u \in H^4(\mathcal{T}_h) \cap H^1(\Omega) \right\} \\ V_h &= \left\{ (u, u_F) : u \in \mathcal{P}^k(T) \, \forall T \in \mathcal{T}, u_F \in \mathcal{P}^k(E) \, \forall E \in \mathcal{F}_h \right\} \end{aligned}$$

and the ones including the null drichlet conditions

$$\begin{aligned} V_0 &= \{ (u, u_F) \in V : \quad u = 0, u_F = 0 \text{ on } \partial\Omega \}, \\ V_{0,h} &= \{ (u, u_F) \in V_h : \quad u = 0, u_F = 0 \text{ on } \partial\Omega \}. \end{aligned}$$

Let $(u, u_F) \in V$

$$\begin{aligned} \sum_T (\nabla^4 u, v) &= \sum_T (\nabla^3 u, \nabla v)_T + \left\langle \partial_n \nabla^2 u, v \right\rangle_{\partial\Omega} \\ &= \sum_T (\nabla^2 u, \nabla^2 v)_T - \langle \partial_n \nabla u, \nabla v \rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v \right\rangle_{\partial T} \end{aligned}$$

4.1.2 Basic DG method

Let $w, v \in H^4(T)$. Using the same method as in equation (3.6) in [2] can we deduce that for every triangle $T \in \mathcal{F}_h$

$$\begin{aligned} (\nabla^4 w, v)_T &= \left\langle \partial_n \nabla^2 w, v \right\rangle_{\partial T} - \left(\nabla \left(\nabla^2 w \right), \nabla v \right)_T \\ &= \left(D^2 w, D^2 v \right)_T + \left\langle \partial_n \nabla^2 w, v \right\rangle_{\partial T} - \langle \partial_n \nabla w, \nabla v \rangle_{\partial T} \\ &= \left(D^2 w, D^2 v \right)_T - \langle \partial_{nt} w, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} w, \partial_n v \rangle_{\partial T} + \left\langle \partial_n \nabla^2 w, v \right\rangle \end{aligned}$$

Keep in mind that this is a results by defining $\nabla = (\partial_n, \partial_t)$ such that $\langle \partial_n \nabla w, \nabla v \rangle_{\partial T} = \langle \partial_{nt} w, \partial_t v \rangle_{\partial T} + \langle \partial_{nn} w, \partial_n v \rangle_{\partial T}$. Thus, letting $u, v \in H^4(T)$ does this hold for local continuity

$$(\nabla^4 u, v)_T = \left(D^2 u, D^2 v \right)_T - \langle \partial_{nt} u, \partial_t v \rangle_{\partial t} - \langle \partial_{nn} u, \partial_n v \rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v \right\rangle_{\partial T}. \tag{15}$$

For global continuity does it end up with so that $v \in \left\{ v \in H^1(\Omega) : v_T \in H^4(T), \ \forall T \in \mathcal{T}_h \right\} \cap C^0(\overline{\Omega})$ such that

$$(\nabla^4 u, v)_\Omega = \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v \right)_T + \sum_{E \in \mathcal{F}^{ext}} \left\langle \partial_n \nabla^2 u, v \right\rangle_E - \langle \partial_{nt} u, \partial_n v \rangle_E + \sum_{E \in \mathcal{F}^{int}} \left\langle \partial_{nn} u, [\![\partial_{n_e} v]\!] \right\rangle_E. \tag{16}$$

(This comes from a similar equation (3.7) given in Gu [2]. What we see is that for (15) and (16) to be equivalent on normal and global form must this be true

$$\sum_{T\in\mathcal{T}_h} -\langle\partial_{nt}u,\partial_t v\rangle_{\partial T} - \langle\partial_{nn}u,\partial_n v\rangle_{\partial T} + \left\langle\partial_n\nabla^2 u, v\right\rangle_{\partial T} = \sum_{E\in\mathcal{F}^{ext}} \left\langle\partial_n\nabla^2 u, v\right\rangle_E - \langle\partial_{nt}u,\partial_n v\rangle_E + \sum_{E\in\mathcal{F}^{int}} \left\langle\partial_{nn}u, \llbracket\partial_{n_e}v\rrbracket\right\rangle_E$$

(a) Here is what is happening in Gu [2]. Let $w_h, v_h \in V_h = \left\{v\in C\left(\overline{\Omega}\right) : v_T = v\mid_T \in \mathcal{P}_2(T) \quad \forall T\in\mathcal{T}_h\right\}$. Anyhow, **assuming** that this equation holds can we introduce the numerical correction term,

$$\sum_{E\in\mathcal{F}_h} \tau_h \left\langle\llbracket\partial_{n_e}w_h\rrbracket, \llbracket\partial_{n_e}v_h\rrbracket\right\rangle_E$$

Do some research on the correct stability and symmetry term and why this is necessarry.

Where τ_h is to be determined based on each triangulation.

Keep in mind that the jump is defined as $\llbracket\partial_{ne}v_h\rrbracket = n_e\left(\nabla v_+ - \nabla v_-\right)$. We have now the basic DG method

$$\begin{aligned}\mathcal{A}(w_h,v_h) &= \sum_{T\in\mathcal{T}_h}\left(D^2w_h, D^2v_h\right)_T \\ &+ \sum_{E\in\mathcal{F}_h}\left\langle\left\{\!\!\left\{\partial_{n_en_e}w_h\right\}\!\!\right\}, \llbracket\partial_{n_e}v_h\rrbracket\right\rangle_E + \left\langle\left\{\!\!\left\{\partial_{n_en_e}v_h\right\}\!\!\right\}, \llbracket\partial_{n_e}w\rrbracket\right\rangle_E + \tau_h\left\langle\llbracket\partial_{n_e}w_h\rrbracket, \llbracket\partial_{n_e}v_h\rrbracket\right\rangle_E\end{aligned}. \tag{17}$$

and

$$F(v_h) = (f, v_h)_\Omega. \tag{18}$$

Hence, the discretized numerical problem is to solve

$$\mathcal{A}(w_h,v_h) = F(v_h), \quad w_h, v_h \in V_h = \left\{v\in C\left(\overline{\Omega}\right) : v_T = v\mid_T\in\mathcal{P}_2(T) \quad \forall T\in\mathcal{T}_h\right\}.$$

In Gu [2] (eq 3.10 p.31) they introduce

$$\begin{aligned}\left(D^2v : D^2w\right)_T &= \langle\partial_{nt}v,\partial_t w\rangle_{\partial T} + \langle\partial_{nn}v\partial_n, w\rangle_{\partial T}. \\ &= \sum_{i=1}^{3}\partial_{n_it_i}v\int_{E_i}\partial_t w\,ds + \langle\partial_{nn}v,\partial_n w\rangle_{\partial T} \\ &= \langle\partial_{nn}v,\partial_n w\rangle_{\partial T}\end{aligned}$$

Using the identity $ab + cd = \frac{1}{2}(a+c)(b+d) + \frac{1}{2}(a-c)(b-d)$ they end up with the equations

$$\sum_{T\in\mathcal{T}_h}\left(D^2v : D^2w\right)_T = -\sum_{E\in\mathcal{F}^{int}}\left\{\!\!\left\{\partial_{n_en_e}v\right\}\!\!\right\}\llbracket\partial_{n_e}w\rrbracket - \llbracket\partial_{n_en_e}v\rrbracket\left\{\!\!\left\{\partial_{n_e}w\right\}\!\!\right\}$$

4.1.3 HC0IP Method copied from NGSolve

We consider the Kirchhoff plate equation: Find $w\in H^2$, such that

$$\int\nabla^2 w : \nabla^2 v = \int fv$$

A conforming method requires C^1 continuous finite elements. But there is no good option available, and thus there is no H^2 conforming finite element space in NGSolve.

$$\sum_T\nabla^2w : \nabla^2 v - \int_E\{\nabla^2 w\}_{nn}[\partial_n v] - \int_E\{\nabla^2 v\}_{nn}[\partial_n w] + \alpha\int_E[\partial_n w][\partial_n v]$$

[Baker 77, Brenner Gudi Sung, 2010]

We consider its hybrid DG version, where the normal derivative is a new, facet-based variable:

$$\sum_T\nabla^2w : \nabla^2v - \int_{\partial T}(\nabla^2 w)_{nn}(\partial_n v - \widehat{v_n}) - \int_{\partial T}(\nabla^2 v)_{nn}(\partial_n w - \widehat{w_n}) + \alpha\int_E(\partial_n v - \widehat{v_n})(\partial_n w - \widehat{w_n})$$

5 Cahn Hilliard Equation on a Closed Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a a 2 -phase system such that $c_0(\mathbf{x}, t) : \boldsymbol{\Omega}\times[0,\infty] \rightarrow [0,1]$ and similarly $c_1(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0,\infty] \rightarrow [0,1]$, where $\mathbf{x}$ is a element of some surface $\boldsymbol{\Omega}$ and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$ at any $\mathbf{x}$ at time t. A property of the restriction is that we now can express c_0 using c_1, with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0,\infty] \rightarrow [0,1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase seperation described by a evolutional differential equation [4] using a model based on chemical energy of the substances. However, further development has been done [5] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface Γ, so that $c(\mathbf{x}, t) : \Gamma \times [0,T] \rightarrow [0,1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma \left(M \nabla_\Gamma \left(f_0' - \varepsilon^2 \nabla_\Gamma^2 c \right) \right) = 0 \quad \text{on } \Gamma. \tag{19}$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n} \nabla c)\, \mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2. The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divaded by the total surface area of Γ. Here is the mobility M often derived such that is is dependent on c and is crucial for the result during a possible coarsering event [5]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [5] be nonconvex and nonlinear.

A important observation is that equation (19) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

6 Appendix

6.1 The Space $L^2(\Omega)$

Using the definition from [6] and we let Ω be a an open set in $\mathbb{R}^d$ and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \to \mathbb{R}$. Then is $L^2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^p(\Omega)} = \int_\Omega uv.$$

If the integral is finite do we say that $u, v \in L^p(\Omega)$.

6.2 The Space $H^m(\Omega)$, $m > 1$

Again using the definition from [6]. Let $\alpha = (\alpha_1, \ldots, \alpha_d), \quad \alpha \geq 0$, such that $|\alpha| = \sum_{i=1}^d \alpha_i$. Now we define the space

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall \alpha : |\alpha| \leq m \right\}.$$

Suppose that u, v is measurable functions. We can now define $u \in H^m(\Omega)$ the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)} + \sum_{k=1}^m |u|^2_{H^k(\Omega)} \right), \quad |u|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|^2_{L^2(\Omega)}}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v$$

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