Project Thesis
Bananas and Troika

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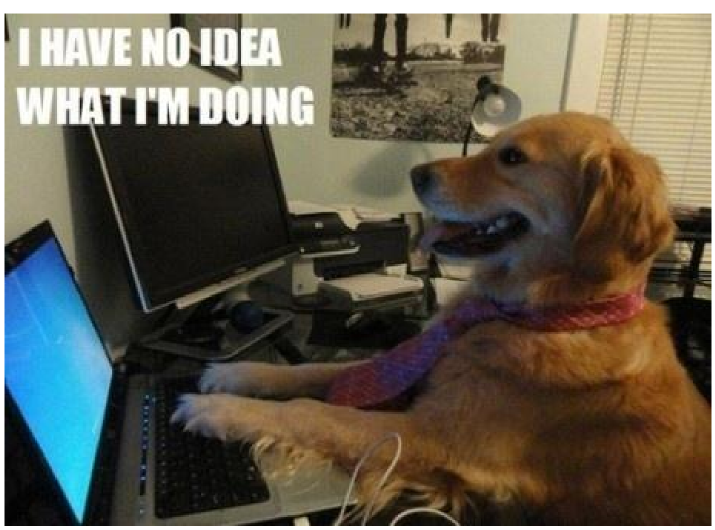

1 Introduction

This guy [1] has done a good job.

2 Cahn Hilliard Equation on a Closed Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a a 2 -phase system such that $c_0(\mathbf{x}, t) : \Omega \times [0, \infty] \to [0, 1]$ and similarly $c_1(\mathbf{x}, t) : \Omega \times [0, \infty] \to [0, 1]$, where $\mathbf{x}$ is a element of some surface Ω and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$ at any $\mathbf{x}$ at time t. A property of the restriction is that we now can express c_0 using c_1, with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \Omega \times [0, \infty] \to [0, 1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase seperation described by a evolutional differential equation [2] using a model based on chemical energy of the substances. However, further development has been done [3] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface Γ, so that $c(\mathbf{x}, t) : \Gamma \times [0, T] \to [0, 1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma \left(M \nabla_\Gamma \left(f_0' - \varepsilon^2 \nabla_\Gamma^2 c \right) \right) = 0 \quad \text{on } \Gamma. \tag{1}$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n} \nabla c)\, \mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2. The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divaded by the total surface area of Γ. Here is the mobility M often derived such that is is dependent on c and is crucial for the result during a possible coarsering event [3]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [3] be nonconvex and nonlinear.

A important observation is that equation (1) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

3 C^0 Interior Penalty Method

In this section do we want to establish a numerical method to fourth order equations. Instead of embarking on the special case of surface PDE described in (1) can weestablish a general numerical theory on $\mathbb{R}^2$ which we can then later generalize on manifolds later.

Assume that we restrict ourself to a surface $\Omega \in \mathbb{R}^2$ and let $f \in L_2(\Omega)$ We want to solve the equation on the form

$$\Delta^2 u = f \quad \text{in } \Omega \tag{2}$$

$$u = \frac{\partial u}{\partial n} = 0 \tag{3}$$

$$. \tag{4}$$

3.1 Weak Formulation

The weak formulation of (2) has the form

$$a(u, v) = \int_\omega f v dx \quad u, v \in V. \tag{5}$$

where

$$a(w, v) = \int_\Omega \nabla^2 w : \nabla^2 v dx. \tag{6}$$

Here do we define $\nabla^2 w : \nabla^2 v$ to be the inner product of the Hessian matrix of w and v, and V is a closed subspace of the sobolev space $H^2(\Omega)$.

3.2 The general C^0 Interior Penalty Method

C^0 Interior penalty method says that

$$a_h(w, v) = \sum_{T \in \mathfrak{T}_h} \int_T \left(\nabla^2 w : \nabla^2 v \right) dx + \sum_{e \in \mathfrak{E}_h^i} \int_e \left\{\!\!\left\{ \frac{\partial w}{\partial n_e} \right\}\!\!\right\} \left[\!\left[\frac{\partial v}{\partial n_e} \right]\!\right] ds$$
$$+ \sum_{e \in \mathfrak{E}_h^i} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial w}{\partial n_e} \right]\!\right] + \sum_{e \in \mathfrak{E}_h^i} \frac{\sigma}{|e|} \int_e \left[\!\left[\frac{\partial w}{\partial n_e} \right]\!\right] \left[\!\left[\frac{\partial v}{\partial n_e} \right]\!\right] ds.$$

where

$$
\begin{aligned}
\left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] &= -n_e \nabla \nu_T, \quad \nu_T = \nu|_T \\
\left\{\!\!\left\{ \frac{\partial^2 u}{\partial n_e^2} \right\}\!\!\right\} &= \frac{\partial^2 u}{\partial n_e^2} \\
\left\{\!\!\left\{ \frac{\partial w}{\partial n_e^2} \right\}\!\!\right\} &= \frac{1}{2}\left(\frac{\partial^2 w_-}{\partial n_e^2} + \frac{\partial^2 w_+}{\partial n_e^2} \right) \\
\left\{\!\!\left\{ \frac{\partial^2 w}{\partial n_e^2} \right\}\!\!\right\} &= \frac{\partial^2 w}{\partial n_e^2}, \quad \text{on edges.}
\end{aligned}
$$

4 Appendix

4.1 $L_2(\Omega)$ space

Using the definition from [4] and we let Ω be a an open set in $\mathbb{R}^d$ and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \to \mathbb{R}$. Then is $L_2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^p(\Omega)} = \int_\Omega uv.$$

References

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[3] Vladimir Yushutin et al. "A computational study of lateral phase separation in biological membranes". In: *International Journal for Numerical Methods in Biomedical Engineering* 35.3 (2019). e3181 cnm.3181, e3181. DOI: https://doi.org/10.1002/cnm.3181. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cnm.3181. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cnm.3181.

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