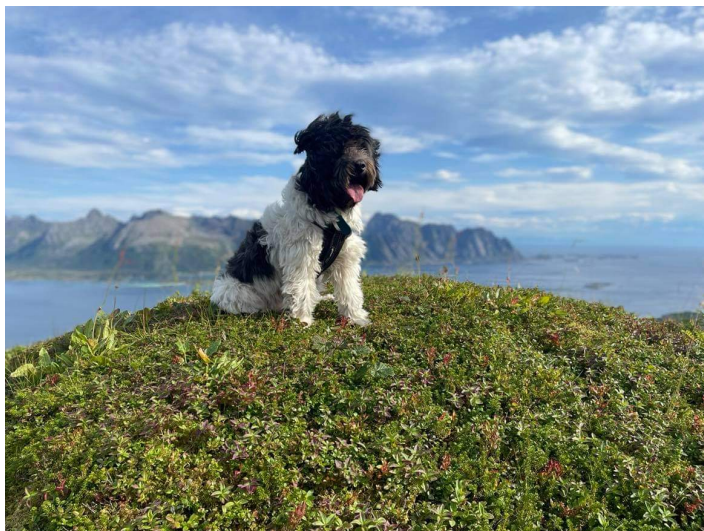


# Project Thesis

## Solving Cahn Hilliard Equation

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# 1 Introduction

PLAN FOR REPORT

1) Introduction

2) DG for poission problem

- classical DG
- (Sability/ apriori error analysis)
- (numerical experiments)
- HDG for Possion Equation

3) Biharmonic Equation (Main part)

- (CIP for biharmonic equation)
- Hybridized CIP for biharmonic equation
- (Stability/ Error Estimate)
- Numerical Experiments
  - \* Manufactured solution
  - \* Convergence rate
  - \* Condition number ( $h^{-4}$ )

4) Cahn-Hilliard Equation

- Combine CIP Biharmonic with Cahn Hilliard

5) Possible Extensions:

- Compare with mixed formulation

## 2 DG for Possion Problem

### 2.1 Possion Problem

Lets define the problem

$$\begin{aligned} -\varepsilon \nabla u &= f && \text{in } \Omega \\ u &= u_D && \text{on } \Gamma_D \\ \partial_n u &= g && \text{on } \Gamma_N \\ \partial_n u + \beta u &= h && \text{on } \Gamma_R \end{aligned}$$

Here is  $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ .

### 2.2 Classical DG

### 2.3 Hybrid DG Method

We want to write this on a weak form. Let the spaces we work on be

$$H^1(\mathcal{T}_h) = \{u \in L^2(\Omega), u \in H^1(T) \forall T \in \mathcal{T}_h\}$$

For the problem to be discontinuous do we define the trial and test function to be  $u \in H^1(\Omega)$  and  $v \in H^1(\mathcal{T}_h)$ . Thus,

$$-\sum_{T \in \mathcal{T}_h} \int_T \varepsilon \nabla^2 u \cdot v dx = \sum_{T \in \mathcal{T}_h} \left\{ \int_T \varepsilon \nabla u \nabla v dx - \int_{\partial T} \varepsilon \cdot \partial_n u \cdot v ds \right\} = \sum_{T \in \mathcal{T}_h} \int_T f \cdot v dx. \quad (1)$$

But we want to introduce the shorter notation equivalently such that

$$\sum_{T \in \mathcal{T}_h} \{ \varepsilon (\nabla u, \nabla v)_T - \varepsilon \langle \partial_n u, v \rangle_{\partial T} \} = \sum_{T \in \mathcal{T}_h} (f, v). \quad (2)$$

Where  $\langle \cdot, \cdot \rangle$  is the surface integral operator. Before we continue we want to introduce an alternative method to integrate using edges. Let  $v_F \in L^2(\mathcal{F}_h)$  for the set of all facets  $\mathcal{F}_h$ . Now the surface integral can be rewritten such that

$$\sum_{T \in \mathcal{T}_h} \varepsilon \langle \partial_n u, v_F \rangle = \sum_{E \in \mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E + \sum_{E \in \mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle. \quad (3)$$

Here we are using the definitions  $n^+$  and  $n^-$  illustrated using figure 1. Let's define some crucial spaces for the DG method

$$V = \{(u, u_F) : u \in H^2(\mathcal{T}_h) \cap H^1(\Omega), u_F \in L^2(\mathcal{F}_h)\}$$

$$V_h = \{(u, u_F) : u \in \mathcal{P}^k(T) \forall T \in \mathcal{T}_h, u_F \in \mathcal{P}^k(E) \forall E \in \mathcal{F}_h\}$$

What is the intuition of a polynomial  $\mathcal{P}^k(E)$  along an edge?

and now including Dirichlet conditions using the previous definition

$$V_D = \{(u, u_F) \in V, u_F = u_D \text{ on } \Gamma_D\} \quad V_{h,D} = \{(u, u_F) \in V_h, u_F = u_D \text{ on } \Gamma_D\}$$

$$V_0 = \{(u, u_F) \in V, u_F = 0 \text{ on } \Gamma_D\} \quad V_{h,0} = \{(u, u_F) \in V_h, u_F = 0 \text{ on } \Gamma_D\}$$

Defining  $(u, u_F) \in V_D$  and  $(v, v_F) \in V_0$ . Now adding (2) and (3) can we easily see that

$$\sum_{T \in \mathcal{T}_h} \{\varepsilon \langle \nabla u, \nabla v \rangle_T\} = \sum_{T \in \mathcal{T}_h} (f, v)_T + \sum_{E \in \mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E + \sum_{E \in \mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle. \quad (4)$$

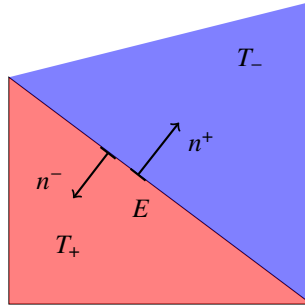


Figure 1: Edge  $E$  shared by the triangles  $T_-$  and  $T_+$  and the normal unit vectors  $n^+$  and  $n^-$ .

Applying the Neumann conditions on  $\Gamma_N$  and  $\Gamma_R$ , can the condition on the exterior facets be rewritten such that

$$\sum_{E \in \mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle = \varepsilon \langle g, v_F \rangle_{\Gamma_N} + \varepsilon \langle h - \beta u, v_F \rangle_{\Gamma_R}$$

Keep in mind that we on the exterior boundaries define the integral so  $\langle f, v_F \rangle_\Gamma = \int_\Gamma f \cdot v_F \cdot n ds$  for an arbitrary Neumann boundary function  $f$  on some surface  $\Gamma$ . Hence (4) ends up being

$$\sum_{T \in \mathcal{T}_h} \varepsilon \langle \nabla u, \nabla v \rangle_T - \sum_{E \in \mathcal{F}^{int}} (\varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E) + \beta \langle \varepsilon u, v_F \rangle_{\Gamma_R} = \sum_{T \in \mathcal{T}_h} (f, v)_T + \langle g, v_F \rangle_{\Gamma_N} + \langle h, v_F \rangle_{\Gamma_R}. \quad (5)$$

According to Lehrenfeld 2010 [1] at page 13 on equation (1.2.7) is (5) equivalent to

$$\sum_{T \in \mathcal{T}_h} (\varepsilon \nabla u, \nabla v)_T - \sum_{T \in \mathcal{T}_h} \langle \varepsilon \partial_n u, [v] \rangle_{\partial T} + \beta \langle \varepsilon u, v_F \rangle_{\Gamma_R} = \sum_{T \in \mathcal{T}_h} (f, v)_T + \langle \varepsilon g, v_F \rangle_{\Gamma_N} + \langle \varepsilon h, v_F \rangle_{\Gamma_R} \quad (6)$$

Where,  $u, u_F \in V_D$  and  $v, v_F \in V_h$ . Here is the jump defined simply as  $[v] = v - v_F$ . Remember that  $v_F = \text{tr}_{\partial T}(v)$ . What we see is for (5) and (6) to be equivalent must this be true.

$$\sum_{T \in \mathcal{T}_h} \langle \varepsilon \partial_n u, [v] \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \langle \varepsilon \partial_n u, v \rangle_{\partial T} - \langle \varepsilon \partial_n u, v_F \rangle_{\partial T} = \sum_{E \in \mathcal{F}^{int}} \varepsilon (\langle \partial_{n^+} u, v_F \rangle_E + \langle \partial_{n^-} u, v_F \rangle_E). \quad (7)$$

why is it true?

Since  $(u, u_F) \in V$  is has to be continious, hence the jump is  $\llbracket u \rrbracket = 0$  for the correct solution. Hence, adding  $-\langle \varepsilon \partial_n v, \llbracket u \rrbracket \rangle_{\partial T}$  for symmetry and  $\tau_h \langle \varepsilon \llbracket u \rrbracket, \llbracket v \rrbracket \rangle_{\partial T}$  for stability with some stabilization parameter  $\tau_h$  for each  $T \in \mathcal{T}_h$ . This can be added to lhs on (6) such that,

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} (\varepsilon \nabla u, \nabla v)_T - \sum_{T \in \mathcal{T}_h} \{ \langle \varepsilon \partial_n u, \llbracket v \rrbracket \rangle_{\partial T} - \langle \varepsilon \partial_n v, \llbracket u \rrbracket \rangle_{\partial T} + \tau_h \langle \varepsilon \llbracket u \rrbracket, \llbracket v \rrbracket \rangle_{\partial T} \} \\ & + \beta \langle \varepsilon u, v_f \rangle_{\Gamma_R} \\ & = \sum_{T \in \mathcal{T}_h} (f, v) + \langle \varepsilon g, v_F \rangle_{\Gamma_N} + \langle \varepsilon h, v_F \rangle_{\Gamma_R} \end{aligned} \quad (8)$$

Finally, we can now construct the discrete system. Let now  $u, u_F \in V_{h,D}$  and  $v, v_F \in V_{h,0}$  be the discretized spaces. Using what we have in (6) can we define

$$\begin{aligned} F(v, v_F) &= \sum_{T \in \mathcal{T}_h} (f, v) + \langle \varepsilon g, v_F \rangle_{\Gamma_N} + \langle \varepsilon h, v_F \rangle_{\Gamma_R} \\ B(u, u_F, v, v_F) &= \sum_{T \in \mathcal{T}_h} (\varepsilon \nabla u, \nabla v)_T - \sum_{T \in \mathcal{T}_h} \{ \langle \varepsilon \partial_n u, \llbracket v \rrbracket \rangle_{\partial T} - \langle \varepsilon \partial_n \llbracket u \rrbracket \rangle_{\partial T} + \tau_h \langle \varepsilon \llbracket u \rrbracket, \llbracket v \rrbracket \rangle_{\partial T} \} + \beta \langle \varepsilon u, v_F \rangle_{\Gamma_R} \\ B(u, u_F, v, v_F) &= F(v, v_F). \end{aligned} \quad (9)$$

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### 3 $C^0$ Interior Penalty Method for Biharmonic Equation

#### 3.1 Introduction of the Boundary Value Problem

In this section do we want to establish a numerical method to fourth order equations. Instead of embarking on the special case of surface PDE described in (16) can we establish a general numerical theory on  $\mathbb{R}^2$ , which we later can generalize on closed surface later. Assume that we restrict ourself to a compact surface  $\Omega \in \mathbb{R}^2$  and let  $f \in L^2(\Omega)$  as defined in 5.1. Let say we want to solve the equation on the form.

$$\begin{aligned} \Delta^2 u - \beta \Delta u + \gamma u &= f \quad \beta, \gamma \geq 0 \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Omega \\ \frac{\partial \Delta u}{\partial n} &= q \quad \text{on } \partial \Omega \end{aligned} \quad (10)$$

For convenience are the boundary condition  $q$  chosen to be defined via a  $\phi \in H^4(\Omega)$  such that  $q = \frac{\partial \Delta \phi}{\partial n}$  so  $\frac{\partial \phi}{\partial n} = 0$ .  $\partial \Omega$ .

#### 3.2 Weak Formulation

We want to rewrite (10) on weak formulation. Now define the Hilbert space

$$V = \left\{ v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega \right\}.$$

It can be shown [2] that a convinient form is to write it as

$$\begin{aligned} a(u, v) &= (f, v)_{L^2(\Omega)} - (q, v)_{L^2(\partial \Omega)} \\ &= \int_{\Omega} D^2 w : D^2 v dx + \int_{\Omega} \nabla w \nabla v dx + \int_{\Omega} \gamma w \cdot v dx. \end{aligned} \quad (11)$$

For all  $\forall v \in V$ , where

$$D^2 w : D^2 v = \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j}.$$

In fact, according to [2] can it be shown that the problem has a unique solution if and only if  $\gamma > 0$ . However, in the case where  $\gamma = 0$  can we provoke a unique solution by introducing the condition

$$\int_{\Omega} f dx = \int_{\partial\Omega} q ds$$

Taking this into account can we expand the solution space such that

$$V^* = \begin{cases} V, & \text{if } \gamma > 0 \\ \{v \in V : v(p^*) = 0\}, & \text{if } \gamma = 0 \end{cases}$$

Where  $p^*$  is a corner in  $\Omega$ . In fact, now all solutions of (11) exists in  $V^*$ .

### 3.3 Construction of $C^0$ Interior Penalty Method

We want to construct a  $C^0$  interior penalty method based on  $C^0$  Lagrange elements. Assume  $\mathcal{T}_h$  be a triangulation of  $\Omega$  and  $V_h$  be the  $\mathcal{P}_2$  Lagrange finite element space associated with  $\mathcal{T}_h$

$$V_h = \left\{ v \in C(\overline{\Omega}) : v_T = v|_T \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h \right\}$$

So that we can earn a similar space for the approximated solution space ,

$$V_h^* = \begin{cases} V_h, & \text{for } \gamma > 0 \\ \{v \in V_h : v(p^*) = 0\} & \text{for } \gamma = 0. \end{cases}$$

Here is  $p^*$  again a corner in  $\Omega$ . Let us now generalize the Hilbert space as well to the approximated solution space by defining

$$H^k(\Omega, \mathcal{T}_h) = \{H^1(\Omega) : v_T \in H^k(T) \quad \forall T \in \mathcal{T}_h\}.$$

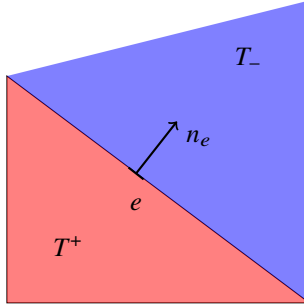


Figure 2: Edge  $e$  shared by the triangles  $T_-$  and  $T_+$  and the normal unit vector  $n_e$ .

Now assume that that  $e \in \mathcal{E}_h^i$  is shared between two triangles  $T_-, T_+ \in \mathcal{T}_h$ . Then we can assume that the unit normal from  $T_-$  to  $T_+$  is described as  $n_e$  as illustrated in figure 2. Finally, we now want to define jumps internally,

$$\begin{aligned} \left[ \left[ \frac{\partial v_h}{\partial n_e} \right] \right] &= \frac{\partial v_{T_+}}{\partial n_e} \Big|_e - \frac{\partial v_{T_-}}{\partial n_e} \Big|_e, \quad \forall v \in H^2(\Omega, \mathcal{T}_h) \\ \left[ \left[ \frac{\partial^2 v_h}{\partial n_e^2} \right] \right] &= \frac{\partial^2 v_{T_+}}{\partial n_e^2} \Big|_e - \frac{\partial^2 v_{T_-}}{\partial n_e^2} \Big|_e \quad \forall v \in H^3(\Omega, \mathcal{T}_h). \end{aligned}$$

And similarly for means internally,

$$\begin{aligned} \left\{ \left\{ \frac{\partial v_{T_-}}{\partial n_e} \right\} \right\} &= \frac{1}{2} \left( \frac{\partial v_{T_+}}{\partial n_e} \Big|_e + \frac{\partial v_{T_-}}{\partial n_e} \Big|_e \right) \quad \forall v \in H^2(\Omega, \mathcal{T}_h) \\ \left\{ \left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\} \right\} &= \frac{1}{2} \left( \frac{\partial^2 v_{T_+}}{\partial n_e^2} \Big|_e + \frac{\partial^2 v_{T_-}}{\partial n_e^2} \Big|_e \right) \quad \forall v \in H^3(\Omega, \mathcal{T}_h), \end{aligned}$$

Let the edges along the boundary be defined as  $e \in \mathcal{E}_h^b$  along a some boundary triangle  $\mathcal{T}_h$ . We can then define the jump and mean as

$$\begin{aligned} \left[ \left[ \frac{\partial v_h}{\partial n_e} \right] \right] &= -\frac{\partial v_T}{\partial n_e} |_e \quad \forall v \in H^2(\Omega, \mathcal{T}_h) \\ \left\{ \left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\} \right\} &= \frac{\partial v_T}{\partial n_e} |_e \quad \forall v \in H^3(\Omega, \mathcal{T}_h) \end{aligned}$$

Using the results from [2] can we formulate the discrete formulation the boundary value problem (10) using  $C^0$  interior penalty method. Our goals is to find a  $u_h \in V_h^*$  such that this is true,

$$\mathcal{A}(u_h, v_h) = (f, v_h)_{L^2(\Omega)} - (q, v_h)_{L^2(\partial\Omega)} \quad \forall v_h \in V_h^*. \quad (12)$$

Where  $w_h, v_h \in V_h$  and

$$\begin{aligned} \mathcal{A}(w_h, v_h) &= \sum_{T \in \mathcal{T}_h} \int_T D^2 w_h : D^2 v_h \\ &+ \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left\{ \frac{\partial^2 w_h}{\partial n_e^2} \right\} \right\} \left[ \left[ \frac{\partial v_h}{\partial n_e} \right] \right] ds \\ &+ \sum_{e \in \mathcal{E}_h} \left\{ \left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\} \right\} \left[ \left[ \frac{\partial w_h}{\partial n_e} \right] \right] ds \\ &+ \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[ \left[ \frac{\partial w_h}{\partial n_e} \right] \right] \left[ \left[ \frac{\partial v_h}{\partial n_e} \right] \right] ds \\ &+ \int_{\Omega} \beta \nabla w_h \cdot \nabla v_h dx + \int_{\Omega} \gamma w_h v_h dx. \end{aligned} \quad (13)$$

The notation  $|e|$  is to describe the length of the edge  $e$  and  $\sigma \geq 1$  is a penalty parameter.

### 3.4 Hybrid DG Biharmonic Equation

Let us again define the problem

$$\begin{aligned} \nabla^4 u &= f \quad \text{in } \Omega \\ \partial_n u &= \partial_n \nabla^2 u = 0 \quad \text{on } \partial\Omega \end{aligned}$$

In fact, we also assume the solvability condtion  $\int_{\Omega} f dx = 0$  to obtain a unique solution according to equation (2.4) in Brenner [3], which also can be related to equation (3.4) in Gu [2]. Anyhow, let us first define the continous space solution

$$\begin{aligned} H^1(\mathcal{T}_h) &= \{v \in L_2(\Omega) : v|_T \in H^1(T) \forall T \in \mathcal{T}_h\} \\ V &= \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\} \end{aligned}$$

Let us now define  $u, v \in V$ .

#### 3.4.1 HDG Method

Let us now define our workspace using the Hilbert spaces

$$\begin{aligned} V &= \{(u, u_F) : u \in H^4(\mathcal{T}_h) \cap H^1(\Omega)\} \\ V_h &= \{(u, u_F) : u \in \mathcal{P}^k(T) \forall T \in \mathcal{T}, u_F \in \mathcal{P}^k(E) \in \mathcal{F}_h\} \end{aligned}$$

and the ones including the null drichlet conditions

$$\begin{aligned} V_0 &= \{(u, u_F) \in V : u = 0, u_F = 0 \text{ on } \partial\Omega\}, \\ V_{0,h} &= \{(u, u_F) \in V_h : u = 0, u_F = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Let  $(u, u_F) \in V$

$$\begin{aligned} \sum_T (\nabla^4 u, v) &= \sum_T (\nabla^3 u, \nabla v)_T + \langle \partial_n \nabla^2 u, v \rangle_{\partial\Omega} \\ &= \sum_T (\nabla^2 u, \nabla^2 v)_T - \langle \partial_n \nabla u, \nabla v \rangle_{\partial T} + \langle \partial_n \nabla^2 u, v \rangle_{\partial T} \end{aligned}$$

### 3.4.2 Basic DG method

Let  $w, v \in H^4(T)$ . Using the same method as in equation (3.6) in [2] can we deduce that for every triangle  $T \in \mathcal{F}_h$

$$\begin{aligned} (\nabla^4 w, v)_T &= \langle \partial_n \nabla^2 w, v \rangle_{\partial T} - (\nabla(\nabla^2 w), \nabla v)_T \\ &= (D^2 w, D^2 v)_T + \langle \partial_n \nabla^2 w, v \rangle_{\partial T} - \langle \partial_n \nabla w, \nabla v \rangle_{\partial T} \\ &= (D^2 w, D^2 v)_T - \langle \partial_{nt} w, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} w, \partial_n v \rangle_{\partial T} + \langle \partial_n \nabla^2 w, v \rangle \end{aligned}$$

Keep in mind that this is a results by defining  $\nabla = (\partial_n, \partial_t)$  such that  $\langle \partial_n \nabla w, \nabla v \rangle_{\partial T} = \langle \partial_{nt} w, \partial_t v \rangle_{\partial T} + \langle \partial_{nn} w, \partial_n v \rangle_{\partial T}$ . Again, using this result and letting  $u, v \in H^4(T)$  can we see that for local continuity is this true

$$(\nabla^4 u, v)_T = (D^2 u, D^2 v)_T - \langle \partial_{nt} u, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v \rangle_{\partial T} + \langle \partial_n \nabla^2 u, v \rangle. \quad (14)$$

For global continuity it ends up with

$$(\nabla^4 u, v)_\Omega = \sum_{T \in \mathcal{T}_h} (D^2 u, D^2 v)_T + \sum_{E \in \mathcal{F}^{ext}} \langle \partial_n \nabla^2 u, v \rangle_E - \langle \partial_{nt} u, \partial_n v \rangle_E + \sum_{E \in \mathcal{F}^{int}} \langle \partial_{nn} u, \llbracket \partial_{ne} v \rrbracket \rangle_E. \quad (15)$$

What we see is that for (14) and (15) to be equivalent on normal and global form must this be true

$$\sum_{T \in \mathcal{T}_h} -\langle \partial_{nt} u, \partial_t v \rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v \rangle_{\partial T} + \langle \partial_n \nabla^2 u, v \rangle_{\partial T} = \sum_{E \in \mathcal{F}^{ext}} \langle \partial_n \nabla^2 u, v \rangle_E - \langle \partial_{nt} u, \partial_n v \rangle_E + \sum_{E \in \mathcal{F}^{int}} \langle \partial_{nn} u, \llbracket \partial_{ne} v \rrbracket \rangle_E$$

## 4 Cahn Hilliard Equation on a Closed Membrane

Let  $c_0$  and  $c_1$  indicate the concentration profile of the substances in a 2-phase system such that  $c_0(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$  and similarly  $c_1(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$ , where  $\mathbf{x}$  is a element of some surface  $\Omega$  and  $t$  is time. However, in the 2 phase problem will we will restrict ourself so that  $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$  at any  $\mathbf{x}$  at time  $t$ . A property of the restriction is that we now can express  $c_0$  using  $c_1$ , with no loss of information. Hence, let us now define  $c = c_0$  so  $c(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$ . It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase separation described by a evolutionary differential equation [4] using a model based on chemical energy of the substances. However, further development has been done [5] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface  $\Gamma$ , so that  $c(\mathbf{x}, t) : \Gamma \times [0, T] \rightarrow [0, 1]$ . Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma (M \nabla_\Gamma (f'_0 - \varepsilon^2 \nabla_\Gamma^2 c)) = 0 \quad \text{on } \Gamma. \quad (16)$$

We define here the tangential gradient operator to be  $\nabla_\Gamma c = \nabla c - (\mathbf{n} \cdot \nabla c) \mathbf{n}$  applied on the surface  $\Gamma$  restricted to  $\mathbf{n} \cdot \nabla_\Gamma c = 0$ .

Lets define  $\varepsilon$  to be the size of the layer between the substances  $c_1$  and  $c_2$ . The density  $\rho$  is simply defined such that  $\rho = \frac{m}{S_\Gamma}$  is a constant based on the total mass divided by the total surface area of  $\Gamma$ . Here is the mobility  $M$  often derived such that is is dependent on  $c$  and is crucial for the result during a possible coarsening event [5]. However, the free energy per unit surface  $f_0 = f_0(c)$  is derived based on the thermodynamical model and should according to [5] be nonconvex and nonlinear.

A important observation is that equation (16) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.



## 5 Appendix

### 5.1 The Space $L^2(\Omega)$

Using the definition from [6] and we let  $\Omega$  be a an open set in  $\mathbb{R}^d$  and  $p \in \mathbb{R}$  such that  $p \geq 1$ . Then we denote  $L^p(\Omega)$  to be the set of measurable function  $u : \Omega \rightarrow \mathbb{R}$  such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p \right)^{\frac{1}{p}}.$$

Now let  $u, v : \Omega \rightarrow \mathbb{R}$ . Then is  $L^2(\Omega)$  a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} uv.$$

If the integral is finite do we say that  $u, v \in L^2(\Omega)$ .

### 5.2 The Space $H^m(\Omega)$ , $m > 1$

Again using the definition from [6]. Let  $\alpha = (\alpha_1, \dots, \alpha_d)$ ,  $\alpha \geq 0$ , such that  $|\alpha| = \sum_{i=1}^d \alpha_i$ . Now we define the space

$$H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall \alpha : |\alpha| \leq m\}.$$

Suppose that  $u, v$  is measurable functions. We can now define  $u \in H^m(\Omega)$  the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left( \|u\|_{L^2(\Omega)}^2 + \sum_{k=1}^m \|u\|_{H^k(\Omega)}^2 \right)^{\frac{1}{2}}, \quad \|u\|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|_{L^2(\Omega)}^2}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u D^\alpha v$$

## References

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