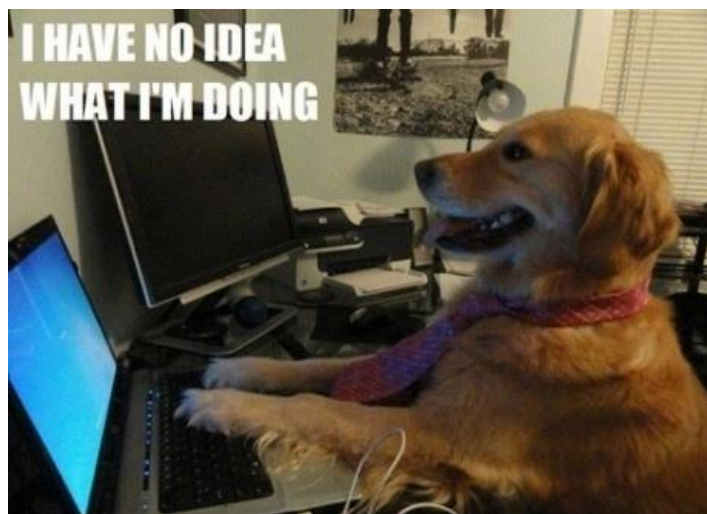


Project Thesis

Solving Cahn Hilliard Equation

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1 Introduction

Introduction

2 Cahn Hilliard Equation on a Closed Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a 2-phase system such that $c_0(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$ and similarly $c_1(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$, where \mathbf{x} is a element of some surface Ω and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$ at any \mathbf{x} at time t . A property of the restriction is that we now can express c_0 using c_1 , with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase separation described by a evolutionary differential equation [1] using a model based on chemical energy of the substances. However, further development has been done [2] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface Γ , so that $c(\mathbf{x}, t) : \Gamma \times [0, T] \rightarrow [0, 1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma (M \nabla_\Gamma (f'_0 - \varepsilon^2 \nabla_\Gamma^2 c)) = 0 \quad \text{on } \Gamma. \quad (1)$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n} \cdot \nabla c) \mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2 . The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divided by the total surface area of Γ . Here is the mobility M often derived such that is is dependent on c and is crucial for the result during a possible coarsening event [2]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [2] be nonconvex and nonlinear.

A important observation is that equation (1) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

3 C^0 Interior Penalty Method

3.1 Introduction of the Boundary Value Problem

In this section do we want to establish a numerical method to fourth order equations. Instead of embarking on the special case of surface PDE described in (1) can we establish a general numerical theory on \mathbb{R}^2 , which we later can generalize on closed surface later. Assume that we restrict ourself to a compact surface $\Omega \in \mathbb{R}^2$ and let $f \in L^2(\Omega)$ as defined in 4.2. Let say we want to solve the equation on the form.

$$\begin{aligned} \Delta^2 u - \beta \Delta u + \gamma u &= f \quad \beta, \gamma \geq 0 \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Omega \\ \frac{\partial \Delta u}{\partial n} &= q \quad \text{on } \partial \Omega \end{aligned} \quad (2)$$

For convenience are the boundary condition q chosen to be defined via a $\phi \in H^4(\Omega)$ such that $q = \frac{\partial \Delta \phi}{\partial n}$ so $\frac{\partial \phi}{\partial n} = 0$. $\partial \Omega$.

3.2 Weak Formulation

We want to rewrite (2) on weak formulation. Now define the Hilbert space

$$V = \left\{ v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega \right\}.$$

It can be shown [3] that a convinient form is to write it as

$$\begin{aligned} a(u, v) &= (f, v)_{L^2(\Omega)} - (q, v)_{L^2(\partial \Omega)} \\ &= \int_\Omega D^2 w : D^2 v \, dx + \int_\Omega \nabla w \cdot \nabla v \, dx + \int_\Omega \gamma w \cdot v \, dx. \end{aligned} \quad (3)$$

For all $\forall v \in V$, where

$$D^2 w : D^2 v = \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j}.$$

Abusing notation can we see this is clearly arise since

$$\begin{aligned}\int_{\Omega} \Delta^2 w \cdot v dx &= - \int_{\Omega} \nabla (\Delta w) \cdot \nabla v dx \\ &= \int_{\Omega} \Delta w \Delta v dx - \int_{\partial\Omega} \nabla v \frac{\partial \Delta w}{\partial n} ds \\ &= (\Delta w, \Delta v)_{L^2(\Omega)} - (q, v)_{L^2(\partial\Omega)}\end{aligned}$$

why is minus sign in front of $(q, v)_{L^2(\partial\Omega)}$ and is it correct to use q in this setting? I also wonder how $(\Delta w, \Delta v)$ appears to be $(D^2 w, D^2 v)$ at some point.

In fact, according to [3] can it be shown that the problem has a unique solution if and only if $\gamma > 0$. However, in the case where $\gamma = 0$ can we provoke a unique solution by introducing the condition

$$\int_{\Omega} f dx = \int_{\partial\Omega} q ds$$

Taking this into account can we expand the solution space such that

$$V^* = \begin{cases} V, & \text{if } \gamma > 0 \\ \{v \in V : v(p^*) = 0\}, & \text{if } \gamma = 0 \end{cases}$$

Where p^* is a corner in Ω . In fact, now all solutions of (3) exists in V^* .

3.3 Construction of C^0 Interior Penalty Method

We want to construct a C^0 interior penalty method based on C^0 Lagrange elements. Assume \mathcal{T}_h be a tringaluation of Ω and V_h be the a \mathcal{P}_2 Lagrange finite element space associated with \mathcal{T}_h

$$V_h = \left\{ v \in C(\overline{\Omega}) : v_T = v|_T \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h \right\}$$

So that we can earn a similar space for the approximated solution space ,

$$V_h^* = \begin{cases} V_h, & \text{for } \gamma > 0 \\ \{v \in V_h : v(p^*) = 0\} & \text{for } \gamma = 0. \end{cases}$$

Here is p^* again a corner in Ω . Let us now generalize the Hilbert space as well to the approximated solution space by defining

$$H^k(\Omega, \mathcal{T}_h) = \{H^1(\Omega) : v_T \in H^k(T) \quad \forall T \in \mathcal{T}_h\}.$$

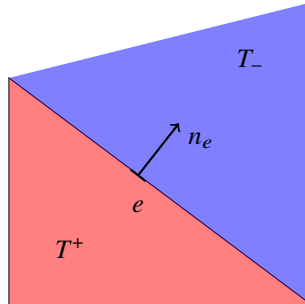


Figure 1: Edge e shared by the triangles T_- and T_+

4 Appendix

4.1 The Space $L^2(\Omega)$

Using the definition from [4] and we let Ω be a an open set in \mathbb{R}^d and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \rightarrow \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p \right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \rightarrow \mathbb{R}$. Then is $L^2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} uv.$$

If the integral is finite do we say that $u, v \in L^2(\Omega)$.

4.2 The Space $H^m(\Omega)$, $m > 1$

Again using the definition from [4]. Let $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha \geq 0$, such that $|\alpha| = \sum_{i=1}^d \alpha_i$. Now we define the space

$$H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall \alpha : |\alpha| \leq m\}.$$

Suppose that u, v is measurable functions. We can now define $u \in H^m(\Omega)$ the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \sum_{k=1}^m \|u\|_{H^k(\Omega)}^2 \right), \quad \|u\|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|_{L^2(\Omega)}^2}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u D^\alpha v$$

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