Project Thesis
Bananas and Troika

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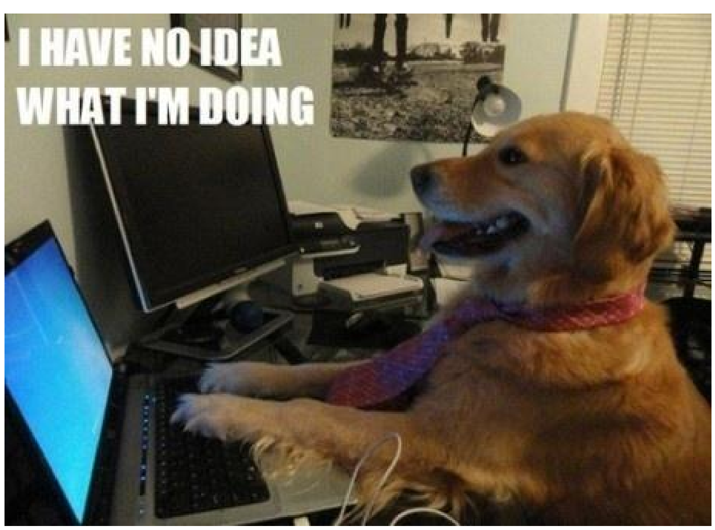

1 Introduction

This guy [1] has done a good job.

2 Cahn Hilliard Equation on a Closed Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a a 2 -phase system such that $c_0(\mathbf{x}, t) : \Omega\times[0,\infty] \to [0,1]$ and similarly $c_1(\mathbf{x}, t) : \Omega \times [0,\infty] \to [0,1]$, where $\mathbf{x}$ is a element of some surface Ω and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t,\mathbf{x}) + c_1(t,\mathbf{x}) = 1$ at any $\mathbf{x}$ at time t. A property of the restriction is that we now can express c_0 using c_1, with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \Omega\times[0,\infty] \to [0,1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase seperation described by a evolutional differential equation [2] using a model based on chemical energy of the substances. However, further development has been done [3] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface Γ, so that $c(\mathbf{x}, t) : \Gamma\times[0,T] \to [0,1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho\frac{\partial c}{\partial t} - \nabla_\Gamma\left(M\nabla_\Gamma\left(f_0' - \varepsilon^2\nabla_\Gamma^2 c\right)\right) = 0 \quad \text{on } \Gamma. \tag{1}$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n}\nabla c)\,\mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n}\cdot\nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2. The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divaded by the total surface area of Γ. Here is the mobility M often derived such that is is dependent on c and is crucial for the result during a possible coarsering event [3]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [3] be nonconvex and nonlinear.

A important observation is that equation (1) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

3 C^0 Interior Penalty Method

In this section do we want to establish a numerical method to fourth order equations. Instead of embarking on the special case of surface PDE described in (1) can we establish a general numerical theory on $\mathbb{R}^2$, which we later can generalize on closed surface later.

Assume that we restrict ourself to a compact surface $\Omega \in \mathbb{R}^2$ and let $f \in L^2(\Omega)$ as defined in 4.2. Let say we want to solve the equation on the form.

$$\begin{aligned} \Delta^2 u - \beta\Delta + \gamma u &= f \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Omega \\ \frac{\partial \Delta u}{\partial n} &= q \quad \text{on } \partial\Omega \end{aligned} \quad . \tag{2}$$

For convinience do we define

3.1 Weak Formulation

The weak formulation of (**??**) has the form

$$a(u,v) = \int_\omega f v\, dx \quad u, v \in V. \tag{3}$$

where

$$a(w,v) = \int_\Omega \nabla^2 w : \nabla^2 v\, dx. \tag{4}$$

Here do we define $\nabla^2 w : \nabla^2 v$ to be the inner product of the Hessian matrix of w and v, and V is a closed subspace of the sobolev space $H^2(\Omega)$.

3.2 The general C^0 Interior Penalty Method

C^0 Interior penalty method says that

$$
\begin{aligned}
a_h(w,v) = \sum_{T\in\mathfrak{T}_h} \int_T \left(\nabla^2 w : \nabla^2 v\right) dx + \sum_{e\in\mathfrak{E}_h^i} \int_e \left\{\frac{\partial w}{\partial n_e}\right\} \left[\!\left[\frac{\partial v}{\partial n_e}\right]\!\right] ds \\
+ \sum_{e\in\mathfrak{E}_h^i} \int_e \left\{\frac{\partial^2 v}{\partial n_e^2}\right\} \left[\!\left[\frac{\partial w}{\partial n_e}\right]\!\right] + \sum_{e\in\mathfrak{E}_h^i} \frac{\sigma}{|e|} \int_e \left[\!\left[\frac{\partial w}{\partial n_e}\right]\!\right] \left[\!\left[\frac{\partial v}{\partial n_e}\right]\!\right] ds
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\left[\!\left[\frac{\partial v}{\partial n_e}\right]\!\right] &= -n_e \nabla v_T, \quad v_T = v|_T \\
\left\{\frac{\partial^2 u}{\partial n_e^2}\right\} &= \frac{\partial^2 u}{\partial n_e^2} \\
\left\{\frac{\partial w}{\partial n_e^2}\right\} &= \frac{1}{2}\left(\frac{\partial^2 w_-}{\partial n_e^2} + \frac{\partial^2 w_+}{\partial n_e^2}\right) \\
\left\{\frac{\partial^2 w}{\partial n_e^2}\right\} &= \frac{\partial^2 w}{\partial n_e^2}, \quad \text{on edges}
\end{aligned}
\tag{6}
$$

4 Appendix

4.1 The Space $L^2(\Omega)$

Using the definition from [4] and we let Ω be a an open set in $\mathbb{R}^d$ and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \to \mathbb{R}$. Then is $L^2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^p(\Omega)} = \int_\Omega uv.$$

If the integral is finite do we say that $u, v \in L^p(\Omega)$.

4.2 The Space $H^m(\Omega)$, $m > 1$

Again using the definition from [4]. Let $\alpha = (\alpha_1, \ldots, \alpha_d), \quad \alpha \geq 0$, such that $|\alpha| = \sum_{i=1}^d \alpha_i$. Now we define the space

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall \alpha : |\alpha| \leq m \right\}.$$

Suppose that u, v is measurable functions. We can now define $u \in H^m(\Omega)$ the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)} + \sum_{k=1}^m |u|^2_{H^k(\Omega)} \right), \quad |u|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|^2_{L^2(\Omega)}}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v$$

References

[1] Edmund Brekke. *Fundamentals of Sensor Fusion.* Not published. Third edition, August 2, 2021.

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[3] Vladimir Yushutin et al. "A computational study of lateral phase separation in biological membranes". In: *International Journal for Numerical Methods in Biomedical Engineering* 35.3 (2019). e3181 cnm.3181, e3181. DOI: https://doi.org/10.1002/cnm.3181. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cnm.3181. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cnm.3181.

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