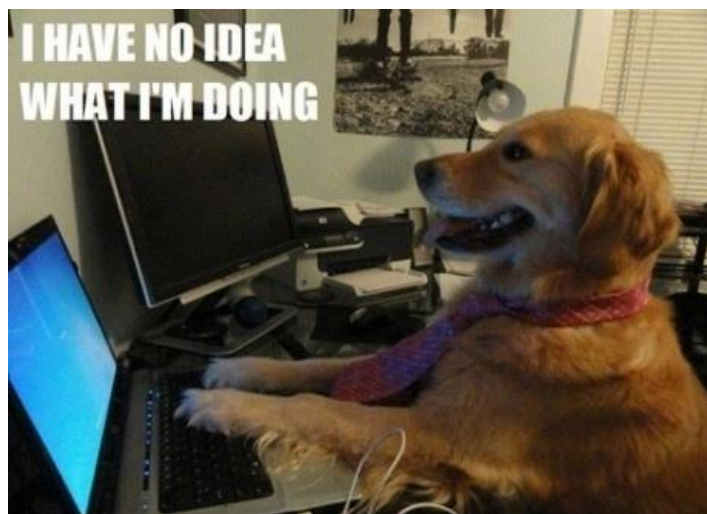


Project Thesis Bananas and Troika

Isak Hammer

March 5, 2022



1 Introduction

This guy [1] has done a good job.

2 Cahn Hilliard Equation on a Cell Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a 2-phase system such that $c_0(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$ and similarly $c_1(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$, where \mathbf{x} is a element of some surface Ω and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$ at any \mathbf{x} at time t . A property of the restriction is that we now can express c_0 using c_1 , with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \Omega \times [0, \infty] \rightarrow [0, 1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase separation described by a evolutional differential equation [2] using a model based on chemical energy of the substances. However, further development has been done [3] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface Γ , so that $c(\mathbf{x}, t) : \Gamma \times [0, T] \rightarrow [0, 1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma (M \nabla_\Gamma (f'_0 - \varepsilon^2 \nabla_\Gamma^2 c)) = 0 \quad \text{on } \Gamma. \quad (1)$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n} \nabla c) \mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2 . The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divided by the total surface area of Γ . Here is the mobility M often derived such that is dependent on c and is crucial for the result during a possible coarsening event [3]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [3] be nonconvex and nonlinear.

A important observation is that equation (1) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

3 C^0 Interior Penalty Method

We want to solve the equation on the form

$$\Delta^2 u = f \quad \text{in } \Omega \quad (2)$$

$$u = \frac{\partial u}{\partial n} = 0 \quad (3)$$

$$. \quad (4)$$

3.1 Weak Formulation

The weak formulation of (2) has the form

$$a(u, v) = \int_\Omega f v dx \quad u, v \in V. \quad (5)$$

where

$$a(w, v) = \int_\Omega \nabla^2 w : \nabla^2 v dx. \quad (6)$$

Here do we define $\nabla^2 w : \nabla^2 v$ to be the inner product of the Hessian matrix of w and v , and V is a closed subspace of the sobolev space $H^2(\Omega)$.

3.2 The general C^0 Interior Penalty Method

C^0 Interior penalty method says that

$$\begin{aligned} a_h(w, v) = & \sum_{T \in \mathfrak{T}_h} \int_T (\nabla^2 w : \nabla^2 v) dx + \sum_{e \in \mathfrak{E}_h^i} \int_e \left\{ \left\{ \frac{\partial w}{\partial n_e} \right\} \right\} \left[\left[\frac{\partial v}{\partial n_e} \right] \right] ds \\ & + \sum_{e \in \mathfrak{E}_h^i} \int_e \left\{ \left\{ \frac{\partial^2 v}{\partial n_e^2} \right\} \right\} \left[\left[\frac{\partial w}{\partial n_e} \right] \right] + \sum_{e \in \mathfrak{E}_h^i} \frac{\sigma}{|e|} \int_e \left[\left[\frac{\partial w}{\partial n_e} \right] \right] \left[\left[\frac{\partial v}{\partial n_e} \right] \right] ds. \end{aligned}$$

where

$$\begin{aligned}\left[\left[\frac{\partial v}{\partial n_e}\right]\right] &= -n_e \nabla \nu_T, \quad \nu_T = \nu|_T \\ \left\{\left\{\frac{\partial^2 u}{\partial n_e^2}\right\}\right\} &= \frac{\partial^2 u}{\partial n_e^2} \\ \left\{\left\{\frac{\partial w}{\partial n_e^2}\right\}\right\} &= \frac{1}{2} \left(\frac{\partial^2 w_-}{\partial n_e^2} + \frac{\partial^2 w_+}{\partial n_e^2} \right) \\ \left\{\left\{\frac{\partial^2 w}{\partial n_e^2}\right\}\right\} &= \frac{\partial^2 w}{\partial n_e^2}, \quad \text{on edges.}\end{aligned}$$

References

- [1] Edmund Brekke. *Fundamentals of Sensor Fusion*. Not published. Third edition, August 2, 2021.
- [2] John W. Cahn and John E. Hilliard. “Free Energy of a Nonuniform System. I. Interfacial Free Energy”. In: *The Journal of Chemical Physics* 28.2 (1958), pp. 258–267. DOI: [10.1063/1.1744102](https://doi.org/10.1063/1.1744102). eprint: <https://doi.org/10.1063/1.1744102>. URL: <https://doi.org/10.1063/1.1744102>.
- [3] Vladimir Yushutin et al. “A computational study of lateral phase separation in biological membranes”. In: *International Journal for Numerical Methods in Biomedical Engineering* 35.3 (2019). e3181 cnm.3181, e3181. DOI: <https://doi.org/10.1002/cnm.3181>. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/cnm.3181>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/cnm.3181>.