# Project Thesis
# Solving Cahn Hilliard Equation

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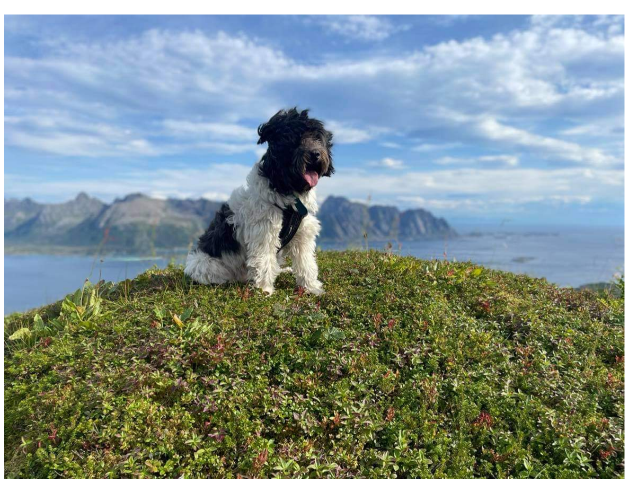

# 1 Introduction

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1) Introduction

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# 2 Biharmonic Equation

## 2.1 Strong form of the Biharmonic Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega$ be its corresponding boundary. Let the fourth order biharmonic equation have the form,

$$\begin{aligned} \Delta^2 u + \alpha u &= f \quad \text{in } \Omega \\ \partial_n u &= g_1(x) \quad \text{on } \partial\Omega\,. \\ \partial_n \nabla^2 u &= g_2(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here is $\Delta^2$ the biharmonic operator, also known as the bilaplacian. We will assume for now that $u \in H^4(\Omega)$, $\alpha$ is a nonnegative constant and $f \in L_2(\Omega)$. We may consider the functions $g_1$ and $g_2$ as time independent boundary conditions. Such problems as (1) are often associated with the Cahn-Hilliard model [1] for phase seperation. As a matter of fact, the major difference is that (1) has no time dependencie. However, depending on how Cahn-Hilliard model is time discretized numerically can (1) naturally arise. I refer to [2] for more informastion on this.

## 2.2 Computational Domains

We may want to define the computational domain. Recall that $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Let $\mathcal{T}_h$ be a triangular mesh of $\Omega$ where every triangle $T \in \mathcal{T}_h$, which is illustrated in figure 1. We define $h$ as the max diameter of the triangle $T$ such that $h = \max_{T\in\mathcal{T}_h} h_T$.

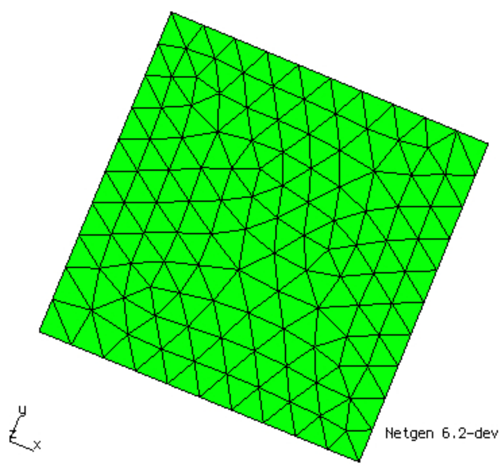


Figure 1: Example of a mesh of $\Omega \subset \mathbb{R}^2$ with triangulation $\mathcal{T}_h$.

We may also define the set of all edges $\mathcal{F}_h$ where every edge is denoted by $E \in \mathcal{F}_h$. However, we will distinguish between the set of external edges $\mathcal{F}_h^{ext}$, which is all edges along $\partial\Omega$, and the interior edges $\mathcal{F}_h^{int}$. Let the edges be denotes as $E \in \mathcal{F}_h$, then the normal vector $n$ is across the edge from $T_+$ to $T_-$, illustraded in figure 2.

## 2.3 Weak Form of Biharmonic Operator in $H^4(T)$

Let $w, v \in H^4(T)$ and $\mathcal{T}_h$ the simplicial triangulation of $\Omega$. Using the same method as in [2, 3] can we deduce that for every triangle $T \in \mathcal{T}_h$ it holds that

$$\begin{aligned} \left(\Delta^2 w, v\right)_T &= \left\langle \partial_n \nabla^2 w, v\right\rangle_{\partial T} - \left(\nabla\left(\nabla^2 w\right), \nabla v\right)_T \\ &= \left(D^2 w, D^2 v\right)_T + \left\langle \partial_n \nabla^2 w, v\right\rangle_{\partial T} - \left\langle \partial_n \nabla w, \nabla v\right\rangle_{\partial T} \qquad (2) \\ &= \left(D^2 w, D^2 v\right)_T - \left\langle \partial_{nt} w, \partial_t v\right\rangle_{\partial T} - \left\langle \partial_{nn} w, \partial_n v\right\rangle_{\partial T} + \left\langle \partial_n \nabla^2 w, v\right\rangle_{\partial T}. \qquad (3) \end{aligned}$$

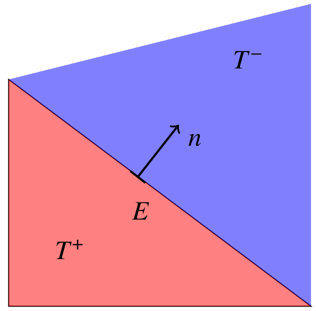


Figure 2: Edge $E \in \mathcal{F}_h$ shared by the triangles $T^+, T^- \in \mathcal{T}_h$ and the normal unit vector $n$.

Note that we in the second step we used that

$$
\begin{aligned}
\left(\nabla\left(\nabla^2 w\right), \nabla v\right)_T &= \sum_{i=1}^{2} \int_T \left(\nabla\nabla w_{x_i}\right) \cdot v_{x_i}\ dx \\
&= \int_T D^2 w : D^2 v\ dx - \int_{\partial T} \left(\partial_n \nabla w\right) \cdot \nabla v\ ds \\
&= \left(D^2 w : D^2 v\right)_T + \langle \partial_n \nabla w, \nabla v\rangle_{\partial T}
\end{aligned}
$$

We denote $D^2$ as the Hessian matrix operator such that

$$
(D^2 u, D^2 v)_\Omega = \int_\Omega D^2 u : D^2 v dx,
$$

where $D^2 u : D^2 v$ is the inner product. Also keep in mind that the last result naturally arise when defining $\nabla = (\partial_n, \partial_t)$ such that

$$
\langle \partial_n \nabla w, \nabla v\rangle_{\partial T} = \langle \partial_{nt} w, \partial_t v\rangle_{\partial T} + \langle \partial_{nn} w, \partial_n v\rangle_{\partial T}\ .
$$

Remark that we have two formulations (2) and (3)

### 2.4 Weak Form Biharmonic Equation in $H^4(\Omega)$

We might want to introduce the full basic weak formulation of (1). Now, let the solution space be on the form,

$$
V = \left\{ v \in H^2(\Omega) : \partial_n v = g_1 \text{ on } \partial\Omega \right\}..
$$

Consider the weak formulation to solve for a $u \in V$ such that

$$
a(u, v) = F(v). \quad \forall v \in V. \tag{4}
$$

By using the identity (2) and utilizing the solution space can we easily do a global summation over the triangulation. Hence, the global weak formulation can be expressed as,

$$
\begin{aligned}
\left(\Delta^2 u, v\right)_\Omega &= \sum_{T\in\mathcal{T}_h} -(\nabla\Delta u, \nabla v)_T + \langle \partial_n(\Delta u), v\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} -(\nabla\Delta u, \nabla v)_T + \langle \partial_n(\Delta u), v\rangle_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} (\Delta u, \Delta v)_T - \underbrace{\langle \partial_n \nabla u, v\rangle_{\partial T}}_{\langle \nabla g_1, v\rangle_{\partial T}} + \underbrace{\langle \partial_n(\Delta u), v\rangle_{\partial T}}_{\langle g_2, v\rangle_{\partial T}} \\
&= (\Delta u, \Delta v)_\Omega - \langle \nabla g_1, v\rangle_{\partial\Omega} + \langle g_2, v\rangle_{\partial\Omega}
\end{aligned}
$$

such that

$$
\begin{aligned}
a(u, v)_\Omega &= \left(D^2 u, D^2 v\right)_\Omega + \alpha(u, v)_\Omega, \\
F(v)_\Omega &= (f, v)_\Omega - \langle g_2, v\rangle_{\partial\Omega} + \langle \nabla g_1, \nabla v\rangle_{\partial\Omega}.
\end{aligned}
$$

In fact, the solution is unique for $\alpha > 0$. However, for $\alpha = 0$ must we assume the solvability condtion,

$$\int_{\Omega} f dx = \int_{\partial\Omega} g_2 ds.$$

This condition easily arise when using the substitution $v = 1$ in (4). To handle this, can we extended the solution space

$$V^* = \begin{cases} V & \alpha > 0 \\ \{v \in V : v(p_*) = 0\}, & \alpha = 0 \end{cases}$$

where $p_*$ is a corner of the polygonal domain $\Omega$. Thus, the unique solution in $v \in V^*$ belongs to $H^3(\Omega)$ and we get the follwing elliptic regularity estimate [3],

$$|u|_{H^3(\Omega)} \leq C_\Omega \left( \|f\|_{L_2(\Omega)} + (1 + \alpha^{\frac{1}{2}}) \cdot \|w\|_{H^4(\Omega)} \right), \quad w \in H^4(\Omega). \tag{5}$$

This regularity estimate may be important for further usecases in terms of error analysis.

# 3 Continious Interior Penalty Method

To solve this numerically do we want to introduce the Continious Interior Penalty Method (CP), which is a Discontinious Galerkin method (DG) using $C^0$ finite elements. There is several reasons why we want to apply $C^0$ instead of the often used $C^1$ finite elements for fourth order problems. First and foremost is the $C^0$ finite elements simpler than obtaining $C^1$ finite elements. Also, compared to other methods similar to the mixed finite element method for the problem (1), CP has in fact preserved the symmetric positive definiteness, which means the stability analysis is more straight forward. Finally and most importantly according to [2] can naive use mixed methods of splitting the boundary conditions of the problem (1) produce wrong solutions if $\Omega$ is nonconvex.

Write about this:

Conformal methods $V_h \subset V$ requires $C^1$. Exists in a good manner in 2D, but does not exist generalization in 3D. Need reference.

Use Bspline as alternative basis. . Less flexible when generating meshes for complicated domains. Need reference.

Write in mixed formulation $\overline{w} = \Delta w$

None-conform discretization of 4th order problem using C0 Elements. Hence CP Method

Let us again define $u, v \in H^4(T)$ and recall the results (3) so

$$\left(\Delta^2 u, v\right)_T = \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v\right\rangle_{\partial T}. \tag{6}$$

For global continuity, let $v \in V = \left\{v \in H^1(\Omega) : v_T \in H^4(T),\ \forall T \in \mathcal{T}_h\right\} \cap C^0(\overline{\Omega})$ and $u \in H^4(\Omega)$ such that,

$$\left(\Delta^2 u, v\right)_\Omega = \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v\right\rangle_{\partial T}. \tag{7}$$

However, this can be simplified to

$$\begin{aligned} \left(\Delta^2 u, v\right)_\Omega &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{E \in \mathcal{F}^{ext}} \left\langle \partial_n \nabla^2 u, v\right\rangle_E - \langle \partial_{nt} u, \partial_t v\rangle_E + \langle \partial_{nn} u, \partial_n v\rangle_E + \sum_{E \in \mathcal{F}^{int}} \langle \partial_{nn} u, [\![\partial_n v]\!]\rangle_E \\ &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{E \in \mathcal{F}^{ext}} \langle g_2, v\rangle_E + \langle n g_2, \nabla_n v\rangle_E + \langle \partial_t g_1, \partial_t v\rangle_E + \sum_{E \in \mathcal{F}^{int}} \langle \partial_{nn} u, [\![\partial_n v]\!]\rangle_E \end{aligned} \tag{8}$$

Where $\mathcal{F}_h^{int}, \mathcal{F}_h^{ext} \subset \mathcal{F}_h$ be the set of interior and exterior facets of the triangulation $\mathcal{T}_h$. Keep in mind that the jump over and edge $E$, visualized in figure 2, is defined as $[\![a]\!] = a^+ - a^-$ and similarly will the mean be defined as $\{\!\{a\}\!\} = \frac{1}{2}(a^+ + a^-)$. The equivalence of (7) and (8) comes from the following argumentation.

$$\begin{aligned} \left(\Delta^2 u, v\right)_\Omega &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - \langle \partial_{nt} u, \partial_t v\rangle_{\partial T} - \langle \partial_{nn} u, \partial_n v\rangle_{\partial T} + \left\langle \partial_n \nabla^2 u, v\right\rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T \\ &\quad + \sum_{E \in \mathcal{F}_h^{ext}} \underbrace{\left\langle \partial_n \nabla^2 u, v\right\rangle_E}_{=\langle g_2, v\rangle_E} - \underbrace{\left\langle \partial_{nt} u, \partial_t v\right\rangle_E}_{=\langle \partial_t g_1, \partial_t v\rangle} - \underbrace{\left\langle \partial_{nn} u, \partial_n v\right\rangle}_{=\langle n g_2, \partial_n v\rangle_E} \\ &\quad + \sum_{E \in \mathcal{F}_h^{int}} \underbrace{\left(\left\langle \partial_{n^+} \nabla^2 u^+, v^+\right\rangle_E + \left\langle \partial_{n^-} \nabla^2 u^+, v^-\right\rangle_E\right)}_{(I)} + \underbrace{\left(\left\langle \partial_{n^+ t} u^+, \partial_t v^+\right\rangle_E + \langle \partial_{n^- t} u^-, \partial_t v^-\rangle_E\right)}_{(II)} + \underbrace{\left(\left\langle \partial_{n^+ n^+} u^+, v^+\right\rangle_E + \langle \partial_{n^- n^-} u^-, v^-\rangle_E\right)}_{(III)} \end{aligned}.$$

Where integration over all interior edges $\forall E \in \mathcal{F}_h^{int}$ is computed in this way:

$$(I) = \left\langle \partial_{n^+}\nabla^2 u^+, v^+\right\rangle_E + \left\langle \partial_{n^-}\nabla^2 u^+, v^-\right\rangle_E = \int_E \llbracket \partial_n \nabla^2 u \cdot v \rrbracket = \int_E \left\{\!\!\left\{ \partial_n \nabla^2 u \right\}\!\!\right\} \underbrace{\llbracket v \rrbracket}_{=0} + \underbrace{\llbracket \partial_n \nabla^2 u \rrbracket}_{=0} \{\!\{ v \}\!\} = 0$$

$$(II) = \left\langle \partial_{n^+ t} u^+, \partial_t v^+\right\rangle_E + \langle \partial_{n^- t} u^-, \partial_t v^-\rangle_E = \int_E \llbracket \partial_{nt} u \cdot \partial_t v \rrbracket = \int_E \{\!\{ \partial_{nt} u \}\!\} \underbrace{\llbracket \partial_t v \rrbracket}_{=0} + \underbrace{\llbracket \partial_{nt} u \rrbracket}_{=0} \{\!\{ \partial_t v \}\!\} = 0 .$$

$$(III) = \left\langle \partial_{n^+ n^+} u^+, \partial_{n^+} v^+\right\rangle_E + \langle \partial_{n^- n^-} u^-, \partial_{n^-} v^-\rangle_E = \int_E \llbracket \partial_{nn} u \cdot \partial_n v \rrbracket = \int_E \{\!\{ \partial_{nn} u \}\!\} \underbrace{\llbracket \partial_n v \rrbracket}_{\neq 0} + \underbrace{\llbracket \partial_{nn} u \rrbracket}_{=0} \{\!\{ \partial_n v \}\!\} = \langle \partial_{nn} u, \llbracket \partial_n v \rrbracket \rangle_E$$

Observe that the cancellations in the term $(I)$ appears of the continuity of $v \in V$ and $u \in H^4(\Omega)$ which makes the jumps zero. For the second term $(II)$ does the terms become zero cancelled because the tangential derivative at the edge has no jump. However, The third term $(III)$ is fairly interesting since the discontinuity in normal vector for $v \in V$ is a jump, while the second term is still continious. It can also be raised that $\{\!\{ \partial_{nn} u \}\!\} = \partial_{nn} u$ holds by the continuity of $H^4(\Omega)$. Anyhow, the definition of jump of should more interesting when we later weakend the continuity of $u$ during discretization. Hence, (7) and (8) is equivalent.

We can finally start defining the fully discrete formulation. Let the basis be a $\mathcal{P}_2$ Lagrange finite element space so,

$$V_h = \left\{ v \in C^0(\Omega) : v_T = v|_T \in P_2(T), \forall T \in \mathcal{T}_h \right\}$$

and

$$V_h^* = \begin{cases} V_h & \text{if } \alpha > 0 \\ \{ v \in V_h : v(p_*) = 0 \} & \text{if } \alpha = 0 \end{cases}$$

Now, if we choose $u \in V_h$ must we take account that the jump is discrete. We have now the final CP formulation. The discretized numerical problem is to solve $w_h \in V_h^*$ such that

$$\mathcal{A}(w_h, v_h) = F(v_h), \quad \forall v_h \in V_h^*. \tag{9}$$

where

$$\begin{aligned} \mathcal{A}(w_h, v_h) = \quad & (\alpha w_h, v_h)_\Omega \\ & + \sum_{T \in \mathcal{T}_h} \left( D^2 w_h, D^2 v_h \right)_T \\ & + \sum_{E \in \mathcal{F}_h^{int}} \langle \{\!\{ \partial_{nn} w_h \}\!\}, \llbracket \partial_n v_h \rrbracket \rangle_E + \langle \{\!\{ \partial_{nn} v_h \}\!\}, \llbracket \partial_n w \rrbracket \rangle_E + \frac{\gamma}{h} \langle \llbracket \partial_n w_h \rrbracket, \llbracket \partial_n v_h \rrbracket \rangle_E \end{aligned} \tag{10}$$

and

$$F(v_h) = (f, v_h)_\Omega + \sum_{\mathcal{F}^{ext}} \langle g_2, v_h \rangle_E + \langle n g_2, \partial_n v_h \rangle_E + \langle \partial_t g_1, \partial_t v_h \rangle_E \tag{11}$$

Notice that the regulation term termined by respectively a global tuning parameter $\gamma > 0$ and edge length $h = |E|$. Another key component to the formulation in (10) after introduction of $w_h, v_h \in V_h^*$ is that we expanded $\langle \partial_{nn} w, \llbracket \partial_n v \rrbracket \rangle_E \to \langle \{\!\{ \partial_{nn} w_h \}\!\}, \llbracket \partial_n v_h \rrbracket \rangle_E$ since we can longer not guarantee a continious jump. For symmetric purposes we also added $\langle \{\!\{ \partial_{nn} v_h \}\!\}, \llbracket \partial_n w_h \rrbracket \rangle_E$.

We may introduce the compact notation of (10).

$$\begin{aligned} \mathcal{A}(w_h, v_h) = \quad & (\alpha w_h, v_h)_\Omega \\ & + \left( D^2 w_h, D^2 v_h \right)_{\mathcal{T}_h} \\ & + \langle \{\!\{ \partial_{nn} w_h \}\!\}, \llbracket \partial_n v_h \rrbracket \rangle_{\mathcal{F}_h} + \langle \{\!\{ \partial_{nn} v_h \}\!\}, \llbracket \partial_n w \rrbracket \rangle_{\mathcal{F}_h} + \frac{\gamma}{h} \langle \llbracket \partial_n w_h \rrbracket, \llbracket \partial_n v_h \rrbracket \rangle_{\mathcal{F}_h} \end{aligned} \tag{12}$$

## 3.1 Error and Stability Analysis of CP

To guarantee convergence and stability we may want to check coercivity and boundedness of the method.

First of all, let us now establish some important inequalites.

$$\textbf{Cauchy-Schwarz inequality: } \|ab\| \leq \|a\|\|b\|$$
$$\textbf{Inverse inequality: } \frac{1}{h}\|\partial_{nn} v_h\|^2_{\mathcal{F}_h} \leq C_j\|\nabla^2 v_h\|^2_{\mathcal{T}_h}$$
$$\textbf{Youngs epsilon inequality: } 2ab = 2\sqrt{\varepsilon}a \cdot \frac{b}{\sqrt{\varepsilon}} \leq \varepsilon a^2 + b^2\frac{1}{\varepsilon}$$

Let the energy norm be on the form

$$\begin{aligned} \|v_h\|^2_h &= \|v_h\|^2_{a_h} = \|w_h\|_\Omega\|v_h\|_\Omega + \|\nabla^2 v_h\|^2_{\mathcal{T}_h} + \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} \\ \|v\|^2_h &= \|v\|^2_{a_h,*} = \|v\|^2_{a_h} + \|h^{\frac{1}{2}} \{\partial_{nn} v\} \|_F, \quad v \in V \oplus V_h \end{aligned}. \tag{13}$$

The method is said to be coercive if $\mathcal{A}_h(v_h, v_h) \geq C\|v_h\|_{a_h}$. Similarly, it is bounded if $\mathcal{A}_h(v_h, u_h) \leq C\|u_h\|^2_{a_h}\|v_h\|^2_{a_h}$ and then, according to Lax Milgram (need reference), the solution does exist and be unique.

### 3.1.1 Coercitivity

Suppose we have the CP problem described in (9). Then is the coercivity be computed such that

$$\begin{aligned} \mathcal{A}(v_h, v_h) &= \alpha\|w_h v_h\|_\Omega + \|D^2 v_h\|^2_{\mathcal{T}_h} + 2\left(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n v_h]\!]\right)_{\mathcal{F}_h} + \frac{\gamma}{h}\| [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} \\ \textit{Cauchy-Schwarz inequality} \quad &\geq \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \|\nabla^2 v_h\|^2_{\mathcal{T}_h} - 2\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\} \|_{\mathcal{F}_h}\|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} + \gamma\|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} \\ \textit{Inverse inequality} \quad &\geq \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \|\nabla^2 v_h\|^2_{\mathcal{T}_h} - 2C_j^{\frac{1}{2}}\|\nabla^2 v_h\|_{\mathcal{T}_h}\|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} + \gamma\|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} \\ \textit{Youngs epsilon inequality} \quad &\geq \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \|\nabla^2 v_h\|^2_{\mathcal{T}_h} - \varepsilon C_j\|\nabla^2 v_h\|^2_{\mathcal{T}_h} - \frac{1}{\varepsilon}\|h^{\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} + \gamma\|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} \\ &= \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \left(1 - \varepsilon C_j\right)\|\nabla^2 v_h\|^2 + \left(\gamma - \frac{1}{\varepsilon}\right)\|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{T}_h} \\ (\varepsilon = \frac{1}{2C_j}) \implies \quad &= \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \frac{1}{2}\|\nabla^2 v_h\|^2_{\mathcal{T}_h} + \underbrace{\left(\gamma - 2C_j\right)}_{\geq \frac{1}{2}}\|h^{\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} \geq C\|v_h\|^2_{a_h} \end{aligned}$$

This holds if $C = \min\{\alpha, 1/2\}$. Observe that for the first inequality is the standard **Cauchy-Schwarz inequality** such that

$$\left(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n v_h]\!]\right)_{\mathcal{F}_h} \geq -\|h^{-\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\} \|_{\mathcal{F}_h}\| \{\!\{\partial_n v_h\}\!\} \|_{\mathcal{F}_h}.$$

On the second inequality the **Inverse inequality** was applied,

$$-\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h} \geq -C_j^{\frac{1}{2}}\|\nabla^2 v_h\|_{\mathcal{T}_h}$$

The next step is then to use the **Youngs epsilon inequality** to be able to seperate the facets and triangulation norms.

$$-2C_j^{\frac{1}{2}}\|\nabla^2 v_h\|_{\mathcal{T}_h}\|h^{\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} \geq -\varepsilon C_j\|\nabla^2 v_h\|^2_{\mathcal{T}_h} - \frac{1}{\varepsilon}\|h^{\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h}$$

The last step was to choose a $\varepsilon$ and $\gamma$ as some positive constant so that the second term is restricted to be multiplied with something bigger than $\frac{1}{2}$. Thus, the term fulfils coercivity of the (13). Hence, the CP method is coercive.

### 3.1.2 Boundedness

We want the CP method to be bounded.

$$\begin{aligned}
\mathcal{A}\left(w_h, v_h\right) &= \left(\alpha w_h, v_h\right)_\Omega + \left(\nabla^2 w_h, \nabla^2 v_h\right)_{\mathcal{T}_h} + \langle \{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!]\rangle_{\mathcal{F}_h} + \langle \{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w]\!]\rangle_{\mathcal{F}_h} + \frac{\gamma}{h}\langle [\![\partial_n w_h]\!], [\![\partial_n v_h]\!]\rangle_{\mathcal{F}_h} \\
\textit{Cauchy-Schwarz inequality} \quad &\leq \alpha \|w_h\|_\Omega \|v_h\|_\Omega + \|\nabla^2 w_h\|_{\mathcal{T}_h} \|\nabla^2 v_h\|_{\mathcal{T}_h} + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} w_h\}\!\}\|_{\mathcal{F}_h} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} \\
&\quad + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\}\|_{\mathcal{F}_h} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!]\|_{\mathcal{F}_h} + \gamma \|h^{-1} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} \|[\![\partial_n w_h]\!]\|_{\mathcal{F}_h} \\
\textit{Inverse inequality} \quad &\leq \alpha \|w_h\|_\Omega \|v_h\|_\Omega + \|\nabla^2 w_h\|_{\mathcal{T}_h} \|\nabla^2 v_h\|_{\mathcal{T}_h} + C_j^{\frac{1}{2}} \|\nabla^2 w_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} + \\
&\quad C_j^{\frac{1}{2}} \|\nabla^2 w_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!]\|_{\mathcal{F}_h} + \gamma \|h^{-1} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} \|[\![\partial_n w_h]\!]\|_{\mathcal{F}_h} \\
\textit{Using (14)} \quad &\leq \alpha \|w_h\|_{a_h} \|v_h\|_{a_h} + \|w_h\|_{a_h} \|v_h\|_{a_h} + 2C_j^{\frac{1}{2}} \|w_h\|_{a_h} \|v_h\|_{a_h} + \gamma \|v_h\|_{a_h} \|w_h\|_{a_h} \\
&\leq \left(\alpha + 1 + 2C_j^{\frac{1}{2}} + \gamma\right) \|v_h\|_{a_h} \|w_h\|_{a_h} \leq K \|v_h\|_{a_h} \|w_h\|_{a_h}
\end{aligned}$$

Thus, the CP method is shown to be bounded. Again, the first step was to apply the **Cauchy-Schwarz inequality** for every term. On the second inequality the **Inverse inequality** was applied so that

$$\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\}\|_{\mathcal{F}_h} \leq C_j^{\frac{1}{2}} \|\nabla^2 v_h\|_{\mathcal{T}_h} \quad \text{and} \quad \|h^{\frac{1}{2}} \{\!\{\partial_{nn} w_h\}\!\}\|_{\mathcal{F}_h} \leq C_j^{\frac{1}{2}} \|\nabla^2 w_h\|_{\mathcal{T}_h}.$$

The second step can we luckily observe that all terms invidually is less than the norm such that

$$\begin{aligned}
\|w_h\|_\Omega \|v_h\|_\Omega &\leq \|w_h\|_{a_h} \|v_h\|_{a_h}, \\
\|\nabla^2 w_h\|_{\mathcal{T}_h} \|\nabla^2 v_h\|_{\mathcal{T}_h} &\leq \|w_h\|_{a_h} \|v_h\|_{a_h}, \\
\|\nabla^2 w_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} &\leq \|w_h\|_{a_h} \|v_h\|_{a_h}, \quad \|\nabla^2 v_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!]\|_{\mathcal{F}_h} \leq \|v_h\|_{a_h} \|w_h\|_{a_h}, \\
\text{and} \quad \gamma \|h^{-1} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} \|[\![\partial_n w_h]\!]\|_{\mathcal{F}_h} &\leq \gamma \|v_h\|_{a_h} \|w_h\|_{a_h}.
\end{aligned} \tag{14}$$

Hence, the CP method is does fulfills the Lax Milgram criteria because it is both bounded and unique.

# 4 HCP Method copied from NGSolve

TODO: Remove this chapter.

We consider the Kirchhoff plate equation: Find $w \in H^2$, such that

$$\int \nabla^2 w : \nabla^2 v = \int f v$$

A conforming method requires $C^1$ continuous finite elements. But there is no good option available, and thus there is no $H^2$ conforming finite element space in NGSolve.

$$\sum_T \nabla^2 w : \nabla^2 v - \int_E \{\nabla^2 w\}_{nn} [\partial_n v] - \int_E \{\nabla^2 v\}_{nn} [\partial_n w] + \alpha \int_E [\partial_n w][\partial_n v]$$

[Baker 77, Brenner Gudi Sung, 2010]

We consider its hybrid DG version, where the normal derivative is a new, facet-based variable:

$$\sum_T \nabla^2 w : \nabla^2 v - \int_{\partial T} (\nabla^2 w)_{nn} (\partial_n v - \widehat{v_n}) - \int_{\partial T} (\nabla^2 v)_{nn} (\partial_n w - \widehat{w_n}) + \alpha \int_E (\partial_n v - \widehat{v_n})(\partial_n w - \widehat{w_n})$$

# 5 Cahn Hilliard Equation on a Closed Membrane

Let $c_0$ and $c_1$ indicate the concentration profile of the substances in a a 2 -phase system such that $c_0(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0, \infty] \to [0, 1]$ and similarly $c_1(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0, \infty] \to [0, 1]$, where $\mathbf{x}$ is a element of some surface $\boldsymbol{\Omega}$ and $t$ is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t, \mathbf{x}) + c_1(t, \mathbf{x}) = 1$ at any $\mathbf{x}$ at time $t$. A property of the restriction is that we now can express $c_0$ using $c_1$, with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x}, t) : \boldsymbol{\Omega} \times [0, \infty] \to [0, 1]$. It has been shown that 2 phase system if thermodynamically unstabl can be evolve into a phase seperation described by a evolutional differential equation [1] using a model based on chemical energy of the substances. However, further development has been done [4] to solve this equation on surfaces. Now assume model that we want to describe is a phase-seperation on a closed membrane surface $\Gamma$, so that $c(\mathbf{x}, t) : \Gamma \times [0, T] \to [0, 1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma \left( M \nabla_\Gamma \left( f_0' - \varepsilon^2 \nabla_\Gamma^2 c \right) \right) = 0 \quad \text{on } \Gamma. \tag{15}$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n} \nabla c) \mathbf{n}$ applied on the surface $\Gamma$ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define $\varepsilon$ to be the size of the layer between the substances $c_1$ and $c_2$. The density $\rho$ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divaded by the total surface area of $\Gamma$. Here is the mobility $M$ often derived such that is is dependent on $c$ and is crucial for the result during a possible coarsering event [4]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [4] be nonconvex and nonlinear.

A important observation is that equation (15) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

# 6 Appendix

## 6.1 The Space $L^2(\Omega)$

Using the definition from [5] and we let $\Omega$ be a an open set in $\mathbb{R}^d$ and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \to \mathbb{R}$. Then is $L^2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^p(\Omega)} = \int_\Omega uv.$$

If the integral is finite do we say that $u, v \in L^p(\Omega)$.

## 6.2 The Space $H^m(\Omega)$, $m > 1$

Again using the definition from [5]. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\quad \alpha \geq 0$, such that $|\alpha| = \sum_{i=1}^d \alpha_i$. Now we define the space

$$H^m(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall \alpha : |\alpha| \leq m\right\}.$$

Suppose that $u, v$ is measurable functions. We can now define $u \in H^m(\Omega)$ the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)} + \sum_{k=1}^m |u|^2_{H^k(\Omega)}\right), \quad |u|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|^2_{L^2(\Omega)}}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u D^\alpha v$$

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