Project Thesis

Solving Biharmonic Equation using Continuous Interior Penalty Method

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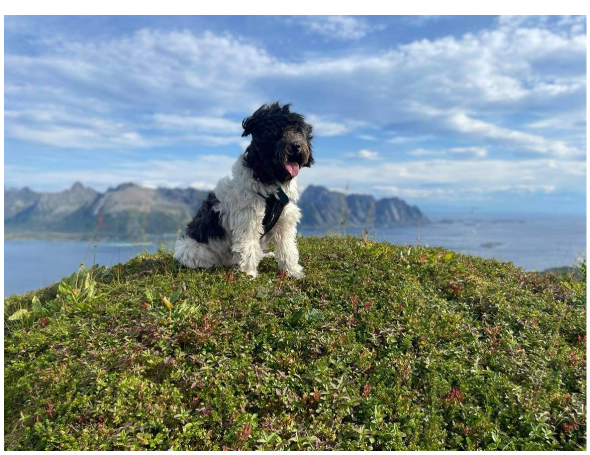

1 Introduction

The biharmonic equation is a fourth order partial differential equation (PDE) which has gained great importance in application such as mathematical modelling of linear elastic theory [1] and phase separation mechanics of two phase systems [2, 3]. However, methods for solving the biharmonic equation analytically is considered extensive and often impossible. Even on very simple plane problems on a unit square often requires advanced computations using integral transforms, variable separations, complex analysis and more. [1]. We therefore tend to lean towards approximating the solution using numerical methods for advanced problems.

There is generally two classes of numerical methods to solve the biharmonic equation. The first class is known as Finite Difference Method (FDM) [4, 5]. Nevertheless, FDM does not handle complex domains well since it generally has strict requirements for the mesh generation. However, some methods have been introduced to solve problems on irregular domains, but is has shown to be relative extensive to implement [5–7].

The second class is denotes as Finite Element Method (FEM). Using this methods implies that there is theoretically no difference on solving problems on a regular or irregular domains, except for taking account for numerical stability and some restrictions on mesh generation [6]. However, a major challenge in FEM is to choose a discrete solution space on the finite elements to approximate the exact solution. We say that a method is conforming if the discrete solution space V_h is subspace of the exact solution space V[8, 9], i.e. $V_h \subseteq V$. In General, for conforming methods requires that for a problem of order $2n$ must the discrete solution space be at least of order $n-1$. Thus, for a biharmonic problem will a conforming FEM method demand at least a basis that is C^1 globally [8]. From this strong continuity conditions rises a lot of complexity when constructing a finite element. In fact, attempts to solve this problems has shown that it arise 21 degrees of freedom in a triangular element [10].

For nonconforming methods, $V_h \nsubseteq V_h$ is the C^1 requirement completely relaxed. This makes the methods more suitable for forth order problems with the cost of more extensive error analysis. In fact, designing nonconformal elements that does converge is rather difficult [8, 10].

A third approach of FEM to solve the biharmonic equation is to solve the problem doing a Mixed FEM method. This method seems promising, because it only require C^0 elements [6, 11]. Though this work well from a numerical point of view, it has shown to have drawbacks by replacing symmetric positive definite continuous problem by a saddle point problem, which is certainly makes the existence and uniqueness proof more challenging [12].

In this report will we focus to work on a fourth approach based on FEM called the Continuous Interior Penalty Method (CP). A major advantage is that the approach preserves the global C^0 continuity and the positive symmetric definiteness, thus makes it attractive to solve the biharmonic equation [13, 14]. In this report will we focus on presenting the derivation of CP and carry out a basic error analysis. We will also present a numerical analysis.

2 Biharmonic Equation

2.1 Strong form of the Biharmonic Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega$ be its corresponding boundary. Let the inhomogeneous fourth order biharmonic equation have the form,

$$\begin{aligned} \Delta^2 u + \alpha u &= f \quad \text{in } \Omega, \\ \partial_n u &= 0 \quad \text{on } \partial\Omega, \\ \partial_n \Delta u &= g(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here is $\Delta^2 = \Delta(\Delta)$ the biharmonic operator, also known as the bilaplacian. We will assume for the strong form that $u \in H^4(\Omega)$, α is a non-negative constant and $f \in L_2(\Omega)$. We may consider the functions $g(x)$ as time independent boundary conditions. Such problems as (1) are often associated with the Cahn-Hilliard model [2] for phase separation. However, depending on how Cahn-Hilliard model is time discretized numerically can (1) naturally arise. I refer to [14] for more information on this.

2.2 Weak Form Biharmonic Equation in $H^4(\Omega)$

We want to introduce the full weak formulation of (1). Now, let the solution space be on the form,

$$V = \left\{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega \right\}.$$

Let $u, v \in V$ then the derivation of the general weak form is,

$$\left(\Delta^2 u, v\right)_\Omega = (\partial_n \Delta u, v)_{\partial\Omega} - (\nabla(\Delta u), \nabla v)_\Omega$$

In fact, the simplest formulation has the form,

$$(\nabla(\Delta u), \nabla v)_\Omega = (\Delta u, \partial_n v)_{\partial\Omega} - (\Delta u, \Delta v)_\Omega,$$

but we don't have boundary condition for Δu. Instead, we can formulate the problem so

why no b.c. for Δu ?

$$\begin{aligned} (\nabla(\Delta u), \nabla v)_\Omega &= \sum_{i=1}^{d} \left(\Delta \partial_{x_i} u, \partial_{x_i} v\right)_\Omega \\ &= \sum_{i=1}^{d} \left(\nabla \cdot \left(\nabla \partial_{x_i} u\right), \partial_{x_i} v\right)_\Omega \\ &= \sum_{i=1}^{d} \left(\partial_n \partial_{x_i} u, \nabla \partial_{x_i} v\right)_{\partial\Omega} - \left(\nabla \partial_{x_i} u, \nabla \partial_{x_i} v\right)_\Omega \\ &= (\partial_n \nabla u, \nabla v)_{\partial\Omega} - \left(D^2 u, D^2 v\right)_\Omega \\ &= (\partial_{nn} u, \partial_n v)_{\partial\Omega} + (\partial_{nt} u, \partial_t v)_{\partial\Omega} - \left(D^2 u, D^2 v\right)_\Omega. \end{aligned}$$

Hence, the boundary condition of Δu is integrated into the formulation. It can be denoted that D^2 is the Hessian matrix operator such that

$$(D^2 u, D^2 v)_\Omega = \int_\Omega D^2 u : D^2 v \, dx,$$

where $D^2 u : D^2 v$ is the inner product and similarly for $\partial_{nn} u = n \cdot D^2 u \cdot n$. Thus, we now have a weak form identity,

$$\left(\Delta^2 u, v\right)_\Omega = \left(D^2 u, D^2 v\right)_\Omega + (\partial_n \Delta u, v)_{\partial\Omega} - (\partial_{nn} u, \partial_n v)_{\partial\Omega} - (\partial_{nt} u, \partial_t v)_{\partial\Omega}. \tag{2}$$

Using weak form identity (2) and the boundary conditions stated in the strong form (1) can we write

$$\begin{aligned}\left(\Delta^2 u, v\right)_\Omega &= \left(D^2 u, D^2 v\right)_\Omega + \underbrace{(\partial_n \Delta u, v)_{\partial\Omega}}_{=(g,v)_{\partial\Omega}} - \underbrace{(\partial_{nn} u, \partial_n v)_{\partial\Omega}}_{=0} - \underbrace{(\partial_{nt} u, \partial_t v)_{\partial\Omega}}_{=0} . \\ &= \left(D^2 u, D^2 v\right)_\Omega + (g, v)_{\partial\Omega}\end{aligned} \tag{3}$$

We can now formulate the full weak formulation by solving for a $u \in V$ such that

$$a(u, v) = F(v). \quad \forall v \in V, \tag{4}$$

where

$$\begin{aligned} a(u, v)_\Omega &= \left(D^2 u, D^2 v\right)_\Omega + \alpha (u, v)_\Omega , \\ F(v)_\Omega &= (f, v)_\Omega - (g, v)_{\partial\Omega} . \end{aligned} \tag{5}$$

In fact, the solution is unique for $\alpha > 0$. However, for $\alpha = 0$ must we assume the solvability condition,

$$\int_\Omega f dx = \int_{\partial\Omega} g ds.$$

This condition easily arise when using the substitution $v = 1$ in (4). To handle this, can we extended the solution space

$$V^* = \begin{cases} V & \alpha > 0 \\ \left\{ v \in V : \int_\Omega v dx = 0 \right\} & \alpha = 0, \end{cases}$$

Thus, the unique solution in $v \in V^*$ belongs to $H^3(\Omega)$ and we get the following elliptic regularity estimate [15],

why does it belong to $H^3(\Omega)$

$$|u|_{H^3(\Omega)} \leq C_\Omega \left(\|f\|_{L_2(\Omega)} + (1 + \alpha^{\frac{1}{2}}) \cdot \|w\|_{H^4(\Omega)} \right), \quad w \in H^4(\Omega) . \tag{6}$$

This regularity estimate may be important for further use cases in terms of error analysis.

3 Continuous Interior Penalty Method

3.1 Introduction

To solve (1) numerically do we want to introduce the Continuous Interior Penalty Method (CP), which is a Discontinuous Galerkin method (DG) using C^0 finite elements. There is several reasons why we want to apply C^0 instead of the often used C^1 finite elements for fourth order problems. First and foremost is the C^0 finite elements simpler than obtaining C^1 finite elements. Also, compared to other methods similar to the mixed finite element method for the problem (1), CP has in fact preserved the symmetric positive definiteness, which means the stability analysis is more straight forward. Finally and most importantly according to [14] can naive use mixed methods of splitting the boundary conditions of the problem (1) produce wrong solutions if Ω is non convex.

Write about this:

Conformal methods $V_h \subset V$ requires C^1. Exists in a good manner in 2D, but does not exist generalization in 3D. Need reference.

Use Bspline as alternative basis. . Less flexible when generating meshes for complicated domains. Need reference.

Write in mixed formulation $\overline{w} = \Delta w$

None-conform discretization of 4th order problem using C0 Elements. Hence CP Method

3.2 Computational Domains

Let $\mathcal{T} = \{T\}$ be a triangulation of $\Omega \subset \mathbb{R}^2$ consisting of triangles T as in figure 2. We may also define the set of all facets $\mathcal{F}_h$, where every facet is denoted by $F \in \mathcal{F}_h$. However, we will distinguish between the set of external facets $\mathcal{F}_h^{ext}$, which is all facets along $\partial\Omega$, and the interior facets $\mathcal{F}_h^{int}$. Let the facets be denotes as $F \in \mathcal{F}_h$, then the normal vector n is across the facets from T^+ to T^-, illustrated in figure 1.

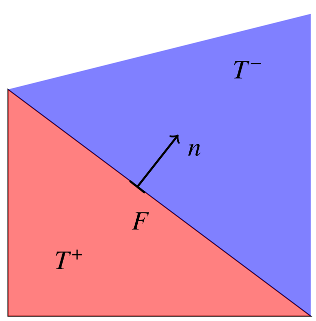

Figure 1: Facet $F \in \mathcal{F}_h$ shared by the triangles $T^+, T^- \in \mathcal{T}_h$ and the normal unit vector n.

A parameter which is useful is the maximum diameter h of the set of triangles $\{T\}$, which we to be defined s.t.

$$\begin{aligned} h_T &= diam\,(T) = \max_{x_1,x_2 \in T} dist(x_1,x_2), \\ h_{min} &= \min_{T \in \mathcal{T}} h_T\,, \\ h_{max} &= \max_{T \in \mathcal{T}} h_T\, := h, \end{aligned} \quad . \tag{7}$$

where the $diam(T)$ is the largest facet for a triangle T. We will also assume mesh conform i.e., if $T_1 \neq T_2$ and $T \cap T_2 \neq \emptyset$, then they share either a vertex or a facet.

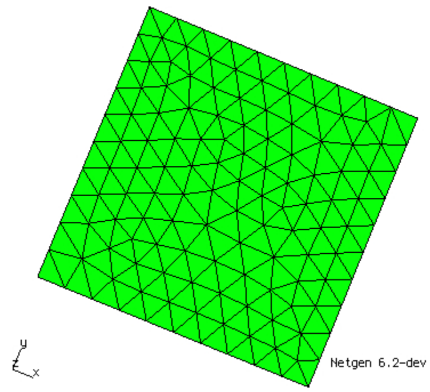

Figure 2: Example of a mesh of $\Omega \subset \mathbb{R}^2$ with triangulation $\mathcal{T}_h$.

Let the chunkiness parameter $c_T := h_T/r_T$, where r_T is the largest ball that be inscribed inside the element T. We can then assume that the mesh is shape-regular i,e., that $c_T \leq c$ independent of T and h. We may also assuming that the mesh is quasi-uniform only if it holds that the mesh is shape regular and $h_{max} \leq ch_{min}$.

Why is quasi-uniform important and how does shape regularity affect the numerical solution?

3.3 Constructing Continuous Interior Penalty Method

Let us assume that $u, v \in H^4(T)$. Using that the weak form identity (2) also holds for a triangle T can we write

$$\left(\Delta^2 u, v\right)_T = \left(D^2 u, D^2 v\right)_T - (\partial_{nt} u, \partial_t v)_{\partial T} - (\partial_{nn} u, \partial_n v)_{\partial T} + (\partial_n \Delta u, v)_{\partial T} . \tag{8}$$

For global continuity, let $v \in V = \left\{v \in H^1(\Omega) : v_T \in H^4(T),\ \forall T \in \mathcal{T}_h\right\}$ and $u \in H^4(\Omega)$ such that,

$$\left(\Delta^2 u, v\right)_\Omega = \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T - (\partial_{nt} u, \partial_t v)_{\partial T} - (\partial_{nn} u, \partial_n v)_{\partial T} + (\partial_n \Delta u, v)_{\partial T} . \tag{9}$$

However, this expression can be written to distinguish integrating over triangles $\mathcal{T}_h$, integrating over exterior facets $\mathcal{F}_h^{ext}$ and then integrate interior facets $\mathcal{F}_h^{int}$.

$$\begin{aligned}\left(\Delta^2 u, v\right)_\Omega &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{F \in \mathcal{F}_h^{ext}} (\partial_n \Delta u, v)_F - (\partial_{nt} u, \partial_t v)_F - (\partial_{nn} u, \partial_n v)_F + \sum_{F \in \mathcal{F}_h^{int}} (\partial_{nn} u, [\![\partial_n v]\!])_F \\ &= \sum_{T \in \mathcal{T}_h} \left(D^2 u, D^2 v\right)_T + \sum_{F \in \mathcal{F}_h^{ext}} (g, v)_F + \sum_{F \in \mathcal{F}^{int}} (\partial_{nn} u, [\![\partial_n v]\!])_F\end{aligned} \tag{10}$$

Keep in mind that any jump over a interior facet $F \subset \mathcal{F}_h^{int}$, visualized in figure 1, is defined as $[\![a]\!] = a^+ - a^-$ and likewise for the mean, $\{\!\{a\}\!\} = \frac{1}{2}(a^+ + a^-)$. The equivalence of (9) and (10) comes from the following argumentation.

$$
\begin{aligned}
\left(\Delta^2 u, v\right)_\Omega &= \sum_{T\in\mathcal{T}_h} \left(D^2u, D^2v\right)_T - (\partial_{nt}u, \partial_t v)_{\partial T} - (\partial_{nn}u, \partial_n v)_{\partial T} + (\partial_n \Delta u, v)_{\partial T} \\
&= \sum_{T\in\mathcal{T}_h} \left(D^2u, D^2v\right)_T \\
&\quad + \sum_{F\in\mathcal{F}_h^{ext}} \underbrace{(\partial_n\Delta u, v)_F}_{=(g,v)_F} - (\partial_{nt}u, \partial_t v)_F - \underbrace{(\partial_{nn}u, \partial_n v)_F}_{=0} \\
&\quad + \sum_{F\in\mathcal{F}_h^{int}} \underbrace{\left(\left(\partial_{n^+}\Delta u^+, v^+\right)_F + \left(\partial_{n^-}\Delta u^+, v^-\right)_F\right)}_{(I)} + \underbrace{\left(\left(\partial_{n^+t}u^+, \partial_t v^+\right)_F + \left(\partial_{n^-t}u^-, \partial_t v^-\right)_F\right)}_{(II)} + \underbrace{\left(\left(\partial_{n^+n^+}u^+, v^+\right)_F + \left(\partial_{n^-n^-}u^-, v^-\right)_F\right)}_{(III)}
\end{aligned}
$$

Where integration over all interior facets $\forall F\in\mathcal{F}_h^{int}$ is computed in this way,

$$
\begin{aligned}
(I) &= \left(\partial_{n^+}\Delta u^+, v^+\right)_F + \left(\partial_{n^-}\Delta u^+, v^-\right)_F = \int_F [\![\partial_n\Delta u\cdot v]\!] = \int_F \{\!\{\partial_n\Delta u\}\!\} \underbrace{[\![v]\!]}_{=0} + \underbrace{[\![\partial_n\Delta u]\!]}_{=0} \{\!\{v\}\!\} = 0 \\
(II) &= \left(\partial_{n^+t}u^+, \partial_t v^+\right)_F + \left(\partial_{n^-t}u^-, \partial_t v^-\right)_F = \int_F [\![\partial_{nt}u\cdot\partial_t v]\!] = \int_F \{\!\{\partial_{nt}u\}\!\} \underbrace{[\![\partial_t v]\!]}_{=0} + \underbrace{[\![\partial_{nt}u]\!]}_{=0} \{\!\{\partial_t v\}\!\} = 0 \\
(III) &= \left(\partial_{n^+n^+}u^+, \partial_{n^+}v^+\right)_F + \left(\partial_{n^-n^-}u^-, \partial_{n^-}v^-\right)_F = \int_F [\![\partial_{nn}u\cdot\partial_n v]\!] = \int_F \{\!\{\partial_{nn}u\}\!\} \underbrace{[\![\partial_n v]\!]}_{\neq 0} + \underbrace{[\![\partial_{nn}u]\!]}_{=0} \{\!\{\partial_n v\}\!\} = (\partial_{nn}u, [\![\partial_n v]\!])_F
\end{aligned}
$$

Observe that the cancellations in the term (I) appears of the continuity of $v\in V$ and $u\in H^4(\Omega)$ which makes the jumps zero. For the second term (II) does the terms become zero cancelled because the tangential derivative at the facet has no jump. However, The third term (III) is fairly interesting since the discontinuity in normal vector for $v\in V$ is a jump, while the second term is still continuous. It can also be raised that $\{\!\{\partial_{nn}u\}\!\} = \partial_{nn}u$ holds by the continuity of $H^4(\Omega)$. Anyhow, the definition of jump of should more interesting when we later weaken the continuity of u during discretization. Hence, (9) and (10) is equivalent.

3.4 Formulation of Continious Interior Penalty Method

We can finally start defining the fully discrete formulation. Let the basis be a $\mathcal{P}_2$ Lagrange finite element space so,

$$V_h = \left\{v\in C^0(\Omega) : v_T = v|_T \in P_2(T), \forall T\in\mathcal{T}_h\right\}$$

and

$$V_h^* = \begin{cases} V_h & \text{if } \alpha > 0 \\ \left\{v\in V_h : \int_\Omega v\,dx = 0\right\} & \text{if } \alpha = 0 \end{cases}$$

Now, if we choose $u\in V_h$ must we take account that the jump is discrete. We have now the final CP formulation. The discretized numerical problem is to solve $w_h\in V_h^*$ such that

$$\mathcal{A}(w_h, v_h) = F(v_h), \quad \forall v_h\in V_h^*. \tag{11}$$

where

$$
\begin{aligned}
\mathcal{A}(w_h, v_h) = \quad & (\alpha w_h, v_h)_\Omega \\
& + \sum_{T\in\mathcal{T}_h} \left(D^2 w_h, D^2 v_h\right)_T \\
& + \sum_{F\in\mathcal{F}_h^{int}} \left(\{\!\{\partial_{nn}w_h\}\!\}, [\![\partial_n v_h]\!]\right)_F + \left(\{\!\{\partial_{nn}v_h\}\!\}, [\![\partial_n w]\!]\right)_F + \frac{\gamma}{h}\left([\![\partial_n w_h]\!], [\![\partial_n v_h]\!]\right)_F
\end{aligned} \tag{12}
$$

and

$$F(v_h) = (f, v_h)_\Omega + \sum_{F \in \mathcal{F}_h^{ext}} -(g, v_h)_F \tag{13}$$

Notice that the regulation term determined by respectively a global tuning parameter $\gamma > 0$. Another key component to the formulation in (12) after introduction of $w_h, v_h \in V_h^*$ is that we expanded $(\partial_{nn} w, [\![\partial_n v]\!])_F \to (\{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!])_F$ since we can longer not guarantee a continuous jump. For symmetric purposes we also added $(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w_h]\!])_F$.

We may introduce the compact notation of (12).

$$\begin{aligned}\mathcal{A}(w_h, v_h) = \quad & (\alpha w_h, v_h)_\Omega \\ & + \left(D^2 w_h, D^2 v_h\right)_{\mathcal{T}_h} \\ & + (\{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h} + (\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w]\!])_{\mathcal{F}_h} + \frac{\gamma}{h}([\![\partial_n w_h]\!], [\![\partial_n v_h]\!])_{\mathcal{F}_h}\end{aligned} \tag{14}$$

3.5 Error and Stability Analysis of CP

To guarantee convergence and stability we may want to check coercivity and boundedness of the method.

First of all, let us now establish some important inequalities.

$$\begin{aligned}\textbf{Cauchy-Schwarz inequality: } & \|ab\| \le \|a\|\|b\| \\ \textbf{Inverse inequality: } & \frac{1}{h}\|\partial_{nn} v_h\|_{\mathcal{F}_h}^2 \le C_j \|D^2 v_h\|_{\mathcal{T}_h}^2 \\ \textbf{Youngs epsilon inequality: } & 2ab = 2\sqrt{\varepsilon}a \cdot \frac{b}{\sqrt{\varepsilon}} \le \varepsilon a^2 + b^2 \frac{1}{\varepsilon}\end{aligned}$$

Let the energy norm be on the form

$$\begin{aligned}\|v_h\|_h^2 = \|v_h\|_{a_h}^2 &= \|w_h\|_\Omega \|v_h\|_\Omega + \|D^2 v_h\|_{\mathcal{T}_h}^2 + \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!]\|_{\mathcal{F}_h} \\ \|v\|_h^2 = \|v\|_{a_h,*}^2 &= \|v\|_{a_h}^2 + \|h^{\frac{1}{2}} \{\partial_{nn} v\}\|_F, \quad v \in V \oplus V_h\end{aligned}. \tag{15}$$

The method is said to be coercive if $\mathcal{A}_h(v_h, v_h) \ge C\|v_h\|_{a_h}$. Similarly, it is bounded if $\mathcal{A}_h(v_h, u_h) \le C\|u_h\|_{a_h}^2 \|v_h\|_{a_h}^2$ and then, according to Lax Milgram (need reference), the solution does exist and be unique.

3.5.1 Coercivity

Suppose we have the CP problem described in (11). Then is the coercivity be computed such that

$$\begin{aligned} \mathcal{A}(v_h, v_h) &= \alpha\|w_h v_h\|_\Omega + \|D^2 v_h\|_{\mathcal{T}_h}^2 + 2(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h} + \frac{\gamma}{h}\|[\![\partial_n v_h]\!]\|_{\mathcal{F}_h}^2 \\ \textit{Cauchy-Schwarz inequality} \quad &\ge \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \|D^2 v_h\|_{\mathcal{T}_h}^2 - 2\|h^{\frac{1}{2}}\{\!\{\partial_{nn} v_h\}\!\}\|_{\mathcal{F}_h}\|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h} + \gamma\|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h}^2 \\ \textit{Inverse inequality} \quad &\ge \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \|D^2 v_h\|_{\mathcal{T}_h}^2 - 2C_j^{\frac{1}{2}}\|D^2 v_h\|_{\mathcal{T}_h}\|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h} + \gamma\|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h}^2 \\ \textit{Youngs epsilon inequality} \quad &\ge \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \|D^2 v_h\|_{\mathcal{T}_h}^2 - \varepsilon C_j\|D^2 v_h\|_{\mathcal{T}_h}^2 - \frac{1}{\varepsilon}\|h^{\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h}^2 + \gamma\|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h}^2 \\ &= \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \left(1 - \varepsilon C_j\right)\|D^2 v_h\|_{\mathcal{T}_h}^2 + \left(\gamma - \frac{1}{\varepsilon}\right)\|h^{-\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h}^2 \\ (\varepsilon = \frac{1}{2C_j}) \implies \quad &= \alpha\|w_h\|_\Omega\|v_h\|_\Omega + \frac{1}{2}\|D^2 v_h\|_{\mathcal{T}_h}^2 + \underbrace{(\gamma - 2C_j)}_{\ge \frac{1}{2}}\|h^{\frac{1}{2}}[\![\partial_n v_h]\!]\|_{\mathcal{F}_h} \ge C\|v_h\|_{a_h}^2 \end{aligned}$$

This holds if $C = \min\{\alpha, 1/2\}$. Observe that for the first inequality is the standard **Cauchy-Schwarz inequality** such that

$$(\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h} \ge -\|h^{-\frac{1}{2}}\{\!\{\partial_{nn} v_h\}\!\}\|_{\mathcal{F}_h}\|\{\!\{\partial_n v_h\}\!\}\|_{\mathcal{F}_h}.$$

On the second inequality the **Inverse inequality** was applied,

$$-\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v\}\!\} \|_{\mathcal{F}_h} \geq -C_j^{\frac{1}{2}} \|D^2 v_h\|_{\mathcal{T}_h}$$

The next step is then to use the **Youngs epsilon inequality** to be able to separate the facets and triangulation norms.

$$-2C_j^{\frac{1}{2}} \|D^2 v_h\|_{\mathcal{T}_h} \|h^{\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h} \geq -\varepsilon C_j \|D^2 v_h\|^2_{\mathcal{T}_h} - \frac{1}{\varepsilon} \|h^{\frac{1}{2}} [\![\partial_n v_h]\!] \|^2_{\mathcal{F}_h}$$

The last step was to choose a ε and γ as some positive constant so that the second term is restricted to be multiplied with something bigger than $\frac{1}{2}$. Thus, the term fulfils coercivity of the (15). Hence, the CP method is coercive.

3.5.2 Boundedness

We want the CP method to be bounded.

$$\begin{aligned}
\mathcal{A}(w_h, v_h) &= (\alpha w_h, v_h)_\Omega + \left(D^2 w_h, D^2 v_h\right)_{\mathcal{T}_h} + (\{\!\{\partial_{nn} w_h\}\!\}, [\![\partial_n v_h]\!])_{\mathcal{F}_h} + (\{\!\{\partial_{nn} v_h\}\!\}, [\![\partial_n w]\!])_{\mathcal{F}_h} \\
&\quad + \frac{\gamma}{h} ([\![\partial_n w_h]\!], [\![\partial_n v_h]\!])_{\mathcal{F}_h} \\
\textit{Cauchy-Schwarz inequality} \quad &\leq \alpha \|w_h\|_\Omega \|v_h\|_\Omega + \|D^2 w_h\|_{\mathcal{T}_h} \|D^2 v_h\|_{\mathcal{T}_h} + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} w_h\}\!\} \|_{\mathcal{F}_h} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} \\
&\quad + \|h^{\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\} \|_{\mathcal{F}_h} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!] \|_{\mathcal{F}_h} + \gamma \|h^{-1} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} \| [\![\partial_n w_h]\!] \|_{\mathcal{F}_h} \\
\textit{Inverse inequality} \quad &\leq \alpha \|w_h\|_\Omega \|v_h\|_\Omega + \|D^2 w_h\|_{\mathcal{T}_h} \|D^2 v_h\|_{\mathcal{T}_h} + C_j^{\frac{1}{2}} \|D^2 w_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} + \\
&\quad C_j^{\frac{1}{2}} \|D^2 w_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!] \|_{\mathcal{F}_h} + \gamma \|h^{-1} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} \| [\![\partial_n w_h]\!] \|_{\mathcal{F}_h} \\
\textit{Using (16)} \quad &\leq \alpha \|w_h\|_{a_h} \|v_h\|_{a_h} + \|w_h\|_{a_h} \|v_h\|_{a_h} + 2C_j^{\frac{1}{2}} \|w_h\|_{a_h} \|v_h\|_{a_h} + \gamma \|v_h\|_{a_h} \|w_h\|_{a_h} \\
&\leq \left(\alpha + 1 + 2C_j^{\frac{1}{2}} + \gamma\right) \|v_h\|_{a_h} \|w_h\|_{a_h} \leq K \|v_h\|_{a_h} \|w_h\|_{a_h}
\end{aligned}$$

Thus, the CP method is shown to be bounded. Again, the first step was to apply the **Cauchy-Schwarz inequality** for every term. On the second inequality the **Inverse inequality** was applied so that

$$\|h^{\frac{1}{2}} \{\!\{\partial_{nn} v_h\}\!\} \|_{\mathcal{F}_h} \leq C_j^{\frac{1}{2}} \|D^2 v_h\|_{\mathcal{T}_h} \quad \text{and} \quad \|h^{\frac{1}{2}} \{\!\{\partial_{nn} w_h\}\!\} \|_{\mathcal{F}_h} \leq C_j^{\frac{1}{2}} \|D^2 w_h\|_{\mathcal{T}_h}.$$

The second step can we luckily observe that all terms invidually is less than the norm such that

$$\begin{aligned}
\|w_h\|_\Omega \|v_h\|_\Omega &\leq \|w_h\|_{a_h} \|v_h\|_{a_h}, \\
\|D^2 w_h\|_{\mathcal{T}_h} \|D^2 v_h\|_{\mathcal{T}_h} &\leq \|w_h\|_{a_h} \|v_h\|_{a_h}, \\
\|D^2 w_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} &\leq \|w_h\|_{a_h} \|v_h\|_{a_h}, \quad \|D^2 v_h\|_{\mathcal{T}_h} \|h^{-\frac{1}{2}} [\![\partial_n w_h]\!] \|_{\mathcal{F}_h} \leq \|v_h\|_{a_h} \|w_h\|_{a_h}, \\
\text{and} \quad \gamma \|h^{-1} [\![\partial_n v_h]\!] \|_{\mathcal{F}_h} \| [\![\partial_n w_h]\!] \|_{\mathcal{F}_h} &\leq \gamma \|v_h\|_{a_h} \|w_h\|_{a_h}.
\end{aligned} \tag{16}$$

Hence, the CP method is does fulfills the Lax Milgram criteria because it is both bounded and unique.

3.6 Apriori Estimates

We will now introduce the notion of an apriori estimate, which can be used the estimate the size of a solution even before we have a solution. However, we will first introduce some basic results from the theory of Galerkin methods.

3.6.1 Galerkin Orthogonality

Let us recall the weak continuous problem (17) and the weak discrete problem (18).

$$\text{Find } u \in V \quad \text{s.t. } a(u, v) = l(v) \quad \forall v \in V. \tag{17}$$

$$\text{Find } u_h \in V_h \quad \text{s.t. } a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h.. \tag{18}$$

In fact, the key property of this problem formulation is that $a(u, v_h) = l(v_h) \quad \forall v_h \in V_h \subset V$. Thus, the Galerkin orthogonality property holds so that,

$$a(u, v_h) - a(u_h, v_h) = a(u - u_h, v_h) = 0, .$$

Keep in mind that we define $u - u_h \in V$

3.6.2 Céa's Lemma

Assume that the problems (17) and (18) satisfied the Lax Milgram criteria,

$$\begin{aligned} a(u, v) \leq C\|u\|\|v\| \quad (V_h), \quad \alpha\|u\|^2 \leq a(u, u) \quad \forall u, v \in V, \\ a(u_h, v_h) \leq C\|u_h\|\|v_h\| \; (V_h), \quad \alpha\|u_h\|^2 \leq a(u_h, u_h) \quad \forall u_h, v_h \in V_h. \end{aligned}.$$

Then the Céa's lemma says that this is satisfied,

$$\|u - u_h\| \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|v_h - v\|. \tag{19}$$

The lemma naturally arise using the following argumentation,

$$\begin{aligned} \alpha\|u - u_h\|^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) - a(u - u_h, v_h - u_h) \\ &\leq C\|u - u_h\|\|u - v_h\| \\ \implies \alpha\|u - u_h\| &\leq C\|v_h - u\| \leq C \inf_{v_h \in V_h} \|v_h - u\|. \end{aligned}. \tag{20}$$

3.6.3 Céa's Lemma for CP Method,

Since we have discrete coercivity, then $V_h \not\subset V$, thus the standard method does not work. Firstly, we want to use the results from 3.5. We have shown that

$$\begin{aligned} &\textit{Discrete coercivity} && \hat{\alpha}\|u_h\|_{a_h}^2 \leq \mathcal{A}(u_h, u_h) \\ &\textit{Boundedness (semi-discrete)} && \mathcal{A}(v, w_h) \leq \widetilde{C}\|v\|_{a_{h,*}}\|w_h\|_{a_h} \quad \forall v \in V_h \oplus H^4(\Omega). \\ &\textit{Boundedness (fully discrete)} && \mathcal{A}(v_h, w_h) \leq \overline{C}\|v\|_{a_{h,*}}\|w_h\|_{a_h} \quad \forall v_h, w_h \in V_h \end{aligned}$$

Let the difference have the form $u - u_h = (u - v_h) + (v_h - u_h)$ and the define identity

$$\|w_h\|_{a_{h,*}} \leq D\|w_h\|_{a_h}, \forall w_h \in V_h.$$

Thus, the norm can now be computed such that

$$\|u - u_h\|_{a_{h,*}} \leq \|u - v_h\|_{a_{h,*}} + \|v_h - u_h\|_{a_{h,*}} \leq \|u - v_h\|_{a_{h,*}} + D\|u_h - v_h\|_{a_h}.$$

Finally, following the same procedure as in (20) we get

$$\begin{aligned} \|u_h - v_h\|_{a_h}^2 \hat{\alpha} &\leq \mathcal{A}(u_h - v_h, u_h - v_h) \\ &= \mathcal{A}_h(u_h - u, u_h - v_h) + \mathcal{A}(u - v_h, u_h - v_h) \\ &\leq \mathcal{A}(u - v_h, u_h - v_h) \\ &\leq \widetilde{C}\|u - v_h\|_{a_{h,*}}\|u - v_h\|_{a_h} \end{aligned}$$

Observe that we now have $\|u_h - v_h\|_{a_h} \leq \frac{\widetilde{C}}{\hat{\alpha}}\|u - v_h\|_{a_{h,*}}$ and $\|u - u_h\|_{a_{h,*}} \leq \left(1 + D\widetilde{C}/\hat{\alpha}\right) \cdot \|u - v_h\|_{a_{h,*}}$. Hence, we have derived a equivalent Céa's lemma for the CP method.

$$\begin{aligned} \|u_h - v_h\|_{a_h} &\leq \frac{\widetilde{C}}{\hat{\alpha}} \inf_{v_h \in V_h} \|v_h - u\|_{a_{h,*}} \\ \|u - u_h\|_{a_{h,*}} &\leq \left(1 + D\widetilde{C}/\hat{\alpha}\right) \cdot \inf_{v_h \in V_h} \|u - v_h\|_{a_{h,*}} \end{aligned}$$

4 Numerical Results

4.1 Manufactured Solution

4.2 Convergence Rate

4.3 Condition Number

5 HCP Method copied from NGSolve

We consider the Kirchhoff plate equation: Find $w \in H^2$, such that

$$\int \nabla^2 w : \nabla^2 v = \int f v$$

A conforming method requires C^1 continuous finite elements. But there is no good option available, and thus there is no H^2 conforming finite element space in NGSolve.

$$\sum_T \nabla^2 w : \nabla^2 v - \int_E \{\nabla^2 w\}_{nn} \, [\partial_n v] - \int_E \{\nabla^2 v\}_{nn} \, [\partial_n w] + \alpha \int_E [\partial_n w] [\partial_n v]$$

[Baker 77, Brenner Gudi Sung, 2010]

We consider its hybrid DG version, where the normal derivative is a new, facet-based variable:

$$\sum_T \nabla^2 w : \nabla^2 v - \int_{\partial T} (\nabla^2 w)_{nn} \, (\partial_n v - \widehat{v_n}) - \int_{\partial T} (\nabla^2 v)_{nn} \, (\partial_n w - \widehat{w_n}) + \alpha \int_E (\partial_n v - \widehat{v_n})(\partial_n w - \widehat{w_n})$$

6 Cahn Hilliard Equation on a Closed Membrane

Let c_0 and c_1 indicate the concentration profile of the substances in a 2 -phase system such that $c_0(\mathbf{x},t) : \boldsymbol{\Omega} \times [0,\infty] \rightarrow [0,1]$ and similarly $c_1(\mathbf{x},t) : \boldsymbol{\Omega} \times [0,\infty] \rightarrow [0,1]$, where $\mathbf{x}$ is a element of some surface $\boldsymbol{\Omega}$ and t is time. However, in the 2 phase problem will we will restrict ourself so that $c_0(t,\mathbf{x}) + c_1(t,\mathbf{x}) = 1$ at any $\mathbf{x}$ at time t. A property of the restriction is that we now can express c_0 using c_1, with no loss of information. Hence, let us now define $c = c_0$ so $c(\mathbf{x},t) : \boldsymbol{\Omega} \times [0,\infty] \rightarrow [0,1]$. It has been shown that 2 phase system if thermodynamically unstable can be evolve into a phase separation described by a evolutional differential equation [2] using a model based on chemical energy of the substances. However, further development has been done [16] to solve this equation on surfaces. Now assume model that we want to describe is a phase-separation on a closed membrane surface Γ, so that $c(\mathbf{x},t) : \Gamma \times [0,T] \rightarrow [0,1]$. Then is the surface Cahn Hilliard equation described such that

$$\rho \frac{\partial c}{\partial t} - \nabla_\Gamma \left(M \nabla_\Gamma \left(f_0' - \varepsilon^2 \nabla_\Gamma^2 c \right) \right) = 0 \quad \text{on } \Gamma. \tag{21}$$

We define here the tangential gradient operator to be $\nabla_\Gamma c = \nabla c - (\mathbf{n}\nabla c)\,\mathbf{n}$ applied on the surface Γ restricted to $\mathbf{n} \cdot \nabla_\Gamma c = 0$.

Lets define ε to be the size of the layer between the substances c_1 and c_2. The density ρ is simply defined such that $\rho = \frac{m}{S_\Gamma}$ is a constant based on the total mass divided by the total surface area of Γ. Here is the mobility M often derived such that is is dependent on c and is crucial for the result during a possible coarsening event [16]. However, the free energy per unit surface $f_0 = f_0(c)$ is derived based on the thermodynamical model and should according to [16] be non convex and nonlinear.

A important observation is that equation (21) is a fourth order equation which makes it more challenging to solve using conventional FEM methods. This clear when writing the equation on the equivalent weak form and second order equations arise.

7 Appendix

7.1 The Space $L^2(\Omega)$

Using the definition from [17] and we let Ω be a an open set in $\mathbb{R}^d$ and $p \in \mathbb{R}$ such that $p \geq 1$. Then we denote $L^p(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that it is equipped in a finite Banach space

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}.$$

Now let $u, v : \Omega \to \mathbb{R}$. Then is $L^2(\Omega)$ a Hilbert space when the inner product is finite such that this exists

$$(u, v)_{L^p(\Omega)} = \int_\Omega uv.$$

If the integral is finite do we say that $u, v \in L^p(\Omega)$.

7.2 The Space $H^m(\Omega)$, $m > 1$

Again using the definition from [17]. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\quad \alpha \geq 0$, such that $|\alpha| = \sum_{i=1}^d \alpha_i$. Now we define the space

$$H^m(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \quad \forall\alpha : |\alpha| \leq m\right\}.$$

Suppose that u, v is measurable functions. We can now define $u \in H^m(\Omega)$ the Banach space is finite .

$$\|u\|_{H^m(\Omega)} = \left(\|u\|_{L^2(\Omega)} + \sum_{k=1}^m |u|^2_{H^k(\Omega)}\right), \quad |u|_{H^k(\Omega)} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha u\|^2_{L^2(\Omega)}}$$

Similarly for the finite Hilbert space

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v$$

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