1 DG for Possion Problem

1.1 Possion Problem

Lets define the problem

$$
\begin{aligned}
-\varepsilon \nabla u &= f \quad \text{in } \Omega \\
u &= u_D \quad \text{on } \Gamma_D \\
\partial_n u &= g \quad \text{on } \Gamma_N \\
\partial_n u + \beta u &= h \quad \text{on } \Gamma_R
\end{aligned}
$$

Here is $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$.

1.2 Classical DG

1.3 Hybrid DG Method

We want to write this on a weak form. Let the spaces we work on be

$$
H^1(\mathcal{T}_h) = \left\{ u \in L^2(\Omega), u \in H^1(T)\, \forall T \in \mathcal{T}_h \right\}
$$

.

For the problem to be discontinuous do we define the trial and test function to be $u \in H^1(\Omega)$ and $v \in H^1(\mathcal{T}_h)$. Thus,

$$
-\sum_{T\in\mathcal{T}_h} \int_T \varepsilon \nabla^2 u \cdot v dx = \sum_{T\in\mathcal{T}_h} \left\{ \int_T \varepsilon \nabla u \nabla v dx - \int_{\partial T} \varepsilon \cdot \partial_n u \cdot v ds \right\} = \sum_{T\in\mathcal{T}_h} \int_T f \cdot v dx. \tag{1}
$$

But we want to introduce the shorter notation equivalently such that

$$
\sum_{T\in\mathcal{T}_h} \left\{ \varepsilon (\nabla u, \nabla v)_T - \varepsilon \langle \partial_n u, v \rangle_{\partial T} \right\} = \sum_{T\in\mathcal{T}_h} (f, v). \tag{2}
$$

Where $\langle \cdot, \cdot \rangle$ is the surface integral operator. Before we contitinue do we want to introduce a alternative method to integrate using edges. Let $v_F \in L^2(\mathcal{F}_h)$ for the set of all facets $\mathcal{F}_h$. Now the surface integral can be rewritten such that

$$
\sum_{T\in\mathcal{T}_h} \varepsilon \langle \partial_n u, v_F \rangle = \sum_{E\in\mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E + \sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle. \tag{3}
$$

Here are we using the definitions n^+ and n^- illustrated using figure 1. Lets define some crucial spaces for the DG method

$$
\begin{aligned}
V &= \left\{ (u, u_F) : u \in H^2(\mathcal{T}_h) \cap H^1(\Omega), u_F \in L^2(\mathcal{F}_h) \right\} \\
V_h &= \left\{ (u, u_F) : u \in \mathcal{P}^k(T)\, \forall T \in \mathcal{T}_h, \quad u_F \in \mathcal{P}^k(E)\, \forall E \in \mathcal{F}_h \right\}
\end{aligned}
$$

and now including drichlet conditions using the previous definition

$$
\begin{aligned}
V_D &= \{ (u, u_F) \in V, u_F = u_D \quad \text{on } \Gamma_D \} \quad & V_{h,D} &= \{ (u, u_F) \in V_h, u_F = u_D \quad \text{on } \Gamma_D \} \\
V_0 &= \{ (u, u_F) \in V, u_F = 0 \quad \text{on } \Gamma_D \} \quad & V_{h,0} &= \{ (u, u_F) \in V_h, u_F = 0 \quad \text{on } \Gamma_D \}
\end{aligned}
$$

Defining $(u, u_F) \in V_D$ and $(v, v_F) \in V_0$. Now adding (2) and (3) can we easily see that

$$
\sum_{T\in\mathcal{T}_h} \left\{ \varepsilon (\nabla u, \nabla v)_T \right\} = \sum_{T\in\mathcal{T}_h} (f, v)_T + \sum_{E\in\mathcal{F}^{int}} \varepsilon \langle \partial_{n^+} u, v_F \rangle_E + \varepsilon \langle \partial_{n^-} u, v_F \rangle_E + \sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle. \tag{4}
$$

Applying the Neumann conditions on Γ_N and Γ_R, can the condition on the exterior facets be rewritten such that

$$
\sum_{E\in\mathcal{F}^{ext}} \varepsilon \langle \partial_n u, v_F \rangle = \varepsilon \langle g, v_F \rangle_{\Gamma_N} + \varepsilon \langle h - \beta u, v_F \rangle_{\Gamma_R}
$$

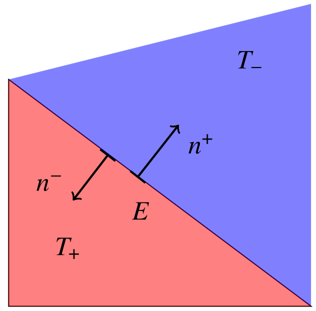

Figure 1: Edge E shared by the triangles T_- and T_+ and the normal unit vectors n^+ and n^-.

Keep in mind that we on the exterior boundaries define the integral so $\langle f, v_F\rangle_\Gamma = \int_\Gamma f \cdot v_F \cdot nds$ for a arbitary neumann boundary function f on some surface Γ. Hence (4) ends up being

$$\sum_{T\in\mathcal{T}_h} \varepsilon\left(\nabla u, \nabla v\right) - \sum_{E\in F^{int}} \left(\varepsilon\left\langle \partial_{n^+} u, v_F\right\rangle_E + \varepsilon\left\langle \partial_{n^-} u, v_F\right\rangle_E\right) + \beta\left\langle \varepsilon u, v_F\right\rangle_{\Gamma_R} = \sum_{T\in\mathcal{T}_h} \left(f, v\right)_T + \left\langle g, v_F\right\rangle_{\Gamma_N} + \left\langle h, v_F\right\rangle_{\Gamma_R}. \quad (5)$$

According to Lehrenfeld 2010 [**lehrenfeld2010**] at page 13 on equation (1.2.7) is (5) equivalent to

$$\sum_{T\in\mathcal{T}_h} \left(\varepsilon\nabla u, \nabla v\right)_T - \sum_{T\in\mathcal{T}_h} \left\langle \varepsilon\partial_n u, [\![v]\!]\right\rangle_{\partial T} + \beta\left\langle \varepsilon u, v_F\right\rangle_{\Gamma_R} = \sum_{T\in\mathcal{T}_h} \left(f, v\right) + \left\langle \varepsilon g, v_F\right\rangle_{\Gamma_N} + \left\langle \varepsilon h, v_F\right\rangle_{\Gamma_R} \quad (6)$$

Where, $u, u_F \in V_D$ and $v, v_F \in V_h$ Here is the jump defined simply as $[\![v]\!] = v - v_F$. Remember that $v_F = tr_{\partial T}\left(v\right)$. What we see is for (5) and (6) to be equivalent must this be true.

$$\sum_{T\in\mathcal{T}_h} \left\langle \varepsilon\partial_n u, [\![v]\!]\right\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \left\langle \varepsilon\partial_n u, v\right\rangle_{\partial T} - \left\langle \varepsilon\partial_n u, v_F\right\rangle_{\partial T} = \sum_{E\in\mathcal{F}^{int}} \varepsilon\left(\left\langle \partial_{n^+} u, v_F\right\rangle_E + \left\langle \partial_{n^-} u, v_F\right\rangle_E\right). \quad (7)$$

Since $(u, u_F) \in V$ is has to be continious, hence the jump is $[\![u]\!] = 0$ for the correct solution. Hence, adding $-\left\langle \varepsilon\partial_n v, [\![u]\!]\right\rangle_{\partial T}$ for symmetry and $\tau_h\left\langle \varepsilon [\![u]\!], [\![v]\!]\right\rangle_{\partial T}$ for stability with some stabilization parameter τ_h for each $T \in \mathcal{T}_h$. This can be added to lhs on (6) such that,

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \left(\varepsilon\nabla u, \nabla v\right)_t - &\sum_{T\in\mathcal{T}_h} \left\{\left\langle \varepsilon\partial_n u, [\![v]\!]\right\rangle_{\partial T} - \left\langle \varepsilon\partial_n v, [\![u]\!]\right\rangle_{\partial T} + \tau_h\left\langle \varepsilon [\![u]\!], [\![v]\!]\right\rangle_{\partial T}\right\}\\ &+ \beta\left\langle \varepsilon u, v_f\right\rangle_{\Gamma_R}\\ &= \sum_{T\in\mathcal{T}_h} \left(f, v\right) + \left\langle \varepsilon g, v_F\right\rangle_{\Gamma_N} + \left\langle \varepsilon h, v_F\right\rangle_{\Gamma_R}\end{aligned} \quad (8)$$

Finally, we can now construct the discrete system. Let now $u, u_F \in V_{h,D}$ and $v, v_F \in V_{h,0}$ be the discretized spaces. Using what we have in (6) can we define

$$F\left(v, v_F\right) = \sum_{T\in\mathcal{T}_h} \left(f, v\right) + \left\langle \varepsilon g, v_F\right\rangle_{\Gamma_N} + \left\langle \varepsilon h, v_F\right\rangle_{\Gamma_R}$$

$$B\left(u, u_F, v, v_F\right) = \sum_{T\in\mathcal{T}_h} \left(\varepsilon\nabla u, \nabla v\right)_t - \sum_{T\in\mathcal{T}_h} \left\{\left\langle \varepsilon\partial_n u, [\![v]\!]\right\rangle_{\partial T} - \left\langle \varepsilon\partial_n [\![u]\!]\right\rangle_{\partial T} + \tau_h\left\langle \varepsilon [\![u]\!], [\![v]\!]\right\rangle_{\partial T}\right\} + \beta\left\langle \varepsilon u, v_F\right\rangle_{\Gamma_R}$$

Hence, the numerical method must solve

$$B\left(u, u_F, v, v_F\right) = F\left(v, v_F\right). \quad (9)$$

2 C^0 Interior Penalty Method for Biharmonic Equation

2.1 Introduction of the Boundary Value Problem

In this section do we want to establish a numerical method to fourth order equations. Instead of embarking on the special case of surface PDE described in (**??**) can we establish a general numerical theory on $\mathbb{R}^2$, which we later can generalize on closed surface later. Assume that we restrict ourself to a compact surface $\Omega \in \mathbb{R}^2$ and let $f \in L^2(\Omega)$ as defined in ??. Let say we want to solve the equation on the form.

$$\begin{aligned} \Delta^2 u - \beta \Delta u + \gamma u = f \quad & \beta, \gamma \geq 0 \\ \frac{\partial u}{\partial n} = 0 \quad & \text{on } \Omega \\ \frac{\partial \Delta u}{\partial n} = q \quad & \text{on } \partial\Omega \end{aligned} . \tag{10}$$

For convenience are the boundary condition q chosen to be defined via a $\phi \in H^4(\Omega)$ such that $q = \frac{\partial \Delta \phi}{\partial n}$ so $\frac{\partial \phi}{\partial n} = 0.\ \partial\Omega$.

2.2 Weak Formulation

We want to rewrite (10) on weak formulation. Now define the Hilbert space

$$V = \left\{ v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \right\}.$$

It can be shown [**gu2012c0**] that a convinient form is to write it as

$$\begin{aligned} a(u, v) &= (f, v)_{L^2(\Omega)} - (q, v)_{L^2(\partial\Omega)} \\ &= \int_\Omega D^2 w : D^2 v dx + \int_\Omega \nabla w \nabla v dx + \int_\Omega \gamma w \cdot v dx. \end{aligned} \tag{11}$$

For all $\forall v \in V$, where

$$D^2 w : D^2 v = \sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j}.$$

In fact, according to [**gu2012c0**] can it be shown that the problem has a unique solution if and only if $\gamma > 0$. However, in the case where $\gamma = 0$ can we provoke a unique solution by introducing the condition

$$\int_\Omega f dx = \int_{\partial\Omega} q ds$$

Taking this into account can we expand the solution space such that

$$V^* = \begin{cases} V, & \text{if } \gamma > 0 \\ \{v \in V : v(p^*) = 0\}, & \text{if } \gamma = 0 \end{cases}$$

Where p^* is a corner in Ω. In fact, now all solutions of (11) exists in V^*.

2.3 Construction of C^0 Interior Penalty Method

We want to construct a C^0 interior penalty method based on C^0 Lagrange elements. Assume $\mathcal{T}_h$ be a triangulation of Ω and V_h be the à $\mathcal{P}_2$ Lagrange finite element space associated with $\mathcal{T}_h$

$$V_h = \left\{ v \in C\left(\overline{\Omega}\right) : v_T = v|_T \in \mathcal{P}_2(T) \quad \forall T \in \mathcal{T}_h \right\}$$

So that we can earn a similar space for the approximated solution space ,

$$V_h^* = \begin{cases} V_h, & \text{for } \gamma > 0 \\ \{v \in V_h : v(p^*) = 0\} & \text{for } \gamma = 0. \end{cases}$$

Here is p^* again a corner in Ω. Let us now generalize the Hilbert space as well to the approximated solution space by defining

$$H^k(\Omega, \mathcal{T}_h) = \left\{ H^1(\Omega) : v_T \in H^k(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

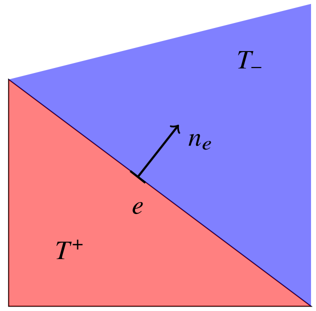

Figure 2: Edge e shared by the triangles T_- and T_+ and the normal unit vector n_e.

Now assume that that $e \in \mathcal{E}_h^i$ is shared between two triangles $T_-, T_+ \in \mathcal{T}_h$. Then we can assume that the unit normal from T_- to T_+ is described as n_e as illustrated in figure 2. Finally, we now want to define jumps internally,

$$\left[\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\right] = \frac{\partial v_{T_+}}{\partial n_e}|_e - \frac{\partial v_{T_-}}{\partial n_e}|_e, \quad \forall v \in H^2\left(\Omega, \mathcal{T}_h\right)$$
$$\left[\!\left[\frac{\partial^2 v_h}{\partial n_e^2} \right]\!\right] = \frac{\partial^2 v_{T_+}}{\partial n_e^2}|_e - \frac{\partial^2 v_{T_-}}{\partial n_e^2}|_e \quad \forall v \in H^3\left(\Omega, \mathcal{T}_h\right).$$

And similarly for means internally,

$$\left\{\!\!\left\{ \frac{\partial v_{T_-}}{\partial n_e} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial v_{T_+}}{\partial n_e}|_e + \frac{\partial v_{T_-}}{\partial n_e}|_e \right) \quad \forall v \in H^2\left(\Omega, \mathcal{T}_h\right)$$
$$\left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_{T_+}}{\partial n_e^2}|_e + \frac{\partial^2 v_{T_-}}{\partial n_e^2}|_e \right) \quad \forall v \in H^3\left(\Omega . \mathcal{T}_h\right),$$

Let the edges along the boundary be defined as $e \in \mathcal{E}_h^b$ along a some boundary triangle $\mathcal{T}_h$. We can then define the jump and mean as

$$\left[\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\right] = -\frac{\partial v_T}{\partial n_e}|_e \quad \forall v \in H^2\left(\Omega, \mathcal{T}_h\right)$$
$$\left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} = \frac{\partial v_T}{\partial n_e}|_e \quad \forall v \in H^3\left(\Omega, \mathcal{T}_h\right)$$

Using the results from [**gu2012c0**] can we formulate the discrete formulation the boundary value problem (10) using C^0 interior penalty method. Our goals is to find a $u_h \in V_h^*$ such that this is true,

$$\mathcal{A}\left(u_h, v_h\right) = \left(f, v_h\right)_{L^2(\Omega)} - \left(q, v_h\right)_{L^2(\partial\Omega)} \quad \forall v_h \in V_h^*. \tag{12}$$

Where $w_h, v_h \in V_h$ and

$$\begin{aligned} \mathcal{A}\left(w_h, v_h\right) = &\sum_{T \in \mathcal{T}_h} \int_T D^2 w_h : D^2 v_h \\ &+ \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w_h}{\partial n_e^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\right] ds \\ &+ \sum_{e \in \mathcal{E}_h} \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial w_h}{\partial n_e} \right]\!\right] ds \\ &+ \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\left[\frac{\partial w_h}{\partial n_e} \right]\!\right] \left[\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\right] ds \\ &+ \int_\Omega \beta \nabla w_h \cdot \nabla v_h dx + \int_\Omega \gamma w_h v_h dx. \end{aligned} \tag{13}$$

The notation $|e|$ is to describe the length of the edge e and $\sigma \geq 1$ is a penalty parameter.