MTH8408A
Méthodes Numériques d'Optimisation et de Contrôle Optimal

Introduction to Variational Calculus and Optimal Control

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The Problem

We seek a *function* $x : [0, 1] \to \mathbb{R}$ that solves the problem

$$\underset{x}{\text{minimize}} \int_0^1 f(t, x(t), \dot{x}(t))\,\mathrm{d}t \quad \text{subject to } x(0) = x_0,\ x(1) = x_1,$$

where we assume that $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous and has continuous partial derivatives.

Under our assumption it is possible to:

1. show that if x is a solution, then it is piecewise $\mathcal{C}^1$,
2. give an expression of the Fréchet differential of the objective,
3. state first-order necessary optimality conditions.

CN1: Euler-Lagrange Conditions

Suppose $x \in \mathcal{C}^1_{\text{PW}}([0,\,1])$ is a local minimizer over all functions in $\mathcal{C}^1_{\text{PW}}([0,\,1])$ satisfying the fixed end-points conditions $x(0) = x_0$ and $x(1) = x_1$.

Then every smooth section of x must satisfy the second-order partial-differential equation

$$\frac{\partial f}{\partial x}(x(t), \dot{x}(t), t) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}}(x(t), \dot{x}(t), t) = 0. \tag{1}$$

Watch out for the *total derivative*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}} = \dot{x}\frac{\partial^2 f}{\partial x \partial \dot{x}} + \ddot{x}\frac{\partial^2 f}{\partial \dot{x}^2} + \frac{\partial^2 f}{\partial t \partial \dot{x}}$$

First Integrals

1. Suppose $\partial f/\partial t = 0$, then

$$\frac{\partial f}{\partial x} - \dot{x}\frac{\partial^2 f}{\partial x \partial \dot{x}} - \ddot{x}\frac{\partial^2 f}{\partial \dot{x}^2} = 0.$$

   Multiplying by $\dot{x}$, this is the same as (check as an exercise)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(x,\dot{x},t) - \dot{x}\frac{\partial f}{\partial \dot{x}}(x,\dot{x},t)\right] = 0, \qquad t \in [0,\ 1],$$

   i.e.,

$$f(x,\dot{x},t) - \dot{x}\frac{\partial f}{\partial \dot{x}}(x,\dot{x},t) = \text{constant}, \qquad t \in [0,\ 1].$$

2. Suppose $\partial f/\partial x = 0$, then

$$\frac{\partial f}{\partial \dot{x}}(x,\dot{x},t) = \text{constant}, \qquad t \in [0,\ 1].$$

Problems in Several Variables

$$\begin{aligned} &\underset{x}{\text{minimize}} \int_0^1 f(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, t)\,\mathrm{d}t \\ &\text{subject to } x(0) = x_0,\ x(1) = x_1, \end{aligned}$$

where $f : \mathbb{R}^{2n+1} \to \mathbb{R}$.

Each smooth section of each x_i must satisfy the Euler-Lagrange equation:

$$\frac{\partial f}{\partial x_i}(x, \dot{x}, t) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}_i}(x, \dot{x}, t) = 0 \qquad i = 1, \ldots, n.$$

Transversality Conditions

When the endpoints or the final time is not fixed, we impose, in addition to the Euler-Lagrange conditions:

1. if t_i is given and $x(t_i)$ is fixed ($i = 0, 1$), then $x(t_i) = x_i$,
2. if t_i is given but $x(t_i)$ is free ($i = 0, 1$), then

$$\frac{\partial f}{\partial \dot{x}}(x(t_i), \dot{x}(t_i), t_i) = 0,$$

3. if t_i is free ($i = 0, 1$) but $x(t_i) = \psi_i(t_i)$, then

$$f(x(t_i), \dot{x}(t_i), t_i) + \left(\dot{\psi}_i(t_i) - \dot{x}(t_i)\right) \frac{\partial f}{\partial \dot{x}}(x(t_i), \dot{x}(t_i), t_i) = 0,$$

4. if both t_i and $x(t_i)$ are free, then

$$f(x(t_i), \dot{x}(t_i), t_i) - \dot{x}(t_i) \frac{\partial f}{\partial \dot{x}}(x(t_i), \dot{x}(t_i), t_i) = 0.$$

Problems with Isoperimetric Constraints

$$
\begin{aligned}
\underset{x}{\text{minimize}} \; & \int_0^1 f(x, \dot{x}, t) \, \mathrm{d}t \\
\text{subject to} \; & \int_0^1 h_i(x, \dot{x}, t) \mathrm{d}t = 0, \quad i = 1, \ldots, m, \\
& x(0) = x_0, \; x(1) = x_1.
\end{aligned}
$$

All previous results apply with f replaced by the Hamiltonian

$$
H(x, \dot{x}, t, \lambda) := f(x, \dot{x}, t) - \sum_{i=1}^{m} \lambda_i h_i(x, \dot{x}, t),
$$

where the λ_i are Lagrange multipliers.

The Problem

$$\begin{aligned}\underset{x,u}{\text{minimize}}\ & h(x(t_1)) + \int_{t_0}^{t_1} f_0(x(t), u(t), t)\,\mathrm{d}t\\ \text{subject to}\ & \dot{x}_i(t) = f_i(x(t), u(t), t), \quad i = 1, \ldots, n,\\ & x_i(t_0) = x_i^0, \quad i = 1, \ldots, n.\end{aligned}$$

Define the Hamiltonian

$$H(x, u, p, t) := f_0(x, u, t) + \sum_{i=1}^{n} p_i f_i(x, u, t),$$

where the $p_i(t)$ are called the *adjoint states*.

CN1: Minimum Principle

If $u^* \in \mathcal{C}_{\mathrm{PW}}([t_0,\ t_1])^m$ is the optimal control and $x^* \in \mathcal{C}^1_{\mathrm{PW}}([t_0,\ t_1])^n$ is the corresponding optimal trajectory, there must exist functions $p_i^* \in \mathcal{C}^1_{\mathrm{PW}}([t_0, t_1])$, $i = 1, \ldots, n$, solving the *adjoint equations*

$$\begin{aligned} \dot{p}_i(t) &= -\frac{\partial H}{\partial x_i}(x^*(t), u^*(t), p(t), t), \\ p_i(t_1) &= \frac{\partial h}{\partial x_i}(x^*(t_1)), \quad i = 1, \ldots, n, \end{aligned}$$

such that

$$u^*(t) = \arg\min_{u \in U} H(x^*(t), u, p^*(t), t), \quad \text{for all } t \in [t_0, t_1],$$

where U is the set of feasible controls.
Moreover, if the system does not depend explicitly on t, i.e., if $\partial H/\partial t = 0$, then

$$H(x^*(t), u^*(t), p^*(t)) = \text{constant}, \quad t \in [t_0,\ t_1].$$