MTH8408A
Méthodes Numériques d'Optimisation et de Contrôle Optimal

Introduction to Optimization

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What is Optimization?

Problems where we must find the configuration of a system, or a selection of parameters such that

- the system operates in an *optimal* way, as defined by a performance measure given by the user
- the configuration / the parameters satisfy prescribed conditions.

Examples:

1. find the most profitable investment given the market conditions
2. find the lightest fuselage subject to aerodynamics constraints
3. find the most scenic route from Montréal to Québec
4. what is the configuration of a water molecule at equilibrium?
5. what is the most fair way to grade students given the difficulty level of classes?
6. what is the trajectory of a ship that must dock into the space station?

Writing an Optimization Problem

More formally, the concept of *performance* and the conditions to be satisfied are described by functions.

A general problem is written in the form

$$\underset{x}{\text{minimize}} \;\; f(x) \quad \text{subject to} \;\; x \in \Omega,$$

where

- x are the *design variables*, or simply *variables*
- f is the *objective function* (it describes performance)
- Ω is the *feasible set* (it specifies restrictions on x)

The design variables live in either $\mathbb{R}^n$ or a function space.

What Do We Learn in this Course?

Solving an optimization problem is not just a matter of writing it down and calling a solver.
Most solvers will simply fail if we write

$$\underset{x\in\mathbb{R}}{\text{minimize}}\ x \quad \text{subject to}\ x^2 = 0.$$

The most important points about numerical methods is to understand

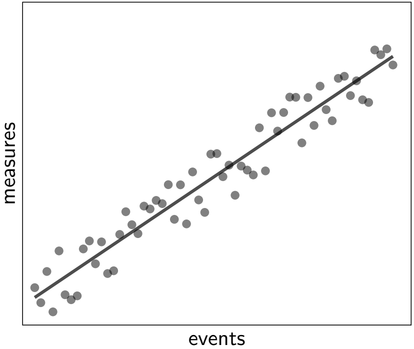
- *why* they fail, or *why* they do not perform well,
- *why* they do not return the solution that we expect.

In this course, we learn

- the mechanisms underlying many optimization algorithms and solvers,
- to write simple yet effective solvers ourselves,
- to model problems in a modeling language,
- to avoid some pitfalls when modeling problems.

One of the Most Famous Optimization Problems

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; \frac{1}{2}\|Ax - b\|^2$$



- Important subproblem inside optimization algorithms,
- good mixture of optimization and linear algebra,
- direct and iterative methods.

A Finance Optimization Problem

Each investment $j = 1, \ldots, n$ has an associated *return* R_j in the next time period.
In general, R_j is a random variable. The *reward* is $\mathbf{E}[R_j]$.

An investor must decide what fraction x_j of a capital to invest in j.
A *portfolio* is such an allocation $(x_1, \ldots, x_n)$.

Markowitz defined the *expected return* as $\sum_j x_j \mathbf{E}[R_j]$.

Investments with high reward also carry some risk. One way to define the *risk* is:

$$\mathbf{E}\left[\sum_j x_j \left(R_j - \mathbf{E}[R_j]\right)^2\right].$$

We want to maximize the reward and at the same time minimize the risk.
That is a *multi-objective* optimization problem. One way to treat solve it to combine both objectives into one.

The Markowitz Portfolio Optimization Problem

Our objective is a weighted combination of reward and risk:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & -\mu \sum_{j=1}^{n} x_j \mathbf{E}[R_j] + \mathbf{E}\left[\sum_{j=1}^{n} x_j \left(R_j - \mathbf{E}[R_j]\right)^2 \right] \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \; j = 1, \ldots, n, \end{aligned}$$

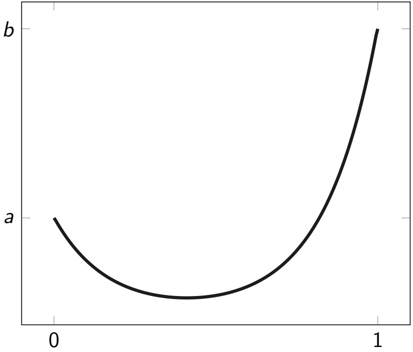
where the parameter $\mu > 0$ balances the risk and reward.

Before solving the problem, we estimate $\mathbf{E}[R_j]$ based on historical data.

A Variational Calculus Problem: The Catenary

A string of length $L > 0$ is suspended from both ends.
What is the shape of the string at rest under gravity?

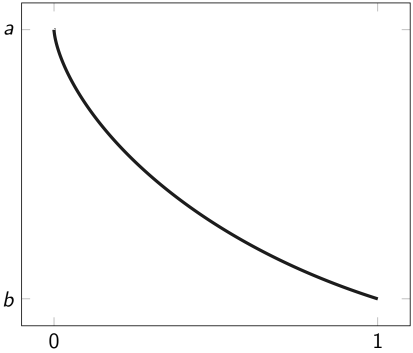
$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \int_0^1 x(t)\sqrt{1+\dot{x}(t)^2}\,\mathrm{d}t \\ \text{subject to} \quad & \int_0^1 \sqrt{1+\dot{x}(t)^2}\,\mathrm{d}t = L, \\ & x(0) = a,\ x(1) = b. \end{aligned}$$



A Variational Calculus Problem: The Brachistochrone

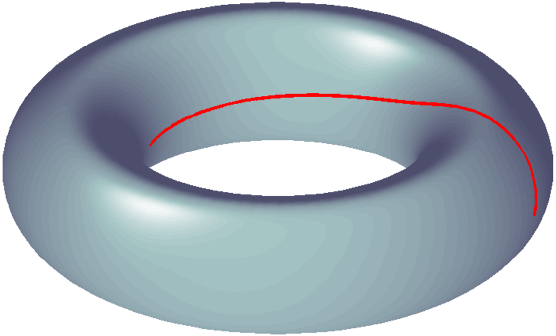
Design a slide with specified height at its ends so a unit mass starting at rest slides down in minimum time, neglecting friction.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \int_0^1 \frac{\sqrt{1+\dot{x}(t)^2}}{\sqrt{x(0)-x(t)}}\,\mathrm{d}t \\ \text{subject to} \quad & x(0)=a,\ x(1)=b. \end{aligned}$$



A Geodesics Problem

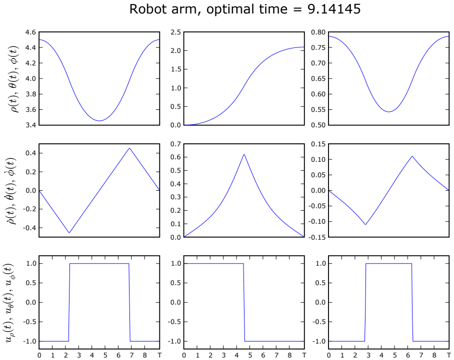
Compute the shortest path between two points on a manifold.



An Optimal Control Problem

Control the motion of a robot arm in minimum time, neglecting the Coriolis force.

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & T \\
\text{subject to} \quad & L\ddot{\rho}(t) = u_\rho(t), \\
& I_\theta(t)\ddot{\theta}(t) = u_\theta(t), \\
& I_\phi(t)\ddot{\phi}(t) = u_\phi(t), \\
& I_\theta(t) = \frac{(L-\rho(t))^3 + \rho(t)^3}{3}\sin^2(\phi(t)), \\
& I_\phi(t) = \frac{(L-\rho(t))^3 + \rho(t)^3}{3}, \\
& \rho(0) = 4.5,\ \theta(0) = 0,\ \phi(0) = \frac{\pi}{4}, \\
& \rho(T) = 4.5,\ \theta(T) = \frac{2\pi}{3},\ \phi(T) = \frac{\pi}{4}
\end{aligned}
$$



Numerous Other Problems

- www.princeton.edu/~rvdb
- www.princeton.edu/~rvdb/tex/trajopt/trajopt_OptEng.pdf
- COPS problems www.mcs.anl.gov/~more/cops/cops3.pdf
- orfe.princeton.edu/~rvdb/ampl/nlmodels

Contact me if you are looking for a problem in a specific area.