Uppsala University

Degree Project D in Computational Science, 30c

Department of Physics and Astronomy
Division of Material stuff?

Holonomic optimal control for qudits

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October 5, 2021

Abstract

1 Engelskt abstrakt

Sammanfattning

Svenskt abstrakt

Contents

Test! Här är några relevant källor! [1],[2],[3]

1 Introduction

2 Background

2.1 Lambda system

A traditional Λ-system is a quantum mechanical system with two decoupled 'ground states' and one 'excited state' coupled to both ground states. Such a state with two ground state could realize a qubit by letting the ground state act as $|0\rangle$ and $|1\rangle$.

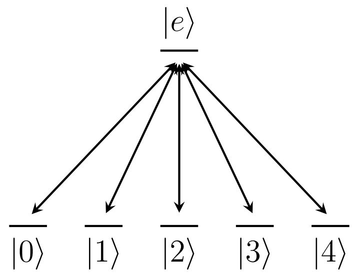

Figure 1: test figure

alfa $= \frac{\pi}{2}\sin^2(\frac{\pi t}{T})$
beta $= \eta(1-\cos\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right))$
alfa' $= \frac{\pi^2}{2T}\sin\left(\frac{2\pi t}{T}\right)$
beta' $= \frac{\eta\pi^2}{2T}\sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right)$

$$\Omega(t) = 2((\frac{\eta\pi^2}{2T}\sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right))\cot\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right)\sin\left(\eta(1-\cos\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right))\right)$$
$$+\eta(1-\cos\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right))\cos\left(\eta(1-\cos\left(\frac{\pi}{2}\sin^2(\frac{\pi t}{T})\right))\right))$$

2.2 Coupled excited states

Hamiltonian of a "double" tripod

$$H(t) = \sum_{i=1}^{2}\sum_{j=1}^{3}\omega_{ij}\,|j\rangle\langle e_i| + \text{h.c} \tag{1}$$

Find dark states $H(t)\,|d\rangle = 0\,|d\rangle$ on form $|d\rangle = \sum_{k=1}^{3} d_k\,|k\rangle$ Result

$$\begin{aligned} d_1 &= \frac{1}{\omega_{11}^*}\left(\omega_{12}^*\omega_{23}^* - \omega_{13}^*\omega_{22}^*\right) \\ d_2 &= \frac{\omega_{13}^*\omega_{21}^*}{\omega_{11}^*} - \omega_{23}^* \\ d_3 &= \omega_{22}^* - \frac{\omega_{12}^*\omega_{21}^*}{\omega_{11}^*} \end{aligned} \tag{2}$$

Bright state $\langle b|d\rangle = 0$

$$\begin{aligned} |b_1\rangle &= \omega_{11}|1\rangle + \omega_{12}|2\rangle + \omega_{13}|3\rangle \\ |b_2\rangle &= \omega_{21}|1\rangle + \omega_{22}|2\rangle + \omega_{23}|3\rangle \end{aligned} \tag{3}$$

Check if $\langle b_1|b_2\rangle = 0$? Holds if $\omega_{11} = -\frac{1}{\omega_{21}^*}\left(\omega_{12}^*\omega_{22} + \omega_{13}\omega_{23}^*\right)$

Change basis,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega_{11} & \omega_{21} & d_1 \\ 0 & 0 & \omega_{12} & \omega_{22} & d_2 \\ 0 & 0 & \omega_{13} & \omega_{23} & d_3 \end{pmatrix} \tag{4}$$

New hamiltonian in $\{|e_1\rangle, |e_2\rangle, |b_1\rangle, |b_2\rangle, |d\rangle\}$ is

$$H_d(t) = T^\dagger H(t) T = \left(\sum_{j=1}^{2}\sum_{k=1}^{3} (\omega_{jk})^2 |b_j\rangle\langle e_j|\right) + [\omega_{11}\omega_{12} + \omega_{12}\omega_{22} + \omega_{13}\omega_{23}]\left(|b_2\rangle\langle e_1| + |b_1\rangle\langle e_2|\right) + \text{h.c} \tag{5}$$

Double check this calculation.

Subspace is $\{|e_1\rangle, |e_2\rangle, |b_1\rangle, |b_2\rangle\}$ since $|d\rangle$ is decoupled.

Simplify notation, $\Omega_j = \sum_{k=1}^{3}\omega_{jk}^2$ and $\Gamma = \omega_{11}\omega_{12} + \omega_{12}\omega_{22} + \omega_{13}\omega_{23}$ now Hamiltonian looks like

$$H_d = \sum_{j=1}^{2}\Omega_j |b_j\rangle\langle e_j| + \Gamma\left(|b_2\rangle\langle e_1| + |b_1\rangle\langle e_2|\right) + \text{ h.c} \tag{6}$$

Now find a "dark path" $|D_i(t)\rangle$ such that $\langle D_i(t)| H_d(t) |D_i(t)\rangle = 0$, for $i = 1, 2$

Set

$$\begin{aligned} |D_1\rangle &= a_1|b_1\rangle + a_2|3\rangle + a_3|e_1\rangle \\ |D_2\rangle &= c_1|b_2\rangle + c_2|3\rangle + c_3|e_2\rangle \end{aligned} \tag{7}$$

and

NEW TRY

$$H(t) = \sum_{i=1}^{2}\sum_{j=1}^{3}\omega_{ij}|j\rangle\langle e_i| + \text{h.c} \tag{8}$$

New dark state

$$|d\rangle = d_1|1\rangle + d_2|2\rangle \tag{9}$$

with $d_1 = -(\omega_{12}^* + \omega_{22}^*)$, $d_2 = (\omega_{11}^* + \omega_{21}^*)$ **There exists no dark state on this form ($d_1 = d_2 = 0$)**

bight state

$$|b\rangle = -d_2^*|1\rangle + d_1^*|2\rangle \tag{10}$$

CRITERA? $\omega_{11}\omega_{22} = \omega_{12}\omega_{21} \implies \frac{\omega_{11}}{\omega_{12}} = \frac{\omega_{21}}{\omega_{22}}$

change basis to $\{e_1, e_2, 3, b, d\}$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_2^* & d_1 \\ 0 & 0 & 0 & d_1^* & d_2 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{11}$$

New hamiltonian

$$H_d = \left(\sum_{j=1}^{2} \omega_{j3} \left|3\right\rangle \left\langle e_j\right| + \sigma_j \left|b\right\rangle \left\langle e_j\right| \right) + \text{h.c} \tag{12}$$

with

$$\begin{aligned} \sigma_1 &= \omega_{12} d_1^* - \omega_{11} d_2^* = -\omega_{12}(\omega_{12} + \omega_{22}) - \omega_{11}(\omega_{11} + \omega_{21}) \\ \sigma_2 &= \omega_{21} d_1^* - \omega_{12} d_2^* = -\omega_{23}(\omega_{12} + \omega_{22}) - \omega_{12}(\omega_{11} + \omega_{21}) \end{aligned}$$

Now find a "dark path" $\left|D_i(t)\right\rangle$ such that $\left\langle D_i(t)\right| H_d(t) \left|D_i(t)\right\rangle = 0$, for $i = 1, 2$
Set

$$\begin{aligned} \left|D_1\right\rangle &= a_1 \left|b\right\rangle + a_2 \left|3\right\rangle + a_3 \left|e_1\right\rangle \\ \left|D_2\right\rangle &= c_1 \left|b\right\rangle + c_2 \left|3\right\rangle + c_3 \left|e_2\right\rangle \end{aligned} \tag{13}$$

Assume that $\left|b_2\right\rangle = \Delta \left|b_1\right\rangle$

$$\begin{aligned} \left|D_1\right\rangle &= \cos\alpha \cos\beta(((e^{-i\varphi}))) \left|b_1\right\rangle - \cos\alpha \sin\beta \left|3\right\rangle - i \sin\alpha \left|e_1\right\rangle \\ \left|D_2\right\rangle &= \cos\alpha \cos\beta(((e^{-i\varphi}))) \left|b_2\right\rangle - \cos\alpha \sin\beta \left|3\right\rangle - i \sin\alpha \left|e_2\right\rangle \end{aligned} \tag{14}$$

then $\left\langle D_1 | D_2\right\rangle = 0 \implies \left\langle b_1 | b_2\right\rangle = -\tan^2\beta \implies \Delta = -i \tan\beta$

$$\begin{aligned} &\left\langle D_i\right| H_d \left|D_i\right\rangle = \left[\sigma_1^* + \sigma_2^* - \sigma_i\right] \Delta_i \cos\beta + \omega_{i3} \sin\beta = 0 \\ &\implies \omega_{i3} = \Delta_i(\sigma_i - \sigma_1^* - \sigma_2^*) \cot\beta \end{aligned} \tag{15}$$

NEW NEW TRY!

$$H(t) = \sum_{i=1}^{2} \sum_{j=1}^{3} \omega_{ij} \left|j\right\rangle \left\langle e_i\right| + \text{h.c} \tag{16}$$

Dark state

$$\begin{aligned} d_1 &= -\left(\omega_{22}^* + \frac{\omega_{12}^* \omega_{23}^*}{\omega_{13}^*} \right) \\ d_2 &= \left(\omega_{21}^* + \frac{\omega_{11}^* \omega_{23}^*}{\omega_{13}^*} \right) \\ d_3 &= \frac{1}{\omega_{13}^*} \left(\omega_{21}^* \omega_{12}^* - \omega_{11}^* \omega_{22}^* \right) \end{aligned} \tag{17}$$

bright state

$$|b\rangle = -d_2^* |1\rangle + d_1^* |2\rangle \tag{18}$$

$$H_d = \left(\sum_{j\ =1}^{2} \omega_{j3} |3\rangle \langle e_j| + \sigma_j |b\rangle \langle e_j| \right) + \text{h.c} \tag{19}$$

$$\sigma_1 = -\frac{\omega_{23}}{\omega_{13}} \left(\omega_{12}^2 + \omega_{11}^2\right) - \omega_{11}\omega_{21} - \omega_{12}\omega_{22}$$

$$\sigma_2 = -\frac{\omega_{23}}{\omega_{13}} \left(\omega_{12}\omega_{22} + \omega_{11}\omega_{21}\right) - \left(\omega_{12}^2 + \omega_{22}^2\right)$$

Then computations follow as before with new coeffcients.

$$H = \sum_{j=1}^{2} \alpha_j |\alpha_j\rangle \langle e_j| + \sum_{i=1}^{2} \omega_{ji} |i\rangle \langle e_j| + \text{ h.c} \tag{20}$$

in new basis

$$H_d = \sum_{j=1}^{2} -\alpha_j^2 - \sum_{i=1}^{2} \omega_{ji}^2 |b_j\rangle \langle e_j| - (\omega_{11}\omega_{21} + \omega_{12}\omega_{22})(|b_2\rangle \langle e_1| + |b_1\rangle \langle e_2|) + \text{ h.c} \tag{21}$$

simplify

$$H_d = \sum_{j=1}^{2} \sigma_j |b_j\rangle \langle e_j| + \Gamma(|b_2\rangle \langle e_1| + |b_1\rangle \langle e_2|) + \text{ h.c} \tag{22}$$

$$\begin{aligned} |D_1\rangle &= \cos\alpha \cos\beta e^{-i\varphi_1} |b_1\rangle - \cos\alpha \sin\beta |\alpha_1\rangle - i \sin\alpha |e_1\rangle \\ |D_2\rangle &= \cos\alpha \cos\beta e^{-i\varphi_2} |b_2\rangle - \cos\alpha \sin\beta |\alpha_2\rangle - i \sin\alpha |e_2\rangle \end{aligned} \tag{23}$$

$$\langle D_1|D_2\rangle = \cos^2\alpha \cos^2\beta e^{+i(\varphi_2-\varphi_1)} \langle b_1|b_2\rangle == 0\textbf{?} \tag{24}$$

holds if

$$\langle b_1|b_2\rangle = \omega_{11}^*\omega_{21} + \omega_{12}^*\omega_{22} = 0 \tag{25}$$

NEW, hopefully last, try
The system

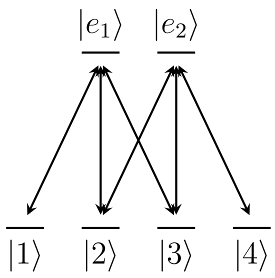

Figure 2: test figure

has a dark state $|d\rangle$ on the form

$$|d\rangle = c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle\,, \quad |c_1|^2 + |c_2|^2 + |c_3|^2 = 1 \tag{26}$$

There exists two orthogonal bright states on the form

$$\begin{aligned} |b_1\rangle &= N_1 \left(x |1\rangle + c_2 |2\rangle + c_3 |3\rangle \right) \\ |b_2\rangle &= N_2 \left(-c_3^* |2\rangle + c_2^* |3\rangle \right) \end{aligned} \tag{27}$$

One can see that $\langle b_1 | b_2 \rangle = \langle d | b_1 \rangle\, 0$, find x such that $\langle d | b_1 \rangle = 0$.
After finding x and normalizing the result is

$$\begin{aligned} |d\rangle &= c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle \\ |b_1\rangle &= \frac{|c_1|}{\sqrt{1 - |c_1|^2}} \left(\left(c_1 - \frac{1}{c_1^*} \right) |1\rangle + c_2 |2\rangle + c_3 |3\rangle \right) \\ |b_2\rangle &= \frac{1}{\sqrt{1 - |c_1|^2}} \left(-c_3^* |2\rangle + c_2^* |3\rangle \right) \end{aligned} \tag{28}$$

The hamiltonian in the space spanned by $\{e_1, e_2, b_1, b_2, 4, d\}$ is given by

$$H_d = \frac{\Omega_1(t)}{2} e^{-i\phi_1} |b_1\rangle \langle e_1| + \frac{\Omega_2(t)}{2} e^{-i\phi_2} |b_2\rangle \langle e_2| + \frac{\Omega_3(t)}{2} |4\rangle \langle e_2| + \text{ h.c} \tag{29}$$

Two dark paths $|D_i\rangle$ exists such that $\langle D_i | H_d | D_i \rangle = 0,\ i = 1, 2$, on the form

$$\begin{aligned} |D_1\rangle &= \alpha_1 |b_1\rangle + \beta_1 |e_1\rangle \\ |D_2\rangle &= \alpha_2 |b_2\rangle + \beta_2 |e_2\rangle + \gamma |4\rangle \end{aligned} \tag{30}$$

A choice would be, given two functions, $u(0) = u(T) = v(0) = v(T) = 0$,

$$\begin{aligned} \alpha_1 &= \cos u(t) \\ \beta_1 &= e^{i(\phi_1 + \frac{\pi}{2})} \sin u(t) \\ \alpha_2 &= \cos u \cos v e^{-i\phi_2} \\ \beta_2 &= -i \sin u \\ \gamma &= -\cos u \sin v \end{aligned} \tag{31}$$

the coeffcients of the dark state could be parametrized by 4 angles, $\chi, \xi, \theta, \varphi$

$$
\begin{aligned}
c_1 &= e^{i\chi} \sin\theta \cos\varphi, \\
c_2 &= e^{i\xi} \sin\theta \sin\varphi, \\
c_3 &= \cos\theta.
\end{aligned} \tag{32}
$$

Now to find the parameters $\Omega_1(t), \Omega_2(t), \Omega_3(t)$, this can be done by reverse engineering by solving the schrödinger equation for $|D_1\rangle, |D_2\rangle$,

$$
\begin{aligned}
i\frac{\partial}{\partial t} |D_1(t)\rangle &= H_d |D_1(t)\rangle \\
i\frac{\partial}{\partial t} |D_2(t)\rangle &= H_d |D_2(t)\rangle
\end{aligned} \tag{33}
$$

a calculation yields

$$
\begin{aligned}
\Omega_1(t) &= -2\dot{u} \\
\Omega_2(t) &= 2\left(\dot{v}\cot u \sin v + \dot{u}\cos v\right) \\
\Omega_3(t) &= 2\left(\dot{v}\cot u \sin v - \dot{u}\sin v\right)
\end{aligned} \tag{34}
$$

Split time evol. into two parts. The relevant part of the time evolution operator is

$$
\begin{aligned}
U_1 &= |d\rangle\langle d| - i|e_1\rangle\langle b_1| - i|e_2\rangle\langle b_2|, \ \phi_1 = \phi_2 = 0 \\
U_2 &= |d\rangle\langle d| + ie^{i\gamma_1}|b_1\rangle\langle e_1| + ie^{i\gamma_2}|b_2\rangle\langle e_2|, \ \phi_1 = -\gamma_1, \ \phi_2 = -\gamma_2
\end{aligned} \tag{35}
$$

The complete operator is then given by

$$
U = U_2 U_1 = |d\rangle\langle d| + e^{i\gamma_1}|b_1\rangle\langle b_1| + e^{i\gamma_2}|b_2\rangle\langle b_2| \tag{36}
$$

Then U is unitary in the computational subspace $\{d, b_1, b_2\}$, since $UU^\dagger = U^\dagger U = \mathbb{1}$

One can set $u(t) = \frac{\pi}{2}\sin^2\frac{\pi t}{T}$ and $v(t) = \eta\left[1 - \cos u(t)\right]$

Now U can be determined by the 6(?) real parameters $\chi, \xi, \theta, \varphi, \gamma_1, \gamma_2$ as $U(\chi, \xi, \theta, \varphi, \gamma_1, \gamma_2)$

References

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[3] Wang Y, Hu Z, Sanders BC, Kais S. Qudits and High-Dimensional Quantum Computing. Frontiers in physics. 2020;8.