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# Universal quantum computation with qudits

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Quantum circuit model has been widely explored for various quantum applications such as Shors algorithm and Grovers searching algorithm. Most of previous algorithms are based on the qubit systems. Herein a proposal for a universal circuit is given based on the qudit system, which is larger and can store more information. In order to prove its universality for quantum applications, an explicit set of one-qudit and two-qudit gates is provided for the universal qudit computation. The one-qudit gates are general rotation for each two-dimensional subspace while the two-qudit gates are their controlled extensions. In comparison to previous quantum qudit logical gates, each primitive qudit gate is only dependent on two free parameters and may be easily implemented. In experimental implementation, multilevel ions with the linear ion trap model are used to build the qudit systems and use the coupling of neighbored levels for qudit gates. The controlled qudit gates may be realized with the interactions of internal and external coordinates of the ion.

**universal qudit gate, qudit circuit, linear ion**

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Quantum circuit model has been explored to undertake intractable computation tasks in regards to classical computers. The primary reason is that the quantum system possesses have different features such as entanglement or quantum correlation [1]. In comparison to the binary logic gates and Boolean algebra in the classical computation theory, the qubit gates and Pauli algebra are critical for the quantum computation based one the qubit system. However, the local qubit operations are not sufficient for synthesizing general global quantum evolutions. Thus some correlated operations are required to construct the universal quantum logic gates which are performed on a small and fixed number of qubits. Specially, global unitary transformations can be implemented using only two-qubit operation at each time [2–5], which has no analog result in the classical reversible logic. For example, three-bit gates are necessary to simulate all reversible Boolean functions [6].

The universal qubit logic may be extended the qudit logic [7–15], where the information unit is qudit system [16]. The qudit state in the $d$-dimensional state space may offer greater flexibility in the storage and processing of quantum information, such as improving the channel capacity [17,18], implementing special quantum gates [19–22], increasing the information security [23–28] and exploring different quantum features [29–34]. The qudit system has also been realized with different physical systems [35–38]. Unfortunately the previous schemes have had to control many freedoms in implementing the evolution of general qudit systems. The primitive qudit gates are more complex than the qubit counterparts be-

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cause of the control of all levels of one qudit system. Herein we present a set of one-qudit and two-qudit gates which are sufficient for the universal qudit computation. These gates are easily implemented using multilevel ions with the linear ion trap model. The controlling parameters are greatly reduced.

We present some primitive quidit gates. In addition we present the universal qudit circuit model for general qudit systems and the associated evolution. Included are the physical implementations of the universal qudit gates.

## 1 Primitive qudit gates

Let $U_d$ be a $d$-dimensional transformation mapping a general qudit state to $|d-1\rangle$ such that

$$U_d(\alpha_0, \alpha_2, \cdots, \alpha_{d-1}) : \sum_{j=0}^{d-1} \alpha_j |j\rangle \mapsto |d-1\rangle. \tag{1}$$

Similar to the qubit case, $U_d$ is not unique in terms of complex parameters $\alpha_0, \cdots, \alpha_{d-1}$. This problem has been addressed elsewhere [7] with probabilistic quantum search algorithm. Here, we define another deterministic unitary transformation to realize $U_d$ with $d-1$ steps. In detail, $U_d$ may be decomposed into

$$U_d = X_{d-1}(a_{d-1}, b_{d-1}) \cdots X_1(a_1, b_1), \tag{2}$$

with

$$X_j(x, y) = \begin{pmatrix} I_{j-1} & & & \\ & \frac{x}{\sqrt{|x|^2+|y|^2}} & \frac{-y}{\sqrt{|x|^2+|y|^2}} & \\ & \frac{y^*}{\sqrt{|x|^2+|y|^2}} & \frac{x^*}{\sqrt{|x|^2+|y|^2}} & \\ & & & I_{d-j-1} \end{pmatrix}, \tag{3}$$

and $a_j = \alpha_j, b_j = \sqrt{\sum_{i=0}^{j-1} \alpha_i^2}$. The new primitive transformations $X_j(x, y)$ in eq. (3) are easily implemented in physics with two freedoms such as the linear ion trap model and linear optics with multiport.

We define the $d$-dimensional phase gate $Z_d$ as an operator

$$Z_d(\theta) = \sum_{j=0}^{d-1} \mathrm{e}^{\mathrm{i}(1-\mathrm{sgn}(d-1-j))\theta} |j\rangle\langle j|, \tag{4}$$

which alters the phase of $|d-1\rangle$ by $\theta$ without affecting other states in the qudit. The sgn denotes the sign function. This seemingly shows that $\{Z_d, X_d\}$ with $X_d = \{X_j(x, y)\}$ is sufficient to simulate all single-qudit unitary operations. Each primitive gate may be implemented by controlling no more than two complex parameters. This decomposition has greatly simplified the physical implementations of qudit gates. If $R_d$ represents either $X_d$ or $Z_d$, then the controlled-qudit gate is defined as:

$$C_2[R_d] = \begin{pmatrix} I_{d^2-d} & \\ & R_d \end{pmatrix} \tag{5}$$

acting on the two-qudit system. The identity operation $I_{d^2-d}$ acts on the substates $|0\rangle|0\rangle, \cdots, |d-2\rangle|d-1\rangle$ while $R_d$ acts on the remaining $d$ substates $|d-1\rangle|0\rangle, \cdots, |d-1\rangle|d-1\rangle$ of one general two-qudit system. These gates are sufficient to construct unitary transformation of $SU(d^n)$ and proved in the next section.

## 2 Universal quantum qudit circuits

Herein we provide the primary result.

**Theorem 1** The following qudit gates set

$$\Gamma = \{X_d, Z_d, C_2[R_d]\} \tag{6}$$

is universal for general quantum computation based on quantum circuit model.

To show the universality of $\Gamma$ we have to address $n$-qudit operations in $SU(d^n)$. Consider an $N$-dimensional unitary transformation $U \in SU(d^n)$ acting on the $n$-qudit state. The following task is to synthesize $U$ with $\Gamma$.

Denote the computation basis of $n$-qudit space $\mathbb{C}^{d^n}$ as:

$$|k\rangle = |k_1, k_2, \cdots, k_n\rangle, \ \ k = 0, \cdots, d^n - 1. \tag{7}$$

$k_1, \ k_2, \cdots, k_n$ is the base-$d$ representation of $k$ and $|k_i\rangle$ denote the states of the $i$th qudit, $i = 1, \cdots, n$. The proof of Theorem 1 is completed by the following subsections.

### 2.1 Eigen-decomposition of $U$

The first step is eigen-decomposition of $U$. For $U \in SU(d^n)$ there exist $N = d^n$ different eigenstates $|E_j\rangle$ with corresponding eigenvalues $\mathrm{e}^{\mathrm{i}\lambda_j}$, $j = 1, 2, \cdots, d^n$. Each eigenstate is represented with the computation basis as:

$$\sum_{j=0}^{N-1} \alpha_j |j\rangle = \sum_{i_1,\cdots,i_n=0}^{d-1} \alpha_{i_1,\cdots,i_n} |i_1, i_2, \cdots, i_n\rangle \tag{8}$$

from special $\alpha_{i_1,\cdots,i_n}$. From the representation theory the unitary matrix $U$ may be rewritten as:

$$U = \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\lambda_j} |E_j\rangle\langle E_j| = \prod_{j=1}^{N} \Upsilon_j \tag{9}$$

with eigenoperators

$$\Upsilon_j = \sum_{s=1}^{N} \mathrm{e}^{\mathrm{i}(1-|\mathrm{sgn}(j-s)|)\lambda_s} |E_s\rangle\langle E_s|, \tag{10}$$

which generate a phase $\lambda_j$ of $|E_j\rangle$ without affecting any other eigenstates, $j = 1, \cdots, N$.

Now the qudit decomposition of $U$ is reduced to synthesize all the eigenoperators $\Upsilon_j$. Notice that $\Upsilon_j$ can be decomposed with two basic transformations [7] as follows:

$$\Upsilon_j = U_{j,N}^{-1} Z_{j,N} U_{j,N}. \tag{11}$$