Uppsala University

Degree Project D in Computational Science, 30c

Department of Physics and Astronomy
Division of Materials Theory

Holonomic optimal control for qudits

Author:
Tomas André

Supervisor:
Erik Sjöqvist
Subject reader:
Martin Almquist

November 4, 2021

Abstract

1 Engelskt abstrakt

Sammanfattning

Svenskt abstrakt

Contents

1 Introduction

Upplägg?

- Introduction
- Background
  - QM introduction, en hastig och lustig introduction till det viktigaste grunderna (Typ vad en ket är osv), kanske hela vägen upp till Adiabatic theorem?
  - QC introduction, några kärn/grund-idéer inom QC samt lite relevant saker för just det jag gör
- 'huvuddel'(Method?) - Funderar på att kombinera den och ha resultaten här också
  - Qutrit exempel med explcita beräkningar
  - Qudit generalizering
  - Något om simulering?
- Discussion & Conclusions

The emerging field of quantum technologies have many promising applications, one of them is quantum computation (QC), which currently is a very active area of research. Quantum computers makes use of some of the quantum mechanical concepts such as superposition, entanglement and interference to design powerful algorithms. These algorithms could be used to solve some hard problems which would not be classically possible, such as prime-number factoring[1] and more[2]. The current quantum computers are very susceptible to decoherence and noise, and thus will not have any commercial use in the near future, but stand as an important proof of concept. The most common model for quantum computation is the circuit model, which are analogous to the classical circuits used for classical computers. Gates turns into unitary transformations and bits into qubits. To achieve the computational advantage its important to construct robust, noise-resilient quantum gates. A good candidate for this is holonomic quantum computation[3][4] and are based on the Berry phase[5] and its non-abelian and/or non-adiabatic generalizations[6][7][8]. These method are only dependent on the geometry of the system and thus resilient to local errors in the dynamical evolution.
The idea that our elements of computation should be limited to two dimensional (qu)bits is sort of an arbitrary choice, it most likely rose out of convenience due to binary logic. So why binary logic? It is simply the easiest non-trivial example, in binary things can be either 1 or 0, `True` or `False`, **on** or **off**. Due to its simplicity its no wonder why this is how the first computer were designed. But are we limited to (qu)bits? As early as 1840 a mechanical trenary calculation machine was built by Thomas Fowler, and in 1958 the first electronic trenary computer was built by the Soviet union **CITATIONS** . Even though it had many advantages over the binary computer it did never see the widespread success. So there is nothing in theory that forbid a higher dimensional computational basis, even more so when it comes to quantum computers where the implementation of the elements of computation already surpass the simplicity of **on** and **off**.

There is already promising results from qudits with $\dim \leq 2$ that show potential [9][10][11], and in the review article [12] a good overview of the field is given and further research into the topic is encouraged.

The idea in this project is to find a new scheme to implement qudits which could be more efficient than some current ones. We do this by expanding the scheme in [13] a, by first showing a specific example of how this would work for the qutrit in Section **section here** and then how the scheme would generalize in Section **section here** .

The report is structured as follows, the background/theory section is split into two parts wherein the first part is aimed at those not well versed in the field of quantum mechanics, it serves as a quick introduction to the most important aspects and as well as the commonly used notation, experienced readers can skip this part. Then follows a part more concerned with quantum computation, quantum information and some of the more advanced quantum mechanical concept that those are built upon. Next up is the "main" section, it start of with a explicit example, the qutrit, and then follows the generalization of this example.

The report ends in a discussion and conclusion.

2 Theoretical background

The Background consists of a quick introduction to the most important quantum mechanical concepts and notation. Readers familiar with quantum mechanics may skip it. The second part explores the fundamentals of quantum computation and

2.1 Basic quantum mechanics, part I

This part offers a quick introduction to the necessary quantum mechanics for readers whom are not familiar with the subject. Quantum mechanics is nothing more than linear algebra with fancy notation and some additional rules. The section contains nothing relevant for later parts of the thesis and can be safely skipped. For a more complete introduction I suggest chapter 1 of Sakurai.

2.1.1 Quantum states and Dirac notation

First lets define what is meant by the term **state**. In classical physics a state would be given by the position and momentum of all its individual constituents. An example would be a system of N particles, the state would be given by $\{(\vec{x}, \vec{p})_i\}_{i=1}^{N}$, where $\vec{x}, \vec{p} \in \mathbb{R}^3$ are the position and momentum in 3 dimensions. In Quantum Mechanics (QM) it more subtle than this, since exact information of the system can not be obtained in the same way. So a state is represented by a normalized vector in a complex vector space. The vector space in which the state vector lives is called a **Hilbert space** and has the following properties. given two vectors u, v in H they satisfy:

The inner product is conjugate symmetric

1. $\langle u, v\rangle = \overline{\langle v, u\rangle} \in \mathbb{C}$

The inner product is linear in the first argument, for constants $a, b \in \mathbb{C}$

2. $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$

The inner product is positive definite

3. $\langle u, u\rangle = 0 \iff u = 0$

These properties can be combined to find some other useful facts that holds, combining the 1st and 2nd property,

$$\langle v, au_1 + bu_2 \rangle = \overline{\langle au_1 + bu_2, v \rangle} = a^* \overline{\langle u_1, v \rangle} + b^* \overline{\langle u_2, v \rangle} = a^* \langle v, u_1 \rangle + b^* \langle v, u_2 \rangle \tag{1}$$

the inner product is anti-linear in the second term. Using the 1st and 3rd property

$$\langle u, u \rangle = \overline{\langle u, u \rangle} \implies \text{Im}(\langle u, u \rangle) = 0 \tag{2}$$

or in words, the inner product of two identical vectors is a real number.

So now we have established that a quantum state is a normalized vector v in a Hilbert space H. Now a property of vector spaces is that any vector can be multiplied by a matrix, and the resulting vector will be a new vector in the same vector space. This is what is meant when a quantum state is **acted** upon. For a matrix A we have that

$$H \ni v \xrightarrow{A} Av = v' \in H \tag{3}$$

acting on a state alters it in various ways.

Now lets go from this linear algebra notation to the Dirac notation commonly used in quantum mechanics, also know as bra-ket notation.

Vectors are replaced by **kets**,$|\rangle$, or **bras**,$\langle|$.
$v \mapsto |\psi\rangle$
$v^\dagger \mapsto \langle\psi|$
and matrices are replaced by operators
$A \mapsto \hat{A}$
The same rules applies to these as for the usual vectors. The label inside the brackets does no in itself have any meanings and are in some sense only just that, labels, but more often than not it is used to represent some property of the state. With this notation the inner product between to states $|\psi\rangle$, $|\varphi\rangle$ is written as

$$\langle\psi|\varphi\rangle = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n = \begin{pmatrix} a_1^* & a_2^* & \dots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{4}$$

from the properties defined earlier the relations $(|\psi\rangle)^\dagger = \langle\psi|$ and $(\langle\psi|\varphi\rangle)^\dagger = \langle\varphi|\psi\rangle$, this suggest that another way to define the states would simply be

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \ \langle\psi| = (|\psi\rangle)^\dagger = (a_1^*, a_2^*, \dots, a_n^*), \ a_1, a_2, \dots, a_n \in \mathbb{C} \tag{5}$$

which is nothing more than a complex vector.

2.1.2 Kets, Bras and Operators

When talking about Quantum mechanics the term **quantum state**, or more likley just state, is mentioned a lot. A quantum mechanical state is represented by a **ket**, an complex-valued vector with either finite or infinite entries. Closely related to the ket is the **bra**, which is the corresponding vector to the ket in the dual space, or more simply, the hermitian conjugate of the ket, see Equation 6.

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \ \langle\psi| = (|\psi\rangle)^\dagger = (a_1^*, a_2^*, \dots, a_n^*), \ a_1, a_2, \dots, a_n \in \mathbb{C} \tag{6}$$

An arbitrary ket can be rewritten as a linear combination of its eigenkets that span the same space, $|\psi\rangle = \sum_i a_i |i\rangle$ The coefficient is know as the probability amplitude, and $|a_i|^2$ corresponds to the probability to find the state in the ith state. The inner product of two quantum states is simply written as $\langle\varphi|\psi\rangle = (\langle\varphi|\psi\rangle)^\dagger \in \mathbb{C}$. An operator can act on a quantum state and corresponds to multiplication with a matrix, and will always yield a new state, ket.

$$\hat{O} |\psi\rangle = |\psi'\rangle \tag{7}$$

Operators are often written in terms of of kets and bras, for example and operator which takes the state $|1\rangle$ and returns the state $|2\rangle$ is written as

$$(|2\rangle \langle 1|) |1\rangle = |2\rangle \langle 1|1\rangle = |2\rangle (\langle 1|1\rangle) = (\langle 1|1\rangle) |2\rangle = 1 |2\rangle = |2\rangle \tag{8}$$

or a identity 2×2 matrix would be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle \langle 1| + |2\rangle \langle 2| \tag{9}$$

and so on. An operator for a physical measurable quantity is called an observable.

2.1.3 Measurements and Observables and Uncertainty

A quantum mechanical measurement "breaks" the superposition of a quantum states and shifts it into a eigenstate of the observable,

$$|\psi\rangle \longrightarrow |i\rangle \tag{10}$$

the probability to find the ith eigenstate is $|a_i|^2$. In quantum mechanics the relation $AB - BA = 0$,where A and B are operators, does not generally hold. This is due to the uncommutative nature of quantum mechanics, it relates to uncertainty but it also required since it a can realize a more complex mathematical structure. The fact that matrices are non-commutative are not surprising, and since operators can be represented by matrices this should not be that confusing. To make it more concrete to which degree two operators commute one can define the commutator on operators A, B as $[A, B] = AB - BA$, which is zero for commuting operators and non-zero otherwise. Observables which don't commute are called incompatible observables, a well know pair of incompatible observables are position and momentum, $\mathbf{x}$ and $\mathbf{p}$, which can not be measured to arbitrary precision, this is due to the fact that $[\mathbf{x}, \mathbf{p}] \neq 0$. So for two incompatible observables the general uncertainty the measurements will be limited by **uncertainty relation here.**

2.1.4 Time evolution and the Schrödinger equation

2.2 Quantum Computation and Quantum Information theory, part 2

2.2.1 The Qubit

A classical bit is a binary system which, so it can occupy two states, either 0 or 1. So with n bits there is 2^n possible states that can be represented, but only one at a time.

A qubit is a quantum state that is in a superposition of $|0\rangle$ and $|1\rangle$, so a general qubit would have the form

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,\ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1. \tag{11}$$

The qubit is no limited to 0 and 1, but can exist in a linear combination of those states. When a measurement is performed the qubit will collapse into $|1\rangle$ or $|2\rangle$ with probability $|\alpha|^2$ and $|\beta|^2$ respectively. A consequence of superposition is that n qubits can represent 2^n states simultaneously.

The combined state of two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by

$$|\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1\,|0\rangle + \beta_1\,|1\rangle) \otimes (\alpha_2\,|0\rangle + \beta_2\,|1\rangle) \tag{12}$$

the tensor product is often omitted and one would write $|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle\,|\psi_2\rangle = |\psi_1, \psi_2\rangle$.

Some of the common qubit gates are the Pauli gates X, Y, Z, the Hadamard gate H, and the T-gate T.

2.2.2 Information stuff

2.2.3 Universal computation

2.2.4 Holonomic Quantum Computation

3 Method

References

[1] Shor PW. Algorithms for quantum computation: discrete logarithms and factoring. IEEE Comput Soc Press. 1994;p. 124–134.

[2] Grover LK. Quantum Mechanics Helps in Searching for a Needle in a Haystack. Physical review letters. 1997;79(2):325–328.

[3] Zanardi P, Rasetti M. Holonomic quantum computation. Physics letters A. 1999;264(2):94–99.

[4] Sjöqvist E, Tong DM, Mauritz Andersson L, Hessmo B, Johansson M, Singh K. Non-adiabatic holonomic quantum computation. New journal of physics. 2012;14(10):103035.

[5] Berry MV. Quantal phase factors accompanying adiabatic changes. Proceedings of the Royal Society of London Series A, Mathematical and physical sciences. 1984;392(1802):45–57.

[6] AHARONOV Y, ANANDAN J. Phase change during a cyclic quantum evolution. Physical review letters. 1987;58(16):1593–1596.

[7] Anandan J. Non-adiabatic non-abelian geometric phase. Physics letters A. 1988;133(4):171–175.

[8] WILCZEK F, ZEE A. Appearance of gauge structure in simple dynamical systems. Physical review letters. 1984;52(24):2111–2114.

[9] Li B, Yu ZH, Fei SM. Geometry of quantum computation with qutrits. Scientific reports. 2013;3(1):2594–2594.

[10] Lu H, Hu Z, Alshaykh MS, Moore AJ, Wang Y, Imany P, et al. Quantum Phase Estimation with Time-Frequency Qudits in a Single Photon. Advanced quantum technologies (Online). 2020;3(2):1900074–n/a.

[11] Luo M, Wang X. Universal quantum computation with qudits. Science China Physics, mechanics & astronomy. 2014;57(9):1712–1717.

[12] Wang Y, Hu Z, Sanders BC, Kais S. Qudits and High-Dimensional Quantum Computing. Frontiers in physics. 2020;8.

[13] Ming-Zhong A, Li S, He R, Xue ZY, Cui JM, Haung YF, et al. Experimental Realization of Nonadiabatic Holonomic Single-Qubit Quantum Gates with Two Dark Paths in a Trapped Ion. Not yet published. 2021;.