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Holonomic optimal control for qudits

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Abstract

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1 Introduction

The emerging field of quantum technologies has many promising applications, one of them is quantum computation (QC), which currently is a very active area of research. Quantum computers make use of some of the quantum mechanical concepts such as superposition, entanglement, and interference to design powerful algorithms. These algorithms could be used to solve some hard problems which would not be classically possible, such as efficient prime-number factoring[1], it can also be used to reduce the time complexity of some commonly used algorithms[2]. Current quantum computers are very susceptible to decoherence and noise, and thus will not have any commercial use soon, but stand as an important proof of concept. The most common model for quantum computation is the circuit model, which is analogous to the classical circuits used for classical computers. Gates are replaced by unitary transformations and bits into qubits. To achieve the computational advantage it is important to construct robust, noise-resilient quantum gates. A good candidate for this is holonomic quantum computation[3, 4] which is based on the Berry phase[5] and its non-abelian and/or non-adiabatic generalizations[6, 7, 8]. These methods are only dependent on the geometry of the system and thus resilient to local errors in the dynamical evolution.
The idea that our elements of computation should be limited to two-dimensional (qu)bits is sort of an arbitrary choice, it most likely rose out of convenience due to binary logic. So why binary logic? It is simply the easiest non-trivial example, in binary things can be either 1 or 0, `True` or `False`, **on** or **off**. Due to its simplicity, it's no wonder why this is how the first computer was designed. But are we limited to (qu)bits? As early as 1840 a mechanical trenary calculation machine was built by Thomas Fowler[9], and in 1958 the first electronic trenary computer was built by the Soviet Union[10]. Even though it had many advantages over the binary computer it never saw the same widespread success. There is nothing in theory that forbids a higher dimensional computational basis, even more so when it comes to quantum computers where the implementation of the elements of computation already surpasses the simplicity of **on** and **off**.

There are already promising qudit results that show potential [11, 12, 13], and in the review article [14] a good overview of the field is given and further research into the topic is encouraged.

The report is structured as follows. The background/theory section is split into two parts where the first part serves as a quick introduction to the most important aspects and as well as the commonly used notation. Then follows a part more concerned with quantum computation, quantum information, and some of the more advanced quantum mechanical concepts that those are built upon. The two main sections contain the idea of this project, to find a new scheme to implement qudits which could be more efficient than some current ones. We do this by expanding the scheme in [15], where first an explicit example is shown for a 3-dimensional qudit, in The Qutrit section, and Generalization, which shows how the scheme can be generalized to arbitrary dimension. This is followed by Discussion and Conclusions, the report ends with a brief outlook on what could be built on further from this project.

2 Theoretical background

The Background consists of a quick introduction to the most important quantum mechanical concepts and notation. The second part explores the fundamentals of quantum computation and

2.1 Basic quantum mechanics, part I

This part offers a quick introduction to the necessary quantum mechanics for readers who are not familiar with the subject. Quantum mechanics is nothing more than linear algebra (with fancy notation and some additional rules). The section contains nothing relevant for later parts of the thesis and can be safely skipped. For a more complete introduction, I suggest chapters 1 and 2 of Sakurai[16].

2.1.1 Quantum states and Dirac notation

First, let's define what is meant by the term **state**. In classical physics, a state would be given by the position and momentum of all its constituents. An example would be a system of N particles, the state would be given by $\{(\vec{x},\vec{p})_i\}_{i=1}^N$, where $\vec{x},\vec{p} \in \mathbb{R}^3$ are the position and momentum in 3 dimensions. In Quantum Mechanics (QM) it is more subtle than this since exact information of the system can not be obtained in the same way. A state is represented by a normalized vector in a complex vector space. The vector space in which the state vector lives is called a **Hilbert space** and has the following properties. Given two vectors u, v in H they satisfy:
The inner product is conjugate symmetric

1. $\langle u, v\rangle = \overline{\langle v, u\rangle} \in \mathbb{C}$

The inner product is linear in the first argument, for constants $a, b \in \mathbb{C}$

2. $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$

The inner product is positive definite

3. $\langle u, u\rangle = 0 \iff u = 0$

These properties can be combined to find some other useful facts that holds, combining the 1st and 2nd property,

$$\langle v, au_1 + bu_2\rangle = \overline{\langle au_1 + bu_2, v\rangle} = a^*\overline{\langle u_1, v\rangle} + b^*\overline{\langle u_2, v\rangle} = a^*\langle v, u_1\rangle + b^*\langle v, u_2\rangle \tag{1}$$

the inner product is anti-linear in the second term. Using the 1st and 3rd property

$$\langle u, u\rangle = \overline{\langle u, u\rangle} \implies \mathrm{Im}(\langle u, u\rangle) = 0 \tag{2}$$

or in words, the inner product of two identical vectors is a real number.

So now we have stated that a quantum state is a normalized vector v in a Hilbert space H. Now a property of vector spaces is that any vector can be multiplied by a matrix, and the resulting vector will be a new vector in the same vector space. This is what is meant when a quantum state is **acted** upon. For a matrix A we have that

$$H \ni v \xrightarrow{A} Av = v' \in H \tag{3}$$

acting on a state alters it in various ways.

Now let's go from this linear algebra notation to the Dirac notation commonly used in quantum mechanics, also known as bra-ket notation. Vectors are replaced by **kets**,$|\rangle$, or **bras**,$\langle|$. $v \mapsto |\psi\rangle$
$v^\dagger \mapsto \langle\psi|$

and matrices are replaced by operators
$A \mapsto \hat{A}$
The same rules apply to these as for the usual vectors. The labels inside the brackets do not in themselves have any meanings and are in some sense only that, labels, but more often than not it is used to represent some property of the state. With this notation the inner product between to states $|\psi\rangle, |\varphi\rangle$ is written as

$$\langle\psi|\varphi\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n = \begin{pmatrix} a_1^* & a_2^* & \dots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{4}$$

from the properties defined earlier the relations $(|\psi\rangle)^\dagger = \langle\psi|$ and $(\langle\psi|\varphi\rangle)^\dagger = \langle\varphi|\psi\rangle$, this suggest that another way to define the states would simply be

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \ \langle\psi| = (|\psi\rangle)^\dagger = (a_1^*, a_2^*, \dots, a_n^*), \ a_1, a_2, \dots, a_n \in \mathbb{C} \tag{5}$$

which is nothing more than a complex vector. A common way to represent kets is as a linear combination of eigenkets, which corresponds to the set of eigenvectors. Thus any ket can be written as a linear combination of these kets if they span the space

$$|\Psi\rangle = \sum_\psi c_\psi |\psi\rangle \tag{6}$$

where the $|a\rangle$s span the space in which $|A\rangle$ lives.
Following are some example calculations to see how to use Dirac-notation, let's consider what the effect of the Pauli-Z operator of a spin-$\frac{1}{2}$ particle is

a) on the up position $|\psi\rangle = |\downarrow\rangle$

b) on an arbitrary superposition of up and down $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C}$

Let's first show how this would be done using the matrix representation and then using dirac notation. Here the kets $\{|\uparrow\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ span the Hilbert space of the particle, they also form an orthonormal basis. The Pauli-Z operator in the matrix representation in this basis is $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have that

a) We simply write out the matrices and vectors and get that
$\sigma_z |\psi\rangle \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \doteq -|\downarrow\rangle$

b) Same as before but with a superposition (linear combination) up and down
$\sigma_z |\psi\rangle \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \doteq \alpha|\uparrow\rangle - \beta|\downarrow\rangle$

The matrix representation works great for small Hilbert spaces, let us do it again now using Dirac notation, here the inner product will be a key aspect, and making use of the orthonormality of $|\uparrow\rangle, |\downarrow\rangle$ it follows that $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$ and $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$. With Dirac notation operators can be expressed as outer products of bras and kets, thus Pauli-Z can be written as $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$.

a) $\sigma_z|\psi\rangle = (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)|\downarrow\rangle = |\uparrow\rangle \underbrace{\langle\uparrow|\downarrow\rangle}_{=0} - |\downarrow\rangle \underbrace{\langle\downarrow|\downarrow\rangle}_{=1} = 0|\uparrow\rangle - 1|\downarrow\rangle = -|\downarrow\rangle.$

b) $\sigma_z|\psi\rangle = (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)(\alpha|\uparrow\rangle + \beta|\downarrow\rangle) = \alpha|\uparrow\rangle \underbrace{\langle\uparrow|\uparrow\rangle}_{=1} - \alpha|\downarrow\rangle \underbrace{\langle\downarrow|\uparrow\rangle}_{=0} + \beta|\uparrow\rangle \underbrace{\langle\uparrow|\downarrow\rangle}_{=0} - \beta|\downarrow\rangle \underbrace{\langle\downarrow|\downarrow\rangle}_{=1} =$
$\alpha|\uparrow\rangle - \beta|\downarrow\rangle.$

Note that the final answer is the same for both cases, but using Dirac notation there was no need to bother with explicit vectors, that is one of the strong advantages of the Dirac notation. These calculations are quite analogous to how a qubit works which will be discussed in the background. Measuring a quantum state $|\psi\rangle = \sum_n c_n|n\rangle$, $c_n \in \mathbb{C}$ it collapses into a single base ket of the measured basis

$$|\psi\rangle \xrightarrow{Measurement} |n\rangle, \tag{7}$$

with probability $|c_n|^2$.

2.1.2 Time evolution and the Schrödinger equation

Say we have a time-dependent state $|\psi(t)\rangle$. How does the system evolve when time goes from t_0 to some later time t_1,

$$|\psi(t_0)\rangle \xrightarrow{t_0 \mapsto t_1} |\psi(t_1)\rangle \tag{8}$$

We can write this as an operator equation $\mathcal{U}(t, t_0)|\psi(t_0)\rangle = |\psi(t)\rangle$. To be a valid operator $\mathcal{U}$ must satisfy some properties. To preserve probability the operator must be unitary, $\mathcal{U}^\dagger\mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1$. The second property that needs to be fulfilled is composition, $\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0)$, which is to say that applying two consecutive time transformations, first between $t_0 \to t_1$ and then $t_1 \to t_2$ is equivalent to applying one transformation between $t_0 \to t_2$. Proceeding let's define how the time evolution operator would act under an infinitesimal change dt, let's propose an operator on the form

$$\mathcal{U}(t_0 + dt, t_0) = \mathbf{1} - i\,\Omega\,dt \tag{9}$$

with Ω being hermitian, $\Omega = \Omega^\dagger$. Ω can be time dependent, and then must be evaluated at t_0. As dt approaches 0 it reduces to an identity operator, which is appropriate. Let's check the two other properties

1. identity

$$\begin{aligned}\mathcal{U}(t_0 + dt, t_0)\mathcal{U}(t_0 + dt, t_0)^\dagger &= (\mathbf{1} - i\,\Omega\,dt)(\mathbf{1} + i\,\Omega\,dt) \\ &= \mathbf{1} + i\,\Omega\,dt - i\,\Omega\,dt + \mathcal{O}(dt^2) \\ &\simeq \mathbf{1}\end{aligned} \tag{10}$$

2. Composition

$$\begin{aligned}\mathcal{U}(t_0+dt_1+dt_2,t_0+dt_1)\mathcal{U}(t_0+dt_1,t_0) &= (\mathbf{1}-i\,\Omega\,dt_2)(\mathbf{1}-i\,\Omega\,dt_1)\\ &= \mathbf{1}-i\,\Omega\,dt_1-i\,\Omega\,dt_2)(\mathbf{1}+\mathcal{O}(dt^2)\\ &\simeq \mathbf{1}-i\,\Omega(dt_1+dt_2)\\ &= \mathcal{U}(t_0+dt_1+dt_2,t_0).\end{aligned} \tag{11}$$

From the operator Ω the Hamiltonian is defined, $\Omega = \frac{H}{\hbar}$, where $\hbar$ is Planck's reduced constant. Thus by definition the Hamiltonian will always be Hermitian. So for an infinitesimal time shift the operator has the form $\mathcal{U}(t_0+dt,t_0) = \mathbf{1} - i\,\frac{H}{\hbar}\,dt$. Time evolution is closely tied to one of Quantum Mechanics postulates, that is that a closed quantum state evolves according to the Schrödinger equation, Equation 12, here it becomes more obvious why it happens that Ω was related to the Hamiltonian. It is common to set $\hbar = 1$, as it is just a numerical shift, so let's do that, which simplifies both the Schrödinger equation and the time evolution operator $\mathcal{U}$.

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle \tag{12}$$

Now it is clear that the time evolution operator is related to the Schrödinger equation in some way, notice that the solution $|\psi(t)\rangle$ can be written in terms of our time evolution operator as $|\psi(t)\rangle = \mathcal{U}(t,t_0)\,|\psi(t_0)\rangle$. So given an initial condition we have solved the Schrödinger equation if we can determine $\mathcal{U}(t,t_0)$. To find the time evolution operator for any time t, we could successively repeat the infinitesimal operator until the desired time is reached $t_0 \mapsto t_0+dt \mapsto t_0+2\,dt \mapsto \cdots \mapsto t-dt \mapsto t$. For the case of a time independent Hamiltonian H it could be written as

$$\lim_{N\to\infty}\left(\mathbf{1}-i\,H\frac{(t-t_0)}{N}\right)^N = \exp\left(-i\,H(t-t_0)\right) \tag{13}$$

For simplicity let us assume that $t_0 = 0$, then the time evolution operator for a finite time for a time independent Hamiltonian could be written as $\mathcal{U}(t,0) = e^{-i\,H\,t}$. If the Hamiltonian has time dependence but it commutes with itself at different times, then an integral can be introduced in the exponential, $\mathcal{U}(t,0) = e^{-i\int_0^t H(t')\,dt'}$, if the Hamiltonian does not commute at different times it becomes a bit more involved and will not be covered here. But the formal solution can be written as $\mathcal{U}(t,0) = \mathbf{T}e^{-i\int_0^t H(t')\,dt'}$., where $\mathbf{T}$ is time-ordering.

To conclude, finding the time evolution operator corresponds to finding the solution to the Schrödinger Equation, which governs the time evolution of quantum systems.

2.2 Quantum Computation and Quantum Information theory, part 2

2.2.1 The Qubit

A classical bit is a two-state (binary) system, which means it can occupy either one of two states, **0** or **1**. So with n bits, there are 2^n possible states that can be represented, but only one at a time.

The general way to define a qubit would be any two dimensional quantum system ($\dim(\mathcal{H}) = 2$). In the orthonormal basis $|0\rangle, |1\rangle$. Where qubits are on the form given in Equation 14, any super position (linear combination) of the basis-kets $|0\rangle, |1\rangle$.

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,\ \alpha,\beta\in\mathbb{C},\ |\alpha|^2+|\beta|^2 = 1. \tag{14}$$

Most of the time it is better to work with the abstract concept of qubits without focusing on the physical implementation, however, see a lot of similarities with spin-$\frac{1}{2}$ particles ($|\downarrow\rangle, |\uparrow\rangle$)

Input		AND	NAND	OR
0	0	0	1	0
0	1	0	1	1
1	0	0	1	1
1	1	1	0	1

Table 1: Some common classical logic gates

or energy levels in a trapped ion ($|g\rangle, |e\rangle$). There is no need to limit the system to a physical 2-dimensional system. Simply choose two appropriate states which form a 2-dimensional subspace.

The coefficients α and β are the probability amplitudes, and tells with what probability one will find either $|0\rangle$ or $|1\rangle$ when measuring. One finds $|0\rangle$,$|1\rangle$ with probability $|\alpha|^2$ and $|\beta|^2$ respectively.

The consequence of superposition is that the qubit is not limited to just either 0 or 1, this in combination with other QM effects is what is used to create classically impossible algorithms[1][2].

The combined state of two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by

$$|\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1 |0\rangle + \beta_1 |1\rangle) \otimes (\alpha_2 |0\rangle + \beta_2 |1\rangle) \tag{15}$$

the tensor product is often omitted and one would write $|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle |\psi_2\rangle = |\psi_1, \psi_2\rangle$. So n qubits is often written as $|x\rangle^{\otimes n}$, $x \in \{0, 1\}$ which is spanned by kets on the form $|x_0 x_1 x_2 ... x_n\rangle$, $x_i \in \{0, 1\}$, representing all possible bit strings of length n.

2.2.2 From Classical Logic Gate to Quantum Gate

A model used in classical computation is the circuit model, which consists of wires and logic gates. The wires carry the bit information, **0**, **1**, and the gates apply different operations. A common gate is the NOT-gate which flips the content of the bit $NOT(\mathbf{0}) = 1$ and $NOT(\mathbf{1}) = 0$, some other common logic gates are shown in Table 1.

To make logic gates suitable for quantum computation the classical logic gate has to be adjusted. A quantum gate has some desired properties

1. Reversibility
2. Conservation of probability

it will turn out that these properties more or less have the same consequence. The first property is not achieved by the classical gates. For example, given the output of the **AND**-gate it is impossible to know the state of the bits that went into the gate. An output of **1** only conveys that the inputs were equal, **1**,**1** or **0**,**0**, but not which one. Adding another output that simply is one of the inputs can make the gate reversible, if you get a **0** as output, and know that the first input was **0**, then the second input must be a **1**. This extends to all inputs and outputs. In the quantum mechanical context, a quantum logic gate is a quantum operator acting on the state of the qubit. $A |\psi\rangle = |\psi'\rangle$, so the equivalence of the classical reversibility would be the existence of an inverse operator A^{-1} such that the operation could be reversed $A^{-1} |\psi'\rangle = A^{-1} A |\psi\rangle = \mathbf{1} |\psi\rangle = |\psi\rangle$

Conservation of probability does not really have any classical analogue due to it being a quantum property. More concretely it means conservation of the inner product. Given the

states $U|\psi\rangle = |\psi'\rangle$ and $U|\varphi\rangle = |\varphi'\rangle$, then the inner product of $|\psi\rangle, |\varphi\rangle$ is the same as for $ket\psi', ket\varphi'$. Thus one can compute

$$\langle\psi'|\varphi'\rangle = \langle\psi| U^\dagger U |\varphi\rangle = \langle\psi|\varphi\rangle \tag{16}$$

and see that this only holds if $U^\dagger U = UU^\dagger = \mathbf{1}$, which happens to be the condition for U to be a unitary operator. From the unitarity it is possible to also see that $U^\dagger = U^{-1}$. The reversibility is also a consequence of the operators being Unitary.

There are many ways to achieve the creation and implementation of quantum gates, some of them will be discussed in more detail in the Section Non-adiabatic Holonomic Quantum Computation, let's see in general how a quantum gate could be implemented. To apply a quantum gate to the qubit means to operate on a quantum state with a unitary operator. The time evolution operator discussed earlier happens to be unitary, and it acts on states according to the Schrödinger equation. Since $U = e^{-i\,H,t}$, the time evolution operator is determined by the Hamiltonian. So by constructing a Hamiltonian, H, that gives U the form of an operator one could in that way apply a quantum gate.

2.2.3 Some important gates and Universal computation

To realize a quantum gate a unitary, U, is applied to the quantum state, a unitary operator has the property that $U^\dagger U = UU^\dagger = \mathbf{1}$, and conserves probability.

Some of the most common gates are the Pauli gates, X, Y, Z, which are equivalent to the Pauli matrices, $\sigma_x, \sigma_y, \sigma_z$. They are defined as

$$X \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

It is more common to use only X and Z since $Y = i\,X\,Z$. The X-gate is somewhat analogous to the classical NOT-gate, with the effect that is swaps the input qubits, $\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$. The Z-gate introduces a phase-flip between the inputs, it does nothing to $|0\rangle$ and reverses the sign of $|1\rangle$, $\begin{cases} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{cases}$.

Superposition is one of the key aspects that make QC able to surpass classical algorithms. The Hadamard gate, H creates a superimposed quantum state, and is widely used all over QC, the gate is defined as

$$H \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{18}$$

Another noteworthy property is that H is its own inverse, $HH = \mathbf{1}$, so it can be used both to open and close superposition. More explicitly, the gate effect is

$$\begin{cases} H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{cases} \quad \begin{cases} H\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |0\rangle \\ H\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |1\rangle \end{cases}. \tag{19}$$

many well known quantum algorithms make heavy use of this operator[2, 1], making it is essential to QC. The concept of the phase flip, Z, can be extended to a phase shift of any arbitrary angle, the gate is denoted by $P(\phi)$ or sometimes simply ϕ. It is defined as

$$P(\phi) \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi}. \end{pmatrix} \tag{20}$$

Then Z reduces to the case $P(\pi) = Z$, some other notable cases of the phase shift are the T and S gates, defined by

$$T = P(\pi/4) \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \; S = T^2 = P(\pi/2) \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}. \tag{21}$$

All gates so far are single-qubit gates, to achieve full universality it turns out that two qubit-gates are enough. The two-qubit gate used is the CNOT, which stands for controlled not, the effect is defined as $\texttt{CNOT}\,|x, y\rangle = |x, x \oplus y\rangle$, where $\oplus$ denotes addition mod 2. The gate takes two inputs, the control, and the target. If the control is $|1\rangle$ then a NOT is applied to the target $|y\rangle \mapsto X\,|y\rangle$, otherwise the gate leaves the inputs unchanged.

2.2.4 Qudit Generalizations

Inherently there is nothing special about bits and binary logic, same goes for qubits. A qudit is a general name for a higher dimensional qubit, often with dimension d, the special case of $d = 3$ is called a *qutrit*, $d = 4$ a *ququart* and so on. A bigger computational basis could even prove to be better in some sense. The information content per unit is higher, N qubits can be reduced down to $\dfrac{N}{\log_2(d)}$ qudits[17], with quickly diminishing returns. The qutrit has a reduction factor of ~ 0.64, the ququart 0.5, with less returns the higher the dimension. Due to more information per unit, fewer operations are required to build quantum gates, minimizing the loss of accuracy from operations. The Toffoli gate can be constructed with only BLANK gates using qutrits, which need BLANK gates with qubits **reference and lookup actual numbers** . The drawback is the increased complexity in the schemes for physical implementation which could come with increased errors. There are already many promising results using qudits[11][12][13], and many more are discussed in [14]. The question is if the benefits of qudits can outweigh the extra cost.

In the following section let, let the dimension d be any integer such that $d \leq 3$, some generalized qudits gates will be defined and discussed, the definitions used are the same used in [14]. The effect of NOT is ambiguous for when the number of basis states are more than 2. So instead of viewing it as NOT, it can be thought of as a permutation of the states $|0\rangle, |1\rangle$. The higher dimensional correspondence of X is defined as

$$X_d = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{22}$$

The effect works as a permutation of basis states, for $d = 3$, the effect would be $|0\rangle \xrightarrow{X_3} |1\rangle \xrightarrow{X_3} |2\rangle \xrightarrow{X_3} |0\rangle$. The higher dimensional Z-gate Z_d is defined as

$$Z_d = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \omega^{d-1} \end{pmatrix} \tag{23}$$

with $\omega = \sqrt[d]{1} = e^{2\pi i/d}$ being the dth roots of unity. It shifts the different basis states a certain amount.

The generalized version of the Hadamard gate for qudits H_d easier to define of how it acts on each basis-ket, the effect is

$$H_d |j\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{ij} |i\rangle , \ j \in \{0, 1, 2, \ldots, d-1\} \tag{24}$$

with ω still the dth root of unity.

Last let us define the generalized qudit T-gate, now d is restricted to prime numbers, such T_d is only defined for prime number dimensions **lookup why and cite** . The gate is defined by

$$T_d = \sum_{k=0}^{d-1} \omega^{\nu_k} |k\rangle \langle k| \tag{25}$$

with ω being the dth root of unity. The exponent ν_k can be recursively defined with $\gamma', z', \varepsilon' \in \mathbb{Z}_d$ restricted by

$$\begin{cases} \nu_0 = 0 \\ \nu_{k+1} = \nu_k + k(2^{-1}\gamma' k + z') + 2^{-1} z' + \varepsilon' \end{cases} \tag{26}$$

The gate that will be implemented in this report is the T_3 with $z' = 1, \gamma' = 2$ and $\varepsilon' = 0$, which explicitly turn out to be

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/9} & 0 \\ 0 & 0 & e^{-2\pi i/9} \end{pmatrix} . \tag{27}$$

Likewise, as for qubits, single qudit universality can be obtained by a combination of some of these gates. The universal set of gates for the qudit is the generalization of the universal set for the qubit, $\{T_m, H_m\}$. Since the T_m-gate is only defined in prime number dimensions universality can not be obtained in this manner in non-prime dimensions.

2.2.5 Non-Adiabatic Holonomic Quantum Computation

Holonomic Quantum Computation (HQC) makes use of holonomies, which is to which extent parallel transport around some closed-loop fails to preserve the transported data. Which in the case of Quantum mechanics means that a periodic system, $H(\mathbf{R}(0)) = H(\mathbf{R}(T))$, where $\mathbf{R}$ is a vector of parameters, gains an additional phase. This was first introduced by M.V. Berry[5] and thus the phase is usually referred to as the Berry phase but is more generally called the geometric phase since the phase arises from the geometry rather than the dynamics of the system. The geometric phase exists also in non-adiabatic and non-abelian contexts, which gives room for more complex structures[6, 7, 8]. The in the adiabatic setting universal computation can be achieved through holonomies by transporting a set of control parameters along a loop(s) in a suitable space[3].

To see how the effect of adibaticy impacts the time evolution let's introduce a time dependent orthonormal ordered basis $|k(t)\rangle \in \mathcal{C}^1, k = 0, 1, 2, \ldots, N$. The following calculation is based on the section on the adiabatic theorem in Sakurai[16].

For an arbitrary state $|\psi(T)\rangle$ after one period T, the Schrödinger equation (12) has a solution on the form

$$|\psi(T)\rangle = U(T,0)\,|\psi(0)\rangle \tag{28}$$

where the general time evolution operator in a periodic system is

$$U(T,0) = \sum_{n,m=1}^{N} \left(\mathbf{T}e^{i\int_0^T(\mathbf{A}(t)-\mathbf{H}(t))dt}\right)_{nm} |n(0)\rangle\,\langle m(0)| \tag{29}$$

where $\mathbf{A}_{nm}(t) = i\,\langle n(t)|\,\frac{\partial}{\partial t}\,|m(t)\rangle$ and $\mathbf{H}_{nm}(t) = \langle n(t)|\,\mathbf{H}(t)\,|m(t)\rangle$, being hermitian $N \times N$ matrices, thus U is unitary. The dynamical time evolution arises from $\mathbf{H}(t)$ while the geometric evolution arises from the non-abelian geometric phase $\mathbf{A}(t)$. In the adiabatic case[18] there will be no mixing of basis states and the qubit state can be encoded in the eigenstates and then be adiabatically transported around a loop in the parameter space. This has the effect of a acting on the state with a holonomy corresponding to a quantum gate[3]. **expand on this**

Due to the long runtimes associated with adiabatic evolution the state is prone to errors and decoherence from outside factors[4]. As a consequence, in the non-adiabatic case (NHQC), it would be possible to construct faster and more robust gates. Since the states are mixing some other way to control $\mathcal{U}$ must be engineered. By removing the dependence on the dynamics of the system, the period time, and energies of the system play no role in the time evolution, explicitly

$$\langle n(t)|\,H(t)\,|m(t)\rangle = 0,\ \forall m,n,t \in [0,T] \tag{30}$$

this means all terms in the dynamical phase in (29) are zero. This is possible given that there exist a two-pulse non commuting loop, $C = C_1 \cup C_2$[4, 19], then the unitary will be reduced to

$$U(T,0) = \sum_{n,m=1}^{N} U_{nm}\,|n(0)\rangle\,\langle m(0)| \tag{31}$$

and can be used to perform any arbitrary quantum gate on the computational space.

2.2.6 Dark states, Bright states and Dark paths

For a given Hamiltonian H it will have eigenvalues for different energies and corresponding eigenkets to those eigenvalues. For a state $|d\rangle$ such that

$$H\,|d\rangle = 0\,|d\rangle\,, \tag{32}$$

then $|d\rangle$ is an eigenstate with eigenvalue 0. We say that $|d\rangle$ is a **dark state** of H. Furthermore we say that any state $|b\rangle$ such that

$$\langle b|d\rangle = 0 \tag{33}$$

is called a **bright state**. A pod-like system with n ground states and m excited states will have $n-m$ darks states and m bright states[20]. The Hamiltonian can be rewritten in the *dark-bright* basis (Morris-Shore basis), an equivalent basis spanned by the bright and dark states. Since the dark states are all individually decoupled from the Hamiltonian this new basis can be further split into two subspaces, one containing the bright states and one containing the dark states. This reduction can be useful since it simplifies the dynamics of the system. The dynamics of a system are closely related to the Hamiltonian of the system. The part of the time evolution affected by the Hamiltonian, for some ordered basis, is $H_{nm}(t) = \langle n|\,H(t)\,|m\rangle$. In the adiabatic case, if the Hamiltonian starts in the dark subspace, it will remain and the

energy will always be zero and it will remain in the dark subspace. Another way to assure the dynamical phase be zero is to evolve the system along a **dark path**, a dark path is a state $|D(t)\rangle$ such that

$$\langle D_n(t)| H(t) |D_m(t)\rangle = 0, \forall t \in [0, T]. \tag{34}$$

thus all elements of the dynamical phase will be zero. The dark state in itself is a dark path. By explicitly finding a dark path the parameters in the Hamiltonian can be reverse-engineered from the Schrödinger equation, enabling more complex Hamiltonians that can fulfill the NHQC criteria.

In [15] they construct and experimentally implement a qubit in an $^{171}Yb^{+}$ ion, using a dark path and an auxiliary state to improve fidelity. They conclude that arbitrary NHQC gates can be constructed with better robustness. We will build upon the scheme presented and create an explicit qutrit example, then speculate how it can be generalized to higher dimensions.

3 The Qutrit

3.1 Construction of the system

In [15] a qubit is implemented by modifying the lambda system, adding an additional level, an auxiliary state, effectively turning it into a tripod. The generalization to implement a qutrit is not trivial but it is clear that one extra ground state is required to make the computational basis larger. Another thing that is needed is to limit the number of dark states to one. The number of dark states is the difference of the number of excited states and the number of ground states[20] (here the auxiliary state is **not** counted as a ground state). Therefore another excited state must be added as well. How these states are coupled is the hard part, we will see that the system given by the Hamiltonian

$$H = \sum_{j=1}^{2} \sum_{i=j}^{3} \omega_{ij} |i\rangle \langle e_j| + \frac{\Omega_a(t)}{2} |a\rangle \langle e_2| + \text{h.c} \tag{35}$$

which is shown in Figure 3.1, will work. It consists of two excited states, $|e_1\rangle, |e_2\rangle$, an auxiliary state $|a\rangle$ and the ground states $|1\rangle, |2\rangle, |3\rangle$ which makes up the computational basis of the qutrit. By changing the basis of the Hamiltonian, it can be rewritten as

$$H_d = \sum_{j=1}^{2} \frac{\Omega_j(t)}{2} e^{-i\phi_j} |b_j\rangle \langle e_j| + \frac{\Omega_a(t)}{2} |a\rangle \langle e_2| + \text{h.c} \tag{36}$$

by a Morris-Shore transformation[21]. We call this new basis the dark state basis. To find this basis find a dark state to the original Hamiltonian, an eigenstate with eigenvalue 0, $H|d\rangle = 0$, then find two bright states, orthogonal to both $|d\rangle$ and each other. This basis can be parametrized with 4 angles, $\theta, \varphi, \chi, \xi$, the dark state is on the form $|d\rangle = \cos\theta |1\rangle + e^{i\chi} \sin\theta \cos\varphi |2\rangle + e^{i\xi} \sin\theta \sin\varphi |3\rangle$, additionally the two bright states are $|b_1\rangle = N_1 \left(-e^{i\xi} \sin\theta \sin\varphi |1\rangle + \cos\theta |2\rangle\right)$ and $|b_2\rangle = N_2 \left(\cos\theta |1\rangle + e^{i\chi} \sin\theta \cos\varphi |2\rangle + \Lambda |3\rangle\right)$, where N_1, N_2 are normalization factors and Λ can be chosen such that $\langle d|b_2\rangle = 0$.

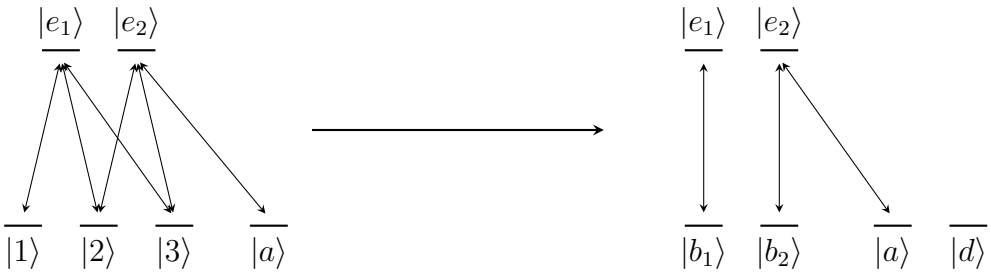

Figure 1: The system given by the Hamiltonian shown in Equation 35 (left) and the transformed system from Equation 36 (right)

These states are bright states which will make up a new orthonormal basis. Explicitly the states are

$$\begin{aligned}
|d\rangle &= \cos\theta\,|1\rangle + e^{i\chi}\sin\theta\cos\varphi\,|2\rangle + e^{i\xi}\sin\theta\sin\varphi\,|3\rangle \\
|b_1\rangle &= \frac{1}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\left(-e^{-i\chi}\sin\theta\cos\varphi\,|1\rangle + \cos\theta\,|2\rangle\right) \\
|b_2\rangle &= \frac{1}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\left(\sin\theta\sin\varphi\cos\theta\,|1\rangle + \frac{e^{i\chi}}{2}\sin^2\theta\sin 2\varphi\,|2\rangle + e^{i\xi}(\sin^2\theta\sin^2\varphi - 1)\,|3\rangle\right)
\end{aligned} \tag{37}$$

The parameters ω_{ij} in the original basis can be determined by replacing the states in H_d by their form in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

Now lets introduce the concept of a dark path, $\langle D(t)|\, H_d\, |D(t)\rangle = 0$, along this path the average energy is always zero. Thus no dynamical phase is accumulated during the time evolution and therefore it follows the conditions required for NHQC.

The following two states satisfy the dark path condition and can be parametrized by two angles $u(t), v(t)$.

$$\begin{aligned}
|D_1(t)\rangle &= \cos u e^{-i\phi_1}\,|b_1\rangle + i\sin u\,|e_1\rangle \\
|D_2(t)\rangle &= \cos u\cos v e^{-i\phi_2}\,|b_2\rangle - i\sin u\,|e_2\rangle - \cos u\sin v\,|a\rangle
\end{aligned} \tag{38}$$

it can easily be verified that $\langle D_i(t)|\, H_d\, |D_j(t)\rangle = 0, i, j = 1, 2$. The angles can be chosen with the constraint that the boundary condition $|D_i(0)\rangle\langle D_i(0)| = |D_i(T)\rangle\langle D_i(T)|, i = 1, 2$. This can be achieved by choosing $u(0) = u(T) = v(0) = v(T) = 0$. A valid choice is $u(t) = \frac{\pi}{2}\sin^2\frac{\pi t}{T}$ and $v(t) = \eta\,[1 - \cos u(t)]$, as for the qubit case[15]. η represents the coupling strength to the auxiliary state $|a\rangle$ and the system reverts into a di-tri-pod(?) structure. Unless mentioned otherwise $\eta = 4.0$, this is the optimal choice for the qubit and shall use it as well[15]. Each dark path starts in the respective bright state and travels along a curve and then returns to the bright state. The choice

Using the Schrödinger equation one can relate the dark path to the Hamiltonian,

$$i\frac{\partial}{\partial t}\,|D_i(t)\rangle = H_d\,|D_i(t)\rangle\,, \tag{39}$$

and thusly one can reverse engineer the time dependent parameters $\Omega_i(t)$ by matching the factors of states. A calculation yields

$$\begin{aligned}
\Omega_1(t) &= -2\dot{u} \\
\Omega_2(t) &= 2\,(\dot{v}\cot u\sin v + \dot{u}\cos v) \\
\Omega_a(t) &= 2\,(\dot{v}\cot u\cos v - \dot{u}\sin v)\,.
\end{aligned} \tag{40}$$

To construct a quantum gate, we make use of the method of multi-pulse single-loops[19], the relevant part of the time evolution operator is

$$\begin{aligned} U_1 &= |d\rangle\langle d| - i\,|e_1\rangle\langle b_1| - i\,|e_2\rangle\langle b_2|\,,\ \phi_1 = \phi_2 = 0 \\ U_2 &= |d\rangle\langle d| + ie^{i\gamma_1}\,|b_1\rangle\langle e_1| + ie^{i\gamma_2}\,|b_2\rangle\langle e_2|\,,\ \phi_1 = -\gamma_1,\ \phi_2 = -\gamma_2 \end{aligned} \tag{41}$$

so the operator for one full loop is

$$U = U_2U_1 = |d\rangle\langle d| + e^{i\gamma_1}\,|b_1\rangle\langle b_1| + e^{i\gamma_2}\,|b_2\rangle\langle b_2|\,. \tag{42}$$

This transformation can be parametrized by 6 real parameters, $U(\chi, \xi, \theta, \varphi, \gamma_1, \gamma_2)$, however it is not enough to construct all gates, for example X_3 requires 2 loops. This is since one loop does not cover all degrees of freedom. The reason for this is elaborated on later in the Generalization section. So the full gate is given by repeating U with another set of parameters. So the full gate $\mathbb{U}$ is given by

$$\mathbb{U} = U(\chi', \xi', \theta', \varphi', \gamma_1', \gamma_2')U(\chi, \xi, \theta, \varphi, \gamma_1, \gamma_2) \tag{43}$$

Now the problem is only a matter of finding all parameters to replicate the desired gate. This is a non-trivial problem since it includes solving a system of 9 non-linear equations containing 12 variables for a gate requiring two loops. Some gates that only require a single loop can with some work be found analytically but the problem quickly becomes too complex to handle.

3.2 Implementation

The implementation is quite straightforward, for a given set of parameters, say that we want to see how the initial state $|\psi(0)\rangle$ evolves with time. Let $|\psi(0)\rangle = (a, b, c)^T$ in the computational basis $\{|1\rangle, |2\rangle, |3\rangle\}$. Then the following equation can be formulated, for some coefficients f_i,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = f_0\,|d\rangle + f_1\,|b_1\rangle + f_2\,|b_2\rangle \tag{44}$$

since at time $t = 0$ the dark paths are in the corresponding bright state. Now expand the dark state and bright state in terms of the original basis. Doing this one can obtain the relation

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} c_1 & -N_1c_2 & N_2c_1 \\ c_2 & N_1c_1 & N_2c_2 \\ c3 & 0 & N_2\Lambda \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{45}$$

So by choosing an initial state, the coefficients can be obtained. Then the system can be solved for any time $t \in T$ using Equation

$$|\psi(t)\rangle = f_0\,|d\rangle + f_1\,|D_1(t)\rangle + f_2\,|D_2(t)\rangle\,, t \in [0, T]. \tag{46}$$

To see the effect of the gates, the probability of finding the system in the standard computational basis states can be plotted against time to see how the gate transform the state from initial to final state, shown in Figure 3.2 and 3.2 for the X_3 and H_3 gates respectively .

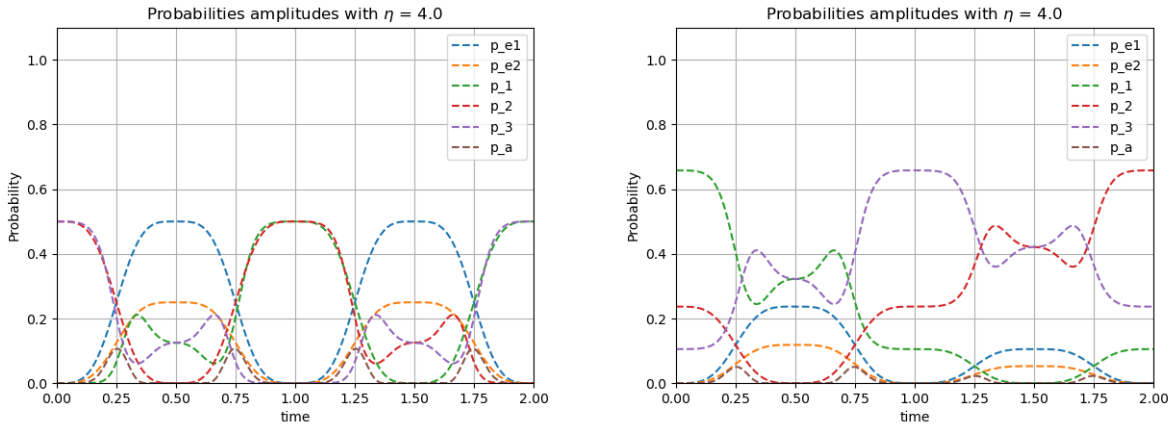

Figure 2: The effect of the X_3-gate on the initial states $\frac{1}{\sqrt{2}}[0,1,1]$ (left) and $\frac{1}{\sqrt{38}}[5,3,2]$ (right). Note that since the plot shows the probabilities, phases can not be seen in the plot.

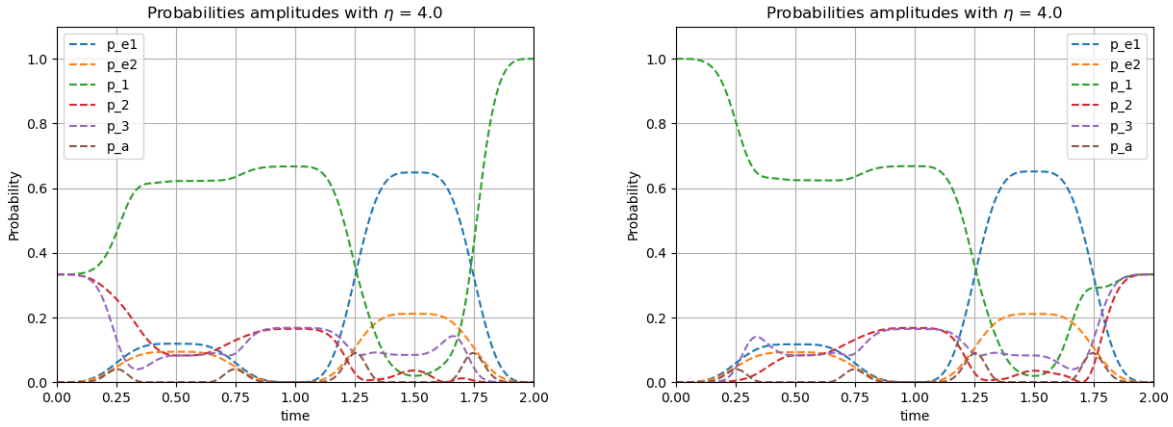

Figure 3: The effect of the H_3-gate on the initial states $\frac{1}{\sqrt{3}}[1,1,1]$ (left) and $[1,0,0]$ (right). Note that since the plot shows the probabilities, phases can not be seen in the plot.

3.3 Important gates, Universality and parameter selection

By specifying the parameters of U the qutrit gates from the Background section can be constructed, a selection of important gates are obtained by the following;

$$X_3 = U(0,0,\frac{\pi}{4},\frac{\pi}{2},0,\pi)U(0,0,\frac{\pi}{2},\frac{\pi}{4},0,\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Z_3 = U(0,0,0,0,\frac{2\pi}{3},\frac{4\pi}{3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

$$T_3 = U(0,0,0,0,\frac{2\pi}{9},\frac{-2\pi}{9}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{9}} & 0 \\ 0 & 0 & e^{\frac{-2\pi i}{9}} \end{pmatrix}$$

$$\begin{aligned} H_3 &= U(6.41010859\cdot 10^{-4}, 6.55568952\cdot 10^{-4}, .475667128, .785362474, 1.58054108, 1.56302702) \\ &\times U(9.81289849\cdot 10^{-3}, 3.56878815\cdot 10^{-18}, 1.18743379, 2.15063745, 9.74301696\cdot 10^{-17}, 1.56882773) \\ &\approx \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \\ 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \end{pmatrix}. \end{aligned} \tag{47}$$

Note that the choice of parameters is not unique and there are multiple ways to create the same unitary. These gates are enough to achieve single-qutrit universality as discussed in Section Theoretical Background. To achieve full universality an additional two-qutrit gate is needed. A common choice for qubits is the CNOT gate[14].

The parameters for a given gate can be specified by solving the non-linear system of equations obtained from by setting the matrix representation of the gate equal to Equation 42 or 43. For the diagonal gates the analytical solution can be easily found by fixing $\theta = \varphi = \chi = \xi = 0$ the basis states reduce to $|d\rangle = |1\rangle, |b_1\rangle = |2\rangle, |b_2\rangle = -|3\rangle$, then all diagonal unitaries can be specified by γ_1 and γ_2 up to a phase factor,

$$U(0,0,0,0,\gamma_1,\gamma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma_1} & 0 \\ 0 & 0 & e^{i\gamma_2} \end{pmatrix}. \tag{48}$$

Analytical solutions are not always easy to find, the H_3-gate gives rise to a system of equation that is non-trivial to solve. In that case an approximative gate $\tilde{U}$ can be found by numerical optimization of the expression $\min||U-\tilde{U}||_F$, where $||\cdot||_F$ is the Frobenius matrix norm given by $||A||_F = \sqrt{\mathrm{Tr}\,[A^\dagger A]}$. Since $\tilde{U} \approx U$ the approximated gates effect will be very close to that of the exact gate.

To assess the robustness of the gate the fidelity is used, a metric that measures how close two quantum states are to each other, $F(|\psi\rangle, |\varphi\rangle) = |\langle\psi|\varphi\rangle|$. The fidelity is averaged by sampling initial states and letting them evolve with time (numerically solving the Schrödinger Equation), and then comparing to the exact solution obtained by multiplying the gate with the initial state. The calculated fidelities are shown in Figure 4.

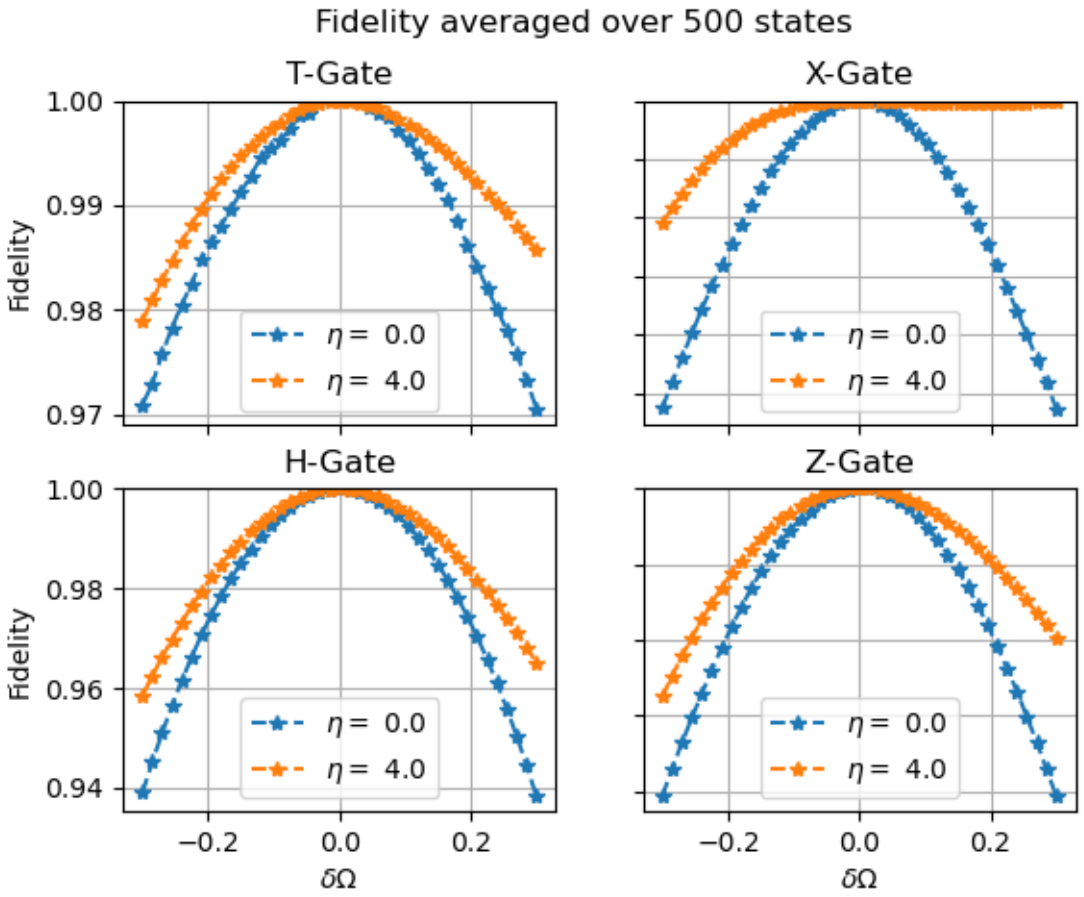

Figure 4: Robustness test, average fidelity of the T_3, X_3, Z_3, and H_3 gates. Average calculated by sampling over 500 randomized initial states with a perturbation of the Ω-pulses, $\Omega \mapsto \Omega(1+\delta)$.

4 Generalization

To generalize the scheme to a qudit with arbitrary dimension n, the same idea from the qutrit case can be extended, n ground states, $m = n-1$ excited states and 1 auxiliary state, therefore the dimension of the Hilbert space is $n + m + 1 = n + 1 - 1 = 2n$. Once again the couplings are not trivial and will be somewhat intricate, but the couplings given by the Hamiltonian in Equation 49, will do the trick.

$$H = \sum_{j=1}^{m} \sum_{i=j}^{n} \omega_{ij} \left|i\right\rangle \left\langle e_j\right| + \frac{\Omega_a(t)}{2} \left|a\right\rangle \left\langle e_m\right| + \text{ h.c} \tag{49}$$

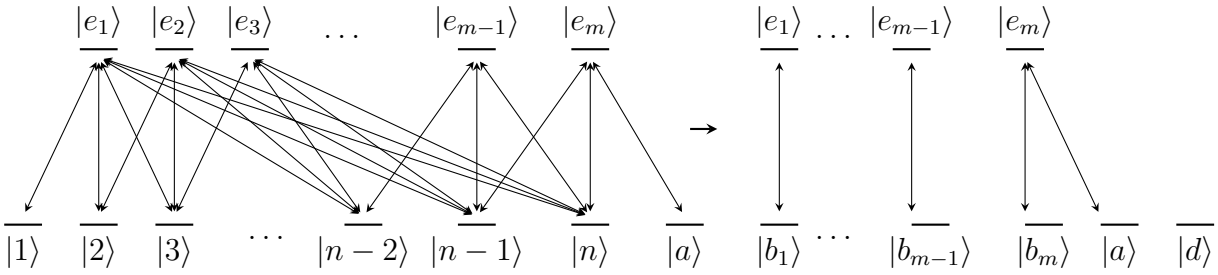

Figure 5: The system given by the Hamiltonian shown in Equation 49 (left) and the transformed system from Equation 54 (right)

The general system for a qudit is given by m excited states $\left|e_i\right\rangle$, $i = 0, 1, \ldots, m$ and n ground states labeled $\left|i\right\rangle$, $i = 1, 2, \ldots, n$ and one auxiliary state $\left|a\right\rangle$. The number of states always follows that $n - m = 1$ to limit the number of dark states to one[20]. The transitions occur only between excited states and ground states. The couplings follow a pattern, the first excited state e_1 is connected to all ground states, e_2 is connected to all ground states except $\left|1\right\rangle$, in general, the excited state $\left|e_i\right\rangle$ is connected to the $(n-i)$th ground states with the highest label. Unless $i = m$, the excited state $\left|e_m\right\rangle$ is connected to the two highest labeled ground states and the auxiliary state $\left|a\right\rangle$. See Equation 49 and Figure 5 for a clarification.

From the standard basis $\{e_1, e_2, \ldots, e_m, 1, 2, \ldots, n, a\}$, it is possible define a new basis, the dark state basis given by $\{e_1, e_2, \ldots, e_m, b_1, b_2, \ldots, b_m, a, d\}$.

Assume a dark state on the form

$$\left|d\right\rangle = c_1 \left|1\right\rangle + c_2 \left|2\right\rangle + c_3 \left|3\right\rangle \cdots + c_n \left|n\right\rangle, \; |c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1 \tag{50}$$

from this dark state one could recursively define $n - 1 = m$ bright states. Starting from

$$\left|b_1\right\rangle = N_1 \left(-c_2 \left|1\right\rangle + c_1 \left|2\right\rangle\right) \tag{51}$$

with N_1 being a normalization factor. Then choose additional bright states on the form

$$\begin{aligned}
\left|b_2\right\rangle &= N_2 \left(c_1 \left|1\right\rangle + c_2 \left|2\right\rangle + \Lambda_1^{(2)} \left|3\right\rangle\right) \\
\left|b_3\right\rangle &= N_3 \left(c_1 \left|1\right\rangle + c_2 \left|2\right\rangle + \Lambda_1^{(3)} \left|3\right\rangle + \Lambda_2^{(3)} \left|4\right\rangle\right) \\
&\vdots \\
\left|b_{m-1}\right\rangle &= N_{m-1} \left(c_1 \left|1\right\rangle + c_2 \left|2\right\rangle + \Lambda_1^{(m-1)} \left|3\right\rangle + \Lambda_2^{(m-1)} \left|4\right\rangle + \cdots + \Lambda_{m-2}^{(m-1)} \left|m\right\rangle\right) \\
\left|b_m\right\rangle &= N_m \left(c_1 \left|1\right\rangle + c_2 \left|2\right\rangle + \Lambda_1^{(m)} \left|3\right\rangle + \Lambda_2^{(m)} \left|4\right\rangle + \cdots + \Lambda_{m-2}^{(m)} \left|m\right\rangle + \Lambda_{m-1}^{(m)} \left|m+1\right\rangle\right).
\end{aligned} \tag{52}$$

By this construction, it is clear that $|b_1\rangle$ is orthogonal to all other bright states.

The coefficients can be chosen in such a way that, in $|b_2\rangle$, the coefficient $\Lambda_1^{(2)}$ can be chosen such that, $\langle d|b_2\rangle = 0$, and in $|b_3\rangle$, the coefficient $\Lambda_1^{(3)}$ can be chosen such that, $\langle b_2|b_3\rangle = 0$ and $\Lambda_2^{(3)}$ such that $\langle d|b_3\rangle = 0$. By recursively repeating this argument one could see that it is possible to chose m bright states, then by normalizing all the N_i can be found, thus we have obtained m orthonormal bright states, $\langle b_i|b_j\rangle = \delta_{ij}$. The coefficients c_i can be parametrized by the euclidean components of the unit-n-sphere and a phase factor.

$$\begin{aligned} c_1 &= \cos(\varphi_1) \\ c_2 &= e^{i\theta_1}\sin(\varphi_1)\cos(\varphi_2) \\ c_3 &= e^{i\theta_2}\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\ &\vdots \\ c_{n-1} &= e^{i\theta_{n-1}}\sin(\varphi_1)\ldots\sin(\varphi_{n-2})\cos(\varphi_{n-1}) \\ c_n &= e^{i\theta_n}\sin(\varphi_1)\ldots\sin(\varphi_{n-2})\sin(\varphi_{n-1}) \end{aligned} \tag{53}$$

c_1 does not need a phase factor since the overall phase of a state is non-measurable and can be chosen such that the first phase factor can be canceled. The remaining Λ coefficients can be expressed in terms of the c_i.

In this newly defined space, the Hamiltonian can be written as

$$H_d = \sum_{i=1}^{m} \frac{\Omega_i(t)}{2} e^{-i\phi_i} |b_i\rangle \langle e_i| + \frac{\Omega_a(t)}{2} |a\rangle \langle e_n| + \text{ h.c} \tag{54}$$

with Ω_i being real-valued time dependent parameters and the ϕ_i time independent phase factors.

With this Hamiltonian m dark paths can be constructed, and must satisfy $\langle D_i(t)| H_d |D_i(t)\rangle = 0, i = 1, 2, \ldots, m$ and $\langle D_i(t)|D_j(t)\rangle = \delta_{ij}$.
The dark paths can be parametrized by two functions $u(t), v(t)$ that satisfy the conditions $u(0) = v(0) = u(T) = v(T) = 0$, will have the form

$$\begin{aligned} |D_i(t)\rangle &= \cos u e^{-i\phi_i} |b_i\rangle + i \sin u |e_i\rangle \,,\ i = 1, 2, \ldots, m-1 \\ |D_m(t)\rangle &= \cos u \cos v e^{-i\phi_n} |b_m\rangle - i \sin u |e_m\rangle - \cos u \sin v |a\rangle \end{aligned} \tag{55}$$

The dark paths start in the bright state and travel along a curve where the expectation value of the energy is constantly 0 and can thus be used non-adiabatically.

By using these states one can reverse engineer the Hamiltonian using the Schrödinger equation to determine Ω_i and Ω_a since

$$i\frac{\partial}{\partial t} |D_i(t)\rangle = H_d |D_i(t)\rangle \,,\ i = 1, 2, \ldots, m \tag{56}$$

a calculation yields

$$\begin{aligned} \Omega_1(t) &= -2\dot{u} \\ \Omega_2(t) &= -2\dot{u} \\ &\vdots \\ \Omega_{m-1}(t) &= -2\dot{u} \\ \Omega_m(t) &= 2\left(\dot{v}\cot u \sin v + \dot{u}\cos v\right) \\ \Omega_a(t) &= 2\left(\dot{v}\cot u \cos v - \dot{u}\sin v\right) \end{aligned} \tag{57}$$

The time evolution is split into k loops, each loop with two pulses, $0 \longrightarrow T/2$ and $T/2 \longrightarrow T$. The relevant part of the time evolution operator for one loop is $U_1(T/2,0) = |d\rangle\langle d| - i\sum_{i=1}^{m}|e_i\rangle\langle b_i|\,,\ \phi_i = 0, i = 1,2,\ldots,n$ and $U_2(T,T/2) = |d\rangle\langle d| + i\sum_{i=1}^{m} e^{i\gamma_i}|b_i\rangle\langle e_i|\,,\ \phi_i = 0, i = 1,2,\ldots,n$ The full operator for one loop is then given by

$$U = U_2U_1 = |d\rangle\langle d| + \sum_{i=1}^{m} e^{i\gamma_i}|b_i\rangle\langle b_i|\,. \tag{58}$$

It is clear that U is unitary in the subspace $\{d, b_1, b_2, \ldots, b_n\}$

The unitary is parametrized by $3m = 3(n-1), n \geq 2$, parameters,

$$U(\varphi_1,\ldots,\varphi_m,\theta_1,\ldots,\theta_m,\gamma_1,\ldots,\gamma_m) \tag{59}$$

Applying the unitary with different parameters in sequence up to k times is enough to create any desirable gate $\mathcal{U} = U^k$. The unitary is controlled by $3(n-1)$ parameters while n dimensional qudit has more degrees of freedom than covered with a single loop.

The qudit state space is equivalent to $SU(n)$, which has dimensionality $\dim(SU(n)) = n^2 - 1$. To cover all degrees of freedom k must satisfy $3(n-1)kgeqn^2 - 1.$ which means $k \geq \frac{n+1}{3}$. Thus the number of loops needed to create any unitary scales linearly at worst since some gates can be created with fewer loops, see Table 2.

In the case of equality $k = \frac{n+1}{3}$, when $n = 3j+2,\ j \in \mathbb{N}$, the fewest amount of loops per dimension is achieved and could potentially be more efficient carriers of information since the same number of loops must be carried out while higher dimension has higher information capacity.

n	$3(n-1)$	n^2-1	k	=
2	3	3	1	✓
3	6	8	2	
4	9	15	2	
5	12	24	2	✓
6	15	35	3	
7	18	48	3	
8	21	63	3	✓
9	24	80	4	
10	27	99	4	
11	30	120	4	✓
12	33	143	5	

Table 2: Table for some dimensions.

The quantum gate $\mathcal{U}$ can be formulated as a linear combination of the dark state and dark paths just as in the qutrit case,

$$|\psi(t)\rangle = f_0|d\rangle + \sum_{i=1}^{m} f_i|D_i(t)\rangle\,, t \in [0,T] \tag{60}$$

where the coefficients can be solved for by choosing an initial state $|\psi(0)\rangle$. This corresponds to one loop, by using $|\psi(T)\rangle$ as an initial state for the next loop. By iterating this method one can simulate $\mathbb{U} = U^k$ where each loop U have a different set of parameters. So in theory this can be implemented and simulated for any dimensional qudit, just as for the case of the qutrit.

5 Discussion and Conclusion

We have shown how to explicitly create a qutrit by expanding the dark path qubit scheme[15] into a higher dimension and how it will generalize in the qudit case and using auxiliary states to improve the robustness of the gates. The qutrit gates have a high fidelity and robustness is improved by the inclusion of the auxiliary state $|a\rangle$ The scaling of number of loops for the qudit gate is linear at worst. In fact, any diagonal gate only requires one loop, by setting $\varphi_1 = \dots \varphi_m = \theta_1 = \dots = \theta_m = 0$ the unitary reduces to the form **show calculation, in appendix maybe?**

$$U(0, \dots, 0, \gamma_1, \dots, \gamma_m) = |1\rangle\langle 1| + \sum_{k=2}^{n} e^{i\gamma_k} |k\rangle\langle k| . \tag{61}$$

Thusly, any phase flip gate, such as Z_m and T_m can be constructed in any dimension using only one loop. Since the gates essential for universality is $\{T_m, H_m\}$, where on of the gates can be obtained by a single loop. Thus only the Hadamard gate will be the most costly part of the scheme as it requires at most $k = \frac{n+1}{3}$ loops in n dimensions.

6 Outlook

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A Extra Calculations

General formula for the $\Lambda_n^{(m)}$ coefficients where $m \geq 2$

$$\Lambda_n^{(m)} = \begin{cases} -\dfrac{\sum_{k=1}^{m-1} |c_k|^2}{c_{m+1}^*} & \text{if} \quad m - n = 1 \\ \\ c_{n+2} & \text{otherwise} \end{cases} \tag{62}$$

and for the normalization

$$N_m = \left(\sum_{k=1}^{m} |c_k|^2 + \left| \frac{1}{c_{m+1}} \sum_{k=1}^{m} |c_k|^2 \right|^2 \right)^{-1/2} \tag{63}$$

B Code

Kod här