20 December 1999

PHYSICS LETTERS A

Physics Letters A 264 (1999) 94–99

www.elsevier.nl/locate/physleta

# Holonomic quantum computation

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Received 16 July 1999; accepted 12 November 1999

Communicated by P.R. Holland

**Abstract**

We show that the notion of generalized Berry phase i.e., non-abelian holonomy, can be used for enabling quantum computation. The computational space is realized by a $n$-fold degenerate eigenspace of a family of Hamiltonians parametrized by a manifold $\mathscr{M}$. The point of $\mathscr{M}$ represents classical configuration of control fields and, for multi-partite systems, couplings between subsystem. Adiabatic loops in the control $\mathscr{M}$ induce non trivial unitary transformations on the computational space. For a generic system it is shown that this mechanism allows for universal quantum computation by composing a generic pair of loops in $\mathscr{M}$. © 1999 Elsevier Science B.V. All rights reserved.

*PACS:* 03.67.Lx; 03.65.Bz

In this Letter we shall speculate about a novel potential application of non-abelian geometric phases (*holonomies*) to Quantum Computation [1]. Ever since their discovery geometric phases in quantum theory have been considered a deep and fascinating subject [2]. This is due on the one hand to their unexpected and ubiquitous role in many physical systems, on the other hand to the elegant formulation they admit in terms of concepts borrowed from differential-geometry and topology [4]. Furthermore, the existence of analog geometric terms associated with *non-abelian* groups e.g., $U(N)$ [6] showed how many of the notions developed in (non-abelian) gauge theory have a scope that extends far beyond the study of fundamental interactions.

We shall show how by encoding quantum information in one of the eigenspaces of a degenerate Hamiltonian $H$ one can in principle achieve the full quantum computational power by using holonomies only. These holonomic computations are realized by moving along loops in a suitable space $\mathscr{M}$ of control parameters labelling the family of Hamiltonians to which $H$ belongs. Attached to each point $\lambda \in \mathscr{M}$ there is a quantum code, and this bundle of codes is endowed with a non-trivial global topology described by a non-abelian gauge field potential $A$. For generic $A$ the associated holonomies will allow for universal quantum computing. In a way, the ideas

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[1] For reviews, see [1,2].

[2] For a review see [3].



PII: S0375-9601(99)00803-8

presented here suggest that gauge fields might play a role also in the arena of information processing.

In the very same way of classical information processing, general quantum computations are realized by networks of elementary bulding blocks. More specifically, the dynamics is obtained by switching on and off *gate* Hamiltonians $\{H_l\}_{l=1}^g$. If in this way any unitary over the state-space can be approximated arbitrarily well, the set of gates $U_l := e^{i H_l}$ is termed *universal* [7,8]. A (universal) quantum computer is defined by the state-space $\mathcal{H} \cong \mathbf{C}^N$ for data encoding and by a (universal) set of quantum gates. A quantum algorithm consists of the given computation $U(T)$ that acts on the quantum state $|\psi\rangle_{\text{in}}$ encoding initial data, its realization as a network of basic gates, along with a (measurement) prescription for extracting the relevant information from $|\psi\rangle_{\text{out}}$. For the aims of this Letter it is worthwile to reformulate this setup in a geometrical fashion.

Any quantum evolution

$$U(T) = \mathbf{T}\exp\left[-i\int_0^T dt H(t)\right]$$

can be associated with a path in a space whose points describe the configurations of suitable 'control fields' $\lambda$, on which the Hamiltonian depends. Indeed any Hamiltonian in $\mathbf{C}^N$ can be written as $H_\lambda = i\sum_{l=1}^{N^2}\Phi_l(\lambda)\,\Gamma_l$ $(\Phi_l \in \mathbf{R})$ where the $\Gamma_l$'s are a basis of the space of anti-hermitean matrices i.e., they are the generators of the Lie algebra $u(N)$. The control parameter space $\mathcal{M}$ is a manifold over which is defined a smooth map $\Phi$ to $u(N)$. If one is able to drive the control field configuration $\lambda \in \mathcal{M}$ through a (smooth) path $\gamma:[0,T] \to \mathcal{M}$ then a family $H(t) := H_{\gamma(t)}$ is defined along with the associated unitary $U_\gamma$. Conversely any smooth family $H(t)$ defines a path in $\mathcal{M} = \mathbf{R}^{N^2}$. Resorting to this language Quantum Computation can be described as the experimenter's capability of generating a small set of $\{\gamma_i\}_{i=1}^g$ of basic paths such that sequences of the corresponding $U_{\gamma_i}$'s approximate with arbitrary good accuracy any unitary transformation on the quantum state-space. It is important to stress the, obvious, fact that the path generation is achieved through a classical control process.

*Non-Abelian holonomies.* In the situation which we are interested in, one deals with $\gamma$'s that are *loops* in $\mathcal{M}$ i.e., $\gamma(T) = \gamma(0)$, and with a family of Hamiltonians $\{H_\lambda\}_{\lambda\in\mathcal{M}}$ with same degeneracy structure i.e., no-level crossing. In the general case $H_\lambda = H_{\gamma(t)}$ has $R$ different eigenvalues $\{\varepsilon_i\}_{i=1}^R$ with degeneracies $\{n_i\}$. If $\Pi_i(\lambda)$ denotes the projector over the eigenspace $\mathcal{H}_i(\lambda) := \mathrm{span}\{|\psi_i^\alpha(\lambda)\rangle\}_{\alpha=1}^{n_i}$, of $H_\lambda$, one has the spectral $\lambda$-dependent resolution $H_\lambda = \sum_{i=1}^R \varepsilon_i(\lambda)\,\Pi_i(\lambda)$.

The state vector evolves according the time-dependent Schrödinger equation $i\,\partial_t|\psi(t)\rangle = H_{\gamma(t)}|\psi(t)\rangle$. We shall restrict ourselves to the case in which the loop $\gamma$'s are *adiabatic* i.e., $\hbar\dot{\gamma}/\gamma \ll \min_{i\neq j}|\varepsilon_j - \varepsilon_i|$. Then it is well known that any initial preparation $|\psi_0\rangle \in \mathcal{H}$ will be mapped, after the period $T$, onto: $|\psi(T)\rangle = U(T)|\psi_0\rangle, U(T) = \oplus_{l=1}^R e^{i\phi_l(T)}\,\Gamma_{A_l}(\gamma)$, where, $\phi_l(T) := \int_0^T d\tau\,\varepsilon_l(\lambda_\tau)$, is the dynamical phase and

$$\Gamma_{A_l}(\gamma) := \mathbf{P}\exp\int_\gamma A_l \in U(n_l), \quad (l = 1,\ldots,R) \quad (1)$$

is called the *holonomy* associated with the loop $\gamma$, (here $\mathbf{P}$ denotes path ordering). In particular when $|\psi_0\rangle \in \mathcal{H}_l$ the final state belongs to the *same* eigenspace. In the following we will drop dynamical phases and focus on the geometrical contribution (1). For $n = 1$ this term is nothing but the celebrated Berry phase, and $A$ is the so-called Bott–Chern connection [5]. For $n_l > 1$ the holonomy $\Gamma_{A_l}(\gamma)$ is sometimes referred to as *non-abelian* geometric phase [6]. The matrix-valued form $A_l$ appearing in Eq. (1) is known as the *adiabatic connection* and it is given by $A_l = \Pi_l(\lambda)\,d\,\Pi_l(\lambda) = \sum_\mu A_{l,\mu}\,d\lambda_\mu$, where [3]

$$(A_{l,\mu})^{\alpha\beta} := \langle\psi_l^\alpha(\lambda)|\,\partial/\partial\lambda^\mu\,|\psi^\beta(\lambda)\rangle \quad (2)$$

$(\lambda_\mu)_{\mu=1}^d$ local coordinates on $\mathcal{M}$. The $A_l$'s are a non-abelian gauge potentials that allow for parallel transport of vectors over $\mathcal{M}$. Indeed the linear mapping (1) of the fiber $\mathcal{H}_l$ onto itself is nothing but the parallel transport of the vector $|\psi_0\rangle$ associated with the connection form $A_l$.

In view of the crucial role played by degeneracy, before moving to the main part of the Letter, we briefly discuss this issue in a geometric fashion by considering the space of Hamiltonians $H$ of a quantum state-space $\mathcal{H} \cong \mathbf{C}^N$.

The control manifold is mapped by $\Phi$ onto a set of Hamiltonians iso-degenerate with $H = H_{\gamma(0)}$. Locally, one has $\Phi(\mathcal{M}) \cong \mathcal{O}(H) \times (\mathbf{R}^R - \Delta_R)$, where $\Delta_R := \{x \in \mathbf{R}^R : i \neq j \Rightarrow x_i \neq x_j\}$, and $\mathcal{O}(H) := \{XHX^\dagger / X \in U(N)\}$ is the orbit of $H$ under the (adjoint) action of $U(N)$. Indeed, any pair of isospectral Hamiltonians belongs to $\mathcal{O}(H)$, moreover once the orbit is given one has still $R$ degrees of freedom (the different eigenvalues) for getting the whole manifold of Hamiltonian with fixed degeneracy structure. By factoring out the the symmetry group of $H$, one finds

$$\mathcal{O}(H) := \frac{U(N)}{U(n_1) \times \cdots \times U(n_R)}, \tag{3}$$

From Eq. (3) it stems that dimension of this manifold reach its maximum (minimum) for the non (maximally) degenerate case $R = N$ ($R = 1$): $d_{\max} = N(N-1) + N = N^2$ ($d_{\min} = 0 + 1 = 1$). This means that the set of non-degenerate Hamiltonians is an *open* submanifold of $\mathbf{R}^{N^2}$, expressing the well-known fact that degeneracy – due to the symmetry constraints that it involves – is a singular case, while non-degeneracy is the generic one. Indeed if one slightly perturbs a non-degenerate Hamiltonian $H$ the resulting operator is, generically, still non-degenerate.

*Universal computation.* The above considerations make clear that the degeneracy requirement for the existence of non-abelian holonomies is rather stringent from a purely geometrical point of view. On the other hand, quite often the physics of the systems under concern provides the required symmetries for having (large) degenerate eigenspaces. Notice that discrete symmetries, like charge conjugation and rotational invariance are rather generic in many-body systems. For example, non-abelian holonomies have been recently shown to play a role in the $SO(5)$ theory of superconductivity [9].

In the following, we will take degeneracy for granted and we will fix our attention to a given $n$-dimensional eigenspace $\mathcal{C}$ of $H$. The state-vectors in $\mathcal{C}$ will be our quantum codewords, and $\mathcal{C}$ will be referred to as the *code*. Clearly, the optimal choice is to take the code to be the largest eigenspace of $H$. Our aim is to perform as many as possible unitary transformations i.e., *computations*, over the code resorting only on the non-abelian holonomies (1) generated by adiabatic loops in $\mathcal{M}$. A first crucial question is:

*How many transformations can be obtained, by Eq. (1), as $\gamma$ varies over the space of loops in $\mathcal{M}$?*

To address this point let us begin by considering the properties of the holonomy map $\Gamma_A$. On the loop space (we set $T = 1$)

$$L_{\lambda_0} := \{\gamma : [0,1] \mapsto \mathcal{M} \,/\, \gamma(0) = \gamma(1) = \lambda_0\} \tag{4}$$

over a point $\lambda_0 \in \mathcal{M}$, there exists a composition law for loop [i.e., $(\gamma_2 \cdot \gamma_1)(t) = \theta(\frac{1}{2} - t)\gamma_1(2t) + \theta(t - \frac{1}{2})\gamma_1(2t-1)$] and a unity element $\gamma_0(t) = \lambda_0$, $t \in [0,1]$. The basic property of map $\Gamma_A : L_{\lambda_0} \mapsto U(n)$ are easily derived from Eq. (1): (i) $\Gamma_A(\gamma_2 \cdot \gamma_1) = \Gamma_A(\gamma_2)\,\Gamma_A(\gamma_1)$; (ii) $\Gamma_A(\gamma_0) = \mathbb{1}$; moreover, by denoting with $\gamma^{-1}$ the loop $t \mapsto \gamma(1-t)$, one has (iii) $\Gamma_A(\gamma^{-1}) = \Gamma_A^{-1}$. This means that by composing loops in $\mathcal{M}$ one obtains a unitary evolution that is the product of the evolutions associated with the individual loops and that staying at rest in the parameter space correspond to no evolution at all. Finally (iii) tells us that for getting the time-reversed evolution one has simply to travel along $\gamma$ with the opposite orientation. Another noteworthy property of $\Gamma_A$ – on which its geometric nature is based – is its invariance under reparametrizations: $\Gamma_A(\gamma \circ \varphi) = \Gamma_A(\gamma)$, where $\varphi$ is any diffeomorphism of $[0,1]$. Physically this means that the evolution map – as long as adiabaticity holds – does not depend on the rate at which $\gamma$ is travelled but just on its geometry. This property is quite non-trivial and, obviously, does not hold for general time-dependent quantum evolutions.

From (i)–(iii) it follows immediately that the set $\mathrm{Hol}(A) := \Gamma_A(L_{\lambda_0})$ is a *subgroup* of $U(n)$ known as the *holonomy group* of the connection $A$. Notice that the distinguished point $\lambda_0$ is not crucial, in that $\Gamma_A(L_{\lambda_0}) \cong \Gamma_A(L_{\lambda_0'})$ provided $\lambda_0$ and $\lambda_0'$ can be connected by a smooth path. When $\mathrm{Hol}(A) = U(n)$, the connection $A$ is called *irreducible*. To our aims the key observation is that irreducibility is the *generic* situation. This result can be stated geometrically by saying that in the space of connections over $\mathcal{M}$, the irreducible ones are an open dense set. The condition of irreducibility can be stated in terms of the *curva-*

*ture* 2-form of the connection $F=\sum_{\mu\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ where

$$F_{\mu\nu}=\partial_\nu A_\mu - \partial_\mu A_\nu - \left[ A_\mu, A_\nu \right]. \tag{5}$$

If the $F_{\mu\nu}$'s linearly span the whole Lie algebra $u(n)$, then $A$ is irreducible [4]. It follows that in the generic case adiabatic connections will provide a mean for realizing universal quantum computation over $\mathscr{C}$. For any chosen unitary transformation $U$ over the code there exists a path $\gamma$ in $\mathscr{M}$ such that $\| \Gamma_A(\gamma) - U \| \le \epsilon$, with $\epsilon$ arbitrarily small. Therefore any computation on the code $\mathscr{C}$ can be realized by driving the control fields configuration $\lambda$ along closed paths $\gamma$ in the control manifold $\mathscr{M}$.

Now we show that the connections associated with non abelian geometric phases are *actually* irreducible. For simplicity in Eq. (3) we set $R=2$, $n_1=1$, $n_2=N-1$ obtaining the $N-1$-dimensional complex projective space

$$\mathscr{O}(H_0) \cong \frac{U(N)}{U(N-1)\times U(1)} \cong \frac{SU(N)}{U(N-1)} \cong \mathbf{CP}^{N-1}. \tag{6}$$

The orbit $\mathscr{O}(H_0)$ of $H_0 \equiv H_{\lambda_0}$ coincides with the manifold of pure states over $\mathbf{C}^N$. When $N=2$ one recovers the original Berry–Simon case, $H_{\mathrm{BS}}=\mathbf{B}\cdot\mathbf{S}$,$(\mathbf{S}:=(\sigma_x,\sigma_y,\sigma_z), \mathbf{B}\in S^2 \cong \mathbf{CP}^1)$, for a spin $\frac{1}{2}$ particle in an external magnetic field $\mathbf{B}$. Here $\mathrm{Hol}(A_{\mathrm{BS}}) = \{\mathrm{e}^{i S_\gamma}\}_\gamma \cong U(1)$, where $S_\gamma$ is the area enclosed by the loop $\gamma$ in the sphere $S^2$. Of course this case, being abelian, has no computational meaning, nevertheless it shows how controllable loops in an external field manifold (the $\mathbf{B}$-space) can be used for generating quantum phases.

For the characterization of the holonomy group we observe first of all that one can identify the control manifold with orbit $\mathscr{O}$. Technically this is due to the fact that the bundle of $N-1$-dimensional 'codes' over $\mathscr{M}$ is vector bundle with structure group $U(N)$. The associated $U(N-1)$-principal bundle is the pull back, through $\Phi$, of $\pi: U(N)/U(1) \mapsto \mathbf{CP}^{N-1}$. The result follows being the latter an universal classifying bundle [11].

For general $N$ the points of $\mathbf{CP}^{N-1}$ are parametrized by the transformations

$$\mathscr{U}(\mathbf{z}) := \mathbf{P}\prod_{\alpha=1}^{N-1} U_\alpha(z_\alpha),$$

where $U_\alpha(z_\alpha) := \exp(z_\alpha |\alpha\rangle\langle N| - \text{h.c.})$. The relevant projectors are given by $\Pi_{\mathbf{z}} = \mathscr{U}(\mathbf{z})\,\Pi\,\mathscr{U}(\mathbf{z})^\dagger$, where $\Pi$ is the projector over the first $N-1$ degenerate eigenstates. By using def. (5) and setting $z_\alpha = z_\alpha^0 + i z_\alpha^1$, one checks that at $\mathbf{z}=0$ the components of the curvature are given by

$$F_{z_\alpha^n, z_\beta^m}(0) = \Pi \left[ \frac{\partial U_\alpha}{\partial z_\alpha^n}, \frac{\partial U_\beta}{\partial z_\beta^m} \right] \Pi \Bigg|_{\mathbf{z}=0}, \tag{7}$$

with $\alpha,\beta = 1,\ldots,N-1,; m,n=0,1$.

Since $\partial U_\alpha / \partial z_\alpha^n = i^n (|\alpha\rangle\langle N| - (-1)^n |N\rangle\langle\alpha|)$, one finds

$$F_{z_\alpha^n, z_\beta^m}(0) = i^{m+n}\left[(-1)^n |\beta\rangle\langle\alpha| - (-1)^m |\alpha\rangle\langle\beta|\right]. \tag{8}$$

From this expression it follows that components of $F$ span the whole $u(N-1)$. As remarked earlier, this result does not depend on the specific point chosen, therefore this example is irreducible i.e., $\mathrm{Hol}(A) \cong U(N-1)$. The general case (3) can be worked out along similar lines it turns out to be irreducible as well. Notice how, for generating control loops for $N$ qubits, one needs to control $2^{N+1}$ real parameters instead of the $2^{2N}$ ones necessary for labelling a generic Hamiltonian.

For practical purposes is relevant the question:

*How many loops should an experimenter be able to generate for getting the whole holonomy group?*

An existential answer is given below by using arguments close to the ones of Ref. [10]. As the non-trivial topology associated with the irreducible gauge-field $A$ allows to map the loop 'alphabet' densely into the group of unitaries over the code, we have the

**Proposition.** *Two generic loops $\gamma_i$ $(i=1,2)$ generate a universal set of gates over $\mathscr{C}$.*

**Proof.** It is known that two generic unitaries $U_1$ and $U_2$ belonging to a subgroup $\mathscr{G}$ of $U(N)$ generate, by composition, a subgroup $G$ dense in $\mathcal{G}$ [10]. In particular if $\mathcal{G}=U(N)$ the $U_i$'s are a universal set of gates. Formally: let $U_{\pm\alpha}:=U_\alpha^{\pm 1}\,(\alpha=0,\pm 1,\pm 2); U_0:=$ , then the set $G$ of transformations obtainable by composing the $U_i$'s (along with their inverses) is given by $U_f:=\mathbf{P}\prod_{p\in\mathbf{N}}U_{f(p)}$, where $f$ is a map from the natural numbers $\mathbf{N}$ to the set $\{0,\pm 1,\pm 2\}$ nonvanishing only for finitely many $p$'s.

From the basic relation (i) it follows that the transformations $U_f$ are generated by composing loops in $L_{\lambda_0}$:

$$U_f=\Gamma_A(\gamma_f),\quad \gamma_f:=\mathbf{P}\prod_{p\in\mathbf{N}}\gamma_{f(p)}. \tag{9}$$

We set $\{U_i:=\Gamma_A(\gamma_i)\}_{i=1}^2$, then $\overline{G}=\mathrm{Hol}(A)=U(N)$, the latter relation follows from irreducibility of $A$. □

Of course, this result does not provide an explicit recipe for obtaining the desired transformations, nevertheless it is conceptually quite remarkable. It shows that, even though adiabatic holonomies are a very special class of quantum evolutions, they still provide the full computational power for processing quantum information.

On the other hand, our result is not completely surprising. Indeed it is important to stress that the parameters $\lambda$, for a multi-partite system, will contain in general external fields as well as couplings between sub-systems. For example if $\mathcal{H}=\mathbf{C}^2\otimes\mathbf{C}^2$ is two-qubits space a possible basis for $u(4)$ is given by $i\,\sigma_\mu\otimes\sigma_\nu$, where $\sigma_0:=\mathbb{1}$ and $\{\sigma_i\}_{i=1}^3$ are the Pauli matrices. Then, for $ij\neq 0$ the $\Gamma_{ij}$'s describe non-trivial interactions between the two qubits, while for $ij=0$ the corresponding generators are single qubit operators. Only the control fields associated with these latter $\Gamma_{ij}$'s can be properly interpreted as external fields while the others generate true entanglement among subsystems. Moreover, quite often, the parameters $\lambda$ are indeed quantum degrees of freedom, which are considered frozen in view of the adiabatic decoupling [3]. In this case the generation of loops $\gamma$ is on *its own* a problem of quantum control.

So far, we have been concerned just with existential issues of unitary evolutions. In the following we shall briefly address the associated problem of computational complexity. A detailed discussion of this point is given elsewhere [12]. It is widely recognized that a crucial ingredient that provides quantum computing with its additional power is *entanglement*. This means that the computational state-space has to be multi-partite e.g., $\mathcal{H}=(\mathbf{C}^2)^{\otimes N}$, and the computations are, efficiently, obtained by composing local gates that act non trivially over a couple of subsystems at most [7,8]. In general the degenerate eigenspaces in which we perform our holonomic computations do not have any preferred tensor product structure. Once one of these structures has been chosen over a $N$-qubit code $\mathcal{C}$ i.e., an isomorphism $\varphi:\mathcal{C}\mapsto(\mathbf{C}^2)^{\otimes N}$ has been selected, any unitary transformation over $\mathcal{U}$ can written as a suitable sequence of CNOT's and single qubit transformations. In the $\mathbf{CP}^N$ model discussed above it can be proven, by explicit computations, that one can constructively get any single-qubit and two-qubit gate as well by composing elementary holonomic loops restricted with suitable 2-dimensional manifolds. The point, bearing on the complexity issue, is that the number of such elementary loops scales exponentially as a function of the qubit number.

A possible way out is given by considering a system that is multipartite from the outset and a special form of the Hamiltonian family $H(\lambda)$. The latter is given by a sum, over all the possible pairs $(i,j)$ of subsystems, of Hamiltonian families $\{H(\mu_{ij})\}$. Suppose that the dependence on the local control parameters $\mu_{ij}$ is such that one can holonomically generate any transformation on a two-qubit subspace $\mathcal{C}_{ij}\subset\mathcal{H}_i\otimes\mathcal{H}_j$ e.g., a $U(8)/U(4)\times U(4)$-model, then one can *efficiently* generate any unitary over the computational subspace $\otimes_{(i,j)}\mathcal{C}_{ij}$ by using holonomies only [12].

*An example.* Let $\mathcal{H}:=\mathrm{span}\{|n\rangle\}_{n\in\mathbf{N}}$ be the Fock space of a single bosonic mode, $H_0=\hbar\,\omega\,n(n-1)$ $(n:=a^\dagger a,[a,a^\dagger]=1)$. Hamiltonians of this kind can arise in quantum optics when one considers higher order non-linearities. By construction the space $\mathcal{C}=\mathrm{span}\{|0\rangle,|1\rangle\}$ is a two-fold degenerate eigenspace of $H_0$ i.e., $H_0\,\mathcal{C}=0$. Consider the two-parameter isospectral family of Hamiltonians $H_{\lambda\mu}:=U_{\lambda\mu}\,H_0\,U_{\lambda\mu}^\dagger,(\lambda,\mu\in\mathbf{C})$ where

$$U_{\lambda,\mu}:=\exp(\lambda\,a^\dagger-\overline{\lambda}\,a)\exp\left(\mu\,a^{2\dagger}-\overline{\mu}\,a^2\right). \tag{10}$$

The first (second) factor in this equation is nothing but the unitary transformation from the Fock vacuum

$|0\rangle$ to the familiar coherent (squezeed) state basis. If $\Pi$ denotes the projector over the degenerate eigenspace of $H_f$, one gets $A = \Pi\, U_{\lambda\mu}^{-1}\, dU_{\lambda\mu}\, \Pi = A_\lambda d\lambda + A_\mu d\mu -$ h.c., where (at $\lambda = \mu = 0$) $A_\lambda := -\Pi\, a^\dagger\, \Pi$, $A_\mu := -\Pi\, a^{2\dagger}\, \Pi$. From this relations the explicit matrix form of $A$ can be immediately computed and irreducibility for the single-qubit space $\mathscr{C}$ verified.

This example, at the formal level, can be easily generalized. (i) Choose an Hamiltonian $H$ belonging to a representation $\rho$ of some dynamical (Lie) algebra $\mathscr{A}$, (ii) Build a $k$-fold degenerate $H_f := f(H)$, (iii) Consider the orbit of $\mathscr{O}(H_f) = f(\mathscr{O}(H))$ under the inner automorphisms of $\mathscr{A}$. In point (ii) $f$ is a smooth real-valued map such that $f(\varepsilon_i) = E, (i = 1,\dots,k)$, with the $\varepsilon_i$'s belong to some subset of the spectrum of $H$. In the present case one has $\mathscr{A} := \{a, a^\dagger, a^2, a^{\dagger\,2}, n := a^\dagger\, a, \mathbb{1}\}$, $f(z) = z\,(z-1)$, and $\rho$ is the bosonic Fock representation.

*Conclusions.* In this Letter we have shown how the notion of non-abelian holonomy (generalized Berry phase) might in principle provide a a novel way for implementing universal quantum computation. The quantum space (the code) for encoding information is realized by a degenerate eigenspace of an Hamiltonian belonging of a smooth iso-degenerate family parametrized by points of a control manifold $\mathscr{M}$. The computational bundle of eigenspaces over $\mathscr{M}$ is endowed by a non-trivial holonomy associated with a generalized Berry connection $A$. Loops in $\mathscr{M}$ induce unitary transformations over the code attached to a distinguished point $\lambda_0 \in \mathscr{M}$. We have shown that, in the generic i.e., irreducible, case universal quantum computation can then be realized by composing in all possible ways a pair of adiabatic loops.

The required capability of generating loops by changing coupling constants, along with the necessity of large degenerate eigenspace, makes evident that from the experimental point of view the scheme we are analysing is exceptionally demanding like any other proposal for quantum computing. However we think that the connection between a differential-geometric concept like that of non-abelian holonomy and the general problematic of quantum information processing is non-trivial and quite intriguing. The individuation of promising physical systems for implementing the 'gauge-theoretic' quantum computer we have been discussing in this Letter is still an open problem that will require a deal of further investigations.

## Acknowledgements

The authors thank H. Barnum, G. Segre and L. Faoro for discussions, A. Uhlmann for useful correspondence. P.Z. is supported by Elsag, a Finmeccanica Company.

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