Uppsala University

Degree Project D in Computational Science, 30c

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Holonomic optimal control for qudits

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October 20, 2021

Abstract

1 Engelskt abstrakt

Sammanfattning

Svenskt abstrakt

Contents

1 Introduction

Upplägg?

- Introduction
- Background
  - QM introduction, en hastig och lustig introduction till det viktigaste grunderna (Typ vad en ket är osv), kanske hela vägen upp till Adiabatic theorem?
  - QC introduction, några kärn/grund-idéer inom QC samt lite relevant saker för just det jag gör
- 'huvuddel'(Method?) - Funderar på att kombinera den och ha resultaten här också
  - Qutrit exempel med explcita beräkningar
  - Qudit generalizering
  - Något om simulering?
- Discussion & Conclusions

The emerging field of quantum technologies have many promising applications, one of them is quantum computation (QC), which currently is a very active area of research. Quantum computers makes use of some of the quantum mechanical concepts such as superposition, entanglement and interference to design algorithm. These algorithms could be used to solve some hard problems which would not be classically possible, such as prime-number factoring[1] and more[2]. The current quantum computers are very susceptible to decoherence and noise, and thus will not have any commercial use in the near future, but stand as an important proof of concept. The most common model for quantum computation is the circuit model, which are analogous to the classical circuits used for classical computers. Gates turns into unitary transformations and bits into qubits. To achieve the computational advantage its important to construct robust, noise-resilient quantum gates. A good candidate for this is holonomic quantum computation[3][4] and are based on the Berry phase[5] and its non-abelian and/or non-adiabatic generalizations[6][7][8]. These method are only dependent on the geometry of the system and thus resilient to local errors in the dynamical evolution.
The idea that our elements of computation should be limited to two dimensional (qu)bits is sort of an arbitrary choice, it most likely rose out of convenience due to binary logic. So why binary logic? It is simply the easiest non-trivial example, in binary things can be either 1 or 0 `True` or `False`, **on** or **off**. Due to its simplicity its no wonder why this is how the first computer were designed. But are we limited to (qu)bits? As early as 1840 a mechanical trenary calculation machine was built by Thomas Fowler, and in 1958 the first electronic trenary computer was built by the Soviet union **CITATIONS** . Even though it had many advantages over the binary computer it did never see the widespread success. So there is nothing in theory that forbid a higher dimensional computational basis, even more so when it comes to quantum computers where the implementation of the elements of computation already surpass the simplicity of **on** and **off**.

There is already promising results from qudits with $\dim \leq 2$ that show potential [9][10][11], and in the review article [12] a good overview of the field is given and further research into the topic is encouraged.

The idea in this project is to find a new scheme to implement qudits which could be more efficient than some current ones, a specific example of how this would work for the qutrit is shown in Section **section here** and then it is shown how the scheme would generalize.
The report is structured as follows, the background/theory section is split into two parts wherein the first part is aimed at those not well versed in the field of quantum mechanics, it serves as a quick introduction to the most important aspects and as well as the commonly used notation, experienced readers can skip this part. Then follows a part more concerned with quantum computation, quantum information and some of the more advanced quantum mechanical concept that those are built upon. Next up is the "main" section, it start of with a explicit example, the qutrit, and then follows the generalization of this example.

The report ends in a discussion and conclusion.

2 Theoretical background

The Background consists of a quick introduction to the most important quantum mechanical concepts and notation. Readers familiar with quantum mechanics may skip it. The second part explores the fundamentals of quantum computation and

2.1 Basic quantum mechanics, part I

This part offers a quick introduction to the necessary quantum mechanics for readers whom are not familiar with the subject. The section contains nothing relevant for later parts of the thesis and can be safely skipped. For a more complete introduction I suggest chapter 1 of Sakurai.

2.1.1 Kets, Bras and Operators

When talking about Quantum mechanics the term **quantum state**, or more likley just state, is mentioned a lot. A quantum mechanical state is represented by a **ket**, an complex-valued vector with either finite or infinite entries. Closely related to the ket is the **bra**, which is the corresponding vector to the ket in the dual space, or more simply, the hermitian conjugate of the ket, see Equation 1.

$$|\psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \ \langle\psi| = (|\psi\rangle)^\dagger = (a_1^*, a_2^*, \dots, a_n^*), \ a_1, a_2, \dots, a_n \in \mathbb{C} \tag{1}$$

An arbitrary ket can be rewritten as a linear combination of its eigenkets that span the same space, $|\psi\rangle = \sum_i a_i |i\rangle$ The coefficient is know as the probability amplitude, and $|a_i|^2$ corresponds to the probability to find the state in the ith state. The inner product of two quantum states is simply written as $\langle\varphi|\psi\rangle = (\langle\varphi|\psi\rangle)^\dagger \in \mathbb{C}$. An operator can act on a quantum state and corresponds to multiplication with a matrix, and will always yield a new state, ket.

$$\hat{O}\,|\psi\rangle = |\psi'\rangle \tag{2}$$

Operators are often written in terms of of kets and bras, for example and operator which takes the state $|1\rangle$ and returns the state $|2\rangle$ is written as

$$(|2\rangle\,\langle 1|)\,|1\rangle = |2\rangle\,\langle 1|1\rangle = |2\rangle\,(\langle 1|1\rangle) = (\langle 1|1\rangle)\,|2\rangle = 1\,|2\rangle = |2\rangle \tag{3}$$

or a identity 2×2 matrix would be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle \langle 1| + |2\rangle \langle 2| \tag{4}$$

and so on. An operator for a physical measurable quantity is called an observable.

2.1.2 Measurements and Observables and Uncertainty

A quantum mechanical measurement "breaks" the superposition of a quantum states and shifts it into a eigenstate of the observable,

$$|\psi\rangle \longrightarrow |i\rangle \tag{5}$$

the probability to find the ith eigenstate is $|a_i|^2$. In quantum mechanics the relation $AB - BA = 0$,where A and B are operators, does not generally hold. This is due to the uncommutative nature of quantum mechanics, it relates to uncertainty but it also required since it a can realize a more complex mathematical structure. The fact that matrices are non-commutative are not surprising, and since operators can be represented by matrices this should not be that confusing. To make it more concrete to which degree two operators commute one can define the commutator on operators A, B as $[A, B] = AB - BA$, which is zero for commuting operators and non-zero otherwise. Observables which don't commute are called incompatible observables, a well know pair of incompatible observables are position and momentum, $\mathbf{x}$ and $\mathbf{p}$, which can not be measured to arbitrary precision, this is due to the fact that $[\mathbf{x}, \mathbf{p}] \neq 0$. So for two incompatible observables the general uncertainty the measurements will be limited by **uncertainty relation here.**

2.1.3 Time evolution and the Schrödinger equation

2.2 Quantum Computation and Quantum Information theory, part 2

2.2.1 Qubits

2.2.2 Information content

2.2.3 Universal computation

2.2.4 Holonomic Quantum Computation

3 Method

The 'Method' will be split into two sections first on of an explicit example for $n = 3$, the qutrit. The second section will show how the example generalizes to higher dimensions.

3.1 The Qutrit

A system given by the Hamiltonian

$$H = \sum_{j=1}^{2} \sum_{i=j}^{3} \omega_{ij} |i\rangle \langle e_j| + \frac{\Omega(t)_a}{2} |a\rangle \langle e_2| + \text{h.c} < \tag{6}$$

which is shown in Figure 3.1. It consists of two excited states, $|e_1\rangle, |e_2\rangle$, an auxiliary state $|a\rangle$ and the ground states $|1\rangle, |2\rangle, |3\rangle$ which makes up the computational basis of the qutrit. By a transformation of basis the Hamiltonian can be rewritten as

$$H_d = \sum_{j=1}^{2} \frac{\Omega_j(t)}{2} e^{-i\phi_j} |b_j\rangle \langle e_j| + \frac{\Omega_a(t)}{2} |a\rangle \langle e_2| + \text{h.c} \tag{7}$$

by a Morris-Shore transformation[13]. Now one can find a dark state to this Hamiltonian, an eigenstate with eigenvalue 0, $H|d\rangle = 0$. In this basis for 3 fixed angles, $\theta, \varphi,$, there exists a normalized dark state $|d\rangle = \cos\theta|1\rangle + e^{i\chi}\sin\theta\cos\varphi|2\rangle + e^{i\xi}\sin\theta\sin\varphi|3\rangle$, additionally one can determine two states $|b_1\rangle = N_1\left(-e^{i\xi}\sin\theta\sin\varphi|1\rangle + \cos\theta|2\rangle\right)$ and $|b_2\rangle = N_2(\cos\theta|1\rangle + e^{i\chi}\sin\theta\cos\varphi|2\rangle + \Lambda|3\rangle)$, where N_1, N_2 are normalization factors and Λ can be chosen such that $\langle d|b_2\rangle = 0$.

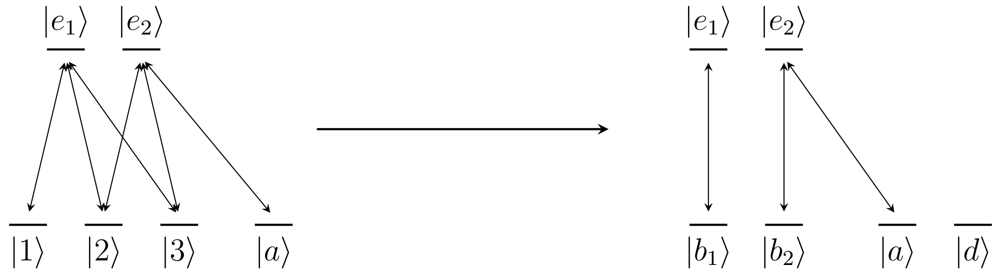


Figure 1: The system given by the Hamiltonian shown in Equation 6 (left) and the transformed system from Equation 7 (right)

These states are bright states which will make up a new orthonormal basis. Explicitly the states are

$$\begin{aligned}
|d\rangle &= \cos\theta|1\rangle + e^{i\chi}\sin\theta\cos\varphi|2\rangle + e^{i\xi}\sin\theta\sin\varphi|3\rangle \\
|b_1\rangle &= \frac{1}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\left(-e^{i\xi}\sin\theta\sin\varphi|1\rangle + \cos\theta|2\rangle\right) \\
|b_2\rangle &= \frac{\sin\theta\sin\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}\left(\cos\theta|1\rangle + e^{i\chi}\sin\theta\cos\varphi|2\rangle + \frac{\sin^2\theta\sin^2\varphi - 1}{\sin\theta\sin\varphi}|3\rangle\right)
\end{aligned} \tag{8}$$

The parameters ω_{ij} in the original basis can be determined by replacing the states in H_d by their form in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

Now lets introduce the concept of a dark path, $\langle D(t)| H_d |D(t)\rangle = 0$, along this path the average energy is always zero. Thus no dynamical phase is accumulated during the time evolution.

The following two states satisfy the dark path condition and can be parametrized by two angles $u(t), v(t)$.

$$\begin{aligned}
|D_1(t)\rangle &= \cos u e^{-i\phi_1}|b_1\rangle + i\sin u|e_1\rangle \\
|D_2(t)\rangle &= \cos u\cos v e^{-i\phi_2}|b_1\rangle - i\sin u|e_2\rangle - \cos u\sin v|a\rangle
\end{aligned} \tag{9}$$

it can easily be verified that $\langle D_i(t)| H_d |D_j(t)\rangle = 0, i, j = 1, 2$. The angles can be chosen with the constraint that the boundary condition $|D_i(0)\rangle\langle D_i(0)| = |D_i(T)\rangle\langle D_i(T)|, i = 1, 2$. This can be achieved by choosing $u(0) = u(T) = v(0) = v(T) = 0$. A valid choice is

$u(t) = \frac{\pi}{2}\sin^2\frac{\pi t}{T}$ and $v(t) = \eta\left[1 - \cos u(t)\right]$, as for the 2D case. Unless mentioned otherwise $\eta = 4.0$. Each dark path starts in the respective bright state and travels along a curve and then returns to the bright state. The choice

Using the Schrödinger equation one can relate the dark path to the Hamiltonian,

$$i\frac{\partial}{\partial t}\left|D_i(t)\right\rangle = H_d\left|D_i(t)\right\rangle, \tag{10}$$

and thusly one can reverse engineer the time dependent parameters $\Omega_i(t)$ by matching the factors of states. A calculation yields

$$\begin{aligned}\Omega_1(t) &= -2\dot{u}\\ \Omega_2(t) &= 2\left(\dot{v}\cot u\sin v + \dot{u}\cos v\right)\\ \Omega_a(t) &= 2\left(\dot{v}\cot u\sin v - \dot{u}\sin v\right).\end{aligned} \tag{11}$$

To construct a universal quantum gate, first the use the method of multi-pulse single-loops[14], the relevant part of the time evolution operator is

$$\begin{aligned}U_1 &= |d\rangle\langle d| - i|e_1\rangle\langle b_1| - i|e_2\rangle\langle b_2|,\ \phi_1 = \phi_2 = 0\\ U_2 &= |d\rangle\langle d| + ie^{i\gamma_1}|b_1\rangle\langle e_1| + ie^{i\gamma_2}|b_2\rangle\langle e_2|,\ \phi_1 = -\gamma_1,\ \phi_2 = -\gamma_2\end{aligned} \tag{12}$$

so the operator for one full loop is

$$U = U_2U_1 = |d\rangle\langle d| + e^{i\gamma_1}|b_1\rangle\langle b_1| + e^{i\gamma_2}|b_2\rangle\langle b_2|. \tag{13}$$

This transformation can be parametrized by 6 real parameters, $U(\chi, \xi, \theta, \varphi, \gamma_1, \gamma_2)$, however it is not enough to achieve universality, this is since one loop does not cover all degrees of freedom. The reason for this is elaborated on in a later section **ref sec** . So the full gate is given by repeating U with another set of parameters. So the full gate $\mathbb{U}$ is given by

$$\mathbb{U} = U(\chi', \xi', \theta', \varphi', \gamma_1', \gamma_2')U(\chi, \xi, \theta, \varphi, \gamma_1, \gamma_2) \tag{14}$$

A selection of qutrit gates from [12] can be obtained by the following parameters.

$$\begin{aligned}X_3 &= U(0, 0, \frac{\pi}{2}, \frac{\pi}{4}, \pi, 0)U(0, 0, \frac{\pi}{4}, \frac{\pi}{2}, 0, \pi) = \begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\\ Z_3 &= U(0, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{4\pi}{3}, \frac{4\pi}{3})U(\frac{2\pi}{3}, \frac{2\pi}{3}, \pi, \pi, \frac{2\pi}{3}, \frac{2\pi}{3}) = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{\frac{2\pi i}{3}} & 0\\ 0 & 0 & e^{\frac{4\pi i}{3}}\end{pmatrix}\\ T_3 &= U()U() = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{\frac{2\pi i}{9}} & 0\\ 0 & 0 & e^{\frac{-2\pi i}{9}}\end{pmatrix}\\ H_3 &= U()() = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}}\\ 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}}\end{pmatrix}\end{aligned} \tag{15}$$

which are some of the important gates to acheive universality[12] as discussed in **ref background?** . Note that the choice of parameters are not unique and there are multiple ways to create the same unitary.

The implementation is quite straight forward, for given set of parameters, say that we want to see how the initial state $|\psi(0)\rangle$ evolves with time. Let $|\psi(0)\rangle = (a, b, c)^T$ in the computational basis $\{1, 2, 3\}$. Then the following equation can be formulated, for some coefficients f_i,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = f_0 |d\rangle + f_1 |b_1\rangle + f_2 |b_2\rangle \tag{16}$$

since at time $t = 0$ the dark paths are in the corresponding bright state. Now expand the dark state and bright state in terms of the original basis. Doing this one can obtain the relation

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} c_1 & -N_1 c_2 & N_2 c_1 \\ c_2 & N_1 c_1 & N_2 c_2 \\ c3 & 0 & N_2 \Lambda \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{17}$$

So by choosing an initial state, the coefficients can be obtained. Then the system can be solved for any time $t \in T$ using Equation

$$|\psi(t)\rangle = f_0 |d\rangle + f_1 |D_1(t)\rangle + f_2 |D_2(t)\rangle, t \in [0, T] \tag{18}$$

Some plots here?

3.2 Generalization

$$H = \sum_{j=1}^{m} \sum_{i=j}^{n} \omega_{ij} |i\rangle \langle e_j| + \frac{\Omega_a(t)}{2} |a\rangle \langle e_m| + \text{ h.c} \tag{19}$$

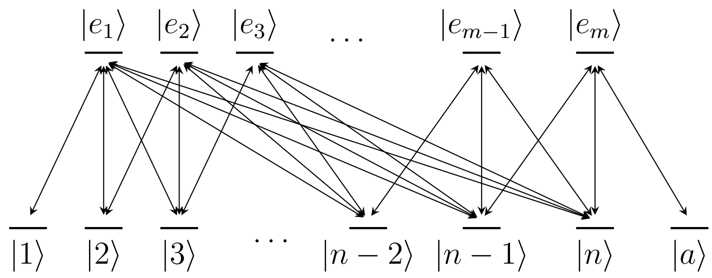


Figure 2: big mess **is this the correct system?**

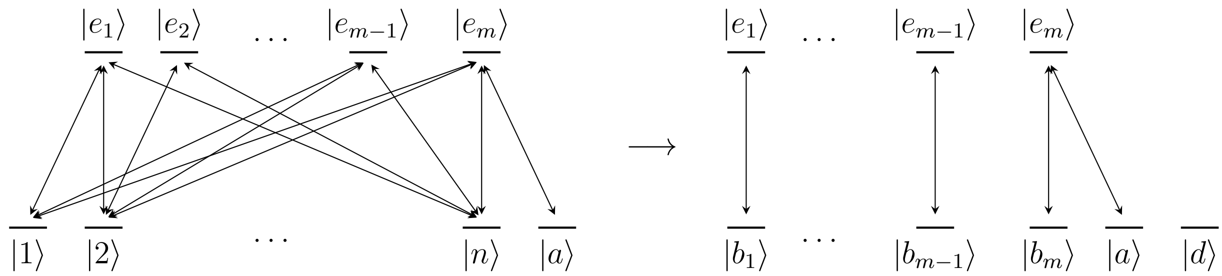


Figure 3: **still wrong mess**

The system is given by m "excited" states $|e_i\rangle$, $i = 0, 1, \dots, m$ and n "ground" states labeled $|i\rangle$, $i = 1, 2, \dots, n$ and one auxiliary state $|a\rangle$. The number of states always follows that $n - m = 1$.(if auxiliary state is counted as a "ground state" it is = 2) So for $m = 5$ excited states there would be $n = 6$ ground states. The transitions (except for auxiliary) occur only between excited states and ground states. the first excited state e_1 is connected to all ground states, e_2 is connected to all but $|1\rangle$, in general the excited state $|e_i\rangle$ is connected to the $(n - i)$th ground states with the highest label. Unless when $i = m$, the excited state $|e_m\rangle$ which is connected to the two highest labeled ground states and the auxiliary state $|a\rangle$.

This is the standard basis $\{e_1, e_2, \dots, e_m, 1, 2, \dots, n, a\}$, it is possible define a new basis, the dark state basis given by $\{e_1, e_2, \dots, e_m, b_1, b_2, \dots, b_m, a, d\}$.

Given a dark state on the form

$$|d\rangle = c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle \cdots + c_n |n\rangle \, , \; |c_1|^2 + |c_2|^2 + \cdots + |c_n|^2 = 1 \tag{20}$$

from this dark state one could recursively define $n - 1 = m$ bright states. Starting from

$$|b_1\rangle = N_1 \left(-c_2 |1\rangle + c_1 |2\rangle\right) \tag{21}$$

with N_1 being a normalization factor. Then choose additional bright states on the form

$$
\begin{aligned}
|b_2\rangle \quad &= N_2 \left(c_1 |1\rangle + c_2 |2\rangle + \Lambda_1^{(2)} |2\rangle \right) \\
|b_3\rangle \quad &= N_3 \left(c_1 |1\rangle + c_2 |2\rangle + \Lambda_1^{(3)} |2\rangle + \Lambda_2^{(3)} |3\rangle \right) \\
&\vdots \\
|b_{m-1}\rangle \quad &= N_{m-1} \left(c_1 |1\rangle + c_2 |2\rangle + \Lambda_1^{(m-1)} |2\rangle + \Lambda_2^{(m-1)} |3\rangle + \cdots + \Lambda_{m-2}^{(m-1)} |m-1\rangle \right) \\
|b_m\rangle \quad &= N_m \left(c_1 |1\rangle + c_2 |2\rangle + \Lambda_1^{(m)} |2\rangle + \Lambda_2^{(m)} |3\rangle + \cdots + \Lambda_{m-2}^{(m)} |m-1\rangle + \Lambda_{m-1}^{(m)} |m\rangle \right).
\end{aligned} \tag{22}
$$

By this construction it is clear that $|b_1\rangle$ is orthogonal to all other bright states.

The coefficients can be chosen in such a way that, in $|b_2\rangle$, the coefficient $\Lambda_1^{(2)}$ can be chosen such that, $\langle d|b_2\rangle = 0$, and in $|b_3\rangle$, the coefficient $\Lambda_1^{(3)}$ can be chosen such that, $\langle b_2|b_3\rangle = 0$ and $\Lambda_2^{(3)}$ such that $\langle d|b_3\rangle = 0$. By recursively repeating this argument one could see that it is possible to chose m bright states, then by normalizing all the N_i can be found, thus we have obtained m orthonormal bright states, $\langle b_i|b_j\rangle = \delta_{ij}$. The coefficients c_i can be parametrized by the euclidean components of the unit-n-sphere and a phase factor.

$$
\begin{aligned}
c_1 &= \cos(\varphi_1) \\
c_2 &= e^{i\theta_1} \sin(\varphi_1)\cos(\varphi_2) \\
c_3 &= e^{i\theta_2} \sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\
&\vdots \\
c_{n-1} &= e^{i\theta_{n-1}} \sin(\varphi_1)\ldots\sin(\varphi_{n-2})\cos(\varphi_{n-1}) \\
c_n &= e^{i\theta_n} \sin(\varphi_1)\ldots\sin(\varphi_{n-2})\sin(\varphi_{n-1})
\end{aligned} \tag{23}
$$

c_1 does not need a phase factor since the overall phase of a state is non-measurable and can be chosen such that the first phase factor can be canceled. The remaining Λ coefficients can be expressed in terms of the c_i.

In this newly defined space the Hamiltonian can be written as

$$
H_d = \sum_{i=1}^{m} \frac{\Omega_i(t)}{2} e^{-i\phi_i} |b_i\rangle\langle e_i| + \frac{\Omega_a(t)}{2} |a\rangle\langle e_n| + \text{ h.c} \tag{24}
$$

with Ω_i being real-valued time dependent parameters and the ϕ_i time independent phase factors.

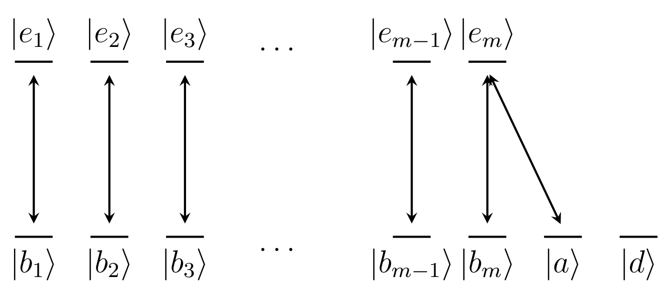


Figure 4: less mess

With this Hamiltonian m dark paths can be constructed, and must fullfill $\langle D_i(t)| H_d |D_i(t)\rangle = 0, i = 1, 2, \ldots, m$ and $\langle D_i(t)|D_j(t)\rangle = \delta_{ij}$.
The dark paths can be parametrized by two functions $u(t), v(t)$ that satisfy the conditions $u(0) = v(0) = u(T) = v(T) = 0$, will have the form

$$\begin{aligned} |D_i(t)\rangle &= \cos u e^{-i\phi_i} |b_i\rangle + i \sin u |e_1\rangle, \; i = 1, 2, \ldots, m-1 \\ |D_m(t)\rangle &= \cos u \cos v e^{-i\phi_n} |b_m\rangle - i \sin u |e_m\rangle - \cos u \sin v |a\rangle \end{aligned} \tag{25}$$

The dark paths start in the bright state and travels along a curve where the expectation value of the energy is constantly 0 and can thus be used non-adiabatically.

By using these states one can reverse engineer the Hamiltonian using the Schrödinger equation to determine Ω_i and Ω_a since

$$i \frac{\partial}{\partial t} |D_i(t)\rangle = H_d |D_i(t)\rangle, \; i = 1, 2, \ldots, m \tag{26}$$

a calculation **actually do the calculation** yields

$$\begin{aligned} \Omega_1(t) &= -2\dot{u} \\ \Omega_2(t) &= -2\dot{u} \\ &\vdots \\ \Omega_{m-1}(t) &= -2\dot{u} \\ \Omega_m(t) &= 2\left(\dot{v} \cot u \sin v + \dot{u} \cos v\right) \\ \Omega_a(t) &= 2\left(\dot{v} \cot u \sin v - \dot{u} \sin v\right) \end{aligned} \tag{27}$$

The time evolution is split into k loops, each loop with two pulses, $0 \longrightarrow T/2$ and $T/2 \longrightarrow T$. The relevant part of the time evolution operator for one loop is $U_1(T/2, 0) = |d\rangle \langle d| - i \sum_{i=1}^{m} |e_i\rangle \langle b_i|, \; \phi_i = 0, i = 1, 2, \ldots, n$ and $U_2(T, T/2) = |d\rangle \langle d| + i \sum_{i=1}^{m} e^{i\gamma_i} |b_i\rangle \langle e_i|, \; \phi_i = 0, i = 1, 2, \ldots, n$ The full operator for one loop is then given by

$$U = U_2 U_1 = |d\rangle \langle d| + \sum_{i=1}^{m} e^{i\gamma_i} |b_i\rangle \langle b_i|. \tag{28}$$

It is clear that U is unitary in the subspace $\{d, b_1, b_2, \ldots, b_n\}$

The unitary is parametrized by $3m = 3(n-1), n \geq 2$, parameters,

$$U(\varphi_1, \ldots, \varphi_m, \theta_1, \ldots, \theta_m, \gamma_1, \ldots, \gamma_m) \tag{29}$$

The following part might not be 100% correct, procced with caution!

Now to perform a quantum gate one needs to apply U k-times, $U_{tot} = U^k$, each time with a different set of parameters. This is due to the fact that just one loop does not have enough degrees of freedom to cover the dimensions of an $n-$dimensional qudit.

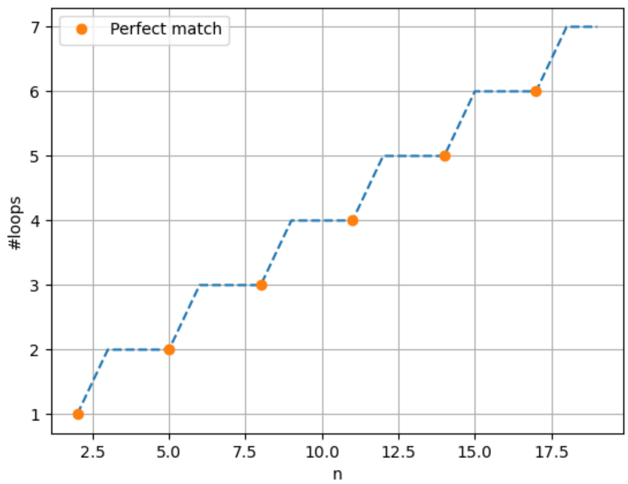
The dimensions of a $n-$dim qudit is given by $\dim(SU(n)) = n^2 - 1$ and each loop only carries $3(n-1)$ degrees of freedom. So the loop needs to be repeated k times such that $3(n-1)k \geq n^2 - 1$. Lets call the dimension n when $\frac{n^2-1}{3(n-1)}$ is an integer, a "perfect" dimension, since no degrees of freedom goes "to waste" so to speak. In Table 1 it seems such that n is perfect when $n = 3p + 2$, for any integer p. A quick calculation shows

$$\frac{n^2 - 1}{3(n-1)} = \frac{(3p+2)^2 - 1}{3((3p+2) - 1)} = \frac{3(3p^2 + 4p + 1)}{3(3p+1)} = \frac{(3p+1)(p+1)}{(3p+1)} = p + 1 \tag{30}$$

and since p is an integer, $p+1$ is also always an integer.
OBS! The T gate is only defined for prime number dimensions... [12]

n	$3(n-1)$	n^2-1	k	is perfect?
2	3	3	1	yes
3	6	8	2	
4	9	15	2	
5	12	24	2	yes
6	15	35	3	
7	18	48	3	
8	21	63	3	yes
9	24	80	4	
10	27	99	4	
11	30	120	4	yes
12	33	143	5	

Table 1: Table for some dimensions.



This seems to hint that for when n is perfect the amount of information has greater efficiency since in higher dimensions more information can be contained but can be executed in the same number of loops as a non-perfect dim.
The quantum gate U can be as a linear combination of the dark state and dark paths

$$|\psi(t)\rangle = f_0 |d\rangle + \sum_{i=1}^{m} f_i |D_i(t)\rangle, t \in [0,T] \tag{31}$$

where the coefficient can be solved for by choosing an initial state $|\psi(0)\rangle$. This corresponds to one loop, by using $|\psi(T)\rangle$ as an initial state for the next loop. By iterating this method one can simulate $\mathbb{U} = U^k$ where each loop U have a different set of parameters.

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