Kuratowski Set Convergence and a Convexity Lemma

1 Setting and notation

Let X be a Banach space (more generally, a metric space suffices for the definitions below). Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of *nonempty* subsets of X.

A *subsequence of indices* is a strictly increasing map $k \mapsto n_k \in \mathbb{N}$, i.e.

$$n_{k+1} > n_k \quad \text{for all } k \in \mathbb{N}.$$

The corresponding subsequence of sets is $(C_{n_k})_{k\in\mathbb{N}}$.

2 Kuratowski upper and lower limits

Definition 1 (Kuratowski upper and lower limits; Kuratowski convergence). *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of X.*

(i) Kuratowski upper limit (outer limit).

$$\limsup_{n\to\infty} C_n := \Big\{ x \in X : \exists\, (n_k)_{k\in\mathbb{N}} \text{ strictly increasing}, \exists\, (x_k)_{k\in\mathbb{N}} \subset X \text{ s.t. } x_k \in C_{n_k} \,\forall k, \text{ and } x_k \to x \Big\}.$$

(ii) Kuratowski lower limit (inner limit).

$$\liminf_{n\to\infty} C_n := \Big\{ x \in X : \exists\, (x_n)_{n\in\mathbb{N}} \subset X \text{ s.t. } x_n \in C_n \,\forall n, \text{ and } x_n \to x \Big\}.$$

(iii) Kuratowski convergence. *We say that (C_n) converges to $C \subset X$ in the sense of Kuratowski if*

$$\liminf_{n\to\infty} C_n \;=\; \limsup_{n\to\infty} C_n \;=\; C.$$

3 Convexity of the Kuratowski upper limit for nested decreasing convex sets

Lemma 1 (Convexity of the Kuratowski upper limit). *Assume that $(C_n)_{n\in\mathbb{N}}$ is nested decreasing, i.e.*

$$C_{n+1} \subseteq C_n \quad \forall n \in \mathbb{N},$$

and that each C_n is convex. Define

$$C := \limsup_{n\to\infty} C_n.$$

Then C is convex.

Proof. Let $c, s \in C$ and let $\lambda \in [0,1]$. We must prove that

$$\lambda c + (1-\lambda)s \in C.$$

Step 1: represent c and s using the definition of $\limsup$

Since $c \in \limsup_{n\to\infty} C_n$, there exist a strictly increasing index sequence $(n_k)_{k\in\mathbb{N}}$ and a sequence $(c_k)_{k\in\mathbb{N}} \subset X$ such that

$$c_k \in C_{n_k} \ \forall k, \qquad c_k \to c.$$

Similarly, since $s \in \limsup_{n\to\infty} C_n$, there exist a strictly increasing index sequence $(\bar{n}_k)_{k\in\mathbb{N}}$ and a sequence $(s_k)_{k\in\mathbb{N}} \subset X$ such that

$$s_k \in C_{\bar{n}_k} \ \forall k, \qquad s_k \to s.$$

Step 2: synchronize indices via $\hat{n}_k := \min(n_k, \bar{n}_k)$

Define

$$\hat{n}_k := \min(n_k, \bar{n}_k) \qquad (k \in \mathbb{N}).$$

Claim 1. *The sequence $(\hat{n}_k)_{k\in\mathbb{N}}$ is strictly increasing.*

Proof. Fix $k \in \mathbb{N}$. There are two cases.

Case 1: $n_k \le \bar{n}_k$. Then $\hat{n}_k = n_k$. Since $n_{k+1} > n_k = \hat{n}_k$ and $\bar{n}_{k+1} > \bar{n}_k \ge n_k = \hat{n}_k$, both n_{k+1} and $\bar{n}_{k+1}$ are strictly larger than $\hat{n}_k$. Hence

$$\hat{n}_{k+1} = \min(n_{k+1}, \bar{n}_{k+1}) > \hat{n}_k.$$

Case 2: $\bar{n}_k \le n_k$. Then $\hat{n}_k = \bar{n}_k$. Since $\bar{n}_{k+1} > \bar{n}_k = \hat{n}_k$ and $n_{k+1} > n_k \ge \bar{n}_k = \hat{n}_k$, again both n_{k+1} and $\bar{n}_{k+1}$ are strictly larger than $\hat{n}_k$, so

$$\hat{n}_{k+1} = \min(n_{k+1}, \bar{n}_{k+1}) > \hat{n}_k.$$

In either case, $\hat{n}_{k+1} > \hat{n}_k$. Therefore $(\hat{n}_k)$ is strictly increasing. $\qquad\square$

Claim 2. *For every $k \in \mathbb{N}$, one has $c_k \in C_{\hat{n}_k}$ and $s_k \in C_{\hat{n}_k}$.*

Proof. By definition, $\hat{n}_k \le n_k$ and $\hat{n}_k \le \bar{n}_k$. Since the family is nested decreasing, for indices $m \ge n$ one has $C_m \subseteq C_n$. Applying this with $(m, n) = (n_k, \hat{n}_k)$ yields $C_{n_k} \subseteq C_{\hat{n}_k}$, hence $c_k \in C_{n_k} \subseteq C_{\hat{n}_k}$. Similarly, $C_{\bar{n}_k} \subseteq C_{\hat{n}_k}$ and thus $s_k \in C_{\bar{n}_k} \subseteq C_{\hat{n}_k}$. $\qquad\square$

Step 3: apply convexity at the synchronized index level

Define

$$t_k := \lambda c_k + (1 - \lambda) s_k \qquad (k \in \mathbb{N}).$$

By the previous claim, $c_k, s_k \in C_{\hat{n}_k}$ for all k. Since each $C_{\hat{n}_k}$ is convex, it follows that

$$t_k \in C_{\hat{n}_k} \qquad \forall k \in \mathbb{N}.$$

Step 4: pass to the limit and conclude via the definition of $\limsup$

Because addition and scalar multiplication are continuous in a normed space and $c_k \to c$, $s_k \to s$, we have

$$t_k = \lambda c_k + (1 - \lambda) s_k \to \lambda c + (1 - \lambda) s.$$

We have thus exhibited a strictly increasing subsequence of indices $(\hat{n}_k)$ and points $t_k \in C_{\hat{n}_k}$ with $t_k \to \lambda c + (1 - \lambda) s$. By the definition of $\limsup_{n\to\infty} C_n$, this implies

$$\lambda c + (1 - \lambda) s \in \limsup_{n\to\infty} C_n = C.$$

Since $c, s \in C$ and $\lambda \in [0, 1]$ were arbitrary, C is convex. $\qquad\square$

Consistency check (dependencies)

The proof uses only: (i) the definition of $\limsup$ to obtain the representing subsequences and convergent point selections; (ii) nestedness $C_{n+1} \subseteq C_n$ to embed C_{n_k} and $C_{\bar{n}_k}$ into $C_{\hat{n}_k}$; (iii) convexity of each C_n to ensure $t_k \in C_{\hat{n}_k}$; and (iv) continuity of affine maps to pass limits through convex combinations, enabling a direct application of the definition of $\limsup$.