Sustainability of Opinion Coordination in Social Networks

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Abstract

The process by which opinions spread through a large social network can be modeled as an interaction between the initial opinion of an agent and their neighbor's opinions. This process can be affected by stubborn agents, who maintain their initial opinion fixed. We characterize under which conditions a group of regular agents can coordinate their opinions and behave like stubborn agents. We show that if an agent has incentives to coordinate his opinion in the first period, he announces the opinion of the coordination in every period. There exist an interval of opinions that can be sustained by the coordinated agents, which depends on their initial opinion and the connectivity of this subgraph. We also study how this coordination impact the convergence speed and characterize the convergence time .

Andrea: Todos los cambios son bienvenidos y sorry el ingles

Victor: we need a catchy title :)

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1 Introduction

Andrea: Hacer la analogía de este modelo a la literatura de carteles incompletos. No todos pueden escoger "precios",solo los invitados a coordinarse y buscamos condiciones bajo las cuales esta coordinación es maximal

2 Coordination Model

Consider a graph $G = (V, E)$ representing a social network. Every node $v \in V$ in the network has an intrinsic opinion $\gamma_0(v)$. At every time step $t \in \mathbb{N}$ and for every $u \in V$, $\gamma_t(u)$ denotes the opinion *declared* by u. We say that (C, β) is a *coordination* if every node $v \in C$ declares the same opinion β, that is, for every $t \in \mathbb{Z}_+$ and for every $u \in C$, $\gamma_t(u) = \beta$. Every node $u \notin C$ updates the opinion according to

$$\gamma_{t+1}(u) = \frac{\eta_u}{\mathsf{deg}(u) + \eta_u} \cdot \gamma_0(u) + \frac{1}{\mathsf{deg}(u) + \eta_u} \sum_{v \in \mathsf{N}(u)} \gamma_t(v), \tag{1}$$

For every $u \in V$, the parameter η_u captures how resistant is u to modify his initial opinion $\gamma_0(u)$. Observe that if $C = \emptyset$ we then recover the classic Friedkin and Johnsen opinion dynamics model []. In what follows, it will be useful to consider the dynamics in a matrix form. Consider the matrix $\mathsf{A} \in \mathsf{R}^{V \times V}$ defined as follows: Let $\mathsf{A}(u, v) = 1/(\mathsf{deg}(u) + \eta_u)$ if $\{u, v\} \in E$ and $u \in V \setminus C$, and zero otherwise, and let $\mathsf{B} \in \mathsf{R}^{V \times V}$ the diagonal matrix given by $\mathsf{B}(u, u) = \eta_u/(\mathsf{deg}(u) + \eta_u)$ for every $u \in V \setminus C$. Then, the dynamics of the declared opinions (1) can be written as

$$\gamma_{t+1} = \mathsf{A}\gamma_t + \beta \mathsf{I}_C + \mathsf{B}\gamma_0, \tag{2}$$

where $\mathsf{I}_C \in \mathsf{R}^{V \times V}$ is such that $\mathsf{I}_C(u, u) = 1$ if $u \in C$, and zero otherwise. In particular, observe that from $t = 1$ every member of the coordination declares the opinion β, that is, for every $u \in C$ we have $\gamma_1(u) = \beta$. In the following, we call γ the *coordination dynamics* for (C, β). A similar dynamics was considered by Ghaderi and Srikant [], to study networks with *stubborn* agents. We later provide a random walk interpretation of the above dynamics to study the evolution and long-run behavior of the process. For notational convenience, in what follows we call $\gamma_0^\beta \in \mathsf{R}^V$ the vector such that $\gamma_0^\beta(u) = \beta$ if $u \in C$ and $\gamma_0^\beta(u) = \gamma_0(u)$ if $u \in V \setminus C$.

Example 1.

Victor: ejemplito con grafo chico y una o dos iteraciones

3 Existence and Behavior of the Long-run Opinions

In what follows we use a result by Ghaderi & Srikant showing the existence and providing a characterization of the long-run opinions of a network under the presence of stubborn agents. To study the dynamics, the authors construct a random walk in an auxiliary graph, and characterize the long-run opinion by studying the stationary distribution of this auxiliary random walk. Observe that in our model, for a coordination (C, β) we have that every $u \in C$ can be seen a fully stubborn agent for the dynamics from $t = 1$. Nevertheless, the value of β can be chosen after the realization of each of the intrinsic opinions of the members in the coordination. For the sake of completeness, we include the random walk construction here and state the results in our context.

An auxiliary random walk. Given a graph G and $C \subseteq V$, consider the auxiliar graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined as follows. We have $\mathcal{V} = V \cup T$ where $T = \{x_v : v \in V \setminus C\}$, that is, the set of nodes $\mathcal{V}$ consists of every node in the original graph and a copy of each node not in C. Furthermore, in the auxiliary graph every node in $V \setminus C$ is connected to its copy, that is, $\mathcal{E} = E \cup \{\{v, x_v\} : v \in V \setminus C\}$. Consider the random walk $(\mathsf{x}_t)_{t \in \mathbb{Z}_+}$ over the graph $\mathcal{G}$ with $\mathsf{P} \in [0,1]^{\mathcal{V} \times \mathcal{V}}$ given by

$$\mathsf{P}(u,v) = \begin{cases} 1/\mathsf{deg}(u) & \text{for every } u \in C \text{ and every } v \in \mathsf{N}(u), \\ 1/(\mathsf{deg}(u) + \eta_u) & \text{for every } u \in V \setminus C \text{ and every } v \in \mathsf{N}(u), \\ \eta_u/(\mathsf{deg}(u) + \eta_u) & \text{for every } u \in V \setminus C \text{ and } v = x_u, \\ 1 & \text{for every } v \in V \text{ and } u = x_v, \end{cases}$$

For every $v \in \mathcal{V}$, consider $\tau_v = \inf\{t \in \mathbb{Z}_+ : \mathsf{x}_t = v\}$, that is, the hitting time of vertex v, and let $\tau = \inf\{t \in \mathbb{Z}_+ : \mathsf{x}_t \in C \cup T\}$ the hitting time of the set of nodes in $C \cup T$. Let $\alpha_C \in \mathsf{R}^{V \times V}$ such that for every $u \in V$ and every $v \in C$ we have $\alpha_C(u,v) = \mathbb{P}_u(\tau_{x_v} = \tau)$. In particular, α_C is a stochastic matrix. Furthermore, consider the quantity given by

$$\Theta_{u,C} = \sum_{v \in C} \mathbb{P}_u(\tau = \tau_v).$$

The value above corresponds to the probability that the random walk $(\mathsf{x}_t)_{t \in \mathbb{Z}_+}$ starting at $u \in V$ hits $C \cup T$ in a vertex that belongs to C. We now state the technical lemma from [] adapted to our context. We include a proof of it in the Appendix.

Lemma 1 ([?]). *Let $G = (V, E)$ be a connected graph and γ_0 a vector of intrinsic opinions. Then, for every coordination (C, β) there exists $\gamma_\infty \in \mathsf{R}^V$ such that $\lim_{t \to \infty} \gamma_t = \gamma_\infty$. Furthermore, we have that $\gamma_\infty = \alpha_C \gamma_0^\beta$.*

That is, for every $v \in V \setminus C$ we have that the limit opinion is given by

$$\gamma_\infty(v) = \sum_{w \in V \setminus C} \gamma_0(w) \alpha_C(v,w) + \beta \Theta_{v,C}. \tag{3}$$

That is, $\gamma_\infty(v)$ is a convex combination of the intrinsic opinions of the nodes in $V \setminus C$, and the opinion β of the coordination. Observe that for every $v \in V \setminus C$, the first term in the equality above is independent of β. We call this term the *effective external opinion*,

$$\tau_{v,C}(\gamma_0) = \sum_{w \in V \setminus C} \gamma_0(w) \alpha_C(v,w). \tag{4}$$

Conditional Expectations. Given a coordination (C, β), by the definition of the hitting probabilities it follows that for each $v \in V \setminus C$, we have that $1 - \Theta_{v,C} = \sum_{w \in V \setminus C} \alpha_C(v,w)$. Consider the probability distribution $f_{v,C}$ over the nodes in $V \setminus C$ such that for each $w \in V \setminus C$ we have

$$f_{v,C}(w) = \frac{\alpha_C(v,w)}{1 - \Theta_{v,C}}.$$

We denote by $\mathsf{E}_{v,C}$ the expectation operator from probability distributation above. Observe that $f_{v,C}$ corresponds to the probability distribution induced by α_C conditional on the random walk

$(\mathsf{x}_t)_{t\in\mathbb{Z}_+}$ starting at v hitting for the first time $C \cup T$ in a vertex of T. Therefore, given a random variable $\mathsf{a} \sim f_{v,C}$, the effective external opinion corresponds to

$$\tau_{v,C}(\gamma_0) = (1 - \Theta_{v,C})\mathsf{E}_{v,C}(\gamma_0(\mathsf{a}))$$

For each $C \subseteq V$, the limit opinion is a function of the intrinsic opinions γ_0 and β and it will be useful in what follows to consider the function mapping a pair $(\gamma_0, \beta) \in \mathsf{R}^V \times [0,1]$ onto $\Omega_C(\gamma_0, \beta) = \alpha_C \gamma_0^\beta \in \mathsf{R}^V$.

Mixing time. A quantity that plays a role in our analysis is the time it requires for the random walk distribution to be very close from the stationary one. Given two probability distributions μ and ν over $\mathcal{V}$, the total variation distance between μ and ν is given by

$$\|\mu - \nu\|_{\mathsf{TV}} = \frac{1}{2}\sum_{u\in\mathcal{V}} |\mu(u) - \nu(u)|.$$

For every every $u \in \mathcal{V}$, we denote by δ_u the probability distribution such that $\delta_u(u) = 1$ and zero otherwise. Then, the mixing time of the random walk $(\mathsf{x}_t)_{t\in\mathbb{Z}_+}$ corresponds to the value

$$t_{\mathsf{mix}}(\mathcal{G}) = \min\Big\{t \in \mathbb{Z}_+ : \max_{u\in\mathcal{V}} \|\mathsf{P}^t\delta_u - \pi\|_{\mathsf{TV}} \le 1/e\Big\}.$$

That is, the amount of time it requires for the distribution induced by the random walk to be within a distance of at most $1/e$ from the stationary distribution, no matter the initial state. The constant is arbitrary, since one could replace its value by ε at cost of a logarithmic factor, $t_{\mathsf{mix}}(\mathcal{G})\log(1/\varepsilon)$.

4 Sustainability in the Long-run

For every node $u \in V$, consider the cost function given by

$$\mathsf{cost}_u(y) = \sum_{v\in\mathsf{N}(u)} (y(v) - y(u))^2,$$

where $y \in \mathsf{R}^V$. That is, it corresponds to the *opinion dispersion* faced by node u respect to its neighbors. In what follows, we study under what conditions it is possible for a coordination to sustain in the long-run, that is, by considering the cost under the limiting opinion. We say that (C, β, γ_0) is *sustainable in the long-run* if for every $u \in C$ we have [1]

$$\mathsf{cost}_u(\Omega_C(\gamma_0, \beta)) \le \mathsf{cost}_u(\Omega_{C-u}(\gamma_0, \beta)).$$

That is, every node in C faces a lower dispersion cost by being in the coordination than being out. In what follows, we say that a pair (C, γ_0) satisfies the *zero opinion contraction condition* if for every $u \in C$, we have that

$$\mathsf{deg}_C(u) + \sum_{v\in\mathsf{N}(u)\setminus C}\left(\frac{\tau_{v,C-u}(\gamma_0)}{\tau_{u,C-u}(\gamma_0)} - 1\right)^2 \ge \sum_{v\in\mathsf{N}(u)\setminus C}\left(\frac{\tau_{v,C}(\gamma_0)}{\tau_{u,C-u}(\gamma_0)}\right)^2.$$

[1] For notational simplicity we denote by $C - u$ the set $C \setminus \{u\}$.

Similarly, we say that a pair (C, γ_0) satisfies the *one opinion contraction condition* if for every $u \in C$, we have that

$$\mathsf{deg}_C(u) + \sum_{v \in \mathsf{N}(u) \setminus C} \left(\frac{\tau_{v,C-u}(\mathsf{e} - \gamma_0)}{\tau_{u,C-u}(\mathsf{e} - \gamma_0)} - 1 \right)^2 \geq \sum_{v \in \mathsf{N}(u) \setminus C} \left(\frac{\tau_{v,C}(\mathsf{e} - \gamma_0)}{\tau_{u,C-u}(\mathsf{e} - \gamma_0)} \right)^2$$

where e is the all ones vector in R^V. In the sequel, we say that (C, γ_0) satisfies the contraction condition if it satisfies at least one of the above conditions. The following is the main theorem of this section.

Theorem 1. *Let $G = (V, E)$ a connected graph and (C, γ_0) satisfying the contraction condition. Then, there exists a non-empty interval $\mathcal{I}(C, \gamma_0) \subseteq [0, 1]$ with $\mathcal{I}(C, \gamma_0) \cap \{0, 1\} \neq \emptyset$ such that for every $\beta \in \mathcal{I}(C, \gamma_0)$ we have that (C, β, γ_0) is sustainable in the long-run.*

Observe that in particular, an extreme opinion in $\{0, 1\}$ makes the coordination sustainable in the long-run as long as (C, γ_0) satisfies the contraction condition. In what follows we show how to prove the theorem above. One particular feature of the above structural result is that whenever a set satisfies the contraction property, it can be either zero contractive or one contractive, but not both. That is, the coordination will be able to sustain one the extreme opinions but not both.

Theorem 2. *Let $G = (V, E)$ a connected graph and (C, γ_0) satisfying the contraction condition. Then, (C, γ_0) is zero contractive and one contractive if and only if $\gamma_0 = \frac{1}{2}\mathsf{e}$.*

4.1 Extreme Opinion Mindsets: Proof of Theorem 1

We study first the conditions under which a node $u \in C$ faces a lower cost by being in the coordination. For every $u \in C$, consider the function

$$f_u(\beta) = \mathsf{cost}_u(\Omega_C(\gamma_0, \beta)) - \mathsf{cost}_u(\Omega_{C-u}(\gamma_0, \beta))$$

In the following, we say that C is *one minded* if for every $u \in C$ we have that $f_u(1) < 0$. Symmetrically, we say that C is *zero minded* if for every $u \in C$ we have that $f_u(0) < 0$.

Proposition 1. *Let $C \subseteq V$ be a non-empty subset of nodes. Then, the following holds.*

(a) If C is one minded, there exists $\beta^1 \in (0, 1)$ such that for every $f_u(\beta) \leq 0$ for every $\beta \in [\beta^1, 1]$.

(b) If C is zero minded, there exists $\beta^0 \in (0, 1)$ such that for every $f_u(\beta) \leq 0$ for every $\beta \in [0, \beta^0]$.

Proof. If C is one minded, by continuity we have that for every $u \in C$ there exists $\beta_u \in (0, 1)$ such that $f_u(\beta) \leq 0$ for every $\beta \in [\beta_u, 1]$. In particular, given $\beta^1 = \max_{u \in C} \beta_u$, we have that for every $u \in C$ it holds that $f_u(\beta) < 0$ for every $\beta \in [\beta^1, 1]$. The proof follows in the same way when C is zero minded. $\square$

In the following proposition we provide a more explicit expression for the costs evaluated in the longrun opinion vectors. Furthermore, we also show some opinion symmetry property of the cost function: the cost we face under certain coordination value is the same if every opinion spin is changed. Having that, we are ready to prove Theorem 1.

Proposition 2. *Let $G = (V, E)$ be a connected graph, $C \subseteq V$ and $\gamma_0 \in [0, 1]^V$. Then, the following holds.*

(*a*) *When* $\beta = 0$*, for every* $v \in V \setminus C$ *we have* $\gamma_\infty(v) = \tau_{v,C}(\gamma_0)$.

(*b*) *For every* $u \in C$*, we have that* $\mathsf{cost}_u(\Omega_C(\gamma_0, 0)) = \sum_{v \in \mathsf{N}(u)\setminus C} \tau^2_{v,C}(\gamma_0)$.

(*c*) *For every* $u \in C$*, we have that*

$$\mathsf{cost}_u(\Omega_{C-u}(\gamma_0, 0)) = \mathsf{deg}_C(u) \cdot \tau^2_{u,C-u}(\gamma_0) + \sum_{v \in \mathsf{N}(u)\setminus C} \Big(\tau_{v,C-u}(\gamma_0) - \tau_{u,C-u}(\gamma_0)\Big)^2.$$

Proof of Proposition 5. Part (a) comes directly from equalities Lemma 1 and equalities (3) and (4). For part (b), observe that for every $v \in \mathsf{N}(u) \cap C$ we have that $\gamma_\infty(u) = \gamma_\infty(v) = 0$. Therefore,

$$\mathsf{cost}_u(\Omega_C(\gamma_0, 0)) = \sum_{v \in \mathsf{N}(u)\setminus C} \gamma^2_\infty(v) = \sum_{v \in \mathsf{N}(u)\setminus C} \tau^2_{v,C}(\gamma_0).$$

Finally, for part (c), observe that for the coordination $(C - u, 0)$ we have that $\gamma_\infty(v) = \tau_{v,C-u}(\gamma_0)$ for every $v \in V \setminus (C - u)$. Therefore,

$$\begin{aligned}\mathsf{cost}_u(\Omega_{C-u}(\gamma_0, 0)) &= \sum_{v \in \mathsf{N}(u)\cap C} \Big(\tau_{u,C-u}(\gamma_0) - 0\Big)^2 + \sum_{v \in \mathsf{N}(u)\setminus C} \Big(\tau_{v,C-u}(\gamma_0) - \tau_{u,C-u}(\gamma_0)\Big)^2 \\ &= \mathsf{deg}_C(u) \cdot \tau^2_{u,C-u}(\gamma_0) + \sum_{v \in \mathsf{N}(u)\setminus C} \Big(\tau_{v,C-u}(\gamma_0) - \tau_{u,C-u}(\gamma_0)\Big)^2. \qquad \square\end{aligned}$$

Proposition 3. *Let* $G = (V, E)$ *be a connected graph,* $C \subseteq V$ *and* $\gamma_0 \in [0, 1]^V$*. Then, for every* $\beta \in [0, 1]$ *and for every* $u \in C$ *we have that*

$$\mathsf{cost}_u(\Omega_C(\gamma_0, \beta)) = \mathsf{cost}_u(\Omega_C(\mathsf{e} - \gamma_0, 1 - \beta)).$$

Proof. Recall that for every $u \in V$, we have that $\widetilde{\gamma}_\infty(v) = \tau_{v,C}(\mathsf{e} - \gamma_0) + (1 - \beta)\Theta_{v,C}$ when the coordination is given by $(C, 1 - \beta, \mathsf{e} - \gamma_0)$. In particular, we have that

$$\tau_{v,C}(\mathsf{e} - \gamma_0) = \sum_{w \in V\setminus C} (1 - \gamma_0(w))\alpha_C(v, w) = 1 - \Theta_{v,C} - \tau_{v,C}(\gamma_0).$$

In particular, it follows that for every $u, v \in V$, we have that

$$\begin{aligned}\widetilde{\gamma}_\infty(v) - \widetilde{\gamma}_\infty(u) &= \tau_{v,C}(\mathsf{e} - \gamma_0) + (1 - \beta)\Theta_{v,C} - \Big(\tau_{u,C}(\mathsf{e} - \gamma_0) + (1 - \beta)\Theta_{u,C}\Big) \\ &= 1 - \Theta_{v,C} - \tau_{v,C}(\gamma_0) + (1 - \beta)\Theta_{v,C} - \Big(1 - \Theta_{u,C} - \tau_{u,C}(\gamma_0) + (1 - \beta)\Theta_{u,C}\Big) \\ &= 1 - \tau_{v,C}(\gamma_0) - \beta\Theta_{v,C} - \Big(1 - \tau_{u,C}(\gamma_0) - \beta\Theta_{u,C}\Big) \\ &= \tau_{u,C}(\gamma_0) + \beta\Theta_{u,C} - (\tau_{v,C}(\gamma_0) + \beta\Theta_{v,C}) = \gamma_\infty(u) - \gamma_\infty(v),\end{aligned}$$

where $\gamma_\infty \in \mathsf{R}^V$ is the limit opinion for the coordination (C, γ_0, β). Therefore, we conclude that

$$\begin{aligned}\mathsf{cost}_u(\Omega_C(\mathsf{e} - \gamma_0, 1 - \beta)) &= \sum_{v \in \mathsf{N}(u)} (\widetilde{\gamma}_\infty(u) - \widetilde{\gamma}_\infty(v))^2 \\ &= \sum_{v \in \mathsf{N}(u)} (\gamma_\infty(u) - \gamma_\infty(v))^2 = \mathsf{cost}_u(\Omega_C(\gamma_0, \beta)). \qquad \square\end{aligned}$$

Proof of Theorem 1. In what follows, we show that if a pair (C, γ_0) satisfies the contraction property, then it is either zero minded or one minded. Suppose first that (C, γ_0) satisfies the zero contraction property. Thanks to Proposition 5, for every $u \in C$ we have that $\tau_{u,C-u}^{-2}(\gamma_0) f_u(0)$ is equal to

$$\sum_{v \in \mathsf{N}(u) \setminus C} \left(\frac{\tau_{v,C}(\gamma_0)}{\tau_{u,C-u}(\gamma_0)} \right)^2 - \mathsf{deg}_C(u) - \sum_{v \in \mathsf{N}(u) \setminus C} \left(\frac{\tau_{v,C-u}(\gamma_0)}{\tau_{u,C-u}(\gamma_0)} - 1 \right)^2 \leq 0,$$

where the last inequality comes from the fact that (C, γ_0) satisfies the zero contraction property. It follows that $f_u(0) \leq 0$ for every $u \in C$, and therefore C is zero minded. By Proposition 4 there exists an interval $[0, \beta^0]$ such that $f_u(\beta) \leq 0$ for every $u \in C$ and for every $\beta \in [0, \beta^0]$. In this case the theorem follows by taking $\mathcal{I}(C, \gamma_0) = [0, \beta^0]$. Now suppose that (C, γ_0) satisfies the one contraction property. Observe that

$$\begin{aligned} f_u(1) &= \mathsf{cost}_u(\Omega_C(\gamma_0, 1)) - \mathsf{cost}_u(\Omega_{C-u}(\gamma_0, 1)) \\ &= \mathsf{cost}_u(\Omega_C(\mathsf{e} - \gamma_0, 0)) - \mathsf{cost}_u(\Omega_{C-u}(\mathsf{e} - \gamma_0, 0)), \end{aligned}$$

where the last equality comes from the symmetry property in Proposition 3. On the other hand, (C, γ_0) satisfies the one contraction property if and only if $(C, \mathsf{e} - \gamma_0)$ satisfies the zero contraction property. Then, it holds that $f_u(1) = \mathsf{cost}_u(\Omega_C(\mathsf{e}-\gamma_0, 0)) - \mathsf{cost}_u(\Omega_{C-u}(\mathsf{e}-\gamma_0, 0)) \leq 0$ and therefore C is one minded. By Proposition 4 there exists an interval $[\beta^1, 1]$ such that $f_u(\beta) \leq 0$ for every $u \in C$ and for every $\beta \in [\beta^1, 1]$. The theorem follows by taking $\mathcal{I}(C, \gamma_0) = [\beta^1, 1]$. $\square$

4.2 An Alternatives Result for the Contraction Property: Proof of Theorem 2

I the following we show that if a pair (C, γ_0) satisfies the contraction property then it should be either zero contractive or one contractive, but not both. That is, the coordination can sustain only one of the extreme opinions in the longrun. This theorem can be seen as an alternatives result, since only one of two possibilities can hold. Given a set $C \subseteq V$ and $u \in C$, consider the function

$$\Phi_u(y, C) = \sum_{v \in \mathsf{N}(u) \setminus C} \tau_{v,C}^2(y) - \sum_{v \in \mathsf{N}(u) \setminus C} \Big(\tau_{v,C-u}(y) - \tau_{u,C-u}(y) \Big)^2 - \mathsf{deg}_C(u) \cdot \tau_{u,C-u}^2(y).$$

Consider the set of vectors given by $\mathcal{F}(C) = \{ y \in \mathsf{R}^V : \Phi_u(y, C) \leq 0 \text{ for every } u \in C \}$. From the definition of the contraction condition, it follows directly that (C, γ_0) is zero contractive if and only if $\gamma_0 \in \mathcal{F}(C)$.

5 Sustainability in the Long-run: Random Intrinsic Opinions

In what follows, we study under what conditions it is possible for a coordination to sustain in the long-run, that is, by considering the cost under the limiting opinion. The intrinsic opinions are drawn independently and uniformly distributed from $[0, 1]$. We say that (C, β, γ_0) is *sustainable in the long-run* if for every $u \in C$ we have [2]

$$\mathsf{E}_{\gamma_0}(\mathsf{cost}_u(\Omega_C(\gamma_0, \beta))) \leq \mathsf{E}_{\gamma_0}(\mathsf{cost}_u(\Omega_{C-u}(\gamma_0, \beta))).$$

[2]For notational simplicity we denote by $C - u$ the set $C \setminus \{u\}$.

That is, every node in C faces a lower cost by being in the coordination than being out. In what follows, we say that a pair (C, γ_0) satisfies the *zero contraction condition* if for every $u \in C$, we have that

$$\mathsf{deg}_C(u) \geq \max\left\{\|\alpha_{u,C-u}\|_1^2 \Phi_1(u, C), \|\alpha_{u,C-u}\|_2^2 \Phi_2(u, C)\right\},$$

where the $\Phi_1(u, C)$ and $\Phi_2(u, C)$ are the graph parameters given by

$$\Phi_1(u, C) = \sum_{v \in \mathsf{N}(u) \setminus C} \|\alpha_{v,C}\|_1^2 - \sum_{v \in \mathsf{N}(u) \setminus C} \|\alpha_{v,C-u} - \alpha_{u,C-u}\|_1^2.$$

$$\Phi_2(u, C) = \sum_{v \in \mathsf{N}(u) \setminus C} \|\alpha_{v,C}\|_2^2 - \sum_{v \in \mathsf{N}(u) \setminus C} \|\alpha_{v,C-u} - \alpha_{u,C-u}\|_2^2.$$

Similarly, we say that a pair (C, γ_0) satisfies the *one opinion contraction condition* if for every $u \in C$, we have that

$$\mathsf{deg}_C(u) + \sum_{v \in V \setminus C} \left(\frac{\tau_{v,C-u}(\mathsf{e} - \gamma_0)}{\tau_{u,C-u}(\mathsf{e} - \gamma_0)} - 1\right)^2 \geq \sum_{v \in V \setminus C} \left(\frac{\tau_{v,C}(\mathsf{e} - \gamma_0)}{\tau_{u,C-u}(\mathsf{e} - \gamma_0)}\right)^2.$$

where e is the all ones vector in R^V. In the sequel, we say that (C, γ_0) satisfies the contraction condition if it satisfies at least one of the above conditions. The following is the main theorem of this section.

Victor: me da la sensacion de que un conjunto no puede zer zero contractive y one contractive a la vez

Theorem 3. *Let $G = (V, E)$ a connected graph and (C, γ_0) satisfying the contraction condition. Then, there exists a non-empty interval $\mathcal{I}(C, \gamma_0) \subseteq [0, 1]$ with $\mathcal{I}(C, \gamma_0) \cap \{0, 1\} \neq \emptyset$ such that for every $\beta \in \mathcal{I}(C, \gamma_0)$ we have that (C, β, γ_0) is sustainable in the long-run.*

Observe that in particular, an extreme opinion in $\{0, 1\}$ makes the coordination sustainable in the long-run as long as (C, γ_0) satisfies the contraction condition. In what follows we show how to prove the theorem above.

Victor: aqui discutir un poco el teorema anterior

5.1 Extreme Opinion Minded Sets: Proof of Theorem 3

We study first the conditions under which a node $u \in C$ faces a lower cost by being in the coordination. For every $u \in C$, consider the function

$$f_u(\beta) = \mathsf{E}_{\gamma_0}(\mathsf{cost}_u(\Omega_C(\gamma_0, \beta))) - \mathsf{E}_{\gamma_0}(\mathsf{cost}_u(\Omega_{C-u}(\gamma_0, \beta))).$$

In the following, we say that C is *one minded* if for every $u \in C$ we have that $f_u(1) < 0$. Symmetrically, we say that C is *zero minded* if for every $u \in C$ we have that $f_u(0) < 0$.

Proposition 4. *Let $C \subseteq V$ be a non-empty subset of nodes. Then, the following holds.*

(a) *If C is one minded, there exists $\beta^1 \in (0, 1)$ such that for every $f_u(\beta) \leq 0$ for every $\beta \in [\beta^1, 1]$.*

(b) *If C is zero minded, there exists $\beta^0 \in (0, 1)$ such that for every $f_u(\beta) \leq 0$ for every $\beta \in [0, \beta^0]$.*

Proof. If C is one minded, by continuity we have that for every $u \in C$ there exists $\beta_u \in (0, 1)$ such that $f_u(\beta) \leq 0$ for every $\beta \in [\beta_u, 1]$. In particular, given $\beta^1 = \max_{u \in C} \beta_u$, we have that for every $u \in C$ it holds that $f_u(\beta) < 0$ for every $\beta \in [\beta^1, 1]$. The proof follows in the same way when C is zero minded. $\square$

In the following propositions we provide a more explicit expression for the costs evaluated in the longrun opinion vectors. Having that, we are ready to prove Theorem 1. One [property that will be useful in what comes next is the following. Consider a random vector $\xi \in \mathsf{R}^m$ where every cordinate is independently and uniformly distributed over $[0,1]$, and consider a vector a vector $x \in \mathsf{R}^m$. Then, we have that

$$\mathsf{E}_\xi(\langle \xi, x\rangle)^2 = \frac{1}{12}\|x\|_2^2 + \|x\|_1^2.$$

Lemma 2. *Let $G=(V,E)$ a connected graph and γ_0 drawn independently and uniformly from $[0,1]$. Then, the following holds.*

(*a*) *For every $v \in V \setminus C$ we have that $\mathsf{E}_{\gamma_0}(\tau_{v,C}(\gamma_0)) = \mathsf{E}_{\gamma_0}(\tau_{v,C}(\mathsf{e}-\gamma_0)) = \frac{1}{2}(1-\Theta_v(C))$.*

(*b*) *For every $v \in V \setminus C$ we have that $\mathsf{E}_{\gamma_0}(\tau^2_{v,C}(\gamma_0)) = \frac{1}{12}\|\alpha_{v,C}\|_2^2 + \mathsf{E}^2_{\gamma_0}(\tau_{v,C}(\gamma_0))$.*

(*c*) *For every $u \in C$ and every $v \in V \setminus (C-u)$ we have that*

$$\mathsf{E}_{\gamma_0}\Big(\tau_{v,C-u}(\gamma_0) - \tau_{u,C-u}(\gamma_0)\Big)^2 = \frac{1}{12}\|\alpha_{v,C-u} - \alpha_{u,C-u}\|_2^2 + \frac{1}{2}(\Theta_{u,C-u} - \Theta_{v,C-u})^2.$$

(*d*) *For every $v \in V \setminus C$ we have that $\mathsf{E}_{\gamma_0}(\gamma_\infty(v)) = \frac{1}{2} + \left(\beta - \frac{1}{2}\right)\Theta_v(C)$.*

Proof of Lemma 2. □

Proposition 5. *Let $G=(V,E)$ be a connected graph, $C \subseteq V$ and $\gamma_0 \in [0,1]^V$. Then, the following holds.*

(*a*) *When $\beta = 0$, for every $v \in V \setminus C$ we have $\gamma_\infty(v) = \tau_{v,C}(\gamma_0)$.*

(*b*) *For every $u \in C$, we have that*

$$\mathsf{E}_{\gamma_0}(\mathsf{cost}_u(\Omega_C(\gamma_0,0))) = \sum_{v \in \mathsf{N}(u)\setminus C} \mathsf{E}_{\gamma_0}(\tau^2_{v,C}(\gamma_0))$$

(*c*) *For every $u \in C$, we have that $\mathsf{cost}_u(\Omega_{C-u}(\gamma_0,0))$ is equal to*

$$\mathsf{deg}_C(u)\cdot \mathsf{E}_{\gamma_0}(\tau^2_{u,C-u}(\gamma_0)) + \sum_{v\in \mathsf{N}(u)\setminus C} \mathsf{E}_{\gamma_0}\Big(\tau_{v,C-u}(\gamma_0) - \tau_{u,C-u}(\gamma_0)\Big)^2.$$

Proof of Proposition 5. □

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Proof of Theorem 1. In what follows, we show that if a pair (C,γ_0) satisfies the contraction property, then it is either zero minded or one minded. Suppose first that (C,γ_0) satisfies the zero contraction property. Thanks to Proposition 5, for every $u \in C$ we have that $\tau^{-2}_{u,C-u}(\gamma_0) f_u(0)$ is equal to

$$\sum_{v\in V\setminus C}\left(\frac{\tau_{v,C}(\gamma_0)}{\tau_{u,C-u}(\gamma_0)}\right)^2 - \mathsf{deg}_C(u) - \sum_{v \in V\setminus C}\left(\frac{\tau_{v,C-u}(\gamma_0)}{\tau_{u,C-u}(\gamma_0)} - 1\right)^2 \le 0,$$

where the last inequality comes from the fact that (C,γ_0) satisfies the zero contraction property. It follows that $f_u(0) \le 0$ for every $u \in C$, and therefore C is zero minded. By Proposition 4 there exists an interval $[0,\beta^0]$ such that $f_u(\beta) \le 0$ for every $u \in C$ and for every $\beta \in [0,\beta^0]$. In this case the theorem follows by taking $\mathcal{I}(C,\gamma_0) = [0,\beta^0]$. □

6 Random Networks

In the following we analyze long-run sustainability coordination over randomly generated networks. Furthermore, we consider the intrinsic opinions to be randomly and independently distributed.

6.1 A Mixture between Geometric and Erdos-Renyi Random Graphs

6.2 Graph Expansion and the Contraction Property

Recall that Ω_C is a linear function in (γ_0, β), and therefore f_u is a quadratic function on β. It will be convenient to express the cost function in matrix form. For every $u \in V$, consider the matrix $\mathsf{A}_u \in \mathsf{R}^{V \times V}$ such that $\mathsf{A}_u(u, v) = -1$ when $v \in \mathsf{N}(u)$, and zero otherwise; and let $\mathsf{D} \in \mathsf{R}^{V \times V}$ the diagonal matrix such that $\mathsf{D}(u, u) = \mathsf{deg}(u)$ for every $u \in V$. Then, for every $y \in \mathsf{R}^V$ we have that

$$\mathsf{cost}_u(y) = \frac{1}{2} y^\top \mathsf{L}_u y + \frac{\eta_u}{2} (\gamma_0(u) - y(u))^2, \tag{5}$$

where $\mathsf{L}_u = \mathsf{D} - \mathsf{A}_u$, and we call this matrix the *Laplacian* for node $u \in V$. In particular, the matrix $\mathsf{L} = \sum_{u \in V} \mathsf{L}_u$ is known as the Laplacian of the graph G.

Example 2.

References