Sustainability of Opinion Coordination in Social Networks

Andrea Canales * Victor Verdugo †‡

Abstract

The process by which opinions spread through a large social network can be modeled as an interaction between the initial opinion of an agent and their neighbor's opinions. This process can be affected by stubborn agents, who maintain their initial opinion fixed. We characterize under which conditions a group of regular agents can coordinate their opinions and behave like stubborn agents. We show that if an agent has incentives to coordinate his opinion in the first period, he announces the opinion of the coordination in every period. There exist an interval of opinions that can be sustained by the coordinated agents, which depends on their initial opinion and the connectivity of this subgraph. We also study how this coordination impact the convergence speed and characterize the convergence time .

Andrea: Todos los cambios son bienvenidos y sorry el ingles

Victor: Otra opcion de titulo es *Robust Sustainability of Beliefs Coordination*

*Institute of Social Sciences, Universidad de O'Higgins. andrea.canales@uoh.cl
†Department of Mathematics, London School of Economics and Political Science. v.verdugo@lse.ac.uk
‡Institute of Engineering Sciences, Universidad de O'Higgins. victor.verdugo@uoh.cl

Contents

1 Introduction

Andrea: Hacer la analogía de este modelo a la literatura de carteles incompletos. No todos pueden escoger "precios",solo los invitados a coordinarse y buscamos condiciones bajo las cuales esta coordinación es maximal

2 Local Interaction and Coordination Model

Consider a graph $G = (V, E)$ representing a social network. Every node $v \in V$ in the network has an intrinsic opinion $\gamma_0(v)$. At every time step $t \in \mathbb{N}$ and for every $u \in V$, $\gamma_t(u)$ denotes the opinion *declared* by u. We say that (C, β) is a *coordination* if every node $v \in C$ declares the same opinion β, that is, for every $t \in \mathbb{Z}_+$ and for every $u \in C$, $\gamma_t(u) = \beta$. Every node $u \notin C$ updates the opinion according to

$$\gamma_{t+1}(u) = \frac{\eta_u}{\mathsf{deg}(u) + \eta_u} \cdot \gamma_0(u) + \frac{1}{\mathsf{deg}(u) + \eta_u} \sum_{v \in \mathsf{N}(u)} \gamma_t(v), \tag{1}$$

For every $u \in V$, the parameter η_u captures how resistant is u to modify his initial opinion $\gamma_0(u)$. Observe that if $C = \emptyset$ we then recover the classic Friedkin and Johnsen opinion dynamics model []. In what follows, it will be useful to consider the dynamics in a matrix form. Consider the matrix $\mathsf{A} \in \mathsf{R}^{V \times V}$ defined as follows: Let $\mathsf{A}(u, v) = 1/(\mathsf{deg}(u) + \eta_u)$ if $\{u, v\} \in E$ and $u \in V \setminus C$, and zero otherwise, and let $\mathsf{B} \in \mathsf{R}^{V \times V}$ the diagonal matrix given by $\mathsf{B}(u, u) = \eta_u/(\mathsf{deg}(u) + \eta_u)$ for every $u \in V \setminus C$. Then, the dynamics of the declared opinions (1) can be written as

$$\gamma_{t+1} = \mathsf{A}\gamma_t + \beta \mathsf{I}_C + \mathsf{B}\gamma_0, \tag{2}$$

where $\mathsf{I}_C \in \mathsf{R}^{V \times V}$ is such that $\mathsf{I}_C(u, u) = 1$ if $u \in C$, and zero otherwise. In particular, observe that from $t = 1$ every member of the coordination declares the opinion β, that is, for every $u \in C$ we have $\gamma_1(u) = \beta$. In the following, we call γ the *coordination dynamics* for (C, β). A similar dynamics was considered by Ghaderi and Srikant [], to study networks with *stubborn* agents. We later provide a random walk interpretation of the above dynamics to study the evolution and long-run behavior of the process.

Example 1.

Victor: ejemplito con grafo chico y una o dos iteraciones

3 Existence and Behavior of the Long-run Opinions

In what follows we use a result by Ghaderi & Srikant showing the existence and providing a characterization of the long-run opinions of a network under the presence of stubborn agents. To study the dynamics, the authors construct a random walk in an auxiliary graph, and characterize the long-run opinion by studying the stationary distribution of this auxiliary random walk. Observe that in our model, for a coordination (C, β) we have that every $u \in C$ can be seen a fully stubborn agent for the dynamics from $t = 1$. Nevertheless, the value of β can be chosen after the realization of each of the intrinsic opinions of the members in the coordination. For the sake of completeness, we include the random walk construction here and state the results in our context.

An auxiliary random walk. Given a graph G and $C \subseteq V$, consider the auxiliar graph, $\mathcal{G}_\mathcal{C} = (\mathcal{V}_\mathcal{C}, \mathcal{E}_\mathcal{C})$ defined as follows. We have $\mathcal{V}_C = V \cup T_{V\setminus C}$ where $T_{V\setminus C} = \{x_v : v \in V \setminus C\}$, that is, the set of nodes $\mathcal{V}_\mathcal{C}$ consists of every node in the original graph and a copy of each node not in C. Furthermore, in the auxiliary graph every node in $V \setminus C$ is connected to its copy, that is, consider

$$\mathcal{E}_C = E \cup \Big\{\{v, x_v\} : v \in V \setminus C\Big\}.$$

Consider the random walk $(\mathsf{x}_t)_{t\in\mathbb{Z}_+}$ over the graph $\mathcal{G}$ with $\mathsf{P} \in [0,1]^{\mathcal{V}\times\mathcal{V}}$ given by

$$\mathsf{P}(u,v) = \begin{cases} 1/\mathsf{deg}(u) & \text{for every } u \in C \text{ and every } v \in \mathsf{N}(u), \\ 1/(\mathsf{deg}(u) + \eta_u) & \text{for every } u \in V \setminus C \text{ and every } v \in \mathsf{N}(u), \\ \eta_u/(\mathsf{deg}(u) + \eta_u) & \text{for every } u \in V \setminus C \text{ and } v = x_u, \\ 1 & \text{for every } v \in V \text{ and } u = x_v, \end{cases}$$

For every $v \in \mathcal{V}_C$, consider $\tau_v = \inf\{t \in \mathbb{Z}_+ : \mathsf{x}_t = v\}$, that is, the hitting time of vertex v, and let

$$\tau_C = \inf\Big\{t \in \mathbb{Z}_+ : \mathsf{x}_t \in C \cup T_{V\setminus C}\Big\}$$

be the hitting time of the set of nodes in $C \cup T_{V\setminus C}$. For every $v \in V$, let $\alpha_{v,C} \in \mathsf{R}^V$ be the vector such that for every $w \in V$ we have $\alpha_{v,C}(w) = \mathbb{P}_v(\tau_{x_w} = \tau)$. In particular, $\alpha_{v,C}$ is a probability distribution. Furthermore, consider the quantity given by

$$\Theta_{u,C} = \sum_{v\in C} \mathbb{P}_u(\tau_C = \tau_v).$$

The value above corresponds to the probability that the random walk $(\mathsf{x}_t)_{t\in\mathbb{Z}_+}$ starting at $u \in V$ hits $C \cup T_{V\setminus C}$ in a vertex that belongs to C. We now state the technical lemma from [] adapted to our context. We include a proof of it in the Appendix.

Lemma 1 (**[?]**). *Let $G = (V, E)$ be a connected graph and γ_0 a vector of intrinsic opinions. Then, for every coordination (C, β) there exists $\gamma_\infty \in \mathsf{R}^V$ such that $\lim_{t\to\infty} \gamma_t = \gamma_\infty$. Furthermore, we have that for every $v \in V$ the limit opinion is given by*

$$\gamma_\infty(v) = \sum_{w\in V\setminus C} \gamma_0(w)\alpha_{v,C}(w) + \beta\Theta_{v,C}. \tag{3}$$

That is, $\gamma_\infty(v)$ is a convex combination of the intrinsic opinions of the nodes in $V \setminus C$, and the opinion β of the coordination. Observe that for every $v \in V \setminus C$, the first term in the equality above is independent of β. We call this term the *effective external opinion*,

Victor: insertar un mono aca con el grafo auxiliar

$$\tau_{v,C}(\gamma_0) = \sum_{w\in V\setminus C} \gamma_0(w)\alpha_{v,C}(w). \tag{4}$$

For each $C \subseteq V$, the limit opinion is a function of the intrinsic opinions γ_0 and β and it will be useful in what follows to consider the function mapping a pair $(\gamma_0, \beta) \in [0,1]^V \times [0,1]$ onto $\Omega(C, \beta, \gamma_0) = \alpha_C\gamma_0^\beta \in \mathsf{R}^V$.

Conditional Expectations. Given a coordination (C, β), by the definition of the hitting probabilities it follows that for each $v \in V \setminus C$, we have that $1 - \Theta_{v,C} = \sum_{w\in V\setminus C} \alpha_{v,C}(w)$. Consider the probability distribution $f_{v,C}$ over the nodes in $V \setminus C$ such that for each $w \in V \setminus C$ we have $f_{v,C}(w) = \frac{\alpha_{v,C}(w)}{1-\Theta_{v,C}}$. We denote by $\mathsf{E}_{v,C}$ the expectation operator from probability distributation above. Observe that $f_{v,C}$ corresponds to the probability distribution induced by α_C conditional on the random walk $(\mathsf{x}_t)_{t\in\mathbb{Z}_+}$ starting at v hitting for the first time $C \cup T$ in a vertex of T. Therefore, given a random variable $\mathsf{a} \sim f_{v,C}$, the effective external opinion corresponds to

$$\tau_{v,C}(\gamma_0) = (1 - \Theta_{v,C})\mathsf{E}_{v,C}(\gamma_0(\mathsf{a}))$$

Mixing time. A quantity that plays a role in our analysis is the time it requires for the random walk distribution to be very close from the stationary one. Given two probability distributions μ and ν over $\mathcal{V}$, the total variation distance between μ and ν is given by

$$\|\mu - \nu\|_{\mathsf{TV}} = \frac{1}{2}\sum_{u\in\mathcal{V}} |\mu(u) - \nu(u)|.$$

For every every $u \in \mathcal{V}$, we denote by δ_u the probability distribution such that $\delta_u(u) = 1$ and zero otherwise. Then, the mixing time of the random walk $(\mathsf{x}_t)_{t\in\mathbb{Z}_+}$ corresponds to the value

$$t_{\mathsf{mix}}(\mathcal{G}) = \min\Big\{t \in \mathbb{Z}_+ : \max_{u\in\mathcal{V}} \|\mathsf{P}^t\delta_u - \pi\|_{\mathsf{TV}} \le 1/e\Big\}.$$

That is, the amount of time it requires for the distribution induced by the random walk to be within a distance of at most $1/e$ from the stationary distribution, no matter the initial state. The constant is arbitrary, since one could replace its value by ε at cost of a logarithmic factor, $t_{\mathsf{mix}}(\mathcal{G})\log(1/\varepsilon)$.

4 Coordination Sustainability in the Long-run

For every node $u \in V$ and $v \in \mathsf{N}(u)$, consider an *interaction cost* function $\mathsf{cost}(y(u), y(v))$ that captures the cost incurred by u and v, where the vector y represents the *state* or *configuration* of the graph. That is, $\mathsf{cost}(y(u), y(v))$ corresponds to the cost incurred by u and v when they are at opinion states $y(u)$ and $y(v)$ respectively. In what follows we assume that the cost satisfies the following properties,

(1) $\mathsf{cost}(x, x) = 0$ for every $x \in \mathsf{R}$,

(2) $\mathsf{cost}(x, y) = \mathsf{cost}(y, x)$ for every $x, y \in \mathsf{R}$,

(3) $\mathsf{cost}(x, y) \leq \mathsf{cost}(x, z) + \mathsf{cost}(z, y)$ for every $x, y, z \in \mathsf{R}$, and

(4) $\mathsf{cost}(x, y) = \mathsf{cost}(1 - x, 1 - y)$ for every $x, y \in [0, 1]$.

Given a node $u \in V$ and a graph configuration y, we denote by $\mathsf{cost}_u(y)$ the total interaction cost between u and its neighbors in $\mathsf{N}(u)$ at configuration y, that is,

Victor: maybe the only costs satisfying these four properties are $f(|x - y|)$ with f convex

$$\mathsf{cost}_u(y) = \sum_{v \in \mathsf{N}(u)} \mathsf{cost}(y(u), y(v)).$$

We study under what conditions it is possible for a coordination to *sustain* in the long-run. Given a triplet (C, β, γ_0), for each node $u \in C$ decides between joining the coordination, or do not join. If the join, their opinion state remains unchanged equal to β. If they do not join the coordination, they update their opinion according to the weighted opinion dynamics. We say that (C, β, γ_0) is *sustainable in the long-run* if for every $u \in C$ we have [1] for every $u \in C$ we have that

$$\mathsf{cost}_u(\Omega(C, \beta, \gamma_0)) \leq \mathsf{cost}_u(\Omega(C - u, \beta, \gamma_0)).$$

That is, joining the coordination (C, β, γ_0) is a best response for every $u \in C$.

Extreme Opinions. Of particular interest is the long-run sustainability of a coordination with an extreme opinion, that is $\beta \in \{0, 1\}$. We say that set $C \subseteq V$ is ω-cohesive if for every $u \in C$ we have that $\mathsf{deg}_C(u) \geq \omega \cdot \mathsf{deg}(u)$. This notion was introduced by Morris **[?]** to study best response configurations in local interaction frameworks. In what follows, we say that a pair (C, γ_0) is ε-divergent if for every $u \in C$ we have that

$$\frac{1}{\mathsf{deg}(u)} \sum_{v \in \mathsf{N}(u) \setminus C} \frac{\mathsf{cost}(\tau_{v,C}(\gamma_0), \tau_{v,C-u}(\gamma_0))}{\mathsf{cost}(0, \tau_{u,C-u}(\gamma_0))} \leq 2\varepsilon.$$

The above notions allows us to find sufficient conditions that guarantee the sustainability of extreme opinions in the long-run. The following is the first main result of this section.

Victor: el setting de morris es 0 o 1

Theorem 1. *Let $G = (V, E)$ be a connected graph and let $C \subseteq V$ a $(1/2 + \varepsilon)$-cohesive set. Then, for every $\gamma_0 \in \mathsf{R}^V$ the following holds.*

(a) *If (C, γ_0) is ε-divergent, there exists an interval $\mathcal{I}_\varepsilon(C, \gamma_0) \subseteq [0, 1]$ with $0 \in \mathcal{I}_\varepsilon(C, \gamma_0)$ such that (C, β, γ_0) is sustainable in the long-run for every $\beta \in \mathcal{I}_\varepsilon(C, \gamma_0)$.*

[1] For notational simplicity we denote by $C - u$ the set $C \setminus \{u\}$.

(b) *If $(C, \mathsf{e} - \gamma_0)$ is ε-divergent, there exists an interval $\mathcal{I}_\varepsilon(C, \gamma_0) \subseteq [0, 1]$ with $1 \in \mathcal{I}_\varepsilon(C, \gamma_0)$ such that (C, β, γ_0) is sustainable in the long-run for every $\beta \in \mathcal{I}_\varepsilon(C, \gamma_0)$.*

Morris characterizes the contagion threshold of local interaction models in terms of the cohesiveness of blabla [**?**]. Recently, Chandrasekhar et al study a model of incomplete information for social learning. They show that blabla.In the rest of this section we prove Theorem 1. Later, in Section 5 we study the conditions that guarantee sustainability in the long-run of extreme opinions, in randomly generated graphs. We come back to all these concepts in that section.

Victor: hablar de los clanes de Xandri aca

Example 2.

Victor: ejemplo aca del caso que en que C es un cluster, es decir, desconectado del resto, ahi la condicion se cumple

4.1 Sustaining an Extreme Opinion: Proof of Theorem 1

Victor: dar intuicion, algun ejemplo para mostrar que C no puede ser tan chico en general, la idea es ver cuando grande tiene que ser

We study first the conditions under which a node $u \in C$ faces a lower cost by being in the coordination. For every $u \in C$, consider the function

$$f_u(\beta) = \mathsf{cost}_u(\Omega(C, \beta, \gamma_0)) - \mathsf{cost}_u(\Omega(C - u, \beta, \gamma_0))$$

In the following, we say that (C, γ_0) is *one minded* if for every $u \in C$ we have that $f_u(1) < 0$. Symmetrically, we say that (C, γ_0) is *zero minded* if for every $u \in C$ we have that $f_u(0) < 0$. The following simple observation allows us to restrict attention on the cost behavior at the extreme opinions.

Proposition 1. *Let $C \subseteq V$ be a non-empty subset of nodes and $\gamma_0 \in \mathsf{R}^V$. Then, the following holds.*

(a) *If C is one minded, there exists $\beta^1 \in (0, 1)$ such that for every $f_u(\beta) \leq 0$ for every $\beta \in [\beta^1, 1]$.*

(b) *If C is zero minded, there exists $\beta^0 \in (0, 1)$ such that for every $f_u(\beta) \leq 0$ for every $\beta \in [0, \beta^0]$.*

Proof. If C is one minded, by continuity we have that for every $u \in C$ there exists $\beta_u \in (0, 1)$ such that $f_u(\beta) \leq 0$ for every $\beta \in [\beta_u, 1]$. In particular, given $\beta^1 = \max_{u \in C} \beta_u$, we have that for every $u \in C$ it holds that $f_u(\beta) < 0$ for every $\beta \in [\beta^1, 1]$. The proof follows in the same way when C is zero minded. □

In the following proposition we provide a more explicit expression for the costs evaluated in the longrun opinion vectors. Furthermore, we also show some opinion symmetry property of the cost function: the cost we face under certain coordination value is the same if every opinion spin is changed. Having that, we are ready to prove Theorem 1.

Proposition 2. *Let $G = (V, E)$ be a connected graph, $C \subseteq V$ and $\gamma_0 \in [0, 1]^V$. Then, the following holds.*

(a) *When $\beta = 0$, for every $v \in V \setminus C$ we have $\Omega_v(C, 0, \gamma_0) = \tau_{v,C}(\gamma_0)$.*

(b) *For every $u \in C$, we have that $\mathsf{cost}_u(\Omega(C, 0, \gamma_0)) = \sum_{v \in \mathsf{N}(u) \setminus C} \mathsf{cost}(\tau_{v,C}(\gamma_0), 0)$.*

(c) *For every $u \in C$, we have that $\mathsf{cost}_u(\Omega(C - u, \gamma_0, 0))$ is equal to*

$$\mathsf{deg}_C(u) \cdot \mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0) + \sum_{v \in \mathsf{N}(u) \setminus C} \mathsf{cost}(\tau_{v,C-u}(\gamma_0), \tau_{u,C-u}(\gamma_0)).$$

Proof of Proposition 2. Part (a) comes directly from equalities Lemma 1 and equalities (3) and (4). For part (b), observe that for every $v \in \mathsf{N}(u) \cap C$ we have that $\Omega_u(C, 0, \gamma_0) = \Omega_v(C, 0, \gamma_0) = 0$. Therefore,

$$\mathsf{cost}_u(\Omega(C, 0, \gamma_0)) = \sum_{v \in \mathsf{N}(u) \setminus C} \mathsf{cost}(0, \Omega_v(C, 0, \gamma_0)) = \sum_{v \in \mathsf{N}(u) \setminus C} \mathsf{cost}(0, \tau_{v,C}(\gamma_0)),$$

where the last equality comes from part (a). Finally, for part (c) we have that $\mathsf{cost}_u(\Omega_{C-u}(\gamma_0, 0))$ is equal to

$$\begin{aligned} &= \sum_{v \in \mathsf{N}(u) \cap C} \mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0) + \sum_{v \in \mathsf{N}(u) \setminus C} \mathsf{cost}(\tau_{v,C-u}(\gamma_0), \tau_{u,C-u}(\gamma_0)) \\ &= \mathsf{deg}_C(u) \cdot \mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0) + \sum_{v \in \mathsf{N}(u) \setminus C} \mathsf{cost}(\tau_{v,C-u}(\gamma_0), \tau_{u,C-u}(\gamma_0)). \end{aligned}$$ □

Proposition 3. *Let $G = (V, E)$ be a connected graph, $C \subseteq V$ and $\gamma_0 \in [0,1]^V$. Then, for every $u \in C$ we have that*

$$\mathsf{cost}_u(\Omega(C, 0, \gamma_0)) = \mathsf{cost}_u(\Omega(C, 1, \mathsf{e} - \gamma_0)).$$

Proof. Recall that for every $u \in V$, we have that $\Omega_u(C, 1, \mathsf{e} - \gamma_0) = \tau_{v,C}(\mathsf{e} - \gamma_0) + \Theta_{v,C}$ when the coordination is given by $(C, 1, \mathsf{e} - \gamma_0)$. Furthermore, we have that

$$\begin{aligned} \tau_{v,C}(\mathsf{e} - \gamma_0) + \Theta_{v,C} &= \sum_{w \in V \setminus C} (1 - \gamma_0(w)) \alpha_C(v, w) + \Theta_{v,C} \\ &= 1 - \Theta_{v,C} - \tau_{v,C}(\gamma_0) + \Theta_{v,C} = 1 - \tau_{v,C}(\gamma_0). \end{aligned}$$

Therefore, we have $\Omega_u(C, 1, \mathsf{e} - \gamma_0) = 1 - \Omega_u(C, 0, \gamma_0)$. In particular, for every $u, v \in V$ we have that

$$\begin{aligned} \mathsf{cost}(\Omega_u(C, 1, \mathsf{e} - \gamma_0), \Omega_v(C, 1, \mathsf{e} - \gamma_0)) &= \mathsf{cost}(1 - \Omega_u(C, 0, \gamma_0), 1 - \Omega_v(C, 0, \gamma_0)) \\ &= \mathsf{cost}(\Omega_u(C, 0, \gamma_0), \Omega_v(C, 0, \gamma_0)), \end{aligned}$$

where the last equality comes from property (4) of the cost function. Therefore, we conclude that

$$\begin{aligned} \mathsf{cost}_u(\Omega(C, 1, \mathsf{e} - \gamma_0,)) &= \sum_{v \in \mathsf{N}(u)} \mathsf{cost}(\Omega_u(C, 1, \mathsf{e} - \gamma_0), \Omega_v(C, 1, \mathsf{e} - \gamma_0)) \\ &= \sum_{v \in \mathsf{N}(u)} \mathsf{cost}(\Omega_u(C, 0, \gamma_0), \Omega_v(C, 0, \gamma_0)) = \mathsf{cost}_u(\Omega(C, 0, \gamma_0)). \end{aligned}$$ □

Proof of Theorem 1. In what follows, let (C, γ_0) be a pair ε-divergent. We show next that (C, γ_0) is Victor: arreglar esto
zero minded. Observe that since cost satisfies symmetry and the triangular inequality properties (2) and (3), we have that for every $u \in C$ and for every $v \in \mathsf{N}(u)$,

$$\mathsf{cost}(\tau_{v,C}(\gamma_0), 0) \leq \mathsf{cost}(\tau_{v,C}(\gamma_0), \tau_{v,C-u}(\gamma_0)) + \mathsf{cost}(\tau_{v,C-u}(\gamma_0), \tau_{u,C-u}(\gamma_0)) + \mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0).$$

The above inequality together with Proposition 2, implies that for every $u \in C$ we have that

$$\begin{aligned} \frac{f_u(0)}{\mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0)} &\leq \sum_{v \in \mathsf{N}(u) \setminus C} \frac{\mathsf{cost}(\tau_{v,C}(\gamma_0), \tau_{v,C-u}(\gamma_0))}{\mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0)} - \mathsf{deg}_C(u) + \mathsf{deg}(u) - \mathsf{deg}_C(u) \\ &\leq 2\varepsilon \cdot \mathsf{deg}(u) + \mathsf{deg}(u) - 2\mathsf{deg}_C(u) = (2\varepsilon + 1)\mathsf{deg}(u) - 2\mathsf{deg}_C(u), \end{aligned}$$

where the second inequality comes from the fact that (C, γ_0) is ε-divergent. Finally, since C is $(1/2+\varepsilon)$-cohesive we conclude that

$$\frac{f_u(0)}{2 \cdot \mathsf{cost}(\tau_{u,C-u}(\gamma_0), 0)} \leq \left(\frac{1}{2}+\varepsilon\right) \mathsf{deg}(u) - \mathsf{deg}_C(u) \leq 0.$$

It follows that $f_u(0) \leq 0$ for every $u \in C$, and therefore C is zero minded. By Proposition 1 there exists an interval $[0, \beta^0]$ such that $f_u(\beta) \leq 0$ for every $u \in C$ and for every $\beta \in [0, \beta^0]$. In this case the theorem follows by taking $\mathcal{I}_\varepsilon(C, \gamma_0) = [0, \beta^0]$. Now suppose that $(C, \mathsf{e} - \gamma_0)$ is ε-divergent. Observe that by Proposition 3 we have that

$$\begin{aligned} f_u(1) &= \mathsf{cost}_u(\Omega(C, 1, \gamma_0)) - \mathsf{cost}_u(\Omega(C-u, 1, \gamma_0)) \\ &= \mathsf{cost}_u(\Omega(C, 0, \mathsf{e}-\gamma_0)) - \mathsf{cost}_u(\Omega(C-u, 0, \mathsf{e}-\gamma_0)). \end{aligned}$$

On the other hand, since $(C, \mathsf{e}-\gamma_0)$ is ε-divergent, by the previous case we conclude that $f_u(1) \leq 0$ and therefore C is one minded. By Proposition 1 there exists an interval $[\beta^1, 1]$ such that $f_u(\beta) \leq 0$ for every $u \in C$ and for every $\beta \in [\beta^1, 1]$. The theorem follows by taking $\mathcal{I}_\varepsilon(C, \gamma_0) = [\beta^1, 1]$. □

5 Random Networks and Robust Sustainability

In the following we analyze long-run sustainability coordination over randomly generated networks. The first random graph family we consider is given by a mixture between the classic Erdos-Renyi model and the Geometric model. This family has been studied in the context of social learning on networks blabla aca. We show that under certain regime of the random graph model, with high probability there exists a set that is center contractive.

5.1 Cohesiveness and Center Contractiveness

Recall that a set $C \subseteq V$ is said to be ω-cohesive if for every $u \in C$ we have that $\mathsf{deg}_C(u) \geq \omega \cdot \mathsf{deg}(u)$. This notion was introduced by Morris **[?]** in a more general interaction framework in order to study contagion processes. In what follows we say that a node $u \in V$ is (ω, k)-*cohesive* if there exists a ω-cohesive set $C \subseteq V$ of size at most k such that $u \in C$. We say that a graph G is (μ, ω, k)-cohesive if the fraction of nodes that are (ω, k)-cohesive is at least μ. Recently, Chandrasekhar et al [] study the existence of $1/2$-cohesive sets in a family of randomly generated graphs. They call such a set a *clan*. In the next section we go back to this point.

Example 3.

Victor: ejemplo aca de un $(1/2 - eps, 1/eps)$-cohesive graph, tomar el del libro-paper de Young

We remark that this notion is closely related to the *close-knit graphs* introduced by Young **[?]** in order to study the long-run behavior of dynamics associated to local interactions systems. In particular, one can show that (ω, k)-close-knit graphs are $(1, \omega, k)$-cohesive graphs. The following lemma relates the cohesiveness property and the fact of being center contractive. It will be useful late in our analysis. Recall that given $C \subseteq V$, the value $\xi_C(u)$ is given by $\sum_{v \in \mathsf{N}(u) \setminus C} \frac{\Theta_{u,C-u} - \Theta_{v,C-u}}{1 - \Theta_{u,C-u}}$. We denote by $\xi_C = \max_{u \in C} \xi_C(u)$ the maximum of this quantity over every node in C, and we call it the *shifting parameter* of C.

Proposition 4. *Let $G = (V, E)$ be a connected graph. If $C \subseteq V$ is $(1/2 + \xi_C)$-cohesive then it is center contractive as well.*

Proof. If C is $(1/2 + \xi_C)$-cohesive we have that $\mathsf{deg}_C(u) \geq (1/2 + \xi_C)\mathsf{deg}(u)$ for every $u \in C$. In particular, since $\xi_C \geq \xi_C(u)$ for all $u \in C$, it follows that $\mathsf{deg}_C(u) \geq (1/2 + \xi_C(u))\mathsf{deg}(u)$ and therefore C is center contractive. □

In the following, we say that graph is (ω, k)-*centered contractive* if at least an $\omega \in [0, 1]$ fraction of the nodes belong to a center contractive set of at most k. In particular, if there exists a center contractive set of size k the graph is $(k/n, k)$-center contractive. On the other hand, every connected graph of n nodes is $(1, n)$-center contractive since the set of nodes is clearly center contractive. The goal is to study when we can guarantee that a large enough fraction of the nodes belong to a center contractive set of small enough size. In the next section we consider a particular family of random graphs and we provide a deeper study on this trade-off.

5.2 Geometric Random Graphs

In the following we consider a random graph model that captures certain features of real world networks. Consider a Poisson point process with uniform intensity $n \in \mathbb{Z}_+$ on the unit square $[0, 1]^2$ and denote its point set by V. This process exhibits two key aspects.

(a) For every Lebesgue measurable set $\Omega \subseteq [0, 1]^2$, we have that $|V \cap \Omega| \sim \mathsf{Poisson}(n\nu(\Omega))$, where ν is the Lebesgue measure in R^2.

(b) For every pair of disjoint Lebesgue measurable sets $\Omega_1, \Omega_2 \subseteq [0, 1]^2$, we have that $|V \cap \Omega_1|$ is independent from $|V \cap \Omega_2|$.

In particular, from the first property we have that $\mathsf{E}(|V|) = n$, that is, the process generates n points on expectation. Consider the random graph $(V_n, E_{n,\rho})$ where V_n is the point set of the above Poisson point process, and

$$E_{n,\rho} = \Big\{\{u, v\} \subseteq V : u \in B_\infty(v, \rho) \text{ and } u \neq v\Big\},$$

where ρ is a *radius* value in $[0, 1]$, and $B_1(v, \rho)$ is the ℓ_∞ ball centered at v and with radius ρ. That is, every node in V is connected to every other node that is within a radius ρ. We denote by $\mathsf{Geom}(n, \rho)$ such random graph, and we denote by $\mathbb{P}_{n,\rho}$ the probability distribution of the random variable $\mathsf{Geom}(n, \rho)$. The notation extends to the expectation and higher moments as well.

Center Contractiveness. It is well known that geometric random graphs exhibit a *threshold behavior* regarding its connectivity []. More specifically, there exists a constant value $\theta \in [0, 1]$ such that for every $\varepsilon > 0$ and $\rho \geq \theta(1 + \varepsilon)\sqrt{\log(n)/n}$, the graph $\mathsf{Geom}(n, \rho)$ is connected with high probability [2]. And on the other hand, for every $\varepsilon > 0$ and $\rho \leq \theta(1 - \varepsilon)\sqrt{\log(n)/n}$, the graph $\mathsf{Geom}(n, \rho)$ is not connected with high probability. In the following we assume that $\rho > \rho_n = \theta\sqrt{\log(n)/n}$. The main result of this section is the following.

Victor: Incluir un mono de ejemplo, debe haber algun generador online o en python seguro

Theorem 2. *For every $n \geq n_0$ and for every radius $\rho \in (\rho_n, 1)$, the random geometric graph $\mathsf{Geom}(n, \rho)$ is $(1 - \gamma_n, 4n\rho^2 f^2)$-center contractive with high probability.*

That is, with high probability, the fraction of nodes in the random graph that belong to a center contractive set is equal to $1 - \gamma_n$. And furthermore, with high probability, the size of the center contractive set is $4n\rho^2 f^2$. Together with Theorem **??** we get the following corollary.

Victor: Creo que hay un tradeoff entre la probabilidad de un nodo de ser cubierto por un center contractive, y el numero total de nodos que son cubiertos por algun center contractive. Idealmente, queremos que casi todos sean cubiertos con alta probabilidad, pero eso dependera de cuan fuertes son las cotas en los lemas que vienen.

[2]We say that an event holds *with high probability* if it occurs with a probability of at least $1 - 1/n^k$ where $k \geq 2$.

Corollary 1. *Consider $n \geq n_0$ and a radius $\rho \in (\rho_n, 1)$. Then, with high probability, a $1-\gamma_n$ fraction of the nodes in the the random graph $\mathsf{Geom}(n,\rho)$ belongs to a set of at size at most $4n\rho^2 f^2$ that is robustly sustainable in the long-run.*

In the rest of section we prove Theorem 2.

Victor: incluir aca una breve discusion sobre como se compara este resultado con el Theorem 2 de Xandri

5.3 Contractiveness in Geometric Random Graphs: Proof of Theorem 2

The first property we state is that whenever a graph is sampled according to the geometric model, for every subset of nodes the shifting parameter is vanishing with high probability. In what follows, given $n \in \mathbb{Z}_+$ and $\rho \in (\rho_n, 1)$, for a node $u \in V_n$ we denote by $C_u = E(\mathsf{Geom}(n,\rho)) \cap B_1(u, \rho f)$ the set of nodes connected to u within a radius of ρf, with $f \in (0,1)$. Recall that $\mathsf{E}(C_u) = 4n\rho^2 f^2$. In what follows, we will study the probability of a sequence of events, and for clarity we define them before going the precise statements of our technical lemmas. Consider $n \in \mathbb{Z}_+$ and a radius $\rho \in (\rho_n, 1)$. For every $u \in V_n$, consider the following events.

(a) Ω_u is the event in which C_u is center contractive.

(b) Γ_u be the event in which $|C_u| = \Theta(n\rho^2 f^2)$.

(c) $\mathcal{H}_u$ is the event in which $\xi_{C_u} \leq 1/q(n)$.

(d) $\mathcal{K}_u$ is the event in which C_u is $(1/2 + 1/q(n))$-cohesive.

(e) $\mathcal{F}_u$ is the event in which for every $v \in \mathsf{N}(u) \setminus C_u$ we have that

$$|\Theta_{v,C_u-u} - \Theta_{u,C_u-u}| \leq \frac{1}{q(n)} O(1 - \Theta_{u,C_u-u}).$$

(f) $\mathcal{J}_u$ is the event in which $\Theta_{u,C_u-u} = O(\rho^2 f^2)$.

In order to prove the theorem above, we follow a similar strategy used by Chandrasekhar et al [] to bound the share of nodes belonging to *clans*. The following technical result is at the core of the argument to show the previous result, and can be of independent interest.

Theorem 3. *Let $n \geq n_3$, a radius $\rho \in (\rho_n, 1)$ and $u \in V_n$ such that $|C_u| = \Theta(n\rho^2 f^2)$. Then, the set C_u is ω-cohesive with probability at least $f(\rho, \omega, n)$.*

Before proving this result we show how to prove Theorem 2 We need a few more technical lemmas that we state next. We leave the proof of Theorem 3 in Section 5.4.

Lemma 2. *Let $n \geq n_3$, a radius $\rho \in (\rho_n, 1)$ and $u \in V_n$ such that $|C_u| = \Theta(n\rho^2 f^2)$. Then, $\xi_{C_u} \leq 1/q(n)$ with probability at least $1 - \kappa_n$.*

Proof. That is, we show next that $\mathbb{P}(\mathcal{H}_u | \Gamma_u) \geq 1 - \kappa_n$. In the following, we show that $\mathbb{P}_{n,\rho}(\mathcal{F}_u \cap \mathcal{J}_u | \Gamma_u) \geq 1 - \nu_n$. □

Victor: completar aca. La idea es ver que si C_u tiene un radio suficientemente grande entonces el hitting probability the C es harto mas grande que la diferencia entre los hitting partiendo de un punto u o un punto v vecino a u. Idealmente nos basta con $\rho = \Theta(\rho_n)$, pero si no lo agrandamos un poco. La diea tambien es que el grafo al ser random, no deberia cambiar mucho

Lemma 3. *Let $n \geq n_3$, a radius $\rho \in (\rho_n, 1)$ and $u \in V_n$ such that $|C_u| = \Theta(n\rho^2 f^2)$. Then, the set C_u is center contractive with probability at least $1 - \beta_n$, that is, $\mathbb{P}_{n,\rho}(\Omega_u | \Gamma_u) \geq 1 - \beta_n$.*

Proof of Lemma 3. In order to conclude, thanks to Proposition 4 it is enough to lower bound the probability that C_u is $(1/2 + 1/q(n))$-cohesive when $|C_u| = \Theta(n\rho^2 f^2)$. This comes from the following fact: When $\xi_{C_u} \leq 1/q(n)$, we have that $(1/2 + 1/q(n))$-cohesiveness implies $(1/2 + \xi_{C_u})$-cohesiveness. In particular, this implies that

$$\mathbb{P}_{n,\rho}(\Omega_u|\Gamma_u) \geq \mathbb{P}_{n,\rho}(\Omega_u \cap \mathcal{H}_u|\Gamma_u) \cdot \mathbb{P}_{n,\rho}(\mathcal{H}_u|\Gamma_u) \geq \mathbb{P}_{n,\rho}(\mathcal{K}_u \cap \mathcal{H}_u|\Gamma_u) \cdot \mathbb{P}_{n,\rho}(\mathcal{H}_u|\Gamma_u).$$

By Theorem 3 we have that with probability at least $f(\rho, 1/2+1/q(n), n)$ the set C_u is $(1/2+1/1(n))$-cohesive when $|C_u| = \Theta(n\rho^2 f^2)$, that is, $\mathbb{P}_{n,\rho}(\mathcal{K}_u|\Gamma_u) \geq f(\rho, 1/2 + 1/q(n), n)$. Together with the union bound and Lemma 2 we get that

$$\mathbb{P}_{n,\rho}(\mathcal{K}_u \cap \mathcal{H}_u|\Gamma_u) \geq 1 - \mathbb{P}_{n,\rho}(\mathcal{K}_u^c|\Gamma_u) - \mathbb{P}_{n,\rho}(\mathcal{H}_u^c|\Gamma_u) \geq f(\rho, 1/2 + 1/q(n), n) - \kappa_n.$$

Finally, overall we get that

$$\mathbb{P}_{n,\rho}(\Omega_u|\Gamma_u) \geq \Big(f(\rho, 1/2 + 1/q(n), n) - \kappa_n\Big)(1 - \kappa_n) = 1 - \beta_n. \qquad \square$$

Lemma 4. *Let $n \geq n_3$, a radius $\rho \in (\rho_n, 1)$ and $u \in V_n$. Then, we have that $2n\rho^2 f^2 \leq |C_u| \leq 6n\rho^2 f^2$ with probability at least $1 - 2e^{-n\rho^2 f^2/3}$.*

Proof. That is, we show next that $\mathbb{P}_{n,\rho}(\Gamma_u) \geq 1 - 2e^{-n\rho^2 f^2/3}$. In our analysis we use the following concentration bound for a Poisson random variable. The proof of this fact can be found in the Appendix.

Claim 1. *Consider a random variable $X \sim Poisson(\lambda)$ with $\lambda > 0$. Then, $\mathbb{P}(|X - \lambda| > \lambda/2) \leq 2e^{-\lambda/12}$.*

Using the claim for the random variable $|C_u| \sim \text{Poisson}(4n\rho^2 f^2)$ we have that

$$\mathbb{P}_{n,\rho}\Big(|C_u - 4n\rho^2 f^2| > 2n\rho^2 f^2\Big) \leq 2e^{-n\rho^2 f^2/3},$$

which concludes the proof. $\square$

Proof of Theorem 2. Fix a node $u \in V_n$. In order to conclude the result it is enough to show that $\mathbb{P}_{n,\rho}(\Omega_u \cap \Gamma_u) \geq 1 - \gamma_n$. The theorem follows by conditioning and using Lemmas 3 and 4,

$$\mathbb{P}_{n,\rho}(\Omega_u \cap \Gamma_u) = \mathbb{P}_{n,\rho}(\Omega_u|\Gamma_u) \cdot \mathbb{P}_{n,\rho}(\Gamma_u) \geq (1 - \beta_n)\Big(1 - 2e^{-n\rho^2 f^2}\Big) = 1 - \gamma_n. \qquad \square$$

5.4 Cohesiveness in Geometric Random Graphs: Proof of Theorem 3

Extender (pero a la vez simplificar) el analisis de Chandrasekhar et al entre pags 44-48, y tambien necesitamos estudiar cual es probabilidad de que efectivamente un nodo este en un conjunto 1/2-cohesive. Para eso hay que estudiar la cota inferior, idealmente tenemos probabilidad tendiendo a 1 simptoticamente, sino prob constante esta bien igual.

6 Cohesiveness and Spectral Gap

It will be convenient to express the cost function in matrix form. For every $u \in V$, consider the matrix $\mathsf{A}_u \in \mathsf{R}^{V \times V}$ such that $\mathsf{A}_u(u,v) = -1$ when $v \in \mathsf{N}(u)$ and zero otherwise; and let $\mathsf{D} \in \mathsf{R}^{V \times V}$ the diagonal matrix such that $\mathsf{D}(u,u) = \mathsf{deg}(u)$ for every $u \in V$. Then, for every $y \in \mathsf{R}^V$ we have that

$$\mathsf{cost}_u(y) = \frac{1}{2} y^\top \mathsf{L}_u y,$$

where $\mathsf{L}_u = \mathsf{D} - \mathsf{A}_u$ and we call this matrix the *local Laplacian* for node $u \in V$. In particular, the matrix $\mathsf{L} = \sum_{u \in V} \mathsf{L}_u$ is known as the *graph Laplacian* of G.

Spectral Gap and Cheeger's Inequality.

Example 4.

References

7 Appendix

Proof of Lemma 1. □

Proof of Claim 1. □