Opinion Dynamics with Limited Information

Dimitris Fotakis[1,2], Vardis Kandiros[2], Vasilis Kontonis[3], and Stratis Skoulakis[2(✉)]

[1] Yahoo Research, New York, NY, USA
dfotakis@oath.com, fotakis@cs.ntua.gr
[2] National Technical University of Athens, Athens, Greece
{vkandiros,sskoul}@corelab.ntua.gr
[3] University of Wisconsin-Madison, Madison, WI, USA
kontonis@cs.wisc.edu

Abstract. We study opinion formation games based on the Friedkin-Johnsen (FJ) model. We are interested in simple and natural variants of the FJ model that use limited information exchange in each round and converge to the same stable point. As in the FJ model, we assume that each agent i has an intrinsic opinion $s_i \in [0, 1]$ and maintains an expressed opinion $x_i(t) \in [0, 1]$ in each round t. To model limited information exchange, we assume that each agent i meets with one random friend j at each round t and learns only $x_j(t)$. The amount of influence j imposes on i is reflected by the probability p_{ij} with which i meets j. Then, agent i suffers a disagreement cost that is a convex combination of $(x_i(t) - s_i)^2$ and $(x_i(t) - x_j(t))^2$.

An important class of dynamics in this setting are *no regret* dynamics. We show an exponential gap between the convergence rate of no regret dynamics and of more general dynamics that do not ensure no regret. We prove that no regret dynamics require roughly $\Omega(1/\varepsilon)$ rounds to be within distance ε from the stable point x^* of the FJ model. On the other hand, we provide an opinion update rule that does not ensure no regret and converges to x^* in $\tilde{O}(\log^2(1/\varepsilon))$ rounds. Finally, we show that the agents can adopt a simple opinion update rule that ensures no regret and converges to x^* in $\text{poly}(1/\varepsilon)$ rounds.

1 Introduction

The study of *Opinion Formation* has a long history [22]. Opinion Formation is a *dynamic process* in the sense that socially connected people (family, friends, colleagues) exchange information and this leads to changes in their expressed opinions over time. Today, the advent of the internet and social media makes

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the study of opinion formation in large social networks even more important; realistic models of how people form their opinions are of great practical interest for prediction, advertisement etc. In an attempt to formalize the process of opinion formation, several models have been proposed (see e.g. [8,15,20]). The common assumption underlying all these models, which dates back to DeGroot [8], is that opinions evolve through a form of repeated averaging of information collected from the agents' social neighborhoods.

Our work builds on the model proposed by Friedkin and Johnsen [15]. The FJ model is a variation on the DeGroot model capturing the fact that consensus on the opinions is rarely reached. According to FJ model each person i has a public opinion $x_i \in [0,1]$ and an internal opinion $s_i \in [0,1]$, which is private and invariant over time. There also exists a weighted graph $G(V,E)$ representing a social network where V stands for the persons ($|V| = n$) and E their social relations. Initially, all nodes start with their internal opinion and at each round t, update their public opinion $x_i(t)$ to a weighted average of the public opinions of their neighbors and their internal opinion,

$$x_i(t) = \frac{\sum_{j \in N_i} w_{ij} x_j(t-1) + w_{ii} s_i}{\sum_{j \in N_i} w_{ij} + w_{ii}}, \tag{1}$$

where $N_i = \{j \in V : (i,j) \in E\}$ is the set of i's neighbors, the weight w_{ij} associated with the edge $(i,j) \in E$ measures the extent of the influence that j poses on i and the weight $w_{ii} > 0$ quantifies how susceptible i is in adopting opinions that differ from her internal opinion s_i.

The FJ model is one of most influential models for opinion formation. It has a very simple update rule, making it plausible for modeling natural behavior and its basic assumptions are aligned with empirical findings on the way opinions are formed [25]. At the same time, it admits a unique stable point $x^* \in [0,1]^n$ to which it converges with a *linear* rate [16]. The FJ model has also been studied under a game theoretic viewpoint. Bindel et al. considered its update rule as the minimizer of a quadratic disagreement cost function and based on it they defined the following opinion formation game [5]. Each node i is a selfish agent whose strategy is the public opinion x_i that she expresses incurring her a disagreement cost

$$C_i(x_i, x_{-i}) = \sum_{j \in N_i} w_{ij}(x_i - x_j)^2 + w_{ii}(x_i - s_i)^2 \tag{2}$$

Note that the FJ model is the *simultaneous best response dynamics* and its stable point x^* is the unique Nash equilibrium of the above game. In [5] they quantified its inefficiency with respect to the total disagreement cost. They proved that the *Price of Anarchy* (PoA) is 9/8 in case G is undirected and $w_{ij} = w_{ji}$. They also provided PoA bounds in the case of unweighted Eulerian directed graphs. We remark that in [5] an alternative framework for studying the way opinions evolve was introduced. The opinion formation process can be described as the *dynamics* of an opinion formation game. This framework is much more comprehensive since different aspects of the opinion formation process can be easily captured by defining suitable games. Subsequent works [3,4,9] considered variants of the above game and studied the convergence properties of the *best response dynamics*.

1.1 Motivation and Our Setting

Many recent works study the Nash equilibrium x^* of the opinion formation game defined in [5] under various perspectives. In [6] they extended the bounds for PoA in more general classes of directed graphs, while recently introduced influence maximization problems [1,17], which are defined with respect to x^*. The reason for this scientific interest is evident: the equilibrium x^* is considered as an appropriate way to model the final opinions formed in a social network, since the *well established* FJ model converges to it.

Our work is motivated by the fact that there are notable cases in which the FJ model is not an appropriate model for the dynamic of the opinions, due to the large amount of information exchange that it implies. More precisely, at each round its update rule (1) requires that every agent learns all the opinions of her social neighbors. In today's large social networks where users usually have several hundreds of friends it is highly unlikely that, each day, they learn the opinions of all their social neighbors. In such environments it is far more reasonable to assume that individuals randomly meet a small subset of their acquaintances and these are the only opinions that they learn. Such information exchange constraints render the FJ model unsuitable for modeling the opinion formation process in such large networks and therefore, it is not clear whether x^* captures the limiting behavior of the opinions. In this work we ask:

Question 1. Is the equilibrium x^* an adequate way to model the final formed opinions in large social networks? Namely, are there simple variants of the FJ model that require limited information exchange and converge fast to x^*? Can they be justified as natural behavior for selfish agents under a game-theoretic solution concept?

To address these questions, one could define precise dynamical processes whose update rules require limited information exchange between the agents and study their convergence properties. Instead of doing so, we describe the opinion formation process in such large networks as *dynamics* of a suitable opinion formation game that captures these information exchange constraints. This way we can precisely define which *dynamics* are *natural* and, more importantly, to study general classes of *dynamics* (e.g. no regret dynamics) without explicitly defining their update rule. The opinion formation game that we consider is a variant of the game in [5] based on interpreting the weight w_{ij} as a measure of how frequently i meets j.

Definition 1. *For a given opinion vector $x \in [0,1]^n$, the disagreement cost of agent i is the random variable $C_i(x_i, x_{-i})$ defined as follows:*

- *Agent i meets one of her neighbors j with probability $p_{ij} = w_{ij} / \sum_{j \in N_i} w_{ij}$.*

– *Agent i suffers cost* $C_i(x_i, x_{-i}) = (1-\alpha_i)(x_i - x_j)^2 + \alpha_i(x_i - s_i)^2$, *where* $\alpha_i = w_{ii}/(\sum_{j \in N_i} w_{ij} + w_{ii})$.

Note that the expected disagreement cost of each agent in the above game is the same as the disagreement cost in [5] (scaled by $\sum_{j \in N_i} w_{ij} + w_{ii}$). Moreover its Nash equilibrium, with respect to the expected disagreement cost, is x^*. This game provides us with a general template of all the *dynamics* examined in this paper. At round t, each agent i selects an opinion $x_i(t)$ and suffers a disagreement cost based on the opinion of the neighbor that she randomly met. At the end of round t, she is informed only about the opinion and the index of this neighbor and may use this information to update her opinion in the next round. Obviously different update rules lead to different *dynamics*, however all of these respect the information exchange constraints: at every round each agent learns the opinion of *just one* of her neighbors. Question 1 now takes the following more concrete form.

Question 2. Can the agents update their opinions according to the limited information that they receive such that the produced opinion vector $x(t)$ converges to the equilibrium x^*? How is the convergence rate affected by the limited information exchange? Are there dynamics that ensure that the cost that the agents experience is minimal?

In what follows, we are mostly concerned about the dependence of the rate of convergence on the distance ε from the equilibrium x^*. Thus, we shall suppress the dependence on other parameters such as the size of the graph, n. We remark that the dependence of our dynamics on these constants is in fact rather good (see Sect. 2), and we do this only for clarity of exposition.

Definition 2 (Informal). *We say that a dynamics converges slowly resp. fast to the equilibrium x^* if it requires* $\mathrm{poly}(1/\varepsilon)$ *resp.* $\mathrm{poly}(\log(1/\varepsilon))$ *rounds to be within error ε.*

1.2 Contribution

The major contribution of the paper is proving an exponential separation on the convergence rate of *no regret dynamics* and the convergence rate of more general dynamics produced by update rules that do not ensure no regret.

No regret dynamics are produced by update rules that ensure no regret to any agent that adopts them. In our setting such an update rule must ensure that the total disagreement cost of an agent that adopts it is close to the total disagreement cost that she would experience by selecting the best fixed opinion in hindsight. The latter must hold even if the identities and the opinions of the neighbors that the agent meets are chosen adversarially. We prove that if all the agents adopt an update rule that ensures *no regret*, then there exists an instance of the game such that the produced opinion vector $x(t)$ requires roughly $\Omega(1/\varepsilon)$ rounds to be ε-close to x^*. The reason is that by definition such update rules only depend on the opinions that the agent observes and don't take into account the weights w_{ij} of the outgoing edges (see Sect. 5). We call the update rules with the latter property, *graph oblivious*. In Sect. 5 we use a novel information theoretic argument to prove the aforementioned lower bound for this more general class.

In Sect. 6, we present a simple update rule whose resulting dynamics converges fast, i.e. the opinion vector $x(t)$ is ε-close to x^* in $O(\log^2(1/\varepsilon))$ rounds. The reason that the previous lower bound doesn't apply is that this rule is not *graph oblivious* and it does not ensure no regret to the agents that adopt it. In fact there is a very simple example with two agents, in which the first follows the rule while the second selects her opinions adversarially, where the first agent experiences regret.

We introduce an intuitive *no regret* update rule and we show that if all agents adopt it, the resulting opinion vector $x(t)$ converges to x^*. Our rule is a *Follow the Leader algorithm*, meaning that at round t, each agent updates her opinion to the minimizer of total disagreement cost that she experienced until round $t-1$. It also has a very simple form: it is roughly the time average of the opinions that the agent observes. In Sect. 3, we bound its convergence rate and show that in order to achieve ε distance from x^*, $\text{poly}(1/\varepsilon)$ rounds are sufficient. In view of our lower bound this rate is close to best possible. In Sect. 4, we prove its *no regret* property. This can be derived by the more general results in [19]. However, we give a short and simple proof that may be of interest.

In conclusion, our results reveal that the equilibrium x^* is a robust choice for modeling the limiting behavior of the opinions of agents since, even in our limited information setting, there exist simple and natural dynamics that converge to it. The convergence rate crucially depends on whether the agents act selfishly, i.e. they are only concerned about their individual disagreement cost.

1.3 Related Work

There exists a large amount of literature concerning the FJ model. Many recent works [3,4,9] bound the inefficiency of equilibrium in variants of opinion formation game defined in [5]. In [16] they bound the convergence time of the FJ model in special graph topologies. In [4] they proved that best response converges to PNE for a variant of the opinion formation game, in which social relations depend on the expressed opinions. Convergence results in other discretized variants of the FJ model can be found in [11,28]. In [13] they provide convergence results for a limited information variant of the FJ model. Although the considered variant is very similar to ours, their convergence results are much weaker, since they concern the expected value of the opinion vector.

Other works that relate to ours concern the convergence properties of dynamics based on no regret learning algorithms. In [12,14,27] it is proved that in a finite n-person game, if each agent updates her mixed strategy according to a no regret algorithm, the resulting *time-averaged* strategy vector converges to Coarse Correlated Equilibrium. The convergence properties of no regret dynamics for games with infinite strategy spaces were considered in [10]. They proved that for a large class of games with concave utility functions (socially concave games), the time-averaged strategy vector converges to Pure Nash Equilibrium. More recent work investigates a stronger notion of convergence of no regret dynamics. In [7] they show that, in n-person finite generic games that admit unique Nash equilibrium, the strategy vector converges *locally* and fast to it. They also provide conditions for *global* convergence. Our results fit in this line of research since we show that for a game with *infinite* strategy space, the strategy vector (and not the time-averaged) converges to the Nash equilibrium x^*.

No regret dynamics in limited information settings have recently received substantial attention from the scientific community since they provide realistic models for the practical applications of game theory. Kleinberg et al. in [23] treated load-balancing in distributed systems as a repeated game and analyzed the convergence properties of no regret learning algorithms under the *full information assumption* that each agent learns the load of every machine. In a subsequent work [24], the same authors consider the same problem in a *limited information setting* ("bulletin board model"), in which each agent learns the load of just the machine that served him. Most relevant to ours are the works [7,21,26], where they examine the convergence properties of no regret learning algorithms when the agents observe their payoffs with some additive zero-mean random noise.

2 Our Results and Techniques

As previously mentioned, an instance of the game in [5] is also an instance of the game of Definition 1. Following the notation introduced earlier we have that if $j \in N_i$ then $p_{ij} = w_{ij}/\sum_{j\in N_i} w_{ij}$ and 0 otherwise. Moreover since $w_{ii} > 0$ by the definition of the game in [5], $\alpha_i = w_{ii}/(\sum_{j\in N_i} w_{ij} + w_{ii}) > 0$. If an agent i does not have outgoing edges ($N_i = \emptyset$) then $p_{ij} = 0$ for all j. Therefore $\sum_{j=1}^n p_{ij} = 0$, $\alpha_i = 1$ if $N_i = \emptyset$ and $\sum_{j=1}^n p_{ij} = 1$, $\alpha_i \in (0,1)$ otherwise. For simplicity we adopt the following notation for an instance of the game of Definition 1.

Definition 3. *We denote an instance of the opinion formation game of Definition 1 as $I = (P, s, \alpha)$, where P is a $n \times n$ matrix with non-negative elements p_{ij}, with $p_{ii} = 0$ and $\sum_{j=1}^n p_{ij}$ is either 0 or 1, $s \in [0,1]^n$ is the internal opinion vector, $\alpha \in (0,1]^n$ the self confidence vector.*

An instance $I = (P, s, \alpha)$ is also an instance of the FJ model, since by the update rule (1) $x_i(t) = (1-\alpha_i)\sum_{j\in N_i} p_{ij}x_j(t-1) + \alpha_i s_i$. It also defines the opinion vector $x^* \in [0,1]^n$ which is the stable point of the FJ model and the Nash equilibrium of the game in [5].

Definition 4. *For a given instance $I = (P, s, \alpha)$ the equilibrium $x^* \in [0,1]^n$ is the unique solution of the following linear system, for every $i \in V$, $x_i^* = (1-\alpha_i)\sum_{j\in N_i} p_{ij}x_j^* + \alpha_i s_i$.*

It is easy to check that the above definition of x^* is equivalent to defining it as the PNE of the game in [5] or as the stable point of the FJ model. The fact that the above linear system always admits a solution follows by standard matrix norm properties.

Throughout the paper we study *dynamics* of the game of Definition 1. We denote as W_i^t the neighbor that agent i met at round t, which is a random variable whose probability distribution is determined by the instance $I = (P, s, \alpha)$ of the game, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$. Another parameter of an instance I that we often use is $\rho = \min_{i \in V} \alpha_i$.

In Sect. 3, we examine the convergence properties of the opinion vector $x(t)$ when all agents update their opinions according to the *Follow the Leader* principle. Since each agent i must select $x_i(t)$, before knowing which of her neighbors she will meet and what opinion her neighbor will express, this update rule says "*play the best according to what you have observed*". For a given instance (P, s, a) of the game dynamics $x(t)$ is defined in Dynamics 1 and Theorem 1 shows its convergence rate to x^*.

Dynamics 1 Follow the Leader dynamics

Initially $x_i(0) = s_i$ for all agents i.
At round $t \geq 0$ each agent i:
  Meets neighbor with index W_i^t, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$.
  Suffers cost $(1 - \alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + \alpha_i(x_i(t) - s_i)^2$ and learns $x_{W_i^t}(t)$.
  Updates her opinion as follows:

$$x_i(t+1) = \operatorname*{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t} [(1 - \alpha_i)(x - x_{W_i^\tau}(\tau))^2 + \alpha_i(x - s_i)^2] \tag{3}$$

Theorem 1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (3) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C\sqrt{\log n}\frac{(\log t)^{3/2}}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} \alpha_i$ and C is a universal constant.

In Sect. 4 we argue that, apart from its simplicity, update rule (3) ensures no regret to any agent that adopts it and therefore the FTL dynamics can be considered as *natural dynamics* for selfish agents. Since each agent i selfishly wants to minimize the disagreement cost that she experiences, it is natural to assume that she selects $x_i(t)$ according to a *no regret algorithm* for the *online convex optimization problem* where the adversary chooses a function $f_t(x) = (1 - \alpha_i)(x - b_t)^2 + \alpha_i(x - s_i)^2$ at each round t. In Theorem 2 we prove that *Follow the Leader* is a no regret algorithm for the above OCO problem. We remark that this does not hold, if the adversary can pick functions from a different class (see e.g. chapter 5 in [18]).

Theorem 2. *Consider the function* $f : [0,1]^2 \mapsto [0,1]$ *with* $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ *for some constants* $s, \alpha \in [0,1]$*. Let* $(b_t)_{t=0}^{\infty}$ *be an arbitrary sequence with* $b_t \in [0,1]$*. If* $x_t = \operatorname{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ *then for all* $t \geq 0$*,* $\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + O(\log t)$*.*

On the positive side, the FTL dynamics converges to x^* and its update rule is simple and ensures no regret to the agents. On the negative side, its convergence rate is outperformed by the rate of FJ model. For a fixed instance $I = (P, s, \alpha)$, the FTL dynamics converges with rate $\widetilde{O}(1/t^{\min(\rho,1/2)})$ while FJ model converges with rate $O(e^{-\rho t})$ [16].

Question 3. Can the agents update their opinions according to other no regret algorithms such that the resulting dynamics converges fast to x^*?

The answer is no. In Sect. 5, we prove that fast convergence cannot be established for any opinion vector, produced by a no regret algorithm for the above online convex problem. The reason that FTL dynamics converges slowly is that rule (3) only depends on the opinions of the neighbors that agent i meets, α_i, and s_i. This is by definition true for any update rule based on a no regret algorithm (see Sects. 4 and 5). As already mentioned, we call this larger class of update rules *graph oblivious*, and we prove that fast convergence cannot be established for *graph oblivious dynamics*.

Definition 5 (graph oblivious update rule). *A graph oblivious update rule* A *is a sequence of functions* $(A_t)_{t=0}^{\infty}$ *where* $A_t : [0,1]^{t+2} \mapsto [0,1]$*.*

Definition 6 (graph oblivious dynamics). *Let a graph oblivious update rule* A*. For a given instance* $I = (P, s, \alpha)$ *the rule* A *produces a graph oblivious dynamics* $x_A(t)$ *defined as follows:*

- *Initially each agent* i *selects her opinion* $x_i^A(0) = A_0(s_i, \alpha_i)$
- *At round* $t \geq 1$*, each agent* i *updates her opinion as follows:*

$$x_i^A(t) = A_t(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), s_i, \alpha_i),$$

where W_i^t *is the neighbors that* i *meets at round* t*.*

Theorem 3 states that for any graph oblivious dynamics there exists an instance $I = (P, s, \alpha)$, where roughly $\Omega(1/\varepsilon)$ rounds are required to achieve convergence within error ε.

Theorem 3. *Let* A *be a* graph oblivious *update rule, which all agents use to update their opinions. For any* $c > 0$ *there exists an instance* $I = (P, s, a)$ *such that* $\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = \Omega(1/t^{1+c})$*, where* $x_A(t)$ *denotes the opinion vector produced by* A *for the instance* $I = (P, s, \alpha)$*.*

To prove Theorem 3, we show that graph oblivious rules whose dynamics converge fast imply the existence of estimators for Bernoulli distributions with "small" sample complexity. The key part of the proof lies in Lemma 6, in which it is proven that such estimators cannot exist.

In Sect. 6, we present a simple update rule that achieves error rate $\mathrm{e}^{-\widetilde{O}(\sqrt{t})}$. This update rule is a function of the opinions and the indices of the neighbors that i met, s_i, α_i and the i-th row of the matrix P. Obviously this rule is not *graph oblivious*, due to its dependency on the i-th row and the indices. However it reveals that slow convergence is not a generic property of the limited information dynamics, but comes with the assumption that agents act selfishly.

3 Convergence Rate of FTL Dynamics

In this section we present the high level idea for proving Theorem 1. We note that since the produced opinion vector $x(t)$ is a random vector, the convergence metric used in Theorem 1 is $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$, where the expectation is taken over the random meeting of the agents. At first notice that update rule (3) can be equivalently written as

$$x_i(t) = (1 - \alpha_i) \sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)/t + \alpha_i s_i,$$

where $W_i(\tau)$ is the neighbor that i met at round τ. Using the fact that $x_i^* = (1 - \alpha_i) \sum_{j \in N_i} p_{ij} x_j^* + \alpha_i s_i$, one can prove that

$$|x_i(t) - x_i^*| \leq (1 - \alpha_i) \sum_{j \in N_i} \left| \frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j] x_j(\tau)}{t} - p_{ij} x_j^* \right|$$

Now assume that $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}| = 0$ for all $t \geq 1$, then we easily get that $\|x(t) - x^*\|_\infty \leq e(t)$ where $e(t)$ satisfies the recursive equation $e(t) = (1 - \rho) \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$ and $\rho = \min_{i \in V} \alpha_i$. It follows that $\|x(t) - x^*\|_\infty \leq 1/t^\rho$. Obviously the latter assumption does not hold, however since W_i^τ are independent random variables with $\mathbf{P}\left[W_i^\tau = j\right] = p_{ij}$, $|\frac{\sum_{\tau=0}^{t-1} \mathbf{1}[W_i^\tau = j]}{t} - p_{ij}|$ tends to 0 with probability 1. In Lemma 1 we use this fact to obtain a similar recursive equation for $e(t)$ and then in Lemma 2 we upper bound its solution.

Lemma 1. *Let $e(t)$ the solution of the recursion $e(t) = \delta(t) + (1 - \rho) \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$ where $e(0) = \|x(0) - x^*\|_\infty$, $\delta(t) = \sqrt{\ln(\pi^2 n t^2 / 6p)/t}$ and $\rho = \min_{i \in V} \alpha_i$. Then,*

$$\mathbf{P}\left[\textit{for all } t \geq 1,\ \|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - p$$

Lemma 2. *Let $e(t)$ be a function satisfying the recursion $e(t) = \delta(t) + (1 - \rho) \sum_{\tau=0}^{t-1} e(\tau)/t$ and $e(0) = \|x(0) - x^*\|_\infty$ where $\delta(t) = \sqrt{\ln(Dt^{2.5})/t}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then $e(t) \leq \sqrt{2\ln(D)} \frac{(\ln t)^{3/2}}{t^{\min(\rho,\, 1/2)}}$.*

4 Follow the Leader Ensures No Regret

In this section we provide rigorous definitions of *no regret* algorithms and explain why update rule (3) ensures no regret to any agent that repeatedly plays the game of Definition 1. Based on the cost that the agents experience, we consider an appropriate *Online Convex Optimization* problem. This problem is a "game" played between an adversary and a player. *At round* $t \geq 0$,

1. *the player selects a value* $x_t \in [0, 1]$.
2. *the adversary observes the* x_t *and selects a* $b_t \in [0, 1]$
3. *the player receives cost* $f(x_t, b_t) = (1 - \alpha)(x_t - b_t)^2 + \alpha(x_t - s)^2$.

where s, α are constants in $[0, 1]$. The goal of the player is to pick x_t based on the history $(b_0, \ldots, b_{t-1})$ in a way that the total cost that she suffers during the "game" is minimized. Generally, different OCO problems can be defined by a set of functions $\mathcal{F}$ that the adversary chooses from and a feasibility set $\mathcal{K}$ from which the player picks her value (see [18] for an introduction to the OCO framework). In our case the feasibility set is $\mathcal{K} = [0, 1]$ and the set of functions is $\mathcal{F}_{s,\alpha} = \{x \mapsto (1 - \alpha)(x - b)^2 + \alpha(x - s)^2 : b \in [0, 1]\}$. As a result, each selection of the constants s, α leads to a different OCO problem.

Definition 7. *An algorithm* A *for the OCO problem with* $\mathcal{F}_{s,\alpha}$ *and* $\mathcal{K} = [0, 1]$ *is a sequence of functions* $(A_t)_{t=0}^{\infty}$ *where* $A_t : [0, 1]^t \mapsto [0, 1]$.

Definition 8. *An algorithm* A *is no regret for the OCO problem with* $\mathcal{F}_{s,\alpha}$ *and* $\mathcal{K} = [0, 1]$ *if and only if for all sequences* $(b_t)_{t=0}^{\infty}$, *if* $x_t = A_t(b_0, \ldots, b_{t-1})$ *then for all* t, $\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + o(t)$.

Informally speaking, if the player selects the value x_t according to a *no regret algorithm* then she does not regret not playing any fixed value no matter what the choices of the adversary are. Theorem 2 states that *Follow the Leader* i.e. $x_t = \mathrm{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ is a no regret algorithm for all the OCO problems with $\mathcal{F}_{s,\alpha}$.

Returning to the dynamics of the game in Definition 1, it is natural to assume that each agent i selects $x_i(t)$ by a no regret algorithm A_i for the OCO problem with $\mathcal{F}_{s_i,\alpha_i}$, since

$$\frac{1}{t} \sum_{\tau=0}^{t} f_i(x_i(\tau), x_{W_i^\tau}(\tau)) \leq \frac{1}{t} \min_{x \in [0,1]} \sum_{\tau=0}^{t} f_i(x, x_{W_i^\tau}(\tau)) + \frac{o(t)}{t}$$

The latter means that the time averaged total disagreement cost that she suffers is close to the time averaged cost by expressing the best fixed opinion and this holds regardless of the opinions of the neighbors that i meets. Meaning that even if the other agents selected their opinions maliciously, her total experienced cost would still be in a sense minimal. Under this perspective update rule (3) is a rational choice for selfish agents and as a result FTL dynamics is a *natural* limited information variant of the FJ model.

We now present the key steps for proving Theorem 2. We first prove that a similar strategy that also takes into account the value b_t admits no regret.

Lemma 3. *Let* $(b_t)_{t=0}^{\infty}$ *be an arbitrary sequence with* $b_t \in [0,1]$. *Then for all* t, $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \le \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$, *where* $y_t = \mathrm{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, b_\tau)$.

Obviously the player cannot know b_t before selecting her value, however we can now understand why *Follow the Leader* admits no regret. Since the cost incurred by the sequence y_t is at most that of the best fixed value, we can compare the cost of x_t with that of y_t. Since the functions in $\mathcal{F}_{s,\alpha}$ are quadratic, the extra term $f(x, b_t)$ that y_t takes into account doesn't change dramatically the minimizer of the total sum i.e. x_t, y_t are relatively close.

Lemma 4. *For all* $t \ge 0$, $f(x_t, b_t) \le f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.

5 Lower Bound for Graph Oblivious Dynamics

In this section we prove that *graph oblivious dynamics* cannot converge much faster than FTL dynamics (Dynamics 1). The reason that this class is of particular interest is that it contains the dynamics produced by any no regret algorithm of the online convex optimation problem presented in the previous section.

Definition 9 (no regret dynamics). *Consider a collection of no regret algorithms such that for each* $(s,\alpha) \in [0,1]^2$ *a no regret algorithm* $A_{s,\alpha}$[1] *for the OCO problem with* $\mathcal{F}_{s,\alpha}$ *and* $\mathcal{K} = [0,1]$, *is selected. For a given instance* $I = (P, s, \alpha)$ *this selection produces the no regret dynamics* $x(t)$ *defined as follows:*

- *Initially each agent* i *selects her opinion* $x_i(0) = A_0^{s_i,\alpha_i}(s_i, \alpha_i)$
- *At round* $t \ge 1$, *each agent* i *selects her opinion,*

$$x_i(t) = A_t^{s_i,\alpha_i}(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), s_i, \alpha_i),$$

where W_i^t *is the neighbor that* i *meets at round* t.

Such a selection of no regret algorithms can be encoded as a graph oblivious update rule. Specifically, the function $A_t : \{0,1\}^{t+2} \mapsto [0,1]$ is defined as $A_t(b_0, \ldots, b_{t-1}, s, \alpha) = A_{s,\alpha}^t(b_0, \ldots, b_{t-1})$. Thus, Theorem 3 applies. For example if agents use the Online Gradient Descent[2] to update their opinion i.e. $x_i(t+1) = x_i(t) - 1/\sqrt{t}(x_i(t) - (1-\alpha_i)x_{W_i^t}(t) - \alpha_i s_i)$. Then we are ensured that fast convergence cannot be established in the respective no regret dynamics. The rest of the section is dedicated to prove Theorem 3. In Lemma 5 we show that any graph oblivious update rule A can be used as an estimator of the parameter $p \in [0,1]$ of a Bernoulli random variable. Before proceeding we briefly introduce some definitions and notation.

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[1] These s, α are scalars in $[0,1]$ and should not be confused with the internal opinion vector s and the self confidence vector α of an instance $I = (P, s, \alpha)$.

[2] Online Gradient Descent is an influential no regret algorithm proposed by Zinkevic in [29] for the general OCO problem, where the adversary can select any convex function with bounded gradient. The latter directly implies that it also ensures no regret in our simpler OCO problem with $\mathcal{F}_{s_i,\alpha_i}$ and $\mathcal{K} = [0,1]$.

Definition 10. *An estimator* $\theta = (\theta_t)_{t=1}^{\infty}$ *is a sequence of functions* θ_t *where* $\theta_t : \{0,1\}^t \mapsto [0,1]$.

Perhaps the first estimator that comes to one's mind is the *sample mean*, that is $\theta_t = \sum_{i=1}^{t} X_i/t$. To measure the efficiency of an estimator we define the *risk*, which corresponds to the expected error of an estimator.

Definition 11. *Let* P *be a Bernoulli distribution with mean* p *and* P^t *be the corresponding* t*-fold product distribution. The risk of an estimator* $\theta = (\theta_t)_{t=1}^{\infty}$ *is* $\mathbf{E}_{(X_1,\ldots,X_t)\sim P^t}\left[|\theta_t(X_1,\ldots,X_t) - p|\right]$ *or* $\mathbf{E}_p\left[|\theta_t - p|\right]$ *for brevity.*

Obviously since p is unknown, any meaningful estimator $\theta = (\theta_t)_{t=1}^{\infty}$ must guarantee that for all $p \in [0,1]$, $\lim_{t\to\infty} \mathbf{E}_p\left[|\theta_t - p|\right] = 0$. For example, *sample mean* has error rate $\mathbf{E}_p\left[|\theta_t - p|\right] \le \frac{1}{2\sqrt{t}}$.

Lemma 5. *Let* A *a graph oblivious update rule such that for all instances* $I = (P, s, \alpha)$, $\lim_{t\to\infty} t^{1+c}\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = 0$. *Then there exists an estimator* $\theta_A = (\theta_t^A)_{t=1}^{\infty}$ *such that for all* $p \in [0,1]$, $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t^A - p|\right] = 0$.

Now in order to prove Theorem 3 we just need to prove the following claim.

Claim. For any estimator θ there exists p such that $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p\right] > 0$.

The above claim states that for any estimator $\theta = (\theta_t)_{t=1}^{\infty}$, we can inspect the functions $\theta_t : \{0,1\}^t \mapsto [0,1]$ and then choose a $p \in [0,1]$ such that the function $\mathbf{E}_p\left[|\theta_t - p|\right] = \Omega(1/t^{1+c})$. The claim follows by Lemma 6, which states something significantly stronger: for *almost all* $p \in [0,1]$, any estimator θ cannot achieve rate $o(1/t^{1+c})$.

Lemma 6. *Let* $\theta = (\theta_t)_{t=1}^{\infty}$ *be a Bernoulli estimator with error rate* $\mathbf{E}_p\left[|\theta_t - p|\right]$. *For any* $c > 0$, *if we select* p *uniformly at random in* $[0,1]$ *then with probability 1,* $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] > 0$.

6 Limited Information Dynamics with Fast Convergence

In this section we provide an update rule that is not *graph oblivious* and converges exponentially fast to x^*. This rule is based on asynchronous distributed minimization algorithms [2] and depends not only on the opinions of the neighbors that an agent i meets, but also on the i-th row of matrix P.

In this case each agent stores the *most recent* opinions of the neighbors that she meets in an array and then updates her opinion to their weighted sum (each agent knows row i of P). For a given instance $I = (P, s, \alpha)$ we call the produced dynamics *Row Dependent dynamics* (Dynamics 2). Now the problem is that the opinions of the neighbors that she keeps in her array are *outdated*, i.e. a neighbor of agent i may have changed opinion since their last meeting. The good news are that as long as this outdatedness is bounded we can still achieve fast convergence (Lemma 7). By bounded outdatedness we mean that there exists a number B such that all agents have met all their neighbors at least once from $t - B$ to t.

Dynamics 2 Row Dependent dynamics

1: Initially $x_i(0) = s_i$ for all agent i.
2: Each agent i keeps an array M_i of length $|N_i|$, randomly initialized.
3: At round $t \geq 0$ each agent i:
  4: Meets neighbor with index W_i^t, $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$.
  5: Suffers cost $(1-\alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + \alpha_i(x_i(t) - s_i)^2$ and learns $(x_{W_i^t}(t), W_i^t)$.
  6: Updates her array M_i and opinion:

$$M_i[W_i^t] \leftarrow x_{W_i^t}(t), \quad x_i(t+1) = (1-\alpha_i) \sum_{j \in N_i} p_{ij} M_i[j] + \alpha_i s_i$$

Lemma 7. *Let $\rho = \min_i \alpha_i$, and $\pi_{ij}(t)$ be the most recent round before round t, that agent i met her neighbor j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t)$ then, for all $t \geq kB$, $\|x(t) - x^*\|_\infty \leq (1-\rho)^k$.*

In Dynamics 2 there does not exist a fixed B that satisfies Lemma 7. However we can select a length value such that the requirements hold with high probability. Observe that agent i simply needs to wait to meet the neighbor j with the smallest weight p_{ij}. Therefore, after $\log(1/\delta)/\min_j p_{ij}$ rounds, agent i met all her neighbors at least once with probability at least $1-\delta$. In order to hold this for all agents, we shall roughly take $B = 1/\min_{p_{ij}>0} p_{ij}$.

Theorem 4. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 1 with equilibrium $x^* \in [0,1]^n$ and let $\rho = \min_{i \in V} \alpha_i$. The opinion vector $x(t) \in [0,1]^n$ produced by Row Dependent dynamics, after t rounds satisfies* $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2\exp(-\rho \min_{ij} p_{ij} \sqrt{t}/(4\ln(nt)))$.

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