Random Geometric Graphs.

Josep Díaz

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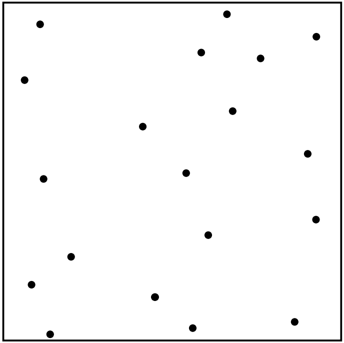
- *Random Euclidean Graphs*,
- *Random Proximity Graphs*,
- Random Geometric Graphs.

Random Euclidean Graphs

Choose a sequence $V = \{x_i\}_{i=1}^n$ of independent and uniformly distributed (iid) points on $[0,1]^d$, and consider the weighted complete graph on V, where the weight of an edge is its Euclidean distance.



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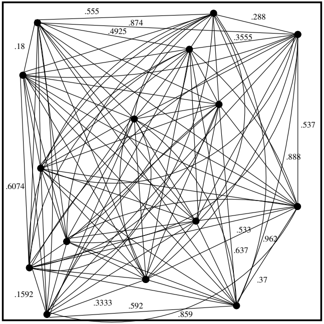
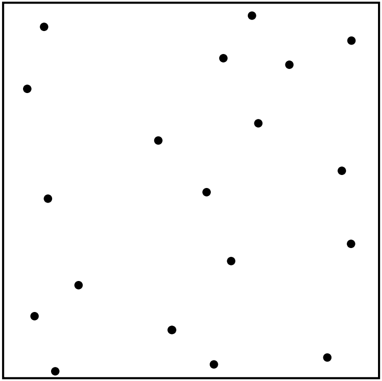


Good references: *M. Steele:* Probability theory and combinatorial optimization. *SIAM (1997).*
J.E. Yukich: Probability theory of classical Euclidean optimization problems. *Lecture Notes in Math., Springer (1998).*

Random Proximity Graphs.

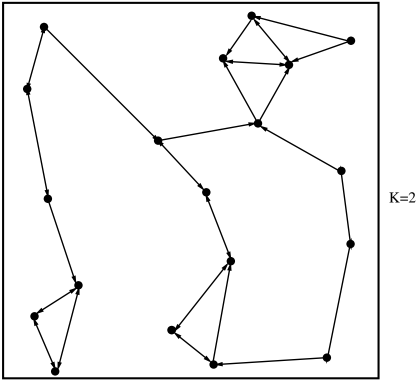
Choose a sequence $V = \{x_i\}_{i=1}^n$ of independent and uniformly distributed points on $[0,1]^2$ and a given k (could be a Poisson point process with intensity 1). Consider the graph $G_{n,k}$ with vertex set V, and each $v \in V$ connects to its k nearest neighbors. These graphs are also denoted k-NNG



K=2

Random Proximity Graphs.

Choose a sequence $V = \{x_i\}_{i=1}^{n}$ of i.i.d. points on $[0,1]^2$ and a given k. Consider the graph $G_{n,k}$ with vertex set V, and each $v \in V$ connects to its k nearest neighbors.



Connectivity of Undirected Random Proximity graphs

Xue, Kumar, (2004)
$G_{n,k}$ is connected aas if $k \geq 0.074 \log n$
$G_{n,k}$ is disconnected aas if $k \leq 5.1774 \log n$
Conjecture: threshold for connectivity $k = (1 + o(1)) \log n$.

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Connectivity for directed Random Proximity.
Balister, Bollobas, Sarkar, Walters, (2006)
$G_{n,k}$ is disconnected aas if $k \geq 0.7209 \log n$
$G_{n,k}$ is connected aas if $k \leq 0.9967 \log n$

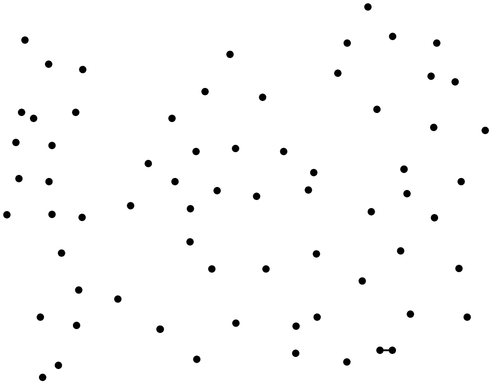
Birth of giant component in undirected k-NNG.

Shang-Hua Teng, F. Yao (2004)
Empirically, the birth of a giant component occurs at $k = 3$
Analytically the 213-NNG has a giant component.

Open problem: The giant component appears at 3-NNG

Random geometric graphs (RGG)

$G(n, r)$: Choose a sequence $\mathcal{V}_n = \{x_i\}_{i=1}^n$ of independent and uniformly distributed points on $[0, 1]^d$, given a fixed $r(n) > 0$, connect two points if their ℓ_p-distance is at most r.



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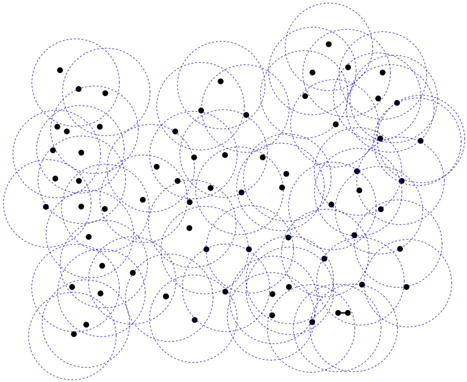

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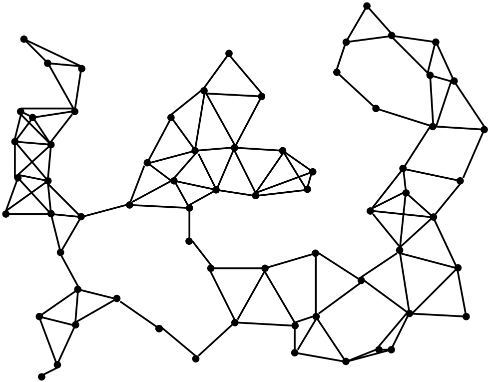


Same result starting from a Poisson point process in the square $\sqrt{n} \times \sqrt{n}$.

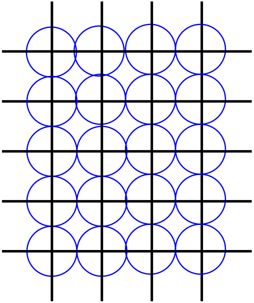
E.N. Gilbert: Random Plane Networks. J. Industrial Appl. Math *(1961).*

M.D. Penrose: Random Geometric Graphs *OUP (2003).*

The deterministic version: Unit Disk Graphs.

Intersection model: An undirected graph is a unit disk graph (UDG) if its vertices can be put in one-to-one correspondence with circles of equal radius in the plane in such a way that two vertices are joined by an edge iff the corresponding circles intersect (wlog we can assume the radius of each disk is 1).
Golumbic: Algorithmic graph theory and perfect graphs *(1980).*

There are other equivalent models as the *containment model.*
Clark, Colbourn, Johnson (1990).

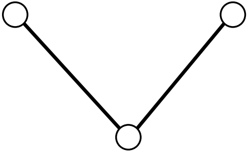


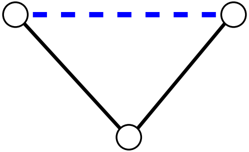
The recognition problem for UDG.

Given an undirected graph G, decide whether G is UDG.

Breu, Kirkpatrick (1998). The recognition problem for UDG is NP-hard.

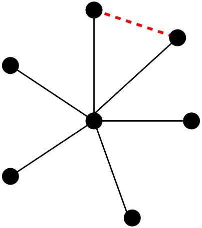
An important difference between RGG and $G_{n,p}$.





An important difference between RGG and $G_{n,p}$.

Given a RGG, not all graphs are feasible to be subgraphs of the RGG:



Basic facts about RGG

In $[0,1]^2$ *and the* ℓ_2*-norm*.
Recall: if $\forall k \geq 1$, a rv X has $\mathbf{E}[X]_k = \mu^k$ for $\mu = \Theta(1)$ then asymptotically X is a Poisson with mean μ.

Let $X =$ the number of *isolated vertices* in $G(n, r(n))$.
Consider the indicator r.v. $X_i = 1$ iff vertex x_i is isolated. Then

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{Pr}[X_i = 1] = n(1 - \pi r(n)^2)^{n-1} \sim ne^{-\pi r(n)^2 n - O(r(n)^4 n)}$$

Let $\mu = ne^{-\pi r^2 n}$, then
- if $\mu = o(1) \Leftrightarrow \mathbf{E}[X] = o(1)$
- if $\mu = \Omega(1) \Rightarrow \mathbf{E}[X] \sim \mu$.

Connectivity of RGG

On $[0,1]^2$ and ℓ_2, the asymptotic behavior of $\mu = ne^{-\pi r(n)^2 n}$ characterizes the connectivity of $G(n, r(n))$:

1. If $\mu \to 0$ then $G(n, r(n))$ is aas *connected*.
2. If $\mu = \Theta(1)$ then $G(n, r(n))$ has a component of size $\Theta(n)$ and X follows a Poisson distribution with parameter μ.
3. If $\mu \to \infty$ then $G(n, r(n))$ is aas *disconnected*.

Connectivity: when $\mathbf{E}[X] = \Theta(1) \Rightarrow r_c = \sqrt{\frac{\ln n - \ln \mu}{\pi n}}$.
r_c is the connectivity radius.

$\mathcal{C}$ = event $G(n, r(n))$ is connected
$\mathcal{D}$ = event $G(n, r(n))$ is disconnected

$$\mathbf{Pr}\left[\mathcal{C}\right] \sim \mathbf{Pr}\left[X = 0\right] \sim e^{-\mu} = \Theta(1)$$

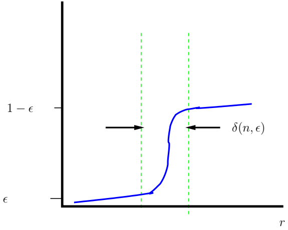
$$\mathbf{Pr}\left[\mathcal{D}\right] \sim \mathbf{Pr}\left[X > 0\right] \sim 1 - e^{-\mu}.$$

M.D. Penrose (1997)
P. Gupta, P.R. Kumar (1998)
M.J. Apple, R.P. Russo (2002)
M.D. Penrose (1999).

Monotone graph properties in RGG

A graph property is *monotone* if it is preserved when edges are added to the graph. Ex: connectivity, Hamiltoniacity.

Goel, Rai, Krinashmachari (2004) Any monotone graph property of a random geometric graph has a sharp threshold at $r = \Theta(\frac{\log n}{\alpha n})^{1/d}$.



where $\delta = o(1)$ (In $d = 2$, $\delta = r_c \ln^{1/4} n$)

Connectivity RGG

$r_c \sim \left(\sqrt{\frac{\ln n}{\pi n}}\right)$ is a sharp threshold for the connectivity of $G(n, r(n))$ on $[0, 1)^2$.

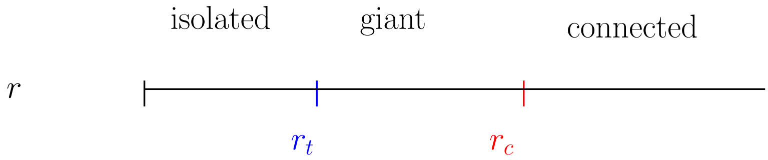
In general for any ℓ_p, $1 \leq p \leq \infty$, with any $d \geq 2$

$$\left(\frac{\ln n}{\alpha_{p,d}\, n}\right)^{1/d},$$

for constant $\alpha_{p,d}$.

Basic facts of RGG.

The *thermodynamical limit* $r_t = \sqrt{\frac{\lambda_c}{n}}$, where whp there appears a giant component in $G(n, r(n))$.
Experimentally for dim=2, $\lambda_c = 2.0736$
For $r_i \leq r_t$, then Prob. v is isolated $= 1 - o(1)$. *Most vertices are isolated*.



Feasible subgraphs of a RGG.

Given a feasible graph $\tilde{G}$ of size k, how many copies of $\tilde{G}$ will appear aas as induced subgraph in $G(n, r(n))$, as $r(n)$ evolves ? How many of those copies form a component in $G(n, r(n))$?

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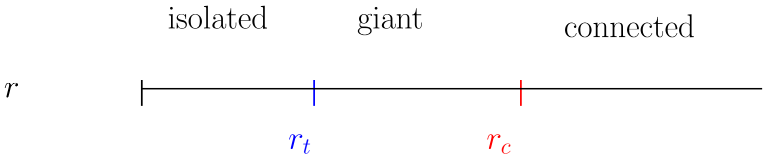
What will be the probability of existence of components of size $i \geq 2$ at connectivity threshold r_c? How do they look like?

Induced components.

Ch. 3 of M. Penrose RGG, *2002*

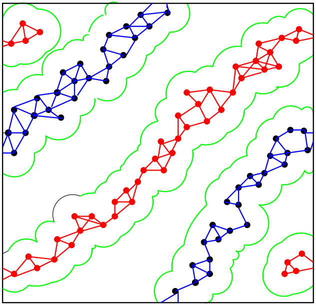


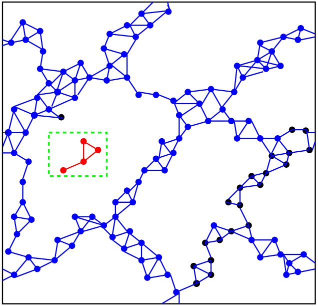
Given a feasible G' of size k:
Let $J = \#$ induced copies of G' in $G(n, r(n))$, which are a *component*.

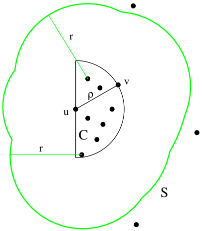
For $r(n) < r_t$, then $\mathbf{E}[J] = \Theta(r(n)^{2(k-1)} n^k)$,
Therefore, aas G' appear at $G(n, r(n))$ as induced component.

Components above r_t

Recall if $\mu = \Theta(1) \Rightarrow$ the probability $G(n, r(n))$ has some component of size > 1, other than a giant component, is $o(1)$.
D., Mitsche, Pérez (2007) gave more accurate bounds for that probability:
Given a component of size i, at r_c the probability of existence of the component is $\Theta(\frac{1}{\log^{i-1} n})$, aas.
Moreover, aas the components of size i will be cliques.



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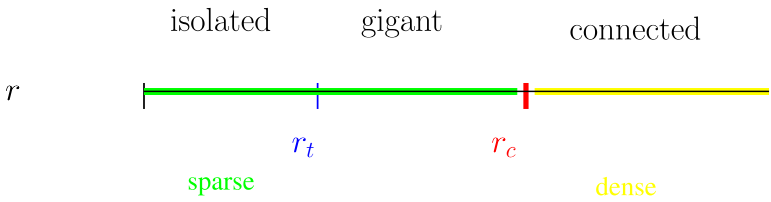
Subconnectivity and superconnectivity regimes



Let $D =$ be a rv counting the degree in $G(n, r(n))$. Then, the *average degree* $\mathbf{E}[D] = \pi r(n)^2(n-1)$ which at r_c is $\Theta(\ln n)$. As $n \to \infty$, the value of $r(n)$ in the ratio $\frac{nr(n)^2}{\ln n}$ gives us an indication whether $\mathbf{E}[D]$ grows slower or faster than $\ln n$.

Subconnectivity and superconnectivity regimes

$G(n, r(n))$ is:
Sparse if $nr(n)^2 = o(\ln n)$ and $nr(n)^2 = n^{o(1)}$ as $n \to \infty$.

$$n^{-c} << nr(n)^2 << \ln n \ \forall c > 0$$

Then asymptotically $G(n, r(n))$ is disconnected and $\mathbf{E}[D]$ grows slower than $\ln n$

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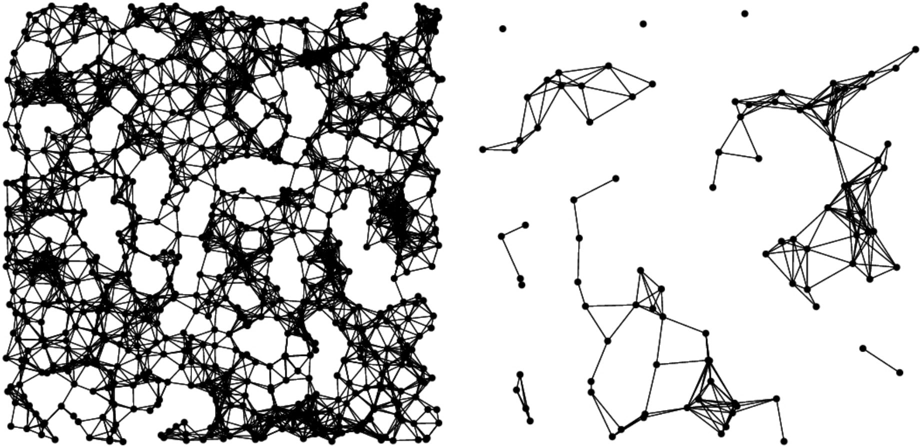
Dense if $nr(n)^2 / \ln n \to \infty$ as $n \to \infty$.

$$nr(n)^2 >> \ln n$$

Then asymptotically $G(n, r(n))$ is connected and $\mathbf{E}[D]$ grows faster than $\ln n$.

Dense and Sparse RGG on $[0, 1]^2$.



Chromatic number and clique number

Given $G(n, r(n))$ the chromatic number $\chi(G(n, r(n)))$ is the minimum number of colors needed to color the vertices of $G(n, r(n))$, such that no edge shares the same color. The clique number $\omega(G(n, r(n)))$ is the size of the largest clique in $G(n, r(n))$.

Clark, Colbourn, Johnson, (1990).

The problem of finding the chromatic number is NPC on UDG

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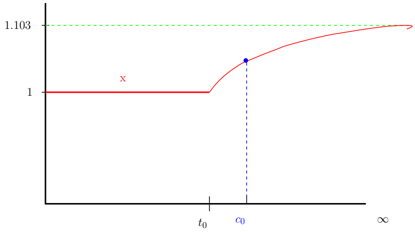
The chromatic and clique numbers on RGG:
$\chi(G(n, r_c)) \sim \Theta(\log n)$ and $\omega(G(n, r_c)) \sim \Theta(\log n)$.

Chromatic and clique numbers

McDiarmid, Muller (2008)
(Version for ℓ_2 and $[0,1]^2$)
Given $G(n, r(n))$, there is a constant $0 < t_0 < \infty$ and a continuos increasing function $x : [0, \infty] \to [0, \infty)$, with $x(t) = 1.103$ as $t \to \infty$ s.t.
if $nr^2 / \ln n \to t$ as $n \to \infty$, then $\frac{\chi(G(n,r(n)))}{\omega(G(n,r(n)))} \to x(t)$ aas



Hamiltonicity

Given $G(n, r(n))$ a *Hamiltonian cycle* is a simple cycle that visits every node of $G(n, r(n))$ exactly once.
$G(n, r(n))$ is *Hamiltonian* if it contains a Hamiltonian cycle.

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Itai, Papadimitriou, Szwarafiter, (1982):
The problem of deciding if a UDG is Hamiltonian is NPC.

Hamiltonicity

M. Penrose RGG (2003):
Open problem: At the exact point where $G(n, r(n))$ becomes 2-connected, it also becomes Hamiltonian, aas

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D., Mitsche, Pérez (2007):
The property that $G(n, r(n))$ is Hamiltonian exhibits a sharp threshold at $r(n) = \sqrt{\frac{\ln n}{\pi n}(1 + o(1))}$.
In fact: for any $\epsilon > 0$,

- if $r = \sqrt{\frac{\log n}{(\pi+\epsilon)n}}$, then a.a.s. $\mathcal{G}$ contains no Hamiltonian cycle,
- if $r = \sqrt{\frac{\log n}{(\pi-\epsilon)n}}$, then a.a.s. $\mathcal{G}$ contains a Hamiltonian cycle.

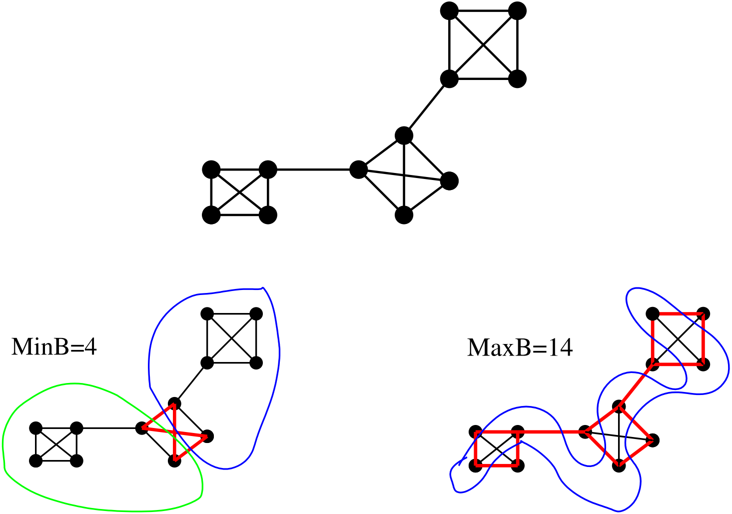
Bisection

Given $G(n, r(n))$ we want to partition the nodes into two equal cardinality subsets, such that:

Min-Bisection: it minimizes the number of edges crossing between subsets.

Max-Bisection: it maximizes the number of edges crossing between subsets.

Max and Min Bisection



Bisection

Max-Bisection:
D., Kaminski. (2007): Max-Bisection for UDG is NPC
(Also Max-Cut for UDG is NPC)

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Min-Bisection:

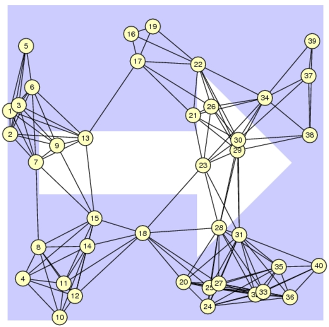
Arora, Karger, Karpinski STOC *(1995)*: APX for UDG

Open problem: Is Min-Bisection NPC for UDG?

D., Penrose, Petit, Serna (1999):
If Min-Bisection is NPC when restricted to planar graphs with $\Delta = 4$, then Min-Bisection is NPC for UDG.

Min-Bisection on dense RGG on $[0, 1]^2$

D., Penrose, Petit, Serna (1999):
The projection algorithm



For dense $G(n, r(n))$, the size of the Min-Bisection is $\geq cn^2 r(n)^2$.
Aas, the the ratio of approximation for the projection algorithm is

$$\frac{\text{Min-Bis}(G(n, r(n)))}{n^2 r(n)^3} \sim 0.265$$

Aas. the projection algorithm is a constant approximation for Min-Bisection on dense RGG.

A generalization of Min-Bisection. Given a $G(n, r(n))$ on $[0, 1]^2$ and a $\beta \in (0, 1/2]$, a β-balanced Min-Cut is a partition of the vertices into 2 subsets, each containing at least βn vertices, such that it minimizes the number of edges across the partition.
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D., Grandoni, Marchetti (2006)
Given a $G(n, r(n))$, for at he connectivity threshold, the β-balanced Min-Cut of $G(n, r(n))$ contains $\Omega(\min\{\beta nc \log n, \sqrt{\beta nc^3 \log^3 n}\})$ edges, aas.

Model for Dynamic Random Geometric.

Models for simulating mobility in Ad Hoc networks:

- Random walk model (R.Guerin-87)
- Random waypoint model (Broch, Maltz, Johnson,Hu, Jetcheva-98)
- The Realistic model (Jardosh, Belding, Almeroth,Suri (2003))

The random walk model for RGG

D., Mitsche, Pérez (2007) (SODA-2008)
In $[0,1)^2$, let $s = s(n) \in \mathbb{R}^+$ and $M = m(n) \in \mathbb{N}$, at $t = 0$ distribute n vertices on $[0,1)^2$ iid.
Take $r(n) = r_c$ and construct an initial random geometric graph $G(n, r_c)$.
Each v chooses independently a random angle $\alpha \in [0, 2\pi)$, advancing in that direction a length s (*step*).
After each step, each vertex independently flips a coin, and with prob $1/m$ they choose a new direction α, otherwise it continues in the same direction.

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At $t = k$, let $G_k(n, r_c, s, m)$ be the resulting random geometric graph.

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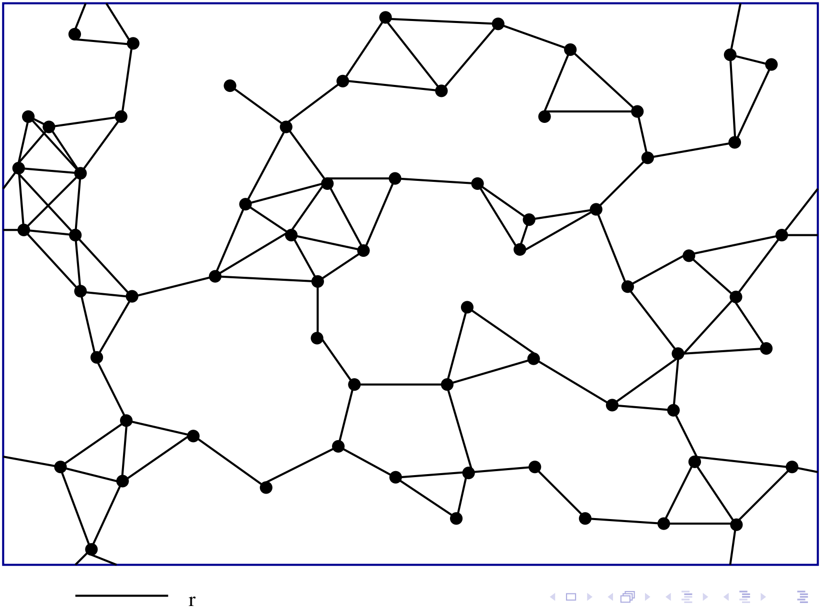
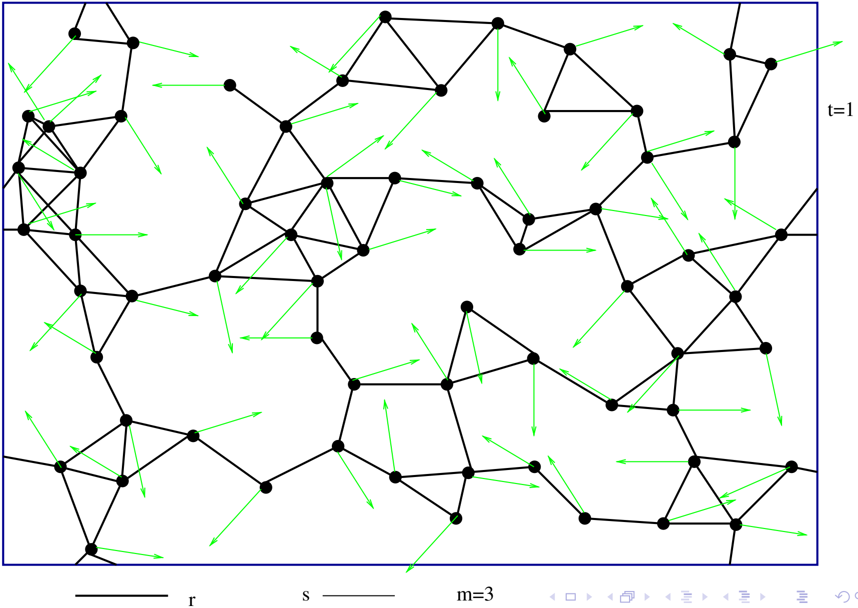
At $t = k$, let $G_k(n, r_c, s, m)$ be the resulting random geometric graph.
How long the connectivity and disconnectivity periods will last?

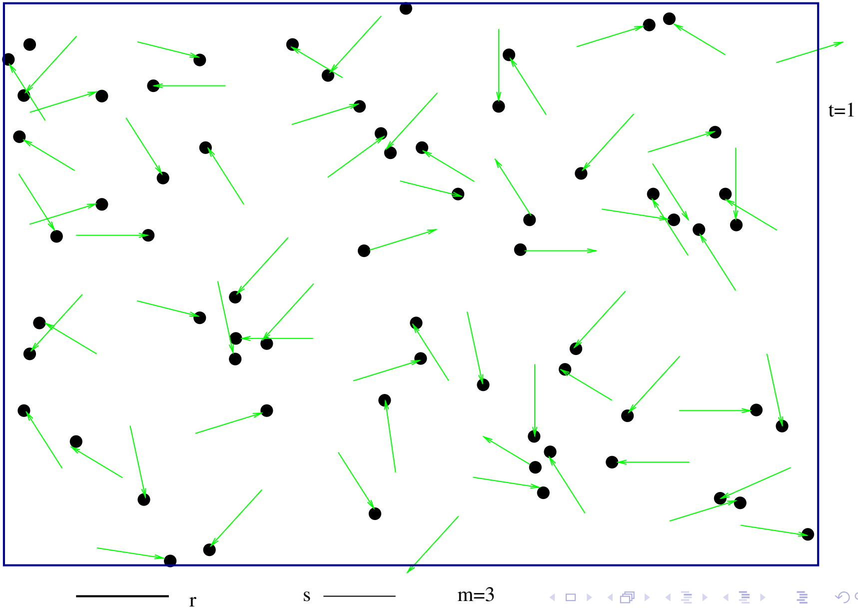
Example: Initial distribution and RGG



Example: Choose initial direction



Example: Move one step s



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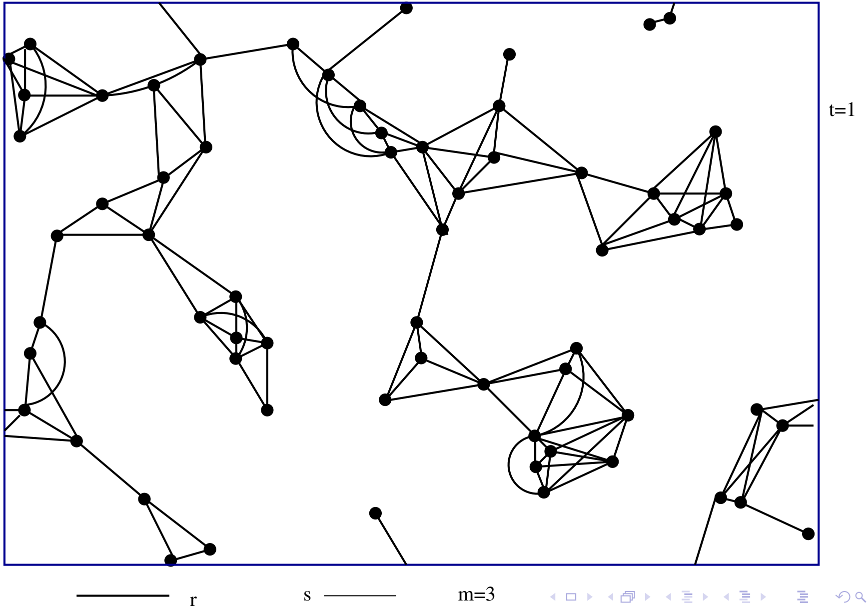
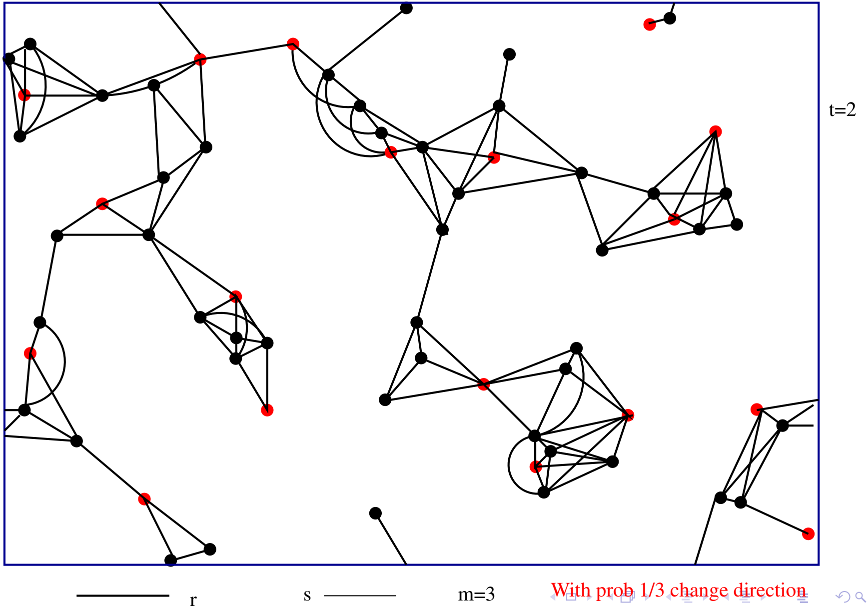


Example: step 2 step s



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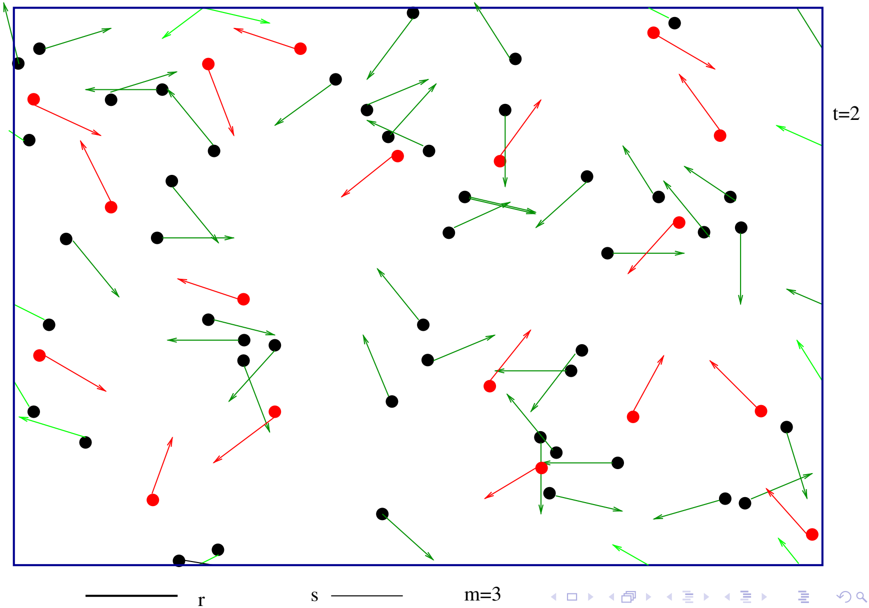

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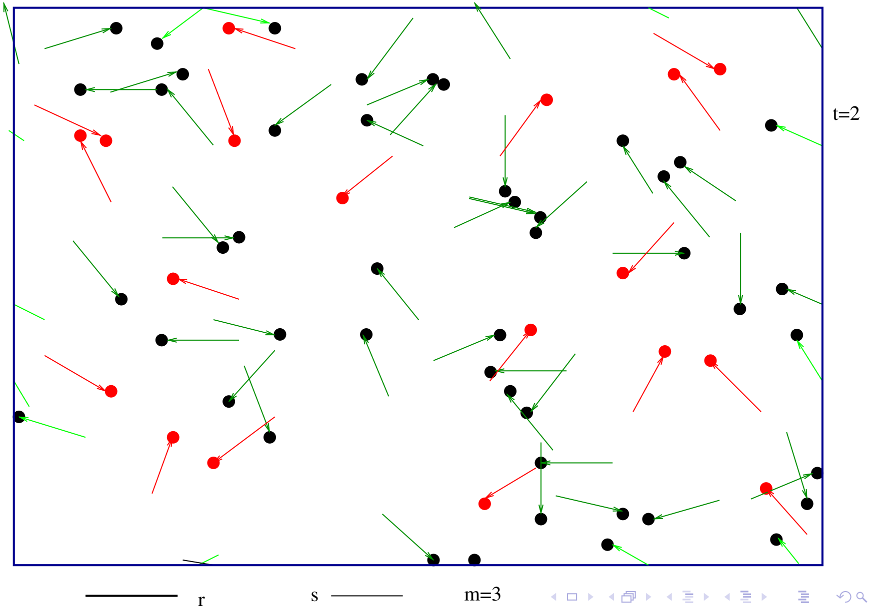
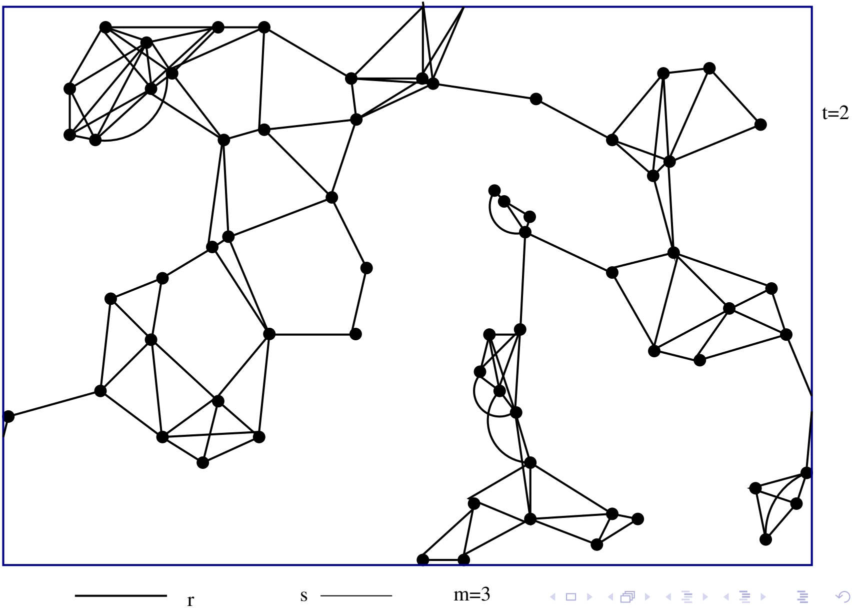


Example: step 3 step s



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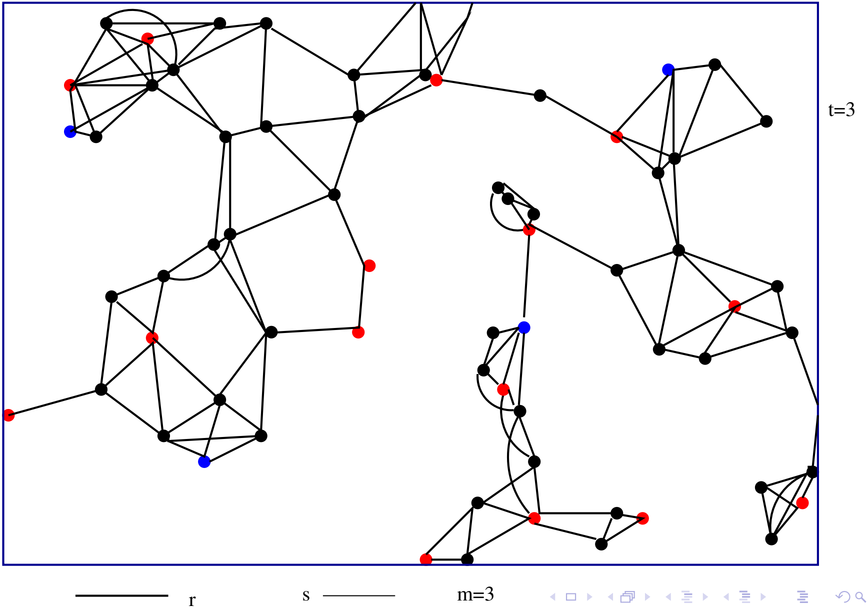

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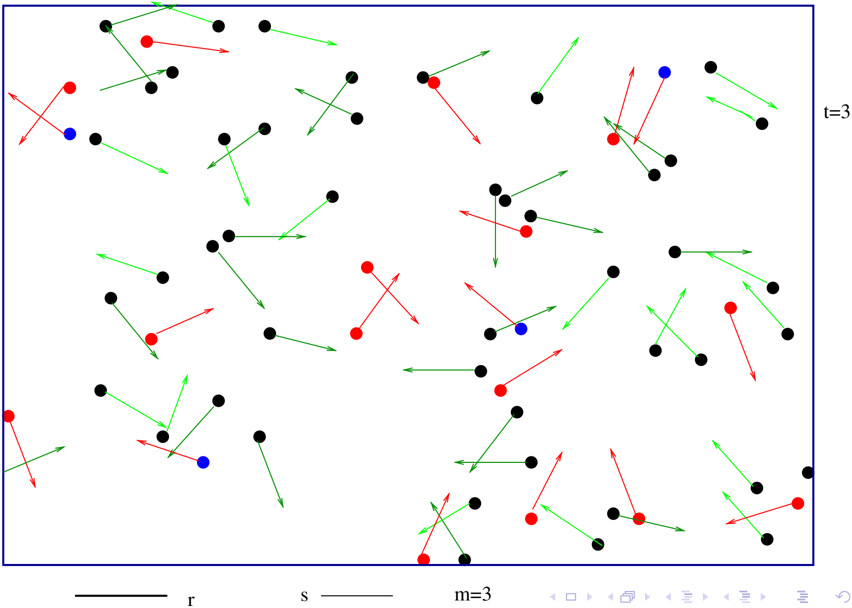

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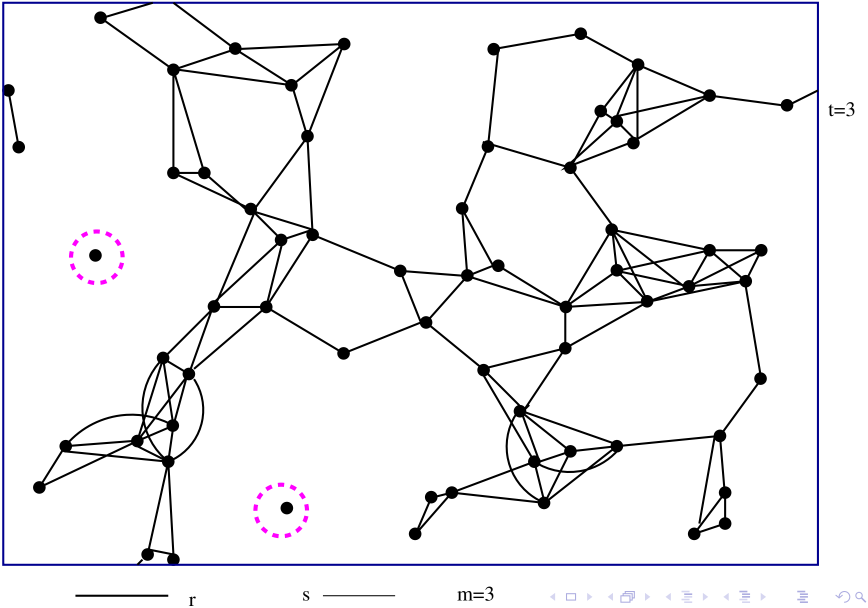

Main result

$\mathcal{C}_k$ = event that $G_k(n, r, s, m)$ is connected at step k,
$\mathcal{D}_k = \overline{\mathcal{C}_k}$ = the event that $G_k(n, r, s, m)$ is disconnected at step k,
$L_k(\mathcal{C})$ = the number of consecutive steps that $G(n, r(n))$ remains connected starting at step k,
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$$\mathbf{E}\left[L_k(\mathcal{C}) \mid \mathcal{D}_{k-1} \wedge \mathcal{C}_k\right] \sim \begin{cases} \infty & \text{if } s = \text{very small }, \\ \Theta(1) & \text{if } s = \text{mid small}, \\ \text{1step} & \text{if } s = \text{could wrap around torus}, \end{cases}$$

Formally:

$$\mathbf{E}\left[L_k(\mathcal{C}) \mid \mathcal{D}_{k-1} \wedge \mathcal{C}_k\right] \sim \begin{cases} \frac{\pi}{4srn} & \text{if } srn = O(1), \\ \frac{1}{(1-E^{-4srn/\pi})} & \text{if } srn = \Theta(1), \\ 1 & \text{if } srn = \Omega(1), \end{cases}$$

Similar expression for $\mathbf{E}\left[L_k(\mathcal{D}) \mid \mathcal{C}_{k-1} \wedge \mathcal{D}_k\right]$

Glimpse of proof

The initial uniform distribution stays invariant as t evolves: At any fixed step $k \in \mathbb{Z}$, the vertices are distributed over the torus $[0, 1)^2$ independently and u.a.r. Consequently for any $k \in \mathbb{Z}$, $G_k(n, r, s, m)$ has the same distribution as $G(n, r(n))$.

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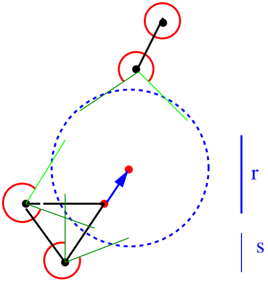
$$\mathbf{E}[L_k(\mathcal{C}) \mid \mathcal{D}_{k-1} \wedge \mathcal{C}_k] = \frac{\mathbf{Pr}[C_k]}{\mathbf{Pr}[\mathcal{D}_{k-1} \wedge \mathcal{C}_k]}$$

$$\mathbf{E}[L_k(\mathcal{D}) \mid \mathcal{C}_{k-1} \wedge \mathcal{D}_k] = \frac{\mathbf{Pr}[D_k]}{\mathbf{Pr}[\mathcal{C}_{k-1} \wedge \mathcal{D}_k]}$$

Characterization of $G_k(n, r, s, m)$ at consecutive steps k and $k + 1$

Birth isolated component



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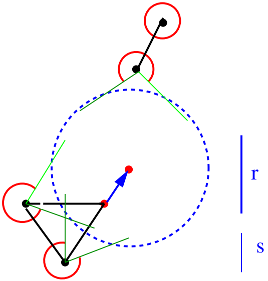
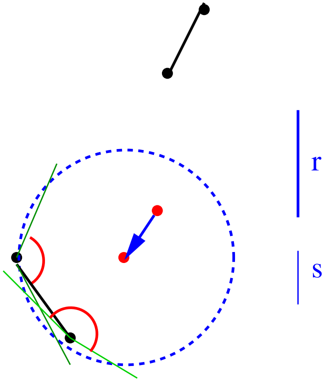


Death isolated component

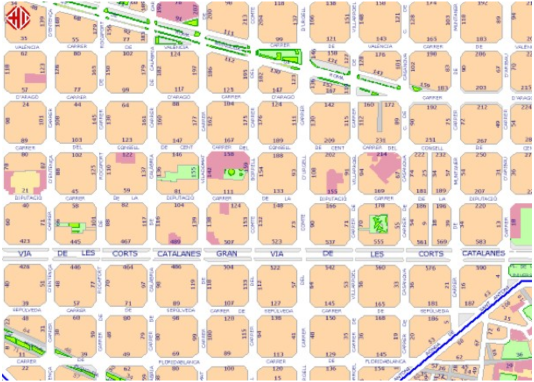


Open problems for dynamic random geometric graphs.

1. The unit square $I = [0, 1]^2$.
2. How is the behavior with different values of r, among the same set of nodes?
3. How about moving m consecutive steps in the same direction before all them synchronously change direction?
4. Waypoint direction and existence of obstacles.
5. Could these models could help to develop new, more efficient, decentralized protocols?

The walkers problem: A particular case of random geometric graphs.

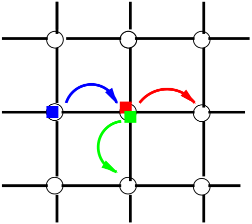
GOAL: Study the connectivity of the ad-hoc network established between the agents randomly scattered in the streets of a city (as a toroidal grid), when they move randomly through the streets:

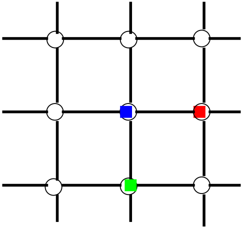


The walkers problem on the toroidal grid.

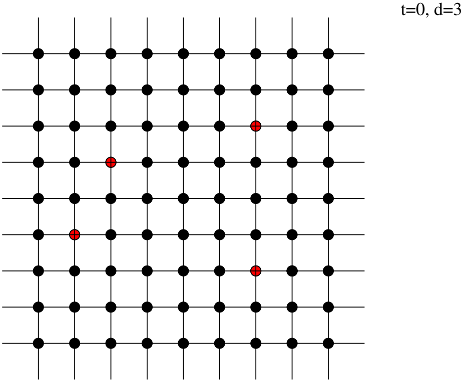
D., Pérez, Serna, Wormald. (2008)
Given a set W of w *walkers*, which at each step can move N/S/E/W on the edges of a toroidal grid T_N with $N = n^2$ vertices. Two walkers have direct communication if their distance is $\leq d$. We wish to study the *evolution of connectivity* in $G_t[W]$ as the walkers move.

Example.



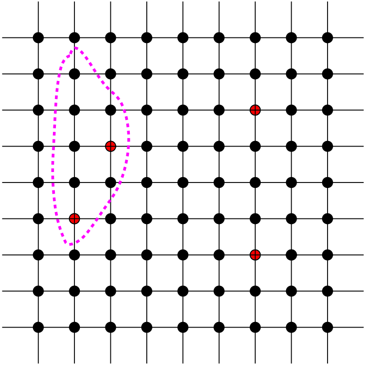


Example.



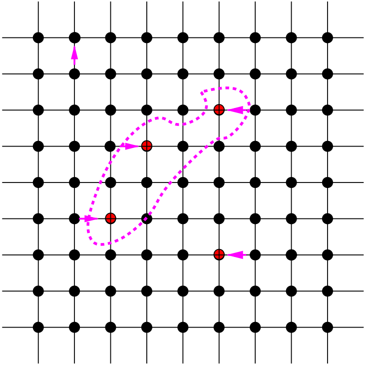
Example.

t=0, d=3



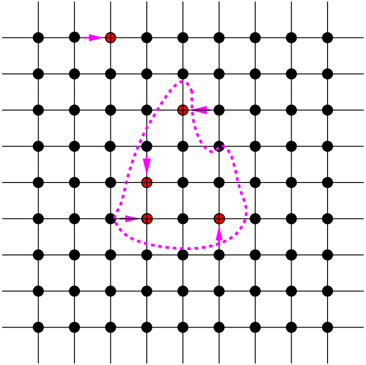
Example.

t=1, d=3

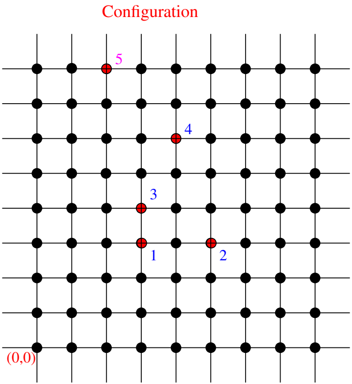


Example.

t=2, d=3



Example.



a={(3,3),(3,5),(4,3),(6,4),(8,3)}

Dynamic setting.

- The system always reaches a state representing a connected component, within finite expected time.
- The initial uniform distribution stays invariant as t evolves, therefore we need to consider only the case $\mu = \Theta(1)$.

Main Results

Interesting case $d = o(n)$,
Let $\mu = N(1 - e^{-w/N})e^{\pi d^2 w/N}$.
Expected value of the distribution of isolated components

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Let $\mu = N(1 - e^{-w/N})e^{\pi d^2 w/N}$.
Expected value of the distribution of isolated components
Compute the average periods of connectivity and non-connectivity
$L(\mathcal{C})$ and $L_k(\mathcal{D})$, of $G[W]$

$$\mathbf{E}\left[L(\mathcal{C})\right] \sim \begin{cases} \frac{1}{\mu w/N} & \text{if } dw/N = o(1), \\ \frac{1}{(1-e^{\mu(1-e^{-w/N})})} & \text{if } dw/N = \Theta(1), \\ \frac{1}{(1-e^{-\mu})}1 & \text{if } dw/N = (1), \end{cases}$$

Same result for the tree, the cycle and the r dimensional grid.

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Open problem: Extend the walkers problem to the r-dimensional cube, for $d = \omega(1)$.

Thank you.