Interest Rate Curves

Advanced Financial Modeling

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Suggested References

Everything will be covered during the course and provided to you as slides, notes or Python notebooks.
If you want more, check the following:

- D. Brigo, F. Mercurio (2006) Interest Rate Models - Theory and Practice (2nd edition)
- L. Andersen, V. Piterbarg (2010) Interest Rate Models. Vol.1, Vol.2, Vol.3.

1. No-Arbitrage Pricing
2. Single Curve Context
3. Building Multiple Curves
4. Curve Sensitivities
5. Convexity Adjustments

Basic Definitions, again

Different types of compounding interest rates:

- **Simple compounding:**

$$L(t, T) = \frac{1 - P(t, T)}{\tau(t, T)P(t, T)}$$

- **Continuous compounding**

$$R(t, T) = -\frac{\ln P(t, T)}{\tau(t, T)}$$

from which we can derive the bond-price

$$P(t, T) = e^{-R(t,T)\tau(t,T)}$$

Basic Definitions, and again

The instantaneous rate $r(t)$ is instead the limit of $R(t, T)$ for $T \to t$, i.e.

$$r(t) = \lim_{\Delta t \to 0} R(t, t + \Delta t)$$

Its continuously compounded representation is called **Bank account** or *money-market account* and it is defined by

$$B(t) = B(0)e^{\int_0^t r(u)du}, \qquad B(0) = 1$$

Which represent a free-risk investment at time 0 of the amount B(0) that evolves according to the differential equation

$$dB(t) = r(t)B(t)dt$$

Building Blocks - Floating Rate

Recall that the simply compounded spot interest rate between t and T is denoted by $L(t, T)$ and represent the constant rate at which the amount $P(t, T)$ at time t is equivalent to one unit at time T.

$$L(t, T) = \frac{P(T, T) - P(t, T)}{\tau(t, T)P(t, T)}, \qquad P(T, T) = 1$$

Where $P(t, T)$ represent a zero-coupon bond price which is given by

$$P(t, T) = \frac{1}{1 + \tau(t, T)L(t, T)}$$

Building Blocks - Forward

Starting from the definition of forward rate agreement (FRA), by absence of arbitrage we obtain the following formula of the value of the FRA at time t

$$\mathbf{FRA}(t,T,S,\tau,N,K) = N\Big[P(t,S)\tau(T,S)K - P(t,T) + P(t,S)\Big]$$

we can define the forward rate $F(t,T,S)$ with $t \leq T \leq S$ as the fixed rate that makes the contract fair at inception

$$F(t,T,S) = \frac{1}{\tau(T,S)}\left[\frac{P(t,T)}{P(t,S)} - 1\right] \tag{1}$$

The instantaneous forward interest rate $f(t,T)$ is a forward rate with maturity with $T + \Delta T$, where ΔT is small

$$f(t,T) = f(t,T,T+\Delta T) = -\frac{\partial lnP(t,T)}{\partial T}$$

Building Blocks - Swap

At time t the value of the floating leg is equal to the discounted expectation of future LIBOR fixing rates. For a single payment:

$$\mathbf{FL}(t, T_{k-1}, T_k) = \tau_k P(t, T_k)\mathbb{E}^{T_k}[L(T_{k-1}, T_k)]$$

Under the forward measure $\mathbb{Q}^{T_k}$ the expected value of future LIBOR rate is the forward rate observed in t, fixed in T_{k-1}, to be exchanged in T_k

$$\mathbf{FL}(t, T_{k-1}, T_k) = \tau_k P(t, T_k)F(t, T_{k-1}, T_k)$$

The floating leg net present value is obtained by summing the value of all flows

$$\mathbf{FL}(t, T_a, \dots, T_b) = \sum_{k=a+1}^{b} \tau_k P(t, T_k)F(t, T_{k-1}, T_k)$$

Building Blocks - Swap

Now consider the swap's fixed leg with K being the fixed rate paid on each payment date $T_c, \ldots, T_d$., the net present value is equal to the discounted future cash flows

$$\mathbf{FIX}(t, T_c, \ldots, T_d) = K \sum_{j=c+1}^{d} \tau_j P(t, T_j)$$

Now we can exploit the price of a payer swap as the difference between the floating leg and the fixed leg

$$\mathbf{IRS}(t, T_a, \ldots, T_b, T_c, \ldots, T_d) = \sum_{k=a+1}^{b} \tau_k P(t, T_k) F(t, T_{k-1}, T_k) - K \sum_{j=c+1}^{d} \tau_j P(t, T_j)$$

Building Blocks - Swap

We can define the swap rate S as the fair rate K that makes the value of the swap equal to zero at inception

$$S_{a,b,c,d}(t) = \frac{\sum_{k=a+1}^{b} \tau_k P(t, T_k) F(t, T_{k-1}, T_k)}{\sum_{j=c+1}^{d} \tau_j P(t, T_j)}$$

Single Curve

At the foundation of interest rate trading there are two curves:

- Discounting-curve is used to discount the future cash-flows in order to obtain their present value
- Forwarding-curve (or estimation, projection) is used to obtain the expected value of the floating coupons at future times.

In a market context where

- Interbank credit/liquidity issues did not matter for pricing, basis swap spreads were negligible (and neglected).
- The collateral did not matter for pricing.

Libor rates were a good proxy for risk free rates and could be adopted both for discounting and forwarding purposes.

Single Curve

Curve Building

- Select one finite set of the most convenient (i.e. liquid) vanilla interest rate instruments traded in real time on the market, with increasing maturities.
- Interpolate/bootstrap a single yield curve for each currency using the selected instruments.
- Use the same curve to compute cash flows and discount factors.
- Hedge[1] the resulting delta risk using the necessary hedge ratios of the same set of vanillas.

[1] details in section [4]

Single Curve Inconsistencies

The pre-crisis approach outlined above is no longer consistent, at least in this simple formulation, with the present market approach:

- It does not take into account the market information carried by the basis swap spreads;
- It does not take into account that the interest rate market is segmented into sub-areas corresponding to instruments with different underlying rate tenors, characterized, in principle, by different dynamics (like correlated processes);
- It does not include collateral agreements and funding rates associated with the bootstrapping instruments.

Modern Market Practice

Curve Building

- Select multiple separated sets of vanilla interest rate, typically short-term cash and 1M, 3M, 6M, 12M tenors;
- For each currency, build the discount-curve associated with the risk free proxy and multiple separated forwarding curves.
- compute the relevant forward rates and the corresponding cash flows from the forwarding curve with the appropriate tenor,
- compute the relevant discount factors from the discounting curve with the appropriate funding characteristics.

Modern Market Practice

Judging a curve construction, main criteria:

- The yield curve should be able to price back the instruments which are used to construct it;
- Implied forward rates should be continuous;
- The interpolation used should be as local as possible;
- The hedge should be local.

Overnight Rates

Nowadays the majority of interest rate derivatives are collateralized, for this reason the common curve used for discounting is the OIS curve (C_d). The OIS rate (ESTR for Euro) is defined as the following geometric sum

$$r_{ois}(t,T) = \left[\prod_{i=t}^{T-\frac{1}{360}} \left(1 + \frac{r_{ON(i)}}{360}\right) - 1 \right]$$

where r_{ONj} is the reference rate from T_i to $T_i + 1day$ and n is the number of days in the calculation period.

Cash Deposits

The first stripping instrument is a simple spot cash deposit with 1 day maturity, a zero coupon bond in which a counter-part lends a nominal amount (N) to another counter-part, which at maturity pays the notional amount plus the $r_{ois}(t, t+1)$ rate accrued over this period. The payoff is given by

$$\phi_{Deposit}(t+1) = N(1 + r_{ois}(t, t+1)\tau_{1day})$$

and its t-value is

$$\begin{aligned}\phi_{Deposit}(t) &= P_d(t, t+1)\mathbb{E}^{t+1}\left[\frac{\phi_{Deposit}(t+1)}{P_d(t+1, t+1)}\right] \\ &= NP_d(t, t+1)(1 + r_{ois}(t, t+1)\tau_{1day}) = N\end{aligned}$$

Following that

$$P_d(t, T_i) = \frac{1}{1 + r_{ois}(t, T_i)\tau_i}$$

OIS Forwards

Starting from the rates of this quoted pillars is possible to get all the corresponding discount factors, identified by the unitary zero-coupon bonds $P_d(t, T_i)$ associated with $r_{ois}(t, T_i)$

$$
\begin{aligned}
P_d(t, T_i) &= e^{-\int_t^{T_i} r_{ois}(s, s+1d)ds} \\
&= e^{-r_{ois}(t, T_i)(T_i - t)}
\end{aligned}
$$

and consequently forward rates:

$$
F_d(t, T_{i-1}, T_i) = \frac{1}{\tau(T_{i-1}, T_i)} \left(\frac{P_d(t, T_{i-1})}{P_d(t, T_i)} - 1 \right) \tag{2}
$$

OIS Swaps

We can exploit the price of a OIS payer swap as the difference between the floating leg and the fixed leg:

$$\mathbf{Swap}_{OIS}(t, T_a, T_b) = \sum_{k=a+1}^{b} \tau_k P_d(t, T_k) F_d(t, T_{k-1}, T_k) - K \sum_{k=a+1}^{b} \tau_k P_k(t, T_k)$$

Terms in the telescopic sum cancel each other:

$$\mathbf{Swap}_{OIS}(t, T_a, T_b) = [P_d(t, T_a) - P_d(t, T_b)] - K \sum_{k=a+1}^{b} \tau_k P_k(t, T_k)$$

OIS

From here it's easy to derive the value of K which makes the contract fair in t, i.e. the fixed rate s.t. $\mathbf{IRS}_{OIS} = 0$, the so called OIS-SWAP rate, given by:

$$K = S_{a,b}(t) = \frac{P_d(t, T_a) - P_d(t, T_b)}{A_{a,b}(t)}$$

Where $A_{a,b}(t)$ is the *annuity factor* of the OIS curve.

Forwarding Curve

In constructing the OIS Discount Curve, both discounting factors and forward rates are compounded starting from the same curve (C_d), which means that we still worked in a single curve framework. Things are different when we deal with forward curves with different tenors (e.g. 1M, 3M, 6M, 12M):

- we will need the C_d curve we just build for discounting, which we have assumed to be the best proxy of the risk-free rate
- then, we will define an additional curve C_x for the forward rates with the desired tenor x

Forwarding Curve

The basic forward curve is usually bootstrapped using a selection from the following market instruments:

- Deposit contracts, covering the window from today up to 1Y;
- Forward or Futures contracts,covering the window from 1M up to 2Y;
- IRS contracts, covering the window from 2Y up to 60Y.

The selection is generally done according to the principle of maximum liquidity.

Multi Curve FRA

FRA contracts are, in most cases, regulated by a CSA, so if we assume continuous mark-to-market and collateral posting, then we could neglect counterparty default risk, and discount our cashflow using the risk free discount factor $P_d(t, T_n)$. Hence the t-value of the FRA (with tenor $6m$) will be given by

$$FRA_{6m} = P_d(t, T_n)\frac{k - F_{6m}(t, T_n, T_S)}{1 + \tau_{6m}F_{6m}(t, T_n, T_S)}\tau_{6m}$$

where the FRA rate, k, that makes the contract fair, is given by

$$k = F_{6m}(t, T_n, T_S) = \frac{1}{\tau_{6m}}\left(\frac{P_{6m}(t, T_n)}{P_{6m}(t, T_S)} - 1\right)$$

Multi Curve Swap

In 6M SWAPs the semi-annual index is exchanged against a fixed coupon, if the two legs (floating vs fixed) have the same payment frequencies (6m), we have that:

$$Swap_{6m}(t, T_a, T_b, K) = FIX(t, T_a, T_b, K) - FLT_{6m}(t, T_a, T_b)$$

$$= K \sum_{k=a+1}^{b} \tau_k P_d(t, T_k) - \sum_{k=a+1}^{b} \tau_k P_d(t, T_k) F_{6m}(t, T_a, T_b)$$

$$= K A_{a,b}(t) - \sum_{k=a+1}^{b} \tau_k P_d(t, T_k) F_{6m}(t, T_a, T_b)$$

Multi Curve Swap

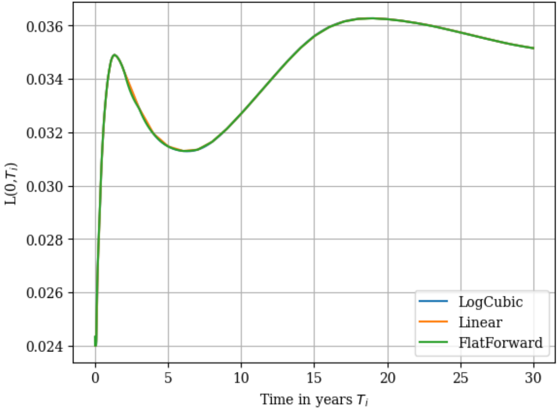
And the strike that makes the contract fair is:

$$K = \frac{\sum_{k=a+1}^{b} \tau_k P_d(t, T_k) F_{6m}(t, T_a, T_b)}{A_{a,b}(t)}$$

Where $A_{a,b}(t)$ is the annuity factor computed from the discounting curve.

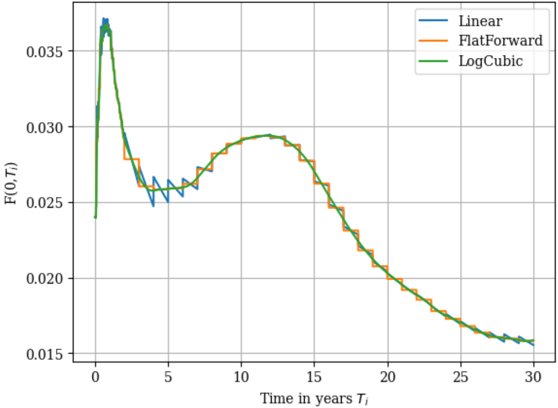
Choice of Interpolation

The choice of the interpolation function does not seems to be important for zero rates:



Choice of Interpolation

Instead, interpolation has an high importance in determining the smoothness of our forward rates structure:



Finite Differences

The finite difference approximation with step size h is given by

$$\frac{dV(p)}{dp} \approx \frac{V(p+h) - V(p)}{h};$$

or

$$\frac{dV(p)}{dp} \approx \frac{V(p) - V(p-h)}{h} \quad \text{(one-sided)};$$

or

$$\frac{dV(p)}{dp} \approx \frac{V(p+h) - V(p-h)}{2h} \quad \text{(two-sided)}$$

- It is typically used for black-box pricing functions
- There is a non-trivial trade-off between convergence and numerical accuracy.

Zero and Par

The present value of a product is by definition the discounted sum of its future cash flows (the amounts A_{cf}):

$$V = \sum_{cf} A_{cf} \cdot D(t_{cf})$$

Use the chain rule to compute the zero sensitivity:

$$\frac{\partial V}{\partial r_i} = \sum_{cf} \frac{\partial V}{\partial D_{cf}} \frac{\partial D_{cf}}{\partial r} \frac{\partial r}{\partial r_i}$$

And the par sensitivity:

$$\frac{\partial P}{\partial x_i} = \sum_j \frac{\partial P}{\partial r_j} \frac{\partial r_j}{\partial x_i} = \sum_{cf} \frac{\partial V}{\partial D_{cf}} \frac{\partial D_{cf}}{\partial r} \sum_j \frac{\partial r}{\partial r_j} \frac{\partial r_j}{\partial x_i}$$

Curve Jacobian

Consider the interest rate derivative pricing function V_{ir} as a function of yield curve parameters (e.i. zero rates) z:

$$V_{\text{ir}} = V_{\text{ir}}(z)$$

Model parameters z are derived from market quotes R of the calibration basket:

$$z = z(R);$$

This gives the mapping:

$$R \mapsto z \mapsto V_{\text{ir}} = V_{\text{ir}}(z(R))$$

The interest rate delta becomes

$$\Delta R = 1\text{bp} \cdot \frac{dV_{\text{ir}}}{dz} z(R) \cdot \frac{dz}{dR} R$$

Curve Jacobian

Consider $H(z, R)$, the q-dimensional objective function of the yield curve calibration problem:

- $z = [z_1, \ldots, z_q]^\top$: yield curve parameters (e.g., zero rates or forward rates);
- $R = [R_1, \ldots, R_q]^\top$: market quotes (par rates) for calibration instruments (e.i. swaps and FRAs);
- Set $r = q$, i.e., the same number of market quotes as model parameters:

$$H_k(z, R) = \text{ImpliedRate}_k(z) - R_k$$

Curve Jacobian

If the pair $(\bar{z}, \bar{R})$ solves the calibration problem $H(\bar{z}, \bar{R}) = 0$ and $\frac{dH}{dz}(\bar{z}, \bar{R})$ is invertible, then we get the Jacobian method for risk calculation:

$$\Delta R = 1\text{bp} \cdot \frac{dV_{\text{Swap}}}{dz} z(R) \cdot \begin{bmatrix} \frac{d}{dz}\text{ImpliedRate}_1(z) \\ \vdots \\ \frac{d}{dz}\text{ImpliedRate}_q(z) \end{bmatrix}^{-1}$$

Algorithmic Differentiation (AD)

Consider a calculation represented by the function $F : \mathbb{R}^n \to \mathbb{R}$ that produces a scalar result z out of an input X in dimension n. Assume F may be broken down into a sequence of sub-calculations $G : \mathbb{R}^n \to \mathbb{R}^m$, $K : \mathbb{R}^m \to \mathbb{R}^p$, and $H : \mathbb{R}^p \to \mathbb{R}$, such that:

$$F(X) = H\{K[G(X)]\}$$

where G, K, and H are simple enough that their Jacobians are known analytically.

Algorithmic Differentiation (AD)

Jacobian matrices of F, G, K, and H are related by the chain rule:

$$\left(\frac{\partial F}{\partial X}\right)^T_{n\times 1} = \left(\frac{\partial G}{\partial X}\right)^T_{n\times m} \left(\frac{\partial K}{\partial Y}\right)^T_{m\times p} \left(\frac{\partial H}{\partial Z}\right)^T_{p\times 1}$$

Because matrix products are associative, we can also accumulate the differentials in reverse order:

$$\left(\frac{\partial F}{\partial X}\right)^T_{n\times 1} = \left(\frac{\partial G}{\partial X}\right)^T_{n\times m} \left[\left(\frac{\partial K}{\partial Y}\right)^T_{m\times p} \left(\frac{\partial H}{\partial Z}\right)^T_{p\times 1}\right]_{m\times 1}$$

AD illustration by A. Savine

	Evaluation	Adjoint Differentiation
	S_0	
	r	
	y	
	σ	
	K	
	T	
1	$DF = \exp(-rT)$	$\bar{r} - = \overline{DF}TDF, \bar{T} - = \overline{DF}rDF$
2	$F = S_0 \exp[(r-y)T]$	$\overline{S_0} + = \bar{F}\frac{F}{S_0}, \bar{r} + = \bar{F}TF, \bar{y} - = \bar{F}TF, \bar{T} + = \bar{F}(r-y)F$
3	$std = \sigma\sqrt{T}$	$\bar{\sigma} + = \overline{std}\sqrt{T}, \bar{T} + = \overline{std}\sigma/2\sqrt{T}$
4	$d = \frac{\log\left(\frac{F}{K}\right)}{std}$	$\bar{F} + = \bar{d}\frac{1}{F\,std}, \bar{K} - = \bar{d}\frac{1}{Kstd}$ $\overline{std} - = \bar{d}\,d/std$
5	$d_1 = d + \frac{std}{2}$	$\bar{d} + = \overline{d_1}, \overline{std} + = \frac{\overline{d_1}}{2}$
6	$d_2 = d - \frac{std}{2}$	$\bar{d} + = \overline{d_2}, \overline{std} - = \frac{\overline{d_2}}{2}$
7	$nd_1 = N(d_1)$	$\overline{d_1} + = \overline{nd_1}n(d_1)$
8	$nd_2 = N(d_2)$	$\overline{d_2} + = \overline{nd_2}n(d_2)$
9	$\nu = DF(Fnd_1 - Knd_2)$	$\overline{DF} + = \bar{\nu}(Fnd_1 - Knd_2)$ $\bar{F} + = \bar{\nu}DFnd_1$ $\overline{nd_1} + = \bar{\nu}DFF$ $\bar{K} + = -\bar{\nu}DFnd_2$ $\overline{nd_2} + = -\bar{\nu}DFK$
	$C = \nu$	$\bar{\nu} = \bar{C}$

Gentle Introduction

Introduction

Let us consider a basic interest rate payoff function, which pays a percentage of a notional N, and the percentage paid will be determined by the Libor rate $L(T_{i-1}, T_i)$, at time T_i:

$$
\begin{aligned}
V(t_0) &= NM(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{L(T_{i-1}, T_i)}{M(T_i)}\right] \\
&= NP(t_0, T_i)\mathbb{E}^{T_i}[L(T_{i-1}, T_i)]
\end{aligned}
$$

We know that under the forward measure $\mathbb{Q}^{T_i}$ the price of such a contract is given by:

$$
V(t_0) = NP(t_0, T_i)F(t_0, T_{i-1}, T_i)
$$

Different Numeraire?

Suppose now that we consider the same contract, however, the payment will take place at some earlier time $T_{i-1} < T_i$, the current value of the contract is then given by:

$$V(t_0) = NM(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{L(T_{i-1}, T_i)}{M(T_{i-1})}\right]$$

When changing measures, to the T_{i-1} forward measure, we work with the following Radon-Nikodym derivative:

$$\left.\frac{d\mathbb{Q}^{T_{i-1}}}{d\mathbb{Q}}\right|_{\mathcal{F}(t_{i-1})} = \frac{P(T_{i-1}, T_{i-1})}{P(t_0, T_{i-1})}\frac{M(t_0)}{M(T_{i-1})}$$

so that

$$V(t_0) = NM(t_0)\mathbb{E}^{T_{i-1}}\left[\frac{P(t_0, T_{i-1})}{P(T_{i-1}, T_{i-1})}\frac{M(T_{i-1})}{M(t_0)}\frac{L(T_{i-1}, T_i)}{M(T_{i-1})}\right]$$

Different Measure!

Therefore

$$V(t_0) = NP(t_0, T_{i-1})\mathbb{E}^{T_{i-1}}[L(T_{i-1}, T_i)] \tag{3}$$

Although the Libor rate $L(T_{i-1}, T_i)$ is a martingale under the T_i forward measure, it is not a martingale under the T_{i-1} forward measure:

$$\mathbb{E}^{T_{i-1}}[L(T_{i-1}, T_i)] \neq \mathbb{E}^{T_i}[L(T_{i-1}, T_i)]$$

The difference between the two expectations is commonly referred to as **convexity**.

Convexity Correction

Remember

$$\frac{d\mathbb{Q}^{T_i}}{d\mathbb{Q}^{T_{i-1}}}\bigg|_{\mathcal{F}(t_{i-1})} = \frac{P(T_{i-1}, T_i)}{P(t_0, T_i)} \frac{P(t_0, T_{i-1})}{P(T_{i-1}, T_{i-1})}$$

By the change of measure technique, we can simplify the equation [3], such that:

$$\begin{aligned} V(t_0) &= NP(t_0, T_{i-1})\mathbb{E}^{T_{i-1}}[L(T_{i-1}, T_i)]; \\ &= NP(t_0, T_{i-1})\mathbb{E}^{T_i}\left[L(T_{i-1}, T_i)\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)} \frac{P(T_{i-1}, T_{i-1})}{P(t_0, T_{i-1})}\right]; \\ &= N\mathbb{E}^{T_i}\left[L(T_{i-1}, T_i)\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)}\right] \end{aligned}$$

Convexity Correction

Then by simply adding and subtracting $L(T_{i-1}, T_i)$ we get:

$$V(t_0) = N\mathbb{E}^{T_i}[L(T_{i-1}, T_i)] + N\mathbb{E}^{T_i}\left[L(T_{i-1}, T_i)\left(\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)} - 1\right)\right]$$

With the second term being the convexity correction between the two maturities

$$\begin{aligned} cc(T_{i-1}, T_i) &= \mathbb{E}^{T_i}\left[L(T_{i-1}, T_i)\left(\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)} - 1\right)\right] \\ &= P(t_0, T_i)\mathbb{E}^{T_i}\left[\frac{L(T_{i-1}, T_i)}{P(T_{i-1}, T_i)}\right] - F(t_0; T_{i-1}, T_i) \end{aligned}$$

Convexity Correction

Recall the definition of the simple Libor rate $L(T_{i-1}, T_i)$:

$$P(T_{i-1}, T_i) = \frac{1}{1 + \tau_i L(T_{i-1}, T_i)}$$

Then the expectation inside the $cc(T_{i-1}, T_i)$ equation can be written as follows:

$$\begin{aligned} \mathbb{E}^{T_i}\left[\frac{L(T_{i-1}, T_i)}{P(T_{i-1}, T_i)}\right] &= \mathbb{E}^{T_i}\left[L(T_{i-1}, T_i) + \tau_i L^2(T_{i-1}, T_i)\right] \\ &= F(t_0, T_{i-1}, T_i) + \tau_i \mathbb{E}^{T_i}[L^2(T_{i-1}, T_i)] \end{aligned}$$

Convexity Correction

Exercise:

Solve the convexity equation, assume a log-normal distribution for the underlying rate L.

$$cc(T_{i-1}, T_i) = P(t_0, T_i)\mathbb{E}^{T_i}\left[\frac{L(T_{i-1}, T_i)}{P(T_{i-1}, T_i)}\right] - F(t_0; T_{i-1}, T_i)$$

Hint:

Consider the dynamics

$$dL_i(t) = \sigma L_i(t) dW(t)$$

and apply Ito's Lemma to $L_i^2(t)$, what can you say about the drift of the SDE?