Lectures on Volatility

Advanced Financial Modeling

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2025

Suggested References

Everything will be covered during the course and provided to you as slides, notes or Python notebooks.

If you want more, check the following:

- J. Gatheral (2006) The Volatility Surface
- E. Derman, M.B. Miller (2016) The Volatility Smile.
- R. Rebonato (2004) Volatility and Correlation.
- C. Crispoldi, G. Wigger, P. Larkin (2015) SABR and SABR LIBOR Market Models in Practice.

Introduction

Moments of a distribution

Given a random variable X we can define its k centered moment as

$$M_k = \mathbb{E}\left[(X - \mathbb{E}[X])^k\right]$$

- Mean $= \mathbb{E}[X]$
- Variance $= M_2$
- Skewness $= \frac{M_3}{M_2^{1.5}}$
- Kurtosis $= \frac{M_4}{M_2^2}$

Introduction

What is volatility?

We can broadly define volatility as the level of dispersion within a series of values.
Let X be a random variable, the second centered moment of its distribution, also called variance is

$$Var[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

volatility is denoted by σ and is given by

$$\sigma(X) = \sqrt{Var[X]}$$

Realized and Implied Volatility

A first characterization of the variable can be made between:

1. **Realized-Historical volatility:** past volatility recorded for a given time series

$$\sigma_R = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(S_i - \hat{S})^2}$$

   where $\hat{S}$ denotes the sample mean for n observations.

2. **Future-Implied volatility:** defined as the value of σ_I that matches model to market prices (e.i. Black or Bachelier volatility). Implied volatility reflects the fair volatility expected by risk neutral investors in the market.

Realized and Implied Volatility

Black model, despite some well known restrictions and practical inconsistencies, represents the world market standard for option pricing and volatility trading

$$V^{mkt} = Opt(S, K, r, 0, T, \sigma) = \textbf{Black}(S, K, r, T, \sigma_{blk})$$

But the model suggests that

- implied volatility is constant and nothing more than a convenient input parameter,
- implied volatility does not carry any information on the real underlying volatility process.

Realized and Implied Volatility

So, if we assume the dynamics of F to be

$$dF(t, T_1, T_2) = \sigma_2(t) F(t, T_1, T_2) dW_t$$

under the log-normality assumption we can write

$$Cpl^{Black}(0, T_1, T_2, K) = P(0, T_2)\tau \mathbf{Black}(K, F_2(0), v_2(T_1))$$
$$v_2(T_1)^2 = \int_0^{T_1} \sigma_2^2(t) dt$$

Realized and Implied Volatility

Does that holds true in reality? Does there exist a single volatility parameter $v_2(T_1)$ such that both equations hold?

$$C(0, T_1, T_2, K_1) = P(0, T_2)\tau\mathbf{Black}(K_1, F_2(0), v_2(T_1))$$

$$C(0, T_1, T_2, K_2) = P(0, T_2)\tau\mathbf{Black}(K_2, F_2(0), v_2(T_1))$$

Clearly not... But

- Can we price options with the **Black** model?
- Can we hedge options with the **Black** model?

Realized and Implied Volatility

Rebonato:

implied volatility σ_{blk} is just **the wrong number to put in the wrong formula to get the right price of plain-vanilla options** and does not carry any information about the market volatility dynamics.

This means that:

as we are trying to hedge derivatives we need to account for the dependence of implied volatility to changes of the underlying variable. Why are we still using Black? Are we insane?

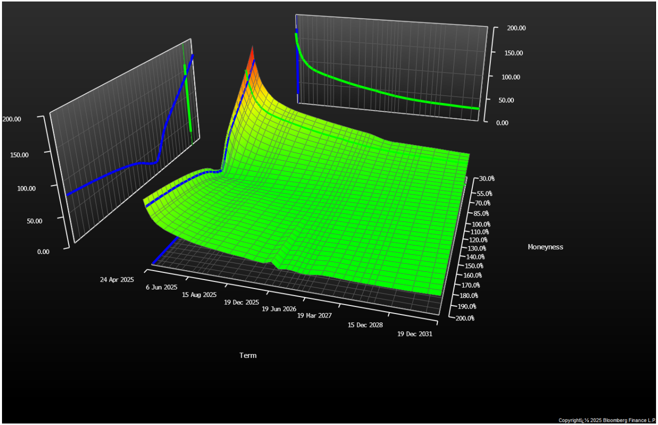
Smile and Skew

The smile derive from the empirical observation of the market data and its manifestation could be justified in different ways

- supply-demand effect for strikes distant from the ATM, *they are like cheap lottery tickets* [1],
- a leptokurtic behaviour of returns that may be different to the assumed standard Gaussian,
- the high correlation of certain maturities to Central Banks politics.

[1] E. Derman, M.B. Miller (2014)

Smile and Skew (and Surfaces)



Smile and Skew

The smile makes Delta a problematic quantity

Consider an option with underlying a single forward rate $F(t, T_{i-1}, T_i)$, in presence of the *smile* the volatility $\sigma^B(T_{i-1})$ is now dependent to $F(t, T_{i-1}, T_i)$. In the case of delta hedging we have that

$$
\begin{aligned}
\Delta &= \frac{\partial Blk(F_t, K, \sigma^B(T_{i-1}))}{\partial F_t} \\
&= N(d_1) + \frac{\partial Blk(F_t, k, \sigma^B(T_{i-1}))}{\partial \sigma^B(T_{i-1})} \frac{\partial \sigma^B(T_{i-1})}{\partial F_t} \\
&= N(d_1) + \mathcal{V}_{Blk} \frac{\partial \sigma^B(T_{i-1})}{\partial F_t}
\end{aligned}
$$

Smile and Skew

What is $\frac{\partial \sigma^B(T_{j-1})}{\partial F_t}$?

There are two popular extreme cases traders work with

- if $\partial \sigma^B(T_{j-1})/\partial F_t = 0$ then we have $\Delta = N(d_1)$ and the smile is said to be a *Sticky-smile* (e.g. Black, Bachelier), meaning that the implied volatility doesn't change for movement of the underlying
- if $\partial \sigma^B(T_{j-1})/\partial F_t \neq 0$, we have a *Floating-smile*, which means that the smile will move accordingly to the movements of the underlying.

1. Volatility Smile
2. **Local, Stochastic and Implied Volatility**
3. Gaussian Models and Skew Dynamics
4. SABR

Models Relation

Gyongy's Lemma Intuition

Consider the following stochastic process

$$d\epsilon_t = \beta(t,\omega)dt + \delta(t,\omega)dW_t \tag{1}$$

with β, δ being adapted processes, and $\delta\delta^T$ being positively defined. We can define another process:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2}$$

whose solution has the same component-wise distribution if

$$b(t,x) = \mathbb{E}\left[\beta_t | \epsilon_t = x\right] \tag{3}$$

$$\sigma^2(t,x) = \mathbb{E}\left[\delta_t^2 | \epsilon_t = x\right] \tag{4}$$

Local and Stochastic Volatility

Consider two models for the underlying:

$$dS_t = rS_t dt + \sigma(t, S_t) S_t dW_t$$
$$d\tilde{S}_t = r\tilde{S}_t dt + \sigma(t, \omega) \tilde{S}_t dW_t$$

where the first equation refers to the local volatility model and the second to the stochastic volatility model.

Local and Stochastic Volatility

We can now exploit the relationship between the drifts using the Gyongy interpretation of parameters

$$r - \frac{1}{2}\sigma^2(t, S_t) = \mathbb{E}\left[r - \frac{1}{2}\sigma^2(t,\omega) \middle| \ln \frac{\tilde{S}_t}{\tilde{S}_0} = \ln \frac{S_t}{S_0} \right]$$

$$\sigma^2(t, s) = \mathbb{E}\left[\sigma^2(t,\omega) \middle| \tilde{S}_t = S_t \right]$$

It is possible to interpret the information carried by the local variance model as the average of a process whose volatility is stochastic.

Local and Implied Volatility

What about their relation with implied volatility?
We know that the payoff of a vanilla Call option is given by

$$C_{K,T}(S_T, T) = (S_T - K)^+$$

with Black and Scholes (BS) PDE

$$\frac{\partial C_{BS}}{\partial t} = rC_{BS} - rS\frac{\partial C_{BS}}{\partial S} - \frac{1}{2}\sigma_{BS}^2 S^2 \frac{\partial^2 C_{BS}}{\partial S^2}$$

we can replace the 2^{nd} derivative with Γ

$$\frac{\partial C_{BS}}{\partial t} = rC_{BS} - rS\frac{\partial C_{BS}}{\partial S} - \frac{1}{2}\sigma_{BS}^2 S^2 \Gamma_{BS}$$

Local and Implied Volatility

A tale of two options and a Delta hedged portfolio

Let's construct a portfolio long one option $C_{K,T}$ and short another option C_{BS}. We will assume that the first option process is driven by a local volatility model:

$$\pi = C_{K,T} - C_{BS}$$
$$\frac{\partial \pi}{\partial t} = \frac{\partial C_{K,T}}{\partial t} - \frac{\partial C_{BS}}{\partial t}$$

Remember, $C_{K,T}$ PDE is

$$\frac{\partial C_{K,T}}{\partial t} = rC_{K,T} - rS\frac{\partial C_{K,T}}{\partial S} - \frac{1}{2}\sigma^2(t,S)S^2\frac{\partial^2 C_{K,T}}{\partial S^2}$$

Local and Implied Volatility

Substitute the two PDE to get

$$\frac{\partial \pi}{\partial t} = r(C_{K,T} - C_{BS}) - rS\left(\frac{\partial C_{K,T}}{\partial S} - \frac{\partial C_{BS}}{\partial S}\right) - \frac{1}{2}S^2\left(\sigma^2(t,S)\frac{\partial^2 C_{K,T}}{\partial S^2} - \sigma^2_{BS}\Gamma_{BS}\right)$$

recall that the two S processes are different, but here:

- We know the value of both at $t0$ (equal).
- we collect the differences in P&L with the Γ terms, because the drift terms are the same.
- you can assume that both options have been Δ-hedged.

Local and Implied Volatility

We can now write the equation in terms of portfolio

$$\frac{\partial \pi}{\partial t} = r\pi - rS\frac{\partial \pi}{\partial S} - \frac{1}{2}S^2\left(\sigma^2(t,S)\frac{\partial^2 C_{K,T}}{\partial S^2} - \sigma_{BS}^2\Gamma_{BS}\right)$$

add and subtract $\sigma^2(t,s)\Gamma_{BS}$ inside the parenthesis

$$\begin{aligned}\frac{\partial \pi}{\partial t} = r\pi - rS\frac{\partial \pi}{\partial S} &- \frac{1}{2}S^2\sigma^2(t,S)\left(\frac{\partial^2 C_{K,T}}{\partial S^2} - \Gamma_{BS}\right)\\ &- \frac{1}{2}S^2\Gamma_{BS}(\sigma^2(t,S) - \sigma_{BS}^2)\end{aligned}$$

Local and Implied Volatility

Substitute in terms of the portfolio Γ and set

$$\frac{1}{2}S^2\Gamma_{BS}(\sigma^2(t,S)-\sigma_{BS}^2)=f$$

we can write the portfolio PDE as

$$\frac{\partial\pi}{\partial t}=r\pi-rS\frac{\partial\pi}{\partial S}-\frac{1}{2}S^2\sigma^2(t,S)\frac{\partial^2\pi}{\partial S^2}-f(t,S)$$

by Feynman-Kac thorem

$$\pi_0=\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rT}\pi(T,S_T)\right]+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T rdu}f(t,S_t)dt\right]$$

Local and Implied Volatility

Portfolio π is long and short an option of the same maturity

$$\pi_0 = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rT} S^2 \Gamma_{BS}(\sigma^2(t,S) - \sigma_{BS}^2)dt\right] = 0$$

this is the expected value of the lifetime P&L of the portfolio as we know that

- terminal payoff is the same, $\pi_T = 0$
- initial value is the same as option prices reflects the market price, $\pi_0 = 0$

Local and Implied Volatility

Therefore

$$\sigma_{BS}^2 = \frac{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rT} S^2 \Gamma_{BS} \sigma^2(t,S) dt\right]}{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rT} S^2 \Gamma_{BS} dt\right]}$$

implied variance is expressed as the weighted expectation of local variance, where the weight is $w(t,S) = e^{-rT} S^2 \Gamma_{BS}$

$$\sigma_{BS}^2 = \frac{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T w(t,S) \sigma^2(t,S) dt\right]}{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T w(t,S) dt\right]}$$

Time-Dependent and Implied Volatility

Let's see what happens when σ is a deterministic function of time

$$dS_t = rS_t dt + \sigma(t) S_t dW_t$$

if we repeat previous derivation we get

$$f = \frac{1}{2} S^2 \Gamma_{BS} (\sigma^2(t,S) - \sigma^2_{K,T}(t))$$

$$\begin{aligned}
\pi_0 &= \frac{1}{2} \mathbb{E}^{\mathbb{Q}} \left[\int_0^T e^{-rT} S^2 \Gamma_{BS} (\sigma^2(t,S) - \sigma^2_{K,T}(t)) dt \right] \\
&= e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[S^2 \Gamma_{BS} (\sigma^2(t,S) - \sigma^2_{K,T}(t)) \right] = 0
\end{aligned}$$

Time-Dependent and Implied Volatility

We assume that the P&L will be zero over time:

$$\pi_0 = \mathbb{E}^{\mathbb{Q}}\left[S^2\Gamma_{BS}(\sigma^2(t,S)\right] - \sigma^2_{K,T}(t)\mathbb{E}^{\mathbb{Q}}\left[S^2\Gamma_{BS}\right] = 0$$

Then:

$$\sigma^2_{K,T}(t) = \frac{\mathbb{E}^{\mathbb{Q}}\left[S^2\Gamma_{BS}\sigma^2(t,S)\right]}{\mathbb{E}^{\mathbb{Q}}\left[S^2\Gamma_{BS}\right]}$$

Time-Dependent and Implied Volatility

To see the relation between the time dependent volatility and the constant implied volatility:

$$\ln S_T = \ln S_0 + rT - \frac{1}{2}\int_0^T \sigma^2_{K,T}(t)dt + \int_0^T \sigma_{K,T}(t)dW_t$$

Calculate mean and variance:

$$\mathbb{E}[\ln S_T] = \ln S_0 + rT - \frac{1}{2}\int_0^T \sigma^2_{K,T}(t)dt$$

$$Var[\ln S_T] = \int_0^T \sigma^2_{K,T}(t)dt$$

Time-Dependent and Implied Volatility

By Ito-Isometry we know that:

$$\mathbb{E}[\ln S_T] = \ln S_0 + \left(r - \frac{1}{2}\sigma_{BS}^2\right) T$$
$$Var[\ln S_T] = \sigma_{BS}^2 T$$

Finally, comparing the two variances:

$$\sigma_{BS}^2 = \frac{1}{T}\int_0^T \sigma_{K,T}^2(t)dt$$

We can express Black volatility as the expectation of time dependent volatility.

Summary

- The empirical evidence provided by econometric studies is a departure from the lognormal distribution for most of the Assets.
- Traders use Black as a quoting mechanism and use it against one of his main assumptions, reality comes back with the calculation of vol-sensitivities.
- In order to avoid P&L swings traders must choose models who best represent the dynamics of their assets. Hence, local volatility, stochastic volatility, jumps models are brought on the stage.
- Statistical analysis and traders' tricks are two faces of the same coin: the inadequate modeling of the underlying detected by econometric analysis is reflected in troubles when traders use models in their hedging strategies.

1. Volatility Smile

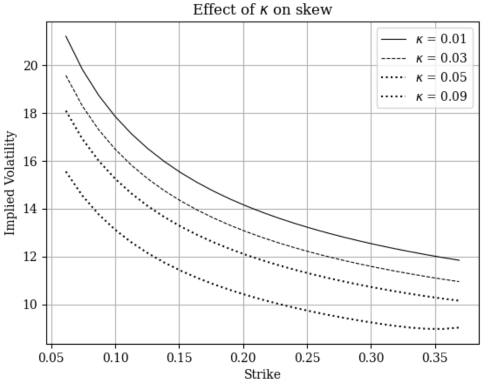
2. Local, Stochastic and Implied Volatility

3. **Gaussian Models and Skew Dynamics**

4. SABR

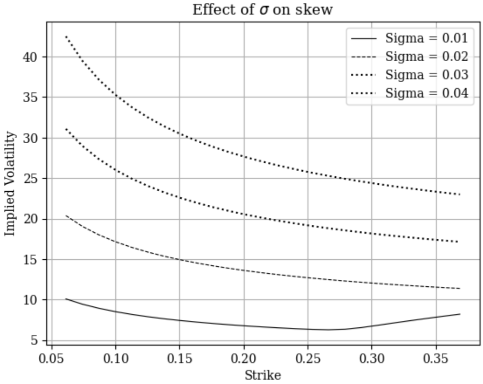
One Factor Gaussian Rate

Keeping the σ parameter fixed and varying κ



One Factor Gaussian Rate

Keeping the κ parameter fixed and varying σ



One Factor Gaussian Rate

Remark

Consider the model

$$dr(t) = [\theta(t) - \kappa r(t)]dt + \sigma(t)dW(t),$$

where κ and σ are positive constants, with θ chosen to fit the currently observed yield curve in the market

$$\theta(t) = \frac{\partial}{\partial t} f^M(0,t) + \kappa f^M(0,t) + \frac{\sigma(t)^2}{2\kappa}(1 - e^{-2\kappa t}),$$

- Caplet volatilities increase as σ increases and decrease as κ increases,
- Such features are indeed consistent with the theoretical meaning of the model parameters.

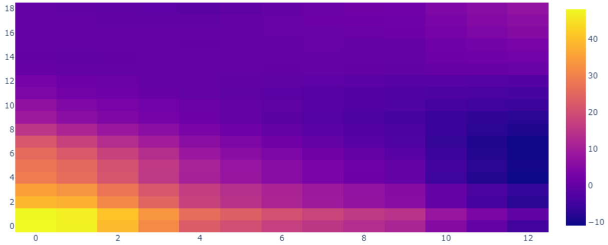
Calibration Issues

It's important to understand how the implied volatility structures vary due to changes in model parameters. For a single factor Gaussian short rate model, we can conclude that:

- Despite being different in nature, the impact of the short-rate parameters on volatility smile is similar.
- A widely accepted market practice is to keep the mean reversion parameter fixed to a coherent value, while calibrating the σ parameter to implied volatilities.
- Model-smile is not so flexible and cannot reproduce complex volatility structures.

Calibration Issues

Consider the following calibration of a Gaussian Short Rate model with time dependent volatility (see *calibration.ipynb*). By plotting errors (in bps) we can clearly see all the limitations of such models in terms of model volatility:



Dynamics

Among the stochastic volatility models available for sure one of the most accepted and utilized in the market is the SABR (*Stochastic Alpha-Beta-Rho*) model by Hagan et al. (2002)
The model is described by the following processes

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_F(t), & \text{with } F_t = F(t, T_{j-1}, T_j) \\ d\alpha_t = \nu \alpha_t dZ_\alpha(t) \end{cases}$$

Each of these parameters has a **direct impact on the smile**

$$\rho = \langle dW_F(t), dZ_\alpha(t) \rangle$$

Hagan Approximation

The price of a plain-vanilla option written on the forward rate F_t, with expiry T_n and strike k can be obtain by putting in the Black formula the implied volatility given by:

$$\sigma(F_t, k) = \frac{\alpha_t}{(kF_t)^{\frac{1-\beta}{2}} \left[1 + \frac{(1-\beta)^2}{24} \ln^2(\frac{F_t}{k}) + \frac{(1-\beta)^4}{1920} \ln^4(\frac{F_t}{k}) + \dots\right]}$$

$$\left(\frac{z}{x(z)}\right) \left[1 + \frac{(1-\beta)^2}{24} \frac{\alpha^2}{(kF_k)^{1-\beta}} T_n + \right.$$

$$\left. + \frac{1}{4} \frac{\alpha_t \beta \rho \nu}{(kF_k)^{\frac{1-\beta}{2}}} T_n + \frac{2-3\rho^2}{24} \nu^2 T_n + \dots \right]$$

Hagan Approximation

where

$$z = \frac{\nu}{\alpha_t}(kF_t)^{\frac{1-\beta}{2}} \ln\left(\frac{F_t}{k}\right),$$

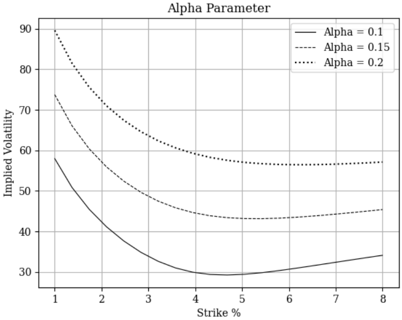
$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right)$$

In the case of an option ATM $(F_t = k)$, the formulation can be simplified to

$$\sigma_{ATM} = \frac{\alpha_t}{F_t^{1-\beta}}\left[1 + \left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{F_t^{2-2\beta}} + \frac{1}{4}\frac{\alpha_t\beta\rho\nu}{(kF_k)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\nu^2 + \dots\right)\right.$$

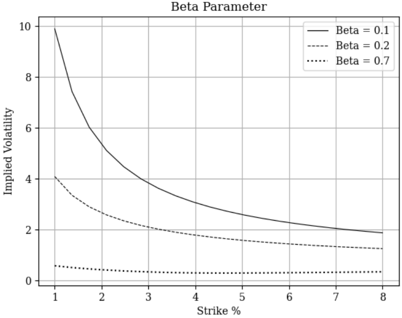
α Parameter

The α parameter is the value at time zero of the stochastic volatility process. Changes of this value causes the vertical movement of the smile, with little or no effect on its shape:



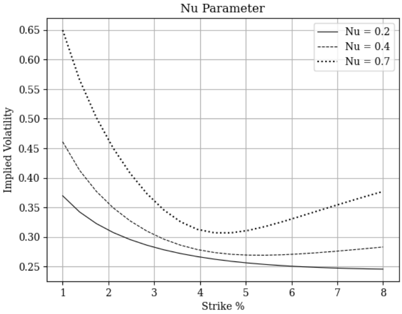
β Parameter

The β parameter is the constant elasticity of variance. Moving from values of β between 0 and 1 the model switches from a normal-like to a lognormal-like behaviour:



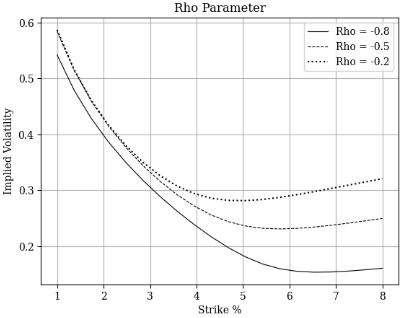
ν Parameter

The ν parameter is the volatility of α. It has influence on the curvature of the smile:



ρ Parameter

The ρ parameter is the correlation between the forward rate and the volatility processes. Its value can vary between -1 and +1:



Revisited Black Sensitivities

The SABR delta Δ_{SABR} is the first derivative with respect to the forward rate F_t

$$\Delta_{SABR} = \Delta_{Blk} + \mathcal{V}_{Blk} \frac{\partial \sigma^B(T_{j-1})}{\partial \alpha_t} \frac{\rho \nu}{F_t^\beta}$$

The SABR Vega ($\mathcal{V}_{SABR}$) is defined by the change of the option value due to infinitesimal change in α_t

$$\mathcal{V}_{SABR} = \mathcal{V}_{Blk} \left(\frac{\partial \sigma^B(T_{j-1})}{\partial \alpha_t} + \frac{\partial \sigma^B(T_{j-1})}{\partial F_t} \frac{\rho F_t^\beta}{\nu} \right)$$

Skew and Curvature Sensitivities

The sensitivity with respect to β or ρ is called *vanna* and represent the risk of changes in the smile skewness.

$$vanna = \begin{cases} \frac{\partial Blk(F_t, k, \sigma^B(T_{j-1}), w)}{\partial \rho} = \mathcal{V}_{Blk} \frac{\partial \sigma^B(T_{j-1})}{\partial \rho} \\ \frac{\partial Blk(F_t, k, \sigma^B(T_{j-1}), w)}{\partial \beta} = \mathcal{V}_{Blk} \frac{\partial \sigma^B(T_{j-1})}{\partial \beta} \end{cases}$$

The sensitivity with respect to ν is instead called *volga* and express the risk of a change in the smile curvature.

$$volga = \frac{\partial Blk(F_t, k, \sigma^B(T_{j-1}), w)}{\partial \nu} = \mathcal{V}_{Blk} \frac{\partial \sigma^B(T_{j-1})}{\partial \nu}$$