Interest Rate Models

Advanced Financial Modeling

Andrea Carapelli

Università di Siena

2024

Suggested References

Everything will be covered during the course and provided to you as slides, notes or Python notebooks.
If you want more, check the following:

- D. Brigo, F. Mercurio (2006) Interest Rate Models - Theory and Practice (2nd edition)
- L. Andersen, V. Piterbarg (2010) Interest Rate Models. Vol.1, Vol.2, Vol.3.
- T. Bjork (2009) Arbitrage Theory in Continuous Time (3rd edition).

Framework

Heath, Jarrow and Morton (1992) assumed that, for a fixed a maturity T, the instantaneous forward rate $f(t,T)$ evolves, under a given measure, according to the following diffusion process:

$$df(t,T) = \mu(t,T)dt + \sigma_f(t,T)dW_t$$

- The framework directly models the dynamics of instantaneous forward rates
- The framework provides an explicit relation between the volatility of the instantaneous forward rates and arbitrage-free drift
- The framework can be used to derive a limited number of models, like extended short rate models and Libor Market Models (LMM)

Risk Neutral Dynamics

Under the HJM framework the short rate $r(t)$ is defined as the limit of the instantaneous forward $r(t) = f(t,t)$. We will start by modelling $P(t,T)$ (same maturity of $f(t,T)$), as the instantaneous forward $f(t,T)$ is not a traded asset

$$dP(t,T) = r_t P(t,T)dt + \sigma_P(t,T)P(t,T)dW_t \tag{1}$$

Starting from this process now we have to infer the dynamics of $f(t,T)$.

Risk Neutral Dynamics

We can compute $d \ln P(t, T)$ by Ito and obtain

$$d \ln P(t, T) = r_t dt + \sigma_P(t, T) dW_t - \frac{1}{2}\sigma_P(t, T)^2 dt$$

take derivative with respect to T as we are interested in the instantaneous forward rate dynamics

$$\frac{\partial}{\partial T} d \ln P(t, T) = \frac{\partial}{\partial T}\sigma_P(t, T) dW_t - \frac{1}{2} 2\sigma_P(t, T)\frac{\partial}{\partial T}\sigma_P(t, T) dt$$

Risk Neutral Dynamics

Derive the instantaneous forward rate dynamics under the risk neutral measure

$$df(t, T) = \sigma_P(t, T)\frac{\partial}{\partial T}\sigma_P(t, T)dt - \frac{\partial}{\partial T}\sigma_P(t, T)dW_t$$

with the **diffusion** coefficient of the instantaneous forward being

$$\sigma_f(t, T) = -\frac{\partial}{\partial T}\sigma_P(t, T)$$

$$\int_t^T \sigma_f(t, u)du + C = -\sigma_P(t, T)$$

Risk Neutral Dynamics

At maturity $\sigma(T, T) = 0, \; C = 0.$
We can write

$$\sigma_P(t, T) = -\int_t^T \sigma_f(t, u) du$$

substitute to obtain the **drift** coefficient of the forward

$$\mu(t, T) = \sigma_f(t, T) \int_t^T \sigma_f(t, u) du$$

Finally, we get the full dynamics of $f(t, T)$ under the risk neutral measure $\mathbb{Q}$

$$df(t, T) = \left(\sigma_f(t, T) \int_t^T \sigma_f(t, u) du \right) dt + \sigma_f(t, T) dW_t \qquad (2)$$

Forward Dynamics

The dynamics of $B(t)$ are known

$$\frac{B(0)}{B(t)} = e^{-\int_0^t r_u du}$$

and we defined $P(t,T)$ process in [1], we can write

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = e^{-\frac{1}{2}\int_0^t \sigma_P^2(u,T)du + \int_0^t \sigma_P(u,T)dW_u} \tag{3}$$

Forward Dynamics

Seems like the Radon-Nikodym derivative

If we compare the Radon-Nikodym derivative with the equation [3] we see that we established a connection as $y_u = \sigma_P(u, T)$. Following the Girsanov theorem we can write

$$dW_t^T = dW_t - \sigma_P(t, T)dt = dW_t + \int_t^T \sigma_f(t, u)dudt$$

Forward Dynamics

Now we can replace the new Brownian process in the Risk Neutral HJM [2] and obtain the dynamics under the new T-Forward probability measure

$$
\begin{aligned}
df(t,T) = &\left(\sigma_f(t,T)\int_t^T \sigma_f(t,u)du\right)dt \\
&+ \sigma_f(t,T)\left(dW_t^T - \int_t^T \sigma_f(t,u)dudt\right)
\end{aligned}
$$

the drift term can be simplified as $f(t,T)$ is a martingale under the $\mathbb{Q}^T$ forward measure

$$
df(t,T) = \sigma_f(t,T)dW_t^T \tag{4}
$$

Forward Dynamics

But how does this relates to the multi-curve framework?

- We have shown that it is possible to model a single T-maturity forward as a martingale using the zero coupon bond $P(t, T)$ numeraire.
- In reality, you would have to model multiple forwards with maturity $T_1, T_2, \ldots, T_f$.

You don't want to model each one of them under a different probability measure. Typically, you can use the **longest maturity** $P(t, T_f)$ where $T_f > T$ where T_f is the numeraire of the **Terminal forward measure**.

Terminal Forward Dynamics

From Girsanov theorem we know that

$$dW_t^{T_f} = dW_t - \sigma_P(t, T_f)dt = dW_t + \int_t^{T_f} \sigma_f(t,u)dudt$$

we can change the measure of the Risk Neutral equation [2]

$$df(t,T) = -\left(\sigma_f(t,T)\int_T^{T_f}\sigma_f(t,u)du\right)dt + \sigma_f(t,T)dW_t^T \quad (5)$$

Different Measures, same Driver

We can easily see the difference between probability measures in the HJM framework:

- $\mathbb{Q} \rightarrow$ the drift integral will be defined with the remaining maturity,
- $\mathbb{Q}^T \rightarrow$ drift will be zero,
- $\mathbb{Q}^{T_f} \rightarrow$ the drift integral will be defined between T (maturity of the modelled forward) and T_f (maturity of the terminal numeraire).

1. The HJM Framework
2. Extended Short Rate Models
3. Gaussian Dynamics

Short Rate Models

General theory of short rate models assumes that the instantaneous spot rate evolves under the real-world measure according to $\mathbb{Q}_0$

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^{\mathbb{Q}_0}(t)$$

where μ and σ are well behaved functions and $W^{\mathbb{Q}_0}(t)$ is a $\mathbb{Q}_0$ Brownian motion. It is possible to show[1] the existence of a stochastic process λ, if

$$dP(t, T) = \mu^T(t, r(t))dt + \sigma^T(t, r(t))dW^{\mathbb{Q}_0}(t)$$

then

$$\frac{\mu^T(t, r(t)) - r(t)P(t, T)}{\sigma^T(t, r(t))} = \lambda(t)$$

[1] see T. Bjork (2009).

Short Rate Models

Moreover, there exists a measure $\mathbb{Q}$ that is equivalent to $\mathbb{Q}_0$ and is defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_0} = e^{-\frac{1}{2}\int_0^t \lambda^2(u)du - \int_0^t \lambda(u)dW^{\mathbb{Q}_0}(u)}$$

As a consequence, the process r evolves under $\mathbb{Q}$ according to

$$dr(t) = [\mu(t, r(t)) - \lambda(t)\sigma(t, r(t))]dt + \sigma(t, r(t))dW(t) \qquad (6)$$

where $W(t) = W^{\mathbb{Q}_0}(t) + \int_0^t \lambda(u)du$ is a Brownian motion under $\mathbb{Q}$.

Why Extended Models

- A classic problem with standard models is their endogenous nature.
- If we observe the initial zero-coupon bond curve $T \to P^M(0, T)$ from the market, and we wish our model to fit this curve, the only way is to force model parameters to produce a model curve as close as possible to the market.
- Although the values $P^M(0, T)$ are actually observed only at a finite number of maturities, three parameters are not enough to reproduce satisfactorily a given term structure.

Why Extended Models

Endogenous models cannot:

- Reproduce satisfactorily the initial yield curve;
- Reproduce realistic volatility structures.

To improve this situation, exogenous term structure models are usually considered:

- Such models are built by suitably modifying the above endogenous models, by including a new input, the market term structure;
- The basic strategy that is used to transform an endogenous model into an exogenous model is the inclusion of "time-varying" parameters, derived from the HJM framework.

Derivation of Extended Models

A computation that will be helpful in the following slides is the instantaneous absolute volatility of instantaneous forward rates in affine models:

$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T)$$

$$df(t,T) = -\frac{\partial}{\partial T}\ln A(t,T) + \frac{\partial}{\partial T}B(t,T)r(t)$$

so

$$df(t,T) = (\dots)dt + \frac{\partial}{\partial T}B(t,T)\sigma(t,r(t))dW(t)$$

where $\sigma(t,r(t))$ is the diffusion coefficient in the short rate dynamics.

Derivation of Extended Models

It follows that the absolute volatility of the instantaneous forward rate $f(t, T)$ at time t in a short rate model with an affine term structure is

$$\sigma_f(t, T) = \frac{\partial B(t, T)}{\partial T} \sigma(t, r(t)) \tag{7}$$

where $B(t, T)$ and $\sigma(t, r(t))$ are respectively the B function of the chosen short rate model and the volatility term.

Derivation of Extended Models

We know the dynamics of the HJM under the risk neutral measure $\mathbb{Q}$ being

$$df(t,T) = \left(\sigma_f(t,T) \int_t^T \sigma_f(t,u)du \right) dt + \sigma_f(t,T)dW_t$$

Substitute $\sigma_f(t,T)$ in the equation with the B function of the Vasicek model to get

$$df(t,T) = \sigma^2 e^{-\kappa(T-t)} \left(\int_t^T e^{-\kappa(u-t)} du \right) dt + \sigma e^{-\kappa(T-t)} dW_t$$

Derivation of Extended Models

Solve and integrate to obtain

$$
\begin{aligned}
\int_0^t df(s,T) &= \frac{\sigma^2}{\kappa} \int_0^t \left(e^{-\kappa(T-s)} - e^{-2\kappa(T-s)} \right) ds \\
&\quad + \sigma \int_0^t e^{-\kappa(T-s)} dW_s \\
f(t,T) - f(0,T) &= \frac{\sigma^2}{\kappa} \left(\frac{e^{-\kappa(T-s)}}{\kappa} - \frac{e^{-2\kappa(T-s)}}{2\kappa} \right) \Bigg|_{s=0}^{s=t} \\
&\quad + \sigma \int_0^t e^{-\kappa(T-s)} dW_s
\end{aligned}
$$

Derivation of Extended Models

as we know $r(t) = f(t,t)$, we can substitute to get

$$f(t,t) = f(0,t) + \frac{\sigma^2}{\kappa}\left(\frac{1-e^{-\kappa t}}{\kappa} - \frac{1-e^{-2\kappa t}}{2\kappa}\right) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s$$

$$r(t) = f(0,t) + \frac{\sigma^2}{2\kappa^2}\left(1 - 2e^{-\kappa t} + e^{-2\kappa t}\right) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s \quad (8)$$

Derivation of Extended Models

we know that ???

$$dr(t) = df(t,T)\Big|_{T=t} + \frac{\partial}{\partial T} f(t,T)\Big|_{T=t} dt$$

and get

$$dr(t) = \kappa\left(\frac{1}{\kappa}\frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{\kappa^2}\left(e^{-\kappa t} - e^{-2\kappa t}\right) - \sigma \int_0^t e^{-\kappa(t-s)} dW_s\right) dt + \sigma dW_t$$

Derivation of Extended Models

and obtain

$$dr(t) = \kappa\left(\frac{1}{\kappa}\frac{\partial}{\partial t}f(0,t) + \frac{\sigma^2}{\kappa^2}\left(e^{-\kappa t} - e^{-2\kappa t}\right) - \sigma\int_0^t e^{-\kappa(t-s)}dW_s\right)dt + \sigma dW_t$$

where we factor out the κ as we notice that the term

$$\sigma\int_0^t e^{-\kappa(t-s)}dW_s$$

is the same as in equation [8].

Derivation of Extended Models

if we isolate the term and we substitute

$$dr(t) = \kappa \left(\frac{1}{\kappa} \frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{2\kappa^2} \left(1 - e^{-2\kappa t}\right) - r(t) + f(0,t) \right) dt + \sigma dW_t$$

we can conclude that this model will match the initial term structure, setting

$$\theta(t) = f(0,t) + \frac{1}{\kappa} \frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{2\kappa^2} \left(1 - e^{-2\kappa t}\right)$$

we have the dynamics for the extended short rate model

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma dW_t$$

One Factor Gaussian Short Rate (HW)

We have shown that $r(t)$ evolves under the risk neutral measure according to

$$dr(t) = [\theta(t) - \kappa r(t)]dt + \sigma dW(t)$$

now consider κ and σ to be positive constants, with θ chosen to fit the currently observed yield curve in the market [2]

$$\theta(t) = \frac{\partial}{\partial t} f^M(0,t) + \kappa f^M(0,t) + \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})$$

where $\frac{\partial}{\partial t} f^M(0,t)$ denotes partial derivative of f^M with respect to its second argument and $f^M(0,t)$ being the market instantaneous forward rate at time 0 for maturity t

[2] section [2]

One Factor Gaussian Short Rate (HW) -Conditional Mean and Variance

The equation can be integrated to obtain

$$r(t) = r(s)e^{-\kappa(t-s)} + \alpha(t) - \alpha(s)e^{-\kappa(t-s)} + \sigma \int_s^t e^{-\kappa(t-u)} dW(u)$$

where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2\kappa^2}(1 - e^{-\kappa t})^2$$

$r(t)$ conditional on $r(s)$ with $t > s$ is Normally distributed with mean and variance given respectively by

$$\mathbb{E}[r(t)|r(s)] = r(s)e^{-\kappa(t-s)} + \alpha(t) - \alpha(s)e^{-\kappa(t-s)}$$

$$Var[r(t)|r(s)] = \frac{\sigma^2}{2\kappa}[1 - e^{-2\kappa(t-s)}]$$

One Factor Gaussian Short Rate (HW) - Bonds

It's possible to obtain the price for zero-coupon bonds, allowing us to compute a new term structure as a function of a given short rate. It's shown that

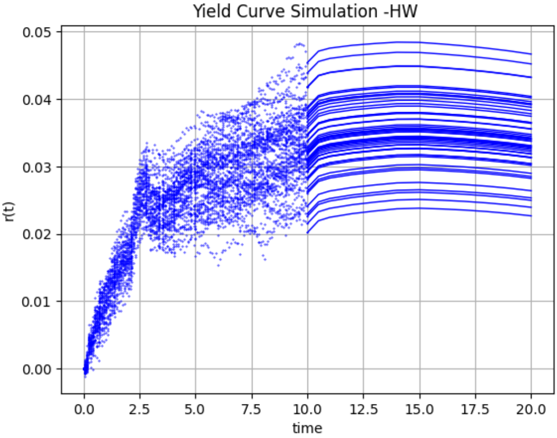
$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{9}$$

where

$$B(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)} exp\left(B(t,T)f^M(0,t) - \frac{\sigma^2}{4\kappa}(1-e^{-2\kappa t})B(t,T)^2\right)$$

Yield Curve Dynamics



Two Factors Gaussian Short Rate (G2++)

Remarks:

- It's not easy to find a realistic market situation that can be correctly represented by a one factor model;
- In such models it's impossible to find any feasible opportunities to capture interest rates correlation;
- A model with two or more factors, capable to include forward rates correlation, produces better results and is ideal to explain an acceptable portion of interest rates variability.

Two Factors Gaussian Short Rate (G2++)

The Gaussian two factor model defines the short rate as the sum of two correlated Gaussian random variables, under the risk neutral measure $\mathbb{Q}$ the interest rate dynamics are

$$r(t) = x(t) + y(t) + \varphi(t), \qquad r(0) = r_0$$

With $\varphi(t)$ being a deterministic function of time, chosen to exactly fit the initial term structure of interest rates [3], $\varphi(0) = r_0$

$$\begin{aligned}\varphi(t) =& f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \\ &+ \frac{\eta^2}{2b^2}(1 - e^{-bt2}) + \rho\frac{\sigma\eta}{ab}(1 - e^{-at})(1 - e^{-bt})\end{aligned}$$

[3] similar procedure of section [2]

Two Factors Gaussian Short Rate (G2++)

and the two processes $x(t)$ and $y(t)$ defined as

$$
\begin{aligned}
dx(t) &= -ax(t)dt + \sigma dW_1(t), \qquad x(0) = 0; \\
dy(t) &= -by(t)dt + \eta dW_2(t), \qquad y(0) = 0
\end{aligned}
$$

where W_1 and W_2 are two different Brownian motion with correlation ρ:

$$dW_1(t)dW_2(t) = \rho dt$$

Two Factors Gaussian Short Rate (G2++)

Integrate to obtain the two stochastic differential equations solutions

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_1(u)$$
$$y(t) = y(s)e^{-b(t-s)} + \eta \int_s^t e^{-b(t-u)} dW_2(u)$$

the short rate evaluated at t_0 for a given maturity T is expressed as

$$r_0(T) = \sigma \int_0^T e^{-a(T-u)} dW_1(u) + \eta \int_0^T e^{-b(T-u)} dW_2(u) + \varphi(T)$$

letting $T \to \infty$ the two factors reverts to their initial value $x(\infty), y(\infty) \to 0$ and the $r(T)$ process mean reverts to $\varphi(t)$.

Two Factors Gaussian Short Rate (G2++)

By the properties of stochastic integral of deterministic functions we obtain mean, variance and covariance from the bivariate Gaussian distribution of $r(t)$ conditional on $r(s)$ with $s < t$ as

$$\mathbb{E}^{\mathbb{Q}}[x(t)|x(s)] = x(s)e^{-a(t-s)}, \quad \mathbb{E}^{\mathbb{Q}}[y(t)|y(s)] = y(s)e^{-b(t-s)}$$

$$Var^{\mathbb{Q}}[x(t)] = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}], \quad Var_s^{\mathbb{Q}}[y(t)] = \frac{\eta^2}{2b}[1 - e^{-2b(t-s)}]$$

$$cov_s[x(t), y(t)] = \rho\frac{\sigma\eta}{a+b}[1 - e^{-(a+b)(t-s)}]$$

Two Factors Gaussian Short Rate (G2++)

The two processes can also be rewritten in terms of two independent Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$ applying a Cholesky decomposition

$$dW_1(t) = d\tilde{W}_1,$$
$$dW_2(t) = \rho d\tilde{W}_1 + \sqrt{1-\rho^2}\, d\tilde{W}_2.$$

Substituting in the two equation for $x(t)$ and $y(t)$ by integration we get

$$x(t) = x(s)e^{-a(t-s)} + \sigma \int_0^t e^{-a(t-u)} d\tilde{W}_1(u)$$
$$y(t) = y(s)e^{-b(t-s)} + \eta\rho \int_0^t e^{-b(t-u)} d\tilde{W}_1(u) + \eta\sqrt{1-\rho^2} \int_0^t e^{-b(t-u)} d\tilde{W}_2(u)$$

Two Factors Gaussian Short Rate (G2++)

We obtain the discount bond term structure

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]$$

where the integral in the exponential $I(t,T) = \int_t^T r_s ds$ is shown[4] to be Normally distributed with mean given by

$$\mathbb{E}_t^{\mathbb{Q}}[I(t,T)] = \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{1-e^{-b(T-t)}}{b}y(t)$$

[4] see D. Brigo, F. Mercurio (2006) Interest Rate Models.

Two Factors Gaussian Short Rate (G2++)

and variance

$$
\begin{aligned}
Var^{\mathbb{Q}}[I(t,T)] = & \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] \\
& \frac{\eta^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right] \\
& 2\rho\frac{\sigma\eta}{ab}\left[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b}\right. \\
& \left. - \frac{e^{-(a+b)(T-t)} - 1}{a+b}\right]
\end{aligned}
$$

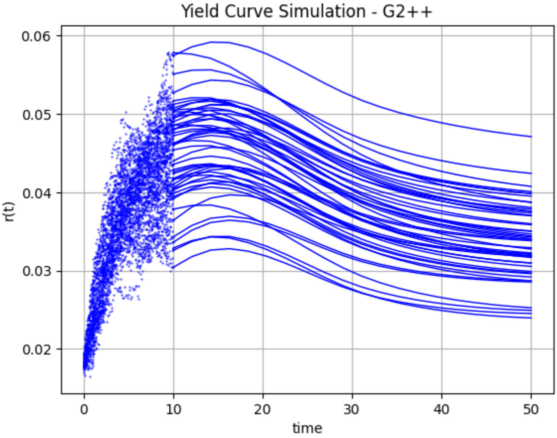
Two Factors Gaussian Short Rate (G2++)

With the zero coupon bond price at time t being defined as

$$P(t, T) = \frac{P(0, T)}{P(0, t)} e^{A(t,T)}$$

$$A(t, T) = \frac{1}{2}[var(t, T) - var(0, T) + var(0, t)] - \mathbb{E}_t^{\mathbb{Q}}[I(t, T)]$$

Yield Curve Dynamics



Yield Curve Dynamics

Exercise:

Replicate the plot above for the Hull and White model

Hint:

Start from the file *"short_rates.ipynb"* and write your own Monte-Carlo loop without using QuantLib. Use the function on slide 47 for mean and variance and compute future rate curves as presented on slide 48.