2.3 Solve the G2++ Process

If we integrate we obtain the two stochastic differential equations solutions

$$\begin{aligned} x(t) =& x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_1(u), \\ y(t) =& y(s)e^{-b(t-s)} + \eta \int_s^t e^{-b(t-u)} dW_2(u); \end{aligned}$$

then the short rate evaluated at t_0 for a given maturity T is expressed as

$$r_0(T) = \sigma \int_0^T e^{-a(T-u)} dW_1(u) + \eta \int_0^T e^{-b(T-u)} dW_2(u) + \varphi(T);$$

letting $T \to \infty$ the two factors reverts to their initial value $x(\infty), y(\infty) \to 0$ and the $r(T)$ process mean reverts to $\varphi(t)$.

By the properties of stochastic integral of deterministic functions we obtain mean, variance and covariance from the bivariate Normal distribution of $r(t)$ conditional on $r(s)$ with $s < t$ as

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[x(t)|x(s)] = x(s)e^{-a(t-s)}, &\quad \mathbb{E}^{\mathbb{Q}}[y(t)|y(s)] = y(s)e^{-b(t-s)}; \\ Var^{\mathbb{Q}}[x(t)] = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}], &\quad Var_s^{\mathbb{Q}}[y(t)] = \frac{\eta^2}{2b}[1 - e^{-2b(t-s)}]; \\ cov_s[x(t), y(t)] = \rho \frac{\sigma\eta}{a+b}&[1 - e^{-(a+b)(t-s)}]. \end{aligned}$$

The two processes can also be rewritten in terms of two independent Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$ applying a Cholesky decomposition

$$\begin{aligned} dW_1(t) &= d\tilde{W}_1, \\ dW_2(t) &= \rho d\tilde{W}_1 + \sqrt{1-\rho^2} d\tilde{W}_2. \end{aligned}$$

Substituting in the two equation for $x(t)$ and $y(t)$ by integration we get

$$\begin{aligned} x(t) &= x(s)e^{-a(t-s)} + \sigma \int_0^t e^{-a(t-u)} d\tilde{W}_1(u), \\ y(t) &= y(s)e^{-b(t-s)} + \eta\rho \int_0^t e^{-b(t-u)} d\tilde{W}_1(u) + \eta\sqrt{1-\rho^2} \int_0^t e^{-b(t-u)} d\tilde{W}_2(u). \end{aligned}$$

2.4 xVA Simulation

a) If the default of a counterparty happens after the final payment of derivative T, the value at time t is simply

$$1_{\tau > T} V(t, T)$$

. If the default occurs before the maturity time $\tau < T$:

1. We receive/pay all the payments until the default time: $1_{\tau \le T} V(t, \tau)$;
2. Depending on the counterparty, we may be able to recover some of the future payments, assuming the recovery fraction to be R the value yields: $1_{\tau \le T} R \max(V(\tau; T), 0)$;
3. On the other hand, if we owe the money to the counterparty that has defaulted we cannot keep the money but we need to pay it completely back: $1_{\tau \le T} \min(V(\tau; T), 0)$.

Thus, when including all the components, a price of a *risky* derivative is given by:

$$\begin{aligned} V_D(t_0, T) = \mathbb{E}^Q \big[& 1_{\tau > T} V(t_0, T) + 1_{\tau \le T} V(t_0, \tau) \\ & + D(t_0, \tau)\, 1_{\tau \le T} R \max(V(\tau, T), 0) \\ & + D(t_0, \tau)\, 1_{\tau \le T} \min(V(\tau, T), 0) \,|\, \mathcal{F}_t \big] \end{aligned}$$

Since $x = \max(x,0) + \min(x,0)$, the simplified equation reads:

$$\begin{aligned} V_D(t_0,T) = \mathbb{E}^Q \big[& 1_{\tau > T} V(t_0,T) + 1_{\tau \le T} V(t_0,\tau) \\ & + D(t_0,\tau)\, 1_{\tau \le T} V(\tau;T) \\ & + D(t_0,\tau)\, 1_{\tau \le T} (R-1) \max(V(\tau;T),0) \,|\, \mathcal{F}_t \big] \end{aligned}$$

We immediately note that the first three terms in the expression above yield:

$$\begin{aligned} & \mathbb{E}^Q \big[1_{\tau > T} V(t_0,T) + 1_{\tau \le T} V(t_0,\tau) + D(t_0,\tau) 1_{\tau \le T} V(\tau,T) \big] \\ & = \mathbb{E}^Q \big[1_{\tau > T} V(t_0,T) + 1_{\tau \le T} V(t_0,T) \big] \\ & = V(t_0). \end{aligned}$$

The value of the risky derivative $V_D(t)$ is:

$$\begin{aligned} V_D(t_0) &= V(t_0) + \mathbb{E}^Q \left[1_{\tau \le T} \left(\mathrm{R}(\tau) - 1 \right) D(t,\tau)\, V(\tau)^+ \,|\, \mathcal{F}_t \right] \\ &= V(t_0) - \mathbb{E}^Q \left[1_{\tau \le T} \, \mathrm{LGD}(\tau) D(t,\tau)\, V(\tau)^+ \,|\, \mathcal{F}_t \right] \\ &= V(t_0) - \mathrm{uCVA}(t_0). \end{aligned}$$

b) Starting from the uCVA equation:

$$\mathrm{uCVA}(t) = 1_{\tau > t} \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}^Q \left[\left(e^{-\int_t^{t_{i-1}} \lambda(s) ds} - e^{-\int_t^{t_i} \lambda(s) ds} \right) \mathrm{LGD}(t_{i-1}) \cdot D(t, t_{i-1}) \cdot \mathrm{PV}^+(t_{i-1}) \right].$$

Assuming :

- finite number of timesteps N,
- constant loss given default,
- independence between default rates and interest rates,
- deterministic hazard rates in t_0.

$$\mathrm{uCVA}_{sw}(t_0) = \mathrm{LGD} \sum_{i=1}^{N} \left(e^{-\int_t^{t_{i-1}} \lambda(s) ds} - e^{-\int_t^{t_i} \lambda(s) ds} \right) \mathbb{E}^Q \left[D(t, t_{i-1}) \mathrm{PV}^+(t_{i-1}) \right]$$

with $\mathbb{E}^Q \left[D(t, t_{i-1}) \mathrm{PV}^+(t_{i-1}) \right]$ being the swaption part.