Advanced Financial Modeling

2024

Exercises

1. Forwards

Suppose that we are in a single curve framework, the term structure of interest rates is defined by the following function:

$$L(0,t) = 1\% + 2\%(e^{-t/3})$$

Assume that at time $t = 0$ you enter a payer FRA with notional $100K$, the rate reset in $t = 2$ and flows are exchanged in $t = 3$.

a) Suppose that, as soon as we execute the trade all the term structure has a parallel shift of $+1\%$. Compute the value of the FRA after the shift

b) Did you make or lose money? Motivate the answer.

2. Interest Rate Swaps

Suppose that we are in a single curve framework, the term structure of interest rates is defined by the following function:

$$L(0,t) = 1.3\% + 1.1\%(t)$$

you enter a 3y receiver vanilla swap 1y forward, fixed and floating payments are annual, and the notional is 100 mln.

a. what is the fixed rate of the swap?

b. what is the relation between FRAs and Swaps? In this case, can you say how single forwards NPV are distributed across the life of the swap?

c. Is the position short or long interest rates? What is the difference with the multi curve framework?

d. Suppose that, as soon as we execute the trade, all the swap rate curve has a parallel shift of -1%. Compute the value of the derivative after the shift.

3. Bond Volatility

Show that an option on a Coupon Bond can be priced as a portfolio of options on Zero Coupon Bonds. Explain clearly when and why do you need this result.

4. OIS Derivatives

Starting from the definition of overnight rate in the EU:

$$r_{ois}(t,T) = \left[\prod_{i=t}^{T-\frac{1}{365}} \left(1 + \frac{r_{ON_i}}{360}\right) - 1 \right]$$

with the corresponding discount factors given by $P_d(t,T_i)$:

$$\begin{aligned} P_d(t,T_i) &= e^{-\int_t^{T_i} r_{ois}(s,s+1d)ds} \\ &= e^{-r_{ois}(t,T_i)(T_i - t)} \end{aligned}$$

a) What is an Overnight Indexed Swap (OIS), and how is it defined?

b) How can you price the floating leg and fixed leg of an OIS? Explain the methodology for pricing the floating leg based on realized overnight rates and the fixed leg based on predetermined fixed rates.

c) Under what conditions can the floating leg equation be simplified? How?

d) Define the annuity factor.

5. Convexity Adjustments

Assume a log-normal distribution for the underlying rate L:

$$dL_i(t) = \sigma L_i(t) dW(t)$$

a) Consider a basic interest rate payoff function, which pays a percentage of a notional N, and the percentage paid will be determined by the Libor rate $L(T_{i-1}, T_i)$. What happen if the payment date is different from the natural T_i? Prove the derivation for $cc(T_{i-1}, T_i)$ in question (b).

b) Solve the convexity equation:

$$cc(T_{i-1}, T_i) = P(t_0, T_i) E^{T_i} \left[\frac{L(T_{i-1}, T_i)}{P(T_{i-1}, T_i)} \right] - F(t_0; T_{i-1}, T_i)$$

hint: apply Ito's Lemma to $L_i^2(t)$, what can you say about the drift of this SDE?

6. Forward HJM Process

Define the HJM dynamics of $f(t, T)$ under the risk neutral measure $\mathbb{Q}$

$$df(t,T) = \left(\sigma_f(t,T) \int_t^T \sigma_f(t,u) du \right) dt + \sigma_f(t,T) dW_t$$

Recall the change of numeraire equality

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(t,T)}{P(0,T)} \frac{B(0)}{B(t)}$$

a) Show that $f(t, T)$ is a martingale under the $\mathbb{Q}^T$ forward measure.

b) What is the difference between forward measure $\mathbb{Q}^T$ and terminal forward measure $\mathbb{Q}^{T_f}$?

7. Solve the G2++ Process

The Gaussian two factor model defines the short rate as the sum of two correlated Gaussian random variables, under the risk neutral measure $\mathbb{Q}$ the interest rate dynamics are

$$r(t) = x(t) + y(t) + \varphi(t), \qquad r(0) = r_0.$$

With $\varphi(t)$ being a deterministic function of time, chosen to exactly fit the initial term structure of interest rates, $\varphi(0) = r_0$

$$\begin{aligned} \varphi(t) =& f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \\ &+ \frac{\eta^2}{2b^2}(1 - e^{-bt2}) + \rho \frac{\sigma\eta}{ab}(1 - e^{-at})(1 - e^{-bt}), \end{aligned}$$

and the two processes $x(t)$ and $y(t)$ defined as

$$\begin{aligned} dx(t) &= -ax(t)dt + \sigma dW_1(t), \qquad x(0) = 0; \\ dy(t) &= -by(t)dt + \eta dW_2(t), \qquad y(0) = 0; \end{aligned}$$

where W_1 and W_2 are two different Brownian motion with correlation ρ

a) Compute conditional expectations $\mathbb{E}^{\mathbb{Q}}[x(t)|x(s)]$ and $\mathbb{E}^{\mathbb{Q}}[y(t)|y(s)]$, what is the value of r_0?

b) Compute factor variances and covariance.

c) What is the difference with the one factor model?

8. xVA Simulation

a) Define briefly Counterparty Credit Risk and CVA. Show that, at valuation time t, with $\tau > t$ (no default happened yet), the price of a derivative under counterparty risk is:

$$V^d(t) = V(t) - \mathbb{E}^Q \left[1_{t<\tau\leq T} \cdot \mathrm{LGD}(\tau) \cdot D(t,\tau) \cdot \mathrm{PV}^+(\tau) \right]$$

b) Write the uCVA equation for an interest rate swap

c) Define netting and collateral. Which components of the collateral value are stochastic variable?

d) Why FVA exists? Define the analogy between credit and funding spread in FVA ad DVA.