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1 Short Rate Extension

We know that affine term-structure models are interest-rate models where the continuously compounded spot rate $R(t,T)$ is an affine function in the short rate $r(t)$

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t), \tag{1}$$

where α and β are deterministic functions of time. If this happens, the model is said to possess an affine term structure. This relationship is always satisfied when the zero–coupon bond price can be written in the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{2}$$

A computation that will be helpful in the following is the instantaneous absolute volatility of instantaneous forward rates in affine models. Since in general

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} lnP(t,T), \\ df(t,T) &= -\frac{\partial}{\partial T} lnA(t,T) + \frac{\partial}{\partial T} B(t,T)r(t), \end{aligned}$$

so if we substitute $r(t)$ we obtain

$$df(t,T) = (\dots)dt + \frac{\partial}{\partial T} B(t,T)\sigma(t,r(t))dW(t),$$

where $\sigma(t,r(t))$ is the diffusion coefficient in the short rate dynamics.
It follows that the absolute volatility of the instantaneous forward rate $f(t,T)$ at time t in a short rate model with an affine term structure is

$$\sigma_f(t,T) = \frac{\partial}{\partial T} B(t,T)\sigma(t,r(t)) \tag{3}$$

where $B(t,T)$ and $\sigma(t,r(t))$ are respectively the B function of the chosen short rate model and the volatility term.
Vasicek (1977) assumed that the instantaneous spot rate under the real-world measure evolves as an Ornstein-Uhlenbeck process with constant coefficients. We know that the standard Vasicek model under the risk neutral measure is given by

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW_t \tag{4}$$

where $r(0) = r_0, \kappa, \theta$ are positive constants and the short rate r is mean reverting, for t going to infinity, to the value θ. We know that the main problem of Vasicek lies in its limited number of parameters, therefore, the model won't be able to fit the initial term structure. We will show that replacing θ with the following function

$$\theta(t) = f(0,t) + \frac{1}{\kappa}\frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{2\kappa^2}(1 - e^{-2\kappa t}) \tag{5}$$

allow the model to fit the initial term structure. Starting from eq.[3], the B function in Vasicek is given by

$$B(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right] \tag{6}$$

then

$$\frac{\partial}{\partial T} B(t,T) = e^{-\kappa(T-t)} \tag{7}$$

and $\sigma(t,r(t)) = \sigma$, therefore

$$\sigma_f(t,T) = \sigma e^{-\kappa(T-t)} \tag{8}$$

Substitute in the risk neutral HJM and get

$$df(t,T) = \sigma^2 e^{-\kappa(T-t)}\left(\int_t^T e^{-\kappa(u-t)}du\right)dt + \sigma e^{-\kappa(T-t)}dW_t \tag{9}$$

$$df(t,T) = \sigma^2 e^{-\kappa(T-t)} \left(-\frac{1}{\kappa} e^{-\kappa(u-t)} \Big|_{u=t}^{u=T} \right) dt + \sigma e^{-\kappa(T-t)} dW_t$$
$$df(t,T) = \sigma^2 e^{-\kappa(T-t)} \left(-\frac{1}{\kappa} \left(e^{-\kappa(T-t)} - 1 \right) \right) dt + \sigma e^{-\kappa(T-t)} dW_t$$
$$df(t,T) = \sigma^2 e^{-\kappa(T-t)} \left(\frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)} \right) \right) dt + \sigma e^{-\kappa(T-t)} dW_t$$
$$df(t,T) = \frac{\sigma^2}{\kappa} \left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)} \right) dt + \sigma e^{-\kappa(T-t)} dW_t$$

and integrate to obtain

$$\int_0^t df(s,T) = \frac{\sigma^2}{\kappa} \int_0^t \left(e^{-\kappa(T-s)} - e^{-2\kappa(T-s)} \right) ds + \sigma \int_0^t e^{-\kappa(T-s)} dW_s$$
$$f(t,T) - f(0,T) = \frac{\sigma^2}{\kappa} \left(\frac{e^{-\kappa(T-s)}}{\kappa} - \frac{e^{-2\kappa(T-s)}}{2\kappa} \right) \Big|_{s=0}^{s=t} + \sigma \int_0^t e^{-\kappa(T-s)} dW_s$$
$$f(t,T) - f(0,T) = \frac{\sigma^2}{\kappa} \left(\frac{e^{-\kappa(T-t)} - e^{-\kappa T}}{\kappa} - \frac{e^{-2\kappa(T-t)} - e^{-2\kappa T}}{2\kappa} \right) + \sigma \int_0^t e^{-\kappa(T-s)} dW_s$$

as we know $r(t) = f(t,t)$, we can substitute to get

$$f(t,t) = f(0,t) + \frac{\sigma^2}{\kappa} \left(\frac{1 - e^{-\kappa t}}{\kappa} - \frac{1 - e^{-2\kappa t}}{2\kappa} \right) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s$$

$$r(t) = f(0,t) + \frac{\sigma^2}{2\kappa^2} \left(1 - 2e^{-\kappa t} + e^{-2\kappa t} \right) + \sigma \int_0^t e^{-\kappa(t-s)} dW_s \tag{10}$$

The differential of the short rate is equal to the differential of the forward with respect to the first parameter and the differential of the second parameter, both evaluated at small t.

$$dr(t) = df(t,T)\Big|_{T=t} + \frac{\partial}{\partial T} f(t,T)\Big|_{T=t} dt \tag{11}$$

where (la prima mettendo T=t in df)(la seconda facendo la detivata di f)

$$df(t,T)\Big|_{T=t} = \sigma dW_t$$
$$\frac{\partial}{\partial T} f(t,T)\Big|_{T=t} = \frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{\kappa} \left(e^{-\kappa t} - e^{-2\kappa t} \right) - \kappa\sigma \int_0^t e^{-\kappa(t-s)} dW_s$$

and get

$$dr(t) = \kappa \left(\frac{1}{\kappa} \frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{\kappa^2} \left(e^{-\kappa t} - e^{-2\kappa t} \right) - \sigma \int_0^t e^{-\kappa(t-s)} dW_s \right) dt + \sigma dW_t \tag{12}$$

where we factor out the κ as we notice that the term $\sigma \int_0^t e^{-\kappa(t-s)} dW_s$ is the same as in equation [10],

$$\sigma \int_0^t e^{-\kappa(t-s)} dW_s = r(t) - f(0,t) - \frac{\sigma^2}{2\kappa^2} \left(1 - 2e^{-\kappa t} + e^{-2\kappa t} \right) \tag{13}$$

if we isolate the term and we substitute in [12] we get

$$dr(t) = \kappa \left(\frac{1}{\kappa} \frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{2\kappa^2} \left(1 - e^{-2\kappa t} \right) - r(t) + f(0,t) \right) dt + \sigma dW_t \tag{14}$$

as we define this settings from HJM we can conclude that this model will match the initial term structure, setting

$$\theta(t) = f(0,t) + \frac{1}{\kappa} \frac{\partial}{\partial t} f(0,t) + \frac{\sigma^2}{2\kappa^2} \left(1 - e^{-2\kappa t} \right) \tag{15}$$

we have the dynamics for the extended short rate model

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma dW_t \tag{16}$$