Introduction to CCR and xVA

Advanced Financial Modeling

Andrea Carapelli

Università di Siena

2024

UNIVERSITÀ DI SIENA 1240

Suggested References

Everything will be covered during the course and provided to you as slides, notes or Python notebooks.
If you want more, check the following:

- D. Brigo, F. Mercurio (2006) Interest Rate Models - Theory and Practice (2nd edition).
- J. Gregory (2020) The xVA Challenge.

Counterparty Credit Risk

- CCR is related to the probabilty that the counterparty will default prior to the expiration of the contract and will not make all the remaining payments.
- A derivative contract with a defaultable counterparty is less worthy than a contract with a risk-free counterparty
- Trades are charged an additional premium, which is added to the fair price of the derivative, due to the different costs and adverse scenarios carried by the contract

$$V^D(t) = V(t) + xVA(t)$$

Credit Adjustment CVA-DVA

Credit Value Adjustment is charged to risky counterparties and is a negative quantity to be added to fair prices.
The unilateral CVA charge can be approximated as

$$\mathbf{uCVA} \approx LGD_{ctp} \text{ x } PD_{ctp} \text{ x } EPE$$

Debt Value Adjustment is simply a CVA seen from the counterpart perspective.
The unilateral DVA charge can be approximated as

$$\mathbf{uDVA} \approx LGD_{self} \text{ x } PD_{self} \text{ x } ENE$$

Credit Adjustment CVA-DVA

$$\mathrm{uCVA}(t) = \mathbb{E}^{Q}\left[1_{t<\tau\leq T}\cdot \mathrm{LGD}(\tau)\cdot D(t,\tau)\cdot \mathrm{PV}^{+}(\tau)\,|\,\mathcal{G}_t\right] \tag{1}$$

Making time partition arbitrarily fine, by taking the limit $n\to\infty$, and setting

$$X(t_{i-1}) := \mathrm{LGD}(t_{i-1})\cdot D(t,t_{i-1})\cdot \mathrm{PV}^{+}(t_{i-1}),$$

we can write (1) as

$$\begin{aligned}\mathrm{uCVA}(t) &= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}^{Q}\left[1_{t_{i-1}<\tau\leq t_i}\cdot X(t_{i-1})\,|\,\mathcal{G}_t\right]\\ &= \frac{1_{\tau>t}}{Q(\tau>t\,|\,\mathcal{F}_t)}\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}^{Q}\left[(1_{\tau>t_{i-1}}-1_{\tau>t_i})\cdot X(t_{i-1})\,|\,\mathcal{F}_t\right],\end{aligned}$$

Credit Adjustment CVA-DVA

Assuming independence between the payoff and default probabilities we can write:

$$\mathrm{uCVA}(t) = 1_{\tau > t} \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}^{Q}\left[\left(e^{-\int_t^{t_{i-1}} \lambda(s)ds} - e^{-\int_t^{t_i} \lambda(s)ds}\right) \tilde{X}(t_i)\right].$$

We choose t_i typically at the beginning or the end of the interval $[t_{i-1}, t_i]$ for "anticipated" respectively "postponed" approximations of CVA.

Funding Adjustment FVA

Exposures that give rise to counterparty risk also need to be funded and therefore give rise to additional costs. Such funding costs are generally recognised via the Funding Value Adjustment (FVA).

$$\mathbf{FCA} \approx R_{borrow} \times SP_{self} \times SP_{ctp} \times EPE$$

$$\mathbf{FBA} \approx R_{lend} \times SP_{self} \times SP_{ctp} \times ENE$$

FVA is generally seen by banks as an internal cost of financing their uncollateralised derivatives portfolios.

$$\mathbf{FVA} = FBA + FCA$$

FVA

A definition of FVA can be given by:

$$
\begin{aligned}
&\mathrm{FVA}(t) = \\
&\sum_{i=1}^{n} f_b(t, t_{i-1}, t_i)\tau_i \mathbb{E}^Q \left(S_C(t, t_{i-1}) S_B(t, t_{i-1}) \mathrm{PV}(t_i)^+ D(t, t_i) \right) \\
&+ \sum_{i=1}^{n} f_l(t, t_{i-1}, t_i)\tau_i \mathbb{E}^Q \left(S_C(t, t_{i-1}) S_B(t, t_{i-1}) \mathrm{PV}(t_i)^- D(t, t_i) \right)
\end{aligned}
$$

Credit and Funding Analogy

Positive Exposure:

- Is at risk when a counterparty defaults,
- Is also the amount that has to be funded when the counterparty does not default.

Negative Exposure:

- Is associated with own default,
- Is also a funding benefit.

Two common framework to avoid a double-counting of funding benefits:

$$V^D(t) \approx V(t) - [CVA(t) - DVA(t)] - FCA(t)$$

$$V^D(t) \approx V(t) - CVA(t) + [FBA(t) - FCA(t)]$$

Remarks

- Derivatives pricing under CCR requires the valuation of future risk exposures, and it accounts in the xVAs the corresponding costs or benefits with respect to the market.
- The most numerically expensive element is the evaluation of Expected Exposures and sensitivities,
- Quantifying exposure is complex due to the long periods involved, the many different market variables that may influence the future value, and risk mitigants such as netting and collateral.
- A Cross-Asset Modelling is required, sometimes assumptions must be done to increase the practical tractability of default probabilities and recovery rates.

Pricing Exposures

We define the exposure of a derivative $E(t)$, the positive exposure $PE(t)$ and the negative exposure $NE(t)$ at some instant t as

$$E(t) = V(t)$$
$$PE(t) = max(V(t), 0)$$
$$NE(t) = min(V(t), 0)$$

the expectations in t_0, under $\mathbb{Q}$-measure, are given by

$$EE(t_0, t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{N(t_0)}{N(t)} E(t) | \mathcal{F}(t_0)\right],$$

$$EPE(t_0, t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{N(t_0)}{N(t)} PE(t) | \mathcal{F}(t_0)\right],$$

$$ENE(t_0, t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{N(t_0)}{N(t)} NE(t) | \mathcal{F}(t_0)\right],$$

Pricing Exposures

xVA as Exotic Options

When we include default risk into the risk neutral valuation we obtain a payoff depending on the underlying of the basic derivative and on the default of the counterparty. The latter feature transforms our payoff, e.i. $PE(t)$, $NE(t)$, in a credit-hybrid derivative.

- A key aspect will be volatility, of relevant contracts and margin.
- Risk Neutral valuations for path dependent options and accurate sensitivities are relatively complex to price.

Pricing Exposures

At valuation time t, the value of the unilateral CVA of a derivative contract expiring at time T, assuming that the counterparty has not defaulted yet, i.e. on $\tau \leq t$ where τ is the the time to default of the counterparty is

$$uCVA(t) = -LGD(\tau) \sum_{i=1}^{m} EPE(t, t_i) dPD(t_{i-1}, t_i),$$

vice versa, the value of the unilateral DVA is

$$uDVA(t) = LGD_{bank}(\tau) \sum_{i=1}^{m} ENE(t, t_i) dPD_{bank}(t_{i-1}, t_i).$$

Simulation of Exposure

At the core of exposure calculation there is an options-pricing-type problem, the most common and generic approach for quantifying exposures is Monte Carlo simulation. Due to the different nature of different underlyings and other components that drive xVA, the main complexities are

- **Model choice**: it is necessary to choose a model for each underlying risk factor, usually across multiple asset classes.
- **Calibration**: model calibration to market and historical data is generally challenging given the large number of risk factors
- **Numerical implementation**: model implementation will impact performance and accuracy
- **RAM and DataBases**: Production-level implementations require high computational power, especially for sensitivities

IRS Exposure

Consider a vanilla interest rate swap, the most classic linear derivative that allows two counterparties to exchange fixed for floating payments flows. The pricing equation for a payer swap under a single-curve framework, assuming same payment dates for both legs, can be simplified as

$$V(t) = N \sum_{k=i+1}^{m} \tau_k P(t, T_k)(L(t; T_{k-1}, T_k) - K), \tag{2}$$

where N is the notional amount, $T_{i+1}, \ldots T_m$ are the payment dates, $\tau_k = T_k - T_{k-1}$ and $L(t; T_{k-1}, T_k)$ is the forward rate.

IRS Exposure

We choose to model this simplified market with a one factor Gaussian short rate model. We showed it is Normally distributed with conditional mean and variance given respectively by

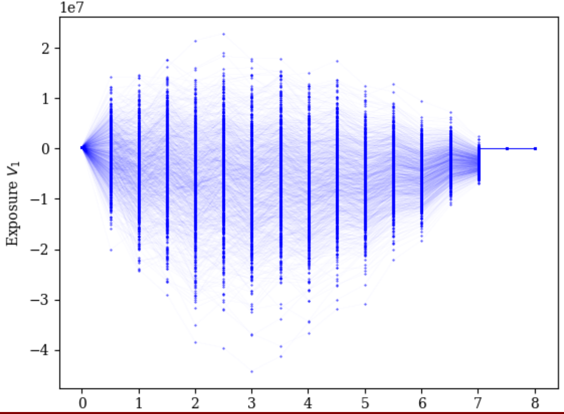
$$\mathbb{E}[r(t)|r(s)] = r(s)e^{-\kappa(t-s)} + \alpha(t) - \alpha(s)e^{-\kappa(t-s)},$$
$$Var[r(t)|r(s)] = \frac{\sigma^2}{2\kappa}[1 - e^{-2\kappa(t-s)}].$$

and zero coupon bond prices

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)},$$
$$B(t, T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa},$$
$$A(t, T) = \frac{P(0, T)}{P(0, t)} exp\left(B(t, T)f^M(0, t) - \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t})B(t, T)^2\right).$$

IRS Exposure

We are interested in $\sum_{i=1}^{m} ENE(t, t_i)$ and $\sum_{i=1}^{m} EPE(t, t_i)$ along the time grid discretization m (V_1 receiver swap with $7y$ maturity and fair strike)



Netting

Netting is a traditional way to mitigate counterparty risk where there may be a large number of transactions of both positive and negative value with either

- **Single counterparty:** it's a bilateral netting across different deal exposures of the same counterparty netting set.
- **Multiple counterparties:** it's a multilateral netting across all deal exposures belonging to a certain set.

Netting

Portfolios may contain a large number of transactions, which may partially offset (hedge) one another.
In general, netting can be seen as a method of aggregating obligations whilst keeping market risk constant, but reducing

- settlement risk
- counterparty risk,
- operational risk
- liquidity risk
- systemic risk

Netting

For a specific netting set, container of n trades, we define its exposure $E_{ns}(t)$, positive exposure $PE_{ns}(t)$ and negative exposure $NE_{ns}(t)$ at some instant t as

$$E_{ns}(t) = \sum_{i=1}^{n} V_i(t)$$

$$PE_{ns}(t) = max\left(\sum_{i=1}^{n} V_i(t), 0\right)$$

$$NE_{ns}(t) = min\left(\sum_{i=1}^{n} V_i(t), 0\right)$$

Netting

Jensen Inequality

$max(0, x)$ is a convex function, hence for a sample of T elements

$$max\left(0, \frac{1}{T}\sum_{i=1}^{T} x_i\right) \leq \frac{1}{T}\sum_{i=1}^{T} max(0, x_i)$$

which immediately leads, for a specific netting set, container of n trades, to

$$|CVA_{ns}(t)| \leq \left|\sum_{i=1}^{n} CVA_i\right|$$

Collateral

- The future exposure could increase to a relatively large, unmanageable level.
- A way to mitigate this problem is to have a contractual feature in the transaction that permits a risk-mitigating action to reduce a large exposure.
- On an exchange, margin-posting represents a daily settlement,
- whereas in over-the-counter (OTC) derivatives it represents a collateralisation process.

Collateral

There are two fundamentally different types of margin C

- **Initial Margin:** the initial margin is referred to separately as independent amount. Initial margin is not netted against variation margin amounts and may also be segregated.
- **Variation Margin:** depends on future exposuresand it is defined in the CSA by the credit support amount.

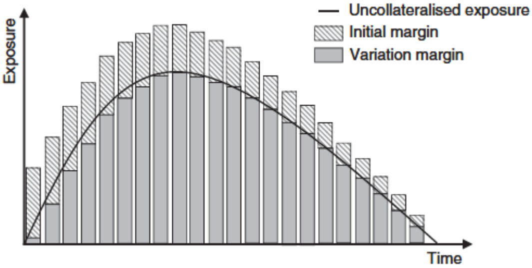
$$E(t) = V(t) - C$$

$$PE(t) = max(V(t) - C, 0)$$

$$NE(t) = min(V(t) - C, 0)$$

Collateral

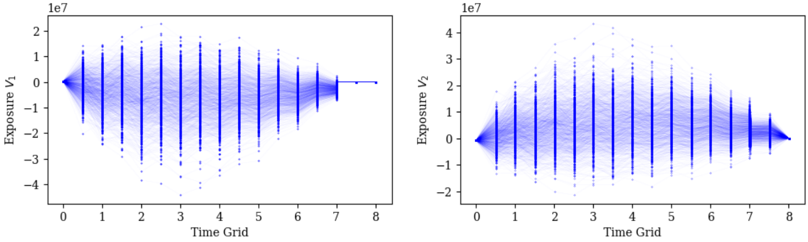
Impact of margin on a typical exposure profile



source: J. Gregory (2020) The xVA Challenge.

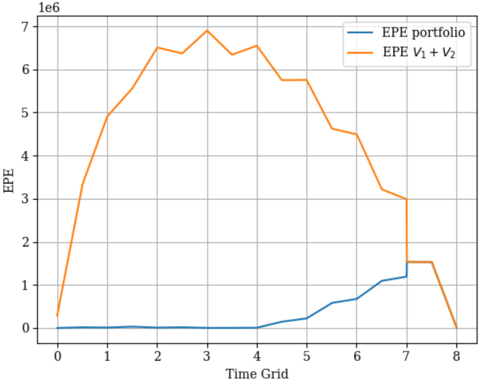
IRS Netting-Portfolio

Let V_1 be the previously defined receiver swap, and let V_2 be a payer swap, with $80M$ notional and $8y$ maturity:



IRS Netting-Portfolio

The two contracts offset (hedge) each other exposure as they are written on the same underlying, they have similar size, strike and maturity



Poisson Process

Time Homogeneous Poisson

A "time homogeneous Poisson process" $\{M_t, t \geq 0\}$ is a unit-jump increasing, right continuous process with stationary independent increments and null initial condition, $M_0 = 0$. It turns out these seemingly general requirements imply strong facts on the process distribution and on its jump times. Indeed, let $\tau^1, \ldots, \tau^m$ be the first, second etc. jump times of M. There exists a positive constant $\hat{\gamma}$ such that

$$\mathbb{Q}\{M_t = 0\} = \mathbb{Q}\{\tau^1 > t\} = e^{-\hat{\lambda} t}$$

Poisson Process

Time Inhomogeneous Poisson

Having not defaulted (jumped) before t, the (risk neutral) probability of defaulting (jumping) in the next dt instants is

$$\mathbb{Q}(\tau \in [t, t + dt] | \tau > t) = \lambda(t) dt$$

where the "probability" dt factor λ is assumed here for simplicity to be strictly positive and is in general called intensity or hazard rate. Define also the further quantity

$$\Lambda(t) = \int_0^t \lambda(u) du$$

as the cumulated intensity, or also Hazard function.

Pricing Default

Default Event

Default is represented by means of the default time. The default time, typically denoted by τ , is a random time that can be modeled in several ways

Intensity Models

Describe default by means of an exogenous jump process, the default time τ is the first jump time of a Poisson process. The Poisson process can have deterministic or stochastic (Cox) intensity.

Pricing Default

Intensity as Discount Factors

By the properties of the exponential distribution $\mathbb{Q}(\xi \leq x) = 1 - e^{-x}$ we can show that

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\Lambda(\tau) > \Lambda(t)) = \mathbb{Q}(\xi > \Lambda(t)) = e^{-\int_0^t \lambda(u)du}$$

if intensity is stochastic the corresponding expression, or survival probability at t, is

$$\mathbb{Q}(\tau > t)\mathbb{E}[e^{-\int_0^t \lambda(u)du}]$$

this is just the **price of a zero coupon bond** in an interest rate model with short rate r replaced by λ.

Managing xVA

- An xVA portfolio will experience sensitivity to every single market parameter for the underlying transactions in every currency, asset class, and product type.
- there will be sensitivities specific to xVA, such as where volatility risk arises from non-volatility-sensitive products.
- We can compare xVA management to managing a book of options. For example, representing swap exposure as a series of European swaptions

Managing xVA

Where is the risk?

$$\partial xVA \approx \frac{\partial xVA}{\partial S} dS + \frac{\partial xVA}{\partial E} dE + \frac{\partial xVA}{\partial t} dt + \frac{\partial^2 xVA}{\partial S \partial E} dSdE$$

where exposure components dE can be broken down in

- Interest rate risk, to multiple curves, including basis.
- Interest rate volatility risk.
- FX spot, cross currency basis risk, and FX volatilty risk.

while spread components dS are credit delta and cross-gamma effects.

Cross-Gamma and Wrong-Way-Risk

- Wrong-way risk describes the dependence of counterparty default on the portfolio exposure.
- Cross-gamma refers to the joint move of two variables. Even if the two variables are hedged or explained independently, their joint move may have a material impact. Cross-gamma components are typically hard to hedge.
- A simple example could be a simultaneous move where interest rates go down and credit spreads widen, this can cause significant P&L movements.
- Default intensities must be allowed to be stochastic in order to account for the correlation between interest rates and spread components.

Monte Carlo Sensitivities

Classic Finite Differences

With $V(\theta)$ continuous

$$V(\theta + \Delta\theta) = V(\theta) + \frac{\partial V}{\partial \theta}\Delta\theta + \frac{1}{2}\frac{\partial^2 V}{\partial \theta^2}\Delta\theta^2 + \dots$$

with central differential

$$\frac{\partial V}{\partial \theta} = \frac{V(\theta + \Delta\theta) - V(\theta - \Delta\theta)}{2\Delta\theta} + \mathcal{O}(\Delta\theta^2)$$

Monte Carlo Sensitivities

Every xVA is typically sensitive to many different market variables

- In order to compute all xVA risks with finite differences, simulations must be repeated thousands of times with inputs bumped one by one.
- This "brute force" solution is not viable, calculation speed is too slow and there is a massive hardware, development, and maintenance costs.
- The alternative was to implement crude approximations, at the expense of accuracy

Monte Carlo Sensitivities

Algorithmic Differentiation

- Also known as automatic differentiation (AD), allows one to produce derivative sensitivities to calculation code, automatically, analytically. In Machine Learing it is known under the name **backpropagation**.
- AD can compute sensitivities way faster than numerical bumping and finite differences.
- AD operates directly on analytics and can be applied to a single trade or a portfolio of trades. It has quickly become an essential part of quantitative finance.

AAD

Is a direct consequence of the **chain rule**. Suppose we have a function $y = h(g(f(x)))$, we can compute the derivative $\partial y/\partial x$ starting from the input value x

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial f}\frac{\partial h}{\partial g}\frac{\partial y}{\partial h} = \frac{\partial y}{\partial x}$$

and we can also work backwards

$$\frac{\partial y}{\partial h}\frac{\partial h}{\partial g}\frac{\partial g}{\partial f}\frac{\partial f}{\partial x} = \frac{\partial y}{\partial x}$$