This is a written exam. You have 4 hours to complete it. Make sure to write your name and student number on each sheet you hand in.

1. (10 points) Mark the correct answer(s) to the following multiple-choice questions. If there are multiple possibilities, check all of them.

   (a) Consider the system $x^+ = -0.5x$. Which of the following sets are invariant?
   - ◯ $\{x \mid x^4 \leq 5\}$
   - ◯ $\{x \mid x^3 \leq 5\}$
   - ◯ $\{x \mid -1 \leq x \leq 2\}$
   - ◯ $\{x \mid -1/2 \leq x \leq 2\}$

   (b) Consider the system
   $$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} 0.5x_1 \\ 1.5x_2 \end{bmatrix},$$
   which of the following sets are invariant?
   - ◯ $\{x \mid x_2 = 0,\ x_1 \leq 10\}$
   - ◯ $\{x \mid x_1 = 0,\ x_2 \leq 10\}$
   - ◯ $\{x \mid x_2 = x_1\}$

   (c) Consider the following MPC problem,
   $$\begin{aligned} \mathbf{min} \quad & \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k \\ \mathbf{s.t.} \quad & x_0 = x \\ & x_{k+1} = Ax_k + Bu_k,\ k \in \mathbb{N}_{[0,N-1]} \end{aligned}$$
   which of the following statements is/are true?
   - ◯ The MPC control law $\kappa_N(x)$ is quadratic
   - ◯ The MPC control law $\kappa_N(x)$ is linear
   - ◯ If the closed-loop system $x^+ = Ax + B\kappa_N(x)$ is stable for $N = 5$, then it is also stable for $N = 6$
   - ◯ The closed-loop system will be stable if $x^+ = Ax$ is unstable and $N \geq \mathbf{rank}(A)$

   (d) Consider the MPC control law for the linear system $x^+ = Ax + Bu$
   $$\begin{aligned} V^\star(x) \quad &= \min_u \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N), \\ & \mathbf{s.t.} x_0 = x \\ & \quad x_{k+1} = Ax_k + Bu_k \\ & \quad x_k \in X,\ u_k \in U \\ & \quad x_N \in X_f \end{aligned}$$
   where K is a matrix such that the set $X_f \subset X$ is invariant for $x^+ = (A + BK)x$, $KX_f \subset U$, and the function $\ell(x, u)$ is positive definite. Which of the following additional conditions will ensure asymptotic stability of the closed-loop system?
   - ◯ $(A + BK)^T P(A + BK) - P \preceq 0$
   - ◯ $x^T[(A + BK)^T P(A + BK) - P]x \leq -\ell(x, Kx)$ for all $x \in X_f$
   - ◯ X_f is a control invariant set

(e) Which of the following statements implies that $S = \{x \mid x^T P x \leq 1\}$, $P \succeq 0$ is an invariant set for the system $x^+ = Ax$?

- ◯ $A^T P A \succeq P$
- ◯ $A^T P A \preceq P$
- ◯ $A^T P A \succeq 0$
- ◯ $A^T P A \preceq 0$

(f) Define the control law $\kappa(x)$ as

$\kappa(x) := \arg\min\{\|u\|_2 \mid f(x,u) \in C_\infty\}$ where C_∞ is the maximal control invariant set for the system $x^+ = f(x,u)$ subject to the constraints $x \in X$, $u \in U$.
Is the system $x^+ = f(x, \kappa(x))$ necessarily asymptotically stable?

- ◯ Yes
- ◯ No

(g) Let x_0 be a state that is *not* in the maximal control invariant set $\mathcal{C}_\infty$ for the system $x^+ = f(x,u)$ under the constraints $(x,u) \in X \times U$ controlled by an MPC controller $\kappa_N(x)$ obtained by solving an optimization problem with horizon N and no terminal constraint. Is it possible that the N-step sequence $\{x_0, x_1, \ldots, x_N\}$ is in X and $\{\kappa_N(x_0), \kappa_N(x_1), \ldots, \kappa_N(x_{N-1})\}$ is in U, where $x_{k+1} = f(x_k, \kappa_N(x_k))$?

- ◯ Yes
- ◯ No

(h) Let x_0 be in the maximal control invariant set for the system $x^+ = f(x,u)$ subject to constraints $(x,u) \in X \times U$. Is it possible that there exists a $u_0 \in U$ such that $f(x_0, u_0) \notin X$?

- ◯ Yes
- ◯ No

(i) Consider the following four MPC problems.

$$
\begin{aligned}
V_1(x) \quad &= \min \sum_{k=0}^{\infty} x_k^\top Q x_i + u_k^\top R u_k \\
&\textbf{s.t.}\, x_0 = x \\
&\qquad x_{k+1} = A x_k + B u_k,\ k \in \mathbb{N}_{[0,N-1]} \\
&\qquad (x_k, u_k) \in X \times U,\ k \in \mathbb{N}_{[0,N-1]}
\end{aligned}
$$

$$
\begin{aligned}
V_2(x) \quad &= \min \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + V_1(x_N) \\
&\textbf{s.t.}\, x_0 = x \\
&\qquad x_{k+1} = A x_k + B u_k,\ k \in \mathbb{N}_{[0,N-1]} \\
&\qquad (x_k, u_k) \in X \times U,\ k \in \mathbb{N}_{[0,N-1]}
\end{aligned}
$$

$$
\begin{aligned}
V_3(x) \quad &= \min \sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k \\
&\textbf{s.t.}\, x_0 = x \\
&\qquad x_{k+1} = A x_k + B u_k,\ k \in \mathbb{N}_{[0,N-1]}
\end{aligned}
$$

$$
\begin{aligned}
V_4(x) \quad &= \min \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + V_3(x_N) \\
&\textbf{s.t.}\, x_0 = x \\
&\qquad x_{k+1} = A x_k + B u_k,\ k \in \mathbb{N}_{[0,N-1]} \\
&\qquad (x_k, u_k) \in X \times U,\ k \in \mathbb{N}_{[0,N-1]} \\
&\qquad x_N \in X_f
\end{aligned}
$$

with $Q \succ 0$, $R \succ 0$ and $X_f \subseteq X$ a positive invariant set for the system $x^+ = Ax + B\kappa(x)$, where $\kappa(x)$ is the control law defined by the MPC problem on the bottom left. Mark all correct statements.

- ◯ $V_1(x) \leq V_2(x)$
- ◯ $V_1(x) \geq V_2(x)$
- ◯ $V_1(x) = V_2(x)$
- ◯ $V_3(x) \leq V_1(x)$
- ◯ $V_3(x) \geq V_1(x)$
- ◯ $V_3(x) = V_1(x)$
- ◯ $V_3(x) \leq V_4(x),\ \forall x \in \textbf{dom}\, V_4$
- ◯ $V_3(x) \geq V_4(x),\ \forall x \in \textbf{dom}\, V_4$
- ◯ $V_3(x) = V_4(x),\ \forall x \in \textbf{dom}\, V_4$

2. (4 points) Consider the following quadratically constrained quadratic program:

$$\begin{aligned} \mathbf{min} \quad & \frac{1}{2} z^\top Q z + c^\top z \\ \mathbf{s.t.} \quad & z^\top z \leq \alpha \end{aligned} \tag{1}$$

Where $Q \succ 0$ is a positive definite matrix.

(a) Give functions f, g so that problem (1) is equivalent to

$$\mathbf{min}\, f(z) + g(z).$$

Hint: you may want to use the indicator function.

(b) Give the steps of the gradient projection algorithm for the functions you gave in part (a) of this question.

(c) Suppose that you apply gradient projection with stepsize $\gamma = 1/L_f$ where L_f is the Lipschitz constant of f. Give a bound on the ratio $\frac{\|z^\nu - z^\star\|}{\|z^0 - z^\star\|}$, where z^ν is the ν-th iterate and $z^\star$ is the optimizer, in the following cases

i. $Q = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$, $q = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\alpha = 4$

ii. $Q = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$, $q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\alpha = 20$

iii. $Q = \begin{bmatrix} 200 & 0 \\ 0 & 2 \end{bmatrix}$, $q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\alpha = 20$

(d) Give functions f, g and matrix L so that problem (1) is equivalent to

$$\begin{aligned} \mathbf{min} \quad & f(z) + g(w) \\ \mathbf{s.t.} \quad & Lw = z \end{aligned}$$

Hint: you may want to use the indicator function.

(e) Give the three steps of the AMA for the functions and data you gave in part (d) of this question.

(f) Give the three steps of the ADMM for the functions and data you gave in part (d) of this question.

3. (2 points) Consider the linear system

$$x^+ = \begin{bmatrix} 0.5 & 0 \\ 4 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.3 & 0.2 \\ -0.6 & 0.9 \end{bmatrix} u$$

with constraints on the input $\|u\|_\infty \leq 1$.

(a) What is the maximal control invariant set for this system? Justify your answer.

(b) Consider the following standard MPC optimization problem, and let $\kappa_N(x)$ be the resulting receding-horizon control law.

$$\begin{aligned}
\mathbf{min} \quad & \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + x_N^\top P_f x_N \\
\mathbf{s.t.} \quad & x_0 = x \\
& x_{k+1} = A x_k + B u_k, && k \in \mathbb{N}_{[0,N-1]} \\
& u_k \in U, && k \in \mathbb{N}_{[0,N-1]} \\
& x_N \in X_f
\end{aligned}$$

Describe how to choose a terminal control law, K_f, terminal weights P_f and terminal set X_f so that the closed-loop system $x^+ = Ax + B\kappa_N(x)$ has a maximal invariant set equal to that given in part a and is asymptotically stable.

4. (4 points) Consider the following simple optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & ax \\ \textbf{s.t.} \quad & 1 - x = 0, \end{aligned}$$

where $a > 0$ is some parameter.

(a) Give the optimal values for the primal variable x and the dual variable λ.

(b) Consider the l_1 penalty function.

i. Write down the expression for this penalty function and sketch its plot for several values of the penalty coefficient c. For example $c \in \{20a, 10a, 5a\}$.

ii. Which value of x minimizes this penalty function for these values of c? How does it relate to the minimizer of the original problem?

iii. Does this relation hold for all $c \in \mathbb{R}$? If so, prove it. If not, give the values of c for which it does hold.

(c) consider the quadratic penalty function.

i. Write down the expression for this penalty function

ii. Which value of x minimizes this penalty function (as a function of c)?

iii. Does the relation you found in question 4-(b)-ii still hold, for any or all $c \in \mathbb{R}$? Show why (not).

(d) consider the augmented lagrangian.

i. Write down the expression for the augmented Lagrangian related to this optimization problem.

ii. Start from some value $c^0 = 10$ for the penalty coefficient and a value $\lambda^0 = 20$ for the Lagrange multiplier. Compute the first update of the Augmented Lagrangian Method.

iii. With this value of λ, Does the relation you found in question 4-(b)-ii still hold, for any or all $c \in \mathbb{R}$? Show why (not).