OPTIMIZATION - OPTIMALISATIE

EXERCISE SESSION 3: DUALITY

Exercise 1 (**Duality in linear programming).**

You are in charge of deciding on the crops to use on a small farm. There are 110 acres of unused land, and you can decide to plant potatoes, sugar beets, or cotton. Each of the crops has different expected returns and requirements, as listed in Table 1.

	Potatoes	Sugar beets	Cotton
Expected returns (€/acre)	1,800	2,500	2,000
Fertilizer use (kg/acre)	200	120	60
Water use (m^3/acre)	3,000	5,000	4,000

Table 1. Crop properties.

You have 20,000 kg of fertilizer and a water budget of 360,000 cubic meters. Divide the 110 acres of land across the three different crops, maximizing the expected returns.

(1a) Write the problem as a linear program. Use the form

$$\begin{aligned} \underset{x}{\textbf{maximize}} \quad & c^\top x \\ \textbf{subject to} \quad & Ax \le b \\ & x \ge 0. \end{aligned} \tag{1.P}$$

(1b) Is the problem convex? Does strong duality hold?

(1c) Rewrite the problem from Task (1a) as a minimization problem. Write down the Lagrangian function $\mathcal{L}(x, \lambda, \mu)$ for that problem. What conditions do the Lagrange multipliers λ and μ have to satisfy for the lower bound property of the Lagrangian to hold?

Hint: Recall the lower bound property: for any feasible value $\tilde{x}$, $\mathcal{L}(\tilde{x}, \lambda, \mu)$ is no larger than the objective function value at $\tilde{x}$.

(1d) Write down the Lagrangian dual function $q(\lambda, \mu) = \mathbf{inf}_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu)$. Give an explicit (piecewise) expression for q. Under what condition on λ and μ is its value finite?

Hint: Write down the optimality conditions of the unconstrained minimization problem in the definition of q to eliminate the infimum.

(1e) Write down the Lagrangian dual problem of the original linear program. Impose the conditions on the Lagrange multipliers you derived in Task (1c) and Task (1d) as constraints, and use those to eliminate μ. Write the dual problem as a linear program in standard form.

(1f) Solve both the primal and dual linear programs from Tasks (1a) and (1e) using `scipy.optimize.linprog`. How many acres should you plant for each crop? Compare the objective values of the primal and dual solutions.

(1g) Use the dual solution to answer the following questions:

- Is the constraint that limits the amount of water active?
- The water price is €0.30/m^3. Should you buy more water?
- Perform a sensitivity analysis: Which of the following will have the largest influence on the total returns?
  (a) Adding 0.01 acres of land.
  (b) Adding 1 kg of fertilizer.
  (c) Adding 1 m^3 of water.

Exercise 2 **(Duality in nonlinear programming).**

Consider the following problem, with the objective function and constraint set visualized in Figure 1.

$$\begin{aligned} \underset{x_1, x_2}{\textbf{minimize}} \quad & x_1^4 - 2x_2^2 - x_2 \\ \textbf{subject to} \quad & x_1^2 + x_2^2 + x_2 \leq 0 \end{aligned} \tag{2.P}$$

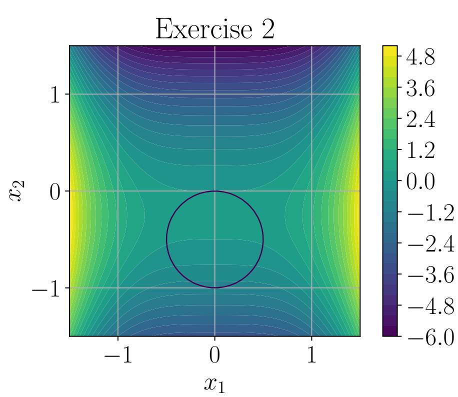

Figure 1. Objective and boundary of the constraint set for problem (2.P).

(2a) Is this problem convex?

(2b) Do any solutions exist?

(2c) Write down the Lagrangian dual problem.

(2d) Is the Lagrangian dual function convex? Is it concave?

(2e) Find the optimal solutions of the primal and the dual problem, and compare their optimal values. Do they match?

Exercise 3 (More duality in nonlinear programming).

Discuss the convexity of the following problems. Then write down and solve a dual problem.

(3a)

$$\begin{aligned} \underset{x_1,x_2,x_3}{\textbf{minimize}} \quad & x_1 - 4x_2 + x_3^4 \\ \textbf{subject to} \quad & x_1 + x_2 + x_3^2 \leq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

(3b)

$$\begin{aligned} \underset{x_1,x_2}{\textbf{minimize}} \quad & x_1^2 + \tfrac{1}{2}x_2^2 + x_1x_2 - 2x_1 - 3x_2 \\ \textbf{subject to} \quad & x_1 + x_2 \leq 1 \end{aligned}$$

(3c)

$$\begin{aligned} \underset{x_1,x_2,x_3}{\textbf{minimize}} \quad & x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 - x_3 \\ \textbf{subject to} \quad & x_1 + x_2 + x_3 \leq 1 \\ & x_3 \leq 1 \end{aligned}$$