# OPTIMIZATION - OPTIMALISATIE

## EXERCISE SESSION 5: OPTIMALITY CONDITIONS FOR UNCONSTRAINED AND CONSTRAINED OPTIMIZATION

**Exercise 1** (**Optimality Conditions Unconstrained Optimization).**

Let the function

$$f(x) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$$

be given.

(a) Compute the gradient of $f$.

(b) Compute the Hessian of $f$.

(c) Show that $f$ has only one stationary point.

(d) Show that the stationary point is neither a maximum nor a minimum, but a saddle point.

(e) Sketch the contour lines of $f$.

**Exercise 2** (**Optimality Conditions Constrained Optimization).**

Regard the following minimization problem:

$$\min_{x\in\mathbb{R}^2} x_2^4 + (x_1+2)^4 \ \text{ s.t. } \begin{cases} -x_1^2 - x_2^2 \geq -8 \\ x_1 - x_2 = 0. \end{cases}$$

(2a) How many scalar decision variables, how many equality, and how many inequality constraints does this problem have?

(2b) Sketch the feasible set $\Omega \subset \mathbb{R}^2$.

(2c) Bring this problem into the NLP standard form

$$\min_{x\in\mathbb{R}^n} f(x) \text{ s.t. } \begin{cases} h(x) = 0 \\ g(x) \leq 0 \end{cases}$$

by defining the functions $f, g, h$ appropriately.

(2d) Is this optimization problem convex? Justify.

(2e) Write down the Lagrangian function of this optimization problem.

(2f) A feasible point of the problem is $\bar{x} = (2,2)^\top$. What is the active set $\mathcal{A}(\bar{x})$ at this point?

(2g) Is the linear independence constraint qualification (LICQ) satisfied at $\bar{x}$? Justify.

(2h) An optimal solution of the problem is $x^* = (-1, -1)^\top$. What is the active set $\mathcal{A}(x^*)$ at this point?

(2i) Is the linear independence constraint qualification (LICQ) satisfied at $x^*$? Justify.

(2j) Describe the tangent cone $T_\Omega(x^*)$ (the set of feasible directions) to the feasible set at this point $x^*$, <u>by a set definition formula with explicitly computed numbers</u>.

(2k) Compute the gradient of the Lagrangian and find the multiplier vectors $\lambda^*$, $\mu^*$ such that the above point $x^*$ satisfies the KKT conditions.

(2l) Describe the critical cone $C(x^*, \mu^*)$ at the point $(x^*, \lambda^*, \mu^*)$ <u>in a set definition using explicitly computed numbers</u>.

(2m) Formulate the second order necessary conditions for optimality (SONC) for this problem and test if they are satisfied at $(x^*, \lambda^*, \mu^*)$. Can you prove whether $x^*$ is a local or even global minimizer?

**Exercise 3 (Degenerate Optimization Problem).** Regard the following minimization problem:

$$\min_{x \in \mathbb{R}^2} -x_1 \text{ s.t. } \begin{cases} -x_2 \leq 0 \\ x_2 + x_1^5 \leq 0. \end{cases}$$

(3a) How many scalar decision variables, how many equality, and how many inequality constraints does this problem have?

(3b) Sketch the feasible set $\Omega \subset \mathbb{R}^2$.

(3c) Determine the unique solution of the problem.

(3d) Bring this problem into the NLP standard form

$$\min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } \begin{cases} h(x) = 0 \\ g(x) \leq 0 \end{cases}$$

by defining the functions $f, g, h$ appropriately.

(3e) Write down the Lagrangian function of this optimization problem.

(3f) Draw the gradients of the objective function and of the active constraints at the solution $x^*$ in your sketch.

(3g) Write down the Lagrange gradient. Does a Lagrange multiplier vector exist to satisfy $\nabla \mathcal{L}_x(x^*, \mu^*) = 0$? Justify.

(3h) Can a constraint qualification hold at the optimal solution? Justify.

(3i) Bonus: Cancel the objective gradient out of the Lagrange gradient optimality condition and try to find multipliers such that the sum of the constraint gradients vanishes. Both multipliers are not allowed to be zero!
This describes a generalization of the KKT conditions the so-called Fritz-John conditions.