Soc Choice Welfare (2004) 23: 91–102
DOI: 10.1007/s00355-003-0238-6

Social Choice and Welfare

The Dodgson ranking and its relation to Kemeny's method and Slater's rule

Christian Klamler

Institute of Public Economics, University of Graz, Universitaetsstr. 15, 8010 Graz, Austria (e-mail: christian.klamler@uni-graz.at)

Received: 13 November 2002/Accepted: 31 March 2003

Abstract. In this paper we provide a binary extension of Dodgson's non-binary preference aggregation rule. This new aggregation rule is then compared to two other rules which, as Dodgson's rule, are also explicitly based on distance functions, namely Kemeny's and Slater's rule. It is shown that the alternative which is top ranked by any of those rules can occur at any position in the Dodgson ranking.

1 Introduction

A Condorcet winner is an alternative that beats all other alternatives by a simple majority. However, simple majority rule does not always determine such a Condorcet winner, let alone a complete social ranking of alternatives, because of voting cycles. Various methods have been devised to overcome Condorcet's paradox, some of those explicitly based on the use of distance functions. The most prominent of such rules come from Dodgson [3], Kemeny [6] and Slater [15][1]. All of them essentially use the same way of measuring distances, however, as has been discussed in Fishburn [4], they are operating on different informational levels. In this paper we will provide a comparison of those rules.

I am very grateful to Nick Baigent, Daniel Eckert and the participants of the CEPET 2002 Workshop in Udine and the Graz University Research Seminar for their comments on earlier drafts of this paper. It goes without saying that any remaining errors are exclusively the author's responsibility.

[1] Recently it has been shown by Baigent and Klamler [2] that another famous procedure, namely the transitive closure, belongs to that class.

In the absence of a Condorcet winner, a Dodgson winner is the alternative from the top cycle[2] which is "closest" to being a Condorcet winner. As Dodgson's rule is non-binary, one purpose of this paper is to provide a binary extension of that rule. The derived Dodgson ranking is based on the alternatives' distances from being a Condorcet winner. This provides an intuitively very reasonable way of aggregating individual preferences. Hence it can be taken as an alternative to existing binary aggregation rules.

The second purpose of this paper is then to compare Dodgson's rule with two prominent binary aggregation rules also based on distance functions, Kemeny's and Slater's rule. In that respect we will show that the top ranked alternatives by those rules could be found at any position in the Dodgson ranking. That is, although all of these rules are extensions of simple majority rule and all of them rely on the same underlying distance function, it is sometimes the case that, even if no individual changes its rankings of the alternatives, a simple change in the aggregation rule might lead to opposite outcomes. Such insight is not obtained from the usual evaluations of different rules to overcome Condorcet's paradox. Those either take place in an axiomatic framework (Fishburn [4]) or have been based on comparisons of choice sets (Laffond et al. [7]).

Recently, Ratliff [9, 10] has investigated the relationship between the non-binary Dodgson rule and Kemeny's rule and positional rules, respectively. Saari and Merlin [13] present a discussion of Kemeny's rule, and Saari [12] provides an insightful overview over different procedures based on pairwise votes using a geometric approach. The paper is also in some relation to Baigent [1], who discusses the idea of being furthest from majoritarian choice, i.e., furthest from being a Condorcet winner. He shows that for a very wide class of social choice functions it will sometimes be the case that an alternative that is furthest from majoritarian choice will be in the choice set.

The striking difference of Dodgson's rule to many other aggregation rules is that it uses certain profile information, i.e., positions of the alternatives in the individual rankings do matter. We will show that it is exactly this additional information which is mainly responsible for the inconsistencies occurring between Dodgson's rule and other rules (which do not take into account such information).

The paper is structured as follows. Section 2 presents the formal framework. In Sect. 3 we will introduce and discuss the idea of a certain agreement among individuals in their rankings. Dodgson's rule and its binary extension will be defined in Sect. 4. Section 5 then compares it to Kemeny's rule, Sect. 6 compares it to Slater's rule and Sect. 7 concludes the paper.

---

[2] The top cycle is the smallest subset of the set of alternatives such that every alternative in the top cycle is strictly majority preferred to all alternatives outside the top cycle.

2 Formal framework

Let X denote a finite set of $n \geq 4$ alternatives and H denote a finite set of h individuals. A preference $R \subseteq X \times X$ is a binary relation on X. For all $A_j, A_m \in X$, the weak preference of A_j over A_m will be denoted by $A_j \succsim_R A_m$. The symmetric and asymmetric part of R will be written as $\sim_R$ and $\succ_R$ respectively. Whenever there is no danger of confusion, subscripts will be dropped. Let $\mathscr{B}$ be the set of all complete binary relations on X, $\mathscr{W} \subset \mathscr{B}$ the set of all weak orders (complete and transitive binary relations) on X and $\mathscr{L} \subset \mathscr{W}$ the set of all linear orders (complete, transitive and asymmetric binary relations) on X. Let $r(T)$ be a linear ranking of any subset $T \subseteq X$, then $\rho(r(T))$ denotes the special permutation of $r(T)$ such that it reverses the whole ranking, e.g. for $T = \{A_1, A_2, A_3\}$, if $r(T)$ has A_1 strictly preferred to A_2 strictly preferred to A_3, i.e. $A_1 \succ A_2 \succ A_3$ then $\rho(r(T))$ is $A_3 \succ A_2 \succ A_1$. Furthermore, for all $A' \in X \backslash T$, $A' \succ r(T)$ means that for all $A \in T, A' \succ A$, e.g. if $r(T)$ is $A_1 \succ A_2 \succ A_3$ then $A' \succ r(T)$ means $A' \succ A_1 \succ A_2 \succ A_3$.

Lists of individual (strict) preferences (also called profiles) will be written as $u = (L_1^u, L_2^u, \ldots, L_h^u) \in \mathscr{L}^h$ where $L_i^u \in \mathscr{L}$ is individual i's preference on X in profile u.

For all $A_j, A_m \in X$, the majority margin of A_j over A_m in profile $u \in \mathscr{L}^h$ is denoted by $a_{j,m}^u = |\{i \in H : A_j \succsim_{L_i^u} A_m\}| - |\{i \in H : A_m \succsim_{L_i^u} A_j\}|$.[3] As the discussed rules are methods to overcome problems resulting from simple majority rule (SMR) we define SMR as a function $v : \mathscr{L}^h \to \mathscr{B}$ such that for all $u \in \mathscr{L}^h$ and all $A_j, A_m \in X$, $A_j \succsim_{v(u)} A_m$ if and only if $a_{j,m}^u \geq 0$. That is, an alternative A_j is at least as good as alternative A_m if and only if there are at least as many individuals weakly preferring A_j over A_m than there are individuals weakly preferring A_m over A_j.

Furthermore, let $\mathbb{Z}$ be the set of all integers, then, using SMR, we can assign to any profile $u \in \mathscr{L}^h$ a point in pairwise space $\mathbb{Z}^{\binom{n}{2}}$ denoted by the vector $w^u = (a_{1,2}^u, \ldots, a_{j,m}^u, \ldots, a_{n-1,n}^u) \in \mathbb{Z}^{\binom{n}{2}}$ where $j, m \in \{1, 2, \ldots, n\}$, $j < m$.

Finally, use will be made of concepts measuring the distance between binary relations and profiles, respectively. Let $\mathbb{R}$ be the set of all real numbers. The Kemeny distance function on $\mathscr{B}$ will be defined as $\delta : \mathscr{B} \times \mathscr{B} \to \mathbb{R}_+$ such that for all $R, R' \in \mathscr{B}$, $\delta(R, R') = \frac{|(R-R') \cup (R'-R)|}{2}$.[4] Distance on the set of (linear) profiles will be measured by distance function $d : \mathscr{L}^h \times \mathscr{L}^h \to \mathbb{R}_+$ such that for all $u, u' \in \mathscr{L}^h$, $d(u, u') = \sum_{i=1}^{h} \delta(L_i^u, L_i^{u'})$.

[3] Whenever there is no danger of confusion the superscript will be dropped.

[4] As we are exclusively concerned with linear orders, the division by 2 is for the convenience of being able to talk about distance values and numbers of pairwise switches interchangeably.

Table 1. Rankings not allowed in a 4 – 1-agreement profile

$A_4 \succ A_1 \succ A_2 \succ A_3$	$A_3 \succ A_4 \succ A_1 \succ A_2$
$A_4 \succ A_1 \succ A_3 \succ A_2$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$A_2 \succ A_4 \succ A_1 \succ A_3$	$A_3 \succ A_2 \succ A_4 \succ A_1$

3 Agreement profiles

Saari [11] has shown that for any vector $z \in \mathbb{Z}^{\binom{n}{2}}$, such that all entries are either even or odd, there exists a profile $u \in \mathscr{L}^h$ such that $w^u = z$, i.e., any point in pairwise space can be obtained using SMR. This has been extended by Ratliff [9] in the sense that for any point in pairwise space there always exists a profile such that only switches in adjacent alternatives in the individual rankings are necessary to determine the Dodgson winner. This might not go far enough in determining the problems that might occur between Dodgson's rule and other aggregation rules. Hence we will show that any point in pairwise space can be obtained from a profile that has, for at least one pair of alternatives $A_j, A_m \in X$, a certain agreement in the positions of those alternatives in the preference rankings of the individuals that prefer A_m over A_j.[5] Later sections will use such agreement profiles to derive the main results of this paper.

Definition 3.1. *For all $A_j, A_m \in X$, let $\bar{\mathscr{L}}_{j,m} \subseteq \mathscr{L}$ be the set of all $L \in \mathscr{L}$ such that $A_j \succ_L A_m$. A profile $u \in \mathscr{L}^h$, satisfies m–j-agreement if and only if for some pair $A_m, A_j \in X$, it is the case that for all $i \in H$ such that $A_m \succ_{L_i^u} A_j$, for all $R \in \bar{\mathscr{L}}_{j,m}$, $\delta(L_i^u, R) \geq n-2$.*

Example 3.2. To provide some intuition for the above definition consider $X = \{A_1, A_2, A_3, A_4\}$. There are $n! = 4! = 24$ possible linear orderings of the alternatives in set X. However, agreement profiles do not allow some of these linear orders. For example, a *4 – 1*-agreement profile is not allowed to contain rankings that have A_4 and A_1 adjacent whenever A_4 is preferred to A_1. In an adjacency situation only one pairwise switch would be necessary to get A_1 above A_4, hence violating the distance requirement in Definition 3.1. The linear orders banned from such a profile are stated in Table 1.

It will be of use to us to have profiles of linear orders that have a certain structure in the sense that, in addition to the agreement as stated in Definition 3.1, these profiles lead also to specific pairwise margins.

[5] There is a huge literature on so called domain restrictions mainly used in the discussion about possibility results for SMR. See Sen and Pattanaik [14] or Gaertner [5].

Definition 3.3. *For any two pairs* $(A_p, A_q), (A_m, A_j) \in X^2$, *a* $p{-}q$-portion with $m{-}j$-agreement *is a profile of linear orders such that* $a_{p,q} = 2$ *and for all pairs* $\{A_k, A_l\} \neq \{A_p, A_q\}$, $a_{k,l} = 0$ *and the profile satisfies* $m{-}j$*-agreement.*

Lemma 3.4. *Let* $|X| \geq 4$. *Given a pair* $A_m, A_j \in X$, *for all pairs* $A_p, A_q \in X$ *there exists a* $p{-}$q*-portion with* $m{-}$j*-agreement.*

Proof. Case 1: Let $\{m, j\} \cap \{p, q\} = \emptyset$ and $T = X \backslash \{A_p, A_q, A_m, A_j\}$. For any linear ranking on $T, r(T)$, a $p{-}q$-portion with $m{-}j$-agreement is then the following pair of linear orders on X:

$$A_m \succ A_p \succ A_q \succ r(T) \succ A_j$$

$$A_j \succ \rho(r(T)) \succ A_p \succ A_q \succ A_m$$

Obviously such a portion gives $a_{p,q} = 2$ and zero for all other pairwise margins.
Case 2: Let $m = p, j \neq p, q$ and $T = X \backslash \{A_q, A_m, A_j\}$. Then by just substituting A_m for A_p we get the following pair of linear orders, providing us with the desired pairwise margins:

$$A_m \succ A_q \succ r(T) \succ A_j$$

$$A_j \succ \rho(r(T)) \succ A_m \succ A_q$$

The same argumentation can be used for $m = q, j \neq p, q$ which obviously leads to pairwise margins in the desired form.

Case 3: For $j = p$ or $j = q$ and $m \neq p, q$ just interchange A_m and A_j in the argumentation under case 2.

Case 4: Let $m = p$ and $j = q$. We have to be careful to preserve the required agreement in the positions of A_m and A_j. Hence we first take the following pair of linear orders, where $T = X \backslash \{A_m, A_j, A_n\}$:

$$A_m \succ r(T) \succ A_j \succ A_n$$

$$A_n \succ A_m \succ \rho(r(T)) \succ A_j$$

Such a portion not only has $a_{m,j} = 2$ but also $a_{m,s} = 2$ and $a_{s,j} = 2$ for all $A_s \in T$. However, by cases 1 to 3 we know that there exist $s{-}m$- and $j{-}s$-portions with $m{-}j$-agreement. By adding those portions we get a profile where $a_{m,j} = 2$ and $a_{k,l} = 0$ for all $\{A_k, A_l\} \neq \{A_m, A_j\}$.

We are now able to show that for any vector in $\mathbb{Z}^{\binom{n}{2}}$, with all entries having the same parity, there exists a profile which, by using SMR, will exactly determine such a vector as the vector of pairwise margins.

Proposition 3.5. *If there are more than three alternatives, then for any $z \in \mathbb{Z}^{\binom{n}{2}}$, with all entries having the same parity, there exists a profile $u \in \mathcal{L}^h$ such that u satisfies m–j-agreement and $w^u = z$.*

Proof. Case 1: All the entries in z are even. The proof follows immediately from Lemma 3.4.

Case 2: All entries in z are odd. Take any linear order with A_m on top and A_j on bottom. This implies that all pairwise margins are either 1 or -1. The proof now follows immediately from Lemma 3.4 and the fact that by adding (or subtracting) any multiples of 2 to (from) either 1 or -1 we are able to get any desired odd number.

Hence, in this section we have shown that any desired point in pairwise space $\mathbb{Z}^{\binom{n}{2}}$ (with all entries having the same parity) can be obtained from a particular profile on X and some certain agreement among the individuals over at least one pair of alternatives can be guaranteed.[6] It is exactly this fact that will be used to derive later results concerning the differences of the investigated rules.

4 The Dodgson ranking

Dodgson [3] suggested a non-binary procedure which, in the absence of a Condorcet winner, always chooses the alternative(s) in the top cycle of X that is (are) "closest" from being a Condorcet winner. His concept of distance is based on the number of inversions of pairs in the individual preferences. Hence profile information is essential for applying Dodgson's rule. A natural binary extension of Dodgson's non-binary rule is to first group the alternatives into different levels (of cycles and singletons) and then rank the alternatives within each level based on Dodgson's concept of distance.[7] The different levels can be seen as a sequence of subsets of X. Level 1 is the top cycle set of X, level 2 is the top cycle set of all alternatives not in level 1, level 3 is the top cycle set of all alternatives not in levels 1 and 2, and so on.

Based on a definition by Miller et al. [8] we define, for any $T \subseteq X$ and any profile $u \in \mathcal{L}^h$, the top cycle set $T^* \subseteq T$ as a non-empty subset of T such that (i) for all $A \in T^*$ and any $A' \in T \backslash T^*$ we have $A \succ_{v(u)} A'$ and (ii) no proper subset of T^* meets condition (i).[8] Then the different levels can be seen as a finite sequence of top cycle sets, $\left\langle T_j^* \right\rangle_1^t$, such that

[6] It is of course clear that for any pair different from (A_m, A_j) we can insure any desired number of adjacent positions in a set of linear orders that satisfies m–j-agreement.

[7] I am very grateful to an anonymous referee for pointing out to me the importance of this grouping in the case of majority cycles not including all alternatives in X.

[8] A top cycle set could of course only contain one alternative. If there is only one alternative in level 1, then this alternative is the Condorcet winner.

T_1^* is the top cycle set of X
T_j^* is the top cycle set of $X \backslash \bigcup_{i=1}^{j-1} T_i^*$
$T_t^* = \emptyset$

Now, for all $A_j \in X$ let $\Gamma(A_j) \subset \mathscr{L}^h$ denote the non-empty set of all profiles for which A_j is the Condorcet winner, i.e. $a_{j,m} > 0$ for all $A_m \in X \backslash \{A_j\}$. The *Dodgson distance* of an alternative will then formally be determined by the function $\Delta^u : X \to \mathbb{R}_+$ such that for all $u \in \mathscr{L}^h$ and all $A \in X$, $\Delta^u(A) = \min_{u' \in \Gamma(A)} d(u, u')$.

Hence, we are now able to provide a binary extension of Dodgson's non-binary aggregation procedure.

Definition 4.1. *For all $u \in \mathscr{L}^h$, the Dodgson ranking $D \in \mathscr{W}$ is such that for all $A_j, A_m \in X$, $A_j \succsim_D A_m$ if and only if $A_j \in T_r^*$ and $A_m \in T_s^*$, $r < s$, and (if $T_r^* = T_s^*$) $\Delta^u(A_j) \leq \Delta^u(A_m)$.*

Intuitively such a Dodgson ranking seems to be a very attractive solution to the problem of voting cycles. The method insures that alternatives are higher ranked in the Dodgson ranking whenever they are either on a higher level or, if they are on the same level, of smaller distance from being a Condorcet winner. Moreover it implies the existence of a Dodgson loser, i.e., the alternative on the lowest level that is considered furthest away from being a Condorcet winner.[9] This can be seen as an unattractive alternative for choice. The question arises whether any of the other rules devised to overcome Condorcet's paradox will pick such an alternative. As those methods are extensions of SMR this could be seen as undesirable because it goes against the majoritarian legitimacy inherited from SMR.

5 Kemeny vs. Dodgson

The rule devised by Kemeny [6] also explicitly uses distances. It is based on the degree of agreement among the individuals on certain rankings. Therefore we have to find the ranking for which the sum of the distances to every single individual's preference is minimal. Obviously such a ranking might not be unique.

Definition 5.1. *For all $u \in \mathscr{L}^h$, $K \in \mathscr{W}$ is the Kemeny ranking if and only if for all $R \in \mathscr{W}$, $\sum_{i=1}^{h} \delta(L_i^u, K) \leq \sum_{i=1}^{h} \delta(L_i^u, R)$.*

Recently Saari and Merlin [13] have provided a simple description of Kemeny's rule, indicating that Kemeny's rule is based on a different informational level requiring only information about pairwise margins. They show that a

[9] For a further discussion see Baigent [1].

Table 2. Preference profile

Nr.	Ranking	Nr.	Ranking
9	$A_1 \succ A_2 \succ A_3 \succ A_4$	5	$A_3 \succ A_4 \succ A_2 \succ A_1$
5	$A_1 \succ A_3 \succ A_2 \succ A_4$	5	$A_4 \succ A_2 \succ A_1 \succ A_3$
5	$A_2 \succ A_4 \succ A_3 \succ A_1$	5	$A_4 \succ A_3 \succ A_1 \succ A_2$

Table 3. Tallies and margins

	Tallies	Margins		Tallies	Margins
$A_1 \succ A_2$	19,15	4	$A_2 \succ A_3$	19,15	4
$A_1 \succ A_3$	19,15	4	$A_2 \succ A_4$	19,15	4
$A_1 \succ A_4$	14,20	−6	$A_3 \succ A_4$	19,15	4

Kemeny ranking is the weak order with the smallest sum of margins in the pairs that have to be switched to arrive exactly at that weak order.[10] A Kemeny winner is then the top alternative in a Kemeny ranking. What is striking is that, although both methods are devised to serve the same purpose (namely to overcome Condorcet's problem) and are using the same way of measuring differences between binary relations (Kemeny distance), it is possible that outcomes will be very different. It can even be shown that for some preference profiles the Kemeny winner will be the Dodgson loser.

Example 5.2. Consider $|X| = 4$, $|H| = 34$, and the following *4*−1-agreement profile $u \in \mathscr{L}^{34}$ given in Table 2, where numbers determine how many voters have each linear ranking. From Table 2 we obtain, by using SMR, the pairwise tallies and margins which are presented in Table 3. As can be clearly seen, the above profile leads to a cycle $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_1$ which includes all alternatives.

From Table 3 we are able to calculate the Dodgson distances, i.e., the distance (or necessary number of switches) of each alternative from being a Condorcet winner. As the above profile is a *4*−1-agreement profile, we need 4 individuals switching their rankings in *two* pairs each to make A_1 a Condorcet winner. For all other pairs there exist enough adjacent switching possibilities. The Dodgson distance (number of pairwise switches) for every $A \in X$, denoted by $\Delta(A)$, is given in Table 4.

Hence, from Table 4 and the fact that all alternatives belong to the top cycle, the Dodgson ranking is $A_2 \succ A_3 \sim A_4 \succ A_1$ with A_1 being the Dodgson looser. On the other hand, deriving the Kemeny ranking we see that for the

[10] It has to be emphasised again, that there might be more than one Kemeny ranking for certain profiles.

Table 4. Dodgson distances

$\Delta(A_1)$	8
$\Delta(A_2)$	3
$\Delta(A_3)$	6
$\Delta(A_4)$	6

linear order $A_1 \succ A_2 \succ A_3 \succ A_4$ we only need to switch $A_4 \succ A_1$ giving a margin of *6*. Obviously, other rankings with A_1 above A_4 need more switches. Every ranking where A_1 is not on top requires at least two switches by either moving A_1 below A_2 and A_3 or A_4 above A_2 and A_3 with added margins of at least *8*. Hence, A_1, the Dodgson looser, is the Kemeny winner.

Theorem 5.3. *If there are more than three alternatives, then there exists a profile such that the unique Kemeny winner is the unique Dodgson loser.*

Proof. The proof is based on creating a profile, $u \in \mathscr{L}^h$, $2 < |H| = h < \infty$, such that $w^u = (a^u_{1,2}, \ldots, a^u_{j,m}, \ldots, a^u_{n-1,n}) \in \mathbb{Z}^{\binom{n}{2}}$ has a special form and u satisfies $n-1$-agreement. Let $l, k \in \mathbb{Z}_+$ be two positive integers having the same parity, with $l, k \geq 3$. Let $a_{1,n} = -k$ and $a_{p,q} = l$ for all $p, q \in \{1, 2, 3, \ldots, n-1, n\}$, $p < q, p \neq 1, q \neq n$. Such a $u \in \mathscr{L}^h$ will determine a simple majority relation with a cycle and by Proposition 3.5 such a profile exists.

Dodgson part. As the determined cycle is including all alternatives in X, there is only one level (the top cycle) to be considered. Therefore the Dodgson ranking is based exclusively on the Dodgson distance in this case. By $n-1$-agreement, $\Delta^u(A_1) = (n-2)\left(\frac{k}{2}+1\right)$ for k being even and $\Delta^u(A_1) = \frac{(n-2)(k+1)}{2}$ for k being odd. For A_1 to become the Dodgson loser, this number has to be larger than that for any other alternative in X, therefore (for l even) it has to be larger than $(n-2)(\frac{l}{2}+1)$ (for l odd, the number is $\frac{(n-2)(l+1)}{2}$). Hence, for both the even and the odd case we get the following necessary condition for A_1 to be the unique Dodgson loser, namely $k > l$.

Kemeny part. For A_1 to be the Kemeny winner, it has to be the top alternative in the Kemeny ranking. Using Saari and Merlin's [13] method to determine the Kemeny ranking, we get a value of k for the ranking $A_1 \succ A_2 \succ \cdots \succ A_n$ to be the Kemeny ranking. We have to compare this value with any ranking that has A_1 not on top. Any such ranking with A_1 above A_n is obviously not a Kemeny ranking as in addition to the pair (A_1, A_n) another pair has to be switched. For any ranking with A_n above A_1, A_n has to be moved above at least $n-2-s$ alternatives and A_1 has to be moved below at least s alternatives $(2 \leq s \leq n-2)$. It is obvious that there have to be at least $n-2$ switches, therefore, for any ranking with A_1 not on top we get a sum of margins of $(n-2)l$. Hence the necessary condition for A_1 to be the Kemeny winner is $k < (n-2)l$.

Therefore we have to find an n–1-agreement profile $u \in \mathscr{L}^h$ that determines the vector $w^u = (l, l, l, \ldots, l, -k, l, \ldots, l, l)$ where $a_{1,n} = -k$ and all other pairwise margins are equal to l and where $l < k < (n-2)l$. By Proposition 3.5 such a profile exists and this proves the theorem.

We have shown that such an extreme dissimilarity can indeed occur. However, it can also be shown that there is no consistency between the Kemeny winner and the Dodgson ranking, i.e., the Kemeny winner can actually occur at any position in the Dodgson ranking.

Theorem 5.4. *If there are more than three alternatives, then the unique Kemeny winner can occur at any position in the Dodgson ranking.*

Proof. Assume, for some n–1-agreement profile $u \in \mathscr{L}^h$, $2 < |H| = h < \infty$, $w^u = (l, l, \ldots, l, -k, l, \ldots, l, l)$ where $a_{1,n} = -k$ and all other pairwise margins are equal to l and $3 \leq l < k < (n-2)l$. By Theorem 5.3 we then know that A_1 is the Kemeny winner but the Dodgson loser. From the argumentation in the proof of Theorem 5.3 it is clear that an increase in any other margin besides $a_{1,n}$ would not change A_1 from being the Kemeny winner and would still let the majority cycle contain all alternatives in X. For the above point in pairwise space, w^u, we get the Dodgson distances as stated in Table 5.

Hence, to insure position s for alternative A_1 in the Dodgson ranking, $1 \leq s \leq n$, we just have to increase the Dodgson distances of n–s alternatives above $\Delta(A_1)$. We know that $l < k < (n-2)l$ and will assume the special case $k = l + 2 < (n-2)l$ and k,l being odd. Hence, $\Delta(A_1) = (n-2)\frac{l+3}{2}$. Thus, for $A_1 \succ_D A_n$ we need $\Delta(A_n) > (n-2)\frac{l+3}{2}$. As $l \geq 3$, multiply all $a_{q,n}$ with $q \in \{2, 3, \ldots, n-1\}$ by 2 to obtain $\Delta(A_n) = (n-2)\frac{2l+1}{2} > \Delta(A_1)$. Hence A_1 is still the Kemeny winner but now has handed over the bottom position in the Dodgson ranking to alternative A_n. In general for A_1 being in position s in the Dodgson ranking, let $a_{q,n} = 2l$ for all $q \in \{2, 3, \ldots, n-1\}$ and for all $A_p \neq A_n, p \geq s$, insure that $\Delta(A_p) > (n-2)\frac{l+3}{2}$. Therefore, for any A_p assume $a_{q,p} = j$ for $q < p$. We get $\Delta(A_p) = (p-1)\frac{j+1}{2}$. Therefore it is necessary to have $(p-1)\frac{j+1}{2} > (n-2)\frac{l+3}{2}$. This requires the condition $j > \frac{n-2}{p-1}(l+3) - 1$ to hold for A_p to be ranked above A_1 in the Dodgson ranking. Such a condition can be satisfied for any $n \geq 4, p \in \{2, 3, \ldots, n-1\}$ and any $l \geq 3$ such

Table 5. Dodgson distances

	k,l odd	k,l even
$\Delta(A_1)$	$(n-2)\frac{k+1}{2}$	$(n-2)\frac{k+2}{2}$
$\Delta(A_2)$	$\frac{l+1}{2}$	$\frac{l+2}{2}$
$\Delta(A_3)$	$2\frac{l+1}{2}$	$2\frac{l+2}{2}$
$\vdots$	$\vdots$	$\vdots$
$\Delta(A_{n-1})$	$(n-2)\frac{l+1}{2}$	$(n-2)\frac{l+2}{2}$
$\Delta(A_n)$	$(n-2)\frac{l+1}{2}$	$(n-2)\frac{l+2}{2}$

that l is odd. By Proposition 3.5 there always exists a profile which determines those pairwise margins and thus the theorem is proved for the odd case. To show that this holds also for l being even is left to the reader.

6 Slater vs. Dodgson

The third rule explicitly using distances has been devised by Slater [15]. A Slater ranking is the weak order that is of shortest Kemeny distance from the simple majority relation. Obviously, also the Slater ranking might be non-unique. Of all the discussed rules in this paper, the Slater ranking requires the least information. Only the simple majority relation is necessary, but not any information about pairwise margins or individual rankings. Based on this modest information it then uses the same distance function as the above rules to determine a transitive social preference.

Definition 6.1. *For all* $u \in \mathscr{L}^h$, $S \in \mathscr{W}$ *is the Slater ranking if and only if for all* $R \in \mathscr{W}$, $\delta(v(u), S) \leq \delta(v(u), R)$.

Theorem 6.2. *If there are more than three alternatives, then there exists a profile such that the unique Slater winner is the unique Dodgson loser.*

Proof. Consider again the $n{-}1$-agreement profile used in the proof of Theorem 5.3 which leads to the following point in pairwise space, $w^u = (l, l, \ldots, l, -k, l, \ldots, l, l)$, $l \geq 3$, where (for both k and l having the same parity) $a_{1,n} = -k$ and $a_{p,q} = l$ for all $p, q \in \{1, 2, 3, \ldots, n-1, n\}$, $p < q, p \neq 1, q \neq n$.

Dodgson part. As has been shown in the proof of Theorem 5.3 for A_1 to be a Dodgson loser we require $k > l$.

Slater part. As the Slater ranking is the weak order of closest Kemeny distance to the original simple majority relation we see that for the ranking $A_1 \succ A_2 \succ \cdots \succ A_n$ one pairwise switch (between A_n and A_1) is required. Any other ranking with A_1 above A_n but with A_1 not on top will obviously be of larger distance away from the simple majority relation. In any ranking with A_n above A_1, A_n has to be moved above at least $n-2-s$ alternatives and A_1 has to be moved below at least s alternatives $(2 \leq s \leq n-2)$. It is obvious that there have to be at least $n-2$ switches, which guarantees that, for $n \geq 4$, A_1 will be the Slater winner for any positive integer values of l and k.

Theorem 6.3. *If there are more than three alternatives, then the Slater winner can occur at any position in the Dodgson ranking.*

Proof. The proof is obvious from the proof of Theorem 5.4, in the sense that as long as in the above vector w^u, $a_{1,n}$ is negative and $a_{p,q}$ is positive for all

$p, q \in \{1, 2, 3, \ldots, n-1, n\}$, $p < q, p \neq 1, q \neq n$, the Slater winner does not change and we could show in the proof of Theorem 5.4 that this can be held fixed although A_1 can take any position in the Dodgson ranking.

7 Conclusion

This paper has introduced a new and reasonable aggregation rule to determine a social weak order in the presence of majority cycles. Based on Dodgson's non-binary rule which uses the idea of distance from being a Condorcet winner, it provides an attractive alternative to other binary aggregation rules. We then compared the Dodgson ranking with Kemeny's and Slater's rule, emphasizing two interesting aspects. First, we show that different aggregation rules can lead to opposite outcomes, giving insight that cannot be provided by other kinds of comparing aggregation rules. Second, Kemeny's and Slater's rules sometimes give doubtful results in the sense that the top alternatives in their rankings are furthest from being Condorcet winners.

References

[1] Baigent N (1983) Some reservation on strictly majoritarian social choice. Discussion Paper Series, University of Essex

[2] Baigent N, Klamler C (2004) Transitive closure, proximity and intransitivities. Econ Theory 23: 175–181

[3] Dodgson C (1876) A method of taking votes on more than two issues. In: Black D (1958) The theory of committees and elections. Cambridge University Press, London

[4] Fishburn P (1977) Condorcet social choice functions. SIAM J Appl Math 33: 469–489

[5] Gaertner W (1998) The theory of social choice since 1951. Jahrb Nationaloekon Statist 217/6: 657–680

[6] Kemeny J (1959) Mathematics without numbers. Daedalus 88: 577–591

[7] Laffond G et al. (1995) Condorcet choice correspondences: A set-theoretical comparison. Math Soc Sci 30: 23–35

[8] Miller NR et al. (1990) The structure of the banks set. Publ Choice 66: 243–251

[9] Ratliff TC (2001) A comparison of Dodgson's method and Kemeny's rule. Soc Choice Welfare 18: 79–90

[10] Ratliff TC (2002) A comparison of Dodgson's method and the Borda count. Econ Theory 20: 357–372

[11] Saari DG (1995) Basic geometry of voting. Springer, Berlin Heidelberg New York

[12] Saari DG (2000) Mathematical structure of voting paradoxes: I. Pairwise votes. Econ Theory 15: 1–53

[13] Saari DG, Merlin VR (2000) A geometric examination of Kemeny's rule. Soc Choice Welfare 17: 403–438

[14] Sen AK, Pattanaik PK (1969) Necessary and sufficient conditions for rational choice and majority decision. J Econ Theory 1: 178–202

[15] Slater P (1961) Inconsistencies in a schedule of paired comparisons. Biometrica 48: 303–312