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The Logarithmic Contributions to the $O(\alpha_s^3)$ Asymptotic Massive Wilson Coefficients and Operator Matrix Elements in Deeply Inelastic Scattering

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Abstract

We calculate the logarithmic contributions to the massive Wilson coefficients for deep-inelastic scattering in the asymptotic region $Q^2 \gg m^2$ to 3-loop order in the fixed-flavor number scheme and present the corresponding expressions for the massive operator matrix elements needed in the variable flavor number scheme. Explicit expressions are given both in Mellin-N space and z-space.

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1 Introduction

The heavy flavor corrections to deep-inelastic structure functions amount to sizeable contributions, in particular in the region of small values of the Bjorken variable x. Starting from lower values of the virtuality, over a rather wide kinematic range, their scaling violations are very different from those of the massless contributions. Currently the heavy flavor corrections are known in semi-analytic form to 2–loop (NLO) order [1]. The present accuracy of the deep-inelastic data reaches the order of 1% [2]. It therefore requires the next-to-next-to-leading order (NNLO) corrections for precision determinations of both the strong coupling constant $\alpha_s(M_Z^2)$ [3] and the parton distribution functions (PDFs) [4,5], as well as the detailed understanding of the heavy flavor production cross sections in lepton–nucleon scattering [6]. The precise knowledge of these quantities is also of central importance for the interpretation of the physics results at the Large Hadron Collider, LHC, [7].

In the kinematic region at HERA, where the twist-2 contributions to the deep-inelastic scattering (DIS) dominate cf. [8][2], i.e. $Q^2/m^2 \gtrsim 10$, with $m = m_c$ the charm quark mass, it has been proven in Ref. [10] that the heavy flavor Wilson coefficients factorize into massive operator matrix elements (OMEs) and the massless Wilson coefficients. The massless Wilson coefficients for the structure function $F_2(x, Q^2)$ are known to 3-loop order [11]. In the region $Q^2 \gg m^2$, where $Q^2 = -q^2$, with q the space-like 4–momentum transfer and m the heavy quark mass, the power corrections $O((m^2/Q^2)^k), k \geq 1$ to the heavy quark structure functions become very small.

In Ref. [12] a series of fixed Mellin moments N up to $N = 10, ..., 14$, depending on the respective transition, has been calculated for all the OMEs at 3–loop order[3]. Also the moments of the transition coefficients needed in the variable flavor scheme (VFNS) have been calculated. Here, the massive OMEs for given total spin N were mapped onto massive tadpoles which have been computed using `MATAD` [14].

In the present paper, we calculate the logarithmic contributions to the unpolarized massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ to 3–loop order and the massive OMEs needed in the VFNS. These include the logarithmic terms $\log(Q^2/m^2)$. In the following, we set the factorization and renormalization scales equal $\mu_F = \mu_R \equiv \mu$ and exhibit the $\log(m^2/\mu^2)$ dependence on the Wilson coefficients, besides their dependence on the virtuality Q^2. The logarithmic contributions are determined by the lower order massive OMEs [15–19], the mass- and coupling constant renormalization constants, and the anomalous dimensions [20, 21], as has been worked out in Ref. [12]. For the structure function $F_L(x, Q^2)$ the asymptotic heavy flavor Wilson coefficients at $O(\alpha_s^3)$ were calculated in [22]. They are also presented here, for inclusive hadronic final states. In this case the corrections, however, become effective only at much higher scales of Q^2 [10] compared to the case of $F_2(x, Q^2)$. We first choose the fixed flavor number scheme to express the heavy flavor contributions to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$. This scheme has to be considered as the genuine scheme in quantum field theoretic calculations since the initial states, the twist–2 *massless* partons can, at least to a good approximation, be considered as LSZ-states. The representations in the VFNS can be obtained using the respective transition coefficients within the appropriate regions, where one single heavy quark flavor becomes effectively massless. Here, appropriate matching scales have to be applied, which vary in dependence on the observable considered, cf. [23].

Two of the OMEs, $A_{qq,Q}^{\mathsf{PS}}(N)$ and $A_{qg,Q}(N)$, have already been calculated completely including the constant contribution in Ref. [24]. They and the corresponding massive Wilson coefficients

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[2]For higher order corrections to the gluonic contributions in the threshold region, cf. [9].

[3]For the corresponding contributions in case of transversity see [13].

contribute first at 2– and 3–loop order, respectively. For these quantities we also derive numerical results. The quantities being presented in the present paper derive from OMEs which were computed in terms of generalized hypergeometric functions [25] and sums thereof, prior to the expansion in the dimensional variable $\varepsilon = D - 4$, cf. [26–28]. Finally, they are represented in terms of nested sums over products of hypergeometric terms and harmonic sums, which can be calculated using modern summation techniques [29–33]. They are based on a refined difference field of [34] and generalize the summation paradigms presented in [35] to multi-summation. The results of this computation can be expressed in terms of nested harmonic sums [36, 37]. The corresponding representations in z-space are obtained in terms of harmonic polylogarithms [38]. Here, the variable z denotes the partonic momentum fraction. The results in Mellin N-space can be continued to complex values of N as has been described in Refs. [26, 39].

It is the aim of the present paper to provide a detailed documentation of formulae both in N- and z-space for all logarithmic contributions to the heavy flavor Wilson coefficients of the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ and the massive OMEs needed in the variable flavor number scheme up to $O(\alpha_s^3)$. Here, we refer to a minimal representation, i.e. we use all the algebraic relations between the harmonic sums and the harmonic polylogarithms, respectively, leading to a minimal number of basic functions. Based on the known Mellin moments [12] we also perform numerical comparisons between the different contributions to the Wilson coefficients and massive OMEs at $O(\alpha_s^3)$ referring to the parton distributions [5].

The paper is organized as follows. In Section 2, we summarize the basic formalism. The Wilson coefficients $L_{q,2}^{\sf PS}$ and $L_{g,2}^{\sf S}$ are discussed in Section 3. As they are known in complete form we also present numerical results. In Section 4, the logarithmic contributions to the Wilson coefficients $H_{q,2}^{\sf PS}$ and $H_{g,2}^{\sf S}$,[4] are derived. The corresponding Wilson coefficients for the longitudinal structure function $F_L(x, Q^2)$ in the asymptotic region are presented in Section 5. In Section 6, we compare the different loop contributions to the massive Wilson coefficients and OMEs for a series of Mellin moments in dependence on the virtuality Q^2. Section 7 contains the conclusions. In Appendix A the massive OMEs needed in the VFNS are given in Mellin N-space. The asymptotic heavy flavor Wilson coefficients contributing to the structure function $F_2(x, Q^2)$ are presented in z-space in Appendix B, retaining all contributions except for the 3-loop constant part of the unrenormalized OMEs $a_{ij}^{(3)}$ being not yet known. Likewise, in Appendix C and D, the asymptotic heavy flavor Wilson coefficients for the structure function $F_L(x, Q^2)$ and the massive OMEs are given in z-space.

2   The heavy flavor Wilson coefficients in the asymptotic region

We consider the heavy flavor contributions to the inclusive unpolarized structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in deep-inelastic scattering, cf. [41, 42], in case of single electro-weak gauge-boson exchange at large virtualities Q^2. At higher orders in the strong coupling constant these corrections receive both contributions from massive and massless partons in the hadronic final state, which is summed over completely. In the latter case, the heavy flavor corrections are also due to virtual contributions. We consider the situation in which the contributions to the twist-2 operators dominate in the Bjorken limit. Here, no transverse momentum effects of the initial state contribute. In the present paper, we consider only heavy flavor contributions

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[4]The expressions for the non-singlet Wilson-coefficient, are presented elsewhere together with the OME for transversity [40].

due to N_F massless and one massive flavor of mass m.[5] The Wilson coefficients are calculable perturbatively and are denoted by

$$\mathcal{C}_{i,(2,L)}^{\mathsf{S,PS,NS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) . \tag{1}$$

Here, x denotes the Bjorken variable, the index i refers to the respective initial state on-shell parton $i = q, g$ being a quark or gluon, and $\mathsf{S, PS, NS}$ label the flavor singlet, pure–singlet and non–singlet contributions, respectively. In the twist-2 approximation the Bjorken variable x and the parton momentum fraction z are identical. Representations in momentum fraction space are therefore also called z-space representation in what follows.

The massless flavor contributions in (1) may be identified and separated in the Wilson coefficients into a purely light part $C_{i,(2,L)}$, and a heavy part by :

$$\mathcal{C}_{i,(2,L)}^{\mathsf{S,PS,NS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{i,(2,L)}^{\mathsf{S,PS,NS}}\left(x, N_F, \frac{Q^2}{\mu^2}\right)$$
$$+ H_{i,(2,L)}^{\mathsf{S,PS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) + L_{i,(2,L)}^{\mathsf{S,PS,NS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) . \tag{2}$$

The heavy flavor Wilson coefficients are defined by $L_{i,j}$ and $H_{i,j}$, depending on whether the exchanged electro-weak gauge boson couples to a light (L) or heavy (H) quark line. From this it follows that the light flavor Wilson coefficients $C_{i,j}$ depend on N_F light flavors only, whereas $H_{i,j}$ and $L_{i,j}$ may contain light flavors in addition to the heavy quark, indicated by the argument $N_F + 1$. The perturbative series of the heavy flavor Wilson coefficients read

$$H_{g,(2,L)}^{\mathsf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=1}^{\infty} a_s^i H_{g,(2,L)}^{(i),\mathsf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) , \tag{3}$$

$$H_{q,(2,L)}^{\mathsf{PS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=2}^{\infty} a_s^i H_{q,(2,L)}^{(i),\mathsf{PS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) , \tag{4}$$

$$L_{g,(2,L)}^{\mathsf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=2}^{\infty} a_s^i L_{g,(2,L)}^{(i),\mathsf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) , \tag{5}$$

$$L_{q,(2,L)}^{\mathsf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=2}^{\infty} a_s^i L_{q,(2,L)}^{(i),\mathsf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) . \tag{6}$$

Here, we defined $a_s = \alpha_s/(4\pi)$. At leading order, only the term $H_{g,(2,L)}$ contributes via the photon–gluon fusion process, [44–49],

$$\gamma^* + g \to Q + \overline{Q} . \tag{7}$$

At $O(a_s^2)$, the terms $H_{q,(2,L)}^{\mathsf{PS}}$, $L_{q,(2,L)}^{\mathsf{S}}$ and $L_{g,(2,L)}^{\mathsf{S}}$ contribute as well. They result from the processes

$$\gamma^* + q(\bar{q}) \to q(\bar{q}) + X , \tag{8}$$
$$\gamma^* + g \to q(\bar{q}) + X , \tag{9}$$

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[5]At 3–loop order there are also contributions by graphs carrying heavy quark lines of different mass. These are dealt with elsewhere [43].

where X may contain heavy flavor contributions. $L^S_{q,(2,L)}$ can be split into the flavor non-singlet and pure-singlet contributions

$$L^S_{q,(2,L)} = L^{NS}_{q,(2,L)} + L^{PS}_{q,(2,L)}, \tag{10}$$

and at $O(a_s^2)$ only the non-singlet term contributes. The pure-singlet term emerges at 3–loop order.

The heavy quark contribution to the structure functions $F_{(2,L)}(x,Q^2)$ for one heavy quark of mass m and N_F light flavors is then given by, cf. [15], in case of pure photon exchange[6]

$$\frac{1}{x}F^{Q\bar{Q}}_{(2,L)}(x,N_F+1,Q^2,m^2) =$$

$$\sum_{k=1}^{N_F} e_k^2 \left\{ L^{NS}_{q,(2,L)}\left(x,N_F+1,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,N_F) + f_{\bar{k}}(x,\mu^2,N_F)\right] \right.$$

$$+\frac{1}{N_F}L^{PS}_{q,(2,L)}\left(x,N_F+1,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F)$$

$$\left. +\frac{1}{N_F}L^{S}_{g,(2,L)}\left(x,N_F+1,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \right\}$$

$$+ e_Q^2 \left[H^{PS}_{q,(2,L)}\left(x,N_F+1,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \right.$$

$$\left. +H^{S}_{g,(2,L)}\left(x,N_F+1,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \right], \tag{11}$$

The meaning of the argument (N_F+1) in Eqs. (11) in the massive Wilson coefficients shall be interpreted as N_F *massless* and one *massive flavor*. N_F denotes the number of massless flavors. The symbol $\otimes$ denotes the Mellin convolution,[7]

$$[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \, \delta(x - x_1 x_2) A(x_1) B(x_2) . \tag{12}$$

The charges of the light quarks are denoted by e_k and that of the heavy quark by e_Q. The scale μ^2 is the factorization scale, and $f_k, f_{\bar{k}}, \Sigma$ and G are the quark, anti-quark, flavor singlet and gluon distribution functions, with

$$\Sigma(x,\mu^2,N_F) = \sum_{k=1}^{N_F} \left[f_k(x,\mu^2,N_F) + f_{\bar{k}}(x,\mu^2,N_F)\right] . \tag{13}$$

An important part of the kinematic region in case of heavy flavor production in DIS is located at larger values of Q^2, cf. e.g. [54,55]. As has been shown in Ref. [10], the heavy flavor Wilson coefficients $H_{i,j}$, $L_{i,j}$ factorize in the limit $Q^2 \gg m^2$ into massive operator matrix elements A_{ki} and the massless Wilson coefficients $C_{i,j}$, if one heavy quark flavor and N_F light flavors are considered. The massive OMEs are process independent quantities and contain all the mass dependence except for the power corrections $\propto (m^2/Q^2)^k$, $k \geq 1$. The process dependence is implied by the massless Wilson coefficients. This allows the analytic calculation of the NLO

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[6]For the heavy flavor corrections in case of $W^\pm$-boson exchange up to $O(\alpha_s^2)$ see [50–53].
[7]Note that the heavy flavor threshold in the limit $Q^2 \gg m^2$ is again x and not $x(1+4m^2/Q^2)$, which is the case retaining also power corrections.

heavy flavor Wilson coefficients, [10, 17]. Comparing these asymptotic expressions with the exact LO and NLO results obtained in Refs. [44–47, 49] and [1], respectively, one finds that this approximation becomes valid in case of $F_2^{Q\overline{Q}}$ for $Q^2/m^2 \gtrsim 10$. These scales are sufficiently low and match with the region analyzed in deeply inelastic scattering for precision measurements. In case of $F_L^{Q\overline{Q}}$, this approximation is only valid for $Q^2/m^2 \gtrsim 800$, [10]. For the latter case, the 3–loop corrections were calculated in Ref. [22]. This difference is due to the emergence of terms $\propto (m^2/Q^2)\ln(m^2/Q^2)$, which only vanish slowly in the limit $Q^2/m^2 \to \infty$.

In order to derive the factorization formula, one considers the inclusive Wilson coefficients $\mathcal{C}_{i,j}^{\mathsf{S,PS,NS}}$, which have been defined in Eq. (1). After applying the light cone expansion (LCE) [56] to the partonic tensor, or the forward Compton amplitude, corresponding to the respective Wilson coefficients, one arrives at the factorization relation,

$$
\mathcal{C}_{j,(2,L)}^{\mathsf{S,PS,NS,asymp}}\left(N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) =
$$
$$
\sum_i A_{ij}^{\mathsf{S,PS,NS}}\left(N, N_F + 1, \frac{m^2}{\mu^2}\right) C_{i,(2,L)}^{\mathsf{S,PS,NS}}\left(N, N_F + 1, \frac{Q^2}{\mu^2}\right) + O\left(\frac{m^2}{Q^2}\right). \tag{14}
$$

Here, μ refers to the factorization scale between the heavy and light contributions in $\mathcal{C}_{j,i}$ and 'asymp' denotes the limit $Q^2 \gg m^2$. The $C_{i,j}$ are the light Wilson coefficients, cf. [11], taken at $N_F + 1$ flavors. This can be inferred from the fact that in the LCE the Wilson coefficients describe the singularities for very large values of Q^2, which can not depend on the presence of a quark mass. The mass dependence is given by the OMEs A_{ij}, between partonic states. Eq. (14) accounts for all mass effects but corrections which are power suppressed, $(m^2/Q^2)^k, k \geq 1$. This factorization is only valid if the heavy quark coefficient functions are defined in such a way that all radiative corrections containing heavy quark loops are included. Otherwise, (14) would not show the correct asymptotic Q^2–behavior, [15, 19]. An equivalent way of describing Eq. (14) is obtained by considering the calculation of the massless Wilson coefficients. Here, the initial state collinear singularities are given by evaluating the massless OMEs between off–shell partons, leading to transition functions Γ_{ij}. The Γ_{ij} are given in terms of the anomalous dimensions of the twist–2 operators and transfer the initial state singularities to the bare parton–densities due to mass factorization, cf. e.g. [10, 15]. In the case at hand, something similar happens: The initial state collinear singularities are transferred to the parton densities except for those which are regulated by the quark mass and described by the OMEs. Instead of absorbing these terms into the parton densities as well, they are used to reconstruct the asymptotic behavior of the heavy flavor Wilson coefficients. Here,

$$
A_{ij}^{\mathsf{S,NS}}\left(N, N_F + 1, \frac{m^2}{\mu^2}\right) = \langle j|O_i^{\mathsf{S,NS}}|j\rangle = \delta_{ij} + \sum_{i=k}^{\infty} a_s^k A_{ij}^{(k),\mathsf{S,NS}} \tag{15}
$$

are the operator matrix elements of the local twist–2 operators between on–shell partonic states $|j\rangle$, $j = q, g$.

Let us now derive the explicit expressions for the massive Wilson coefficients in the asymptotic region. One may split Eq. (14) into parts by considering the different N_F contributions. We define

$$
\tilde{f}(N_F) \equiv \frac{f(N_F)}{N_F}. \tag{16}
$$

This is necessary in order to separate the different types of contributions in Eq. (11), weighted by the electric charges of the light and heavy flavors, respectively. Since we would like to derive

the heavy flavor part, we define as well for later use

$$\hat{f}(N_F) \;\equiv\; f(N_F + 1) - f(N_F) , \tag{17}$$

where $\hat{\tilde{f}}(N_F) \equiv [\widetilde{\hat{f}(N_F)}]$. The following Eqs. (18)–(22) are the same as Eqs. (2.31)–(2.35) in Ref. [15]. We present these terms here again, however, since Ref. [15] contains a few inconsistencies regarding the $\tilde{f}$–description. Contrary to the latter reference, the argument corresponding to the number of flavors stands for all flavors, light or heavy. The separation for the NS–term is obtained by

$$C_{q,(2,L)}^{\text{NS}}\left(N, N_F, \frac{Q^2}{\mu^2}\right) + L_{q,(2,L)}^{\text{NS}}\left(N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \;=\;$$
$$A_{qq,Q}^{\text{NS}}\left(N, N_F + 1, \frac{m^2}{\mu^2}\right) C_{q,(2,L)}^{\text{NS}}\left(N, N_F + 1, \frac{Q^2}{\mu^2}\right) . \tag{18}$$

Here and in the following, we omit the index "asymp" to denote the asymptotic heavy flavor Wilson coefficients. For the remaining terms, we suppress the arguments N, Q^2/μ^2 and m^2/μ^2 for brevity, all of which can be inferred from Eqs. (2, 14). Additionally, we will suppress from now on the index S and label only the NS and PS terms explicitly. The contributions to $L_{i,j}$ read

$$C_{q,(2,L)}^{\text{PS}}(N_F) + L_{q,(2,L)}^{\text{PS}}(N_F + 1) \;=\; \left[A_{qq,Q}^{\text{NS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1) + A_{Qq}^{\text{PS}}(N_F + 1)\right]$$
$$\times N_F \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1) C_{q,(2,L)}^{\text{NS}}(N_F + 1)$$
$$+ A_{gq,Q}(N_F + 1) N_F \tilde{C}_{g,(2,L)}(N_F + 1) , \tag{19}$$

$$C_{g,(2,L)}(N_F) + L_{g,(2,L)}(N_F + 1) \;=\; A_{gg,Q}(N_F + 1) N_F \tilde{C}_{g,(2,L)}(N_F + 1)$$
$$+ A_{qg,Q}(N_F + 1) C_{q,(2,L)}^{\text{NS}}(N_F + 1)$$
$$+ \left[A_{qg,Q}(N_F + 1) + A_{Qg}(N_F + 1)\right] N_F \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) . \tag{20}$$

The terms $H_{i,j}$ are given by

$$H_{q,(2,L)}^{\text{PS}}(N_F + 1) \;=\; A_{Qq}^{\text{PS}}(N_F + 1)\left[C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1)\right]$$
$$+ \left[A_{qq,Q}^{\text{NS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1)\right]\tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1)$$
$$+ A_{gq,Q}(N_F + 1)\tilde{C}_{g,(2,L)}(N_F + 1) , \tag{21}$$

$$H_{g,(2,L)}(N_F + 1) \;=\; A_{gg,Q}(N_F + 1)\tilde{C}_{g,(2,L)}(N_F + 1) + A_{qg,Q}(N_F + 1)\tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1)$$
$$+ A_{Qg}(N_F + 1)\left[C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1)\right] . \tag{22}$$

Expanding the above relations up to $O(a_s^3)$, we obtain, using Eqs. (16, 17), the heavy flavor Wilson coefficients in the asymptotic limit, cf. [12] :

$$L_{q,(2,L)}^{\text{NS}}(N_F + 1) \;=\; a_s^2\left[A_{qq,Q}^{(2),\text{NS}}(N_F + 1)\,\delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F)\right]$$
$$+ \; a_s^3\left[A_{qq,Q}^{(3),\text{NS}}(N_F + 1)\,\delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1)\right.$$

$$+ \hat{C}^{(3),\text{NS}}_{q,(2,L)}(N_F) \Bigg] , \tag{23}$$

$$L^{\text{PS}}_{q,(2,L)}(N_F + 1) = a_s^3 \Bigg[A^{(3),\text{PS}}_{qq,Q}(N_F + 1)\, \delta_2 + A^{(2)}_{gq,Q}(N_F + 1)\ N_F \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1)$$

$$+ N_F \hat{\tilde{C}}^{(3),\text{PS}}_{q,(2,L)}(N_F) \Bigg] , \tag{24}$$

$$L^{\text{S}}_{g,(2,L)}(N_F + 1) = a_s^2 A^{(1)}_{gg,Q}(N_F + 1) N_F \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1)$$

$$+ a_s^3 \Bigg[A^{(3)}_{qg,Q}(N_F + 1)\, \delta_2 + A^{(1)}_{gg,Q}(N_F + 1)\ N_F \tilde{C}^{(2)}_{g,(2,L)}(N_F + 1)$$

$$+ A^{(2)}_{gg,Q}(N_F + 1)\ N_F \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1)$$

$$+ A^{(1)}_{Qg}(N_F + 1)\ N_F \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) + N_F \hat{\tilde{C}}^{(3)}_{g,(2,L)}(N_F) \Bigg] , \tag{25}$$

$$H^{\text{PS}}_{q,(2,L)}(N_F + 1) = a_s^2 \Bigg[A^{(2),\text{PS}}_{Qq}(N_F + 1)\, \delta_2 + \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) \Bigg] \tag{26}$$

$$+ a_s^3 \Bigg[A^{(3),\text{PS}}_{Qq}(N_F + 1)\, \delta_2 + \tilde{C}^{(3),\text{PS}}_{q,(2,L)}(N_F + 1)$$

$$+ A^{(2)}_{gq,Q}(N_F + 1)\, \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1) + A^{(2),\text{PS}}_{Qq}(N_F + 1)\, C^{(1),\text{NS}}_{q,(2,L)}(N_F + 1) \Bigg] ,$$

$$H^{\text{S}}_{g,(2,L)}(N_F + 1) = a_s \Bigg[A^{(1)}_{Qg}(N_F + 1)\, \delta_2 + \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1) \Bigg]$$

$$+ a_s^2 \Bigg[A^{(2)}_{Qg}(N_F + 1)\, \delta_2 + A^{(1)}_{Qg}(N_F + 1)\, C^{(1),\text{NS}}_{q,(2,L)}(N_F + 1)$$

$$+ A^{(1)}_{gg,Q}(N_F + 1)\, \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1) + \tilde{C}^{(2)}_{g,(2,L)}(N_F + 1) \Bigg]$$

$$+ a_s^3 \Bigg[A^{(3)}_{Qg}(N_F + 1)\, \delta_2 + A^{(2)}_{Qg}(N_F + 1)\, C^{(1),\text{NS}}_{q,(2,L)}(N_F + 1)$$

$$+ A^{(2)}_{gg,Q}(N_F + 1)\, \tilde{C}^{(1)}_{g,(2,L)}(N_F + 1)$$

$$+ A^{(1)}_{Qg}(N_F + 1) \Big\{ C^{(2),\text{NS}}_{q,(2,L)}(N_F + 1) + \tilde{C}^{(2),\text{PS}}_{q,(2,L)}(N_F + 1) \Big\}$$

$$+ A^{(1)}_{gg,Q}(N_F + 1)\, \tilde{C}^{(2)}_{g,(2,L)}(N_F + 1) + \tilde{C}^{(3)}_{g,(2,L)}(N_F + 1) \Bigg] , \tag{27}$$

with $\delta_2 = 1$ for F_2 and $\delta_2 = 0$ for F_L. Again, the argument $(N_F + 1)$ in the massive OMEs signals that these functions depend on N_F massless and one massive flavor, while the setting of N_F in the massless Wilson coefficients is a functional one. The above equations include radiative corrections due to heavy quark loops to the Wilson coefficients. Therefore, in order to compare e.g. with the calculation in Refs. [1], these terms still have to be subtracted. Since the light flavor Wilson coefficients were calculated in the $\overline{\text{MS}}$–scheme, the same scheme has to be used for the massive OMEs. It should also be thoroughly used for renormalization to derive consistent results in QCD analyses of deep-inelastic scattering data and to be able to compare to other analyses of hard scattering data directly. This requests special attendance w.r.t. the choice of the scheme in which a_s is defined, cf. [12].

The renormalized massive OMEs depend on the ratio m^2/μ^2, while the scale ratio in the massless Wilson coefficients is μ^2/Q^2. The latter are pure functions of the momentum fraction z, or the Mellin variable N, if one sets $\mu^2 = Q^2$. The mass dependence on the heavy flavor Wilson coefficients in the asymptotic region derives from the unrenormalized massive OMEs

$$\hat{A}^{(3)}_{ij}(\varepsilon) = \frac{1}{\varepsilon^3} \hat{a}^{(3),3}_{ij} + \frac{1}{\varepsilon^2} \hat{a}^{(3),2}_{ij} + \frac{1}{\varepsilon} \hat{a}^{(3),1}_{ij} + \hat{a}^{(3),0}_{ij} , \tag{28}$$

applying mass, coupling constant, and operator-renormalization, as well as mass factorization,

cf. Ref. [12]. The renormalized massive OMEs obey then the general structure

$$A_{ij}^{(3)}\left(\frac{m^2}{Q^2}\right) = a_{ij}^{(3),3}\ln^3\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),2}\ln^2\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),1}\ln\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),0} . \tag{29}$$

The subsequent calculations will be performed in the $\overline{\text{MS}}$ scheme for the coupling constant and the on-shell scheme for the heavy quark mass m. The transition to the scheme in which m is renormalized in the $\overline{\text{MS}}$-scheme is described in Ref. [12]. The strong coupling constant is obtained as the *perturbative* solution of the equation

$$\frac{da_s(\mu^2)}{d\ln(\mu^2)} = -\sum_{l=0}^{\infty}\beta_l a_s^{l+2}(\mu^2) \tag{30}$$

to 3–loop order, where β_k are the expansion coefficients of the QCD β–function and μ^2 denotes the renormalization scale. For simplicity we identify the factorization (μ_F) and renormalization (μ_R) scales from now on. In the subsequent sections we present explicit expressions of the asymptotic heavy flavor Wilson coefficients in Mellin-N space. They depend on the logarithms

$$L_Q = \ln\left(\frac{Q^2}{\mu^2}\right) \quad \text{and} \quad L_M = \ln\left(\frac{m^2}{\mu^2}\right) , \tag{31}$$

where $\mu \equiv \mu_F = \mu_R$.

Besides the Wilson coefficients (23–27) the massive OMEs are important themselves to establish the matching conditions in the variable flavor number scheme in describing the process of a single massive quark becoming massless[8] at large enough scales μ^2, [12,15]. Here, the PDFs for $N_F + 1$ massless quarks are related to the former N_F massless quarks process independently. The corresponding relations to 3–loop order read, cf. also [15][9] :

$$
\begin{aligned}
f_k(N_F+1,\mu^2) + f_{\bar{k}}(N_F+1,\mu^2) &= A_{qq,Q}^{\mathsf{NS}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(N_F,\mu^2) + f_{\bar{k}}(N_F,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Sigma(N_F,\mu^2) \\
&\quad + \tilde{A}_{qg,Q}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes G(N_F,\mu^2) \tag{32}
\end{aligned}
$$

$$
f_{Q+\bar{Q}}(N_F+1,\mu^2) = A_{Qq}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Sigma(N_F,\mu^2) + A_{Qg}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes G(N_F,\mu^2) \tag{33}
$$

$$
G(N_F+1,\mu^2) = A_{gq,Q}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes \Sigma(N_F,\mu^2) + A_{gg,Q}^{S}\left(N_F,\frac{\mu^2}{m^2}\right) \otimes G(N_F,\mu^2) . \tag{34}
$$

$$
\begin{aligned}
\Sigma(N_F+1,\mu^2) &= \left[A_{qq,Q}^{\mathsf{NS}}\left(N_F,\frac{\mu^2}{m^2}\right) + N_F\tilde{A}_{qq,Q}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right) + A_{Qq}^{\mathsf{PS}}\left(N_F,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes \Sigma(N_F,\mu^2) \\
&\quad + \left[N_F\tilde{A}_{qg,Q}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right) + A_{Qg}^{\mathsf{S}}\left(N_F,\frac{\mu^2}{m^2}\right)\right] \otimes G(N_F,\mu^2) \tag{35}
\end{aligned}
$$

---

[8]For the VFNS in case of both the bottom and charm quarks transmuting into massless states, see [43].

[9]Here, we have corrected some typographical errors in (33–35) in [15], in accordance with the appendix of Ref. [15].

Here, the N_F-dependence of the OMEs is understood as functional and μ^2 denotes the matching scale, which for the heavy-to-light transitions is normally much larger than mass scale m^2, [23]. We will present the corresponding OMEs in Appendix A. The results of the calculations being presented in the subsequent sections have been obtained making mutual use of the packages `HarmonicSums.m` [58] and `Sigma.m` [29].

3 The Wilson Coefficients $L_{q,2}^{\mathsf{PS}}$ and $L_{g,2}^{\mathsf{S}}$

The OMEs for these Wilson coefficients have been calculated in [24]. They contribute for the first time at 3– and 2–loop order, respectively, and stem from processes in which the virtual electro-weak gauge boson couples to a massless quark. As a shorthand notation we also define the function

$$\tilde{\gamma}_{qg}^0 = -4\frac{N^2 + N + 2}{N(N+1)(N+2)} \tag{36}$$

denoting the kinetic part of the leading order anomalous dimensions separating off the corresponding color factor.

In Mellin-N space the Wilson coefficient $L_{q,2}^{\mathsf{PS}}$ reads :

$$
\begin{aligned}
L_{q,2}^{\mathsf{PS}} = {}&\tfrac{1}{2}\left[1 + (-1)^N\right] \\
&\times a_s^3 \Bigg\{ C_F N_F T_F^2 \Bigg[-\frac{32 P_4 L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_Q\Bigg[\frac{64 P_6}{27(N-1)N^4(N+1)^4(N+2)^3} \\
&- \frac{256 P_1 (-1)^N}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{2(\tilde{\gamma}_{qg}^0)^2(N+2)L_M^2}{3(N-1)} \\
&+ \Bigg[\frac{64(N^2+N+2)(8N^3+13N^2+27N+16)}{9(N-1)N^2(N+1)^3(N+2)} - \frac{64(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)}\Bigg]L_M \\
&+ \frac{512 S_{-2}}{3(N-1)N(N+1)(N+2)}\Bigg] - \frac{32 P_4 L_M^2}{9(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \Bigg[-\frac{32 P_7}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64 P_2 S_1}{3(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\frac{32}{3}\left(S_1^2 - S_2\right)\Bigg]L_M \\
&- \frac{32 P_9}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{32 P_8 S_1}{81(N-1)N^4(N+1)^4(N+2)^3} \\
&- \frac{16 P_3 S_1^2}{27(N-1)N^3(N+1)^3(N+2)^2} - \frac{16 P_5 S_2}{27(N-1)N^3(N+1)^3(N+2)^2} \\
&+ \frac{32 L_Q^3(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} \\
&+ \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\Bigg[-\frac{64}{27}S_1^3 + \frac{32}{9}S_2 S_1 + \frac{160 S_3}{27} + \frac{256\zeta_3}{9}\Bigg]\Bigg] \\
&+ N_F \hat{\tilde{C}}_{2,q}^{\mathsf{PS},(3)}(N_F) \Bigg\} \,,
\end{aligned}
\tag{37}
$$

with the polynomials

$$P_1 = 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \tag{38}$$

$$P_2 = N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \tag{39}$$

$$P_3 = N^7 - 37N^6 - 248N^5 - 799N^4 - 1183N^3 - 970N^2 - 580N - 168 \tag{40}$$

$$P_4 = 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \tag{41}$$

$$P_5 = 49N^7 + 185N^6 + 340N^5 + 287N^4 + 65N^3 + 62N^2 - 196N - 168 \tag{42}$$

$$P_6 = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3$$
$$+1864N^2 + 1248N + 432 \tag{43}$$

$$P_7 = 143N^{10} + 838N^9 + 1995N^8 + 1833N^7 - 1609N^6 - 5961N^5 - 7503N^4 - 6928N^3$$
$$-4024N^2 - 816N + 144 \tag{44}$$

$$P_8 = 176N^{10} + 973N^9 + 1824N^8 - 948N^7 - 10192N^6 - 19173N^5 - 20424N^4 - 16036N^3$$
$$-7816N^2 - 1248N + 288 \tag{45}$$

$$P_9 = 1717N^{13} + 16037N^{12} + 66983N^{11} + 161797N^{10} + 241447N^9 + 216696N^8 + 86480N^7$$
$$-67484N^6 - 170003N^5 - 165454N^4 - 81976N^3 - 15792N^2 - 1008N - 864 \ . \tag{46}$$

For the massless 3-loop Wilson coefficients $C_{i,j}^k$ we refer to Ref. [11]. Here and in the following, their expression will be kept symbolically. The corresponding z-space expressions are given in Appendix B.

Likewise the Wilson coefficient $L_{g,2}^{\mathsf{S}}$ is given by :

$$L_{g,2}^{\mathsf{S}} = \tfrac{1}{2}\left[1+(-1)^N\right]\left\{a_s^2 T_F^2 N_F\left\{L_M\left[\frac{4}{3}\tilde{\gamma}_{qg}^0 S_1 - \frac{16(N^3-4N^2-N-2)}{3N^2(N+1)(N+2)}\right]\right.\right.$$

$$\left.-\frac{4}{3}\tilde{\gamma}_{qg}^0 L_Q L_M\right\} + a_s^3\left\{N_F T_F^3\left[L_M^2\left[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 - \frac{64(N^3-4N^2-N-2)}{9N^2(N+1)(N+2)}\right] - \frac{16}{9}\tilde{\gamma}_{qg}^0 L_Q L_M^2\right]\right.$$

$$+C_A N_F T_F^2\left[\left[\frac{64(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\right]L_Q^3 + \left[-\frac{64(-1)^N(N^3+4N^2+7N+5)}{3(N+1)^3(N+2)^3}\right.\right.$$

$$+\frac{8P_{25}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^4-7N^3+5N^2-17N-13)S_1}{9(N-1)N(N+1)^2(N+2)}$$

$$+L_M\left[\frac{64(N^2+N+1)(N^2+N+2)}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\right] + \tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^2 + \frac{4S_2}{3} + \frac{8}{3}S_{-2}\right]\right]L_Q^2$$

$$+\left[-\frac{32(8N^4-7N^3+5N^2-17N-13)S_1^2}{9(N-1)N(N+1)^2(N+2)} + \frac{128(-1)^N(N^3+4N^2+7N+5)S_1}{3(N+1)^3(N+2)^3}\right.$$

$$-\frac{32P_{24}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^N P_{18}}{9(N-1)N^2(N+1)^4(N+2)^4}$$

$$-\frac{16P_{32}}{27(N-1)N^3(N+1)^4(N+2)^4} + L_M^2\left[\frac{64(N^2+N+1)(N^2+N+2)}{3(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$\left.+\frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\right] + \frac{32(8N^4+13N^3-22N^2-9N-26)S_2}{9(N-1)N(N+1)(N+2)^2} + \frac{128(N^2+N-1)S_3}{9N(N+1)(N+2)}$$

$$+\frac{64(8N^5+15N^4+6N^3+11N^2+16N+16)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} + L_M\left[\frac{32P_{26}}{9(N-1)N^3(N+1)^3(N+2)^3}\right.$$

$$-\frac{128(-1)^N(N^3+4N^2+7N+5)}{3(N+1)^3(N+2)^3} - \frac{64(2N-1)(N^3+9N^2+7N+7)S_1}{9(N-1)N(N+1)^2(N+2)}$$

$$+\tilde{\gamma}_{qg}^0\left[-\frac{8}{3}S_1^2+\frac{8S_2}{3}+\frac{16}{3}S_{-2}\right]\right]-\frac{128\left(N^2+N+3\right)S_{-3}}{3N(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[\frac{8}{9}S_1^3-8S_2S_1+\frac{32}{3}S_{2,1}\right]$$

$$+\frac{256S_{-2,1}}{3N(N+1)(N+2)}+\frac{(N-1)\left[\frac{64}{3}S_{-2}S_1-32\zeta_3\right]}{N(N+1)}\right]L_Q+\frac{16P_{12}S_1^2}{81N(N+1)^3(N+2)^3}$$

$$+\frac{8P_{39}}{243(N-1)N^5(N+1)^5(N+2)^5}+\frac{512}{9}\frac{\left(N^2+N+1\right)\left(N^2+N+2\right)}{(N-1)N^2(N+1)^2(N+2)^2}\zeta_3$$

$$+\frac{8P_{36}S_1}{243(N-1)N^4(N+1)^4(N+2)^4}+L_M^3\left[-\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$\left.-\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\right]-\frac{16P_{13}S_2}{81N(N+1)^3(N+2)^3}+\frac{64\left(5N^4+38N^3+59N^2+31N+20\right)S_3}{81N(N+1)^2(N+2)^2}$$

$$-\frac{32\left(121N^3+293N^2+414N+224\right)S_{-2}}{81N(N+1)^2(N+2)}+L_M^2\left[-\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)}{3(N+1)^3(N+2)^3}\right.$$

$$+\frac{8P_{25}}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32\left(8N^4-7N^3+5N^2-17N-13\right)S_1}{9(N-1)N(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^2\right.$$

$$\left.\left.+\frac{4S_2}{3}+\frac{8}{3}S_{-2}\right]\right]+\frac{128\left(5N^2+8N+10\right)S_{-3}}{27N(N+1)(N+2)}$$

$$+\frac{\left(5N^4+20N^3+41N^2+49N+20\right)\left[\frac{32}{81}S_1^3-\frac{32}{27}S_2S_1+\frac{128}{27}S_{2,1}\right]}{N(N+1)^2(N+2)^2}$$

$$+L_M\left[\frac{32\left(2N^5+21N^4+27N^3+11N^2+25N-14\right)S_1^2}{9(N-1)N(N+1)^2(N+2)^2}+\frac{16P_{27}S_1}{27(N-1)N^3(N+1)^3(N+2)^3}\right.$$

$$+\frac{128(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{3(N+1)^3(N+2)^3}-\frac{16}{3}\tilde{\gamma}_{qg}^0S_2S_1-\frac{64(-1)^NP_{14}}{9(N-1)N^2(N+1)^3(N+2)^4}$$

$$+\frac{16P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4}-\frac{32\left(2N^5+21N^4+51N^3+23N^2-11N-14\right)S_2}{9(N-1)N(N+1)^2(N+2)^2}$$

$$+\frac{64S_3}{3(N+2)}-\frac{64\left(2N^5+21N^4+36N^3-7N^2-68N-56\right)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}+\frac{\frac{256}{3}S_{-2,1}-\frac{128}{3}S_{-3}}{N(N+1)(N+2)}$$

$$+\frac{(N-1)\left(\frac{64}{3}S_{-2}S_1-32\zeta_3\right)}{N(N+1)}\right]+\tilde{\gamma}_{qg}^0\left[\frac{1}{27}S_1^4-\frac{2}{9}S_2S_1^2+\left[\frac{16}{9}S_{2,1}-\frac{40S_3}{27}\right]S_1\right.$$

$$\left.+\frac{64}{9}\zeta_3S_1+\frac{1}{9}S_2^2+\frac{14S_4}{9}+\frac{32}{9}S_{-4}+\frac{32}{9}S_{3,1}-\frac{16}{9}S_{2,1,1}\right]\right]$$

$$+C_FN_FT_F^2\left[\left[\frac{16\left(N^2+N+1\right)\left(N^2+N+2\right)\left(3N^4+6N^3-N^2-4N+12\right)}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{16}{9}\tilde{\gamma}_{qg}^0S_1\right]L_Q^3\right.$$

$$+\left[-\frac{4P_{31}}{9(N-1)N^4(N+1)^4(N+2)^3}+\frac{16P_{21}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\left[\frac{20S_2}{3}-4S_1^2\right]\right.$$

$$+L_M\left[\frac{8\left(N^2+N+2\right)P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2}+\frac{8}{3}\tilde{\gamma}_{qg}^0S_1\right]\right]L_Q^2+\left[\frac{16L_M^2\left(N^2+N+2\right)^3}{(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$-\frac{16P_{22}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{64(-1)^NP_{37}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}$$

$$+\frac{4P_{42}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}-\frac{8P_{30}S_1}{9(N-1)N^4(N+1)^4(N+2)^3}$$

$$+\frac{16P_{23}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}+L_M\left[-\frac{16P_{28}}{3(N-1)N^4(N+1)^4(N+2)^3}\right.$$

$$+\frac{16P_{17}S_1}{3(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\left(\frac{16S_2}{3}-\frac{16}{3}S_1^2\right)\right]-\frac{256\left(N^2+N+1\right)S_3}{3N(N+1)(N+2)}$$

$$+\frac{64P_{16}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}+\tilde{\gamma}_{qg}^0\left[\frac{8}{3}S_1^3-8S_2S_1-\frac{32}{3}S_{2,1}\right]$$

$$+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}+\frac{64(N-1)\zeta_3}{N(N+1)}\right]L_Q-\frac{64}{9}\frac{\left(N^2+N+2\right)P_{10}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$+\frac{8\left(215N^4+481N^3+930N^2+748N+120\right)S_1^2}{81N^2(N+1)^2(N+2)}+\frac{P_{40}}{243(N-1)N^6(N+1)^6(N+2)^5}$$

$$-\frac{4P_{35}S_1}{243(N-1)N^5(N+1)^5(N+2)^2}+L_M^3\left[\frac{8\left(N^2+N+2\right)P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\right]$$

$$+L_M^2\left[\frac{4P_{29}}{9(N-1)N^4(N+1)^4(N+2)^3}-\frac{16P_{20}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^2-\frac{4S_2}{3}\right]\right]$$

$$+\frac{8\left(109N^4+291N^3+478N^2+324N+40\right)S_2}{27N^2(N+1)^2(N+2)}+\frac{\left(10N^3+13N^2+29N+6\right)\left[-\frac{16}{81}S_1^3-\frac{16}{27}S_2S_1\right]}{N^2(N+1)(N+2)}$$

$$+\frac{32\left(5N^3-16N^2+N-6\right)S_3}{81N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[-\frac{1}{27}S_1^4-\frac{2}{9}S_2S_1^2-\frac{8}{27}S_3S_1-\frac{64}{9}\zeta_3S_1-\frac{1}{9}S_2^2+\frac{14S_4}{9}\right]$$

$$+L_M\left[-\frac{8P_{19}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}-\frac{16P_{33}S_1}{27(N-1)N^4(N+1)^4(N+2)^3}\right.$$

$$+\frac{64(-1)^NP_{38}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3}+\frac{8\left(N^2+N+2\right)P_{11}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}$$

$$+\frac{4P_{41}}{135(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3}+\frac{64P_{15}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}$$

$$+\tilde{\gamma}_{qg}^0\left(\frac{8}{3}S_1^3-\frac{8}{3}S_2S_1-\frac{16}{3}S_{2,1}\right)+\frac{\frac{512}{3}S_1S_{-2}+\frac{256}{3}S_{-3}-\frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}+\frac{(N-1)\left(64\zeta_3-\frac{64S_3}{3}\right)}{N(N+1)}\right]\right]$$

$$+N_F\hat{\tilde{C}}_{2,g}^{\mathsf{S},(3)}(N_F)\Bigg\}\Bigg\}\,,\tag{47}$$

where

$$P_{10}\ =\ 3N^6+9N^5-N^4-17N^3-38N^2-28N-24\tag{48}$$

$$P_{11}\ =\ 47N^6+141N^5+59N^4-117N^3+2N^2+84N+72\tag{49}$$

$$P_{12}\ =\ 65N^6+455N^5+1218N^4+1820N^3+1968N^2+1460N+448\tag{50}$$

$$P_{13}\ =\ 139N^6+1093N^5+3438N^4+5776N^3+5724N^2+3220N+752\tag{51}$$

$$P_{14}\ =\ 9N^7+71N^6+214N^5+320N^4+275N^3+215N^2+160N+32\tag{52}$$

$$P_{15}\ =\ N^8+8N^7-2N^6-60N^5-23N^4+108N^3+96N^2+16N+48\tag{53}$$

$$P_{16}\ =\ N^8+8N^7-2N^6-60N^5+N^4+156N^3+24N^2-80N-240\tag{54}$$

$$P_{17}\ =\ 3N^8+8N^7-2N^6-24N^5+15N^4+88N^3+152N^2+96N+48\tag{55}$$

$$P_{18}\ =\ 5N^8-8N^7-137N^6-436N^5-713N^4-672N^3-407N^2-192N-32\tag{56}$$

$$P_{19}\ =\ 7N^8+4N^7-90N^6-224N^5-21N^4+388N^3+608N^2+336N+144\tag{57}$$

$$P_{20}\ =\ 10N^8+46N^7+105N^6+139N^5+87N^4-17N^3+50N^2+84N+72\tag{58}$$

$$P_{21}\ =\ 19N^8+70N^7+63N^6-41N^5-192N^4-221N^3-142N^2-60N-72\tag{59}$$

$$P_{22} = 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 \tag{60}$$

$$P_{23} = 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 \tag{61}$$

$$P_{24} = 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 \tag{62}$$

$$P_{25} = 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 \tag{63}$$

$$P_{26} = 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 \tag{64}$$

$$P_{27} = 55N^9 + 336N^8 + 218N^7 - 2180N^6 - 6529N^5 - 9764N^4 - 9368N^3 - 6032N^2 \\ -2448N - 576 \tag{65}$$

$$P_{28} = N^{11} - 56N^9 - 236N^8 - 373N^7 + 82N^6 + 1244N^5 + 2330N^4 + 2560N^3 + 1712N^2 \\ +896N + 288 \tag{66}$$

$$P_{29} = 33N^{11} + 231N^{10} + 662N^9 + 1254N^8 + 1801N^7 + 2759N^6 + 5440N^5 + 9884N^4 \\ +12512N^3 + 9200N^2 + 5184N + 1728 \tag{67}$$

$$P_{30} = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 \\ +12976N^3 + 8016N^2 + 4608N + 1728 \tag{68}$$

$$P_{31} = 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 - 4645N^6 - 8936N^5 - 11980N^4 \\ -12352N^3 - 8272N^2 - 4800N - 1728 \tag{69}$$

$$P_{32} = 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4 \\ -6840N^3 + 120N^2 + 2432N + 960 \tag{70}$$

$$P_{33} = 121N^{11} + 988N^{10} + 3554N^9 + 6972N^8 + 7131N^7 - 846N^6 - 14806N^5 - 25354N^4 \\ -26096N^3 - 16752N^2 - 8352N - 2592 \tag{71}$$

$$P_{34} = 27N^{12} + 441N^{11} + 2206N^{10} + 5360N^9 + 7445N^8 + 8555N^7 + 18766N^6 + 44852N^5 \\ +67572N^4 + 63960N^3 + 39632N^2 + 15648N + 2880 \tag{72}$$

$$P_{35} = 2447N^{12} + 16902N^{11} + 59649N^{10} + 125860N^9 + 128761N^8 - 36530N^7 - 248341N^6 \\ -304460N^5 - 162188N^4 - 11724N^3 + 29160N^2 + 19440N + 7776 \tag{73}$$

$$P_{36} = 3361N^{12} + 23769N^{11} + 62338N^{10} + 59992N^9 - 63303N^8 - 317823N^7 - 585520N^6 \\ -640602N^5 - 430132N^4 - 167536N^3 - 27648N^2 + 9504N + 5184 \tag{74}$$

$$P_{37} = 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} - 19377N^{10} - 58253N^9 - 26543N^8 + 170601N^7 \\ +362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 117072N - 25920 \tag{75}$$

$$P_{38} = 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} - 47693N^9 - 125723N^8 - 86079N^7 \\ +36437N^6 + 22559N^5 - 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 \tag{76}$$

$$P_{39} = 3180N^{15} + 38835N^{14} + 188728N^{13} + 456665N^{12} + 460954N^{11} - 406761N^{10} - 1972948N^9 \\ -2827653N^8 - 1857970N^7 + 109786N^6 + 1302824N^5 + 1092456N^4 \\ +265888N^3 - 227616N^2 - 194688N - 44928 \tag{77}$$

$$P_{40} = 28503N^{17} + 297639N^{16} + 1232041N^{15} + 2461407N^{14} + 2169615N^{13} + 662941N^{12} \\ +2110979N^{11} + 5346653N^{10} + 2021366N^9 - 7290864N^8 - 11721384N^7 - 3689680N^6 \\ +15676192N^5 + 32276800N^4 + 31869312N^3 + 18809856N^2 + 6856704N + 1244160 \tag{78}$$

$$P_{41} = 75N^{18} + 3330N^{17} + 35497N^{16} + 175010N^{15} + 486862N^{14} + 966996N^{13} + 2037362N^{12} \\ +3604404N^{11} - 1625689N^{10} - 29506022N^9 - 78753403N^8 - 107977014N^7 - 71548880N^6 \\ +18344016N^5 + 89016048N^4 + 92657952N^3 + 58942080N^2 + 25505280N + 5598720 \tag{79}$$

$$P_{42} = 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} - 3216546N^{12} \\ -1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 + 25361392N^7 + 9982840N^6 \\ -10154128N^5 - 26098704N^4 - 26761536N^3 - 17642880N^2 - 8087040N - 1866240 \,. \tag{80}$$

In all the representations of the massive Wilson coefficients and OMEs in N–space we apply algebraic reduction [59]. The 2–loop term in (47) is purely multiplicative and induced by renormalization only, while the 3–loop contributions require the calculation of massive OMEs. The above Wilson coefficients depend on the harmonic sums

$$S_1, S_{-2}, S_2, S_{-3}, S_3, S_{-4}, S_4, S_{-2,1}, S_{2,1}, S_{3,1}, S_{2,1,1}, \tag{81}$$

apart of those defining the massless 3–loop Wilson coefficients [11][10]. The harmonic sums are defined recursively by, cf. [36, 37],

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad b, a_i \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}, N \geq 1, S_\emptyset = 1 . \tag{82}$$

In the above $\zeta_l = \sum_{k=1}^{\infty} 1/k^l, l \in \mathbb{N}, l \geq 2$ denote the Riemann ζ-values, which are convergent harmonic sums in the limit $N \to \infty$. In the constant part of the other Wilson coefficients it is expected that more complicated multiple zeta values emerge, which have been dealt with in [61].

In Eq. (47) denominator terms $\propto 1/(N-2)$ occur. They cancel in the complete expression and the rightmost singularity is located at $N = 1$ as expected for this Wilson coefficient. Let us now consider both the small- and large-x dominant terms for both Wilson coefficients. Those of the massless parts were given in [11] before. Both Wilson coefficients contain terms $\propto 1/(N-1)$. For simplicity we consider the choice of scale $Q^2 = \mu^2$ here. The expansion of the heavy flavor contribution, subtracting the massless 3–loop Wilson coefficients, denoted by $\widehat{L}_i$, around $N = 1(x \to 0)$ and in the limit $N \to \infty(x \to 1)$, setting $Q^2 = \mu^2$, are given by

$$\widehat{L}_{q,2}^{\text{PS}}(N \to 1) \propto \frac{1}{N-1}C_F T_F^2 N_F\left\{\frac{1024}{27}\zeta_3 - \frac{64}{729}\left[54L_M^3 - 81L_M^2 + 342L_M + 500\right]\right\} \tag{83}$$

$$\widehat{L}_{g,2}^{\text{S}}(N \to 1) \propto \frac{1}{N-1}\left\{C_A T_F^2 N_F\left\{\frac{512}{27}\zeta_3 - \frac{16}{729}\left[108L_M^3 + 540L_M^2 + 54L_M + 3091\right]\right\}\right.$$

$$\left.+C_F T_F^2 N_F\left\{\frac{1024}{27}\zeta_3 - \frac{32}{729}\left[108L_M^3 - 864L_M^2 + 1314L_M - 1091\right]\right\}\right\} \tag{84}$$

$$\widehat{L}_{q,2}^{\text{PS}}(N \to \infty) \propto -\frac{64}{27}C_F T_F^2 N_F\frac{\ln^3(\bar{N})}{N^2} \tag{85}$$

$$\widehat{L}_{g,2}^{\text{S}}(N \to \infty) \propto -\frac{4}{27}\frac{(N-2)\ln^4(\bar{N})}{N^2}(C_A - C_F)T_F^2 N_F . \tag{86}$$

The corresponding limits for the contributions of the massless Wilson coefficients behave like

$$N_F\hat{\tilde{C}}_{2,q}^{\text{PS},(3)}(N_F)(N \to 1) \propto \frac{4}{N-1}C_F T_F^2 N_F\left[\frac{22112}{729} - \frac{32}{9}\zeta_2 + \frac{128}{27}\zeta_3\right] [11] \tag{87}$$

$$N_F\hat{\tilde{C}}_{2,g}^{\text{S},(3)}(N_F)(N \to 1) \propto \frac{4}{N-1}\left\{C_A T_F^2 N_F\left[-\frac{572}{729} + \frac{160}{27}\zeta_2 + \frac{64}{27}\zeta_3\right]\right.$$

$$\left.+C_F T_F^2 N_F\left[\frac{45468}{729} - \frac{512}{27}\zeta_2 + \frac{128}{27}\zeta_3\right]\right\} [11] \tag{88}$$

$$N_F\hat{\tilde{C}}_{2,q}^{\text{PS},(3)}(N_F)(N \to \infty) \propto \frac{\ln^3(\bar{N})}{N^3}C_F T_F^2 N_F\frac{64}{27} \tag{89}$$

---

[10]For the algebraically reduced representations see [60].

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$$N_F \hat{\bar{C}}_{2,g}^{\mathsf{S},(3)}(N_F)(N \to \infty) \quad \propto \quad \frac{\ln^4(\bar{N})}{N}\left[\frac{68}{27}C_F T_F^2 N_F + \frac{28}{27}C_A T_F^2 N_F\right], \tag{90}$$

where $\bar{N} = N\exp(\gamma_E)$ and γ_E denotes the Euler-Mascheroni constant.
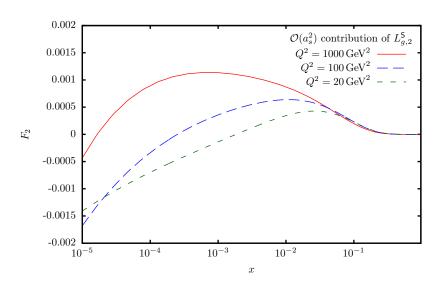
Figure 1: The $O(a_s^2)$ contribution by $L_{g,2}^{\mathsf{S}}$ to the structure function $F_2(x,Q^2)$.
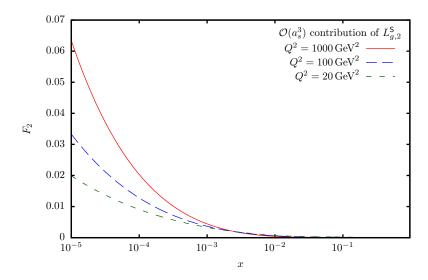
Figure 2: The $O(a_s^3)$ contribution by $L_{g,2}^{\mathsf{S}}$ to the structure function $F_2(x,Q^2)$.

While the expression for $L_{q,2}^{\mathsf{PS}}$ is the same in the $\overline{\mathrm{MS}}$– and on-mass-shell scheme to $O(a_s^3)$, $L_{g,2}^{\mathsf{S}}$, in its 3-loop contribution, changes by the term

$$\hat{L}_{g,2}^{\mathsf{S},(3),\overline{\mathrm{MS}}}(N) \quad = \quad L_{g,2}^{\mathsf{S},(3),\mathrm{OMS}}(N) + a_s^3 \frac{32}{3}C_F T_F^2 N_F\left[3L_M - 4\right]$$

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$$\times \left[\frac{(N^2 + N + 2)}{N(N+1)(N+2)} S_1 + \frac{(N^3 - 4N^2 - N - 2)}{N^2(N+1)(N+2)} \right] \tag{91}$$

setting $Q^2 = \mu^2$. Here, we have identified the logarithms L_M in both schemes symbolically. In applications, either the on-shell or the $\overline{\text{MS}}$ mass has to be used here. The corresponding expression in z–space reads

$$
\begin{aligned}
L_{g,2}^{\text{S},(3),\overline{\text{MS}}}(z) &= L_{g,2}^{\text{S},(3),\text{OMS}}(z) + a_s^3 \frac{32}{3} C_F T_F^2 N_F \left[3L_M - 4\right] \\
&\times \left[(2z^2 - 2z + 1)\left[H_0(z) + H_1(z)\right] + 8z^2 - 8z + 1 \right],
\end{aligned}
\tag{92}
$$

with $H_{\vec{a}}(z)$ harmonic polylogarithms, see Eq. (592).
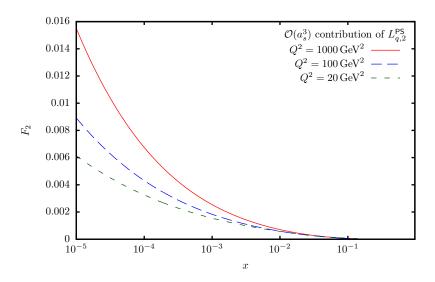
Figure 3: The $O(a_s^3)$ contribution by $L_{q,2}^{\text{PS}}$ to the structure function $F_2(x, Q^2)$.

x	$Q^2 = 20\text{GeV}^2$	$Q^2 = 100\text{GeV}^2$	$Q^2 = 1000\text{GeV}^2$
10^{-4}	1.946	3.200	5.340
10^{-3}	1.141	1.702	2.526
10^{-2}	0.641	0.825	1.040
10^{-1}	0.400	0.409	0.412

Table 1: Values of the structure function $F_2(x, Q^2)$ in the low x region using the PDF-parameterization [5].

In Figure 1 we illustrate the contribution of $L_{q,2}^{\mathsf{PS}}$ to the structure function $F_2(x,Q^2)$ using the PDFs of Ref. [5], cf. Eq. (11). Likewise Figures 2 and 3 show the corresponding contributions by $L_{g,2}^{\mathsf{S}}$ at $O(a_s^2)$ and $O(a_s^3)$, respectively. Note that the $O(a_s^2)$-terms, cf. also Ref. [19] are smaller than those at $O(a_s^3)$, which is caused by terms $\propto 1/z$ in the 3-loop contribution to $L_{g,2}^{\mathsf{S}}$, which are absent at 2-loop order.

These contributions emerging on the 2- and 3-loop level are minor compared to the values of the structure function $F_2(x,Q^2)$, for which typical values are given in Table 1.[11] A global comparison of all heavy flavor contributions up to 3-loop order can presently only be performed using the known number of Mellin moments, cf. [12], given in Section 6.

4 The Logarithmic Contributions to $H_{q,2}^{\mathsf{PS}}$ and $H_{g,2}^{\mathsf{S}}$ to $O(a_s^3)$

The pure-singlet Wilson coefficient $H_{q,2}^{\mathsf{PS}}$, except for the constant part $a_{Qq}^{\mathsf{PS},(3)}$ of the unrenormalized operator matrix element in the on-shell scheme can be expressed by harmonic sums and rational functions in N only. As before we reduce to a basis eliminating the algebraic relations [59]. It is given by :

$$H_{q,2}^{\mathsf{PS}} = \tfrac{1}{2}[1 + (-1)^N]$$

$$\times \left\{ a_s^2 \left\{ C_F T_F \left[-\frac{4L_M^2\left(N^2 + N + 2\right)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{\left(4S_1^2 - 12S_2\right)\left(N^2 + N + 2\right)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \right.$$

$$+ \frac{4L_Q^2\left(N^2 + N + 2\right)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(-1)^N P_{45}}{3(N-1)N^2(N+1)^3(N+2)^3}$$

$$+ \frac{8P_{75}}{3(N-1)N^4(N+1)^4(N+2)^3} + \frac{8P_{57}S_1}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$+ L_Q\left[-\frac{8S_1\left(N^2 + N + 2\right)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{8P_{57}}{(N-1)N^3(N+1)^3(N+2)^2} \right]$$

$$\left. \left. + \frac{64S_{-2}}{(N-1)N(N+1)(N+2)} - \frac{8\left(N^2 + 5N + 2\right)\left(5N^3 + 7N^2 + 4N + 4\right)L_M}{(N-1)N^3(N+1)^3(N+2)^2} \right] \right\}$$

$$+ a_s^3 \left\{ C_F^2 T_F \left[L_Q^3\left[\frac{8\left(N^2 + N + 2\right)^2\left(3N^2 + 3N + 2\right)}{3(N-1)N^3(N+1)^3(N+2)} - \frac{32\left(N^2 + N + 2\right)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] \right. \right.$$

$$+ L_Q^2\left[\frac{\left(24S_1^2 - 24S_2\right)\left(N^2 + N + 2\right)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{4P_{73}}{(N-1)N^4(N+1)^4(N+2)^2} \right.$$

$$\left. + \frac{8P_{62}S_1}{(N-1)N^3(N+1)^3(N+2)^2} \right] + L_Q\left[\frac{\left(104S_1 S_2 - \frac{56}{3}S_1^3\right)\left(N^2 + N + 2\right)^2}{(N-1)N^2(N+1)^2(N+2)} \right.$$

$$- \frac{16\left(N^2 + N - 22\right)S_3\left(N^2 + N + 2\right)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{\left(128S_{-3} - 256S_{-2,1} - 384\zeta_3\right)\left(N^2 + N + 2\right)}{(N-1)N^2(N+1)^2(N+2)}$$

$$- \frac{4P_{67}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2} - \frac{64(-1)^N P_{98}}{15(N-2)(N-1)^3 N^3(N+1)^5(N+2)^4(N+3)^3}$$

$$+ \frac{4P_{100}}{15(N-1)^3 N^5(N+1)^5(N+2)^4(N+3)^3} + \frac{128(-1)^N P_{45}S_1}{3(N-1)N^2(N+1)^3(N+2)^3}$$

---

[11]Note that the kinematic region at small x probed at HERA is limited to values $x \geq Q^2/(yS)$, with $S \simeq 10^5 \mathrm{GeV}^2$ and $y \in [0,1]$.

$$-\frac{8P_{79}S_1}{3(N-1)N^4(N+1)^4(N+2)^3}+\frac{512S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)}+\frac{4P_{65}S_2}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$+L_M^2\left[\frac{16\left(N^2+N+2\right)^2S_1}{(N-1)N^2(N+1)^2(N+2)}-\frac{4\left(N^2+N+2\right)^2\left(3N^2+3N+2\right)}{(N-1)N^3(N+1)^3(N+2)}\right]$$

$$+L_M\left[\frac{32\left(N^2+5N+2\right)\left(5N^3+7N^2+4N+4\right)S_1}{(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$\left.-\frac{8\left(N^2+5N+2\right)\left(3N^2+3N+2\right)\left(5N^3+7N^2+4N+4\right)}{(N-1)N^4(N+1)^4(N+2)^2}\right]$$

$$+\frac{32P_{69}S_{-2}}{(N-2)(N-1)N^3(N+1)^3(N+2)(N+3)}\right]-\frac{8(3N+2)\left(N^2+N+2\right)S_1^3}{3(N-1)N^3(N+1)(N+2)}$$

$$+\frac{4P_{74}S_1^2}{(N-1)N^4(N+1)^4(N+2)^3}-2\frac{\left(N^2+N+2\right)\zeta_2}{(N-1)N^4(N+1)^4(N+2)}P_{55}-\frac{4P_{94}}{(N-1)N^5(N+1)^6(N+2)^3}$$

$$-\frac{4}{3}\frac{\left(N^2+N+2\right)^2\left(3N^2+3N+2\right)\zeta_3}{(N-1)N^3(N+1)^3(N+2)}+4\frac{\left(N^2+N+2\right)\left(5N^4+4N^3+N^2-10N-8\right)\zeta_2}{(N-1)N^3(N+1)^3(N+2)}S_1$$

$$+\frac{4P_{91}S_1}{(N-1)N^5(N+1)^5(N+2)^3}+L_M^3\left[\frac{4\left(N^2+N+2\right)^2\left(3N^2+3N+2\right)}{3(N-1)N^3(N+1)^3(N+2)}\right.$$

$$\left.-\frac{16\left(N^2+N+2\right)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\right]-\frac{8\left(N^2+N+2\right)\left(3N^4+9N^3+15N^2+11N-2\right)S_1S_2}{(N-1)N^3(N+1)^3(N+2)}$$

$$-\frac{4P_{76}S_2}{(N-1)N^4(N+1)^4(N+2)^3}+L_M^2\left[-\frac{4\left(13N^2+5N-6\right)S_1\left(N^2+N+2\right)^2}{(N-1)N^3(N+1)^3(N+2)}\right.$$

$$\left.+\frac{\left(24S_2-8S_1^2\right)\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{4P_{51}\left(N^2+N+2\right)}{(N-1)N^4(N+1)^4(N+2)}\right]$$

$$-\frac{8\left(N^2+N+2\right)\left(3N^4+48N^3+43N^2-22N-8\right)S_3}{3(N-1)N^3(N+1)^3(N+2)}+\frac{32\left(N^2-3N-2\right)\left(N^2+N+2\right)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)}$$

$$+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\frac{2}{3}S_1^4-12S_2S_1^2+\left[\frac{16S_3}{3}+32S_{2,1}\right]S_1+\frac{16}{3}\zeta_3S_1+18S_2^2-12S_4\right.$$

$$+32S_{3,1}-64S_{2,1,1}+\left(4S_1^2-12S_2\right)\zeta_2\right]+L_M\left[\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\frac{8}{3}S_1^3-24S_2S_1\right.\right.$$

$$\left.-\frac{80S_3}{3}+32S_{2,1}+96\zeta_3\right]-\frac{4P_{60}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2}-\frac{4P_{92}}{(N-1)N^5(N+1)^5(N+2)^3}$$

$$\left.\left.+\frac{8P_{77}S_1}{(N-1)N^4(N+1)^4(N+2)^3}-\frac{4P_{59}S_2}{(N-1)N^3(N+1)^3(N+2)^2}\right]\right]$$

$$+C_FT_F^2\left[\left[\frac{32\left(N^2+N+2\right)^2L_Q^3}{9(N-1)N^2(N+1)^2(N+2)}-\frac{32P_{66}L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2}\right.\right.$$

$$+L_Q\left[\frac{32\left(N^2+N+2\right)^2L_M^2}{3(N-1)N^2(N+1)^2(N+2)}+\left[\frac{64\left(N^2+N+2\right)\left(8N^3+13N^2+27N+16\right)}{9(N-1)N^2(N+1)^3(N+2)}\right.\right.$$

$$\left.-\frac{64\left(N^2+N+2\right)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\right]L_M-\frac{256(-1)^NP_{45}}{9(N-1)N^2(N+1)^3(N+2)^3}$$

$$
+\frac{64P_{82}}{27(N-1)N^4(N+1)^4(N+2)^3}+\frac{512S_{-2}}{3(N-1)N(N+1)(N+2)}\Bigg]
$$

$$
-\frac{128\left(N^2+N+2\right)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)}+\frac{16\left(N^2+N+2\right)\left(7N^4+16N^3+32N^2+19N+2\right)S_1^2}{3(N-1)N^3(N+1)^3(N+2)}
$$

$$
-\frac{32\left(11N^5+26N^4+57N^3+142N^2+84N+88\right)L_M^2}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{32\zeta_2 P_{48}}{9(N-1)N^3(N+1)^2(N+2)^2}
$$

$$
+\frac{32P_{95}}{81(N-1)N^5(N+1)^5(N+2)^4}-\frac{32P_{68}S_1}{27(N-1)N^3(N+1)^4(N+2)}
$$

$$
+L_M\Bigg[\frac{\left[\frac{32}{3}S_1^2-32S_2\right]\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{64P_{80}}{27(N-1)N^4(N+1)^4(N+2)^3}
$$

$$
+\frac{64P_{57}S_1}{3(N-1)N^3(N+1)^3(N+2)^2}\Bigg]+\frac{16P_{61}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}
$$

$$
+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[-\frac{64}{9}S_1^3-\frac{32}{3}S_2S_1-\frac{32}{3}\zeta_2 S_1+\frac{160S_3}{9}+\frac{128\zeta_3}{9}\right]\Bigg]\Bigg]
$$

$$
+N_F T_F^2 C_F\Bigg[\frac{32\left(N^2+N+2\right)^2 L_Q^3}{9(N-1)N^2(N+1)^2(N+2)}-\frac{32P_{66}L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2}
$$

$$
+\Bigg[\frac{\left[-\frac{16}{3}S_1^2-\frac{16S_2}{3}\right]\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{32\left(8N^3+13N^2+27N+16\right)S_1\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^3(N+2)}
$$

$$
-\frac{256(-1)^N P_{45}}{9(N-1)N^2(N+1)^3(N+2)^3}+\frac{32P_{84}}{27(N-1)N^4(N+1)^4(N+2)^3}
$$

$$
+\frac{512S_{-2}}{3(N-1)N(N+1)(N+2)}\Bigg]L_Q-\frac{32\left(N^2+N+2\right)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)}
$$

$$
-\frac{16}{9}\frac{\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}P_{63}-\frac{32P_{87}}{3(N-1)N^5(N+1)^5(N+2)^4}
$$

$$
+L_M^2\Bigg[\frac{32P_{64}}{9(N-1)N^3(N+1)^3(N+2)^2}-\frac{32\left(N^2+N+2\right)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)}\Bigg]
$$

$$
+L_M\Bigg[\frac{\left[-\frac{16}{3}S_1^2-\frac{80S_2}{3}\right]\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{32P_{78}}{27(N-1)N^4(N+1)^4(N+2)^3}
$$

$$
+\frac{32P_{63}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\Bigg]+\frac{64\left(N^2+5N+2\right)\left(5N^3+7N^2+4N+4\right)S_2}{3(N-1)N^3(N+1)^3(N+2)^2}
$$

$$
+\frac{\left(N^2+N+2\right)^2\left[\frac{64S_3}{3}+\frac{16}{3}S_1\zeta_2+\frac{32\zeta_3}{9}\right]}{(N-1)N^2(N+1)^2(N+2)}\Bigg]
$$

$$
+C_A C_F T_F\Bigg[L_Q^3\Bigg[-\frac{16S_1\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)}
$$

$$
-\frac{8\left(11N^4+22N^3-23N^2-34N-12\right)\left(N^2+N+2\right)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2}\Bigg]
$$

$$
+L_Q^2\Bigg[-\frac{16\left(5N^2-1\right)S_1\left(N^2+N+2\right)^2}{(N-1)^2N^3(N+1)^3(N+2)}+\frac{\left(16S_1^2-16S_2-32S_{-2}\right)\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}
$$

$$+\frac{16(-1)^N\left(N^5+9N^4+24N^3+36N^2+32N+8\right)\left(N^2+N+2\right)}{(N-1)N^3(N+1)^4(N+2)^3}$$

$$+\frac{8P_{85}}{9(N-1)^2N^4(N+1)^3(N+2)^3}\Bigg]+L_Q\Bigg[\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\Bigg[-\frac{40}{3}S_1^3+40S_2S_1$$

$$-144S_{-3}+96S_{-2,1}\Bigg]+\frac{4P_{47}S_1^2\left(N^2+N+2\right)}{3(N-1)^2N^3(N+1)^3(N+2)^2}+48\frac{\left(N^2+N+6\right)\left(N^2+N+2\right)\zeta_3}{(N-1)N^2(N+1)^2(N+2)}$$

$$+\frac{32S_{-2}S_1\left(N^2+N+2\right)}{N^2(N+1)^2}+\frac{4P_{52}S_2\left(N^2+N+2\right)}{3(N-1)^2N^3(N+1)^3(N+2)^2}+\frac{32(-1)^NP_{93}}{9(N-1)^2N^4(N+1)^5(N+2)^4}$$

$$-\frac{8\left(13N^2+13N+62\right)S_3\left(N^2+N+2\right)}{3(N-1)N^2(N+1)^2(N+2)}-\frac{8P_{97}}{27(N-1)^2N^5(N+1)^5(N+2)^4}$$

$$-\frac{32(-1)^NP_{72}S_1}{3(N-1)N^3(N+1)^4(N+2)^3}-\frac{8P_{88}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^3}$$

$$+\frac{16P_{71}S_{-2}}{3(N-1)^2N^3(N+1)^3(N+2)^2}\Bigg]+\frac{8\left(N^2+N+2\right)\left(N^3+8N^2+11N+2\right)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2}$$

$$+\frac{4\left(N^2+N+2\right)P_{43}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3}+8\left(N^2+N+2\right)\frac{(-1)^N\zeta_2}{(N-1)N^3(N+1)^4(N+2)^3}P_{44}$$

$$-\frac{8}{9}\frac{\left(N^2+N+2\right)\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2}P_{49}+\frac{4}{9}\frac{\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3}P_{89}$$

$$+\frac{8P_{99}}{3(N-1)^2N^6(N+1)^6(N+2)^5}-\frac{4}{3}\frac{\left(N^2+N+2\right)\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2}P_{53}S_1$$

$$-\frac{8\left(N^2+N+2\right)P_{70}S_1}{(N-1)N^2(N+1)^5(N+2)^4}+L_M^3\Bigg[\frac{16S_1\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)}$$

$$+\frac{8\left(11N^4+22N^3-23N^2-34N-12\right)\left(N^2+N+2\right)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2}\Bigg]+\frac{4P_{86}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^3}$$

$$-\frac{8\left(N^2+N+2\right)\left(3N^3-12N^2-27N-2\right)S_1S_2}{(N-1)N^2(N+1)^3(N+2)^2}-\frac{16\left(N^2+N+2\right)P_{50}S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$+\frac{32(-1)^N\left(N^2+N+2\right)\left(N^4+2N^3+7N^2+22N+20\right)S_{-2}}{(N-1)N(N+1)^4(N+2)^3}$$

$$+L_M^2\Bigg[\frac{\left(16S_2+32S_{-2}\right)\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{8P_{54}S_1\left(N^2+N+2\right)}{3(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$-\frac{16(-1)^N\left(N^5+9N^4+24N^3+36N^2+32N+8\right)\left(N^2+N+2\right)}{(N-1)N^3(N+1)^4(N+2)^3}$$

$$-\frac{8P_{83}}{9(N-1)^2N^4(N+1)^3(N+2)^3}\Bigg]+\frac{\left(N^2-N-4\right)\left(N^2+N+2\right)}{(N-1)N(N+1)^3(N+2)^2}\Bigg[-64(-1)^NS_1S_{-2}$$

$$-32(-1)^NS_{-3}+64S_{-2,1}-32(-1)^NS_1\zeta_2-24(-1)^N\zeta_3\Bigg]$$

$$+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\Bigg[-\frac{2}{3}S_1^4-20S_2S_1^2-32(-1)^NS_{-3}S_1+\left(64S_{-2,1}-\frac{160S_3}{3}\right)S_1$$

$$-\frac{8}{3}\left(-7+9(-1)^N\right)\zeta_3S_1-2S_2^2+S_{-2}\left(-32(-1)^NS_1^2-32(-1)^NS_2\right)-36S_4-16(-1)^NS_{-4}$$

$$+16S_{3,1}+32S_{-2,2}+32S_{-3,1}+16S_{2,1,1}-64S_{-2,1,1}+\left(-4\left(-3+4(-1)^N\right)S_1^2\right.$$

$$-4\left(-1+4(-1)^N\right)S_2-8\left(1+2(-1)^N\right)S_{-2}\right)\zeta_2\Bigg]+L_M\Bigg[\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\Bigg[-\frac{8}{3}S_1^3$$

$$+40S_2S_1 + 32\big(1+(-1)^N\big)S_{-2}S_1 + 16(-1)^N S_{-3} - 32S_{2,1} + 12\big(-9+(-1)^N\big)\zeta_3\bigg]$$

$$+\frac{4\big(17N^4 - 6N^3 + 41N^2 - 16N - 12\big)S_1^2\big(N^2 + N + 2\big)}{3(N-1)^2N^3(N+1)^2(N+2)} + \frac{4P_{56}S_2\big(N^2 + N + 2\big)}{3(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$+\frac{8\big(31N^2 + 31N + 74\big)S_3\big(N^2 + N + 2\big)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{16\big(7N^2 + 7N + 10\big)S_{-3}\big(N^2 + N + 2\big)}{(N-1)N^2(N+1)^2(N+2)}$$

$$-\frac{128\big(N^2 + N + 1\big)S_{-2,1}\big(N^2 + N + 2\big)}{(N-1)N^2(N+1)^2(N+2)} + \frac{\big(N^2 - N - 4\big)\big(N^2 + N + 2\big)32(-1)^N S_{-2}}{(N-1)N(N+1)^3(N+2)^2}$$

$$-\frac{64(-1)^N P_{81}}{9(N-1)N^3(N+1)^5(N+2)^4} + \frac{8P_{96}}{27(N-1)^2N^5(N+1)^5(N+2)^4}$$

$$+\frac{64(-1)^N P_{46}S_1}{3(N-1)N^2(N+1)^3(N+2)^3} - \frac{8P_{90}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^3}$$

$$+\frac{16P_{58}S_{-2}}{(N-1)N^3(N+1)^3(N+2)^2}\bigg]\bigg] + a_{Qq}^{\mathsf{PS},(3)} + \tilde{C}_{2,q}^{\mathsf{PS},(3)}(N_F+1)\bigg\}\bigg\}\,, \tag{93}$$

with the polynomials

$$
\begin{align}
P_{43} &= N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \tag{94}\\
P_{44} &= 2N^6 + 7N^5 + 31N^4 + 82N^3 + 86N^2 + 32N + 8 \tag{95}\\
P_{45} &= 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \tag{96}\\
P_{46} &= 5N^6 + 29N^5 + 78N^4 + 118N^3 + 114N^2 + 72N + 16 \tag{97}\\
P_{47} &= 5N^6 + 135N^5 + 327N^4 + 329N^3 + 220N^2 - 176N - 120 \tag{98}\\
P_{48} &= 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24 \tag{99}\\
P_{49} &= 11N^6 + 6N^5 + 75N^4 + 68N^3 - 200N^2 - 80N - 24 \tag{100}\\
P_{50} &= 11N^6 + 29N^5 - 7N^4 - 25N^3 - 56N^2 - 72N - 24 \tag{101}\\
P_{51} &= 16N^6 + 35N^5 + 33N^4 - 11N^3 - 41N^2 - 36N - 12 \tag{102}\\
P_{52} &= 17N^6 - 57N^5 - 213N^4 - 175N^3 - 140N^2 + 64N + 72 \tag{103}\\
P_{53} &= 17N^6 + 27N^5 + 75N^4 + 149N^3 - 20N^2 - 80N - 24 \tag{104}\\
P_{54} &= 17N^6 + 51N^5 + 51N^4 + 89N^3 + 40N^2 - 80N - 24 \tag{105}\\
P_{55} &= 38N^6 + 108N^5 + 151N^4 + 106N^3 + 21N^2 - 28N - 12 \tag{106}\\
P_{56} &= 73N^6 + 189N^5 + 45N^4 + 31N^3 - 238N^2 - 412N - 120 \tag{107}\\
P_{57} &= N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \tag{108}\\
P_{58} &= 2N^7 + 14N^6 + 37N^5 + 102N^4 + 155N^3 + 158N^2 + 132N + 40 \tag{109}\\
P_{59} &= 3N^7 - 15N^6 - 153N^5 - 577N^4 - 854N^3 - 652N^2 - 408N - 128 \tag{110}\\
P_{60} &= 5N^7 + 19N^6 + 61N^5 + 197N^4 + 266N^3 + 212N^2 + 136N + 32 \tag{111}\\
P_{61} &= 5N^7 + 37N^6 + 188N^5 + 643N^4 + 925N^3 + 742N^2 + 460N + 120 \tag{112}\\
P_{62} &= 7N^7 + 21N^6 + 5N^5 - 117N^4 - 244N^3 - 232N^2 - 192N - 80 \tag{113}\\
P_{63} &= 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \tag{114}\\
P_{64} &= 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \tag{115}\\
P_{65} &= 9N^7 + 15N^6 - 103N^5 - 575N^4 - 998N^3 - 948N^2 - 696N - 256 \tag{116}\\
P_{66} &= 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \tag{117}\\
P_{67} &= 25N^7 + 91N^6 + 101N^5 - 195N^4 - 546N^3 - 556N^2 - 520N - 224 \tag{118}\\
P_{68} &= 62N^7 + 329N^6 + 986N^5 + 1790N^4 + 2242N^3 + 1653N^2 + 650N + 96 \tag{119}\\
P_{69} &= N^8 + 8N^7 + 8N^6 - 14N^5 - 53N^4 - 82N^3 + 60N^2 + 104N + 96 \tag{120}\\
P_{70} &= 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \tag{121}
\end{align}
$$

$$
\begin{array}{rll}
P_{71} &=& 6N^8 - 42N^7 - 241N^6 - 579N^5 - 307N^4 + 477N^3 + 602N^2 + 492N + 168 & (122) \\
P_{72} &=& 10N^8 + 71N^7 + 244N^6 + 497N^5 + 698N^4 + 720N^3 + 512N^2 + 248N + 48 & (123) \\
P_{73} &=& 19N^9 + 86N^8 + 144N^7 - 38N^6 - 535N^5 - 1016N^4 - 1180N^3 - 872N^2 & \\
&& -416N - 96 & (124) \\
P_{74} &=& N^{10} + 15N^9 + 105N^8 + 361N^7 + 660N^6 + 828N^5 + 814N^4 + 384N^3 & \\
&& -112N^2 - 128N - 32 & (125) \\
P_{75} &=& 6N^{10} + 49N^9 + 197N^8 + 472N^7 + 833N^6 + 1469N^5 + 2142N^4 + 1904N^3 & \\
&& +1040N^2 + 432N + 96 & (126) \\
P_{76} &=& 11N^{10} + 123N^9 + 541N^8 + 1273N^7 + 1806N^6 + 1672N^5 + 1006N^4 + 320N^3 & \\
&& -16N^2 - 64N - 32 & (127) \\
P_{77} &=& 19N^{10} + 143N^9 + 412N^8 + 426N^7 - N^6 + 159N^5 + 1066N^4 + 1552N^3 & \\
&& +1456N^2 + 848N + 224 & (128) \\
P_{78} &=& 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 & \\
&& +16N^2 + 672N + 288 & (129) \\
P_{79} &=& 60N^{10} + 397N^9 + 1073N^8 + 1111N^7 + 623N^6 + 4328N^5 + 12432N^4 + 15944N^3 & \\
&& +12704N^2 + 6816N + 1728 & (130) \\
P_{80} &=& 67N^{10} + 383N^9 + 867N^8 + 696N^7 - 755N^6 - 2391N^5 - 3027N^4 - 2744N^3 & \\
&& -1256N^2 - 48N + 144 & (131) \\
P_{81} &=& 77N^{10} + 646N^9 + 2553N^8 + 6903N^7 + 14498N^6 + 22898N^5 + 24861N^4 & \\
&& +17068N^3 + 7040N^2 + 1760N + 192 & (132) \\
P_{82} &=& 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 & \\
&& +1864N^2 + 1248N + 432 & (133) \\
P_{83} &=& 118N^{10} + 675N^9 + 1588N^8 + 1652N^7 + 326N^6 + 357N^5 + 876N^4 & \\
&& +1672N^3 + 3440N^2 + 2544N + 576 & (134) \\
P_{84} &=& 127N^{10} + 740N^9 + 1737N^8 + 1308N^7 - 1592N^6 - 2226N^5 + 1386N^4 & \\
&& +3064N^3 + 3040N^2 + 2496N + 864 & (135) \\
P_{85} &=& 151N^{10} + 708N^9 + 1156N^8 + 464N^7 - 967N^6 + 372N^5 + 3672N^4 & \\
&& +5236N^3 + 6152N^2 + 3792N + 864 & (136) \\
P_{86} &=& 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 & \\
&& +9608N^3 + 6512N^2 + 2432N + 384 & (137) \\
P_{87} &=& 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 & \\
&& -904N^3 - 592N^2 - 208N - 32 & (138) \\
P_{88} &=& 118N^{11} + 529N^{10} + 1264N^9 + 3846N^8 + 11353N^7 + 23684N^6 + 32793N^5 & \\
&& +31801N^4 + 22836N^3 + 10448N^2 + 2592N + 432 & (139) \\
P_{89} &=& 127N^{11} + 820N^{10} + 2197N^9 + 1890N^8 - 1847N^7 - 1960N^6 + 3843N^5 & \\
&& +9730N^4 + 13632N^3 + 10688N^2 + 4944N + 864 & (140) \\
P_{90} &=& 136N^{11} + 1039N^{10} + 3100N^9 + 3534N^8 - 1295N^7 - 6352N^6 - 8421N^5 & \\
&& -11729N^4 - 7644N^3 + 1376N^2 + 1920N + 144 & (141) \\
P_{91} &=& 7N^{12} + 47N^{11} + 123N^{10} + 76N^9 - 598N^8 - 2178N^7 - 3626N^6 & \\
&& -3933N^5 - 3254N^4 - 1608N^3 - 144N^2 + 112N + 32 & (142) \\
P_{92} &=& 37N^{12} + 305N^{11} + 1017N^{10} + 1462N^9 + 592N^8 + 408N^7 + 4064N^6 & \\
&& +9645N^5 + 12222N^4 + 10280N^3 + 6064N^2 + 2192N + 352 & (143)
\end{array}
$$

$$
\begin{aligned}
P_{93} =\ & 242N^{12} + 1853N^{11} + 6173N^{10} + 12711N^9 + 18608N^8 + 17040N^7 - 302N^6 \\
& -24986N^5 - 32225N^4 - 20010N^3 - 7904N^2 - 2016N - 288
\end{aligned}
\tag{144}
$$

$$
\begin{aligned}
P_{94} =\ & 5N^{13} + 27N^{12} - 97N^{11} - 1410N^{10} - 5754N^9 - 12428N^8 - 16530N^7 \\
& -14531N^6 - 7956N^5 - 1038N^4 + 2176N^3 + 1632N^2 + 448N + 32
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
P_{95} =\ & 119N^{13} + 1897N^{12} + 12595N^{11} + 48221N^{10} + 124877N^9 + 239946N^8 \\
& +345670N^7 + 356234N^6 + 253043N^5 + 129982N^4 + 55768N^3 + 20112N^2 \\
& +5616N + 864
\end{aligned}
\tag{146}
$$

$$
\begin{aligned}
P_{96} =\ & 686N^{14} + 8408N^{13} + 39228N^{12} + 89257N^{11} + 113445N^{10} + 109336N^9 \\
& +76360N^8 - 109649N^7 - 393915N^6 - 482272N^5 - 376932N^4 - 263440N^3 \\
& -155472N^2 \\
& -56448N - 8640
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
P_{97} =\ & 1790N^{14} + 15938N^{13} + 56250N^{12} + 90805N^{11} + 43917N^{10} - 38450N^9 - 42314N^8 \\
& -169217N^7 - 616623N^6 - 992860N^5 - 964980N^4 - 697072N^3 - 376464N^2 \\
& -127872N - 19008
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
P_{98} =\ & 30N^{16} + 397N^{15} + 1996N^{14} + 3786N^{13} - 3905N^{12} - 30084N^{11} - 44372N^{10} \\
& +5100N^9 + 71344N^8 + 27709N^7 - 104744N^6 - 146534N^5 - 30293N^4 + 77346N^3 \\
& +33768N^2 - 23544N - 3888
\end{aligned}
\tag{149}
$$

$$
\begin{aligned}
P_{99} =\ & 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} \\
& -17182N^{10} - 36527N^9 - 27469N^8 - 11770N^7 + 5554N^6 + 32640N^5 + 46456N^4 \\
& +34528N^3 + 14816N^2 + 3584N + 384
\end{aligned}
\tag{150}
$$

$$
\begin{aligned}
P_{100} =\ & 1245N^{18} + 19980N^{17} + 133282N^{16} + 461805N^{15} + 787161N^{14} + 185392N^{13} \\
& -1368400N^{12} - 225082N^{11} + 6978631N^{10} + 13143336N^9 + 5808466N^8 \\
& -11433627N^7 - 19928573N^6 - 12013164N^5 + 1462668N^4 + 8209584N^3 \\
& +6906384N^2 + 2980800N + 544320 \ .
\end{aligned}
\tag{151}
$$

The Wilson coefficient $H^{\sf S}_{g,2}$, except for the constant contribution $a^{(3)}_{Qg}$, has a similar structure. It is given by :

$$
H^{\sf S}_{g,2} = \tfrac{1}{2}[1 + (-1)^N]
$$

$$
\times \left\{ a_s T_F \left\{ -\tilde{\gamma}^0_{qg} L_Q - \frac{4(N^3 - 4N^2 - N - 2)}{N^2(N+1)(N+2)} + \tilde{\gamma}^0_{qg} S_1 + \tilde{\gamma}^0_{qg} L_M \right\} \right.
$$

$$
+ a_s^2 \left\{ T_F^2 \left[\frac{4}{3}\tilde{\gamma}^0_{qg} L_M^2 - \frac{4}{3}\tilde{\gamma}^0_{qg} L_Q L_M + \left[\frac{4}{3}\tilde{\gamma}^0_{qg} S_1 - \frac{16(N^3 - 4N^2 - N - 2)}{3N^2(N+1)(N+2)} \right] L_M \right] \right.
$$

$$
+ C_F T_F \left[\left[\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} + 2\tilde{\gamma}^0_{qg} S_1 \right] L_Q^2 + \left[-\frac{4P_{111}}{N^3(N+1)^3(N+2)} \right. \right.
$$

$$
+ \frac{4(3N^4 + 2N^3 - 9N^2 - 16N - 12)S_1}{N^2(N+1)^2(N+2)} + L_M \left[-\frac{4(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \right.
$$

$$
\left. -4\tilde{\gamma}^0_{qg} S_1 \right] + \tilde{\gamma}^0_{qg} (4S_2 - 4S_1^2) \right] L_Q - \frac{2(9N^4 + 6N^3 - 15N^2 - 28N - 20)S_1^2}{N^2(N+1)^2(N+2)}
$$

$$
+ \frac{16(-1)^N P_{252}}{15(N-2)(N-1)^2 N^2(N+1)^4(N+2)^4(N+3)^3} + L_M^2 \left[\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \right.
$$

$$+2\tilde{\gamma}_{qg}^0 S_1\Bigg] + \frac{4P_{265}}{15(N-1)^2N^4(N+1)^4(N+2)^4(N+3)^3} + \frac{4P_{106}S_1}{N^3(N+1)^3(N+2)}$$

$$+L_M\Bigg[\frac{4P_{111}}{N^3(N+1)^3(N+2)} - \frac{4(3N^4+2N^3-9N^2-16N-12)S_1}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\Big[4S_1^2-4S_2\Big]\Bigg]$$

$$+\frac{2(11N^4+42N^3+43N^2-4N-12)S_2}{N^2(N+1)^2(N+2)} + \frac{16P_{103}S_{-2}}{(N-2)N^2(N+1)^2(N+2)(N+3)}$$

$$+\tilde{\gamma}_{qg}^0\Big[2S_1^3-2S_2S_1-4S_{2,1}\Big] + \frac{128S_1S_{-2}+64S_{-3}-128S_{-2,1}}{N(N+1)(N+2)} + \frac{(N-1)\Big[48\zeta_3-16S_3\Big]}{N(N+1)}\Bigg]$$

$$+C_AT_F\Bigg[\Bigg[\Bigg[\frac{16(N^2+N+1)(N^2+N+2)}{(N-1)N^2(N+1)^2(N+2)^2} + 2\tilde{\gamma}_{qg}^0 S_1\Bigg]L_Q^2 + \Bigg[-\frac{32(-1)^N(N^3+4N^2+7N+5)}{(N+1)^3(N+2)^3}$$

$$+\frac{8P_{160}}{(N-1)N^3(N+1)^2(N+2)^3} + \frac{16(N^2+1)(N^2-4N-1)S_1}{(N-1)N(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\Big[-2S_1^2+2S_2+4S_{-2}\Big]\Bigg]L_Q$$

$$-\frac{16P_{165}S_1}{(N-1)N^3(N+1)^2(N+2)^3} - \frac{4(2N^5-3N^4-5N^3-3N^2-33N-6)S_1^2}{(N-1)N(N+1)^2(N+2)^2}$$

$$-\frac{16(-1)^N P_{167}}{3(N-1)N^2(N+1)^4(N+2)^4} - \frac{8P_{239}}{3(N-1)N^4(N+1)^4(N+2)^4}$$

$$+\frac{32(-1)^N(N^3+4N^2+7N+5)S_1}{(N+1)^3(N+2)^3} + L_M^2\Bigg[-\frac{16(N^2+N+1)(N^2+N+2)}{(N-1)N^2(N+1)^2(N+2)^2} - 2\tilde{\gamma}_{qg}^0 S_1\Bigg]$$

$$+\frac{4P_{105}S_2}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{8(3N^2+3N+2)S_3}{N(N+1)(N+2)} + L_M\Bigg[\frac{32(-1)^N(N^3+4N^2+7N+5)}{(N+1)^3(N+2)^3}$$

$$-\frac{8P_{161}}{(N-1)N^3(N+1)^2(N+2)^3} - \frac{32(2N+3)S_1}{(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0\Big[-2S_1^2-2S_2-4S_{-2}\Big]\Bigg]$$

$$+\frac{16(N^5-N^4-5N^3+3N^2+14N+12)S_{-2}}{(N-1)N(N+1)^2(N+2)^2} - \frac{16(N^2+N+4)S_{-3}}{N(N+1)(N+2)}$$

$$+\frac{(N-1)\Big[16S_{-2}S_1-16S_{-2,1}\Big]}{N(N+1)} + \frac{(N^2-N-4)16(-1)^N S_{-2}}{(N+1)^2(N+2)^2} - \frac{6(5N^2+5N-6)\zeta_3}{N(N+1)(N+2)}$$

$$+\tilde{\gamma}_{qg}^0\Big[-8S_1S_2-4(-1)^N S_{-2}S_1-2(-1)^N S_{-3}+4S_{2,1}-\frac{3}{2}(-1)^N\zeta_3\Big]\Bigg]\Bigg\}$$

$$+a_s^3\Bigg\{T_F^3\Bigg[\frac{16}{9}\tilde{\gamma}_{qg}^0 L_M^3 - \frac{16}{9}\tilde{\gamma}_{qg}^0 L_Q L_M^2 + \Bigg[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 - \frac{64(N^3-4N^2-N-2)}{9N^2(N+1)(N+2)}\Bigg]L_M^2 - \frac{16\tilde{\gamma}_{qg}^0\zeta_3}{9}\Bigg]$$

$$+C_AT_F^2\Bigg[\Bigg[\frac{64(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\Bigg]L_Q^3 + \Bigg[-\frac{64(-1)^N(N^3+4N^2+7N+5)}{3(N+1)^3(N+2)^3}$$

$$+\frac{8P_{194}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^4-7N^3+5N^2-17N-13)S_1}{9(N-1)N(N+1)^2(N+2)}$$

$$+L_M\Bigg[\frac{64(N^2+N+1)(N^2+N+2)}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\Bigg] + \tilde{\gamma}_{qg}^0\Big[-\frac{4}{3}S_1^2+\frac{4S_2}{3}+\frac{8}{3}S_{-2}\Big]\Bigg]L_Q^2$$

$$+\Bigg[-\frac{32(8N^4-7N^3+5N^2-17N-13)S_1^2}{9(N-1)N(N+1)^2(N+2)} + \frac{128(-1)^N(N^3+4N^2+7N+5)S_1}{3(N+1)^3(N+2)^3}$$

$$-\frac{32P_{186}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^N P_{170}}{9(N-1)N^2(N+1)^4(N+2)^4}$$

<div align="center">24</div>

$$-\frac{16P_{233}}{27(N-1)N^3(N+1)^4(N+2)^4} + L_M^2\left[\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1\right]$$

$$+\frac{32\left(8N^4+13N^3-22N^2-9N-26\right)S_2}{9(N-1)N(N+1)(N+2)^2} + \frac{64\left(8N^5+15N^4+6N^3+11N^2+16N+16\right)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}$$

$$+\frac{128\left(N^2+N-1\right)S_3}{9N(N+1)(N+2)} + L_M\left[-\frac{128(-1)^N\left(N^3+4N^2+7N+5\right)}{3(N+1)^3(N+2)^3} + \frac{32P_{195}}{9(N-1)N^3(N+1)^3(N+2)^3}\right.$$

$$\left.-\frac{64(2N-1)\left(N^3+9N^2+7N+7\right)S_1}{9(N-1)N(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\left[-\frac{8}{3}S_1^2 + \frac{8S_2}{3} + \frac{16}{3}S_{-2}\right]\right] - \frac{128\left(N^2+N+3\right)S_{-3}}{3N(N+1)(N+2)}$$

$$+\tilde{\gamma}_{qg}^0\left[\frac{8}{9}S_1^3 - 8S_2S_1 + \frac{32}{3}S_{2,1}\right] + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1)\left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3\right]}{N(N+1)}\Bigg]L_Q$$

$$+\frac{32\left(N^3+8N^2+11N+2\right)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{152}S_1^2}{27N(N+1)^3(N+2)^3} + \frac{4}{9}\frac{P_{201}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^3}$$

$$+\frac{8P_{264}}{81(N-1)N^5(N+1)^5(N+2)^5} + \frac{32}{3}\left(2N^4+3N^3+10N^2+37N+35\right)\frac{(-1)^N\zeta_2}{(N+1)^3(N+2)^3}$$

$$+\frac{32}{9}\frac{\left(9N^5-4N^4+N^3+92N^2+42N+28\right)\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{8P_{249}S_1}{81(N-1)N^4(N+1)^4(N+2)^4}$$

$$-\frac{32}{9}\frac{\left(5N^4+8N^3+17N^2+43N+20\right)\zeta_2}{N(N+1)^2(N+2)^2}S_1 + L_M^3\left[-\frac{448\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$\left.-\frac{56}{9}\tilde{\gamma}_{qg}^0 S_1\right] + \frac{8P_{192}S_2}{3(N-1)N^3(N+1)^3(N+2)^3} - \frac{32\left(3N^3-12N^2-27N-2\right)S_1S_2}{3N(N+1)^2(N+2)^2}$$

$$+\frac{256\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + L_M^2\left[\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)}{(N+1)^3(N+2)^3}\right.$$

$$-\frac{8P_{199}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32\left(8N^5+9N^4-57N^3-31N^2+25N-26\right)S_1}{9(N-1)N(N+1)^2(N+2)^2}$$

$$\left.+\tilde{\gamma}_{qg}^0\left[-\frac{20}{3}S_1^2 - 4S_2 - 8S_{-2}\right]\right] + \frac{128(-1)^N\left(N^4+2N^3+7N^2+22N+20\right)S_{-2}}{3(N+1)^3(N+2)^3}$$

$$+\frac{\left(N^2-N-4\right)}{(N+1)^2(N+2)^2}\left[-\frac{256}{3}(-1)^N S_1S_{-2} - \frac{128}{3}(-1)^N S_{-3} + \frac{256}{3}S_{-2,1} - \frac{128}{3}(-1)^N S_1\zeta_2\right.$$

$$\left.-32(-1)^N\zeta_3\right] + \tilde{\gamma}_{qg}^0\left[\frac{2}{9}S_1^4 + \frac{20}{3}S_2S_1^2 + \frac{32}{3}(-1)^N S_{-3}S_1 + \left(\frac{160S_3}{9} - \frac{64}{3}S_{-2,1}\right)S_1 + \frac{2}{3}S_2^2\right.$$

$$+\frac{8}{9}\left(-2+9(-1)^N\right)\zeta_3 S_1 + S_{-2}\left(\frac{32}{3}(-1)^N S_1^2 + \frac{32}{3}(-1)^N S_2\right) + 12S_4$$

$$+\frac{16}{3}(-1)^N S_{-4} - \frac{16}{3}S_{3,1} - \frac{32}{3}S_{-2,2} - \frac{32}{3}S_{-3,1} - \frac{16}{3}S_{2,1,1} + \frac{64}{3}S_{-2,1,1} + \left[\frac{2}{3}\left(-3+8(-1)^N\right)S_1^2\right.$$

$$\left.+\frac{2}{3}\left(-3+8(-1)^N\right)S_2 + \frac{4}{3}\left(1+4(-1)^N\right)S_{-2}\right]\zeta_2\right] + L_M\left[\frac{64\left(N^5+9N^4+3N^3+N^2+26N-4\right)S_1^2}{9(N-1)N(N+1)^2(N+2)^2}\right.$$

$$+\frac{128(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{3(N+1)^3(N+2)^3} + \frac{16P_{198}S_1}{27(N-1)N^3(N+1)^3(N+2)^3} + \frac{64(N-1)S_{-2}S_1}{3N(N+1)}$$

$$-\frac{64(-1)^N P_{157}}{9(N-1)N^2(N+1)^3(N+2)^4} + \frac{16P_{242}}{27(N-1)N^4(N+1)^4(N+2)^4} + \frac{64\left(7N^2+7N+8\right)S_3}{9N(N+1)(N+2)}$$

$$-\frac{64P_{104}S_2}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{64\left(2N^5+21N^4+36N^3-7N^2-68N-56\right)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2}$$

$$-\frac{128S_{-3}}{3N(N+1)(N+2)} - \frac{128S_{-2,1}}{3(N+2)} + \frac{\left(N^2-N-4\right)\frac{128}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2} - \frac{16\left(3N^2+3N-2\right)\zeta_3}{N(N+1)(N+2)}$$

$$+\tilde{\gamma}_{qg}^0\left[-\frac{8}{9}S_1^3 - \frac{40}{3}S_2 S_1 - \frac{32}{3}(-1)^N S_{-2}S_1 - \frac{16}{3}(-1)^N S_{-3} - 4(-1)^N\zeta_3\right]\Bigg]\Bigg]$$

$$+C_A T_F^2 N_F\Bigg[\left[\left[\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\right]L_Q^3 + \left[-\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)}{3(N+1)^3(N+2)^3}\right.\right.$$

$$+\frac{8P_{194}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32\left(8N^4-7N^3+5N^2-17N-13\right)S_1}{9(N-1)N(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^2 + \frac{4S_2}{3}\right.$$

$$\left.+\frac{8}{3}S_{-2}\right]\Bigg]L_Q^2 + \left[-\frac{32\left(8N^4-7N^3+5N^2-17N-13\right)S_1^2}{9(N-1)N(N+1)^2(N+2)} + \frac{128(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{3(N+1)^3(N+2)^3}\right.$$

$$-\frac{16P_{187}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^N P_{170}}{9(N-1)N^2(N+1)^4(N+2)^4} + \frac{128\left(N^2+N-1\right)S_3}{9N(N+1)(N+2)}$$

$$-\frac{8P_{248}}{27(N-1)N^4(N+1)^4(N+2)^4} + \frac{32\left(8N^4+13N^3-22N^2-9N-26\right)S_2}{9(N-1)N(N+1)(N+2)^2}$$

$$+\frac{64\left(8N^5+15N^4+6N^3+11N^2+16N+16\right)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} - \frac{128\left(N^2+N+3\right)S_{-3}}{3N(N+1)(N+2)}$$

$$+\tilde{\gamma}_{qg}^0\left[\frac{8}{9}S_1^3 - 8S_2 S_1 + \frac{32}{3}S_{2,1}\right] + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1)\left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3\right]}{N(N+1)}\Bigg]L_Q$$

$$+\frac{16\left(N^3+8N^2+11N+2\right)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{102}S_1^2}{3N(N+1)^3(N+2)^3} + \frac{4}{9}\frac{P_{175}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$+\frac{16P_{261}}{3(N-1)N^5(N+1)^5(N+2)^5} + \frac{16}{9}\frac{\left(9N^5-14N^4-19N^3+52N^2+12N+8\right)\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16P_{166}S_1}{3N(N+1)^4(N+2)^4} + \frac{16}{9}\frac{\left(5N^4+32N^3+47N^2+28N+20\right)\zeta_2}{N(N+1)^2(N+2)^2}S_1$$

$$+L_M^3\left[-\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\right] + \frac{8P_{191}S_2}{3(N-1)N^3(N+1)^3(N+2)^3}$$

$$-\frac{16\left(3N^3-12N^2-27N-2\right)S_1 S_2}{3N(N+1)^2(N+2)^2} + \frac{128\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$+L_M^2\left[\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)}{3(N+1)^3(N+2)^3} - \frac{8P_{172}}{9(N-1)N^2(N+1)^3(N+2)^3}\right.$$

$$\left.-\frac{32\left(5N^4+20N^3+47N^2+58N+20\right)S_1}{9N(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^2 - \frac{4S_2}{3} - \frac{8}{3}S_{-2}\right]\right]$$

$$+\frac{\left(N^4+2N^3+7N^2+22N+20\right)\left[\frac{64}{3}(-1)^N S_{-2} + \frac{32}{3}(-1)^N\zeta_2\right]}{(N+1)^3(N+2)^3}$$

$$+\frac{\left(N^2-N-4\right)\left[-\frac{128}{3}(-1)^N S_1 S_{-2} - \frac{64}{3}(-1)^N S_{-3} + \frac{128}{3}S_{-2,1} - \frac{64}{3}(-1)^N S_1\zeta_2 - 16(-1)^N\zeta_3\right]}{(N+1)^2(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\left[\frac{1}{9}S_1^4 + \frac{10}{3}S_2 S_1^2 + \frac{16}{3}(-1)^N S_{-3}S_1 + \left[\frac{80S_3}{9} - \frac{32}{3}S_{-2,1}\right]S_1 + \frac{4}{9}\left(-7+9(-1)^N\right)\zeta_3 S_1 + \frac{1}{3}S_2^2\right.$$

$$+S_{-2}\left[\frac{16}{3}(-1)^N S_1^2 + \frac{16}{3}(-1)^N S_2\right] + 6S_4 + \frac{8}{3}(-1)^N S_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} - \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1}$$

$$+\frac{32}{3}S_{-2,1,1} + \left[\frac{4}{3}\left(-1+2(-1)^N\right)S_1^2 + \frac{4}{3}\left(-1+2(-1)^N\right)S_2 + \frac{8}{3}(-1)^N S_{-2}\right]\zeta_2\right]$$

$$
+L_M\left[-\frac{16\left(10N^4+43N^3+106N^2+131N+46\right)S_1^2}{9N(N+1)^2(N+2)^2}+\frac{16P_{140}S_1}{27N(N+1)^3(N+2)^3}\right.
$$

$$
-\frac{64(-1)^N\left(7N^5+43N^4+117N^3+166N^2+107N+16\right)}{9(N+1)^4(N+2)^4}+\frac{8P_{246}}{27(N-1)N^4(N+1)^4(N+2)^4}
$$

$$
-\frac{16P_{122}S_2}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{64\left(5N^2+8N+10\right)S_{-2}}{9N(N+1)(N+2)}+\frac{\left(N^2-N-4\right)\frac{64}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2}
$$

$$
\left.\left.+\tilde\gamma_{qg}^0\left[-\frac{8}{9}S_1^3-\frac{8}{3}S_2S_1-\frac{16}{3}(-1)^N S_{-2}S_1-\frac{40S_3}{9}-\frac{8}{3}\left(2+(-1)^N\right)S_{-3}-\frac{16}{3}S_{2,1}+\frac{16}{3}S_{-2,1}\right]\right]\right]
$$

$$
+C_F^2 T_F\left[-\frac{\left(15N^4+6N^3-25N^2-32N-28\right)S_1^4}{3N^2(N+1)^2(N+2)}+\frac{2P_{159}S_1^2}{N^3(N+1)^4(N+2)}\right.
$$

$$
-\frac{4\left(3N^5-47N^4-147N^3-93N^2+8N+12\right)S_1^3}{3N^3(N+1)^2(N+2)}-4\frac{\left(3N^4+14N^3+43N^2+48N+20\right)\zeta_2}{N^2(N+1)^2(N+2)}S_1^2
$$

$$
-\frac{2\left(5N^4-14N^3+53N^2+120N+28\right)S_2S_1^2}{N^2(N+1)^2(N+2)}-\frac{2P_{211}S_1}{N^5(N+1)^5(N+2)}-\frac{4P_{118}S_2S_1}{N^3(N+1)^3(N+2)}
$$

$$
-\frac{8\left(3N^4+90N^3+83N^2-44N-4\right)S_3S_1}{3N^2(N+1)^2(N+2)}-\frac{16\left(3N^4+2N^3+19N^2+28N+12\right)S_{2,1}S_1}{N^2(N+1)^2(N+2)}
$$

$$
-\frac{8P_{132}\zeta_2 S_1}{N^3(N+1)^3(N+2)}-\frac{\left(11N^4+142N^3+147N^2-32N-12\right)S_2^2}{N^2(N+1)^2(N+2)}-\frac{P_{247}}{N^6(N+1)^6(N+2)}
$$

$$
+16\frac{(-1)^N\left(N^2+N+2\right)\zeta_2}{N(N+1)^4(N+2)}+\frac{2}{3}\left(N^2+N+2\right)\frac{\left(153N^4+306N^3+165N^2+12N+4\right)\zeta_3}{N^3(N+1)^3(N+2)}
$$

$$
+L_Q^3\left[\frac{2\left(N^2+N+2\right)\left(3N^2+3N+2\right)^2}{3N^3(N+1)^3(N+2)}-\frac{16\left(N^2+N+2\right)S_1\left(3N^2+3N+2\right)}{3N^2(N+1)^2(N+2)}\right.
$$

$$
\left.-\frac{8}{3}\tilde\gamma_{qg}^0 S_1^2\right]+L_M^3\left[-\frac{2\left(N^2+N+2\right)\left(3N^2+3N+2\right)^2}{3N^3(N+1)^3(N+2)}+\frac{16\left(N^2+N+2\right)S_1\left(3N^2+3N+2\right)}{3N^2(N+1)^2(N+2)}\right.
$$

$$
\left.+\frac{8}{3}\tilde\gamma_{qg}^0 S_1^2\right]-\frac{2P_{182}S_2}{N^4(N+1)^4(N+2)}+\frac{4\left(21N^5+217N^4+415N^3+351N^2+152N-4\right)S_3}{3N^2(N+1)^3(N+2)}
$$

$$
-\frac{16\left(N^2-3N-2\right)\left(3N^2+3N+2\right)S_{2,1}}{N^3(N+1)^2(N+2)}+L_M^2\left[\frac{32\left(N^3+5N^2+6N+4\right)S_1^2}{N^2(N+1)^2(N+2)}\right.
$$

$$
+\frac{2P_{138}S_1}{N^3(N+1)^3(N+2)}-\frac{32(-1)^N\left(N^2+N+2\right)}{N(N+1)^4(N+2)}-\frac{P_{178}}{N^4(N+1)^4(N+2)}
$$

$$
-\frac{16\left(N^2+N+2\right)\left(3N^2+3N+2\right)S_2}{N^2(N+1)^2(N+2)}-\frac{32\left(N^2+N+2\right)S_{-2}}{N^2(N+1)^2(N+2)}+\tilde\gamma_{qg}^0\left[8S_1^3-16S_2S_1\right.
$$

$$
\left.\left.-16S_{-2}S_1-8S_3-8S_{-3}+16S_{-2,1}\right]\right]+L_Q^2\left[\frac{32\left(N^3+5N^2+6N+4\right)S_1^2}{N^2(N+1)^2(N+2)}+\frac{2P_{138}S_1}{N^3(N+1)^3(N+2)}\right.
$$

$$
-\frac{32(-1)^N\left(N^2+N+2\right)}{N(N+1)^4(N+2)}-\frac{P_{178}}{N^4(N+1)^4(N+2)}+L_M\left[-\frac{2\left(N^2+N+2\right)\left(3N^2+3N+2\right)^2}{N^3(N+1)^3(N+2)}\right.
$$

$$
+\frac{16\left(N^2+N+2\right)S_1\left(3N^2+3N+2\right)}{N^2(N+1)^2(N+2)}+8\tilde\gamma_{qg}^0 S_1^2\right]-\frac{16\left(N^2+N+2\right)\left(3N^2+3N+2\right)S_2}{N^2(N+1)^2(N+2)}
$$

$$
\left.\left.-\frac{32\left(N^2+N+2\right)S_{-2}}{N^2(N+1)^2(N+2)}+\tilde\gamma_{qg}^0\left[8S_1^3-16S_2S_1-16S_{-2}S_1-8S_3-8S_{-3}+16S_{-2,1}\right]\right]\right.
$$

$$+\frac{P_{190}\zeta_2}{2N^4(N+1)^4(N+2)} + \frac{16(N^2+N+2)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} + 96\tilde{\gamma}_{qg}^0 \log(2)\zeta_2$$

$$+\frac{(N^2+N+2)(3N^2+3N+2)\Big[6S_4 - 16S_{3,1} + 32S_{2,1,1} + 8S_2\zeta_2 - \frac{16}{3}S_1\zeta_3\Big]}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\Big[\frac{1}{3}S_1^5$$

$$-\frac{2}{3}S_2S_1^3 + \Big(-\frac{16}{3}S_3 - 16S_{2,1}\Big)S_1^2 - \frac{8}{3}\zeta_3S_1^2 + \Big[-3S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1}\Big]S_1$$

$$+\frac{8}{3}S_2S_3 + \Big[-4S_1^3 + 8S_2S_1 + 8S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1}\Big]\zeta_2\Big]$$

$$+L_Q\Bigg[\frac{16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N+1)^2(N+2)} - \frac{2P_{136}S_1^2}{N^3(N+1)^3(N+2)} + \frac{64(-1)^N(N^2+N+2)S_1}{N(N+1)^4(N+2)}$$

$$-\frac{2P_{139}S_1}{N^4(N+1)^3} - \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2S_1}{N^2(N+1)^2(N+2)} - \frac{64(2N^3 + N^2 + 3N - 10)S_{-2}S_1}{N^2(N+1)^2(N+2)}$$

$$-\frac{32(-1)^N P_{266}}{5(N-2)(N-1)^2 N^3(N+1)^5(N+2)^5(N+3)^3} + \frac{2P_{277}}{5(N-1)^2 N^5(N+1)^5(N+2)^5(N+3)^3}$$

$$-48\frac{(9N^4 + 10N^3 + 9N^2 - 8N + 12)\zeta_3}{N^2(N+1)^2(N+2)} + L_M^2\Bigg[\frac{2(N^2+N+2)(3N^2+3N+2)^2}{N^3(N+1)^3(N+2)}$$

$$-\frac{16(N^2+N+2)S_1(3N^2+3N+2)}{N^2(N+1)^2(N+2)} - 8\tilde{\gamma}_{qg}^0 S_1^2\Bigg] + \frac{2P_{147}S_2}{N^3(N+1)^3(N+2)}$$

$$-\frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)S_3}{N^2(N+1)^2(N+2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_{-2}}{(N+1)^3(N+2)^3}$$

$$-\frac{64P_{212}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)} - \frac{64(N^2 + 3N + 4)S_{-3}}{N(N+1)^2(N+2)}$$

$$+\frac{16(N^2+N+2)(3N^2+3N+2)S_{2,1}}{N^2(N+1)^2(N+2)} + L_M\Bigg[-\frac{64(N^3 + 5N^2 + 6N + 4)S_1^2}{N^2(N+1)^2(N+2)}$$

$$-\frac{4P_{138}S_1}{N^3(N+1)^3(N+2)} + \frac{64(-1)^N(N^2+N+2)}{N(N+1)^4(N+2)} + \frac{2P_{178}}{N^4(N+1)^4(N+2)}$$

$$+\frac{32(N^2+N+2)(3N^2+3N+2)S_2}{N^2(N+1)^2(N+2)} + \frac{64(N^2+N+2)S_{-2}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\Big[-16S_1^3 + 32S_2S_1 + 32S_{-2}S_1$$

$$+16S_3 + 16S_{-3} - 32S_{-2,1}\Big]\Bigg] + \frac{128(N-1)(N^2 + 2N + 4)S_{-2,1}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\Big[-8S_1^4 + 24S_2S_1^2 - 48S_{-3}S_1$$

$$+\Big[16S_{2,1} - 32S_{-2,1}\Big]S_1 - 48\zeta_3 S_1 - 16S_2^2 - 32S_{-2}^2 + S_{-2}\Big[32S_1^2 - 32S_2\Big] - 32S_4 - 80S_{-4} - 32S_{3,1}$$

$$+32S_{-2,2} + 64S_{-3,1}\Big]\Bigg] + L_M\Bigg[-\frac{16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N+1)^2(N+2)} + \frac{2P_{136}S_1^2}{N^3(N+1)^3(N+2)}$$

$$-\frac{64(-1)^N(N^2+N+2)S_1}{N(N+1)^4(N+2)} + \frac{2P_{139}S_1}{N^4(N+1)^3} + \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2S_1}{N^2(N+1)^2(N+2)}$$

$$+\frac{64(2N^3 + N^2 + 3N - 10)S_{-2}S_1}{N^2(N+1)^2(N+2)} + \frac{32(-1)^N P_{266}}{5(N-2)(N-1)^2 N^3(N+1)^5(N+2)^5(N+3)^3}$$

$$-\frac{2P_{277}}{5(N-1)^2 N^5(N+1)^5(N+2)^5(N+3)^3} + 48\frac{(9N^4 + 10N^3 + 9N^2 - 8N + 12)\zeta_3}{N^2(N+1)^2(N+2)}$$

$$-\frac{2P_{147}S_2}{N^3(N+1)^3(N+2)} + \frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)S_3}{N^2(N+1)^2(N+2)}$$

$$-\frac{128(-1)^N\left(N^3+4N^2+7N+5\right)S_{-2}}{(N+1)^3(N+2)^3}+\frac{64P_{212}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)}$$

$$+\frac{64\left(N^2+3N+4\right)S_{-3}}{N(N+1)^2(N+2)}-\frac{16\left(N^2+N+2\right)\left(3N^2+3N+2\right)S_{2,1}}{N^2(N+1)^2(N+2)}-\frac{128(N-1)\left(N^2+2N+4\right)S_{-2,1}}{N^2(N+1)^2(N+2)}$$

$$+\tilde{\gamma}_{qg}^0\Big[8S_1^4-24S_2S_1^2+48S_{-3}S_1+\Big[32S_{-2,1}-16S_{2,1}\Big]S_1+48\zeta_3S_1+16S_2^2+32S_{-2}^2$$

$$+S_{-2}\Big[32S_2-32S_1^2\Big]+32S_4+80S_{-4}+32S_{3,1}-32S_{-2,2}-64S_{-3,1}\Big]\Big]\Big]$$

$$+C_A^2T_F\left[\frac{P_{123}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{4P_{176}S_1^3}{9(N-1)N^2(N+1)^3(N+2)^3}\right.$$

$$-\frac{4}{3}\frac{P_{131}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{2P_{215}S_1^2}{3(N-1)N^2(N+1)^4(N+2)^4}$$

$$+\frac{2(N-2)\left(55N^5+347N^4+379N^3+213N^2+326N+120\right)S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{4}{9}\frac{P_{217}S_1\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^3}-\frac{4P_{262}S_1}{3(N-1)N^5(N+1)^5(N+2)^5}$$

$$-\frac{4P_{193}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3}+\frac{16P_{144}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{8}{3}\frac{(-1)^NP_{213}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^4}-\frac{2}{9}\frac{P_{251}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^4}$$

$$-\frac{4\left(11N^4+22N^3-35N^2-46N-24\right)P_{261}}{3(N-1)^2N^6(N+1)^6(N+2)^6}-\frac{4}{9}\frac{\zeta_3P_{148}S_1}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{4}{9}\frac{\left(11N^4+22N^3-35N^2-46N-24\right)\left(9N^5-14N^4-19N^3+52N^2+12N+8\right)\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$+L_Q^3\left[-\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2+\frac{8\left(N^2+N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$\left.-\frac{16\left(N^2+N+1\right)\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{9(N-1)^2N^3(N+1)^3(N+2)^3}\right]+L_M^3\left[\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2\right.$$

$$-\frac{8\left(N^2+N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$\left.+\frac{16\left(N^2+N+1\right)\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{9(N-1)^2N^3(N+1)^3(N+2)^3}\right]$$

$$-\frac{2\left(11N^4+22N^3-35N^2-46N-24\right)P_{191}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^4}$$

$$-\frac{32\left(11N^4+22N^3-35N^2-46N-24\right)\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{16(-1)^N\left(N^4+2N^3+7N^2+22N+20\right)\left(11N^4+22N^3-35N^2-46N-24\right)S_{-2}}{3(N-1)N(N+1)^4(N+2)^4}$$

$$-32\frac{\left(N^2+N+1\right)\left(N^2+N+2\right)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2}S_{-2}+L_M^2\left[-\frac{4P_{127}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$-\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{(N+1)^3(N+2)^3}+\frac{8P_{216}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4}+\frac{16P_{241}}{9(N-1)^2N^4(N+1)^3(N+2)^4}$$

$$-\frac{4\left(N^2+N+2\right)\left(11N^4+22N^3-83N^2-94N-72\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{8\left(N^2+N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[4S_1^3+12S_2S_1+16S_{-2}S_1$$

$$+4S_3+4S_{-3}-8S_{-2,1}\Big]\Big]+L_Q^2\bigg[-\frac{4P_{141}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{(N+1)^3(N+2)^3}-\frac{8P_{218}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$+\frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4}-\frac{16P_{240}}{9(N-1)^2N^4(N+1)^3(N+2)^4}$$

$$+\frac{4\left(N^2+N+2\right)\left(11N^4+22N^3-83N^2-94N-72\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{8\left(N^2+N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\Big[4S_1^3-12S_2S_1-16S_{-2}S_1-4S_3-4S_{-3}+8S_{-2,1}\Big]\bigg]$$

$$+\frac{\left(5N^5-131N^3-58N^2+232N+96\right)}{(N-1)N(N+1)^2(N+2)^3}\Big[\frac{32}{3}(-1)^N S_1S_{-2}+\frac{16}{3}(-1)^N S_1\zeta_2\Big]$$

$$+\frac{\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{(N-1)N^2(N+1)^2(N+2)^2}\Big[\frac{1}{3}S_2^2+\frac{16}{3}(-1)^N S_{-2}S_2$$

$$+6S_4+\frac{8}{3}(-1)^N S_{-4}-\frac{8}{3}S_{3,1}-\frac{16}{3}S_{-2,2}-\frac{16}{3}S_{-3,1}-\frac{8}{3}S_{2,1,1}+\frac{32}{3}S_{-2,1,1}$$

$$+\Big(\frac{8}{3}(-1)^N S_2+\frac{8}{3}(-1)^N S_{-2}\Big)\zeta_2\Big]+\frac{\left(N^2-N-4\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{(N-1)N(N+1)^3(N+2)^3}$$

$$\times\Big[4(-1)^N\zeta_3+\frac{16}{3}(-1)^N S_{-3}-\frac{32}{3}S_{-2,1}\Big]+\frac{P_{130}\Big[\frac{16}{3}(-1)^N S_{-2}S_1^2+\frac{8}{3}(-1)^N\zeta_2 S_1^2\Big]}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{\left(11N^5+34N^4-49N^3-24N^2-68N-48\right)\Big[\frac{16}{3}(-1)^N S_1S_{-3}-\frac{32}{3}S_1S_{-2,1}+4(-1)^N S_1\zeta_3\Big]}{(N-1)N^2(N+1)(N+2)^2}$$

$$+L_Q\bigg[\frac{8\left(47N^4-133N^3+11N^2-155N-58\right)S_1^3}{9(N-1)N(N+1)^2(N+2)}+\frac{8P_{223}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{32(N-2)(N+3)S_{-2}S_1^2}{N(N+1)(N+2)}-\frac{32(-1)^N P_{197}S_1}{3(N-1)N^3(N+1)^4(N+2)^4}+\frac{8P_{257}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4}$$

$$-\frac{8P_{137}S_2S_1}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{32\left(N^2+N+6\right)S_3S_1}{N(N+1)(N+2)}-\frac{16P_{142}S_{-2}S_1}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{32\left(9N^2+9N+22\right)S_{-3}S_1}{N(N+1)(N+2)}-\frac{64\left(3N^2+3N+10\right)S_{-2,1}S_1}{N(N+1)(N+2)}-\frac{384\zeta_3 S_1}{N(N+1)(N+2)}$$

$$-8\frac{P_{117}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^N P_{256}}{9(N-1)^2N^4(N+1)^5(N+2)^5}$$

$$+\frac{4P_{269}}{27(N-1)^2N^5(N+1)^5(N+2)^5}-\frac{8P_{219}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{8P_{149}S_3}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{16P_{220}S_{-2}}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$+\frac{\left(N^3+4N^2+7N+5\right)\left(-64(-1)^N S_1^2+64(-1)^N S_2+192(-1)^N S_{-2}\right)}{(N+1)^3(N+2)^3}$$

$$+\frac{16P_{133}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{32P_{112}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\Big[-2S_1^4+16S_2S_1^2-2S_2^2-12S_{-2}^2-16S_{-2}S_2-4S_4-44S_{-4}-\frac{88}{3}S_{2,1}-16S_{3,1}+56S_{-2,2}$$

$$+64S_{-3,1}-96S_{-2,1,1}\Big]\Big]+\tilde{\gamma}_{qg}^0\Big[-\frac{1}{3}S_1^5-10S_2S_1^3-16(-1)^N S_{-3}S_1^2+\Big[32S_{-2,1}-\frac{80S_3}{3}\Big]S_1^2$$

$$-\frac{4}{3}\left(-7+9(-1)^N\right)\zeta_3 S_1^2-8(-1)^N S_{-4}S_1+\Big[-S_2^2-18S_4+8S_{3,1}+16S_{-2,2}+16S_{-3,1}+8S_{2,1,1}$$

$$-32S_{-2,1,1}\Big]S_1+S_{-2}\Big[-16(-1)^N S_1^3-16(-1)^N S_2 S_1\Big]+\Big[-4\left(-1+2(-1)^N\right)S_1^3-8(-1)^N S_2 S_1$$

$$-4\left(1+2(-1)^N\right)S_{-2}S_1+\frac{11S_2}{3}-2S_3-2S_{-3}+4S_{-2,1}\Big]\zeta_2\Big]+L_M\Big[\frac{128(-1)^N P_{171}S_1}{3(N-1)N^2(N+1)^4(N+2)^4}$$

$$-\frac{8\left(11N^4+26N^3-139N^2-218N+8\right)S_1^3}{9N(N+1)^2(N+2)^2}+\frac{4P_{221}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{8P_{254}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4}-\frac{8P_{124}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{32\left(2N^5-23N^4-32N^3+13N^2+4N-12\right)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)^2}+2\frac{P_{151}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16(-1)^N P_{237}}{9(N-1)N^3(N+1)^5(N+2)^5}-\frac{4P_{268}}{27(N-1)^2N^5(N+1)^5(N+2)^5}$$

$$+\frac{4P_{222}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3}+\frac{\left(N^3+4N^2+7N+5\right)\left(-64(-1)^N S_1^2-64(-1)^N S_2\right)}{(N+1)^3(N+2)^3}$$

$$-\frac{8P_{145}S_3}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{16(-1)^N P_{129}S_{-2}}{3(N-1)N(N+1)^3(N+2)^3}+\frac{16P_{202}S_{-2}}{9(N-1)N^3(N+1)^3(N+2)^3}$$

$$-\frac{16P_{126}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}-\frac{16\left(N^2+N+2\right)\left(11N^4+22N^3+13N^2+2N+24\right)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{(N-1)N^2(N+1)^2(N+2)^2}\Big[-\frac{8}{3}(-1)^N S_{-3}-2(-1)^N\zeta_3\Big]$$

$$+\frac{\left(11N^5+34N^4-49N^3-24N^2-68N-48\right)\Big[-\frac{16}{3}(-1)^N S_1 S_{-2}\Big]}{(N-1)N^2(N+1)(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\Big[2S_1^4+32S_2S_1^2+8\left(8+(-1)^N\right)S_{-3}S_1+\Big[40S_3-16S_{2,1}-80S_{-2,1}\Big]S_1$$

$$+6\left(-5+(-1)^N\right)\zeta_3 S_1+2S_2^2+12S_{-2}^2+S_{-2}\Big[8\left(3+2(-1)^N\right)S_1^2+16S_2\Big]+4S_4+44S_{-4}+16S_{3,1}$$

$$-56S_{-2,2}-64S_{-3,1}+96S_{-2,1,1}\Big]+\frac{16P_{125}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}\Big]\Big]$$

$$+C_F T_F^2\Bigg[\Big[\frac{16\left(N^2+N+1\right)\left(N^2+N+2\right)\left(3N^4+6N^3-N^2-4N+12\right)}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1\Big]L_Q^3$$

$$+\Big[-\frac{16L_M\left(N^2+N+2\right)^3}{(N-1)N^3(N+1)^3(N+2)^2}-\frac{4P_{232}}{9(N-1)N^4(N+1)^4(N+2)^3}$$

$$+\frac{16P_{177}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\Big[\frac{20S_2}{3}-4S_1^2\Big]\Big]L_Q^2+\Big[-\frac{16P_{181}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2}$$

$$-\frac{8P_{229}S_1}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{64(-1)^N P_{259}}{45(N-2)(N-1)^2 N^3(N+1)^4(N+2)^4(N+3)^3}$$

$$+\frac{4P_{275}}{45(N-1)^2 N^5(N+1)^5(N+2)^4(N+3)^3} + L_M^2\left[-\frac{16(N^2+N+2)P_{109}}{3(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$\left.-\frac{16}{3}\tilde{\gamma}_{qg}^0 S_1\right] + L_M\left[\frac{8P_{231}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{32P_{173}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$+\tilde{\gamma}_{qg}^0\left[-\frac{8}{3}S_1^2 - \frac{8S_2}{3}\right]\right] + \frac{16P_{183}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{256(N^2+N+1)S_3}{3N(N+1)(N+2)}$$

$$+\frac{64P_{163}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \tilde{\gamma}_{qg}^0\left[\frac{8}{3}S_1^3 - 8S_2 S_1 - \frac{32}{3}S_{2,1}\right]$$

$$+\frac{\frac{512}{3}S_1 S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)}\right]L_Q - \frac{16(N^4-5N^3-32N^2-18N-4)S_1^2}{3N^2(N+1)^2(N+2)}$$

$$-\frac{16}{9}(N^2+N+2)\frac{\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}P_{113} + \frac{2}{9}\frac{P_{234}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3}$$

$$-\frac{2P_{253}}{3(N-1)N^5(N+1)^6(N+2)^2} + \frac{4P_{244}S_1}{3(N-1)N^5(N+1)^5(N+2)^2}$$

$$-\frac{80}{9}\frac{(N^3+4N^2+11N+6)\zeta_2}{N^2(N+1)(N+2)}S_1 + L_M^3\left[\frac{16(N^2+N+2)P_{113}}{9(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$\left.+\frac{32}{9}\tilde{\gamma}_{qg}^0 S_1\right] + L_M^2\left[-\frac{4P_{227}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{16P_{164}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$\left.+\tilde{\gamma}_{qg}^0\left[\frac{20}{3}S_1^2 - 4S_2\right]\right] - \frac{16P_{110}S_2}{3N^3(N+1)^3(N+2)} + \frac{(3N+2)\left[-\frac{32}{9}S_1^3 - \frac{32}{3}S_2 S_1\right]}{N^2(N+2)}$$

$$-\frac{32(3N^4+48N^3+43N^2-22N-8)S_3}{9N^2(N+1)^2(N+2)} + \frac{128(N^2-3N-2)S_{2,1}}{3N^2(N+1)(N+2)}$$

$$+L_M\left[\frac{16P_{174}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8P_{230}S_1}{9(N-1)N^4(N+1)^4(N+2)^3}\right.$$

$$+\frac{64(-1)^N P_{260}}{45(N-2)(N-1)^2 N^3(N+1)^4(N+2)^4(N+3)^3} + \frac{4P_{274}}{45(N-1)^2 N^5(N+1)^5(N+2)^4(N+3)^3}$$

$$-\frac{16P_{179}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{qg}^0\left[\frac{8}{3}S_1^3 + \frac{8}{3}S_2 S_1\right] + \frac{128(N^2+N+4)S_3}{3N(N+1)(N+2)}$$

$$+\frac{64P_{162}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \frac{\frac{512}{3}S_1 S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)}$$

$$\left.+\frac{64(N-1)\zeta_3}{N(N+1)}\right] + \tilde{\gamma}_{qg}^0\left[-\frac{2}{9}S_1^4 - \frac{4}{3}S_2 S_1^2 + \left[-\frac{16}{9}S_3 - \frac{32}{3}S_{2,1}\right]S_1 - \frac{32}{9}\zeta_3 S_1 - \frac{2}{3}S_2^2 + 4S_4\right.$$

$$\left.-\frac{32}{3}S_{3,1} + \frac{64}{3}S_{2,1,1} + \left[2S_2 - \frac{10}{3}S_1^2\right]\zeta_2\right]\right]$$

$$+C_F N_F T_F^2\left[\left[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 + \frac{16(N^2+N+1)(N^2+N+2)(3N^4+6N^3-N^2-4N+12)}{9(N-1)N^3(N+1)^3(N+2)^2}\right]L_Q^3\right.$$

$$+\left[-\frac{4P_{232}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{16P_{177}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$+ L_M \left[-\frac{8\left(N^2+N+2\right)\left(3N^2+3N+2\right)}{3N^2(N+1)^2(N+2)} - \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1 \right] + \tilde{\gamma}_{qg}^0 \left[\frac{20S_2}{3} - 4S_1^2 \right] \right] L_Q^2$$

$$+ \left[-\frac{16P_{181}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{8P_{229}S_1}{9(N-1)N^4(N+1)^4(N+2)^3} \right.$$

$$+ \frac{64(-1)^N P_{259}}{45(N-2)(N-1)^2 N^3(N+1)^4(N+2)^4(N+3)^3} + \frac{8P_{276}}{45(N-1)^2 N^5(N+1)^5(N+2)^4(N+3)^3}$$

$$+ L_M \left[\frac{8\left(N^2+N+2\right)\left(57N^4+72N^3+29N^2-22N-24\right)}{9N^3(N+1)^3(N+2)} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{3}S_1^2 - 8S_2 \right] \right.$$

$$\left. -\frac{16\left(N^2+N+2\right)\left(29N^2+29N-6\right)S_1}{9N^2(N+1)^2(N+2)} \right] - \frac{256\left(N^2+N+1\right)S_3}{3N(N+1)(N+2)} + \frac{16P_{183}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}$$

$$+ \frac{64P_{163}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{3}S_1^3 - 8S_2 S_1 - \frac{32}{3}S_{2,1} \right]$$

$$+ \frac{\frac{512}{3}S_1 S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)} \right] L_Q - \frac{2}{9}\frac{P_{235}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3}$$

$$- \frac{8\left(N^4-5N^3-32N^2-18N-4\right)S_1^2}{3N^2(N+1)^2(N+2)} - \frac{8}{9}\frac{\left(N^2+N+2\right)\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}P_{108}$$

$$- \frac{4P_{271}}{3(N-1)N^6(N+1)^6(N+2)^5} + \frac{16\left(2N^5-2N^4-11N^3-19N^2-44N-12\right)S_1}{3N^2(N+1)^3(N+2)}$$

$$- \frac{16}{9}\frac{\left(5N^3+11N^2+28N+12\right)\zeta_2}{N^2(N+1)(N+2)}S_1 + L_M^3 \left[\frac{8\left(N^2+N+2\right)P_{108}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1 \right]$$

$$- \frac{8P_{225}S_2}{3(N-1)N^4(N+1)^4(N+2)^3} + \frac{(3N+2)\left[-\frac{16}{9}S_1^3 - \frac{16}{3}S_2 S_1\right]}{N^2(N+2)}$$

$$+ L_M^2 \left[-\frac{4P_{228}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{32\left(5N^3+8N^2+19N+6\right)S_1}{9N^2(N+1)(N+2)} + \tilde{\gamma}_{qg}^0 \left[\frac{4}{3}S_1^2 + \frac{4S_2}{3} \right] \right]$$

$$- \frac{16P_{169}S_3}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{64\left(N^2-3N-2\right)S_{2,1}}{3N^2(N+1)(N+2)}$$

$$+ L_M \left[\frac{8\left(49N^4+122N^3+213N^2+164N+12\right)S_1^2}{9N^2(N+1)^2(N+2)} - \frac{8P_{258}}{9(N-1)N^5(N+1)^5(N+2)^4} \right.$$

$$+ \frac{8P_{146}S_1}{9N^3(N+1)^3(N+2)} - \frac{8P_{184}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[\frac{16}{3}S_1 S_2 - 16S_3 + \frac{16}{3}S_{2,1} \right] \right]$$

$$\left. + \tilde{\gamma}_{qg}^0 \left[-\frac{1}{9}S_1^4 - \frac{2}{3}S_2 S_1^2 - \frac{4}{3}\zeta_2 S_1^2 + \left[-\frac{8}{9}S_3 - \frac{16}{3}S_{2,1} \right] S_1 - \frac{8}{9}\zeta_3 S_1 - \frac{1}{3}S_2^2 + 2S_4 - \frac{16}{3}S_{3,1} + \frac{32}{3}S_{2,1,1} \right] \right]$$

$$+ C_A C_F T_F \left[\frac{2P_{134}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{4P_{180}S_1^3}{9(N-1)N^3(N+1)^3(N+2)^2} \right.$$

$$- \frac{4P_{245}S_1^2}{3(N-1)N^4(N+1)^4(N+2)^4} - \frac{8}{3}\frac{\left(19N^5-11N^4-8N^3-49N^2-17N+18\right)\zeta_2}{(N-1)N^2(N+1)^2(N+2)}S_1^2$$

$$+ \frac{4P_{143}S_2 S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{2}{9}\frac{P_{155}S_1\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$+ \frac{4}{9}\frac{P_{189}S_1\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{4P_{263}S_1}{3(N-1)N^5(N+1)^5(N+2)^5}$$

$$+\frac{4P_{200}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3}+\frac{8P_{153}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16\left(11N^5+45N^4-3N^3-145N^2-176N-20\right)S_{2,1}S_1}{3(N-1)N(N+1)^2(N+2)^2}-\frac{2P_{101}S_2^2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{256(-1)^N\left(N^3+4N^2+7N+5\right)}{(N+1)^3(N+2)^3}+8\frac{(-1)^N\zeta_2}{N^2(N+1)^4(N+2)^3}P_{115}-\frac{2}{9}\frac{P_{188}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$-\frac{1}{18}\frac{P_{208}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^2}+\frac{P_{273}}{3(N-1)N^6(N+1)^6(N+2)^5}$$

$$+L_M^3\left[-\frac{16}{3}\tilde{\gamma}_{qg}^0S_1^2-\frac{8\left(N^2+N+2\right)\left(N^2+N+6\right)\left(7N^2+7N+4\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$\left.-\frac{2\left(N^2+N+2\right)\left(3N^2+3N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)}{9(N-1)N^3(N+1)^3(N+2)^2}\right]+L_Q^3\left[-\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2\right.$$

$$+\frac{8\left(N^2+N+2\right)\left(13N^4+26N^3-43N^2-56N-12\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$\left.-\frac{4\left(N^2+N+2\right)\left(3N^2+3N+2\right)\left(11N^4+22N^3-23N^2-34N-12\right)}{9(N-1)N^3(N+1)^3(N+2)^2}\right]$$

$$-4\frac{\zeta_2P_{107}S_2}{(N-1)N^2(N+1)^2(N+2)^2}-\frac{4P_{243}S_2}{3(N-1)N^4(N+1)^4(N+2)^4}+\frac{4P_{185}S_3}{9(N-1)N^3(N+1)^3(N+2)^2}$$

$$+\frac{4\left(N^2+N+2\right)\left(19N^4+38N^3-22N^2-41N-30\right)S_4}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^NP_{116}S_{-2}}{N^2(N+1)^4(N+2)^3}$$

$$-\frac{16\left(N^2-3N-2\right)\left(11N^4+22N^3-35N^2-46N-24\right)S_{2,1}}{3(N-1)N^3(N+1)^2(N+2)^2}$$

$$-\frac{8\left(N^2+N+2\right)\left(31N^4+62N^3-73N^2-104N-60\right)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+L_Q^2\left[-\frac{8\left(10N^5+6N^4+5N^3-38N^2-17N+2\right)S_1^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{(N+1)^3(N+2)^3}\right.$$

$$-\frac{4P_{204}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}-\frac{16(-1)^N\left(3N^4+11N^3+19N^2+15N+2\right)}{N(N+1)^3(N+2)^3}$$

$$+\frac{P_{238}}{9(N-1)N^4(N+1)^4(N+2)^3}+L_M\left[\frac{22\left(N^2+N+2\right)\left(3N^2+3N+2\right)}{3N^2(N+1)^2(N+2)}\right.$$

$$\left.+\frac{22}{3}\tilde{\gamma}_{qg}^0S_1\right]+\frac{8\left(N^2+N+2\right)\left(23N^4+46N^3-50N^2-73N-18\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\left[4S_1^3-12S_2S_1\right.$$

$$\left.\left.+4S_3+6S_{-2}+4S_{-3}-8S_{-2,1}\right]\right]+L_M^2\left[\frac{8P_{121}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$+\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{(N+1)^3(N+2)^3}-\frac{8P_{203}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}$$

$$-\frac{16(-1)^N\left(3N^4+11N^3+19N^2+15N+2\right)}{N(N+1)^3(N+2)^3}+\frac{P_{224}}{9(N-1)N^4(N+1)^3(N+2)^3}$$

$$+\frac{8\left(N^2+N+2\right)\left(N^4+2N^3+20N^2+19N+30\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\left[-12S_1^3+4S_2S_1+4S_3+6S_{-2}\right.$$

$$\left.\left.+4S_{-3}-8S_{-2,1}\right]\right]+\frac{8\left(N^2+N+2\right)\left(35N^4+70N^3-137N^2-172N-84\right)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{8\left(N^2+N+2\right)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)}-48\tilde{\gamma}_{qg}^0\log(2)\zeta_2+\frac{P_{168}\left[16(-1)^NS_1S_{-2}+8(-1)^NS_1\zeta_2\right]}{N^3(N+1)^3(N+2)^3}$$

$$+\frac{\left(N^4+3N^3+8N^2+12N+4\right)\left[96(-1)^NS_{-2}S_1^2+48(-1)^N\zeta_2S_1^2\right]}{N(N+1)^2(N+2)^2}$$

$$+\frac{\left(3N^5+10N^4+25N^3+38N^2+20N+8\right)\left[16(-1)^NS_2S_{-2}+8(-1)^NS_2\zeta_2\right]}{N^2(N+1)^2(N+2)^2}$$

$$+\frac{\left(N^2+N+2\right)\left(3N^2+3N+2\right)}{N^2(N+1)^2(N+2)}\left[8(-1)^NS_{-2}\zeta_2+8(-1)^NS_{-4}-16S_{-2,2}-16S_{-3,1}+32S_{-2,1,1}\right]$$

$$+\frac{P_{156}\left[8(-1)^NS_{-3}-16S_{-2,1}+6(-1)^N\zeta_3\right]}{N^3(N+1)^3(N+2)^2}$$

$$+\frac{\left(9N^5+28N^4+73N^3+110N^2+44N+8\right)\left[8(-1)^NS_1S_{-3}-16S_1S_{-2,1}+6(-1)^NS_1\zeta_3\right]}{N^2(N+1)^2(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\Bigg[\frac{20}{3}S_2S_1^3+\left[24S_3+16S_{2,1}-24S_{-2,1}\right]S_1^2+8(-1)^NS_{-4}S_1+\Big[8S_2^2+12S_4+8S_{3,1}-16S_{-2,2}$$

$$-16S_{-3,1}-40S_{2,1,1}+32S_{-2,1,1}\Big]S_1+S_{-3}\left[12(-1)^NS_1^2+4(-1)^NS_2\right]+S_2\left[\frac{16S_3}{3}-8S_{-2,1}\right]$$

$$+S_{-2}\left[8(-1)^NS_1^3+24(-1)^NS_2S_1+32\right]+\left[4\big(1+(-1)^N\big)S_1^3+4\big(-1+2(-1)^N\big)S_{-2}S_1-2S_3\right.$$

$$\left.-2S_{-3}+4S_{-2,1}\right]\zeta_2+\left[\frac{1}{3}\big(-11+27(-1)^N\big)S_1^2\right]\zeta_3\Bigg]$$

$$+L_Q\Bigg[\frac{8\left(29N^5+81N^4+117N^3-49N^2-362N-104\right)S_1^3}{3(N-1)N(N+1)^2(N+2)^2}+\frac{4P_{206}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3}$$

$$-\frac{32(-1)^NP_{255}S_1}{15(N-2)(N-1)^2N^2(N+1)^4(N+2)^4(N+3)^3}+\frac{2P_{270}S_1}{45(N-1)^2N^4(N+1)^4(N+2)^4(N+3)^3}$$

$$-\frac{8\left(17N^5+33N^4-43N^3-153N^2-254N-80\right)S_2S_1}{(N-1)N(N+1)^2(N+2)^2}+\frac{32(N-2)(N+3)S_3S_1}{N(N+1)(N+2)}$$

$$-\frac{16P_{158}S_{-2}S_1}{3(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)}+\frac{512S_{-2,1}S_1}{N(N+1)(N+2)}-\frac{24\left(11N^2+11N-10\right)\zeta_3S_1}{N(N+1)(N+2)}$$

$$+2\frac{\zeta_3P_{154}}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^NP_{272}}{45(N-2)(N-1)^3N^3(N+1)^5(N+2)^5(N+3)^3}$$

$$-\frac{2P_{279}}{45(N-1)^3N^5(N+1)^5(N+2)^5(N+3)^3}$$

$$+L_M^2\Bigg[8\tilde{\gamma}_{qg}^0S_1^2+\frac{8\left(N^2+N+2\right)\left(3N^4+6N^3+7N^2+4N+4\right)S_1}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16\left(N^2+N+1\right)\left(N^2+N+2\right)\left(3N^2+3N+2\right)}{(N-1)N^3(N+1)^3(N+2)^2}\Bigg]+\frac{\left(N^2+N+10\right)\left[-64S_{-2}S_1^2-32S_{-3}S_1\right]}{N(N+1)(N+2)}$$

$$-\frac{4P_{207}S_2}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{\left(N^3+4N^2+7N+5\right)\left(128(-1)^NS_2-128(-1)^NS_1^2\right)}{(N+1)^3(N+2)^3}$$

$$+\frac{32P_{135}S_3}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{16(-1)^N\left(3N^5-6N^4-61N^3-124N^2-96N-16\right)S_{-2}}{N(N+1)^3(N+2)^3}$$

$$-\frac{16P_{119}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{32P_{226}S_{-2}}{3(N-2)(N-1)N^3(N+1)^3(N+2)^3(N+3)}$$

$$-\frac{16\left(N^2+N+2\right)\left(31N^2+31N+6\right)S_{2,1}}{3N^2(N+1)^2(N+2)}-\frac{16P_{120}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+L_M\Bigg[\frac{16\left(10N^5+40N^4+121N^3+161N^2+52N+12\right)S_1^2}{3N^2(N+1)^2(N+2)^2}+\frac{4P_{205}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}$$

$$-\frac{128(-1)^N\left(N^3+4N^2+7N+5\right)S_1}{(N+1)^3(N+2)^3}+\frac{32(-1)^N\left(3N^4+11N^3+19N^2+15N+2\right)}{N(N+1)^3(N+2)^3}$$

$$-\frac{2P_{236}}{9(N-1)N^4(N+1)^4(N+2)^3}-\frac{16\left(N^2+N+2\right)\left(4N^2+4N-1\right)S_2}{N^2(N+1)^2(N+2)}+\tilde\gamma_{qg}^0\Big[8S_1^3+8S_2S_1-8S_3$$

$$-12S_{-2}-8S_{-3}+16S_{-2,1}\Big]\Bigg]+\frac{\left(3N^5+8N^4+27N^3+46N^2+20N+8\right)16(-1)^NS_1S_{-2}}{N^2(N+1)^2(N+2)^2}$$

$$+\frac{\left(N^2+N+2\right)\left(3N^2+3N+2\right)\Big[8(-1)^NS_{-3}+6(-1)^N\zeta_3\Big]}{N^2(N+1)^2(N+2)}+\tilde\gamma_{qg}^0\Big[40S_2S_1^2+16(-1)^NS_{-2}S_1^2$$

$$+8(-1)^NS_{-3}S_1-16S_{2,1}S_1+6(-1)^N\zeta_3S_1-8S_2^2+24S_{-2}^2+8S_4+40S_{-4}$$

$$+32S_{3,1}-16S_{-3,1}-32S_{-2,1,1}\Big]\Bigg]+L_M\Bigg[\frac{8\left(3N^5+2N^4-61N^3-112N^2-56N-24\right)S_1^3}{N^2(N+1)^2(N+2)^2}$$

$$-\frac{2P_{209}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{32(-1)^N\left(15N^5+97N^4+260N^3+328N^2+158N-4\right)S_1}{N(N+1)^3(N+2)^4}$$

$$-\frac{2P_{250}S_1}{9(N-1)N^4(N+1)^4(N+2)^4}+\frac{8P_{114}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{16\left(3N^3+2N^2-47N-62\right)S_{-2}S_1}{N(N+1)^2(N+2)}-6\frac{\zeta_3P_{150}}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16(-1)^NP_{267}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^5(N+3)^3}+\frac{2P_{278}}{45(N-1)^2N^5(N+1)^5(N+2)^5(N+3)^3}$$

$$+\frac{2P_{210}S_2}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{\left(N^3+4N^2+7N+5\right)\left(128(-1)^NS_1^2-128(-1)^NS_2\right)}{(N+1)^3(N+2)^3}$$

$$-\frac{16P_{128}S_3}{(N-1)N^2(N+1)^2(N+2)^2}-\frac{16(-1)^N\left(3N^5-6N^4-61N^3-124N^2-96N-16\right)S_{-2}}{N(N+1)^3(N+2)^3}$$

$$-\frac{16P_{214}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)}+\frac{16\left(3N^4+4N^3-9N^2-14N+8\right)S_{-3}}{N^2(N+1)^2(N+2)}$$

$$+\frac{32\left(N^2+N+2\right)\left(10N^4+20N^3+5N^2-5N+6\right)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{16\left(3N^4+10N^3+43N^2+44N-20\right)S_{-2,1}}{N^2(N+1)^2(N+2)}$$

$$+\frac{\left(3N^5+8N^4+27N^3+46N^2+20N+8\right)\Big[-16(-1)^NS_1S_{-2}\Big]}{N^2(N+1)^2(N+2)^2}$$

$$+\frac{\left(N^2+N+2\right)\left(3N^2+3N+2\right)\Big[-8(-1)^NS_{-3}-6(-1)^N\zeta_3\Big]}{N^2(N+1)^2(N+2)}+\tilde\gamma_{qg}^0\Big[-8S_1^4-32S_2S_1^2$$

$$-16\left(1+(-1)^N\right)S_{-2}S_1^2-8\left(1+(-1)^N\right)S_{-3}S_1+\Big[32S_{2,1}-8S_3\Big]S_1$$

$$-6\left(3+(-1)^N\right)\zeta_3S_1+8S_2^2-24S_{-2}^2-8S_4-40S_{-4}-32S_{3,1}+16S_{-3,1}+32S_{-2,1,1}\Big]\Bigg]\Bigg]$$

$$+a_{Qg}^{(3)} + \tilde{C}_{2,g}^{\mathsf{S},(3)}(N_F + 1)\Bigg\}\Bigg\} \,, \tag{152}$$

with the polynomials

$$
\begin{aligned}
P_{101} &= N^6 - 81N^5 - 264N^4 - 185N^3 - 307N^2 - 256N - 204 & (153)\\
P_{102} &= N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 & (154)\\
P_{103} &= N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48 & (155)\\
P_{104} &= N^6 + 21N^5 + 57N^4 + 31N^3 + 26N^2 + 20N + 24 & (156)\\
P_{105} &= 2N^6 - 7N^5 - 41N^4 - 31N^3 - 29N^2 - 22N - 16 & (157)\\
P_{106} &= 2N^6 - 7N^5 - 24N^4 - 35N^3 - 44N^2 - 44N - 16 & (158)\\
P_{107} &= 3N^6 + 5N^5 + 27N^4 + 35N^3 + 6N^2 + 12N + 8 & (159)\\
P_{108} &= 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 & (160)\\
P_{109} &= 3N^6 + 9N^5 + 2N^4 - 11N^3 - 23N^2 - 16N - 12 & (161)\\
P_{110} &= 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 & (162)\\
P_{111} &= 4N^6 + 5N^5 - 10N^4 - 39N^3 - 40N^2 - 24N - 8 & (163)\\
P_{112} &= 6N^6 - 12N^5 + 17N^4 + 106N^3 + 127N^2 + 104N + 84 & (164)\\
P_{113} &= 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12 & (165)\\
P_{114} &= 7N^6 - 93N^5 - 327N^4 - 287N^3 - 316N^2 - 112N - 24 & (166)\\
P_{115} &= 7N^6 - 20N^5 - 176N^4 - 335N^3 - 276N^2 - 116N - 16 & (167)\\
P_{116} &= 7N^6 - 19N^5 - 171N^4 - 325N^3 - 264N^2 - 108N - 16 & (168)\\
P_{117} &= 7N^6 + 21N^5 + 5N^4 - 25N^3 - 204N^2 - 188N - 192 & (169)\\
P_{118} &= 8N^6 + 13N^5 - 111N^4 - 193N^3 - 89N^2 - 56N - 20 & (170)\\
P_{119} &= 9N^6 + 21N^5 + 11N^4 - 5N^3 - 104N^2 - 76N - 144 & (171)\\
P_{120} &= 9N^6 + 39N^5 + 53N^4 + 25N^3 + 94N^2 + 44N + 312 & (172)\\
P_{121} &= 10N^6 + 18N^5 - 111N^4 - 164N^3 - 61N^2 - 16N + 36 & (173)\\
P_{122} &= 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48 & (174)\\
P_{123} &= 11N^6 - 15N^5 - 327N^4 - 181N^3 + 292N^2 - 20N - 48 & (175)\\
P_{124} &= 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N - 240 & (176)\\
P_{125} &= 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72 & (177)\\
P_{126} &= 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N - 108 & (178)\\
P_{127} &= 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N - 48 & (179)\\
P_{128} &= 11N^6 + 35N^5 + 59N^4 + 57N^3 - 38N^2 - 68N + 40 & (180)\\
P_{129} &= 11N^6 + 47N^5 + 7N^4 + 9N^3 + 90N^2 + 28N + 96 & (181)\\
P_{130} &= 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 & (182)\\
P_{131} &= 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 & (183)\\
P_{132} &= 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 & (184)\\
P_{133} &= 16N^6 + 78N^5 - 23N^4 - 228N^3 - 503N^2 - 408N - 228 & (185)\\
P_{134} &= 17N^6 + 111N^5 + 234N^4 + 203N^3 - 89N^2 - 296N - 36 & (186)\\
P_{135} &= 22N^6 + 69N^5 + 71N^4 + 23N^3 - 57N^2 - 68N + 84 & (187)\\
P_{136} &= 23N^6 - 7N^5 - 237N^4 - 593N^3 - 678N^2 - 548N - 200 & (188)\\
P_{137} &= 23N^6 + 9N^5 - 71N^4 - 53N^3 - 184N^2 - 92N - 16 & (189)\\
P_{138} &= 25N^6 + 35N^5 - 55N^4 - 243N^3 - 286N^2 - 204N - 72 & (190)\\
P_{139} &= 29N^6 + 91N^5 + 235N^4 + 405N^3 + 272N^2 + 288N + 120 & (191)
\end{aligned}
$$

$$P_{140} = 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 \tag{192}$$

$$P_{141} = 35N^6 - 15N^5 - 183N^4 - 133N^3 - 356N^2 - 164N - 48 \tag{193}$$

$$P_{142} = 35N^6 - 15N^5 - 101N^4 + 31N^3 + 54N^2 + 164N + 120 \tag{194}$$

$$P_{143} = 44N^6 + 96N^5 + 369N^4 + 290N^3 - 695N^2 - 428N - 108 \tag{195}$$

$$P_{144} = 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 \tag{196}$$

$$P_{145} = 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 \tag{197}$$

$$P_{146} = 57N^6 + 161N^5 - 25N^4 - 193N^3 - 172N^2 - 36N + 48 \tag{198}$$

$$P_{147} = 65N^6 + 199N^5 + 197N^4 - 143N^3 - 330N^2 - 316N - 120 \tag{199}$$

$$P_{148} = 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 \tag{200}$$

$$P_{149} = 80N^6 + 60N^5 + 9N^4 + 230N^3 + 901N^2 + 988N + 1188 \tag{201}$$

$$P_{150} = 81N^6 + 211N^5 - 23N^4 - 355N^3 - 334N^2 - 4N - 344 \tag{202}$$

$$P_{151} = 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 \tag{203}$$

$$P_{152} = 130N^6 + 865N^5 + 2316N^4 + 3811N^3 + 4434N^2 + 2884N + 536 \tag{204}$$

$$P_{153} = 133N^6 + 699N^5 + 1395N^4 + 217N^3 - 880N^2 + 164N + 288 \tag{205}$$

$$P_{154} = 155N^6 + 369N^5 + 211N^4 - 65N^3 - 1002N^2 - 556N - 1416 \tag{206}$$

$$P_{155} = 215N^6 + 429N^5 + 891N^4 + 491N^3 - 2486N^2 - 1436N - 408 \tag{207}$$

$$P_{156} = 3N^7 + 28N^6 + 66N^5 + 90N^4 + 107N^3 + 78N^2 + 36N + 8 \tag{208}$$

$$P_{157} = 9N^7 + 71N^6 + 214N^5 + 320N^4 + 275N^3 + 215N^2 + 160N + 32 \tag{209}$$

$$P_{158} = 21N^7 + 120N^6 - 128N^5 - 1038N^4 - 89N^3 + 2382N^2 + 1636N - 600 \tag{210}$$

$$P_{159} = 81N^7 + 247N^6 + 291N^5 + 277N^4 + 108N^3 - 56N^2 + 20N + 24 \tag{211}$$

$$P_{160} = N^8 + 5N^7 + 10N^6 + 27N^5 + 65N^4 + 112N^3 + 124N^2 + 80N + 32 \tag{212}$$

$$P_{161} = N^8 + 5N^7 + 14N^6 + 23N^5 + 25N^4 + 52N^3 + 56N^2 + 48N + 16 \tag{213}$$

$$P_{162} = N^8 + 8N^7 - 2N^6 - 60N^5 - 23N^4 + 108N^3 + 96N^2 + 16N + 48 \tag{214}$$

$$P_{163} = N^8 + 8N^7 - 2N^6 - 60N^5 + N^4 + 156N^3 + 24N^2 - 80N - 240 \tag{215}$$

$$P_{164} = N^8 + 22N^7 + 111N^6 + 211N^5 + 42N^4 - 281N^3 - 406N^2 - 204N - 72 \tag{216}$$

$$P_{165} = 2N^8 + N^7 - 6N^6 + 26N^5 + 64N^4 + 51N^3 + 54N^2 + 28N + 8 \tag{217}$$

$$P_{166} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \tag{218}$$

$$P_{167} = 2N^8 + 44N^7 + 211N^6 + 485N^5 + 654N^4 + 581N^3 + 391N^2 + 192N + 32 \tag{219}$$

$$P_{168} = 3N^8 + 41N^7 + 136N^6 + 233N^5 + 331N^4 + 360N^3 + 208N^2 + 80N + 16 \tag{220}$$

$$P_{169} = 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 \tag{221}$$

$$P_{170} = 5N^8 - 8N^7 - 137N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 \tag{222}$$

$$P_{171} = 7N^8 + 40N^7 + 110N^6 + 193N^5 + 261N^4 + 313N^3 + 260N^2 + 96N + 16 \tag{223}$$

$$P_{172} = 9N^8 + 54N^7 + 80N^6 - 110N^5 - 645N^4 - 1168N^3 - 1132N^2 - 672N - 160 \tag{224}$$

$$P_{173} = 10N^8 + 46N^7 + 87N^6 + 85N^5 - 75N^4 - 251N^3 - 274N^2 - 132N - 72 \tag{225}$$

$$P_{174} = 11N^8 + 74N^7 + 213N^6 + 281N^5 - 30N^4 - 427N^3 - 446N^2 - 180N - 72 \tag{226}$$

$$P_{175} = 15N^8 + 36N^7 + 50N^6 - 252N^5 - 357N^4 + 152N^3 - 68N^2 + 88N + 48 \tag{227}$$

$$P_{176} = 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 \tag{228}$$

$$P_{177} = 19N^8 + 70N^7 + 63N^6 - 41N^5 - 192N^4 - 221N^3 - 142N^2 - 60N - 72 \tag{229}$$

$$P_{178} = 21N^8 + 42N^7 - 38N^6 - 360N^5 - 631N^4 - 730N^3 - 472N^2 - 216N - 48 \tag{230}$$

$$P_{179} = 23N^8 + 2N^7 - 135N^6 + 29N^5 + 210N^4 - 151N^3 - 350N^2 - 132N - 72 \tag{231}$$

$$P_{180} = 27N^8 - 36N^7 - 956N^6 - 1724N^5 + 187N^4 + 1288N^3 + 70N^2 - 224N - 72 \tag{232}$$

$$P_{181} = 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 \tag{233}$$

$$
\begin{aligned}
P_{182} &= 41N^8 + 5N^7 - 195N^6 - 97N^5 + 326N^4 + 424N^3 + 208N^2 + 72N + 16 & (234)\\
P_{183} &= 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 & (235)\\
P_{184} &= 79N^8 + 196N^7 + 132N^6 + 274N^5 + 465N^4 + 82N^3 + 332N^2 + 456N + 288 & (236)\\
P_{185} &= 105N^8 + 978N^7 + 1688N^6 - 1330N^5 - 5245N^4 - 4672N^3 - 2212N^2 - 544N - 288 & (237)\\
P_{186} &= 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 & (238)\\
P_{187} &= 170N^8 + 369N^7 - 521N^6 - 1393N^5 - 761N^4 - 952N^3 - 544N^2 + 32N + 144 & (239)\\
P_{188} &= 264N^8 + 1407N^7 + 2246N^6 + 1746N^5 + 804N^4 - 1069N^3 - 674N^2 - 92N - 24 & (240)\\
P_{189} &= 283N^8 + 838N^7 + 1482N^6 + 628N^5 - 1497N^4 - 1130N^3 - 772N^2 + 456N + 288 & (241)\\
P_{190} &= 633N^8 + 2532N^7 + 5036N^6 + 6142N^5 + 4275N^4 + 1118N^3 - 176N^2 - 184N - 48 & (242)\\
P_{191} &= N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 & (243)\\
P_{192} &= 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 1172N^3 + 904N^2 + 512N + 128 & (244)\\
P_{193} &= 6N^9 + 93N^8 + 576N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2 \\
&\quad + 1488N + 384 & (245)\\
P_{194} &= 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 & (246)\\
P_{195} &= 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 & (247)\\
P_{196} &= 11N^9 + 78N^8 + 214N^7 + 335N^6 + 383N^5 + 571N^4 + 916N^3 + 876N^2 + 480N + 96 & (248)\\
P_{197} &= 35N^9 + 150N^8 + 232N^7 + 137N^6 + 119N^5 + 661N^4 + 1174N^3 + 876N^2 + 480N + 96 & (249)\\
P_{198} &= 37N^9 + 210N^8 - 52N^7 - 2738N^6 - 7249N^5 - 9368N^4 - 8216N^3 - 5888N^2 \\
&\quad - 2448N - 576 & (250)\\
P_{199} &= 45N^9 + 270N^8 + 820N^7 + 1478N^6 + 1683N^5 + 1996N^4 + 2356N^3 + 2328N^2 \\
&\quad + 1408N + 288 & (251)\\
P_{200} &= 57N^9 + 624N^8 + 1756N^7 + 1092N^6 - 1803N^5 - 1512N^4 + 966N^3 + 1116N^2 \\
&\quad + 920N + 528 & (252)\\
P_{201} &= 69N^9 + 366N^8 + 1124N^7 + 1966N^6 + 2523N^5 + 5228N^4 + 7340N^3 + 5352N^2 \\
&\quad + 3008N + 672 & (253)\\
P_{202} &= 94N^9 + 597N^8 + 1616N^7 + 2410N^6 + 1841N^5 + 1165N^4 + 2191N^3 + 3802N^2 \\
&\quad + 2916N + 648 & (254)\\
P_{203} &= 121N^9 + 696N^8 + 1535N^7 + 1585N^6 + 416N^5 - 749N^4 - 836N^3 + 16N^2 \\
&\quad + 528N + 144 & (255)\\
P_{204} &= 197N^9 + 1242N^8 + 2938N^7 + 3524N^6 + 2713N^5 + 2234N^4 + 3680N^3 + 6176N^2 \\
&\quad + 4080N + 864 & (256)\\
P_{205} &= 439N^9 + 2634N^8 + 6008N^7 + 6694N^6 + 3545N^5 + 736N^4 + 2008N^3 + 6208N^2 \\
&\quad + 5136N + 1152 & (257)\\
P_{206} &= 538N^9 + 3333N^8 + 7802N^7 + 7630N^6 + 458N^5 - 1415N^4 + 7786N^3 + 12340N^2 \\
&\quad + 5592N + 864 & (258)\\
P_{207} &= 664N^9 + 3861N^8 + 9038N^7 + 11830N^6 + 9344N^5 + 3793N^4 + 3874N^3 + 11044N^2 \\
&\quad + 9624N + 2592 & (259)\\
P_{208} &= 891N^9 + 4455N^8 + 16078N^7 + 28774N^6 + 37047N^5 + 45835N^4 + 42192N^3 + 28888N^2 \\
&\quad + 10640N + 1776 & (260)\\
P_{209} &= 923N^9 + 5208N^8 + 11824N^7 + 12854N^6 + 2185N^5 - 7030N^4 + 1436N^3 + 15032N^2 \\
&\quad + 12864N + 3456 & (261)\\
P_{210} &= 965N^9 + 4884N^8 + 10816N^7 + 20810N^6 + 36895N^5 + 40442N^4 + 27692N^3 + 22712N^2
\end{aligned}
$$

$$+14496N + 3456 \tag{262}$$

$$P_{211} = 2N^{10} - 46N^9 - 98N^8 + 282N^7 + 1063N^6 + 1569N^5 + 1275N^4 + 403N^3 - 94N^2$$
$$-108N - 24 \tag{263}$$

$$P_{212} = 2N^{10} + 12N^9 + 24N^8 + 11N^7 - 48N^6 - 151N^5 - 282N^4 - 480N^3 - 664N^2$$
$$-576N - 288 \tag{264}$$

$$P_{213} = 11N^{10} + 44N^9 + 74N^8 + 196N^7 + 31N^6 - 1426N^5 - 3044N^4 - 2762N^3 - 1476N^2$$
$$-480N - 96 \tag{265}$$

$$P_{214} = 11N^{10} + 76N^9 + 138N^8 - 204N^7 - 1041N^6 - 988N^5 + 752N^4 + 1896N^3 + 944N^2$$
$$-384N - 576 \tag{266}$$

$$P_{215} = 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 9218N^4 - 6116N^3 - 72N^2$$
$$-752N - 192 \tag{267}$$

$$P_{216} = 85N^{10} + 425N^9 + 902N^8 + 932N^7 - 521N^6 - 685N^5 + 2022N^4 + 2928N^3 + 968N^2$$
$$-1296N - 576 \tag{268}$$

$$P_{217} = 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2$$
$$-2160N - 864 \tag{269}$$

$$P_{218} = 118N^{10} + 425N^9 + 197N^8 + 86N^7 + 1240N^6 + 2489N^5 + 4401N^4 + 3480N^3 + 524N^2$$
$$-1728N - 864 \tag{270}$$

$$P_{219} = 118N^{10} + 557N^9 + 461N^8 - 94N^7 + 1300N^6 + 3521N^5 + 4509N^4 + 1920N^3$$
$$-1132N^2 - 2376N - 1008 \tag{271}$$

$$P_{220} = 127N^{10} + 536N^9 + 611N^8 + 602N^7 + 1474N^6 + 2099N^5 + 798N^4 - 2301N^3$$
$$-4486N^2 - 3708N - 936 \tag{272}$$

$$P_{221} = 170N^{10} + 883N^9 + 2041N^8 + 2998N^7 - 448N^6 - 5465N^5 + 129N^4 + 6624N^3$$
$$+1132N^2 - 2016N - 864 \tag{273}$$

$$P_{222} = 170N^{10} + 1213N^9 + 3235N^8 + 2794N^7 - 2692N^6 - 3767N^5 - 1293N^4$$
$$-1632N^3 - 5324N^2 - 6240N - 2016 \tag{274}$$

$$P_{223} = 226N^{10} + 317N^9 - 811N^8 + 662N^7 + 4552N^6 + 3857N^5 + 3933N^4 + 2364N^3$$
$$+236N^2 - 1656N - 720 \tag{275}$$

$$P_{224} = 489N^{10} + 2934N^9 + 9364N^8 + 18830N^7 + 18627N^6 + 124N^5 - 19856N^4 - 19296N^3$$
$$-10640N^2 - 2880N - 1152 \tag{276}$$

$$P_{225} = 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3$$
$$-1232N^2 - 736N - 192 \tag{277}$$

$$P_{226} = 11N^{11} + 37N^{10} - 27N^9 - 118N^8 + 21N^7 - 249N^6 - 1097N^5 - 1138N^4 + 552N^3$$
$$+3448N^2 + 3456N + 2016 \tag{278}$$

$$P_{227} = 21N^{11} + 231N^{10} + 1334N^9 + 4086N^8 + 6277N^7 + 1775N^6 - 9488N^5 - 18076N^4$$
$$-18208N^3 - 11344N^2 - 5568N - 1728 \tag{279}$$

$$P_{228} = 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4$$
$$+7328N^3 + 5456N^2 + 3456N + 1152 \tag{280}$$

$$P_{229} = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4$$
$$+12976N^3 + 8016N^2 + 4608N + 1728 \tag{281}$$

$$P_{230} = 51N^{11} + 269N^{10} + 46N^9 - 1934N^8 - 3973N^7 - 875N^6 + 7364N^5 + 14972N^4$$
$$+16768N^3 + 10896N^2 + 5376N + 1728 \tag{282}$$

$$P_{231} = 51N^{11} + 357N^{10} + 1238N^9 + 2586N^8 + 2755N^7 - 1435N^6 - 9212N^5 - 15028N^4$$

$$-15280N^3 - 9808N^2 - 5184N - 1728 \tag{283}$$

$$P_{232} = 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 - 4645N^6 - 8936N^5 - 11980N^4$$
$$-12352N^3 - 8272N^2 - 4800N - 1728 \tag{284}$$

$$P_{233} = 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4$$
$$-6840N^3 + 120N^2 + 2432N + 960 \tag{285}$$

$$P_{234} = 243N^{11} + 1701N^{10} + 5378N^9 + 10350N^8 + 11479N^7 + 1193N^6 - 14684N^5 - 20572N^4$$
$$-16288N^3 - 8944N^2 - 4992N - 1728 \tag{286}$$

$$P_{235} = 333N^{11} + 2331N^{10} + 6556N^9 + 9270N^8 + 5081N^7 - 6701N^6 - 17554N^5 - 20036N^4$$
$$-15680N^3 - 9200N^2 - 5664N - 1728 \tag{287}$$

$$P_{236} = 753N^{11} + 4809N^{10} + 13174N^9 + 20466N^8 + 17717N^7 + 6829N^6 + 3908N^5$$
$$+15304N^4 + 25408N^3 + 20272N^2 + 8448N + 1152 \tag{288}$$

$$P_{237} = 837N^{11} + 7757N^{10} + 30120N^9 + 68575N^8 + 119176N^7 + 191350N^6 + 262979N^5$$
$$+258308N^4 + 163106N^3 + 63360N^2 + 14848N + 1536 \tag{289}$$

$$P_{238} = 1017N^{11} + 6195N^{10} + 14050N^9 + 12738N^8 - 2023N^7 - 5093N^6 + 27548N^5$$
$$+69760N^4 + 80752N^3 + 54064N^2 + 20928N + 3456 \tag{290}$$

$$P_{239} = 3N^{12} + 21N^{11} + 17N^{10} - 202N^9 - 842N^8 - 1924N^7 - 3378N^6 - 5059N^5$$
$$-6008N^4 - 4860N^3 - 2536N^2 - 960N - 192 \tag{291}$$

$$P_{240} = 9N^{12} + 63N^{11} + 38N^{10} - 414N^9 - 1035N^8 - 1341N^7 - 1511N^6 - 2972N^5$$
$$-6011N^4 - 8038N^3 - 6892N^2 - 3432N - 864 \tag{292}$$

$$P_{241} = 9N^{12} + 63N^{11} + 71N^{10} - 381N^9 - 1536N^8 - 2529N^7 - 1946N^6 - 1331N^5$$
$$-2096N^4 - 4036N^3 - 4144N^2 - 2304N - 576 \tag{293}$$

$$P_{242} = 39N^{12} + 585N^{11} + 2938N^{10} + 7136N^9 + 9083N^8 + 7745N^7 + 14668N^6 + 38246N^5$$
$$+59856N^4 + 55560N^3 + 32144N^2 + 12480N + 2304 \tag{294}$$

$$P_{243} = 48N^{12} + 459N^{11} + 2322N^{10} + 8290N^9 + 20159N^8 + 30862N^7 + 28247N^6 + 16109N^5$$
$$+9312N^4 + 7488N^3 + 4064N^2 + 1328N + 192 \tag{295}$$

$$P_{244} = 61N^{12} + 302N^{11} + 531N^{10} + 348N^9 - 349N^8 - 786N^7 + 457N^6 + 2524N^5$$
$$+2012N^4 + 204N^3 - 360N^2 - 240N - 96 \tag{296}$$

$$P_{245} = 92N^{12} + 796N^{11} + 3089N^{10} + 7550N^9 + 10547N^8 + 1029N^7 - 19496N^6$$
$$-24199N^5 - 8960N^4 + 736N^3 + 1744N^2 + 816N + 192 \tag{297}$$

$$P_{246} = 201N^{12} + 1845N^{11} + 6910N^{10} + 12854N^9 + 8915N^8 - 7741N^7 - 17126N^6$$
$$-4294N^5 + 16260N^4 + 22080N^3 + 12416N^2 + 4128N + 576 \tag{298}$$

$$P_{247} = 239N^{12} + 1338N^{11} + 3137N^{10} + 3164N^9 - 983N^8 - 6640N^7 - 8123N^6$$
$$-4526N^5 - 342N^4 + 1232N^3 + 848N^2 + 256N + 32 \tag{299}$$

$$P_{248} = 255N^{12} + 2169N^{11} + 6496N^{10} + 7694N^9 - 127N^8 - 6973N^7 + 4132N^6$$
$$+25502N^5 + 31956N^4 + 22656N^3 + 9632N^2 + 864N - 576 \tag{300}$$

$$P_{249} = 581N^{12} + 7035N^{11} + 37826N^{10} + 112904N^9 + 190293N^8 + 174327N^7 + 92032N^6$$
$$+69438N^5 + 78364N^4 + 44464N^3 + 11520N^2 - 3168N - 1728 \tag{301}$$

$$P_{250} = 825N^{12} + 7363N^{11} + 25396N^{10} + 40686N^9 + 26213N^8 - 12749N^7 - 55498N^6$$
$$-89796N^5 - 110552N^4 - 134960N^3 - 127584N^2 - 64704N - 12672 \tag{302}$$

$$P_{251} = 69N^{13} + 420N^{12} + 794N^{11} - 1357N^{10} - 10401N^9 - 15678N^8 + 532N^7 + 239N^6$$
$$-40018N^5 - 69432N^4 - 69152N^3 - 43792N^2 - 18336N - 3456 \tag{303}$$

$$P_{252} = 76N^{13} + 922N^{12} + 4479N^{11} + 9107N^{10} - 3747N^9 - 52973N^8 - 76133N^7 + 42261N^6$$

$$+199307N^5 + 123839N^4 - 77470N^3 - 84132N^2 - 2160N - 432 \tag{304}$$

$$P_{253} = 295N^{13} + 2387N^{12} + 8005N^{11} + 13687N^{10} + 10883N^9 + 389N^8 - 2641N^7 + 6029N^6$$
$$+11034N^5 + 6644N^4 + 1384N^3 + 80N^2 + 128N + 64 \tag{305}$$

$$P_{254} = 296N^{13} + 2368N^{12} + 10916N^{11} + 27006N^{10} + 23644N^9 - 19764N^8 - 61931N^7$$
$$-63733N^6 - 52001N^5 - 56865N^4 - 38104N^3 + 2664N^2 + 7344N + 432 \tag{306}$$

$$P_{255} = 377N^{13} + 4649N^{12} + 21813N^{11} + 38539N^{10} - 39339N^9 - 272611N^8 - 332971N^7$$
$$+220377N^6 + 801934N^5 + 384958N^4 - 362030N^3 - 297864N^2 - 1080N - 864 \tag{307}$$

$$P_{256} = 859N^{13} + 7376N^{12} + 25294N^{11} + 47088N^{10} + 63868N^9 + 80876N^8 + 63648N^7$$
$$-35856N^6 - 146697N^5 - 157168N^4 - 91320N^3 - 34800N^2 - 8640N - 1152 \tag{308}$$

$$P_{257} = 1211N^{13} + 5680N^{12} + 3338N^{11} - 17355N^{10} - 31517N^9 - 48486N^8 - 139667N^7$$
$$-278026N^6 - 340745N^5 - 269457N^4 - 138568N^3 - 34632N^2 + 9072N + 3888 \tag{309}$$

$$P_{258} = 70N^{14} + 555N^{13} + 1599N^{12} + 1192N^{11} - 4430N^{10} - 13305N^9 - 11835N^8 + 8440N^7$$
$$+35816N^6 + 57126N^5 + 60340N^4 + 44464N^3 + 27808N^2 + 12768N + 2880 \tag{310}$$

$$P_{259} = 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} - 19377N^{10} - 58253N^9 - 26543N^8 + 170601N^7$$
$$+362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 117072N - 25920 \tag{311}$$

$$P_{260} = 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} - 47693N^9 - 125723N^8 - 86079N^7$$
$$+36437N^6 + 22559N^5 - 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 \tag{312}$$

$$P_{261} = 4N^{15} + 50N^{14} + 267N^{13} + 765N^{12} + 1183N^{11} + 682N^{10} - 826N^9 - 1858N^8 - 1116N^7$$
$$+457N^6 + 1500N^5 + 2268N^4 + 2400N^3 + 1392N^2 + 448N + 64 \tag{313}$$

$$P_{262} = 26N^{15} + 314N^{14} + 1503N^{13} + 3222N^{12} + 2510N^{11} + 1996N^{10} + 15041N^9 + 40728N^8$$
$$+54008N^7 + 44956N^6 + 31936N^5 + 30416N^4 + 29568N^3 + 16704N^2 + 5376N + 768 \tag{314}$$

$$P_{263} = 101N^{15} + 1234N^{14} + 6867N^{13} + 21904N^{12} + 40098N^{11} + 32226N^{10} - 22057N^9$$
$$-86972N^8 - 114557N^7 - 111416N^6 - 89204N^5 - 37312N^4 + 13392N^3 + 23040N^2$$
$$+9792N + 1536 \tag{315}$$

$$P_{264} = 390N^{15} + 5121N^{14} + 30556N^{13} + 114173N^{12} + 321958N^{11} + 771597N^{10}$$
$$+1583594N^9 + 2637549N^8 + 3381542N^7 + 3199120N^6 + 2183360N^5 + 1123200N^4$$
$$+489952N^3 + 178176N^2 + 48384N + 6912 \tag{316}$$

$$P_{265} = 75N^{16} + 1245N^{15} + 8291N^{14} + 27609N^{13} + 43437N^{12} + 14221N^{11} - 5995N^{10}$$
$$+182937N^9 + 488696N^8 + 296818N^7 - 452292N^6 - 730430N^5 - 186180N^4$$
$$+259728N^3 + 241056N^2 + 116640N + 25920 \tag{317}$$

$$P_{266} = 115N^{16} + 1838N^{15} + 11829N^{14} + 36114N^{13} + 30900N^{12} - 133946N^{11} - 454068N^{10}$$
$$-457420N^9 + 249211N^8 + 864716N^7 + 312979N^6 - 634466N^5 - 587862N^4$$
$$-19556N^3 + 104832N^2 + 9504N + 1728 \tag{318}$$

$$P_{267} = 185N^{16} + 2988N^{15} + 19694N^{14} + 62954N^{13} + 64470N^{12} - 207876N^{11} - 792388N^{10}$$
$$-861230N^9 + 437231N^8 + 1750616N^7 + 869954N^6 - 1016136N^5 - 1130122N^4$$
$$-96596N^3 + 199872N^2 + 31104N + 1728 \tag{319}$$

$$P_{268} = 939N^{16} + 10527N^{15} + 37207N^{14} + 18679N^{13} - 202006N^{12} - 617170N^{11} - 930025N^{10}$$
$$-882917N^9 - 157123N^8 + 1388549N^7 + 2739376N^6 + 2837500N^5 + 2088640N^4$$
$$+1259696N^3 + 622464N^2 + 211392N + 34560 \tag{320}$$

$$P_{269} = 1155N^{16} + 12417N^{15} + 37693N^{14} - 12293N^{13} - 285754N^{12} - 613900N^{11} - 571735N^{10}$$
$$-134309N^9 + 778901N^8 + 2698745N^7 + 4995724N^6 + 5915740N^5 + 4978144N^4$$
$$+3161840N^3 + 1498752N^2 + 479808N + 76032 \tag{321}$$

$$
\begin{aligned}
P_{270} =\ & 1665N^{16} + 33005N^{15} + 287646N^{14} + 1402624N^{13} + 4031902N^{12} + 6199846N^{11} \\
& + 1054640N^{10} - 16668628N^9 - 37272559N^8 - 38892027N^7 - 17387942N^6 + 3962700N^5 \\
& + 15625800N^4 + 26960688N^3 + 27379296N^2 + 12985920N + 2332800 \quad\quad (322)
\end{aligned}
$$

$$
\begin{aligned}
P_{271} =\ & 87N^{17} + 1099N^{16} + 6055N^{15} + 19019N^{14} + 37119N^{13} + 45159N^{12} + 29583N^{11} - 2639N^{10} \\
& - 30218N^9 - 40778N^8 - 39994N^7 - 35844N^6 - 30808N^5 - 30384N^4 - 28256N^3 \\
& - 16064N^2 - 5248N - 768 \quad\quad (323)
\end{aligned}
$$

$$
\begin{aligned}
P_{272} =\ & 829N^{17} + 13413N^{16} + 83461N^{15} + 226391N^{14} + 55508N^{13} - 1239070N^{12} - 2862466N^{11} \\
& - 1217372N^{10} + 3372689N^9 + 2779147N^8 - 2705687N^7 + 171733N^6 \\
& + 8617302N^5 + 5817902N^4 - 3127236N^3 - 3652560N^2 - 336096N - 25920 \quad\quad (324)
\end{aligned}
$$

$$
\begin{aligned}
P_{273} =\ & 1407N^{17} + 18107N^{16} + 103463N^{15} + 347083N^{14} + 760095N^{13} + 1142715N^{12} + 1220067N^{11} \\
& + 983393N^{10} + 702746N^9 + 533822N^8 + 337702N^7 - 3552N^6 \\
& - 300296N^5 - 332160N^4 - 188128N^3 - 63232N^2 - 13184N - 1536 \quad\quad (325)
\end{aligned}
$$

$$
\begin{aligned}
P_{274} =\ & 95N^{18} + 3940N^{17} + 48989N^{16} + 308380N^{15} + 1166094N^{14} + 2843192N^{13} + 4428234N^{12} \\
& + 3171928N^{11} - 4692053N^{10} - 19875244N^9 - 34305831N^8 - 34774388N^7 - 16392680N^6 \\
& + 11584912N^5 + 30493776N^4 + 29700864N^3 + 18783360N^2 + 8294400N + 1866240 \quad\quad (326)
\end{aligned}
$$

$$
\begin{aligned}
P_{275} =\ & 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} - 3216546N^{12} \\
& - 1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 + 25361392N^7 + 9982840N^6 \\
& - 10154128N^5 - 26098704N^4 - 26761536N^3 - 17642880N^2 - 8087040N - 1866240 \quad\quad (327)
\end{aligned}
$$

$$
\begin{aligned}
P_{276} =\ & 500N^{18} + 8215N^{17} + 56287N^{16} + 201810N^{15} + 361782N^{14} + 98826N^{13} - 759348N^{12} \\
& - 495786N^{11} + 3942186N^{10} + 11896133N^9 + 16709737N^8 + 13315736N^7 + 3779660N^6 \\
& - 7306454N^5 - 14232852N^4 - 13254768N^3 - 8367840N^2 - 3771360N - 855360 \quad\quad (328)
\end{aligned}
$$

$$
\begin{aligned}
P_{277} =\ & 150N^{19} + 2815N^{18} + 24285N^{17} + 131358N^{16} + 511310N^{15} + 1515954N^{14} \\
& + 3372978N^{13} + 5213980N^{12} + 4715522N^{11} + 980739N^{10} - 2709391N^9 - 3741506N^8 \\
& - 4630558N^7 - 5623132N^6 - 2333736N^5 + 3419632N^4 + 5238496N^3 + 3231936N^2 \\
& + 1123200N + 172800 \quad\quad (329)
\end{aligned}
$$

$$
\begin{aligned}
P_{278} =\ & 5410N^{19} + 98215N^{18} + 764965N^{17} + 3280996N^{16} + 8031920N^{15} + 8939378N^{14} \\
& - 7608074N^{13} - 44964380N^{12} - 74768226N^{11} - 57879177N^{10} - 5243187N^9 + 13745888N^8 \\
& - 28158216N^7 - 49672024N^6 + 14757808N^5 + 94650144N^4 + 100507392N^3 \\
& + 53764992N^2 + 15655680N + 1866240 \quad\quad (330)
\end{aligned}
$$

$$
\begin{aligned}
P_{279} =\ & 7060N^{20} + 123495N^{19} + 898682N^{18} + 3394183N^{17} + 6222824N^{16} + 376386N^{15} \\
& - 22032204N^{14} - 39912378N^{13} - 13976964N^{12} + 31985011N^{11} + 4994394N^{10} \\
& - 91499501N^9 - 97243208N^8 + 54501988N^7 + 183103272N^6 + 127073120N^5 \\
& - 20272608N^4 - 88410816N^3 - 62225280N^2 - 21772800N - 3110400 \ . \quad\quad (331)
\end{aligned}
$$

The corresponding expressions in z-space are given in Appendix B.

Note that our result for $H_{g,2}^{\mathsf{S}}$ differs from the one given in Eq. (B.7) in z-space in Ref. [9] by the term

$$
C_F T_F^2 N_F \frac{4(N^2 + N + 2)}{N(N+1)(N+2)} [28\zeta_2 - 69] \quad\quad (332)
$$

in N-space. This result of [9] is based on the calculation carried out in Ref. [12], including the renormalization formulae derived there. We have checked, however, that our result Eq. (152) is in full agreement with Eq. (27) and the moments having been calculated by part of the present authors in Ref. [12]. The corresponding expression in z-space is presented in Appendix B.

5 The Asymptotic Wilson Coefficients for the Longitudinal Structure Function

The Wilson coefficients have been calculated in Ref. [22] for exclusive heavy flavor production, retaining three contributions only. In total also here five Wilson coefficients contribute and the expressions are slightly modified in the inclusive case of the complete structure function $F_L(x, Q^2)$, cf. [12]. In Mellin-N space they read :

$$L_{q,L}^{\mathsf{PS},(3)} = \tfrac{1}{2}[1 + (-1)^N]$$

$$\times \left\{ a_s^3 \left\{ C_F N_F T_F^2 \left[\frac{128 L_Q^2 (N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)} + \frac{128(N^2 + N + 2)L_M^2}{3(N-1)N(N+1)^2(N+2)} \right. \right. \right.$$

$$- \frac{256 L_Q (11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)}{9(N-1)N^2(N+1)^3(N+2)^2} + \left[\frac{256(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^3(N+2)} \right. $$

$$\left. - \frac{256(N^2 + N + 2)S_1}{3(N-1)N(N+1)^2(N+2)} \right] L_M + \frac{64(N^2 + N + 2)[S_1^2 + S_2]}{3(N-1)N(N+1)^2(N+2)}$$

$$\left. - \frac{128(8N^3 + 13N^2 + 27N + 16)S_1}{9(N-1)N(N+1)^3(N+2)} + \frac{128(43N^4 + 105N^3 + 224N^2 + 230N + 86)}{27(N-1)N(N+1)^4(N+2)} \right]$$

$$\left. \left. + N_F \hat{\tilde{C}}_{L,q}^{\mathsf{PS},(3)}(N_F) \right\} \right\}, \tag{333}$$

$$L_{g,L}^{\mathsf{S}} = \tfrac{1}{2}[1 + (-1)^N]$$

$$\times \left\{ a_s^2 \frac{64 N_F T_F^2 L_M}{3(N+1)(N+2)} + a_s^3 \left\{ N_F T_F^3 \frac{256 L_M^2}{9(N+1)(N+2)} + C_A N_F T_F^2 \right[\right.$$

$$\left[\frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128 S_1}{3(N+1)(N+2)} \right] L_Q^2 + \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right.$$

$$+ \frac{64 Q_2}{9(N-1)N(N+1)^3(N+2)^3} + \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N-1)N(N+1)^2(N+2)}$$

$$+ L_M \left[\frac{512(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256 S_1}{3(N+1)(N+2)} \right]$$

$$+ \frac{1}{(N+1)(N+2)} \left[\frac{128}{3} S_1^2 - \frac{128 S_2}{3} - \frac{256}{3} S_{-2} \right] \right] L_Q + \frac{32 Q_5}{27(N-1)N^3(N+1)^4(N+2)^2}$$

$$- \frac{64(56N + 47)S_1}{27(N+1)^2(N+2)} + L_M^2 \left[\frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128 S_1}{3(N+1)(N+2)} \right]$$

$$+ L_M \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} + \frac{128 Q_4}{9(N-1)N^2(N+1)^3(N+2)^3} \right.$$

$$\left. \left. + \frac{256(N^3 - 6N^2 + 2N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\frac{128}{3}S_1^2 - \frac{128 S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \right] \right]$$

$$+ C_F T_F^2 N_F \left[\frac{64(N^2 + N + 2)(N^4 + 2N^3 + 2N^2 + N + 6)L_Q^2}{3(N-1)N^2(N+1)^3(N+2)^2} \right.$$

$$+ \left[\frac{256(-1)^N Q_6}{45(N-2)(N-1)^2 N^2(N+1)^3(N+2)^2(N+3)^3} \right.$$

$$
-\frac{32Q_9}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} - \frac{128Q_1S_1}{3(N-1)N^2(N+1)^3(N+2)^2}
$$

$$
+\frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)} + \frac{64\left(N^2+N+2\right)\left(N^4+2N^3-7N^2-8N-12\right)L_M}{3(N-1)N^2(N+1)^3(N+2)^2}\Bigg]L_Q
$$

$$
+\frac{64\left(N^2+N+2\right)^2L_M^2}{(N-1)N^2(N+1)^3(N+2)^2} - \frac{16Q_7}{(N-1)N^4(N+1)^5(N+2)^2}
$$

$$
+L_M\Bigg[\frac{256(-1)^NQ_6}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}
$$

$$
+\frac{64Q_8}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} - \frac{64Q_3S_1}{3(N-1)N^2(N+1)^3(N+2)^2}
$$

$$
+\frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)}\Bigg]\Bigg]\Bigg] + N_F\,\hat{\tilde{C}}_{L,g}^{\mathsf{S},(3)}(N_F)\Bigg\}\Bigg\}, \tag{334}
$$

with

$$
\begin{aligned}
Q_1 &= 2N^6 + 6N^5 + 7N^4 + 4N^3 + 9N^2 + 8N + 12 & (335)\\
Q_2 &= 3N^6 + 3N^5 - 121N^4 - 391N^3 - 474N^2 - 308N - 80 & (336)\\
Q_3 &= 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 & (337)\\
Q_4 &= 6N^7 + 24N^6 + 47N^5 + 104N^4 + 219N^3 + 316N^2 + 208N + 48 & (338)\\
Q_5 &= 15N^8 + 60N^7 + 572N^6 + 1470N^5 + 2135N^4 + 1794N^3 + 722N^2 - 24N - 72 & (339)\\
Q_6 &= N^{10} - 13N^9 - 39N^8 + 222N^7 + 1132N^6 + 1787N^5 + 913N^4 + 392N^3 + 645N^2 \\
&\quad -324N - 108 & (340)\\
Q_7 &= 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2 \\
&\quad -32N - 16 & (341)\\
Q_8 &= 45N^{13} + 656N^{12} + 4397N^{11} + 17513N^{10} + 43665N^9 + 63005N^8 + 27977N^7 - 71993N^6 \\
&\quad -140386N^5 - 78985N^4 + 25350N^3 + 80460N^2 + 100008N + 38880 & (342)\\
Q_9 &= 95N^{13} + 1218N^{12} + 6096N^{11} + 14484N^{10} + 11570N^9 - 28440N^8 - 117844N^7 \\
&\quad -225884N^6 - 238953N^5 - 83290N^4 + 57660N^3 + 122040N^2 + 182304N + 77760, & (343)
\end{aligned}
$$

$$
L_{q,L}^{\mathsf{NS}} = \tfrac{1}{2}[1+(-1)^N]
$$

$$
\times\Bigg\{a_s^2 C_F T_F\Bigg\{\frac{16L_Q}{3(N+1)} - \frac{8\left(19N^2+7N-6\right)}{9N(N+1)^2} - \frac{16S_1}{3(N+1)}\Bigg\}
$$

$$
+a_s^3\Bigg\{C_F^2 T_F\Bigg[\Bigg[\frac{8\left(3N^2+3N+2\right)}{N(N+1)^2} - \frac{32S_1}{N+1}\Bigg]L_Q^2
$$

$$
+\Bigg[\frac{256(-1)^NQ_{11}}{15(N-2)(N-1)^2N^2(N+1)^4(N+2)^2(N+3)^3}
$$

$$
-\frac{32Q_{12}}{45(N-1)^2N^2(N+1)^4(N+2)^2(N+3)^3} - \frac{16(N+10)(5N+3)S_1}{9N(N+1)^2}
$$

$$
+\frac{512\left(N^4+2N^3-N^2-2N-6\right)S_{-2}}{3(N-2)N(N+1)^2(N+3)} + \frac{1}{N+1}\Bigg[\frac{128}{3}S_1^2 - \frac{512}{3}S_{-2}S_1 - \frac{128S_2}{3} - \frac{256S_3}{3}
$$

$$
-\frac{256}{3}S_{-3} + \frac{512}{3}S_{-2,1} + 256\zeta_3\Bigg]\Bigg]L_Q + \frac{2Q_{10}}{27N^3(N+1)^4} + L_M^2\Bigg[\frac{8\left(3N^2+3N+2\right)}{3N(N+1)^2}
$$

$$-\frac{32S_1}{3(N+1)}\Bigg] + L_M\Bigg[\frac{8\big(3N^4+6N^3+47N^2+20N-12\big)}{9N^2(N+1)^3} + \frac{\frac{64S_2}{3}-\frac{320S_1}{9}}{N+1}\Bigg]$$

$$+\frac{-\frac{896}{27}S_1+\frac{160S_2}{9}-\frac{32S_3}{3}}{N+1}\Bigg] + C_F T_F^2\Bigg[\frac{64L_Q^2}{9(N+1)} + \Bigg[-\frac{64\big(19N^2+7N-6\big)}{27N(N+1)^2}-\frac{128S_1}{9(N+1)}\Bigg]L_Q\Bigg]$$

$$+C_F T_F^2 N_F\Bigg[\frac{128L_Q^2}{9(N+1)} + \Bigg[-\frac{128\big(19N^2+7N-6\big)}{27N(N+1)^2}-\frac{256S_1}{9(N+1)}\Bigg]L_Q\Bigg] + C_F C_A T_F\Bigg[L_Q\Bigg[$$

$$-\frac{128(-1)^N Q_{11}}{15(N-2)(N-1)^2 N^2(N+1)^4(N+2)^2(N+3)^3} - \frac{256\big(N^4+2N^3-N^2-2N-6\big)S_{-2}}{3(N-2)N(N+1)^2(N+3)}$$

$$+\frac{16Q_{13}}{135(N-1)^2 N^2(N+1)^4(N+2)^2(N+3)^3} + \frac{1}{N+1}\Bigg[\frac{256}{3}S_{-2}S_1 + \frac{1088S_1}{9} + \frac{128S_3}{3}$$

$$+\frac{128}{3}S_{-3} - \frac{256}{3}S_{-2,1} - 128\zeta_3\Bigg]\Bigg] - \frac{352L_Q^2}{9(N+1)}\Bigg] + \hat{C}_{L,q}^{\mathsf{NS},(3)}(N_F)\Bigg\}\Bigg\}, \tag{344}$$

with

$$Q_{10} = 219N^6+657N^5+1193N^4+763N^3-40N^2-48N+72 \tag{345}$$

$$Q_{11} = 2N^{11}+41N^{10}+226N^9+556N^8+963N^7+2733N^6+7160N^5+8610N^4+1969N^3$$
$$-2748N^2-864N-216 \tag{346}$$

$$Q_{12} = 180N^{12}+2385N^{11}+11798N^{10}+23030N^9-10466N^8-131068N^7-245294N^6$$
$$-196786N^5-22282N^4+86571N^3+50688N^2-7236N-3888 \tag{347}$$

$$Q_{13} = 2345N^{12}+31510N^{11}+163614N^{10}+380250N^9+208092N^8-794874N^7-1604762N^6$$
$$-833938N^5+451419N^4+584028N^3+113724N^2-36288N+7776, \tag{348}$$

$$H_{q,L}^{\mathsf{PS}} = \tfrac{1}{2}[1+(-1)^N]$$

$$\times\Bigg\{a_s^2 C_F T_F\Bigg\{-\frac{32S_1\big(N^2+N+2\big)}{(N-1)N(N+1)^2(N+2)} + \frac{32L_Q\big(N^2+N+2\big)}{(N-1)N(N+1)^2(N+2)}$$

$$-\frac{32\big(N^5+2N^4+2N^3-5N^2-12N-4\big)}{(N-1)N^2(N+1)^3(N+2)^2}\Bigg\} + a_s^3\Bigg\{C_F^2 T_F\Bigg[\Bigg[\frac{64\big(N^2+N+1\big)\big(N^2+N+2\big)}{(N-1)N^2(N+1)^3(N+2)}$$

$$-\frac{64\big(N^2+N+2\big)S_1}{(N-1)N(N+1)^2(N+2)}\Bigg]L_Q^2 + \Bigg[\frac{128(-1)^N\big(N^2+N+2\big)Q_{16}}{15(N-2)(N-1)^3 N^3(N+1)^4(N+2)^2(N+3)^3}$$

$$-\frac{32Q_{18}}{15(N-1)^3 N^3(N+1)^4(N+2)^2(N+3)^3} + \frac{128\big(2N^5+5N^4+7N^3+2N^2-12N-8\big)S_1}{(N-1)N^2(N+1)^3(N+2)^2}$$

$$+\frac{\big(N^2+N+2\big)\big(64S_1^2-64S_2\big)}{(N-1)N(N+1)^2(N+2)} + \frac{128\big(N^2+N+2\big)S_{-2}}{(N-2)N(N+1)^2(N+3)}\Bigg]L_Q$$

$$-\frac{16\big(N^2+N+2\big)^2 L_M^2}{(N-1)N^2(N+1)^3(N+2)} + \frac{16Q_{17}}{(N-1)N^4(N+1)^5(N+2)^3} - \frac{32\big(N^2+N+2\big)^2 S_2}{(N-1)N^2(N+1)^3(N+2)}$$

$$-\frac{32\big(N^2+5N+2\big)\big(5N^3+7N^2+4N+4\big)L_M}{(N-1)N^3(N+1)^4(N+2)^2}\Bigg] + C_F T_F^2 N_F\Bigg[\frac{128L_Q^2\big(N^2+N+2\big)}{3(N-1)N(N+1)^2(N+2)}$$

$$-\frac{256L_Q\big(11N^5+35N^4+59N^3+55N^2-4N-12\big)}{9(N-1)N^2(N+1)^3(N+2)^2}\Bigg] + C_F T_F^2\Bigg[\frac{128\big(N^2+N+2\big)L_Q^2}{3(N-1)N(N+1)^2(N+2)}$$

$$-\frac{256\big(11N^5+35N^4+59N^3+55N^2-4N-12\big)L_Q}{9(N-1)N^2(N+1)^3(N+2)^2} + \frac{128\big(N^2+N+2\big)L_M^2}{3(N-1)N(N+1)^2(N+2)}$$

$$+\frac{128\left(43N^4+105N^3+224N^2+230N+86\right)}{27(N-1)N(N+1)^4(N+2)}-\frac{128\left(8N^3+13N^2+27N+16\right)S_1}{9(N-1)N(N+1)^3(N+2)}$$

$$+L_M\left[\frac{256\left(8N^3+13N^2+27N+16\right)}{9(N-1)N(N+1)^3(N+2)}-\frac{256\left(N^2+N+2\right)S_1}{3(N-1)N(N+1)^2(N+2)}\right]$$

$$+\frac{\left(N^2+N+2\right)\left[\frac{64}{3}S_1^2+\frac{64S_2}{3}\right]}{(N-1)N(N+1)^2(N+2)}\Bigg]$$

$$+C_FC_AT_F\Bigg[\Bigg[-\frac{32\left(N^2+N+2\right)\left(11N^4+22N^3-23N^2-34N-12\right)}{3(N-1)^2N^2(N+1)^3(N+2)^2}$$

$$-\frac{64\left(N^2+N+2\right)S_1}{(N-1)N(N+1)^2(N+2)}\Bigg]L_Q^2+\Bigg[\frac{128(-1)^NQ_{14}}{(N-1)N^2(N+1)^4(N+2)^3}$$

$$+\frac{64Q_{15}}{9(N-1)^2N^3(N+1)^3(N+2)^3}+\frac{128\left(N^4-N^3-4N^2-11N-1\right)S_1}{(N-1)^2N(N+1)^3(N+2)}$$

$$+\frac{\left(N^2+N+2\right)\left[128S_1^2-128S_2-256S_{-2}\right]}{(N-1)N(N+1)^2(N+2)}\Bigg]L_Q\Bigg]+\tilde{C}_{L,q}^{\mathsf{PS},(3)}\left(N_F+1\right)\Bigg\}\Bigg\},\tag{349}$$

with

$$Q_{14} = N^6+8N^5+30N^4+58N^3+65N^2+42N+8\tag{350}$$

$$Q_{15} = 142N^8+593N^7+801N^6+199N^5-1067N^4-900N^3+976N^2+1128N$$
$$+288\tag{351}$$

$$Q_{16} = N^{10}-13N^9-39N^8+222N^7+1132N^6+1787N^5+913N^4+392N^3+645N^2$$
$$-324N-108\tag{352}$$

$$Q_{17} = N^{10}+8N^9+29N^8+49N^7-11N^6-131N^5-161N^4-160N^3-168N^2$$
$$-80N-16\tag{353}$$

$$Q_{18} = 225N^{12}+2494N^{11}+9980N^{10}+14480N^9-11602N^8-68380N^7-86828N^6$$
$$-15080N^5+67401N^4+60334N^3-312N^2-33912N-12528,\tag{354}$$

and

$$H_{g,L}^{\mathsf{S}} = \tfrac{1}{2}[1+(-1)^N]$$

$$\times\Bigg\{a_sT_F\frac{16}{(N+1)(N+2)}+a_s^2\Bigg\{\frac{64L_MT_F^2}{3(N+1)(N+2)}+C_AT_F\Bigg[\frac{64(-1)^N\left(N^3+4N^2+7N+5\right)}{(N+1)^3(N+2)^3}$$

$$-\frac{32\left(2N^5+9N^4+5N^3-12N^2-20N-8\right)}{(N-1)N^2(N+1)^2(N+2)^3}+\frac{64\left(2N^3-2N^2-N-1\right)S_1}{(N-1)N(N+1)^2(N+2)}$$

$$+L_Q\Bigg[\frac{128\left(N^2+N+1\right)}{(N-1)N(N+1)^2(N+2)^2}-\frac{64S_1}{(N+1)(N+2)}\Bigg]+\frac{32S_1^2-32S_2-64S_{-2}}{(N+1)(N+2)}\Bigg]$$

$$+C_FT_F\Bigg[-\frac{16L_M\left(N^2+N+2\right)}{N(N+1)^2(N+2)}+\frac{16L_Q\left(N^2+N+2\right)}{N(N+1)^2(N+2)}$$

$$+\frac{16Q_{29}}{15(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}+\frac{64(-1)^NQ_{30}}{15(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}$$

$$-\frac{16\left(3N^2+3N+2\right)S_1}{N(N+1)^2(N+2)}+\frac{64(N-1)S_{-2}}{(N-2)(N+1)(N+3)}\Bigg]\Bigg\}$$

$$+a_s^3\Bigg\{\frac{256L_M^2T_F^3}{9(N+1)(N+2)}+C_AT_F^2\Bigg[\Bigg[\frac{256\left(N^2+N+1\right)}{3(N-1)N(N+1)^2(N+2)^2}-\frac{128S_1}{3(N+1)(N+2)}\Bigg]L_Q^2$$

$$
+ \left[\frac{256(-1)^N\left(N^3 + 4N^2 + 7N + 5\right)}{3(N+1)^3(N+2)^3} + \frac{64Q_{22}}{9(N-1)N(N+1)^3(N+2)^3} \right.
$$

$$
+ \frac{256\left(11N^3 - 6N^2 - 8N - 3\right)S_1}{9(N-1)N(N+1)^2(N+2)} + L_M \left[\frac{512\left(N^2 + N + 1\right)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256S_1}{3(N+1)(N+2)} \right]
$$

$$
+ \frac{\frac{128}{3}S_1^2 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \Bigg] L_Q + \frac{32Q_{27}}{27(N-1)N^3(N+1)^4(N+2)^2} - \frac{64(56N + 47)S_1}{27(N+1)^2(N+2)}
$$

$$
+ L_M^2 \left[\frac{256\left(N^2 + N + 1\right)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \right] + L_M \left[\frac{256(-1)^N\left(N^3 + 4N^2 + 7N + 5\right)}{3(N+1)^3(N+2)^3} \right.
$$

$$
\left. + \frac{128Q_{24}}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{256\left(N^3 - 6N^2 + 2N - 3\right)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\frac{128}{3}S_1^2 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \right] \Bigg]
$$

$$
+ C_A T_F^2 N_F \Bigg[\left[\frac{256\left(N^2 + N + 1\right)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \right] L_Q^2
$$

$$
+ \left[\frac{256(-1)^N\left(N^3 + 4N^2 + 7N + 5\right)}{3(N+1)^3(N+2)^3} + \frac{64Q_{22}}{9(N-1)N(N+1)^3(N+2)^3} \right.
$$

$$
\left. + \frac{256\left(11N^3 - 6N^2 - 8N - 3\right)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\frac{128}{3}S_1^2 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \right] L_Q \Bigg]
$$

$$
+ C_A^2 T_F \Bigg[\left[\frac{128S_1^2}{(N+1)(N+2)} + \frac{32\left(11N^4 + 22N^3 - 59N^2 - 70N - 48\right)S_1}{3(N-1)N(N+1)^2(N+2)^2} \right.
$$

$$
\left. - \frac{64\left(N^2 + N + 1\right)\left(11N^4 + 22N^3 - 35N^2 - 46N - 24\right)}{3(N-1)^2N^2(N+1)^3(N+2)^3} \right] L_Q^2
$$

$$
+ \left[-\frac{32\left(59N^4 + 70N^3 - 155N^2 - 118N - 72\right)S_1^2}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256(-1)^N\left(N^3 + 4N^2 + 7N + 5\right)S_1}{(N+1)^3(N+2)^3} \right.
$$

$$
- \frac{64Q_{28}S_1}{9(N-1)^2N^2(N+1)^3(N+2)^3} - \frac{64(-1)^NQ_{25}}{3(N-1)N^2(N+1)^4(N+2)^4}
$$

$$
- \frac{32Q_{31}}{9(N-1)^2N^3(N+1)^3(N+2)^4} + \frac{32\left(11N^4 + 22N^3 - 83N^2 - 94N - 72\right)S_2}{3(N-1)N(N+1)^2(N+2)^2}
$$

$$
+ \frac{64\left(11N^4 + 22N^3 - 59N^2 - 70N - 48\right)S_{-2}}{3(N-1)N(N+1)^2(N+2)^2} + \frac{1}{(N+1)(N+2)} \left[-128S_1^3 \right.
$$

$$
\left. + 384S_2S_1 + 512S_{-2}S_1 + 128S_3 + 128S_{-3} - 256S_{-2,1} \right] \Bigg] L_Q \Bigg]
$$

$$
+ C_F^2 T_F \Bigg[\left[\frac{8\left(N^2 + N + 2\right)\left(3N^2 + 3N + 2\right)}{N^2(N+1)^3(N+2)} - \frac{32\left(N^2 + N + 2\right)S_1}{N(N+1)^2(N+2)} \right] L_Q^2
$$

$$
+ \left[\frac{128(-1)^N\left(N^2 + N + 2\right)Q_{34}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} - \frac{8Q_{39}}{5(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} \right.
$$

$$
- \frac{16\left(9N^4 + 26N^3 + 49N^2 + 48N + 12\right)S_1}{N^2(N+1)^3(N+2)} + L_M \left[\frac{64\left(N^2 + N + 2\right)S_1}{N(N+1)^2(N+2)} \right.
$$

$$
\left. - \frac{16\left(N^2 + N + 2\right)\left(3N^2 + 3N + 2\right)}{N^2(N+1)^3(N+2)} \right] + \frac{256\left(N^2 + N + 2\right)\left(N^4 + 2N^3 - N^2 - 2N - 6\right)S_{-2}}{(N-2)N^2(N+1)^3(N+2)(N+3)}
$$

$$+\frac{\left(N^2+N+2\right)}{N(N+1)^2(N+2)}\left[64S_1^2-256S_{-2}S_1-64S_2-128S_3-128S_{-3}+256S_{-2,1}+384\zeta_3\right]\right]L_Q$$

$$+\frac{16(3N+2)S_1^2}{N^2(N+1)(N+2)}+\frac{8Q_{26}}{N^4(N+1)^5(N+2)}+\frac{16\left(N^4-N^3-20N^2-10N-4\right)S_1}{N^2(N+1)^3(N+2)}$$

$$+L_M^2\left[\frac{8\left(N^2+N+2\right)\left(3N^2+3N+2\right)}{N^2(N+1)^3(N+2)}-\frac{32\left(N^2+N+2\right)S_1}{N(N+1)^2(N+2)}\right]$$

$$+\frac{16\left(N^4+17N^3+17N^2-5N-2\right)S_2}{N^2(N+1)^3(N+2)}+\frac{\left(N^2+N+2\right)\left[-\frac{16}{3}S_1^3-16S_2S_1+\frac{64S_3}{3}\right]}{N(N+1)^2(N+2)}$$

$$+L_M\left[-\frac{128(-1)^N\left(N^2+N+2\right)Q_{34}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3}\right.$$

$$+\frac{8Q_{39}}{5(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3}+\frac{16\left(9N^4+26N^3+49N^2+48N+12\right)S_1}{N^2(N+1)^3(N+2)}$$

$$-\frac{256\left(N^2+N+2\right)\left(N^4+2N^3-N^2-2N-6\right)S_{-2}}{(N-2)N^2(N+1)^3(N+2)(N+3)}+\frac{\left(N^2+N+2\right)}{N(N+1)^2(N+2)}\left[-64S_1^2\right.$$

$$\left.\left.\left.+256S_{-2}S_1+64S_2+128S_3+128S_{-3}-256S_{-2,1}-384\zeta_3\right]\right]\right]$$

$$+C_FT_F^2\left[\frac{64\left(N^2+N+2\right)\left(N^4+2N^3+2N^2+N+6\right)L_Q^2}{3(N-1)N^2(N+1)^3(N+2)^2}+\left[-\frac{128L_M\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^3(N+2)^2}\right.\right.$$

$$+\frac{256(-1)^NQ_{30}}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}-\frac{32Q_{37}}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3}$$

$$\left.-\frac{128Q_{21}S_1}{3(N-1)N^2(N+1)^3(N+2)^2}+\frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)}\right]L_Q$$

$$-\frac{64(N-2)(N+3)\left(N^2+N+1\right)\left(N^2+N+2\right)L_M^2}{3(N-1)N^2(N+1)^3(N+2)^2}-\frac{16Q_{32}}{(N-1)N^4(N+1)^5(N+2)^2}$$

$$+L_M\left[\frac{256(-1)^NQ_{30}}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}+\frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)}\right.$$

$$\left.\left.+\frac{32Q_{38}}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3}-\frac{128Q_{19}S_1}{3(N-1)N^2(N+1)^3(N+2)^2}\right]\right]$$

$$+C_FT_F^2N_F\left[\frac{64\left(N^2+N+2\right)\left(N^4+2N^3+2N^2+N+6\right)L_Q^2}{3(N-1)N^2(N+1)^3(N+2)^2}\right.$$

$$+\left[\frac{256(-1)^NQ_{30}}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}-\frac{128Q_{21}S_1}{3(N-1)N^2(N+1)^3(N+2)^2}\right.$$

$$\left.-\frac{32Q_{37}}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3}+\frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)}-\frac{64\left(N^2+N+2\right)L_M}{3N(N+1)^2(N+2)}\right]L_Q$$

$$\left.\left.+L_M\left[\frac{32\left(N^2+N+2\right)\left(19N^2+7N-6\right)}{9N^2(N+1)^3(N+2)}+\frac{64\left(N^2+N+2\right)S_1}{3N(N+1)^2(N+2)}\right]\right]$$

$$+C_FC_AT_F\left[\left[-\frac{16\left(N^2+N+2\right)\left(11N^4+22N^3-23N^2-34N-12\right)}{3(N-1)N^2(N+1)^3(N+2)^2}-\frac{32\left(N^2+N+2\right)S_1}{N(N+1)^2(N+2)}\right]L_Q^2\right.$$

$$+\left[\frac{32\left(5N^2+5N+2\right)S_1^2}{N(N+1)^2(N+2)}-\frac{256(-1)^NQ_{30}S_1}{15(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}\right.$$

$$
+\frac{128Q_{33}S_1}{15(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3}-\frac{128\left(N^4+2N^3+N^2+12\right)S_{-2}S_1}{(N-2)N(N+1)^2(N+2)(N+3)}
$$

$$
-\frac{64(-1)^N Q_{41}}{45(N-2)(N-1)^3 N^3(N+1)^5(N+2)^3(N+3)^3}+\frac{8Q_{42}}{45(N-1)^3 N^3(N+1)^5(N+2)^3(N+3)^3}
$$

$$
-\frac{128Q_{23}S_{-2}}{3(N-2)N^2(N+1)^3(N+2)(N+3)}+\frac{176\left(N^2+N+2\right)L_M}{3N(N+1)^2(N+2)}
$$

$$
+\frac{\left(N^2+N+2\right)\left[-32S_2+64S_3+64S_{-3}-128S_{-2,1}-192\zeta_3\right]}{N(N+1)^2(N+2)}\Bigg]L_Q
$$

$$
-\frac{16\left(N^3+8N^2+11N+2\right)S_1^2}{N(N+1)^3(N+2)^2}+\frac{16Q_{35}}{(N-1)N^4(N+1)^5(N+2)^4}-\frac{16Q_{20}S_1}{N(N+1)^4(N+2)^3}
$$

$$
+L_M^2\left[\frac{32\left(N^2+N+2\right)S_1}{N(N+1)^2(N+2)}-\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{(N-1)N^2(N+1)^3(N+2)^2}\right]
$$

$$
-\frac{16\left(7N^5+21N^4+13N^3+21N^2+18N+16\right)S_2}{(N-1)N^2(N+1)^3(N+2)^2}+\frac{\left(N^2-N-4\right)64(-1)^N S_{-2}}{(N+1)^3(N+2)^2}
$$

$$
+\frac{\left(N^2+N+2\right)}{N(N+1)^2(N+2)}\left[\frac{16}{3}S_1^3+48S_2S_1+64(-1)^N S_{-2}S_1+\frac{128S_3}{3}+32(-1)^N S_{-3}-64S_{-2,1}\right]
$$

$$
+L_M\left[\frac{64(-1)^N Q_{36}}{5(N-2)(N-1)^2 N^3(N+1)^5(N+2)^3(N+3)^3}\right.
$$

$$
-\frac{8Q_{40}}{45(N-1)^2 N^3(N+1)^5(N+2)^3(N+3)^3}-\frac{16\left(23N^4+92N^3+209N^2+256N+92\right)S_1}{3N(N+1)^3(N+2)^2}
$$

$$
+\frac{64\left(N^2+N+2\right)\left(3N^4+6N^3-7N^2-10N-12\right)S_{-2}}{(N-2)N^2(N+1)^3(N+2)(N+3)}
$$

$$
\left.+\frac{\left(N^2+N+2\right)\left[32S_1^2-128S_{-2}S_1+32S_2-64S_3-64S_{-3}+128S_{-2,1}+192\zeta_3\right]}{N(N+1)^2(N+2)}\right]\Bigg]\Bigg]
$$

$$
+\tilde{C}_{L,g}^{\mathsf{S},(3)}(N_F+1)\Bigg\}\Bigg\}, \tag{355}
$$

with

$$
\begin{aligned}
Q_{19} &= N^6+3N^5-2N^4-9N^3-17N^2-12N-12 & (356)\\
Q_{20} &= N^6+8N^5+23N^4+54N^3+94N^2+72N+8 & (357)\\
Q_{21} &= 2N^6+6N^5+7N^4+4N^3+9N^2+8N+12 & (358)\\
Q_{22} &= 3N^6+3N^5-121N^4-391N^3-474N^2-308N-80 & (359)\\
Q_{23} &= 10N^6+30N^5+N^4-48N^3-89N^2-60N-36 & (360)\\
Q_{24} &= 6N^7+24N^6+47N^5+104N^4+219N^3+316N^2+208N+48 & (361)\\
Q_{25} &= 11N^8+66N^7+106N^6-121N^5-775N^4-1325N^3-1130N^2-552N-96 & (362)\\
Q_{26} &= 12N^8+52N^7+132N^6+216N^5+191N^4+54N^3-25N^2-20N-4 & (363)\\
Q_{27} &= 15N^8+60N^7+572N^6+1470N^5+2135N^4+1794N^3+722N^2-24N-72 & (364)\\
Q_{28} &= 133N^8+430N^7-271N^6-1361N^5+110N^4+2023N^3+1684N^2-12N-144 & (365)\\
Q_{29} &= 26N^9+539N^8+3244N^7+8465N^6+9342N^5+841N^4-5720N^3-2193N^2\\
&\quad +2484N+1404 & (366)\\
Q_{30} &= N^{10}-13N^9-39N^8+222N^7+1132N^6+1787N^5+913N^4+392N^3\\
&\quad +645N^2-324N-108 & (367)
\end{aligned}
$$

$$Q_{31} = 3N^{10} - 48N^9 - 856N^8 - 2702N^7 - 1961N^6 + 2142N^5 + 3122N^4 - 1924N^3$$
$$-5552N^2 - 4032N - 1152 \tag{368}$$

$$Q_{32} = 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3$$
$$-36N^2 - 32N - 16 \tag{369}$$

$$Q_{33} = 35N^{10} + 372N^9 + 1263N^8 + 673N^7 - 5090N^6 - 11596N^5 - 8413N^4 + 2305N^3$$
$$+8049N^2 + 3078N + 108 \tag{370}$$

$$Q_{34} = 2N^{11} + 41N^{10} + 226N^9 + 556N^8 + 963N^7 + 2733N^6 + 7160N^5 + 8610N^4 + 1969N^3$$
$$-2748N^2 - 864N - 216 \tag{371}$$

$$Q_{35} = 2N^{12} + 20N^{11} + 86N^{10} + 192N^9 + 199N^8 - N^7 - 297N^6 - 495N^5 - 514N^4 - 488N^3$$
$$-416N^2 - 176N - 32 \tag{372}$$

$$Q_{36} = 12N^{13} + 143N^{12} + 591N^{11} + 954N^{10} + 371N^9 + 1658N^8 + 11559N^7 + 26626N^6$$
$$+29129N^5 + 14011N^4 - 2374N^3 - 6576N^2 - 1944N - 432 \tag{373}$$

$$Q_{37} = 95N^{13} + 1218N^{12} + 6096N^{11} + 14484N^{10} + 11570N^9 - 28440N^8 - 117844N^7$$
$$-225884N^6 - 238953N^5 - 83290N^4 + 57660N^3 + 122040N^2 + 182304N$$
$$+77760 \tag{374}$$

$$Q_{38} = 185N^{13} + 2582N^{12} + 15584N^{11} + 53036N^{10} + 109190N^9 + 124040N^8 + 12604N^7$$
$$-200836N^6 - 294247N^5 - 116270N^4 + 85260N^3 + 158760N^2 + 193536N$$
$$+77760 \tag{375}$$

$$Q_{39} = 35N^{14} + 465N^{13} + 1962N^{12} - 348N^{11} - 32130N^{10} - 131686N^9 - 280396N^8$$
$$-363984N^7 - 290209N^6 - 122547N^5 + 6730N^4 + 47316N^3 + 11928N^2$$
$$-21600N - 5184 \tag{376}$$

$$Q_{40} = 1255N^{14} + 18165N^{13} + 107824N^{12} + 331744N^{11} + 515430N^{10} + 132498N^9$$
$$-1057432N^8 - 2202648N^7 - 1979173N^6 - 534079N^5 + 350880N^4 - 29088N^3$$
$$-519264N^2 - 382320N - 62208 \tag{377}$$

$$Q_{41} = 11N^{15} - 2N^{14} + 308N^{13} + 5275N^{12} + 24535N^{11} + 52925N^{10} + 50941N^9 - 5977N^8$$
$$-85550N^7 - 191059N^6 - 294877N^5 - 248414N^4 - 64728N^3 + 57636N^2 + 28944N$$
$$+6480 \tag{378}$$

$$Q_{42} = 1255N^{15} + 16338N^{14} + 76085N^{13} + 117654N^{12} - 198422N^{11} - 971844N^{10}$$
$$-1002678N^9 + 1019372N^8 + 3525323N^7 + 3236906N^6 + 272625N^5 - 1523746N^4$$
$$-632844N^3 + 606888N^2 + 635904N + 129600 \ . \tag{379}$$

The expressions in z-space are presented in Appendix C.

As has been outlined for the 2–loop results in Ref. [10] already, the scales at which the asymptotic expressions are dominating are estimated to be $Q^2/m^2 \gtrsim 800$. They are far outside the kinematic region in which the structure function $F_L(x, Q^2)$ can presently be measured in deep-inelastic scattering. The corresponding expressions are therefore of merely theoretical character and cannot be used in current phenomenological analyses.

6 Comparison of Mellin Moments for the Wilson Coefficients and OMEs

In order to compare the relative impact of the different Wilson coefficients on the structure function $F_2(x, Q^2)$ we will consider the Mellin moments for $N = 2$ to 10 in the following, folded

with the moments of the respective parton distribution functions in the flavor singlet case, i.e. the gluon $G(x, Q^2)$ and quark–singlet density $\Sigma(x, Q^2)$ for $N_F = 3$ and characteristic values of Q^2. Since only a series of Mellin moments has been calculated at large momentum transfer Q^2 in Ref. [12], a detailed numerical comparison is only possible in this way at the moment. The numerical results for the moments of the contributing parton densities are given in Table 2. Note that for $N \geq 2$ the moments for the singlet-distribution are mostly larger than those of the gluon. We apply these parton densities to study the relative contributions of the different Wilson coefficients, normalizing to $H_{g,2}^{\mathsf{S}}$ within the respective order in a_s using the following ratios:

$$R\left(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}}\right) = \frac{c_{N_F} L_{g,2}^{\mathsf{S}} G}{c_Q H_{g,2}^{\mathsf{S}} G}$$

$$R\left(L_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}}\right) = \frac{c_{N_F} L_{q,2}^{\mathsf{PS}} \Sigma}{c_Q H_{g,2}^{\mathsf{S}} G}$$

$$R\left(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}}\right) = \frac{c_Q H_{q,2}^{\mathsf{PS}} \Sigma}{c_Q H_{g,2}^{\mathsf{S}} G},$$

where

$$c_{N_F} = \frac{1}{N_F} \sum_{k=0}^{N_F} e_k^2, \qquad c_Q = e_Q^2. \tag{380}$$

In the numerical examples we set $e_Q = e_c = 2/3$.

Q^2	$20\,\mathrm{GeV}^2$				
N	2	4	6	8	10
G	0.4583	0.0044	0.0003	3.62×10^{-5}	7.78×10^{-6}
Σ	0.5417	0.0353	0.0070	2.10×10^{-3}	8.01×10^{-4}
Q^2	$100\,\mathrm{GeV}^2$				
N	2	4	6	8	10
G	0.4819	0.0038	0.0002	3.60×10^{-5}	8.55×10^{-6}
Σ	0.5181	0.0296	0.0056	1.61×10^{-3}	5.97×10^{-4}
Q^2	$1000\,\mathrm{GeV}^2$				
N	2	4	6	8	10
G	0.5042	0.0032	0.0002	3.14×10^{-5}	7.60×10^{-6}
Σ	0.4958	0.0244	0.0043	1.20×10^{-3}	4.32×10^{-3}

Table 2: The moments $N = 2, ..., 10$ of the gluon and quark-singlet momentum density using the parton distribution functions [5].

Q^2		20 GeV^2				
N		2	4	6	8	10
Σ/G		1.1821	7.9967	25.847	57.965	103.06
$O(a_s^2):$	$R(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}})$	0.0387	0.1349	1.7000	-0.2592	-0.1384
	$R(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	-0.2588	-0.3018	1.9946	-1.6153	-1.7222
$O(a_s^3):$	$R(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}})$	0.0829	0.1983	0.6628	-2.5018	-0.5957
	$R(L_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	0.0438	0.1042	0.7483	-5.7476	-2.3762
	$R(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	-0.2259	-0.3472	0.3387	-9.3371	-4.5870
Q^2		100 GeV^2				
Σ/G		1.0753	7.7514	22.797	44.660	69.888
$O(a_s^2)$	$R(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}})$	0.0313	0.0687	0.1071	0.1587	0.2418
	$R(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	-0.2496	-0.3753	-0.5429	-0.6531	-0.6743
$O(a_s^3)$	$R(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}})$	0.0533	0.0853	0.1449	0.2186	0.3195
	$R(L_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	0.0340	0.0378	0.1006	0.2600	0.5828
	$R(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	-0.3062	-0.5165	-0.7070	-0.7471	-0.5637
Q^2		1000 GeV^2				
Σ/G		0.9833	7.5948	20.958	38.236	56.876
$O(a_s^2)$	$R(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}})$	0.0243	0.0420	0.0531	0.0615	0.0687
	$R(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	-0.2837	-0.4085	-0.5690	-0.6707	-0.7237
$O(a_s^3)$	$R(L_{g,2}^{\mathsf{S}}, H_{g,2}^{\mathsf{S}})$	0.0337	0.0392	0.0597	0.0764	0.0907
	$R(L_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	0.0313	0.0209	0.0297	0.0505	0.0828
	$R(H_{q,2}^{\mathsf{PS}}, H_{g,2}^{\mathsf{S}})$	-0.3825	-0.5903	-0.8058	-0.9253	-0.9679

Table 3: Relative impact of the moments $N = 2, ..., 10$ of the individual massive Wilson coefficients, weighted by moments of the corresponding parton distributions [5], at $O(a_s^2)$ and $O(a_s^3)$ normalized to the contribution to $H_{g,2}^{\mathsf{S}}$ for $Q^2 = 20, 100$ and 1000 GeV^2.

Before we discuss quantitative results, a remark on the contributions by the color factor $d_{abc}d_{abc}$ to the massless Wilson coefficients Refs. [62–64] and [11, 57] used in the present analysis, is in order. For $SU(N)$ one obtains

$$d_{abc}d_{abc} = \frac{(N^2 - 1)(N^2 - 4)}{N} \ . \tag{381}$$

It emerges weighted by $1/N_c$ and $1/N_A$ for external quark and gluon lines, respectively, with $N_c = N$ and $N_A = N^2 - 1$. In Refs. [62–64] this group-theoretic expression has been used, while in [11, 57] a factor of 16 has been taken out and was absorbed into the Lorentz structure of the corresponding contribution to the Wilson coefficient. We agree with the N_F-dependence as given in Refs. [62–64]. Furthermore, we note a typographical error in Eq. (4.13) of [11]. Here,

the corresponding term reads correctly[12]

$$c_{2,g}^{(3)}(x) \simeq -932.089 N_F \frac{L_0}{x}..., \qquad \text{with} \ \ L_0 = \ln(x) \ . \tag{382}$$

Also in the pure-singlet case the massless Wilson coefficients contain terms $\propto d_{abc}d_{abc}$, although with a generally different charge-weight factor, cf. [62–64].

Q^2		$20\,\mathrm{GeV}^2$
N		2	4	6	8	10
Σ/G		1.1821	7.9967	25.847	57.965	103.06
$O(a_s):$	$R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.8182	-2.5455	-3.2432	-3.9286
$O(a_s^2):$	$R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.6395	-2.3808	-3.1781	-4.0262
	$R(A_{Qq}^{\mathsf{PS}}, A_{Qg})$	-0.1259	-0.3656	-0.7822	-1.3339	-1.9352
	$R(A_{qq,Q}^{\mathsf{NS}}, A_{Qg})$	-0.0584	-1.1306	-6.3206	-20.735	-49.508
	$R(A_{gq,Q}, A_{Qg})$	0.1843	1.1422	3.9956	9.8073	18.995
$O(a_s^3):$	$R(A_{gg,Q}, A_{Qg})$	-1.0051	-1.3397	-1.8466	-2.4306	-3.0890
	$R(A_{Qq}^{\mathsf{PS}}, A_{Qg})$	-0.1604	-0.4838	-0.9635	-1.5449	-2.1295
	$R(A_{qq,Q}^{\mathsf{NS}}, A_{Qg})$	-0.0404	-0.5832	-2.9406	-9.1817	-21.375
	$R(A_{gq,Q}, A_{Qg})$	0.1473	1.1265	3.7972	8.9925	16.961
	$R(A_{qg,Q}, A_{Qg})$	0.0051	-0.0202	-0.0326	-0.0445	-0.0567
	$R(A_{qq,Q}^{\mathsf{PS}}, A_{Qg})$	0.0534	0.1093	0.2460	0.4678	0.7619
Q^2		$100\,\mathrm{GeV}^2$
Σ/G		1.0753	7.7514	22.797	44.660	69.888
$O(a_s):$	$R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.8182	-2.5455	-3.2432	-3.9286
$O(a_s^2):$	$R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.7746	-2.6448	-3.5805	-4.5771
	$R(A_{Qq}^{\mathsf{PS}}, A_{Qg})$	-0.1884	-0.4246	-0.7627	-1.0887	-1.3514
	$R(A_{qq,Q}^{\mathsf{NS}}, A_{Qg})$	-0.1247	-2.4385	-12.377	-35.460	-74.510
	$R(A_{gq,Q}, A_{Qg})$	0.3131	1.3950	3.9075	7.7692	12.560
$O(a_s^3):$	$R(A_{gg,Q}, A_{Qg})$	-1.0048	-1.6120	-2.3201	-3.0734	-3.8667
	$R(A_{Qq}^{\mathsf{PS}}, A_{Qg})$	-0.2799	-0.5808	-0.9473	-1.2540	-1.4615
	$R(A_{qq,Q}^{\mathsf{NS}}, A_{Qg})$	-0.1772	-2.8698	-13.694	-37.928	-77.874
	$R(A_{gq,Q}, A_{Qg})$	0.3924	1.4140	3.4078	6.0520	8.9253
	$R(A_{qg,Q}, A_{Qg})$	0.0048	-0.0375	-0.0491	-0.0580	-0.0657
	$R(A_{qq,Q}^{\mathsf{PS}}, A_{Qg})$	0.0647	0.0984	0.1726	0.2553	0.3319

Table 4: Relative impact of the moments $N = 2, ..., 10$ of the individual massive OMEs, weighted by moments of the corresponding parton distributions [5], at the different orders in a_s normalized to the contribution to A_{Qg} for $Q^2 = 20$ and $100\,\mathrm{GeV}^2$.

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[12]The expression in the parameterization given at `http://www.liv.ac.uk/~avogt/` is correct, however.

Let us now consider the relative impact of the individual massive Wilson coefficients. The ratios at $O(a_s^2)$ and $O(a_s^3)$ for different values of Q^2 and the moments $N = 2$ to 10 are given in Table 3. One first notes that at low values of Q^2 the moments of $L_{g,2}^{\sf S}$ change sign, which is also the case for $H_{q,2}^{\sf PS}$ in the whole region up to $Q^2 = 1000$ GeV^2. At $O(a_s^2)$ $L_{g,2}^{\sf S}$ is small for low moments and grows 24% for $N = 10$ compared to $H_{g,2}^{\sf S}$ at $Q^2 = 100$ GeV^2, with lower values at higher Q^2. A comparable tendency is observed at $O(a_s^2)$. The fraction $|R(H_{q,2}^{\sf PS}, H_{g,2}^{\sf S})|$ moves between 25% and 170% comparing the moments $N = 2$ to 10 at $Q^2 = 20$ GeV^2 and upper values of $\sim 70\%$ at $Q^2 = 1000$GeV^2.

In the case of the comparison of the massive OMEs we normalize to A_{Qg} with PDFs according to their appearance in the singlet and gluon transitions from $N_F \to N_F + 1$ massless flavors in the variable flavor number scheme, cf. Eqs. (33–35):

$$R(A_{gg,Q}, A_{Qg}) = \frac{A_{gg,Q}\, G}{A_{Qg}\, G} \qquad\qquad R(A_{Qq}^{\sf PS}, A_{Qg}) = \frac{A_{Qq}^{\sf PS}\, \Sigma}{A_{Qg}\, G}$$

$$R(A_{qq,Q}^{\sf NS}, A_{Qg}) = \frac{A_{qq,Q}^{\sf NS}\, \Sigma}{A_{Qg}\, G} \qquad\qquad R(A_{gq,Q}, A_{Qg}) = \frac{A_{gq,Q}\, \Sigma}{A_{Qg}\, G}$$

$$R(A_{qg,Q}, A_{Qg}) = \frac{A_{qg,Q}\, G}{A_{Qg}\, G} \qquad\qquad R(A_{qq,Q}^{\sf PS}, A_{Qg}) = \frac{A_{qq,Q}^{\sf PS}\, \Sigma}{A_{Qg}\, G}\;.$$

These ratios describe the relative impact, within the corresponding order in a_s, of the massive OMEs in the variable flavor number scheme for the flavor singlet contributions.

Q^2		1000 GeV^2				
N		2	4	6	8	10
Σ/G		0.9833	7.5948	20.958	38.236	56.876
$O(a_s):$	$R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.8182	-2.5455	-3.2432	-3.9286
$O(a_s^2)$	$R(A_{gg,Q}, A_{Qg})$	-1.0000	-2.0101	-3.0170	-4.0596	-5.1403
	$R(A_{Qq}^{\sf PS}, A_{Qg})$	-0.2555	-0.4521	-0.6997	-0.8850	-1.0072
	$R(A_{qq,Q}^{\sf NS}, A_{Qg})$	-0.2048	-3.7111	-16.978	-44.037	-85.945
	$R(A_{gq,Q}, A_{Qg})$	0.4603	1.5427	3.5816	6.1302	8.8857
$O(a_s^3):$	$R(A_{gg,Q}, A_{Qg})$	-1.0054	-1.7515	-2.4980	-3.2560	-4.0291
	$R(A_{Qq}^{\sf PS}, A_{Qg})$	-0.3731	-0.6275	-0.9067	-1.0906	-1.1928
	$R(A_{qq,Q}^{\sf NS}, A_{Qg})$	-0.2930	-4.1599	-17.532	-43.424	-82.110
	$R(A_{gq,Q}, A_{Qg})$	0.5934	1.5060	2.9692	4.5100	5.9397
	$R(A_{qg,Q}, A_{Qg})$	0.0054	-0.0469	-0.0561	-0.0624	-0.0675
	$R(A_{qq,Q}^{\sf PS}, A_{Qg})$	0.0727	0.0902	0.1341	0.1721	0.2011

Table 5: The same as Table 4 for $Q^2 = 1000$GeV^2.

The numerical values for different scales of Q^2 are given in Tables 4 and 5. $|A_{gg,Q}/A_{Qg}|$ rises from about 1 to higher values from $N = 2$ to 10, irrespectively of the values of Q^2 and the order in a_s. The smallest contributions are $|A_{qg,Q}|$ and $A_{qq,Q}^{\sf PS}$ contributing the ratios R by $\sim 0.5\%$ to 5% and

form ~ 5 to $\sim 10\%$, respectively, for $N = 2$ and 4, i.e. in the region dominated by lower values of the Bjorken variable x. The OMEs $|A_{Q,q}^{\mathsf{PS}}|$ and $A_{gq,Q}$ have contributions of 16–62% and 14–150%, respectively, for $N = 2$ and 4. Also the flavor non-singlet Wilson coefficient $|A_{qq,Q}^{\mathsf{NS}}|$ contributes in the flavor singlet transitions and is weighted by the distribution Σ here. Its relative impact rises with Q^2 and amounts from $\sim 4\%$ to 370% for the R-ratio considering the lower moments $N = 2$ and $N = 4$ only.

Right after having obtained a series of moments for the massive OMEs at 3–loops in [12], it became clear that the logarithmic contributions are of comparable order to the constant term. Moreover, there is a strong functional dependence w.r.t. N, as displayed in Tables 2–5. To obtain a definite answer, the calculation of the constant parts of the unrenormalized OMEs $a_{ij}^{(3)}$ as a function of $N \in \mathbb{C}$ is necessary. In particular predictions for the range of small values $x \simeq 10^{-4}$ appear to be rather difficult otherwise.

7 Conclusions

We have derived the contributions of the massive Wilson coefficients to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in deep-inelastic scattering and the corresponding massive OMEs to 3–loop order in the asymptotic region $Q^2 \gg m^2$ both in Mellin–N and z–space except for the constant parts a_{ij} of the unrenormalized OMEs, which are not known for all quantities yet. Here, we retained both the scale-dependence due to the virtuality Q^2 and the factorization and renormalization scales μ^2, which were set equal. This allows for scale variation studies in applications. Two of the Wilson coefficients, $L_{q,2}^{\mathsf{PS}}$ and $L_{g,2}^{\mathsf{S}}$, are known in complete form, and the corresponding results for $L_{q,2}^{\mathsf{NS}}$ will be given in [40]. In the variable flavor number scheme being applied to describe the process through which an initially massive quark transmutes into a massless one at high momentum scales, moreover, the matching coefficients A_{ij} are needed. Here, $A_{qq,Q}^{\mathsf{PS}}$ and $A_{qg,Q}$ are known in complete form to 3–loop order and the results for A_{gq} and $A_{qq,Q}^{\mathsf{NS}}$ are given in [65] and [40], respectively.

We have given numerical results for the Wilson coefficients $L_{q,2}^{\mathsf{PS}}$ and $L_{g,2}^{\mathsf{S}}$. Using the available Mellin moments we have performed a numerical comparison of the different Wilson coefficients and operator matrix elements inside the respective order in the coupling constant for the moments $N = 2$ to 10 and in the Q^2 range between 20 and 1000 GeV^2. While some of the quantities studied are of minor importance, several others of the Wilson coefficients and OMEs are of similar size, which is varying in the kinematic range of experimental interest for present and future precision measurements. Even in case of the charm-quark contributions the logarithmic terms are not dominant over the constant contributions in wide kinematic ranges, as. e.g. at HERA.

The expression which were derived in the present paper are available in form of computer-readable files on request via e-mail to `Johannes.Bluemlein@desy.de`.

A   The massive operator matrix elements in N-space

In this appendix we present the massive OMEs in Mellin–space to be used in the matching coefficients in the variable flavor number scheme Eqs. (32–35). The corresponding representations in z–space are given in Appendix D. Thus far the OMEs $A_{qq,Q}^{\mathsf{PS}}$ and $A_{qg,Q}$ are known completely. The other OMEs are presented except for the 3–loop constant part a_{ij} in the unrenormalized OMEs. The OMEs $A_{qq,Q}^{\mathsf{NS}}$ and $A_{gq,Q}^{\mathsf{S}}$ are presented elsewhere [40, 65].

The transition matrix elements are given by $A_{qq,Q}^{\mathsf{PS}}$ and $A_{qg,Q}$:

$$A_{qq,Q}^{\mathsf{PS}} = \tfrac{1}{2}[1 + (-1)^N]$$

$$\times \left\{ a_s^3 C_F N_F T_F^2 \left\{ L_M^2 \left[\frac{32(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} - \frac{32 P_{280}}{9(N-1)N^3(N+1)^3(N+2)^2} \right] \right.\right.$$

$$+ \left[-\frac{32 P_{282}}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64 P_{280} S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right.$$

$$\left. + \frac{(N^2 + N + 2)^2 \left[-\frac{32}{3} S_1^2 - \frac{32 S_2}{3} \right]}{(N-1)N^2(N+1)^2(N+2)} \right] L_M - \frac{32 P_{284}}{243(N-1)N^5(N+1)^5(N+2)^4}$$

$$+ \frac{32 P_{283} S_1}{81(N-1)N^4(N+1)^4(N+2)^3} - \frac{32(N^2 + N + 2)^2}{9(N-1)N^2(N+1)^2(N+2)} L_M^3$$

$$\left.\left.+ \frac{P_{281}\left[-\frac{16}{27} S_1^2 - \frac{16 S_2}{27} \right]}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{(N^2 + N + 2)^2 \left[\frac{80}{27} S_1^3 + \frac{80}{9} S_2 S_1 + \frac{160 S_3}{27} + \frac{256 \zeta_3}{9} \right]}{(N-1)N^2(N+1)^2(N+2)} \right\} \right\}, \quad (383)$$

with

$$\begin{aligned}
P_{280} &= 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 & (384) \\
P_{281} &= 40N^7 + 185N^6 + 430N^5 + 521N^4 + 452N^3 + 404N^2 - 16N - 96 & (385) \\
P_{282} &= 95N^{10} + 712N^9 + 2379N^8 + 4269N^7 + 4763N^6 + 4569N^5 + 3309N^4 + 200N^3 \\
&\quad -808N^2 - 48N + 144 & (386) \\
P_{283} &= 233N^{10} + 1744N^9 + 5937N^8 + 11454N^7 + 14606N^6 + 15396N^5 + 12030N^4 + 3272N^3 \\
&\quad -928N^2 - 96N + 288 & (387) \\
P_{284} &= 1330N^{13} + 13931N^{12} + 66389N^{11} + 187681N^{10} + 354532N^9 + 492456N^8 + 532664N^7 \\
&\quad +423970N^6 + 204541N^5 + 34274N^4 - 11704N^3 - 3408N^2 - 1008N - 864 & (388)
\end{aligned}$$

and

$$A_{qg,Q} = \tfrac{1}{2}[1 + (-1)^N]$$

$$\times \left\{ a_s^3 \left\{ C_F N_F T_F^2 \left[\left[\frac{8(N^2 + N + 2)P_{285}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9} \tilde{\gamma}_{qg}^0 S_1 \right] L_M^3 \right.\right.\right.$$

$$+ \left[\frac{4 P_{291}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{32(5N^3 + 8N^2 + 19N + 6)S_1}{9N^2(N+1)(N+2)} + \tilde{\gamma}_{qg}^0 \left[-\frac{4}{3} S_1^2 - \frac{4 S_2}{3} \right] \right] L_M^2$$

$$+ \left[\frac{16(10N^3 + 13N^2 + 29N + 6)S_1^2}{9N^2(N+1)(N+2)} - \frac{16(103N^4 + 257N^3 + 594N^2 + 524N + 120)S_1}{27N^2(N+1)^2(N+2)} \right.$$

$$+ \frac{4 P_{293}}{27(N-1)N^5(N+1)^5(N+2)^4} + \frac{16(10N^3 + 25N^2 + 29N + 6)S_2}{9N^2(N+1)(N+2)}$$

$$\left. + \tilde{\gamma}_{qg}^0 \left[\frac{4}{9} S_1^3 + \frac{4}{3} S_2 S_1 - \frac{16 S_3}{9} \right] \right] L_M + \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)S_1^2}{81N^2(N+1)^2(N+2)}$$

$$- \frac{64}{9} \frac{(N^2 + N + 2)P_{285}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{P_{295}}{243(N-1)N^6(N+1)^6(N+2)^5}$$

$$- \frac{16(1523N^5 + 5124N^4 + 11200N^3 + 14077N^2 + 7930N + 1344)S_1}{243N^2(N+1)^3(N+2)}$$

$$+ \frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)S_2}{27N^2(N+1)^2(N+2)} + \frac{(10N^3 + 13N^2 + 29N + 6)\left[-\frac{16}{81} S_1^3 - \frac{16}{27} S_2 S_1 \right]}{N^2(N+1)(N+2)}$$

$$
+\frac{32\left(5N^3-16N^2+N-6\right)S_3}{81N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[-\frac{1}{27}S_1^4-\frac{2}{9}S_2S_1^2-\frac{8}{27}S_3S_1-\frac{64}{9}\zeta_3S_1-\frac{1}{9}S_2^2+\frac{14S_4}{9}\right]\right]
$$

$$
+C_AN_FT_F^2\left[L_M^3\left[-\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{8}{9}\tilde{\gamma}_{qg}^0S_1\right]\right.
$$

$$
+\left[\frac{8P_{289}}{9(N-1)N^2(N+1)^3(N+2)^3}+\frac{32\left(5N^4+20N^3+47N^2+58N+20\right)S_1}{9N(N+1)^2(N+2)^2}\right.
$$

$$
+\tilde{\gamma}_{qg}^0\left[\frac{4}{3}S_1^2+\frac{4S_2}{3}+\frac{8}{3}S_{-2}\right]\right]L_M^2+\left[-\frac{32\left(5N^4+20N^3+41N^2+49N+20\right)S_1^2}{9N(N+1)^2(N+2)^2}\right.
$$

$$
+\frac{16P_{292}}{27(N-1)N^4(N+1)^4(N+2)^4}-\frac{32\left(5N^4+26N^3+47N^2+43N+20\right)S_2}{9N(N+1)^2(N+2)^2}
$$

$$
+\frac{16P_{286}S_1}{27N(N+1)^3(N+2)^3}-\frac{64\left(5N^2+8N+10\right)S_{-2}}{9N(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[-\frac{4}{9}S_1^3+\frac{4}{3}S_2S_1-\frac{8S_3}{9}-\frac{16}{3}S_{-3}\right.
$$

$$
\left.\left.-\frac{16}{3}S_{2,1}\right]\right]L_M-\frac{16P_{287}S_1^2}{81N(N+1)^3(N+2)^3}+\frac{8P_{294}}{243(N-1)N^5(N+1)^5(N+2)^5}
$$

$$
+\frac{512}{9}\left(N^2+N+1\right)\left(N^2+N+2\right)\frac{\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{16P_{290}S_1}{243(N-1)N^2(N+1)^4(N+2)^4}
$$

$$
-\frac{16P_{288}S_2}{81N(N+1)^3(N+2)^3}+\frac{64\left(5N^4+38N^3+59N^2+31N+20\right)S_3}{81N(N+1)^2(N+2)^2}
$$

$$
-\frac{32\left(121N^3+293N^2+414N+224\right)S_{-2}}{81N(N+1)^2(N+2)}+\frac{128\left(5N^2+8N+10\right)S_{-3}}{27N(N+1)(N+2)}
$$

$$
+\frac{\left(5N^4+20N^3+41N^2+49N+20\right)\left[\frac{32}{81}S_1^3-\frac{32}{27}S_2S_1+\frac{128}{27}S_{2,1}\right]}{N(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\left[\frac{1}{27}S_1^4-\frac{2}{9}S_2S_1^2\right.
$$

$$
\left.\left.\left.\left.+\left[\frac{16}{9}S_{2,1}-\frac{40S_3}{27}\right]S_1+\frac{64}{9}\zeta_3S_1+\frac{1}{9}S_2^2+\frac{14S_4}{9}+\frac{32}{9}S_{-4}+\frac{32}{9}S_{3,1}-\frac{16}{9}S_{2,1,1}\right]\right]\right\}\right\},\tag{389}
$$

with the polynomials

$$
\begin{array}{rll}
P_{285} & = & 3N^6+9N^5-N^4-17N^3-38N^2-28N-24 \tag{390}\\
P_{286} & = & 94N^6+631N^5+2106N^4+4243N^3+4878N^2+2812N+680 \tag{391}\\
P_{287} & = & 103N^6+694N^5+2148N^4+3991N^3+4494N^2+2704N+680 \tag{392}\\
P_{288} & = & 139N^6+1093N^5+3438N^4+5776N^3+5724N^2+3220N+752 \tag{393}\\
P_{289} & = & 9N^8+54N^7+56N^6-182N^5-717N^4-1120N^3-1012N^2-672N-160 \tag{394}\\
P_{290} & = & 1244N^{10}+10557N^9+40547N^8+90323N^7+114495N^6+49344N^5-69902N^4\\
& & -115200N^3-64352N^2-11264N+864 \tag{395}\\
P_{291} & = & 33N^{11}+231N^{10}+698N^9+1290N^8+1513N^7+1463N^6+2236N^5+5096N^4+7328N^3\\
& & +5456N^2+3456N+1152 \tag{396}\\
P_{292} & = & 99N^{12}+891N^{11}+2902N^{10}+3392N^9-4300N^8-20914N^7-33059N^6-28357N^5\\
& & -11406N^4+3840N^3+7568N^2+4176N+864 \tag{397}\\
P_{293} & = & 159N^{14}+1590N^{13}+7223N^{12}+20982N^{11}+43703N^{10}+65162N^9+62553N^8+30282N^7\\
& & -28286N^6-145968N^5-257720N^4-241760N^3-158112N^2-73728N-17280 \tag{398}\\
P_{294} & = & 3315N^{15}+39780N^{14}+194011N^{13}+471164N^{12}+416251N^{11}-860568N^{10}-3525799N^9\\
& & -6015120N^8-6333994N^7-4373672N^6-1907512N^5-499824N^4-217952N^3\\
& & -264192N^2-160128N-34560 \tag{399}
\end{array}
$$

$$P_{295} = 13923N^{17} + 180999N^{16} + 1064857N^{15} + 3812487N^{14} + 9348807N^{13} + 16391845N^{12}$$
$$+ 20248499N^{11} + 17070917N^{10} + 11536274N^9 + 11303496N^8 + 13846104N^7 + 16104128N^6$$
$$+ 22643488N^5 + 29337472N^4 + 26395008N^3 + 15388416N^2 + 5612544N + 995328 \,. \quad (400)$$

Next we present the OMEs, which are known except for the constant term in the unrenormalized massive OME at 3–loop order, $a_{ij}^{(3)}$. The matrix element A_{Qq}^{PS} is given by :

$$A_{Qq}^{\mathsf{PS}} = \tfrac{1}{2}[1 + (-1)^N]$$

$$\times \left\{ a_s^2 C_F T_F \left\{ -\frac{4L_M^2(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{8S_2(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right.$$

$$\left. + \frac{4P_{312}}{(N-1)N^4(N+1)^4(N+2)^3} - \frac{8(N^2+5N+2)(5N^3+7N^2+4N+4)L_M}{(N-1)N^3(N+1)^3(N+2)^2} \right\}$$

$$+ a_s^3 \left\{ T_F C_F^2 \left[\left[\frac{4(N^2+N+2)^2(3N^2+3N+2)}{3(N-1)N^3(N+1)^3(N+2)} - \frac{16(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] L_M^3 \right. \right.$$

$$+ \left[-\frac{8(5N^2+N-2)S_1(N^2+N+2)^2}{(N-1)N^3(N+1)^3(N+2)} + \frac{16S_2(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right.$$

$$\left. + \frac{4P_{297}(N^2+N+2)}{(N-1)N^4(N+1)^4(N+2)} \right] L_M^2 + \left[\frac{\left[\tfrac{8}{3}S_1^3 - 24S_2 S_1 - \tfrac{80S_3}{3} + 32S_{2,1} + 96\zeta_3 \right](N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right.$$

$$- \frac{4(5N^3+4N^2+9N+6)S_1^2(N^2+N+2)}{(N-1)N^2(N+1)^3(N+2)} - \frac{4P_{321}}{(N-1)N^5(N+1)^5(N+2)^3}$$

$$\left. + \frac{8P_{313}S_1}{(N-1)N^4(N+1)^4(N+2)^3} - \frac{4P_{307}S_2}{(N-1)N^3(N+1)^3(N+2)^2} \right] L_M - \frac{2(N^2+N+2)\zeta_2 P_{304}}{(N-1)N^4(N+1)^4(N+2)}$$

$$- \frac{4(N^2+N+2)(N^4-5N^3-32N^2-18N-4)S_1^2}{(N-1)N^3(N+1)^3(N+2)} + \frac{4P_{323}}{(N-1)N^6(N+1)^6(N+2)^3}$$

$$- \frac{4}{3}\frac{(N^2+N+2)^2(3N^2+3N+2)\zeta_3}{(N-1)N^3(N+1)^3(N+2)} + \frac{4(N^2+N+2)(5N^4+4N^3+N^2-10N-8)\zeta_2}{(N-1)N^3(N+1)^3(N+2)}S_1$$

$$+ \frac{8(N^2+N+2)(2N^5-2N^4-11N^3-19N^2-44N-12)S_1}{(N-1)N^3(N+1)^4(N+2)} - \frac{4(N^2+N+2)P_{303}S_2}{(N-1)N^4(N+1)^4(N+2)}$$

$$+ \frac{(3N+2)(N^2+N+2)\left[-\tfrac{8}{3}S_1^3 - 8S_2 S_1 \right]}{(N-1)N^3(N+1)(N+2)} - \frac{8(N^2+N+2)(3N^4+48N^3+43N^2-22N-8)S_3}{3(N-1)N^3(N+1)^3(N+2)}$$

$$+ \frac{32(N^2-3N-2)(N^2+N+2)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)} + \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\tfrac{2}{3}S_1^4 + 4S_2 S_1^2 \right.$$

$$\left. + \left(\tfrac{16S_3}{3} + 32S_{2,1} \right)S_1 + \tfrac{16}{3}\zeta_3 S_1 + 2S_2^2 - 12S_4 + 32S_{3,1} - 64S_{2,1,1} + (4S_1^2 - 12S_2)\zeta_2 \right] \right]$$

$$+ C_F T_F^2 \left[-\frac{128(N^2+N+2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} + \left[\frac{32(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right. \right.$$

$$\left. - \frac{32P_{298}}{9(N-1)N^3(N+1)^2(N+2)^2} \right] L_M^2 + \left[\frac{\left(-\tfrac{32}{3}S_1^2 - 32S_2 \right)(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right.$$

$$\left. - \frac{64P_{315}}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64P_{309}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right] L_M$$

$$+\frac{16\left(N^2+N+2\right)\left(8N^3+13N^2+27N+16\right)S_1^2}{9(N-1)N^2(N+1)^3(N+2)}+\frac{32}{9}\frac{P_{298}\zeta_2}{(N-1)N^3(N+1)^2(N+2)^2}$$

$$+\frac{32P_{322}}{81(N-1)N^5(N+1)^5(N+2)^4}-\frac{32\left(N^2+N+2\right)\left(43N^4+105N^3+224N^2+230N+86\right)S_1}{27(N-1)N^2(N+1)^4(N+2)}$$

$$+\frac{16P_{310}S_2}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{\left(N^2+N+2\right)^2\left[-\frac{16}{9}S_1^3-\frac{16}{3}S_2S_1-\frac{32}{3}\zeta_2S_1+\frac{160S_3}{9}+\frac{128\zeta_3}{9}\right]}{(N-1)N^2(N+1)^2(N+2)}\Bigg]$$

$$+C_FT_F^2N_F\Bigg[-\frac{16}{9}\frac{P_{326}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}+L_M^2\Bigg[\frac{32P_{327}}{9(N-1)N^3(N+1)^3(N+2)^2}$$

$$-\frac{32\left(N^2+N+2\right)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Bigg]+L_M\Bigg[-\frac{32P_{328}}{27(N-1)N^4(N+1)^4(N+2)^3}$$

$$+\frac{32P_{326}S_1}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{\left(N^2+N+2\right)^2\left[-\frac{16}{3}S_1^2-\frac{80S_2}{3}\right]}{(N-1)N^2(N+1)^2(N+2)}\Bigg]$$

$$-\frac{32P_{329}}{3(N-1)N^5(N+1)^5(N+2)^4}+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\frac{16}{3}\zeta_2S_1+\frac{64S_3}{3}+\frac{32\zeta_3}{9}\right]$$

$$-\frac{32\left(N^2+N+2\right)^2L_M^3}{9(N-1)N^2(N+1)^2(N+2)}+\frac{64\left(N^2+5N+2\right)\left(5N^3+7N^2+4N+4\right)S_2}{3(N-1)N^3(N+1)^3(N+2)^2}\Bigg]$$

$$+C_FC_AT_F\Bigg[\frac{8\left(N^2+N+2\right)\left(N^3+8N^2+11N+2\right)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2}+\frac{4\left(N^2+N+2\right)P_{296}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3}$$

$$-\frac{4}{3}\frac{\left(N^2+N+2\right)P_{301}\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2}S_1-\frac{8\left(N^2+N+2\right)P_{311}S_1}{(N-1)N^2(N+1)^5(N+2)^4}$$

$$-\frac{8\left(N^2+N+2\right)\left(3N^3-12N^2-27N-2\right)S_2S_1}{(N-1)N^2(N+1)^3(N+2)^2}-\frac{8}{9}\frac{\left(N^2+N+2\right)P_{299}\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$+\frac{4}{9}\frac{P_{320}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3}+\frac{8P_{325}}{3(N-1)^2N^6(N+1)^6(N+2)^5}$$

$$+L_M^3\Bigg[\frac{8\left(11N^4+22N^3-23N^2-34N-12\right)\left(N^2+N+2\right)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2}+\frac{16S_1\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)}\Bigg]$$

$$+\frac{4P_{317}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^3}-\frac{16\left(N^2+N+2\right)P_{300}S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$+L_M^2\Bigg[\frac{\left[16S_2+32S_{-2}\right]\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}+\frac{8P_{302}S_1\left(N^2+N+2\right)}{3(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$-\frac{8P_{319}}{9(N-1)^2N^4(N+1)^4(N+2)^3}\Bigg]$$

$$+\frac{\left(N^2+N+2\right)\left(N^4+2N^3+7N^2+22N+20\right)\left[32(-1)^NS_{-2}+16(-1)^N\zeta_2\right]}{(N-1)N(N+1)^4(N+2)^3}$$

$$+\frac{\left(N^2-N-4\right)\left(N^2+N+2\right)}{(N-1)N(N+1)^3(N+2)^2}\Big[-64(-1)^NS_1S_{-2}-32(-1)^NS_{-3}+64S_{-2,1}-32(-1)^NS_1\zeta_2$$

$$-24(-1)^N\zeta_3\Big]+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\Big[-\frac{2}{3}S_1^4-20S_2S_1^2-32(-1)^NS_{-3}S_1+\big(64S_{-2,1}$$

$$-\frac{160S_3}{3}\big)S_1-\frac{8}{3}\big(-7+9(-1)^N\big)\zeta_3S_1-2S_2^2+S_{-2}\big(-32(-1)^NS_1^2$$

$$-32(-1)^N S_2) - 36 S_4 - 16(-1)^N S_{-4} + 16 S_{3,1} + 32 S_{-2,2} + 32 S_{-3,1} + 16 S_{2,1,1}$$

$$-64 S_{-2,1,1} + \left(-4\left(-3 + 4(-1)^N\right) S_1^2 - 4\left(-1 + 4(-1)^N\right) S_2 - 8\left(1 + 2(-1)^N\right) S_{-2}\right)\zeta_2\bigg]$$

$$+ L_M \Bigg[\frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \bigg[-\frac{8}{3} S_1^3 + 40 S_2 S_1 + 32\left(1 + (-1)^N\right) S_{-2} S_1$$

$$+ 16(-1)^N S_{-3} - 32 S_{2,1} + 12\left(-9 + (-1)^N\right)\zeta_3 \bigg] + \frac{8 P_{324}}{27(N-1)^2 N^5(N+1)^5(N+2)^4}$$

$$+ \frac{4\left(17N^4 - 6N^3 + 41N^2 - 16N - 12\right) S_1^2\left(N^2 + N + 2\right)}{3(N-1)^2 N^3(N+1)^2(N+2)} + \frac{4 P_{305} S_2\left(N^2 + N + 2\right)}{3(N-1)^2 N^3(N+1)^3(N+2)^2}$$

$$+ \frac{8\left(31N^2 + 31N + 74\right) S_3\left(N^2 + N + 2\right)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{16\left(7N^2 + 7N + 10\right) S_{-3}\left(N^2 + N + 2\right)}{(N-1)N^2(N+1)^2(N+2)}$$

$$- \frac{128\left(N^2 + N + 1\right) S_{-2,1}\left(N^2 + N + 2\right)}{(N-1)N^2(N+1)^2(N+2)} + \frac{\left(N^2 - N - 4\right)32(-1)^N S_{-2}\left(N^2 + N + 2\right)}{(N-1)N(N+1)^3(N+2)^2}$$

$$- \frac{8 P_{316} S_1}{9(N-1)^2 N^4(N+1)^4(N+2)^2} + \frac{16 P_{306} S_{-2}}{(N-1)N^3(N+1)^3(N+2)^2} \bigg]\Bigg] + a_{Qq}^{\mathrm{PS},(3)} \Bigg\}\Bigg\}, \tag{401}$$

with the polynomials

$$
\begin{array}{lll}
P_{296} & = & N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \hspace{2cm} (402) \\
P_{297} & = & 7N^6 + 15N^5 + 7N^4 - 23N^3 - 26N^2 - 20N - 8 \hspace{2cm} (403) \\
P_{298} & = & 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24 \hspace{2cm} (404) \\
P_{299} & = & 11N^6 + 6N^5 + 75N^4 + 68N^3 - 200N^2 - 80N - 24 \hspace{2cm} (405) \\
P_{300} & = & 11N^6 + 29N^5 - 7N^4 - 25N^3 - 56N^2 - 72N - 24 \hspace{2cm} (406) \\
P_{301} & = & 17N^6 + 27N^5 + 75N^4 + 149N^3 - 20N^2 - 80N - 24 \hspace{2cm} (407) \\
P_{302} & = & 17N^6 + 51N^5 + 51N^4 + 89N^3 + 40N^2 - 80N - 24 \hspace{2cm} (408) \\
P_{303} & = & 27N^6 + 102N^5 + 135N^4 + 56N^3 - 8N^2 - 20N - 8 \hspace{2cm} (409) \\
P_{304} & = & 38N^6 + 108N^5 + 151N^4 + 106N^3 + 21N^2 - 28N - 12 \hspace{2cm} (410) \\
P_{305} & = & 73N^6 + 189N^5 + 45N^4 + 31N^3 - 238N^2 - 412N - 120 \hspace{2cm} (411) \\
P_{306} & = & 2N^7 + 14N^6 + 37N^5 + 102N^4 + 155N^3 + 158N^2 + 132N + 40 \hspace{2cm} (412) \\
P_{307} & = & 3N^7 - 15N^6 - 133N^5 - 449N^4 - 658N^3 - 500N^2 - 296N - 96 \hspace{2cm} (413) \\
P_{308} & = & 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \hspace{2cm} (414) \\
P_{309} & = & 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \hspace{2cm} (415) \\
P_{310} & = & 8N^7 + 37N^6 + 158N^5 + 565N^4 + 796N^3 + 628N^2 + 400N + 96 \hspace{2cm} (416) \\
P_{311} & = & 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \hspace{2cm} (417) \\
P_{312} & = & N^{10} + 8N^9 + 29N^8 + 49N^7 - 11N^6 - 131N^5 - 161N^4 - 160N^3 - 168N^2 - 80N - 16 \hspace{0.3cm} (418)
\end{array}
$$

$$P_{313} = 19N^{10} + 143N^9 + 427N^8 + 567N^7 + 454N^6 + 822N^5 + 1560N^4 + 1784N^3 + 1488N^2$$
$$+ 768N + 192 \tag{419}$$

$$P_{314} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2$$
$$+ 672N + 288 \tag{420}$$

$$P_{315} = 43N^{10} + 320N^9 + 1059N^8 + 1914N^7 + 2431N^6 + 2874N^5 + 2379N^4 + 820N^3 + 352N^2$$
$$+ 336N + 144 \tag{421}$$

$$P_{316} = 136N^{10} + 647N^9 + 1110N^8 - 438N^7 - 2555N^6 - 2106N^5 - 3105N^4 - 3167N^3 + 418N^2$$
$$+ 924N + 72 \tag{422}$$

$$P_{317} = 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3$$

$$
\begin{align}
&\qquad +6512N^2 + 2432N + 384 \tag{423}\\
P_{318} &= 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2\\
&\quad -208N - 32 \tag{424}\\
P_{319} &= 118N^{11} + 793N^{10} + 2281N^9 + 3402N^8 + 2428N^7 + 1457N^6 + 1917N^5 + 2476N^4\\
&\quad +4392N^3 + 4976N^2 + 2832N + 576 \tag{425}\\
P_{320} &= 127N^{11} + 820N^{10} + 2251N^9 + 2196N^8 - 1109N^7 - 934N^6 + 4491N^5 + 9334N^4\\
&\quad +12552N^3 + 9680N^2 + 4656N + 864 \tag{426}\\
P_{321} &= 37N^{12} + 305N^{11} + 1107N^{10} + 2328N^9 + 3520N^8 + 5020N^7 + 7642N^6 + 10519N^5\\
&\quad +10938N^4 + 8248N^3 + 4656N^2 + 1712N + 288 \tag{427}\\
P_{322} &= 248N^{13} + 2599N^{12} + 12793N^{11} + 39593N^{10} + 87182N^9 + 148026N^8 + 196942N^7\\
&\quad +192416N^6 + 128195N^5 + 63406N^4 + 32344N^3 + 15984N^2 + 5616N + 864 \tag{428}\\
P_{323} &= 4N^{14} + 56N^{13} + 443N^{12} + 2139N^{11} + 6049N^{10} + 10762N^9 + 13272N^8 + 11692N^7\\
&\quad +6106N^6 + 339N^5 - 1254N^4 - 72N^3 + 496N^2 + 240N + 32 \tag{429}\\
P_{324} &= 686N^{14} + 6560N^{13} + 25572N^{12} + 43489N^{11} + 9045N^{10} - 72944N^9 - 125240N^8\\
&\quad -156761N^7 - 206883N^6 - 241600N^5 - 250212N^4 - 225808N^3 - 150864N^2\\
&\quad -56448N - 8640 \tag{430}\\
P_{325} &= 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} - 17182N^{10}\\
&\quad -36527N^9 - 27469N^8 - 11770N^7 + 5554N^6 + 32640N^5 + 46456N^4 + 34528N^3 + 14816N^2\\
&\quad +3584N + 384 \tag{431}\\
P_{326} &= 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \tag{432}\\
P_{327} &= 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \tag{433}\\
P_{328} &= 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3\\
&\quad +16N^2 + 672N + 288 \tag{434}\\
P_{329} &= 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4\\
&\quad -904N^3 - 592N^2 - 208N - 32 \ . \tag{435}
\end{align}
$$

The OME A_{Qg} except for the term $a_{Qg}^{(3)}$ reads :

$$
\begin{aligned}
A_{Qg} = \tfrac{1}{2}&[1 + (-1)^N]\\
\times &\left\{ a_s \tilde{\gamma}_{qg}^0 T_F L_M + a_s^2 \left\{ \frac{4}{3} \tilde{\gamma}_{qg}^0 T_F^2 L_M^2 + C_F T_F \left[\frac{4(3N+2)S_1^2}{N^2(N+2)} + \frac{4\left(N^4 - N^3 - 20N^2 - 10N - 4\right)S_1}{N^2(N+1)^2(N+2)} \right.\right.\right.\\
&\left. + \frac{2P_{375}}{N^4(N+1)^4(N+2)} + L_M^2 \left[\frac{2\left(N^2 + N + 2\right)\left(3N^2 + 3N + 2\right)}{N^2(N+1)^2(N+2)} + 2\tilde{\gamma}_{qg}^0 S_1 \right]\right.\\
&\left. + L_M \left[-\frac{4P_{337}}{N^3(N+1)^3(N+2)} + \frac{16S_1}{N^2} + \tilde{\gamma}_{qg}^0 \left[2S_1^2 - 2S_2 \right] \right] + \frac{4\left(N^4 + 17N^3 + 17N^2 - 5N - 2\right)S_2}{N^2(N+1)^2(N+2)}\right.\\
&\left. + \tilde{\gamma}_{qg}^0 \left[\frac{1}{3}S_1^3 + S_2 S_1 - \frac{4S_3}{3} \right] \right] + C_A T_F \left[-\frac{4\left(N^3 + 8N^2 + 11N + 2\right)S_1^2}{N(N+1)^2(N+2)^2} - \frac{4P_{333}S_1}{N(N+1)^3(N+2)^3}\right.\\
&\left. + \frac{4P_{413}}{(N-1)N^4(N+1)^4(N+2)^4} + L_M^2 \left[-\frac{16\left(N^2 + N + 1\right)\left(N^2 + N + 2\right)}{(N-1)N^2(N+1)^2(N+2)^2} - 2\tilde{\gamma}_{qg}^0 S_1 \right]\right.\\
&\left. - \frac{4\left(7N^5 + 21N^4 + 13N^3 + 21N^2 + 18N + 16\right)S_2}{(N-1)N^2(N+1)^2(N+2)^2} + L_M \left[-\frac{8P_{383}}{(N-1)N^3(N+1)^3(N+2)^3} \right.\right.
\end{aligned}
$$

$$-\frac{32(2N+3)S_1}{(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0\left[-2S_1^2 - 2S_2 - 4S_{-2}\right]\right] + \frac{(N^2-N-4)}{(N+1)^2(N+2)^2}16(-1)^N S_{-2}$$

$$+\tilde{\gamma}_{qg}^0\left[-\frac{1}{3}S_1^3 - 3S_2 S_1 - 4(-1)^N S_{-2}S_1 - \frac{8S_3}{3} - 2(-1)^N S_{-3} + 4S_{-2,1}\right]\right]\Bigg\}$$

$$+a_s^3\Bigg\{ T_F^3\left[\frac{16}{9}\tilde{\gamma}_{qg}^0 L_M^3 - \frac{16\tilde{\gamma}_{qg}^0\zeta_3}{9}\right] + C_A T_F^2\left[\frac{32(N^3+8N^2+11N+2)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{334}S_1^2}{3N(N+1)^3(N+2)^3}\right.$$

$$-\frac{16P_{406}S_1}{81(N-1)N^2(N+1)^4(N+2)^4} - \frac{32}{9}\frac{\left(5N^4+8N^3+17N^2+43N+20\right)\zeta_2}{N(N+1)^2(N+2)^2}S_1$$

$$-\frac{32\left(3N^3-12N^2-27N-2\right)S_2 S_1}{3N(N+1)^2(N+2)^2} + \frac{4}{9}\frac{P_{392}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^3}$$

$$+\frac{8P_{427}}{81(N-1)N^5(N+1)^5(N+2)^5} + \frac{32\left(9N^5-4N^4+N^3+92N^2+42N+28\right)\zeta_3}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$+L_M^3\left[-\frac{448(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{56}{9}\tilde{\gamma}_{qg}^0 S_1\right] + \frac{8P_{385}S_2}{3(N-1)N^3(N+1)^3(N+2)^3}$$

$$+\frac{256\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + L_M^2\left[-\frac{8P_{390}}{9(N-1)N^3(N+1)^3(N+2)^3}\right.$$

$$+\frac{32\left(5N^4+20N^3-N^2-14N+20\right)S_1}{9N(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0\left[-4S_1^2-4S_2-8S_{-2}\right]\right]$$

$$+\frac{\left(N^4+2N^3+7N^2+22N+20\right)\left[\frac{128}{3}(-1)^N S_{-2} + \frac{64}{3}(-1)^N\zeta_2\right]}{(N+1)^3(N+2)^3}$$

$$+\frac{\left(N^2-N-4\right)\left[-\frac{256}{3}(-1)^N S_1 S_{-2} - \frac{128}{3}(-1)^N S_{-3} + \frac{256}{3}S_{-2,1} - \frac{128}{3}(-1)^N S_1\zeta_2 - 32(-1)^N\zeta_3\right]}{(N+1)^2(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\left[\frac{2}{9}S_1^4 + \frac{20}{3}S_2 S_1^2 + \frac{32}{3}(-1)^N S_{-3}S_1 + \left[\frac{160S_3}{9} - \frac{64}{3}S_{-2,1}\right]S_1 + \frac{8}{9}\left(-2+9(-1)^N\right)\zeta_3 S_1 + \frac{2}{3}S_2^2\right.$$

$$+S_{-2}\left(\frac{32}{3}(-1)^N S_1^2 + \frac{32}{3}(-1)^N S_2\right) + 12S_4 + \frac{16}{3}(-1)^N S_{-4} - \frac{16}{3}S_{3,1} - \frac{32}{3}S_{-2,2} - \frac{32}{3}S_{-3,1}$$

$$-\frac{16}{3}S_{2,1,1} + \frac{64}{3}S_{-2,1,1} + \left[\frac{2}{3}\left(-3+8(-1)^N\right)S_1^2 + \frac{2}{3}\left[-3+8(-1)^N\right]S_2 + \frac{4}{3}\left(1+4(-1)^N\right)S_{-2}\right]\zeta_2\right]$$

$$+L_M\left[\frac{16P_{415}}{27(N-1)N^4(N+1)^4(N+2)^4} - \frac{32\left(5N^4+23N^3+65N^2+82N+26\right)S_1^2}{9N(N+1)^2(N+2)^2}\right.$$

$$+\frac{16P_{364}S_1}{27N(N+1)^3(N+2)^3} - \frac{32P_{340}S_2}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{64\left(5N^2+8N+10\right)S_{-2}}{9N(N+1)(N+2)}$$

$$+\frac{(N^2-N-4)\frac{128}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0\left[-\frac{4}{3}S_1^3 - \frac{20}{3}S_2 S_1 - \frac{32}{3}(-1)^N S_{-2}S_1 - 8S_3\right.$$

$$\left.\left.\left.-\frac{16}{3}\left(1+(-1)^N\right)S_{-3} - \frac{16}{3}S_{2,1} + \frac{32}{3}S_{-2,1}\right]\right]\right]$$

$$+C_A T_F^2 N_F\left[\frac{16\left(N^3+8N^2+11N+2\right)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{332}S_1^2}{3N(N+1)^3(N+2)^3} + \frac{4}{9}\frac{P_{376}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2}\right.$$

$$-\frac{16P_{371}S_1}{3N(N+1)^4(N+2)^4} + \frac{16P_{424}}{3(N-1)N^5(N+1)^5(N+2)^5} - \frac{16\left(3N^3-12N^2-27N-2\right)S_2 S_1}{3N(N+1)^2(N+2)^2}$$

$$+\frac{16}{9}\frac{\left(5N^4+32N^3+47N^2+28N+20\right)\zeta_2}{N(N+1)^2(N+2)^2}S_1 + \frac{16}{9}\frac{\left(9N^5-14N^4-19N^3+52N^2+12N+8\right)\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$+L_M^3\left[-\frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{8}{9}\tilde\gamma_{qg}^0 S_1\right]+\frac{8P_{384}S_2}{3(N-1)N^3(N+1)^3(N+2)^3}$$

$$+\frac{128\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{9(N-1)N^2(N+1)^2(N+2)^2}+L_M^2\left[-\frac{8P_{374}}{9(N-1)N^2(N+1)^3(N+2)^3}\right.$$

$$-\frac{32\left(5N^4+20N^3+47N^2+58N+20\right)S_1}{9N(N+1)^2(N+2)^2}+\tilde\gamma_{qg}^0\left[-\frac{4}{3}S_1^2-\frac{4S_2}{3}-\frac{8}{3}S_{-2}\right]\right]$$

$$+\frac{\left(N^4+2N^3+7N^2+22N+20\right)\left[\frac{64}{3}(-1)^N S_{-2}+\frac{32}{3}(-1)^N\zeta_2\right]}{(N+1)^3(N+2)^3}$$

$$+\frac{\left(N^2-N-4\right)}{(N+1)^2(N+2)^2}\left[-\frac{128}{3}(-1)^N S_1 S_{-2}-\frac{64}{3}(-1)^N S_{-3}+\frac{128}{3}S_{-2,1}-\frac{64}{3}(-1)^N S_1\zeta_2-16(-1)^N\zeta_3\right]$$

$$+\tilde\gamma_{qg}^0\left[\frac{1}{9}S_1^4+\frac{10}{3}S_2 S_1^2+\frac{16}{3}(-1)^N S_{-3}S_1+\left[\frac{80S_3}{9}-\frac{32}{3}S_{-2,1}\right]S_1+\frac{4}{9}\left(-7+9(-1)^N\right)\zeta_3 S_1+\frac{1}{3}S_2^2\right.$$

$$+S_{-2}\left[\frac{16}{3}(-1)^N S_1^2+\frac{16}{3}(-1)^N S_2\right]+6S_4+\frac{8}{3}(-1)^N S_{-4}-\frac{8}{3}S_{3,1}-\frac{16}{3}S_{-2,2}-\frac{16}{3}S_{-3,1}-\frac{8}{3}S_{2,1,1}$$

$$+\frac{32}{3}S_{-2,1,1}+\left[\frac{4}{3}\left(-1+2(-1)^N\right)S_1^2+\frac{4}{3}\left(-1+2(-1)^N\right)S_2+\frac{8}{3}(-1)^N S_{-2}\right]\zeta_2\right]$$

$$+L_M\left[-\frac{16\left(10N^4+43N^3+106N^2+131N+46\right)S_1^2}{9N(N+1)^2(N+2)^2}+\frac{8P_{416}}{27(N-1)N^4(N+1)^4(N+2)^4}\right.$$

$$+\frac{16P_{357}S_1}{27N(N+1)^3(N+2)^3}-\frac{16P_{345}S_2}{9(N-1)N^2(N+1)^2(N+2)^2}-\frac{64\left(5N^2+8N+10\right)S_{-2}}{9N(N+1)(N+2)}$$

$$+\frac{\left(N^2-N-4\right)}{(N+1)^2(N+2)^2}\frac{64}{3}(-1)^N S_{-2}+\tilde\gamma_{qg}^0\left[-\frac{8}{9}S_1^3-\frac{8}{3}S_2 S_1-\frac{16}{3}(-1)^N S_{-2}S_1-\frac{40S_3}{9}\right.$$

$$\left.\left.-\frac{8}{3}\left(2+(-1)^N\right)S_{-3}-\frac{16}{3}S_{2,1}+\frac{16}{3}S_{-2,1}\right]\right]+C_F^2 T_F\left[-\frac{\left(3N^4+54N^3+139N^2+120N+36\right)S_1^4}{3N^2(N+1)^2(N+2)}\right.$$

$$-\frac{4\left(18N^5-15N^4-180N^3-111N^2-40N-4\right)S_1^3}{3N^3(N+1)^2(N+2)}+\frac{2P_{359}S_1^2}{N^3(N+1)^3(N+2)}$$

$$-\frac{4\left(3N^4+14N^3+43N^2+48N+20\right)\zeta_2}{N^2(N+1)^2(N+2)}S_1^2-\frac{2\left(3N^3+24N^2+27N+10\right)S_2 S_1^2}{N^2(N+1)^2}$$

$$-\frac{4P_{398}S_1}{N^5(N+1)^5(N+2)}-\frac{4P_{355}S_2 S_1}{N^3(N+1)^3(N+2)}-\frac{8\left(9N^4+102N^3+97N^2-36N-12\right)S_3 S_1}{3N^2(N+1)^2(N+2)}$$

$$-\frac{16\left(3N^4+2N^3+19N^2+28N+12\right)S_{2,1}S_1}{N^2(N+1)^2(N+2)}-\frac{8P_{353}\zeta_2 S_1}{N^3(N+1)^3(N+2)}-\frac{P_{414}}{N^6(N+1)^6(N+2)}$$

$$+\frac{2}{3}\left(N^2+N+2\right)\left(153N^4+306N^3+165N^2+12N+4\right)\frac{\zeta_3}{N^3(N+1)^3(N+2)}$$

$$+L_M^3\left[-\frac{2\left(N^2+N+2\right)\left(3N^2+3N+2\right)^2}{3N^3(N+1)^3(N+2)}+\frac{16\left(N^2+N+2\right)S_1\left(3N^2+3N+2\right)}{3N^2(N+1)^2(N+2)}+\frac{8}{3}\tilde\gamma_{qg}^0 S_1^2\right]$$

$$+\frac{2P_{370}S_2}{N^4(N+1)^4(N+2)}+\frac{4P_{362}S_3}{3N^3(N+1)^3(N+2)}-\frac{16\left(N^2-3N-2\right)\left(3N^2+3N+2\right)S_{2,1}}{N^3(N+1)^2(N+2)}$$

$$+L_M^2\left[\frac{4\left(3N^4+14N^3+43N^2+48N+20\right)S_1^2}{N^2(N+1)^2(N+2)}-\frac{8P_{338}S_1}{N^3(N+1)^3(N+2)}+\frac{P_{378}}{N^4(N+1)^4(N+2)}\right.$$

$$-\frac{12\left(N^2+N+2\right)\left(3N^2+3N+2\right)S_2}{N^2(N+1)^2(N+2)}-\frac{32\left(N^2+N+2\right)S_{-2}}{N^2(N+1)^2(N+2)}+\tilde\gamma_{qg}^0\left[4S_1^3-12S_2 S_1-16S_{-2}S_1\right.$$

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$$
\left. -8S_3 - 8S_{-3} + 16S_{-2,1}\right]\right] + \frac{P_{382}\zeta_2}{2N^4(N+1)^4(N+2)} + \frac{16\left(N^2+N+2\right)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} + 96\tilde{\gamma}_{qg}^0\log(2)\zeta_2
$$

$$
+ L_M\left[\frac{64(N+1)S_1^3}{N^2(N+2)} - \frac{8P_{339}S_1^2}{N^3(N+1)^3(N+2)} + \frac{16\left(N^4+10N^3+3N^2-18N-4\right)S_2S_1}{N^2(N+1)^2(N+2)}\right.
$$

$$
+ \frac{8P_{372}S_1}{N^4(N+1)^4(N+2)} - \frac{P_{403}}{2N^5(N+1)^5(N+2)} - \frac{8\left(N^2+5N+2\right)\left(3N^2-N+2\right)S_3}{N^2(N+1)^2(N+2)}
$$

$$
+ \frac{4P_{356}S_2}{N^3(N+1)^3(N+2)} + \frac{32\left(2N^5+4N^4-N^3+N^2-2N+8\right)S_{-2}}{N^3(N+1)^2(N+2)} - \frac{64(N-1)S_{-3}}{(N+1)^2(N+2)}
$$

$$
- \frac{64\left(N^2+N+2\right)S_{2,1}}{N^2(N+1)^2(N+2)} + \frac{\left(N^2-N+2\right)\left[128S_{-2,1}-128S_{-2}S_1\right]}{N^2(N+1)(N+2)}
$$

$$
+ \tilde{\gamma}_{qg}^0\left[2S_1^4 + 4S_2S_1^2 + 32S_{-3}S_1 + \left(24S_3-32S_{2,1}\right)S_1 + 6S_2^2 + 8S_{-2}^2 + 16S_{-2}S_2 + 20S_4 + 40S_{-4}\right.
$$

$$
\left.\left. -8S_{3,1} - 16S_{-2,2} - 32S_{-3,1} + 24S_{2,1,1}\right] + \frac{48(N-1)\left(3N^2+3N-2\right)\zeta_3}{N^2(N+1)^2}\right]
$$

$$
+ \frac{\left(N^2+N+2\right)\left(3N^2+3N+2\right)\left[-S_2^2 + 8\zeta_2S_2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1} - \frac{16}{3}S_1\zeta_3\right]}{N^2(N+1)^2(N+2)}
$$

$$
+ \tilde{\gamma}_{qg}^0\left[-\frac{1}{3}S_1^5 - 2S_2S_1^3 + \left[-\frac{8}{3}S_3 - 16S_{2,1}\right]S_1^2 - \frac{8}{3}\zeta_3S_1^2 + \left[-S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1}\right]S_1\right.
$$

$$
\left.\left.+ \left[-4S_1^3 + 8S_2S_1 + 8S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1}\right]\zeta_2\right]\right] + C_A^2T_F\left[\frac{P_{346}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2}\right.
$$

$$
- \frac{4P_{377}S_1^3}{9(N-1)N^2(N+1)^3(N+2)^3} - \frac{4}{3}\frac{P_{352}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{2P_{400}S_1^2}{3(N-1)N^2(N+1)^4(N+2)^4}
$$

$$
+ \frac{2(N-2)\left(55N^5+347N^4+379N^3+213N^2+326N+120\right)S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2}
$$

$$
- \frac{4}{9}\frac{P_{366}S_1\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4}{9}\frac{P_{402}S_1\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^3}
$$

$$
- \frac{4P_{425}S_1}{3(N-1)N^5(N+1)^5(N+2)^5} - \frac{4P_{386}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3} + \frac{16P_{360}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2}
$$

$$
- \frac{2}{9}\frac{P_{419}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^4} - \frac{4\left(11N^4+22N^3-35N^2-46N-24\right)P_{424}}{3(N-1)^2N^6(N+1)^6(N+2)^6}
$$

$$
- \frac{4\left(11N^4+22N^3-35N^2-46N-24\right)\left(9N^5-14N^4-19N^3+52N^2+12N+8\right)\zeta_3}{9(N-1)^2N^3(N+1)^3(N+2)^3}
$$

$$
+ L_M^3\left[\frac{8}{3}\tilde{\gamma}_{qg}^0S_1^2 - \frac{8\left(N^2+N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\right.
$$

$$
\left.+ \frac{16\left(N^2+N+1\right)\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{9(N-1)^2N^3(N+1)^3(N+2)^3}\right]
$$

$$
- \frac{2\left(11N^4+22N^3-35N^2-46N-24\right)P_{384}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^4} - 32\frac{\left(N^2+N+1\right)\left(N^2+N+2\right)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2}S_{-2}
$$

$$
- \frac{32\left(11N^4+22N^3-35N^2-46N-24\right)\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{9(N-1)^2N^3(N+1)^3(N+2)^3}
$$

$$
+ L_M^2\left[-\frac{4P_{350}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{401}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3}\right.
$$

$$+\frac{16P_{418}}{9(N-1)^2N^4(N+1)^4(N+2)^4}-\frac{4\left(N^2+N+2\right)\left(11N^4+22N^3-83N^2-94N-72\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{8\left(N^2+N+2\right)\left(11N^4+22N^3-59N^2-70N-48\right)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\Big[4S_1^3+12S_2S_1+16S_{-2}S_1+4S_3+4S_{-3}-8S_{-2,1}\Big]\Big]$$

$$+\frac{\left(N^4+2N^3+7N^2+22N+20\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{(N-1)N(N+1)^4(N+2)^4}\Big[-\frac{16}{3}(-1)^NS_{-2}$$

$$-\frac{8}{3}(-1)^N\zeta_2\Big]+\frac{\left(5N^5-131N^3-58N^2+232N+96\right)\Big[\frac{32}{3}(-1)^NS_1S_{-2}+\frac{16}{3}(-1)^NS_1\zeta_2\Big]}{(N-1)N(N+1)^2(N+2)^3}$$

$$+\frac{P_{351}\Big[\frac{16}{3}(-1)^NS_{-2}S_1^2+\frac{8}{3}(-1)^N\zeta_2S_1^2\Big]}{(N-1)N^2(N+1)^2(N+2)^2}+\frac{\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$\times\Big[\frac{1}{3}S_2^2+\frac{16}{3}(-1)^NS_{-2}S_2+6S_4+\frac{8}{3}(-1)^NS_{-4}-\frac{8}{3}S_{3,1}-\frac{16}{3}S_{-2,2}$$

$$-\frac{16}{3}S_{-3,1}-\frac{8}{3}S_{2,1,1}+\frac{32}{3}S_{-2,1,1}+\Big(\frac{8}{3}(-1)^NS_2+\frac{8}{3}(-1)^NS_{-2}\Big)\zeta_2\Big]$$

$$+\frac{\left(N^2-N-4\right)\left(11N^4+22N^3-35N^2-46N-24\right)\Big[\frac{16}{3}(-1)^NS_{-3}-\frac{32}{3}S_{-2,1}+4(-1)^N\zeta_3\Big]}{(N-1)N(N+1)^3(N+2)^3}$$

$$+\frac{\left(11N^5+34N^4-49N^3-24N^2-68N-48\right)\Big[\frac{16}{3}(-1)^NS_1S_{-3}-\frac{32}{3}S_1S_{-2,1}+4(-1)^NS_1\zeta_3\Big]}{(N-1)N^2(N+1)(N+2)^2}$$

$$+\tilde{\gamma}_{qg}^0\Big[-\frac{1}{3}S_1^5-10S_2S_1^3-16(-1)^NS_{-3}S_1^2+\Big[32S_{-2,1}-\frac{80S_3}{3}\Big]S_1^2-\frac{4}{3}\big(-7+9(-1)^N\big)\zeta_3S_1^2$$

$$-8(-1)^NS_{-4}S_1+\Big[-S_2^2-18S_4+8S_{3,1}+16S_{-2,2}+16S_{-3,1}+8S_{2,1,1}-32S_{-2,1,1}\Big]S_1$$

$$+S_{-2}\Big[-16(-1)^NS_1^3-16(-1)^NS_2S_1\Big]+\Big[-4\big(-1+2(-1)^N\big)S_1^3-8(-1)^NS_2S_1$$

$$-4\big(1+2(-1)^N\big)S_{-2}S_1+\frac{11S_2}{3}-2S_3-2S_{-3}+4S_{-2,1}\Big]\zeta_2\Big]$$

$$+L_M\Big[-\frac{8\left(11N^4+26N^3-139N^2-218N+8\right)S_1^3}{9N(N+1)^2(N+2)^2}+\frac{4P_{404}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{8P_{420}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4}-\frac{8P_{347}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{32\left(2N^5-23N^4-32N^3+13N^2+4N-12\right)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)^2}+2\frac{\zeta_3P_{367}}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{4P_{428}}{27(N-1)^2N^5(N+1)^5(N+2)^5}+\frac{4P_{405}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3}$$

$$-\frac{8P_{361}S_3}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{16P_{394}S_{-2}}{9(N-1)N^3(N+1)^3(N+2)^3}+\frac{16P_{348}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16P_{349}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2}-\frac{16\left(N^2+N+2\right)\left(11N^4+22N^3+13N^2+2N+24\right)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{\left(N^2-N-4\right)\left(11N^4+22N^3-35N^2-46N-24\right)\frac{16}{3}(-1)^NS_{-2}}{(N-1)N(N+1)^3(N+2)^3}$$

$$-\frac{\left(11N^5+34N^4-49N^3-24N^2-68N-48\right)\frac{16}{3}(-1)^NS_1S_{-2}}{(N-1)N^2(N+1)(N+2)^2}$$

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$$+\frac{\left(N^2+N+2\right)\left(11N^4+22N^3-35N^2-46N-24\right)\left[-\frac{8}{3}(-1)^N S_{-3}-2(-1)^N \zeta_3\right]}{(N-1)N^2(N+1)^2(N+2)^2}+\tilde{\gamma}_{qg}^0\bigg[2S_1^4+2S_2^2$$

$$+32S_2S_1^2+8\left(8+(-1)^N\right)S_{-3}S_1+\left[40S_3-16S_{2,1}-80S_{-2,1}\right]S_1+6\left(-5+(-1)^N\right)\zeta_3 S_1+12S_{-2}^2$$

$$+S_{-2}\left[8\left(3+2(-1)^N\right)S_1^2+16S_2\right]+4S_4+44S_{-4}+16S_{3,1}-56S_{-2,2}-64S_{-3,1}+96S_{-2,1,1}\bigg]\bigg]\bigg]$$

$$+C_F T_F^2\Bigg[\left[\frac{16\left(N^2+N+2\right)P_{341}}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{32}{9}\tilde{\gamma}_{qg}^0 S_1\right]L_M^3+\Bigg[-\frac{4P_{410}}{9(N-1)N^4(N+1)^4(N+2)^3}$$

$$+\frac{32\left(5N^3+14N^2+37N+18\right)S_1}{9N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[4S_1^2-\frac{4S_2}{3}\right]\Bigg]L_M^2+\Bigg[\frac{16\left(10N^3+31N^2+59N+18\right)S_1^2}{9N^2(N+1)(N+2)}$$

$$-\frac{16\left(29N^4+163N^3+786N^2+592N+192\right)S_1}{27N^2(N+1)^2(N+2)}+\frac{2P_{423}}{27(N-1)N^5(N+1)^5(N+2)^4}$$

$$+\frac{16\left(2N^4+39N^3+42N^2-5N-2\right)S_2}{3N^2(N+1)^2(N+2)}+\tilde{\gamma}_{qg}^0\left[\frac{4}{3}S_1^3+4S_2S_1-8S_3\right]\Bigg]L_M$$

$$-\frac{16\left(N^4-5N^3-32N^2-18N-4\right)S_1^2}{3N^2(N+1)^2(N+2)}-\frac{16}{9}\frac{\left(N^2+N+2\right)P_{341}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$+\frac{2}{9}\frac{P_{411}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3}+\frac{32\left(2N^5-2N^4-11N^3-19N^2-44N-12\right)S_1}{3N^2(N+1)^3(N+2)}$$

$$-\frac{2P_{421}}{3(N-1)N^6(N+1)^6(N+2)^2}-\frac{80}{9}\frac{\left(N^3+4N^2+11N+6\right)\zeta_2}{N^2(N+1)(N+2)}S_1-\frac{16P_{336}S_2}{3N^3(N+1)^3(N+2)}$$

$$+\frac{(3N+2)\left(-\frac{32}{9}S_1^3-\frac{32}{3}S_2S_1\right)}{N^2(N+2)}-\frac{32\left(3N^4+48N^3+43N^2-22N-8\right)S_3}{9N^2(N+1)^2(N+2)}$$

$$+\frac{128\left(N^2-3N-2\right)S_{2,1}}{3N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\bigg[-\frac{2}{9}S_1^4-\frac{4}{3}S_2S_1^2+\left(-\frac{16}{9}S_3-\frac{32}{3}S_{2,1}\right)S_1-\frac{32}{9}\zeta_3 S_1-\frac{2}{3}S_2^2+4S_4$$

$$-\frac{32}{3}S_{3,1}+\frac{64}{3}S_{2,1,1}+\left(2S_2-\frac{10}{3}S_1^2\right)\zeta_2\bigg]\Bigg]$$

$$+C_F T_F^2 N_F\Bigg[\left[\frac{8\left(N^2+N+2\right)P_{335}}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{8}{9}\tilde{\gamma}_{qg}^0 S_1\right]L_M^3+\Bigg[-\frac{4P_{408}}{9(N-1)N^4(N+1)^4(N+2)^3}$$

$$+\frac{32\left(5N^3+8N^2+19N+6\right)S_1}{9N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[\frac{4}{3}S_1^2+\frac{4S_2}{3}\right]\Bigg]L_M^2+L_M\Bigg[\frac{32\left(5N^3+11N^2+22N+6\right)S_1^2}{9N^2(N+1)(N+2)}$$

$$-\frac{32\left(19N^4+77N^3+303N^2+251N+78\right)S_1}{27N^2(N+1)^2(N+2)}+\frac{2P_{422}}{27(N-1)N^5(N+1)^5(N+2)^4}$$

$$+\frac{16P_{369}S_2}{3(N-1)N^3(N+1)^3(N+2)^2}+\tilde{\gamma}_{qg}^0\left[\frac{8}{9}S_1^3+\frac{8}{3}S_2S_1-\frac{56S_3}{9}\right]\Bigg]-\frac{8}{9}\frac{\left(N^2+N+2\right)P_{335}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$-\frac{8\left(N^4-5N^3-32N^2-18N-4\right)S_1^2}{3N^2(N+1)^2(N+2)}-\frac{2}{9}\frac{P_{412}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3}$$

$$+\frac{16\left(2N^5-2N^4-11N^3-19N^2-44N-12\right)S_1}{3N^2(N+1)^3(N+2)}-\frac{16}{9}\frac{\left(5N^3+11N^2+28N+12\right)\zeta_2}{N^2(N+1)(N+2)}S_1$$

$$-\frac{4P_{430}}{3(N-1)N^6(N+1)^6(N+2)^5}-\frac{8P_{407}S_2}{3(N-1)N^4(N+1)^4(N+2)^3}+\frac{(3N+2)\left[-\frac{16}{9}S_1^3-\frac{16}{3}S_2S_1\right]}{N^2(N+2)}$$

$$-\frac{16P_{373}S_3}{9(N-1)N^3(N+1)^3(N+2)^2}+\frac{64\left(N^2-3N-2\right)S_{2,1}}{3N^2(N+1)(N+2)}+\tilde{\gamma}_{qg}^0\left[-\frac{1}{9}S_1^4-\frac{2}{3}S_2S_1^2\right.$$

$$-\frac{4}{3}\zeta_2 S_1^2 + \left[-\frac{8}{9}S_3 - \frac{16}{3}S_{2,1}\right]S_1 - \frac{8}{9}\zeta_3 S_1 - \frac{1}{3}S_2^2 + 2S_4 - \frac{16}{3}S_{3,1} + \frac{32}{3}S_{2,1,1}\right]$$

$$+C_F C_A T_F\left[-\frac{2P_{330}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{388}S_1^3}{9(N-1)N^3(N+1)^3(N+2)^3}\right.$$

$$-\frac{8}{3}\frac{P_{344}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{409}S_1^2}{3(N-1)N^3(N+1)^4(N+2)^4}$$

$$+\frac{4}{3}\frac{P_{354}S_2 S_1^2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4}{9}\frac{P_{363}S_1\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{426}S_1}{3(N-1)N^5(N+1)^5(N+2)^5}$$

$$+\frac{4}{9}\frac{P_{379}S_1\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{8P_{391}S_2 S_1}{3(N-1)N^3(N+1)^3(N+2)^3} + \frac{8P_{368}S_3 S_1}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16\left(11N^5 + 45N^4 - 3N^3 - 145N^2 - 176N - 20\right)S_{2,1}S_1}{3(N-1)N(N+1)^2(N+2)^2} - \frac{2}{9}\frac{P_{381}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}$$

$$-\frac{2\left(N^2 + N + 2\right)\left(N^4 + 2N^3 - 16N^2 - 17N - 6\right)S_2^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{1}{18}\frac{P_{397}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^2}$$

$$+\frac{P_{429}}{3(N-1)N^5(N+1)^6(N+2)^5} + L_M^3\left[-\frac{8\left(N^2 + N + 2\right)\left(N^2 + N + 6\right)\left(7N^2 + 7N + 4\right)S_1}{9(N-1)N^2(N+1)^2(N+2)^2}\right.$$

$$\left.-\frac{16}{3}\tilde{\gamma}_{qg}^0 S_1^2 - \frac{2\left(N^2 + N + 2\right)\left(3N^2 + 3N + 2\right)\left(11N^4 + 22N^3 - 59N^2 - 70N - 48\right)}{9(N-1)N^3(N+1)^3(N+2)^2}\right]$$

$$-\frac{4P_{399}S_2}{3(N-1)N^3(N+1)^4(N+2)^3} - 4\frac{\left(N^2 + N + 2\right)\left(3N^4 + 6N^3 + 7N^2 + 4N + 4\right)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2}S_2$$

$$+\frac{4P_{380}S_3}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{4\left(N^2 + N + 2\right)\left(19N^4 + 38N^3 - 22N^2 - 41N - 30\right)S_4}{(N-1)N^2(N+1)^2(N+2)^2}$$

$$-\frac{16\left(N^2 - 3N - 2\right)\left(11N^4 + 22N^3 - 35N^2 - 46N - 24\right)S_{2,1}}{3(N-1)N^3(N+1)^2(N+2)^2}$$

$$-\frac{8\left(N^2 + N + 2\right)\left(31N^4 + 62N^3 - 73N^2 - 104N - 60\right)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+L_M^2\left[\frac{8P_{331}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8P_{389}S_1}{9(N-1)N^3(N+1)^3(N+2)^3}\right.$$

$$+\frac{P_{396}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{8\left(N^2 + N + 2\right)\left(N^4 + 2N^3 + 8N^2 + 7N + 18\right)S_2}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$\left.+\tilde{\gamma}_{qg}^0\left[-8S_1^3 + 4S_3 + 6S_{-2} + 4S_{-3} - 8S_{-2,1}\right]\right]$$

$$+\frac{8\left(N^2 + N + 2\right)\left(35N^4 + 70N^3 - 137N^2 - 172N - 84\right)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8\left(N^2 + N + 2\right)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)}$$

$$-48\tilde{\gamma}_{qg}^0 \log(2)\zeta_2 + \frac{P_{343}\left[16(-1)^N S_{-2} + 8(-1)^N\zeta_2\right]}{N(N+1)^4(N+2)^3} + \frac{P_{342}\left[32(-1)^N S_1 S_{-2} + 16(-1)^N S_1\zeta_2\right]}{N(N+1)^3(N+2)^3}$$

$$+\frac{\left(3N^5 + 4N^4 + 31N^3 + 62N^2 + 20N + 8\right)\left[16(-1)^N S_{-2}S_1^2 + 8(-1)^N\zeta_2 S_1^2\right]}{N^2(N+1)^2(N+2)^2}$$

$$+\frac{\left(N^2 + N + 2\right)\left(3N^2 + 3N + 2\right)}{N^2(N+1)^2(N+2)}\left[\left[8(-1)^N S_2 + 8(-1)^N S_{-2}\right]\zeta_2 + 16(-1)^N S_{-2}S_2 + 8(-1)^N S_{-4}\right.$$

$$\left.-16S_{-2,2} - 16S_{-3,1} + 32S_{-2,1,1}\right] + \frac{\left(6N^5 + 33N^4 + 66N^3 + 77N^2 + 58N + 16\right)}{N(N+1)^3(N+2)^2}\left[16(-1)^N S_{-3}\right.$$

$$-32S_{-2,1} + 12(-1)^N\zeta_3\Big] + \frac{(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8)}{N^2(N+1)^2(N+2)^2}\Big[16(-1)^N S_1 S_{-3} - 32S_1 S_{-2,1}$$

$$+12(-1)^N S_1\zeta_3\Big] + \tilde{\gamma}_{qg}^0\Big[\frac{2}{3}S_1^5 + 12S_2 S_1^3 + 16(-1)^N S_{-3}S_1^2 + \Big[\frac{88S_3}{3} + 16S_{2,1} - 32S_{-2,1}\Big]S_1^2$$

$$+\frac{4}{3}\big(-5 + 9(-1)^N\big)\zeta_3 S_1^2 + 8(-1)^N S_{-4}S_1 + \Big[2S_2^2 + 12S_4 + 8S_{3,1} - 16S_{-2,2} - 16S_{-3,1} - 40S_{2,1,1}$$

$$+32S_{-2,1,1}\Big]S_1 + S_{-2}\Big[16(-1)^N S_1^3 + 16(-1)^N S_2 S_1 + 32\Big] + \Big[8(-1)^N S_1^3 + 4\big(-1 + 2(-1)^N\big)S_{-2}S_1$$

$$-2S_3 - 2S_{-3} + 4S_{-2,1}\Big]\zeta_2\Big] + L_M\Big[\frac{8\big(11N^5 - 46N^4 - 499N^3 - 866N^2 - 496N - 144\big)S_1^3}{9N^2(N+1)^2(N+2)^2}$$

$$-\frac{4P_{393}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3} - \frac{8(N+3)\big(13N^5 + 114N^4 + 93N^3 + 52N^2 + 8N + 8\big)S_2 S_1}{3(N-1)N^2(N+1)^2(N+2)^2}$$

$$+\frac{8P_{395}S_1}{27(N-1)N^4(N+1)^3(N+2)^2} + \frac{64\big(N^3 - 13N^2 - 14N - 2\big)S_{-2}S_1}{N^2(N+1)^2(N+2)}$$

$$-\frac{6P_{358}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{P_{417}}{54(N-1)N^5(N+1)^4(N+2)^3} + \frac{4P_{387}S_2}{3(N-1)N^3(N+1)^3(N+2)^3}$$

$$-\frac{8P_{365}S_3}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{16\big(3N^5 - 40N^4 - 87N^3 - 54N^2 - 10N + 12\big)S_{-2}}{N^3(N+1)^3(N+2)}$$

$$+\frac{16\big(3N^4 + 8N^3 - 5N^2 - 6N + 8\big)S_{-3}}{N^2(N+1)^2(N+2)} + \frac{16\big(3N^4 + 2N^3 + 39N^2 + 32N - 12\big)S_{-2,1}}{N^2(N+1)^2(N+2)}$$

$$+\frac{48\big(N^2 + N + 2\big)^2 S_{2,1}}{(N-1)N(N+1)(N+2)^2} - \frac{\big(N^2 - N - 4\big)\big(3N^2 + 3N + 2\big)16(-1)^N S_{-2}}{N(N+1)^3(N+2)^2}$$

$$-\frac{\big(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8\big)16(-1)^N S_1 S_{-2}}{N^2(N+1)^2(N+2)^2}$$

$$+\frac{\big(N^2 + N + 2\big)\big(3N^2 + 3N + 2\big)\big(-8(-1)^N S_{-3} - 6(-1)^N\zeta_3\big)}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0\Big[-4S_1^4 - 36S_2 S_1^2$$

$$-8\big(3 + 2(-1)^N\big)S_{-2}S_1^2 - 8(-1)^N S_{-3}S_1 + \big(-32S_3 + 48S_{2,1} + 16S_{-2,1}\big)S_1 - 6\big(-5 + (-1)^N\big)\zeta_3 S_1$$

$$+8S_2^2 - 12S_{-2}^2 - 20S_4 - 8S_{3,1} - 8S_{-2,2} - 24S_{2,1,1} + 32S_{-2,1,1}\Big]\Big]\Big] + a_{Qg}^{(3)}\Big\}\Big\}, \tag{436}$$

where

$$\begin{align}
P_{330} &= N^6 - 93N^5 - 444N^4 - 317N^3 + 329N^2 + 296N + 84 \tag{437}\\
P_{331} &= N^6 - 9N^5 - 120N^4 - 137N^3 + 29N^2 + 56N + 36 \tag{438}\\
P_{332} &= N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \tag{439}\\
P_{333} &= N^6 + 8N^5 + 23N^4 + 54N^3 + 94N^2 + 72N + 8 \tag{440}\\
P_{334} &= 2N^6 + 11N^5 + 8N^4 - 7N^3 + 14N^2 + 12N - 24 \tag{441}\\
P_{335} &= 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \tag{442}\\
P_{336} &= 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \tag{443}\\
P_{337} &= 5N^6 + 15N^5 + 36N^4 + 51N^3 + 25N^2 + 8N + 4 \tag{444}\\
P_{338} &= 5N^6 + 18N^5 + 51N^4 + 84N^3 + 60N^2 + 34N + 12 \tag{445}\\
P_{339} &= 5N^6 + 26N^5 + 97N^4 + 160N^3 + 135N^2 + 79N + 22 \tag{446}\\
P_{340} &= 5N^6 + 42N^5 + 84N^4 + 35N^3 + 40N^2 + 34N + 48 \tag{447}\\
P_{341} &= 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12 \tag{448}\\
P_{342} &= 6N^6 + 47N^5 + 136N^4 + 223N^3 + 256N^2 + 172N + 32 \tag{449}\\
P_{343} &= 9N^6 + 27N^5 - 65N^4 - 319N^3 - 404N^2 - 200N - 40 \tag{450}
\end{align}$$

$$P_{344} = 10N^6 - 6N^5 - 39N^4 - 44N^3 - 97N^2 + 20N + 12 \tag{451}$$

$$P_{345} = 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48 \tag{452}$$

$$P_{346} = 11N^6 - 15N^5 - 327N^4 - 181N^3 + 292N^2 - 20N - 48 \tag{453}$$

$$P_{347} = 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N - 240 \tag{454}$$

$$P_{348} = 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72 \tag{455}$$

$$P_{349} = 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N - 108 \tag{456}$$

$$P_{350} = 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N - 48 \tag{457}$$

$$P_{351} = 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 \tag{458}$$

$$P_{352} = 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 \tag{459}$$

$$P_{353} = 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 \tag{460}$$

$$P_{354} = 17N^6 + 51N^5 + 390N^4 + 359N^3 - 389N^2 - 200N - 84 \tag{461}$$

$$P_{355} = 18N^6 + 87N^5 + 57N^4 - 119N^3 - 131N^2 - 60N - 20 \tag{462}$$

$$P_{356} = 23N^6 + 39N^5 + 75N^4 + 157N^3 + 96N^2 + 70N + 28 \tag{463}$$

$$P_{357} = 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 \tag{464}$$

$$P_{358} = 45N^6 + 135N^5 - 91N^4 - 407N^3 - 214N^2 + 12N - 248 \tag{465}$$

$$P_{359} = 51N^6 + 140N^5 + 227N^4 + 208N^3 - 202N^2 - 96N - 8 \tag{466}$$

$$P_{360} = 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 \tag{467}$$

$$P_{361} = 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 \tag{468}$$

$$P_{362} = 57N^6 + 297N^5 + 519N^4 + 399N^3 + 92N^2 - 68N - 16 \tag{469}$$

$$P_{363} = 67N^6 + 93N^5 + 351N^4 + 259N^3 - 1054N^2 - 556N - 312 \tag{470}$$

$$P_{364} = 76N^6 + 487N^5 + 1692N^4 + 3271N^3 + 3186N^2 + 1516N + 536 \tag{471}$$

$$P_{365} = 77N^6 + 195N^5 + 627N^4 + 977N^3 - 128N^2 - 452N + 432 \tag{472}$$

$$P_{366} = 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 \tag{473}$$

$$P_{367} = 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 \tag{474}$$

$$P_{368} = 97N^6 + 591N^5 + 1311N^4 + 229N^3 - 712N^2 + 308N + 192 \tag{475}$$

$$P_{369} = N^8 + 24N^7 + 62N^6 + 8N^5 - 123N^4 - 128N^3 - 108N^2 - 72N - 48 \tag{476}$$

$$P_{370} = N^8 + 427N^7 + 1133N^6 + 697N^5 - 434N^4 - 636N^3 - 244N^2 - 64N - 16 \tag{477}$$

$$P_{371} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \tag{478}$$

$$P_{372} = 2N^8 + 29N^7 + 179N^6 + 441N^5 + 529N^4 + 332N^3 + 172N^2 + 92N + 24 \tag{479}$$

$$P_{373} = 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 \tag{480}$$

$$P_{374} = 9N^8 + 54N^7 + 56N^6 - 182N^5 - 717N^4 - 1120N^3 - 1012N^2 - 672N - 160 \tag{481}$$

$$P_{375} = 12N^8 + 52N^7 + 132N^6 + 216N^5 + 191N^4 + 54N^3 - 25N^2 - 20N - 4 \tag{482}$$

$$P_{376} = 15N^8 + 36N^7 + 50N^6 - 252N^5 - 357N^4 + 152N^3 - 68N^2 + 88N + 48 \tag{483}$$

$$P_{377} = 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 \tag{484}$$

$$P_{378} = 33N^8 + 132N^7 + 350N^6 + 636N^5 + 685N^4 + 528N^3 + 292N^2 + 128N + 32 \tag{485}$$

$$P_{379} = 121N^8 + 370N^7 + 924N^6 + 358N^5 - 381N^4 + 184N^3 - 1096N^2 - 48N + 144 \tag{486}$$

$$P_{380} = 321N^8 + 1674N^7 + 2360N^6 - 1378N^5 - 6565N^4 - 5992N^3 - 1972N^2 + 128N - 96 \tag{487}$$

$$P_{381} = 507N^8 + 2190N^7 + 3002N^6 + 1692N^5 - 681N^4 - 2554N^3 - 404N^2 + 664N + 192 \tag{488}$$

$$P_{382} = 633N^8 + 2532N^7 + 5036N^6 + 6174N^5 + 4307N^4 + 1182N^3 - 176N^2 - 184N - 48 \tag{489}$$

$$P_{383} = N^9 + 6N^8 + 15N^7 + 25N^6 + 36N^5 + 85N^4 + 128N^3 + 104N^2 + 64N + 16 \tag{490}$$

$$P_{384} = N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 \tag{491}$$

$$P_{385} = 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 1172N^3 + 904N^2 + 512N + 128 \tag{492}$$

$$P_{386} = 6N^9 + 93N^8 + 576N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2$$
$$+1488N + 384 \tag{493}$$

$$P_{387} = 25N^9 - 43N^8 - 424N^7 + 462N^6 + 4345N^5 + 7513N^4 + 6446N^3 + 4020N^2$$
$$+1944N + 480 \tag{494}$$

$$P_{388} = 36N^9 + 156N^8 - 115N^7 - 1116N^6 - 1251N^5 - 78N^4 + 300N^3 + 84N^2 - 128N - 48 \tag{495}$$

$$P_{389} = 40N^9 + 273N^8 + 635N^7 + 613N^6 + 119N^5 - 2N^4 - 314N^3 - 668N^2 + 24N + 144 \tag{496}$$

$$P_{390} = 45N^9 + 270N^8 + 724N^7 + 1262N^6 + 1731N^5 + 2740N^4 + 3484N^3 + 2928N^2$$
$$+1696N + 384 \tag{497}$$

$$P_{391} = 66N^9 + 534N^8 + 1409N^7 + 1080N^6 - 933N^5 - 1116N^4 + 588N^3 + 996N^2$$
$$+736N + 240 \tag{498}$$

$$P_{392} = 69N^9 + 366N^8 + 1100N^7 + 1894N^6 + 2451N^5 + 5276N^4 + 7460N^3 + 5352N^2$$
$$+3008N + 672 \tag{499}$$

$$P_{393} = 80N^9 + 441N^8 + 568N^7 - 592N^6 - 1202N^5 + 2003N^4 + 4106N^3 + 3116N^2$$
$$+2712N + 864 \tag{500}$$

$$P_{394} = 94N^9 + 597N^8 + 1508N^7 + 2086N^6 + 1517N^5 + 1381N^4 + 2731N^3 + 3802N^2$$
$$+2916N + 648 \tag{501}$$

$$P_{395} = 251N^9 + 1586N^8 + 4206N^7 + 6764N^6 + 4008N^5 - 2242N^4 + 13N^3 + 7122N^2$$
$$+6156N + 1944 \tag{502}$$

$$P_{396} = 489N^9 + 2934N^8 + 7636N^7 + 12206N^6 + 6675N^5 - 12692N^4 - 24608N^3 - 16272N^2$$
$$-2864N + 2304 \tag{503}$$

$$P_{397} = 891N^9 + 5751N^8 + 15070N^7 + 21430N^6 + 37623N^5 + 55339N^4 + 44064N^3 + 25144N^2$$
$$+9488N + 1776 \tag{504}$$

$$P_{398} = 10N^{10} + 62N^9 + 407N^8 + 1119N^7 + 1405N^6 + 889N^5 + 240N^4 - 90N^3 - 114N^2$$
$$-48N - 8 \tag{505}$$

$$P_{399} = 36N^{10} + 456N^9 + 2448N^8 + 7171N^7 + 12399N^6 + 13213N^5 + 8997N^4 + 5000N^3 + 2888N^2$$
$$+992N + 112 \tag{506}$$

$$P_{400} = 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 9218N^4 - 6116N^3 - 72N^2$$
$$-752N - 192 \tag{507}$$

$$P_{401} = 85N^{10} + 425N^9 + 830N^8 + 788N^7 - 521N^6 - 325N^5 + 2238N^4 + 2568N^3 + 968N^2$$
$$-1296N - 576 \tag{508}$$

$$P_{402} = 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2$$
$$-2160N - 864 \tag{509}$$

$$P_{403} = 149N^{10} + 793N^9 + 2368N^8 + 5026N^7 + 6853N^6 + 6277N^5 + 5062N^4 + 3168N^3 + 1296N^2$$
$$+400N + 96 \tag{510}$$

$$P_{404} = 170N^{10} + 883N^9 + 1897N^8 + 2710N^7 - 448N^6 - 4745N^5 + 561N^4 + 5904N^3 + 1132N^2$$
$$-2016N - 864 \tag{511}$$

$$P_{405} = 170N^{10} + 1213N^9 + 3091N^8 + 2506N^7 - 2692N^6 - 3047N^5 - 861N^4 - 2352N^3 - 5324N^2$$
$$-6240N - 2016 \tag{512}$$

$$P_{406} = 436N^{10} + 3960N^9 + 15787N^8 + 36343N^7 + 46431N^6 + 17745N^5 - 28270N^4 - 33648N^3$$
$$-11056N^2 - 1936N + 864 \tag{513}$$

$$P_{407} = 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3$$
$$-1232N^2 - 736N - 192 \tag{514}$$

$$
\begin{aligned}
P_{408} \;=\;& 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 + 7328N^3 \\
& +5456N^2 + 3456N + 1152 \tag{515}
\end{aligned}
$$

$$
\begin{aligned}
P_{409} \;=\;& 95N^{11} + 853N^{10} + 3599N^9 + 9245N^8 + 12320N^7 - 282N^6 - 23342N^5 - 26920N^4 \\
& -10832N^3 - 1712N^2 - 416N - 192 \tag{516}
\end{aligned}
$$

$$
\begin{aligned}
P_{410} \;=\;& 129N^{11} + 903N^{10} + 2894N^9 + 5730N^8 + 6505N^7 + 383N^6 - 9464N^5 - 13912N^4 - 11680N^3 \\
& -6640N^2 - 3648N - 1152 \tag{517}
\end{aligned}
$$

$$
\begin{aligned}
P_{411} \;=\;& 243N^{11} + 1701N^{10} + 5378N^9 + 10350N^8 + 11479N^7 + 1193N^6 - 14684N^5 - 20572N^4 \\
& -16288N^3 - 8944N^2 - 4992N - 1728 \tag{518}
\end{aligned}
$$

$$
\begin{aligned}
P_{412} \;=\;& 333N^{11} + 2331N^{10} + 6556N^9 + 9270N^8 + 5081N^7 - 6701N^6 - 17554N^5 - 20036N^4 \\
& -15680N^3 - 9200N^2 - 5664N - 1728 \tag{519}
\end{aligned}
$$

$$
\begin{aligned}
P_{413} \;=\;& 2N^{12} + 20N^{11} + 86N^{10} + 192N^9 + 199N^8 - N^7 - 297N^6 - 495N^5 - 514N^4 - 488N^3 \\
& -416N^2 - 176N - 32 \tag{520}
\end{aligned}
$$

$$
\begin{aligned}
P_{414} \;=\;& 23N^{12} + 138N^{11} - 311N^{10} - 3148N^9 - 7605N^8 - 8462N^7 - 4163N^6 + 246N^5 + 1540N^4 \\
& +1066N^3 + 444N^2 + 120N + 16 \tag{521}
\end{aligned}
$$

$$
\begin{aligned}
P_{415} \;=\;& 111N^{12} + 1035N^{11} + 3634N^{10} + 5168N^9 - 2662N^8 - 21724N^7 - 37157N^6 - 34963N^5 \\
& -19122N^4 - 4560N^3 + 80N^2 + 1008N + 288 \tag{522}
\end{aligned}
$$

$$
\begin{aligned}
P_{416} \;=\;& 201N^{12} + 1845N^{11} + 6742N^{10} + 11990N^9 + 7139N^8 - 8917N^7 - 15710N^6 - 2110N^5 \\
& +16644N^4 + 22080N^3 + 12416N^2 + 4128N + 576 \tag{523}
\end{aligned}
$$

$$
\begin{aligned}
P_{417} \;=\;& 7299N^{12} + 53973N^{11} + 206656N^{10} + 532170N^9 + 820775N^8 + 650149N^7 + 204230N^6 \\
& +189820N^5 + 606016N^4 + 664624N^3 + 372192N^2 + 143424N + 27648 \tag{524}
\end{aligned}
$$

$$
\begin{aligned}
P_{418} \;=\;& 9N^{13} + 72N^{12} + 101N^{11} - 511N^{10} - 2325N^9 - 4428N^8 - 4619N^7 - 3841N^6 - 4462N^5 \\
& -6012N^4 - 6992N^3 - 5296N^2 - 2592N - 576 \tag{525}
\end{aligned}
$$

$$
\begin{aligned}
P_{419} \;=\;& 69N^{13} + 420N^{12} + 794N^{11} - 1501N^{10} - 11265N^9 - 18414N^8 - 4436N^7 - 5017N^6 \\
& -41818N^5 - 65616N^4 - 62960N^3 - 39184N^2 - 17184N - 3456 \tag{526}
\end{aligned}
$$

$$
\begin{aligned}
P_{420} \;=\;& 296N^{13} + 2368N^{12} + 9908N^{11} + 22254N^{10} + 13564N^9 - 31716N^8 - 71723N^7 - 71221N^6 \\
& -44369N^5 - 33249N^4 - 26584N^3 + 4968N^2 + 7344N + 432 \tag{527}
\end{aligned}
$$

$$
\begin{aligned}
P_{421} \;=\;& 385N^{14} + 2567N^{13} + 6877N^{12} + 9235N^{11} + 5375N^{10} - 1207N^9 - 3313N^8 + 905N^7 \\
& +3876N^6 + 1676N^5 + 256N^4 + 1544N^3 + 1616N^2 + 736N + 192 \tag{528}
\end{aligned}
$$

$$
\begin{aligned}
P_{422} \;=\;& 531N^{14} + 5454N^{13} + 25877N^{12} + 77604N^{11} + 159437N^{10} + 205070N^9 + 82971N^8 \\
& -207408N^7 - 490544N^6 - 694320N^5 - 735104N^4 - 562304N^3 - 355584N^2 - 158976N \\
& -34560 \tag{529}
\end{aligned}
$$

$$
\begin{aligned}
P_{423} \;=\;& 1773N^{14} + 18018N^{13} + 80795N^{12} + 214620N^{11} + 371423N^{10} + 398930N^9 + 154773N^8 \\
& -228072N^7 - 435356N^6 - 492936N^5 - 534656N^4 - 453440N^3 - 299712N^2 \\
& -144000N - 34560 \tag{530}
\end{aligned}
$$

$$
\begin{aligned}
P_{424} \;=\;& 4N^{15} + 50N^{14} + 267N^{13} + 765N^{12} + 1183N^{11} + 682N^{10} - 826N^9 - 1858N^8 - 1116N^7 \\
& +457N^6 + 1500N^5 + 2268N^4 + 2400N^3 + 1392N^2 + 448N + 64 \tag{531}
\end{aligned}
$$

$$
\begin{aligned}
P_{425} \;=\;& 26N^{15} + 314N^{14} + 1503N^{13} + 3222N^{12} + 2510N^{11} + 1996N^{10} + 15041N^9 + 40728N^8 \\
& +54008N^7 + 44956N^6 + 31936N^5 + 30416N^4 + 29568N^3 + 16704N^2 + 5376N + 768 \tag{532}
\end{aligned}
$$

$$
\begin{aligned}
P_{426} \;=\;& 28N^{15} + 335N^{14} + 1953N^{13} + 6497N^{12} + 11508N^{11} + 6624N^{10} - 11753N^9 - 27541N^8 \\
& -33352N^7 - 40915N^6 - 40468N^5 - 16628N^4 + 7584N^3 + 10416N^2 + 4032N + 576 \tag{533}
\end{aligned}
$$

$$
\begin{aligned}
P_{427} \;=\;& 435N^{15} + 5436N^{14} + 32317N^{13} + 119006N^{12} + 307057N^{11} + 620328N^{10} + 1065977N^9 \\
& +1575060N^8 + 1889534N^7 + 1704634N^6 + 1113248N^5 + 592440N^4 + 328672N^3
\end{aligned}
$$

$$+165984N^2 + 59904N + 10368 \tag{534}$$

$$P_{428} = 939N^{16} + 10527N^{15} + 47251N^{14} + 101719N^{13} + 66350N^{12} - 155710N^{11}$$
$$- 322813N^{10} - 16829N^9 + 702425N^8 + 1332497N^7 + 1596952N^6 + 1640548N^5$$
$$+ 1506496N^4 + 1099952N^3 + 604032N^2 + 211392N + 34560 \tag{535}$$

$$P_{429} = 1623N^{16} + 20963N^{15} + 119399N^{14} + 394315N^{13} + 831483N^{12} + 1160715N^{11} + 1086519N^{10}$$
$$+ 712841N^9 + 425270N^8 + 337718N^7 + 207634N^6 - 73752N^5 - 261272N^4$$
$$- 200160N^3 - 64672N^2 - 4480N + 1408 \tag{536}$$

$$P_{430} = 87N^{17} + 1099N^{16} + 6055N^{15} + 19019N^{14} + 37119N^{13} + 45159N^{12} + 29583N^{11} - 2639N^{10}$$
$$- 30218N^9 - 40778N^8 - 39994N^7 - 35844N^6 - 30808N^5 - 30384N^4 - 28256N^3$$
$$- 16064N^2 - 5248N - 768 \;. \tag{537}$$

The operator matrix element $A_{gg,Q}$ except for the term $a_{gg,Q}^{(3)}$ is given by :

$$A_{gg,Q} = \tfrac{1}{2}[1 + (-1)^N] \times$$

$$\left\{ a_s \frac{4}{3} T_F L_M + a_s^2 \left\{ \frac{16}{9} T_F^2 L_M^2 + C_A T_F \left[\left[\frac{16(N^2 + N + 1)}{3(N-1)N(N+1)(N+2)} - \frac{8S_1}{3} \right] L_M^2 \right. \right. \right.$$

$$\left. + \left[\frac{16 P_{433}}{9(N-1)N^2(N+1)^2(N+2)} - \frac{80S_1}{9} \right] L_M + \frac{2 P_{451}}{27(N-1)N^3(N+1)^3(N+2)} - \frac{4(56N+47)S_1}{27(N+1)} \right]$$

$$+ C_F T_F \left[\frac{4(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} L_M^2 + \frac{4 P_{445}}{(N-1)N^3(N+1)^3(N+2)} L_M \right.$$

$$\left. \left. - \frac{P_{461}}{(N-1)N^4(N+1)^4(N+2)} \right] \right\}$$

$$+ a_s^3 \left\{ \left[T_F^3 \left[\frac{64}{27} L_M^3 - \frac{64\zeta_3}{27} \right] + C_A T_F^2 \left[\left[\frac{448(N^2+N+1)}{27(N-1)N(N+1)(N+2)} - \frac{224S_1}{27} \right] L_M^3 \right. \right. \right.$$

$$+ \left[\frac{8 P_{441}}{27(N-1)N^2(N+1)^2(N+2)} - \frac{640S_1}{27} \right] L_M^2 + \left[- \frac{2 P_{454}}{27(N-1)N^3(N+1)^3(N+2)} \right.$$

$$\left. - \frac{8 P_{440} S_1}{9(N-1)N^2(N+1)^2(N+2)} \right] L_M + \frac{8 S_1^2}{3(N+1)} - \frac{4}{27} \frac{P_{442} \zeta_2}{(N-1)N^2(N+1)^2(N+2)}$$

$$- \frac{8 P_{460}}{81(N-1)N^4(N+1)^4(N+2)} + \frac{16(328N^4 + 256N^3 - 247N^2 - 175N + 54)S_1}{81(N-1)N(N+1)^2}$$

$$- \frac{448}{27} \frac{(N^2+N+1)\zeta_3}{(N-1)N(N+1)(N+2)} - \frac{8(2N+1)S_2}{3(N+1)} + \frac{560}{27} S_1 \zeta_2 + \frac{224}{27} S_1 \zeta_3$$

$$+ N_F \left[\left[\frac{128(N^2+N+1)}{27(N-1)N(N+1)(N+2)} - \frac{64S_1}{27} \right] L_M^3 + \left[- \frac{4 P_{457}}{81(N-1)N^3(N+1)^3(N+2)} \right. \right.$$

$$\left. - \frac{16 P_{443} S_1}{81(N-1)N^2(N+1)^2(N+2)} \right] L_M + \frac{16 S_1^2}{9(N+1)} - \frac{4}{27} \frac{\zeta_2 P_{437}}{(N-1)N^2(N+1)^2(N+2)}$$

$$+ \frac{32 P_{465}}{243(N-1)N^4(N+1)^4(N+2)} + \frac{32(328N^4 + 256N^3 - 247N^2 - 175N + 54)S_1}{243(N-1)N(N+1)^2}$$

$$\left. \left. \left. - \frac{128}{27} \frac{(N^2+N+1)\zeta_3}{(N-1)N(N+1)(N+2)} - \frac{16(2N+1)S_2}{9(N+1)} + \frac{160}{27} S_1 \zeta_2 + \frac{64}{27} S_1 \zeta_3 \right] \right] \right.$$

$$+C_A^2 T_F \left[\left[\frac{176 S_1}{27} - \frac{352(N^2+N+1)}{27(N-1)N(N+1)(N+2)} \right] L_M^3 + \left[-\frac{2P_{469}}{9(N-1)^2 N^3(N+1)^3(N+2)^3} \right. \right.$$

$$-\frac{8P_{452}S_1}{9(N-1)^2 N^2(N+1)^2(N+2)^2} + \frac{64}{3}S_{-2}S_1 + \frac{64}{3}S_1 S_2 + \frac{(N^2+N+1)\left[-\frac{128}{3}S_2 - \frac{128}{3}S_{-2} \right]}{(N-1)N(N+1)(N+2)}$$

$$\left. + \frac{32S_3}{3} + \frac{32}{3}S_{-3} - \frac{64}{3}S_{-2,1} \right] L_M^2 + \left[\frac{32}{3}S_{-2}^2 - \frac{16P_{456}S_{-2}}{9(N-1)^2 N^3(N+1)^3(N+2)^2} \right.$$

$$+ \frac{32P_{439}S_1 S_{-2}}{9(N-1)N^2(N+1)^2(N+2)} + 64\frac{(2N^4+4N^3+7N^2+5N+6)\zeta_3}{(N-1)N^2(N+1)^2(N+2)}$$

$$+ \frac{P_{479}}{81(N-1)^2 N^4(N+1)^4(N+2)^3} - \frac{4P_{474}S_1}{81(N-1)^2 N^4(N+1)^4(N+2)^2} + \left[\frac{640S_2}{9} - \frac{32S_3}{3} \right] S_1$$

$$+ \frac{16P_{432}S_2}{9(N-1)N^2(N+1)^2(N+2)} + \frac{8(40N^4+80N^3+73N^2+33N+54)S_3}{9N^2(N+1)^2} - 64 S_1 \zeta_3$$

$$\left. + \frac{P_{438}\left(\frac{16}{9}S_{-3} - \frac{32}{9}S_{-2,1} \right)}{(N-1)N^2(N+1)^2(N+2)} \right] L_M - \frac{44S_1^2}{9(N+1)} - \frac{8P_{468}}{243(N-1)N^4(N+1)^4(N+2)}$$

$$+ \frac{4}{27}\frac{P_{470}\zeta_2}{(N-1)^2 N^3(N+1)^3(N+2)^3} - \frac{8(2834N^4+2042N^3-1943N^2-1151N+594)S_1}{243(N-1)N(N+1)^2}$$

$$+ \frac{16}{27}\frac{P_{444}S_1\zeta_2}{(N-1)^2 N^2(N+1)^2(N+2)^2} + \frac{44(2N+1)S_2}{9(N+1)} + \left[-\frac{32}{3}S_1 S_2 - \frac{32}{3}S_{-2}S_1 - \frac{16S_3}{3} \right.$$

$$\left. -\frac{16}{3}S_{-3} + \frac{32}{3}S_{-2,1} \right] \zeta_2 - \frac{176}{27}S_1\zeta_3 + \frac{(N^2+N+1)\left(\left(\frac{64S_2}{3} + \frac{64}{3}S_{-2} \right)\zeta_2 + \frac{352\zeta_3}{27} \right)}{(N-1)N(N+1)(N+2)} \right]$$

$$+C_F^2 T_F \left[\left[\frac{16(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} - \frac{4(N^2+N+2)^2(3N^2+3N+2)}{3(N-1)N^3(N+1)^3(N+2)} \right] L_M^3 \right.$$

$$+ \left[\frac{8(5N^2+N-2)S_1(N^2+N+2)^2}{(N-1)N^3(N+1)^3(N+2)} - \frac{16S_2(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right.$$

$$\left. -\frac{4P_{436}(N^2+N+2)}{(N-1)N^4(N+1)^4(N+2)} \right] L_M^2 + \left[\frac{\left(-\frac{8}{3}S_1^3 + 24S_2 S_1 - 32S_{2,1} \right)(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right.$$

$$+ \frac{4(5N^3+4N^2+9N+6)S_1^2(N^2+N+2)}{(N-1)N^2(N+1)^3(N+2)} + \frac{16(5N^2+5N-14)S_3(N^2+N+2)}{3(N-1)N^2(N+1)^2(N+2)}$$

$$- \frac{4(17N^4+48N^3+69N^2+10N-8)S_2(N^2+N+2)}{(N-1)N^3(N+1)^3(N+2)} - \frac{96\zeta_3(N^2+N+2)}{N^2(N+1)^2}$$

$$+ \frac{\left(-256S_1 S_{-2} - 128S_{-3} + 256S_{-2,1} \right)(N^2+N+2)}{(N-1)N^2(N+1)^2(N+2)} - \frac{2P_{471}}{(N-1)N^5(N+1)^5(N+2)}$$

$$\left. - \frac{8P_{448}S_1}{(N-1)N^4(N+1)^4(N+2)} - \frac{32(N^6+3N^5+N^4-3N^3-26N^2-24N-16)S_{-2}}{(N-1)N^3(N+1)^3(N+2)} \right] L_M$$

$$+ \frac{4(N^2+N+2)(N^4-5N^3-32N^2-18N-4)S_1^2}{(N-1)N^3(N+1)^3(N+2)} - \frac{4}{3}\frac{P_{453}\zeta_3}{(N-1)N^3(N+1)^3(N+2)}$$

$$-2\frac{P_{464}\zeta_2}{(N-1)N^4(N+1)^4(N+2)} - \frac{8(N^2+N+2)(2N^5-2N^4-11N^3-19N^2-44N-12)S_1}{(N-1)N^3(N+1)^4(N+2)}$$

$$- \frac{P_{480}}{(N-1)N^6(N+1)^6(N+2)} - 4\frac{(N^2+N+2)(5N^4+4N^3+N^2-10N-8)\zeta_2}{(N-1)N^3(N+1)^3(N+2)}S_1$$

$$+\frac{4\left(N^2+N+2\right)P_{434}S_2}{(N-1)N^4(N+1)^4(N+2)}+\frac{(3N+2)\left(N^2+N+2\right)\left(\frac{8}{3}S_1^3+8S_2S_1\right)}{(N-1)N^3(N+1)(N+2)}+128\log(2)\zeta_2$$

$$+\frac{8\left(N^2+N+2\right)\left(3N^4+48N^3+43N^2-22N-8\right)S_3}{3(N-1)N^3(N+1)^3(N+2)}-\frac{32\left(N^2-3N-2\right)\left(N^2+N+2\right)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)}$$

$$+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[-\frac{2}{3}S_1^4-4S_2S_1^2+\left[-\frac{16}{3}S_3-32S_{2,1}\right]S_1-\frac{16}{3}\zeta_3S_1-2S_2^2+12S_4\right.$$

$$\left.-32S_{3,1}+64S_{2,1,1}+\left[12S_2-4S_1^2\right]\zeta_2\right]$$

$$+C_FT_F^2\left[\frac{80\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)}L_M^3+\left[\frac{32\left(N^2+N+2\right)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\right.\right.$$

$$+\frac{8P_{449}}{9(N-1)N^3(N+1)^3(N+2)}\right]L_M^2+\left[\frac{\left(\frac{16}{3}S_1^2-16S_2\right)\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\right.$$

$$-\frac{8P_{467}}{27(N-1)N^4(N+1)^4(N+2)}+\frac{32P_{435}S_1}{9(N-1)N^3(N+1)^3(N+2)}\right]L_M$$

$$-\frac{8}{9}\frac{P_{450}\zeta_2}{(N-1)N^3(N+1)^3(N+2)}+\frac{2P_{475}}{9(N-1)N^5(N+1)^5(N+2)}$$

$$+N_F\left[\frac{64\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)}L_M^3+\left[\frac{\left[16S_1^2-\frac{80S_2}{3}\right]\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\right.\right.$$

$$-\frac{4P_{466}}{9(N-1)N^4(N+1)^4(N+2)}-\frac{32\left(5N^5+52N^4+109N^3+90N^2+48N+16\right)S_1}{3(N-1)N^3(N+1)^3(N+2)}\right]L_M$$

$$+\frac{4}{9}\frac{P_{455}\zeta_2}{(N-1)N^3(N+1)^3(N+2)}-\frac{16\left(N^2+N+2\right)\left(8N^3+13N^2+27N+16\right)\left(S_1^2+S_2\right)}{9(N-1)N^2(N+1)^3(N+2)}$$

$$+\frac{2P_{476}}{81(N-1)N^5(N+1)^5(N+2)}+\frac{32\left(N^2+N+2\right)\left(43N^4+105N^3+224N^2+230N+86\right)S_1}{27(N-1)N^2(N+1)^4(N+2)}$$

$$+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\frac{16}{9}S_1^3+\frac{16}{3}S_2S_1+\frac{16}{3}\zeta_2S_1+\frac{32S_3}{9}-\frac{64\zeta_3}{9}\right]\right]$$

$$+\frac{\left(N^2+N+2\right)^2\left(-\frac{16}{3}S_1\zeta_2-\frac{80\zeta_3}{9}\right)}{(N-1)N^2(N+1)^2(N+2)}\right]$$

$$+C_AC_FT_F\left[-\frac{8\left(N^2+N+2\right)\left(N^3+8N^2+11N+2\right)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2}-\frac{4\left(N^2+N+2\right)P_{431}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3}\right.$$

$$-\frac{4}{3}\frac{P_{463}\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2}S_1-\frac{2P_{473}S_1}{9(N-1)N^2(N+1)^5(N+2)^4}$$

$$+\frac{8\left(N^2+N+2\right)\left(3N^3-12N^2-27N-2\right)S_2S_1}{(N-1)N^2(N+1)^3(N+2)^2}+\frac{8}{9}\frac{\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2}P_{462}$$

$$+\frac{4}{9}\frac{P_{477}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3}-\frac{P_{482}}{18(N-1)^2N^6(N+1)^6(N+2)^5}$$

$$+L_M^3\left[-\frac{8\left(11N^4+22N^3-23N^2-34N-12\right)\left(N^2+N+2\right)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2}-\frac{16S_1\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)}\right]$$

$$-\frac{4\left(N^2+N+2\right)P_{458}S_2}{(N-1)^2N^4(N+1)^4(N+2)^3}-\frac{64\left(N^2+N+2\right)\left(N^5+10N^4+9N^3+3N^2+7N+6\right)S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$+L_M^2\left[\frac{\left[-16S_2-32S_{-2}\right]\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}-\frac{2P_{478}}{9(N-1)^2N^4(N+1)^4(N+2)^3}\right.$$

$$\left.-\frac{8P_{459}S_1}{3(N-1)^2N^3(N+1)^3(N+2)^2}\right]-64\log(2)\zeta_2$$

$$+\frac{\left(N^2+N+2\right)\left(N^4+2N^3+7N^2+22N+20\right)\left[-32(-1)^NS_{-2}-16(-1)^N\zeta_2\right]}{(N-1)N(N+1)^4(N+2)^3}$$

$$+\frac{\left(N^2-N-4\right)\left(N^2+N+2\right)}{(N-1)N(N+1)^3(N+2)^2}\left[64(-1)^NS_1S_{-2}+32(-1)^NS_{-3}-64S_{-2,1}+32(-1)^NS_1\zeta_2\right.$$

$$\left.+24(-1)^N\zeta_3\right]+\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\frac{2}{3}S_1^4+20S_2S_1^2+32(-1)^NS_{-3}S_1\right.$$

$$+\left[\frac{160S_3}{3}-64S_{-2,1}\right]S_1+\frac{8}{3}\left(-7+9(-1)^N\right)\zeta_3S_1+2S_2^2+S_{-2}\left[32(-1)^NS_1^2+32(-1)^NS_2\right]$$

$$+36S_4+16(-1)^NS_{-4}-16S_{3,1}-32S_{-2,2}-32S_{-3,1}-16S_{2,1,1}+64S_{-2,1,1}+\left[4\left(-3+4(-1)^N\right)S_1^2\right.$$

$$\left.+4\left(-1+4(-1)^N\right)S_2+8\left(1+2(-1)^N\right)S_{-2}\right]\zeta_2\right]+L_M\left[\frac{\left(N^2+N+2\right)^2}{(N-1)N^2(N+1)^2(N+2)}\left[\frac{8}{3}S_1^3-40S_2S_1\right.\right.$$

$$\left.-32(-1)^NS_{-2}S_1-16(-1)^NS_{-3}+32S_{2,1}-12(-1)^N\zeta_3\right]-\frac{16\left(5N^2+5N-26\right)S_3\left(N^2+N+2\right)}{3(N-1)N^2(N+1)^2(N+2)}$$

$$-\frac{4\left(17N^4-6N^3+41N^2-16N-12\right)S_1^2\left(N^2+N+2\right)}{3(N-1)^2N^3(N+1)^2(N+2)}+\frac{96\left(N^2+N+4\right)S_{-3}\left(N^2+N+2\right)}{(N-1)N^2(N+1)^2(N+2)}$$

$$-\frac{32\left(N^2+N+14\right)S_{-2,1}\left(N^2+N+2\right)}{(N-1)N^2(N+1)^2(N+2)}-\frac{\left(N^2-N-4\right)32(-1)^NS_{-2}\left(N^2+N+2\right)}{(N-1)N(N+1)^3(N+2)^2}$$

$$+\frac{8P_{481}}{27(N-1)^2N^5(N+1)^5(N+2)^4}-\frac{4\left(N^2+N+10\right)\left(5N^2+5N+18\right)\zeta_3}{(N-1)N^2(N+1)^2(N+2)}$$

$$-\frac{8P_{472}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^2}+\frac{64\left(N^4+2N^3+7N^2+6N+16\right)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)}$$

$$\left.\left.\left.-\frac{4P_{446}S_2}{(N-1)^2N^3(N+1)^3(N+2)^2}-\frac{32P_{447}S_{-2}}{(N-1)^2N^3(N+1)^3(N+2)^2}+64S_1\zeta_3\right]\right]\right]+a_{gg,Q}^{(3)}\right\}\right\},\quad(538)$$

with

$$
\begin{aligned}
P_{431} &= N^6+6N^5+7N^4+4N^3+18N^2+16N-8 & (539)\\
P_{432} &= 3N^6+9N^5-113N^4-241N^3-274N^2-152N-24 & (540)\\
P_{433} &= 3N^6+9N^5+22N^4+29N^3+41N^2+28N+6 & (541)\\
P_{434} &= 3N^6+30N^5+15N^4-64N^3-56N^2-20N-8 & (542)\\
P_{435} &= 4N^6+3N^5-50N^4-129N^3-100N^2-56N-24 & (543)\\
P_{436} &= 7N^6+15N^5+7N^4-23N^3-26N^2-20N-8 & (544)\\
P_{437} &= 9N^6+27N^5+161N^4+277N^3+358N^2+224N+48 & (545)\\
P_{438} &= 20N^6+60N^5+11N^4-78N^3-121N^2-72N-108 & (546)\\
P_{439} &= 20N^6+60N^5+11N^4-78N^3-85N^2-36N-108 & (547)\\
P_{440} &= 40N^6+114N^5+19N^4-132N^3-147N^2-70N-32 & (548)\\
P_{441} &= 63N^6+189N^5+367N^4+419N^3+626N^2+448N+96 & (549)\\
P_{442} &= 99N^6+297N^5+631N^4+767N^3+1118N^2+784N+168 & (550)\\
P_{443} &= 136N^6+390N^5+19N^4-552N^3-947N^2-630N-288 & (551)
\end{aligned}
$$

$$P_{444} = N^8 + 4N^7 + 2N^6 + 64N^5 + 173N^4 + 292N^3 + 256N^2 - 72N - 72 \tag{552}$$

$$P_{445} = N^8 + 4N^7 + 8N^6 + 6N^5 - 3N^4 - 22N^3 - 10N^2 - 8N - 8 \tag{553}$$

$$P_{446} = 3N^8 - 14N^7 - 164N^6 - 454N^5 - 527N^4 - 204N^3 - 112N^2 + 80N + 48 \tag{554}$$

$$P_{447} = 3N^8 + 10N^7 + 13N^6 + N^5 + 28N^4 + 81N^3 + 4N^2 - 12N - 32 \tag{555}$$

$$P_{448} = 3N^8 + 23N^7 + 51N^6 + 95N^5 + 142N^4 + 158N^3 + 56N^2 - 32N - 16 \tag{556}$$

$$P_{449} = 15N^8 + 60N^7 + 76N^6 - 18N^5 - 275N^4 - 546N^3 - 400N^2 - 224N - 96 \tag{557}$$

$$P_{450} = 15N^8 + 60N^7 + 86N^6 + 12N^5 - 166N^4 - 378N^3 - 245N^2 - 148N - 84 \tag{558}$$

$$P_{451} = 15N^8 + 60N^7 + 572N^6 + 1470N^5 + 2135N^4 + 1794N^3 + 722N^2 - 24N - 72 \tag{559}$$

$$P_{452} = 23N^8 + 92N^7 + 46N^6 - 88N^5 + 79N^4 + 476N^3 + 428N^2 - 96N - 96 \tag{560}$$

$$P_{453} = 24N^8 + 96N^7 + 93N^6 - 57N^5 - 143N^4 - 79N^3 - 34N^2 - 20N - 8 \tag{561}$$

$$P_{454} = 27N^8 + 108N^7 - 1440N^6 - 4554N^5 - 5931N^4 - 3762N^3 - 256N^2 + 1184N + 480 \tag{562}$$

$$P_{455} = 63N^8 + 252N^7 + 196N^6 - 258N^5 - 551N^4 - 282N^3 - 220N^2 - 80N + 48 \tag{563}$$

$$P_{456} = 131N^8 + 524N^7 + 691N^6 + 239N^5 - 848N^4 - 1483N^3 - 586N^2 + 108N + 360 \tag{564}$$

$$P_{457} = 297N^8 + 1188N^7 + 640N^6 - 2094N^5 - 1193N^4 + 2874N^3 + 5008N^2 + 3360N + 864 \tag{565}$$

$$P_{458} = N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 \tag{566}$$

$$P_{459} = 3N^{10} + 15N^9 + 35N^8 + 50N^7 + 91N^6 + 233N^5 + 255N^4 + 150N^3 - 24N^2$$
$$-184N - 48 \tag{567}$$

$$P_{460} = 3N^{10} + 15N^9 + 3316N^8 + 12778N^7 + 22951N^6 + 23815N^5 + 14212N^4 + 3556N^3$$
$$-30N^2 + 288N + 216 \tag{568}$$

$$P_{461} = 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2$$
$$-32N - 16 \tag{569}$$

$$P_{462} = 18N^{10} + 90N^9 + 119N^8 - 91N^7 - 167N^6 + 101N^5 + 162N^4 - 72N^3 - 504N^2$$
$$-184N - 48 \tag{570}$$

$$P_{463} = 30N^{10} + 150N^9 + 163N^8 - 224N^7 - 586N^6 - 368N^5 - 39N^4 - 78N^3 + 144N^2$$
$$+184N + 48 \tag{571}$$

$$P_{464} = 40N^{10} + 200N^9 + 282N^8 - 66N^7 - 615N^6 - 753N^5 - 509N^4 - 205N^3 - 2N^2$$
$$+68N + 24 \tag{572}$$

$$P_{465} = 63N^{10} + 315N^9 - 1142N^8 - 6260N^7 - 11927N^6 - 12359N^5 - 7235N^4 - 1778N^3$$
$$+15N^2 - 144N - 108 \tag{573}$$

$$P_{466} = 67N^{10} + 335N^9 + 368N^8 - 762N^7 - 3349N^6 - 6669N^5 - 8310N^4 - 7656N^3$$
$$-4648N^2 - 1600N - 288 \tag{574}$$

$$P_{467} = 219N^{10} + 1095N^9 + 1640N^8 - 82N^7 - 2467N^6 - 2947N^5 - 3242N^4 - 4326N^3$$
$$-3466N^2 - 1488N - 360 \tag{575}$$

$$P_{468} = 693N^{10} + 3465N^9 - 11014N^8 - 62668N^7 - 120361N^6 - 125113N^5 - 73393N^4$$
$$-18010N^3 + 165N^2 - 1584N - 1188 \tag{576}$$

$$P_{469} = 3N^{11} + 21N^{10} - 124N^9 - 1014N^8 - 2185N^7 - 2099N^6 - 934N^5 - 2060N^4 - 4632N^3$$
$$-4256N^2 - 2688N - 768 \tag{577}$$

$$P_{470} = 27N^{11} + 189N^{10} + 631N^9 + 1356N^8 + 2155N^7 + 2207N^6 + 211N^5 - 4984N^4 - 8400N^3$$
$$-5824N^2 - 2544N - 576 \tag{578}$$

$$P_{471} = N^{12} + 6N^{11} - 5N^{10} - 80N^9 - 379N^8 - 846N^7 - 1057N^6 - 786N^5 + 84N^4 + 490N^3$$
$$+324N^2 + 152N + 48 \tag{579}$$

$$P_{472} = 15N^{12} + 90N^{11} + 80N^{10} - 452N^9 - 1401N^8 - 2298N^7 - 5002N^6 - 6516N^5 - 1116N^4$$

$$+2960N^3 + 3464N^2 + 2400N + 864 \tag{580}$$

$$P_{473} = 233N^{12} + 2724N^{11} + 13349N^{10} + 34680N^9 + 46703N^8 + 12096N^7 - 69461N^6$$
$$-137724N^5 - 141176N^4 - 91776N^3 - 34832N^2 - 6336N - 2304 \tag{581}$$

$$P_{474} = 310N^{12} + 2058N^{11} + 5939N^{10} + 17235N^9 + 44700N^8 + 93240N^7 + 140861N^6$$
$$+113169N^5 + 14578N^4 - 40374N^3 - 33372N^2 - 12312N - 3888 \tag{582}$$

$$P_{475} = 391N^{12} + 2346N^{11} + 4795N^{10} + 2758N^9 - 2243N^8 + 1150N^7 + 7713N^6$$
$$+4546N^5 - 792N^4 + 1224N^2 + 864N + 288 \tag{583}$$

$$P_{476} = 1593N^{12} + 9558N^{11} + 15013N^{10} - 8758N^9 - 62269N^8 - 82318N^7 - 79041N^6$$
$$-90898N^5 - 70928N^4 - 15872N^3 + 7344N^2 + 5184N + 1728 \tag{584}$$

$$P_{477} = 15N^{13} + 120N^{12} + 530N^{11} + 1562N^{10} + 2042N^9 - 1680N^8 - 9220N^7 - 12524N^6$$
$$-7911N^5 - 5230N^4 - 5880N^3 - 3344N^2 - 2544N - 864 \tag{585}$$

$$P_{478} = 33N^{13} + 264N^{12} + 479N^{11} - 1366N^{10} - 8809N^9 - 23124N^8 - 34351N^7 - 26198N^6$$
$$-3624N^5 + 5240N^4 - 2496N^3 - 7232N^2 - 7104N - 2304 \tag{586}$$

$$P_{479} = 2493N^{13} + 19944N^{12} + 79295N^{11} + 208394N^{10} + 375431N^9 + 531516N^8 + 623697N^7$$
$$+733338N^6 + 963340N^5 + 1047352N^4 + 895648N^3 + 559488N^2 + 222336N + 41472 \tag{587}$$

$$P_{480} = 39N^{14} + 273N^{13} + 741N^{12} + 1025N^{11} + 1343N^{10} + 3479N^9 + 6707N^8 + 6555N^7$$
$$+2258N^6 - 1520N^5 - 1944N^4 - 532N^3 + 280N^2 + 208N + 32 \tag{588}$$

$$P_{481} = 276N^{16} + 3036N^{15} + 13660N^{14} + 30172N^{13} + 22446N^{12} - 50653N^{11} - 171627N^{10}$$
$$-246412N^9 - 204934N^8 - 83791N^7 + 28263N^6 + 43144N^5 - 33372N^4 - 82640N^3$$
$$-79152N^2 - 47232N - 12096 \tag{589}$$

$$P_{482} = 3135N^{19} + 43890N^{18} + 257636N^{17} + 794084N^{16} + 1224418N^{15} + 244448N^{14}$$
$$-2371724N^{13} - 3594388N^{12} - 792201N^{11} + 2719198N^{10} + 2284064N^9 - 85568N^8$$
$$-227344N^7 + 952768N^6 + 1160704N^5 + 807552N^4 + 574464N^3 + 305664N^2$$
$$+104448N + 18432 \ . \tag{590}$$

B The asymptotic Heavy Flavor Wilson Coefficients contributing to $F_2(x, Q^2)$ in z-space

The representation of the Wilson coefficients in momentum fraction or z-space of the contributions being known at present can be obtained in terms of harmonic polylogarithms [38] over the alphabet

$$\mathfrak{A} = \left\{ \frac{dz}{z}, \frac{dz}{1-z}, \frac{dz}{1+z} \right\} \equiv \{f_0(z), f_1(z), f_{-1}(z)\} \tag{591}$$

as iterated integrals

$$H_{b,\vec{a}}(z) = \int_0^z f_b(y) H_{\vec{a}}(y), \quad f_b \in \mathfrak{A}, \quad H_\emptyset = 1, H_{\underbrace{0,\ldots,0}_{k}}(z) := \frac{1}{k!} \ln^k(z). \tag{592}$$

For brevity we will drop the argument z of the harmonic polylogarithms in the following. As a shorthand notation we introduce the Mellin-inversion of (36)

$$\gamma_{qg} = -4 \left[z^2 + (1-z)^2 \right] \ , \tag{593}$$

where the Mellin transform is defined by

$$\mathbf{M}[f(x)](N) = \int_0^1 dx \; x^{N-1} \; f(x) \;. \tag{594}$$

The Wilson coefficients $L_{q,2}^{\mathsf{PS}}$ and $L_{g,2}^{\mathsf{S}}$ are given in z-space by :

$$L_{q,2}^{\mathsf{PS}} =$$

$$a_s^3 \Bigg\{ C_F N_F T_F^2 \Bigg[\left[\frac{64}{9}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{27z} \right] L_Q^3 + \left[-\frac{64}{3}(z+1)H_0^2 \right. $$

$$+ \frac{64}{9}(4z^2-11z-8)H_0 + \frac{32(z-1)(10z^2+33z-2)}{9z} \Bigg] L_Q^2 + L_Q \Bigg[\frac{64}{3}H_0^3(z+1)$$

$$+ \frac{\left[\frac{256}{9}H_{0,-1} - \frac{256}{9}H_{-1}H_0 \right](z+1)^3}{z} - \frac{32}{9}(12z^2-59z-29)H_0^2$$

$$+ \left[\frac{64}{3}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{9z} \right] L_M^2 - \frac{64(z-1)(304z^2+811z+124)}{81z}$$

$$- \frac{64}{27}(60z^2-155z-233)H_0 - \frac{256(3z^2+1)\zeta_2}{9z} + L_M \Bigg[-\frac{64(z-1)(38z^2+47z+20)}{27z}$$

$$+ \frac{128}{9}(2z^2+11z+8)H_0 + \frac{64(z-1)(4z^2+7z+4)H_1}{9z} + (z+1)\left[\frac{128\zeta_2}{3} - \frac{128}{3}H_{0,1} \right] \Bigg] \Bigg]$$

$$+ L_M^3 \Bigg[\frac{32(z-1)(4z^2+7z+4)}{27z} - \frac{64}{9}(z+1)H_0 \Bigg] - \frac{32}{81}(57z^2+367z+295)H_0^2$$

$$- \frac{16(z-1)(118z^2-107z+118)H_1^2}{81z} + L_M^2 \Bigg[-\frac{64}{3}(z+1)H_0^2 + \frac{64}{9}(4z^2-11z-8)H_0$$

$$+ \frac{32(z-1)(10z^2+33z-2)}{9z} \Bigg] + \frac{64(z-1)(3821z^2+6698z+500)}{729z}$$

$$- \frac{32}{27}(68z^3+53z^2+5z-32)\frac{\zeta_3}{z} - \frac{32}{243}(48z^2+5087z+2783)H_0$$

$$- \frac{32(z-1)(256z^2+521z-14)H_1}{81z} - \frac{32(z-1)(38z^2+47z+20)H_0H_1}{27z}$$

$$- \frac{64(42z^3-227z^2-74z+30)H_{0,1}}{81z} + \frac{64}{9}(2z^2+11z+8)H_0H_{0,1}$$

$$- \frac{64}{27}(6z^2+4z-5)H_{0,0,1} + \frac{128(9z^3+7z^2+7z+3)H_{0,1,1}}{27z}$$

$$+ \frac{32}{81}(198z^2-427z-229)\zeta_2 - \frac{64}{27}(6z^2+62z+53)H_0\zeta_2 + \frac{(z-1)(4z^2+7z+4)}{z}\Bigg[\frac{64}{81}H_1^3$$

$$+ \frac{16}{9}H_0H_1^2 - \frac{32}{9}\zeta_2H_1 \Bigg] + L_M \Bigg[\frac{256\zeta_2z^2}{3} - \frac{16}{9}(8z^2+73z+61)H_0^2$$

$$+ \frac{64}{27}(28z^2-221z-107)H_0 + (z+1)\Bigg[-\frac{32}{9}H_0^3 + \frac{128}{3}H_{0,1}H_0 - \frac{256}{3}\zeta_2H_0 + \frac{128}{3}H_{0,1,1}$$

$$- \frac{128\zeta_3}{3} \Bigg] + \frac{32(z-1)(832z^2+1213z+76)}{81z} + \frac{128(z-1)(4z^2-26z+13)H_1}{27z}$$

$$+ \frac{(z-1)(4z^2+7z+4)\left[-\frac{32}{9}H_1^2 - \frac{64}{9}H_0H_1 \right]}{z} - \frac{64(8z^3-3z^2+3z+4)H_{0,1}}{9z} \Bigg]$$

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$$+(z+1)\left[-\frac{8}{27}H_0^4 - \frac{464}{81}H_0^3 - \frac{64}{3}H_{0,1,1}H_0 + \frac{832}{9}\zeta_3 H_0 - \frac{224\zeta_2^2}{15} + \frac{128}{9}H_{0,0,0,1}\right.$$

$$\left.\left.\left.-\frac{64}{9}H_{0,0,1,1} - \frac{256}{9}H_{0,1,1,1} + \left(\frac{64}{3}H_{0,1} - \frac{64}{9}H_0^2\right)\zeta_2\right]\right] + N_F\hat{\tilde{C}}_{2,q}^{\mathsf{PS},(3)}(N_F)\right\} \tag{595}$$

and

$$L_{g,2}^{\mathsf{S}} =$$

$$a_s^2 N_F T_F^2\left\{\left[\gamma_{qg}^0\left[\frac{4H_0}{3} + \frac{4H_1}{3}\right] - \frac{16}{3}\left(8z^2 - 8z + 1\right)\right]L_M - \frac{4}{3}\gamma_{qg}^0 L_Q L_M\right\}$$

$$+a_s^3\left\{N_F T_F^3\left[\left(\gamma_{qg}^0\left[\frac{16H_0}{9} + \frac{16H_1}{9}\right] - \frac{64}{9}\left(8z^2 - 8z + 1\right)\right)L_M^2 - \frac{16}{9}\gamma_{qg}^0 L_Q L_M^2\right]\right.$$

$$+C_A N_F T_F^2\left[-\frac{4}{27}(30z - 13)H_0^4 - \frac{8}{81}\left(32z^2 + 628z - 169\right)H_0^3 - \frac{16}{81}\left(532z^2 + 2586z - 193\right)H_0^2\right.$$

$$-\frac{64}{27}\left(7z^2 + 7z + 5\right)H_{-1}H_0^2 + \frac{16}{3}(2z - 1)\zeta_2 H_0^2 + \frac{8}{243}\left(4641z^2 - 67330z + 3473\right)H_0$$

$$+\frac{32}{27}\left(59z^2 - 50z + 25\right)H_1 H_0 + \frac{64}{27}\left(14z^2 + 11z + 10\right)H_{0,-1}H_0 - \frac{64}{27}\left(7z^2 - 4z + 5\right)H_{0,1}H_0$$

$$+\frac{16}{9}(62z - 19)\zeta_2 H_0 + \frac{32}{9}(38z + 5)\zeta_3 H_0 + L_M^3\left[\frac{16(z-1)\left(31z^2 + 7z + 4\right)}{27z} - \frac{32}{9}(4z + 1)H_0\right.$$

$$\left.-\frac{8}{9}\gamma_{qg}^0 H_1\right] + \frac{16}{81}\left(172z^2 - 163z + 56\right)H_1^2 - \frac{64}{45}(8z + 3)\zeta_2^2$$

$$+\frac{8\left(276317z^3 - 271875z^2 + 11280z - 6182\right)}{729z} - \frac{16}{27}\left(248z^3 - 438z^2 + 33z - 32\right)\frac{\zeta_3}{z}$$

$$+\frac{8\left(28805z^3 - 28460z^2 + 4612z - 1596\right)H_1}{243z} + L_Q^3\left[-\frac{16(z-1)\left(31z^2 + 7z + 4\right)}{27z}\right.$$

$$+\frac{32}{9}(4z + 1)H_0 + \frac{8}{9}\gamma_{qg}^0 H_1\right] + \left(200z^2 + 191z + 112\right)\left[\frac{32}{81}H_{0,-1} - \frac{32}{81}H_{-1}H_0\right]$$

$$-\frac{32}{27}\left(98z^2 + 347z + 16\right)H_{0,1} - \frac{128}{27}\left(7z^2 + 4z + 5\right)H_{0,0,-1} + \frac{16}{27}\left(28z^2 - 202z + 77\right)H_{0,0,1}$$

$$+\frac{32}{9}\left(14z^2 - 15z + 10\right)H_{0,1,1} + \left(2z^2 + 2z + 1\right)\left[-\frac{64}{27}H_{-1}H_0^3 + \frac{64}{9}H_{0,-1}H_0^2 - \frac{128}{9}H_{0,0,-1}H_0\right.$$

$$\left.+\frac{128}{9}H_{0,0,0,-1}\right] - \frac{64}{3}z^2 H_{0,0,0,1} + \frac{32}{81}\left(117z^2 + 1000z - 27\right)\zeta_2 + L_M^2\left[-\frac{16}{3}(8z + 1)H_0^2\right.$$

$$+\frac{16}{9}\left(49z^2 - 136z - 13\right)H_0 + \frac{8\left(1048z^3 - 894z^2 - 87z - 40\right)}{27z} + \frac{16\left(87z^3 - 80z^2 + 13z - 4\right)H_1}{9z}$$

$$+\gamma_{qg}^0\left[-\frac{4}{3}H_1^2 - \frac{8}{3}H_0 H_1\right] + \left(2z^2 + 2z + 1\right)\left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0\right] - \frac{32}{3}(4z + 1)H_{0,1}$$

$$\left.-\frac{64}{3}(z - 2)z\zeta_2\right] + L_Q^2\left[-\frac{16}{3}(8z + 1)H_0^2 + \frac{16}{9}\left(49z^2 - 136z - 13\right)H_0\right.$$

$$+\frac{8\left(1048z^3 - 894z^2 - 87z - 40\right)}{27z} + \frac{16\left(87z^3 - 80z^2 + 13z - 4\right)H_1}{9z} + \gamma_{qg}^0\left[-\frac{4}{3}H_1^2\right.$$

$$\left.-\frac{8}{3}H_0 H_1\right] + \left(2z^2 + 2z + 1\right)\left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0\right] - \frac{32}{3}(4z + 1)H_{0,1}$$

$$+\left[-\frac{16(z-1)\left(31z^2+7z+4\right)}{9z}+\frac{32}{3}(4z+1)H_0+\frac{8}{3}\gamma^0_{qg}H_1\right]L_M-\frac{64}{3}(z-2)z\zeta_2\right]$$

$$+\left(7z^2-7z+5\right)\left[\frac{32}{81}H_1^3+\frac{32}{27}H_0H_1^2+\frac{32}{27}H_0^2H_1-\frac{128}{27}H_{0,1}H_1+\frac{64}{27}\zeta_2H_1\right]$$

$$+L_Q\left[\frac{448}{9}zH_0^3-\frac{16}{9}\left(63z^2-296z-10\right)H_0^2+\frac{32}{3}\left(5z^2+4z+2\right)H_{-1}H_0^2\right.$$

$$-\frac{32}{3}\left(3z^2-4z+2\right)H_1H_0^2-\frac{64}{27}\left(325z^2-832z-31\right)H_0-\frac{64}{3}\left(3z^2+4z+2\right)H_{0,-1}H_0$$

$$+\frac{64}{3}\left(z^2+2z+2\right)H_{0,1}H_0+\frac{128}{3}(z-5)z\zeta_2H_0+\left[-\frac{16(z-1)\left(31z^2+7z+4\right)}{9z}\right.$$

$$+\frac{32}{3}(4z+1)H_0+\frac{8}{3}\gamma^0_{qg}H_1\right]L_M^2-\frac{8\left(25657z^3-23556z^2-969z-412\right)}{81z}$$

$$+\frac{32}{9}\left(105z^3-184z^2+3z-4\right)\frac{\zeta_2}{z}-\frac{32\left(1032z^3-964z^2+65z-20\right)H_1}{27z}$$

$$+\frac{\left(87z^3-80z^2+13z-4\right)\left[-\frac{16}{9}H_1^2-\frac{32}{9}H_0H_1\right]}{z}+\frac{\left(17z^3+7z^2-4z-2\right)\left[\frac{64}{9}H_{-1}H_0-\frac{64}{9}H_{0,-1}\right]}{z}$$

$$-\frac{64\left(9z^3-59z^2-5z+2\right)H_{0,1}}{9z}+\gamma^0_{qg}\left[\frac{8}{9}H_1^3+\frac{16}{3}H_0H_1^2-\frac{32}{3}H_{0,1}H_1\right]+(2z+1)\left[\frac{32}{3}H_0H_{-1}^2\right.$$

$$-\frac{64}{3}H_{0,-1}H_{-1}+\frac{64}{3}H_{0,-1,-1}\right]+\frac{64}{3}\left(z^2+4z+2\right)H_{0,0,-1}-\frac{64}{3}\left(z^2-2z+2\right)H_{0,0,1}$$

$$+\left(2z^2+2z+1\right)\left[\frac{64}{3}H_{-1}H_{0,1}-\frac{64}{3}H_{0,-1,1}-\frac{64}{3}H_{0,1,-1}\right]-\frac{128}{3}\left(3z^2-5z+1\right)H_{0,1,1}$$

$$-\frac{32}{3}\left(4z^2+2z+1\right)H_{-1}\zeta_2-\frac{32}{3}(2z-1)H_1\zeta_2+L_M\left[-\frac{32}{3}(4z+3)H_0^2\right.$$

$$+\frac{32}{9}\left(101z^2-8z+13\right)H_0-\frac{32\left(8z^3-14z^2+11z-8\right)}{3z}+\frac{32\left(47z^3-40z^2-7z-4\right)H_1}{9z}$$

$$+\gamma^0_{qg}\left[-\frac{8}{3}H_1^2-\frac{16}{3}H_0H_1\right]+\left(2z^2+2z+1\right)\left[\frac{64}{3}H_{0,-1}-\frac{64}{3}H_{-1}H_0\right]-\frac{64}{3}(4z+1)H_{0,1}$$

$$-\frac{128}{3}(z-2)z\zeta_2+\frac{32}{3}\left(6z^2-22z+3\right)\zeta_3\right]+L_M\left[\frac{160}{9}H_0^3-\frac{16}{9}\left(131z^2+32z+13\right)H_0^2\right.$$

$$-\frac{32}{3}(z-1)^2H_1H_0^2+\frac{8}{3}\gamma^0_{qg}H_1^2H_0-\frac{16}{27}\left(880z^2+1486z-353\right)H_0-\frac{32\left(47z^3-40z^2-7z-4\right)H_1H_0}{9z}$$

$$-\frac{64}{3}\left(z^2-4z-1\right)H_{0,1}H_0+\frac{128}{3}\left(z^2-3z-1\right)\zeta_2H_0-\frac{16\left(35z^3-28z^2-7z-4\right)H_1^2}{9z}$$

$$+\frac{8\left(4933z^3-5084z^2+113z+92\right)}{27z}+\frac{32}{9}\left(117z^3-32z^2+9z-4\right)\frac{\zeta_2}{z}$$

$$+\frac{16\left(404z^3-476z^2+271z-144\right)H_1}{27z}+\frac{\left(3z^3-7z^2-14z-2\right)\left[\frac{64}{9}H_{-1}H_0-\frac{64}{9}H_{0,-1}\right]}{z}$$

$$-\frac{64\left(35z^3+11z^2+8z+2\right)H_{0,1}}{9z}+(2z+1)\left[\frac{32}{3}H_0H_{-1}^2-\frac{64}{3}H_{0,-1}H_{-1}+\frac{64}{3}H_{0,-1,-1}\right]$$

$$+z^2\left[\frac{32}{3}H_{-1}H_0^2+\frac{64}{3}H_{0,-1}H_0-64H_{0,0,-1}\right]+\frac{64}{3}\left(z^2-4z+1\right)H_{0,0,1}+\left(2z^2+2z+1\right)\left[\frac{64}{3}H_{-1}H_{0,1}\right.$$

$$-\frac{64}{3}H_{0,-1,1}-\frac{64}{3}H_{0,1,-1}\right]+\frac{64}{3}(4z+1)H_{0,1,1}-\frac{32}{3}\left(4z^2+2z+1\right)H_{-1}\zeta_2+\frac{32}{3}\left(4z^2-6z+3\right)H_1\zeta_2$$

$$+\frac{32}{3}\left(6z^2-2z-1\right)\zeta_3\Bigg] + \gamma_{qg}^0\Bigg[\frac{1}{27}H_1^4 - \frac{8}{9}H_{0,1}H_1^2 - \frac{4}{9}H_0^3H_1 + \left[\frac{16}{9}H_{0,0,1} + \frac{16}{9}H_{0,1,1}\right]H_1$$

$$+\frac{16}{3}\zeta_3 H_1 - \frac{8}{9}H_{0,1}^2 + H_0^2\left[\frac{4}{3}H_{0,1} - \frac{2}{9}H_1^2\right] + H_0\left[\frac{4}{27}H_1^3 - \frac{8}{3}H_{0,0,1} + \frac{8}{9}H_{0,1,1}\right] - \frac{8}{9}H_{0,0,1,1}$$

$$-\frac{8}{9}H_{0,1,1,1} + \left[\frac{4}{9}H_1^2 - \frac{8}{9}H_0H_1 + \frac{8}{9}H_{0,1}\right]\zeta_2\Bigg]\Bigg] + C_F N_F T_F^2\Bigg[-\frac{2}{27}\left(56z^2 + 448z - 179\right)H_0^4$$

$$+\frac{4}{81}\left(288z^2 - 6524z + 3259\right)H_0^3 - \frac{4}{81}\left(4096z^2 + 23771z - 21328\right)H_0^2 + 8(12z-5)\zeta_2 H_0^2$$

$$+\frac{56}{243}\left(1491z^2 - 4715z + 17578\right)H_0 - \frac{448}{81}\left(z^2 - z + 2\right)H_1 H_0 + 112(5z-2)\zeta_2 H_0$$

$$+\frac{16}{9}\left(96z^2 + 92z - 109\right)\zeta_3 H_0 + \Bigg[-\frac{16}{3}(2z-1)H_0^2 - \frac{32}{9}\left(6z^2 - z - 4\right)H_0$$

$$+\frac{8\left(124z^3 - 258z^2 + 159z - 16\right)}{27z} + \frac{8}{9}\gamma_{qg}^0 H_1\Bigg]L_M^3 + \frac{8}{81}\left(364z^2 - 373z + 224\right)H_1^2$$

$$-\frac{32}{45}\left(4z^2 - 112z + 47\right)\zeta_2^2 + \frac{2540132z^3 - 7301946z^2 + 4812411z + 34912}{729z}$$

$$-\frac{16}{27}\left(468z^3 - 1949z^2 + 994z - 64\right)\frac{\zeta_3}{z} + \frac{4\left(43898z^3 - 106070z^2 + 58429z + 1296\right)H_1}{243z}$$

$$+L_Q^3\Bigg[\frac{16}{3}(2z-1)H_0^2 + \frac{16}{9}(4z-11)H_0 - \frac{16\left(62z^3 - 147z^2 + 84z - 8\right)}{27z} + \frac{16}{9}\gamma_{qg}^0 H_1\Bigg]$$

$$-\frac{16}{81}\left(580z^2 + 797z - 3196\right)H_{0,1} + \left(16z^2 - 16z + 5\right)\left[\frac{16}{27}H_0^2 H_1 - \frac{32}{27}H_0 H_{0,1}\right]$$

$$+\frac{16}{27}\left(32z^2 - 977z + 388\right)H_{0,0,1} + \left(7z^2 - 7z + 5\right)\left[-\frac{32}{81}H_1^3 - \frac{64}{27}H_{0,1,1}\right]$$

$$-\frac{16}{9}\left(8z^2 + 100z - 41\right)H_{0,0,0,1} + \frac{16}{81}\left(608z^2 + 769z - 3140\right)\zeta_2 + \gamma_{qg}^0\left[-\frac{1}{27}H_1^4 - \frac{8}{27}H_0^3 H_1\right]$$

$$-\frac{64}{9}\zeta_3 H_1 + \frac{8}{9}H_0^2 H_{0,1} - \frac{16}{9}H_0 H_{0,0,1} - \frac{8}{9}H_{0,1,1,1}\Bigg] + L_Q^2\Bigg[-\frac{8}{3}\left(8z^2 + 56z - 31\right)H_0^2$$

$$+\frac{8}{9}\left(244z^2 - 236z + 571\right)H_0 + \frac{4\left(2156z^3 - 7632z^2 + 4977z + 256\right)}{27z}$$

$$+\frac{16\left(130z^3 - 215z^2 + 112z - 8\right)H_1}{9z} + \gamma_{qg}^0\left[-4H_1^2 - \frac{32}{3}H_0 H_1\right] - \frac{16}{3}\left(12z^2 - 8z - 5\right)H_{0,1}$$

$$+\Bigg[-16(2z-1)H_0^2 - \frac{32}{3}\left(6z^2 - z - 4\right)H_0 + \frac{8\left(124z^3 - 258z^2 + 159z - 16\right)}{9z} + \frac{8}{3}\gamma_{qg}^0 H_1\Bigg]L_M$$

$$-\frac{16}{3}\left(4z^2 - 8z + 13\right)\zeta_2 + (2z-1)\left[-16H_0^3 + 32\zeta_2 H_0 - 32H_{0,0,1} + 32\zeta_3\right]\Bigg]$$

$$+L_M^2\Bigg[-8\left(8z^2 + 16z - 9\right)H_0^2 + \frac{8}{9}\left(92z^2 - 160z + 521\right)H_0 - \frac{4}{3}\gamma_{qg}^0 H_1^2$$

$$+\frac{4\left(2372z^3 - 7140z^2 + 4611z + 256\right)}{27z} + \frac{16\left(54z^3 - 127z^2 + 71z - 8\right)H_1}{9z} + \left(4z^2 + 4z - 11\right)\left[\frac{16\zeta_2}{3}\right]$$

$$-\frac{16}{3}H_{0,1}\Bigg] + (2z-1)\left[-16H_0^3 + 32\zeta_2 H_0 - 32H_{0,0,1} + 32\zeta_3\right]\Bigg] + L_Q\Bigg[\frac{64}{3}\left(2z^2 + 15z - 5\right)H_0^3$$

$$-\frac{16}{45}\left(72z^3 + 560z^2 - 2660z + 2975\right)H_0^2 - \frac{64}{3}\left(5z^2 - 4z + 2\right)H_1 H_0^2$$

$$-\frac{16\left(1444z^3 - 5632z^2 + 9213z + 4\right)H_0}{45z} - \frac{32\left(198z^3 - 283z^2 + 140z - 8\right)H_1 H_0}{9z}$$

$$+\frac{128}{3}(z-2)(3z-2)H_{0,-1}H_0 + \frac{32}{3}\left(16z^2 - 8z - 5\right)H_{0,1}H_0 - \frac{32}{3}\left(4z^2 + 84z - 33\right)\zeta_2 H_0$$

$$-\frac{16\left(168z^3 - 253z^2 + 131z - 8\right)H_1^2}{9z} + \left[16(2z-1)H_0^2 + \frac{16}{3}\left(8z^2 - 9\right)H_0\right.$$

$$\left.-\frac{16(z-1)\left(62z^2 - 73z + 8\right)}{9z}\right]L_M^2 - \frac{4\left(269954z^3 - 828996z^2 + 567861z - 11744\right)}{405z}$$

$$+\frac{16}{45}\left(144z^4 + 1600z^3 - 1400z^2 + 3045z - 80\right)\frac{\zeta_2}{z} - \frac{8\left(3080z^3 - 8448z^2 + 5247z + 256\right)H_1}{27z}$$

$$+\frac{\left(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1\right)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2}$$

$$+\frac{16\left(76z^3 - 254z^2 - 329z - 16\right)H_{0,1}}{9z} + \gamma_{qg}^0\left[\frac{8}{3}H_1^3 + 8H_0 H_1^2 + \frac{32}{3}H_{0,1}H_1\right]$$

$$-\frac{128}{3}\left(7z^2 - 14z + 9\right)H_{0,0,-1} - \frac{32}{3}\left(8z^2 - 60z + 15\right)H_{0,0,1} + \frac{32}{3}\left(24z^2 - 20z + 1\right)H_{0,1,1}$$

$$+\frac{64}{3}\left(8z^2 - 6z + 3\right)H_1\zeta_2 + (z+1)^2\left[-\frac{128}{3}H_0 H_{-1}^2 + \left(\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1}\right)H_{-1} - \frac{128}{3}\zeta_2 H_{-1}\right.$$

$$\left.-\frac{256}{3}H_{0,-1,-1}\right] + L_M\left[\frac{16}{3}\left(32z^2 + 44z - 25\right)H_0^2 - \frac{32}{3}\left(8z^2 - 21z + 89\right)H_0\right.$$

$$-\frac{16\left(1006z^3 - 3333z^2 + 2208z + 128\right)}{27z} - \frac{16\left(64z^3 - 198z^2 + 141z - 16\right)H_1}{9z} + \gamma_{qg}^0\left[-\frac{16}{3}H_1^2\right.$$

$$\left.-\frac{32}{3}H_0 H_1\right] + \frac{64}{3}(z-1)(4z+5)H_{0,1} - \frac{64}{3}\left(8z^2 - 3z - 3\right)\zeta_2 + (2z-1)\left[32H_0^3 - 64\zeta_2 H_0 + 64H_{0,0,1}\right.$$

$$\left.-64\zeta_3\right]\right] + \frac{128}{3}\left(4z^2 - 21z + 11\right)\zeta_3 + (2z-1)\left[16H_0^4 - 96\zeta_2 H_0^2 + 64H_{0,0,1}H_0 - 128\zeta_3 H_0 - \frac{32\zeta_2^2}{5}\right.$$

$$\left.+64H_{0,0,1,1}\right]\right] + L_M\left[-\frac{64}{9}\left(14z^2 + 41z - 13\right)H_0^3 - \frac{16}{45}\left(72z^3 + 360z^2 + 2390z - 2835\right)H_0^2\right.$$

$$-\frac{64}{3}z^2 H_1 H_0^2 - \frac{8\left(524z^3 + 29468z^2 - 50797z + 24\right)H_0}{135z} + \frac{16\left(28z^3 - 162z^2 + 123z - 16\right)H_1 H_0}{9z}$$

$$-\frac{128}{3}\left(z^2 - 4z + 2\right)H_{0,-1}H_0 - 64(z-1)(2z+1)H_{0,1}H_0 + \frac{32}{3}\left(36z^2 + 64z - 23\right)\zeta_2 H_0$$

$$+\frac{8\left(44z^3 - 178z^2 + 143z - 16\right)H_1^2}{9z} + \frac{4\left(259856z^3 - 763164z^2 + 514989z - 11456\right)}{405z}$$

$$+\frac{32}{45}\left(72z^4 - 70z^3 + 345z^2 - 1340z + 40\right)\frac{\zeta_2}{z} + \frac{16\left(794z^3 - 3157z^2 + 2114z + 128\right)H_1}{27z}$$

$$+\frac{\left(36z^5 + 155z^4 + 40z^3 - 45z^2 + 20z + 1\right)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} + \frac{16\left(56z^2 + 413z + 16\right)H_{0,1}}{9z}$$

$$+\gamma_{qg}^0\left[\frac{8}{3}H_1^3 + \frac{16}{3}H_0 H_1^2 + \frac{16}{3}H_{0,1}H_1\right] + \frac{128}{3}\left(z^2 - 10z + 3\right)H_{0,0,-1} - \frac{32}{3}\left(8z^2 + 52z - 11\right)H_{0,0,1}$$

$$-\frac{64}{3}\left(2z^2 + 3z - 6\right)H_{0,1,1} + \frac{64}{3}\left(4z^2 - 2z + 1\right)H_1\zeta_2 + (z+1)^2\left[-\frac{128}{3}H_0 H_{-1}^2 + \left(\frac{64}{3}H_0^2\right.\right.$$

$$\left.\left.+\frac{256}{3}H_{0,-1}\right)H_{-1} - \frac{128}{3}\zeta_2 H_{-1} - \frac{256}{3}H_{0,-1,-1}\right] + \frac{32}{3}\left(24z^2 + 82z - 27\right)\zeta_3$$

$$+(2z-1)\left[-16H_0^4 + 96\zeta_2 H_0^2 - 64H_{0,0,1}H_0 + 128\zeta_3 H_0 + \frac{32\zeta_2^2}{5} - 64H_{0,0,1,1}\right]\right]$$

$$+(2z-1)\left[-\frac{4}{3}H_0^5+\frac{16}{3}\zeta_2H_0^3+\frac{176}{3}\zeta_3H_0^2+\frac{64}{5}\zeta_2^2H_0-32H_{0,0,0,0,1}\right.$$
$$\left.+32\zeta_5\right]\right]+N_F\hat{\tilde{C}}_{2,g}^{\mathsf{S},(3)}(N_F)\Bigg\}\tag{596}$$

The pure-singlet Wilson coefficient $H_{q,2}^{\mathsf{PS}}$ reads :

$$H_{q,2}^{\mathsf{PS}} = a_s^2 C_F T_F\Bigg\{\frac{\left[\frac{32}{3}H_{0,-1}-\frac{32}{3}H_{-1}H_0\right](z+1)^3}{z}+\left[\frac{16}{3}H_0^3+32H_{0,1}H_0-32\zeta_2H_0-32H_{0,0,1}\right.$$

$$\left.+16H_{0,1,1}+16\zeta_3\right](z+1)-8z(2z-5)H_0^2+\left[\frac{4(z-1)(4z^2+7z+4)}{3z}-8(z+1)H_0\right]L_M^2$$

$$+\frac{16(z-1)(52z^2-24z-5)}{9z}+\frac{32}{3}(3z^3-3z^2-1)\frac{\zeta_2}{z}-\frac{8}{9}(88z^2+99z-105)H_0$$

$$+L_Q^2\left[8(z+1)H_0-\frac{4(z-1)(4z^2+7z+4)}{3z}\right]+\frac{16(z-1)(4z^2-26z+13)H_1}{9z}$$

$$+\frac{(z-1)(4z^2+7z+4)}{z}\left[-\frac{4}{3}H_1^2-\frac{16}{3}H_0H_1\right]-\frac{16(2z^3-3z^2+3z+4)H_{0,1}}{3z}$$

$$+\left[8(z+1)H_0^2-\frac{8}{3}(8z^2+15z+3)H_0+\frac{16(z-1)(28z^2+z+10)}{9z}\right]L_M$$

$$+L_Q\left[32H_0z^2+(z+1)\left[-16H_0^2-16H_{0,1}+16\zeta_2\right]-\frac{16(z-1)(4z^2-26z+13)}{9z}\right.$$

$$\left.+\frac{8(z-1)(4z^2+7z+4)H_1}{3z}\right]\Bigg\}$$

$$+a_s^3\Bigg\{C_F^2 T_F\left[\left[\frac{32(4z^2+7z+4)H_1(z-1)}{9z}+\frac{184(z-1)}{9}+\frac{32}{9}z(4z+3)H_0\right.\right.$$

$$+(z+1)\left[-\frac{16}{3}H_0^2-\frac{64}{3}H_{0,1}+\frac{64\zeta_2}{3}\right]\right]L_Q^3+\left[-\frac{8}{3}(28z^2+3z+6)H_0^2\right.$$

$$-\frac{8}{9}(4z^2+315z-198)H_0+\frac{4(z-1)(120z^2-289z-36)}{9z}$$

$$-\frac{8(z-1)(4z^2+337z-32)H_1}{9z}+\frac{(z-1)(4z^2+7z+4)\left[-8H_1^2-16H_0H_1\right]}{z}$$

$$-\frac{16(16z^3-15z^2-9z+12)H_{0,1}}{3z}+\frac{32}{3}(14z^2-3z-9)\zeta_2$$

$$+(z+1)\left[16H_0^3+96H_{0,1}H_0-160\zeta_2H_0-32H_{0,0,1}+96H_{0,1,1}-64\zeta_3\right]\right]L_Q^2$$

$$+\left[\frac{(z+1)^3}{z}\left[\frac{128}{3}H_{-1}H_{0,1}-\frac{128}{3}H_{0,-1,1}-\frac{128}{3}H_{0,1,-1}\right]-\frac{64}{3}(z^2+14z+1)\frac{\zeta_2}{z}H_{-1}(z+1)\right.$$

$$+\frac{(36z^4-36z^3-1069z^2+4z-4)(z+1)}{z^2}\left[\frac{32}{45}H_{0,-1}-\frac{32}{45}H_{-1}H_0\right]$$

$$+\frac{(z^2-10z+1)(z+1)}{z}\left[\frac{64}{3}H_0H_{-1}^2-\frac{128}{3}H_{0,-1}H_{-1}+\frac{128}{3}H_{0,-1,-1}\right]+\left[-14H_0^4\right.$$

$$+128H_{-1}H_0^2-64H_{0,1}H_0^2+\left[-224H_{0,0,1}-320H_{0,1,1}\right]H_0-384H_{0,0,0,-1}$$

$$+480H_{0,0,0,1} + 192H_{0,0,1,1} - 224H_{0,1,1,1} + \left[288H_0^2 + 192H_{0,1}\right]\zeta_2\right)(z+1)$$

$$+\frac{16}{9}\left(52z^2 - 22z + 15\right)H_0^3 + \frac{2}{45}\left(288z^3 + 3680z^2 + 1525z - 7695\right)H_0^2$$

$$-\frac{4(z-1)\left(8z^2 - 305z + 20\right)H_1^2}{3z} - \frac{32}{5}(33z + 23)\zeta_2^2 - \frac{4(z-1)\left(19536z^2 - 1103z + 4056\right)}{135z}$$

$$-\frac{8\left(6298z^3 - 32859z^2 + 606z + 48\right)H_0}{135z} + \frac{64(z-1)\left(z^2 + z + 1\right)H_0^2 H_1}{z}$$

$$-\frac{8(z-1)\left(986z^2 - 4747z - 292\right)H_1}{27z} - 32(z-1)\left(6z^2 + 5z + 6\right)\frac{\zeta_2}{z}H_1$$

$$-\frac{8(z-1)\left(32z^2 - 299z + 32\right)H_0 H_1}{3z} + \frac{(z-1)\left(4z^2 + 7z + 4\right)\left[\frac{56}{9}H_1^3 + \frac{80}{3}H_0 H_1^2\right]}{z}$$

$$-\frac{64}{3}\left(2z^2 + 7z - 3\right)H_0 H_{0,-1} + \frac{8\left(536z^3 - 327z^2 - 453z - 96\right)H_{0,1}}{9z}$$

$$+\frac{32\left(32z^3 + 6z^2 - 21z + 12\right)H_0 H_{0,1}}{3z} + \frac{128}{3}\left(2z^2 + z - 9\right)H_{0,0,-1} + 256zH_0 H_{0,0,-1}$$

$$-\frac{32\left(20z^3 + 63z^2 - 12z + 12\right)H_{0,0,1}}{3z} + \frac{16\left(36z^3 - 45z^2 - 27z + 40\right)H_{0,1,1}}{3z}$$

$$+L_M^2\left[-\frac{16\left(4z^2 + 7z + 4\right)H_1(z-1)}{3z} - \frac{92(z-1)}{3} - \frac{16}{3}z(4z + 3)H_0\right.$$

$$+(z+1)\left(8H_0^2 + 32H_{0,1} - 32\zeta_2\right)\right] - \frac{16}{45}\left(72z^3 + 1100z^2 - 545z - 3615\right)\zeta_2$$

$$-\frac{32}{3}\left(56z^2 - 61z - 30\right)H_0\zeta_2 + (z-1)\left[-64H_{0,-1}H_0^2 + 256H_{0,-1,-1}H_0 - 128H_{0,-1}^2\right.$$

$$+128H_{0,-1}\zeta_2\right] + \frac{16\left(219z^2 + 51z - 16\right)\zeta_3}{3z} - 64(3z - 1)H_0\zeta_3$$

$$+L_M\left[\frac{16}{3}\left(8z^2 + 9z + 3\right)H_0^2 - \frac{8}{9}\left(224z^2 - 99z + 81\right)H_0 - \frac{16}{9}(z-1)(30z + 23)\right.$$

$$-\frac{64(z-1)\left(28z^2 + z + 10\right)H_1}{9z} + \left(8z^2 + 15z + 3\right)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right] + (z+1)\left[-\frac{16}{3}H_0^3\right.$$

$$\left.\left.+64\zeta_2 H_0 - 64H_{0,0,1} + 64\zeta_3\right]\right]\right]L_Q - \frac{4}{27}\left(212z^2 + 225z + 207\right)H_0^3$$

$$-\frac{8(z-1)\left(6z^2 - z - 6\right)H_1^3}{9z} + \frac{2}{27}\left(88z^2 - 1899z - 1548\right)H_0^2$$

$$+\frac{4(z-1)\left(584z^2 + 359z + 71\right)H_1^2}{27z} - \frac{4(z-1)\left(24z^2 + 65z + 24\right)H_0 H_1^2}{3z}$$

$$+\frac{16}{15}\left(74z^2 + 69z - 60\right)\zeta_2^2 + \frac{4(z-1)\left(1312z^2 + 3535z + 699\right)}{27z}$$

$$+\frac{4}{27}\left(1532z^3 + 3033z^2 + 1305z - 540\right)\frac{\zeta_2}{z} + \frac{4}{27}\left(2040z^2 + 2015z - 2297\right)H_0$$

$$-\frac{8(z-1)\left(22z^2 + 53z + 10\right)H_0^2 H_1}{3z} + \frac{4(z-1)\left(2844z^2 + 3572z + 753\right)H_1}{27z}$$

$$+\frac{4}{3}(z-1)\left(8z^2 + 97z + 44\right)\frac{\zeta_2}{z}H_1 + \frac{8(z-1)\left(944z^2 + 809z - 370\right)H_0 H_1}{27z}$$

$$+\frac{8\left(4z^3 + 45z^2 + 15z - 12\right)H_0^2 H_{0,1}}{3z} - \frac{8\left(1548z^3 + 1395z^2 - 648z + 370\right)H_{0,1}}{27z}$$

$$-\frac{8}{3}\big(12z^3 + 51z^2 + 33z - 8\big)\frac{\zeta_2}{z}H_{0,1} - \frac{16\big(82z^3 + 135z^2 + 180z + 30\big)H_0 H_{0,1}}{9z}$$

$$-\frac{32}{3}(z-1)H_1 H_{0,1} + \big(4z^2 + 9z + 3\big)\Big[\frac{2}{9}H_0^4 + \frac{16}{3}H_{0,1}^2\Big] + \frac{32\big(9z^2 + 6z - 4\big)H_0 H_{0,1,1}}{3z}$$

$$+\frac{4\big(616z^3 + 1917z^2 + 1017z + 120\big)H_{0,0,1}}{9z} + \frac{64(z+1)\big(z^2 - 5z + 1\big)H_0 H_{0,0,1}}{z}$$

$$-\frac{8\big(188z^3 - 60z^2 + 141z + 72\big)H_{0,1,1}}{9z} - \frac{32\big(14z^3 - 9z^2 - 42z + 6\big)H_{0,0,0,1}}{3z}$$

$$-\frac{16\big(33z^2 + 45z - 8\big)H_{0,0,1,1}}{3z} + \frac{16\big(28z^3 + 21z^2 - 33z - 32\big)H_{0,1,1,1}}{3z}$$

$$-8(z-1)(4z+3)H_0^2\zeta_2 + \frac{2}{9}\big(368z^2 - 1401z - 417\big)H_0\zeta_2 + L_M^3\Big[\frac{16\big(4z^2 + 7z + 4\big)H_1(z-1)}{9z}$$

$$+\frac{92(z-1)}{9} + \frac{16}{9}z(4z+3)H_0 + (z+1)\Big[-\frac{8}{3}H_0^2 - \frac{32}{3}H_{0,1} + \frac{32\zeta_2}{3}\Big]\Big]$$

$$+L_M^2\Big[\frac{8}{3}z(8z-9)H_0^2 + \frac{4}{3}\big(88z^2 + 91z - 37\big)H_0 + \frac{4(z-1)(61z+12)}{3z}$$

$$+\frac{4(z-1)\big(88z^2 + 135z + 40\big)H_1}{3z} + \frac{(z-1)\big(4z^2 + 7z + 4\big)\Big[\frac{8}{3}H_1^2 + \frac{32}{3}H_0 H_1\Big]}{z}$$

$$-\frac{16\big(4z^3 + 30z^2 + 15z - 8\big)H_{0,1}}{3z} - \frac{16}{3}\big(4z^2 - 24z - 21\big)\zeta_2 + (z+1)\Big[-8H_0^3 - 64H_{0,1}H_0$$

$$+48\zeta_2 H_0 + 80H_{0,0,1} - 32H_{0,1,1} - 48\zeta_3\Big]\Big] - \frac{8}{9}\big(180z^2 + 1069z + 245\big)\zeta_3$$

$$-\frac{16}{9}\big(4z^2 - 96z + 27\big)H_0\zeta_3 + \frac{(z-1)\big(4z^2 + 7z + 4\big)}{z}\Big[-\frac{2}{9}H_1^4 - \frac{16}{9}H_0 H_1^3 - \frac{8}{3}H_0^2 H_1^2$$

$$+\frac{16}{3}H_{0,1}H_1^2 - \frac{8}{3}H_0^3 H_1 - \frac{64}{3}H_{0,1,1}H_1 - \frac{16}{9}\zeta_3 H_1 + \Big[\frac{16}{3}H_0 H_1 - \frac{4}{3}H_1^2\Big]\zeta_2\Big]$$

$$+L_M\Big[-\frac{64}{9}\big(8z^2 + 3z + 3\big)H_0^3 + \frac{2}{9}\big(568z^2 - 1071z + 585\big)H_0^2$$

$$+\frac{16}{27}\big(1003z^2 - 1527z - 291\big)H_0 + \frac{8(z-1)\big(364z^2 - 119z + 184\big)H_1 H_0}{9z}$$

$$-\frac{64}{3}\big(10z^2 + 12z + 3\big)H_{0,1}H_0 + 32(2z-1)(4z+5)\zeta_2 H_0 + \frac{4(z-1)\big(308z^2 + 71z + 92\big)H_1^2}{9z}$$

$$+\frac{4(z-1)\big(1128z^2 + 1655z + 180\big)}{27z} - 16\big(12z^3 + 9z^2 + 7z - 8\big)\frac{\zeta_3}{z}$$

$$+\frac{16(z-1)\big(823z^2 - 1088z - 59\big)H_1}{27z} - \frac{16\big(68z^3 - 75z^2 - 159z - 92\big)H_{0,1}}{9z}$$

$$+\frac{(z-1)\big(4z^2 + 7z + 4\big)}{z}\Big[-\frac{8}{9}H_1^3 - \frac{16}{3}H_0 H_1^2 + \frac{32}{3}H_{0,1}H_1\Big]$$

$$+\frac{32}{3}\big(16z^2 + 30z + 27\big)H_{0,0,1} - \frac{16\big(28z^3 + 57z^2 + 9z - 8\big)H_{0,1,1}}{3z}$$

$$-\frac{8}{3}\big(76z^2 - 111z + 207\big)\zeta_2 + (z+1)\Big[\frac{22}{3}H_0^4 - 112\zeta_2 H_0^2 + \big(160H_{0,0,1} + 64H_{0,1,1}\big)H_0$$

$$+224\zeta_3 H_0 - 32H_{0,1}^2 + \frac{576\zeta_2^2}{5} - 256H_{0,0,0,1} + 64H_{0,0,1,1} + 32H_{0,1,1,1}\Big]\Big]$$

$$+(z+1)\Big[16H_{0,1}H_0^3 + \big[32H_{0,1,1} - 96H_{0,0,1}\big]H_0^2 - 80\zeta_2^2 H_0 + \big[256H_{0,0,0,1} - 96H_{0,0,1,1}$$

$$+64H_{0,1,1,1}\Big]H_0 - 64H_{0,1}H_{0,1,1} - 272H_{0,0,0,0,1} + 64H_{0,0,0,1,1} - 64H_{0,0,1,0,1} + 288H_{0,0,1,1,1}$$

$$+192H_{0,1,0,1,1} + 32H_{0,1,1,1,1} + \zeta_2\Big[\frac{20}{3}H_0^3 - 32H_{0,1}H_0 + 128H_{0,0,1} + 16H_{0,1,1} - \frac{368\zeta_3}{3}\Big]$$

$$+\Big[\frac{32}{3}H_{0,1} - \frac{16}{3}H_0^2\Big]\zeta_3 + 240\zeta_5\Big]\Big] + C_F T_F^2\Big[\Big[\Big[\frac{64}{9}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{27z}\Big]L_Q^3$$

$$+\Big[-\frac{64}{3}(z+1)H_0^2 + \frac{64}{9}(4z^2-11z-8)H_0 + \frac{32(z-1)(10z^2+33z-2)}{9z}\Big]L_Q^2$$

$$+\Big[\frac{\big(\frac{256}{9}H_{0,-1} - \frac{256}{9}H_{-1}H_0\big)(z+1)^3}{z} + \frac{64}{3}H_0^3(z+1) - \frac{32}{9}(12z^2-59z-29)H_0^2$$

$$+\Big(\frac{64}{3}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{9z}\Big)L_M^2 - \frac{64(z-1)(304z^2+811z+124)}{81z}$$

$$-\frac{64}{27}(60z^2-155z-233)H_0 - \frac{256(3z^2+1)\zeta_2}{9z} + L_M\Big[-\frac{64(z-1)(38z^2+47z+20)}{27z}$$

$$+\frac{128}{9}(2z^2+11z+8)H_0 + \frac{64(z-1)(4z^2+7z+4)H_1}{9z} + (z+1)\Big[\frac{128\zeta_2}{3} - \frac{128}{3}H_{0,1}\Big]\Big]\Big]L_Q$$

$$+\Big[\frac{128(z-1)(4z^2+7z+4)}{27z} - \frac{256}{9}(z+1)H_0\Big]L_M^3 - \frac{32}{27}(75z^2+100z+64)H_0^2$$

$$-\frac{16(z-1)(22z^2+29z+22)H_1^2}{9z} + \Big[\frac{32(z-1)(142z^2+103z+34)}{27z}$$

$$-\frac{64}{9}(4z^2+26z+11)H_0\Big]L_M^2 - \frac{64(z-1)(461z^2-460z+740)}{243z}$$

$$-\frac{64}{27}(14z^3-62z^2-77z-20)\frac{\zeta_2}{z} - \frac{32}{27}(148z^3+279z^2+111z-16)\frac{\zeta_3}{z}$$

$$+\frac{32}{81}(784z^2-473z+463)H_0 - \frac{32(z-1)(16z^2-161z-254)H_1}{81z}$$

$$+\frac{32(z-1)(74z^2-43z+20)H_0H_1}{27z} + \frac{(z-1)(4z^2+7z+4)\big[\frac{64}{27}H_1^3 + \frac{16}{9}H_0H_1^2 - \frac{32}{9}H_0^2H_1\big]}{z}$$

$$-\frac{64(70z^3-19z^2+104z-10)H_{0,1}}{27z} - \frac{64(2z^3+z^2-2z+4)H_0H_{0,1}}{9z}$$

$$+\frac{64(10z^3+16z^2+z+4)H_{0,0,1}}{9z} + \frac{128(3z^3+12z^2+12z+1)H_{0,1,1}}{9z}$$

$$+(8z^2+15z+3)\Big[\frac{16}{27}H_0^3 + \frac{32}{9}\zeta_2H_0\Big] + (z+1)\Big[-\frac{8}{9}H_0^4 + \frac{64}{3}H_{0,1}H_0^2 - \frac{32}{3}\zeta_2H_0^2$$

$$+\Big[-\frac{128}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1}\Big]H_0 + \frac{1216}{9}\zeta_3H_0 + 32\zeta_2^2 + \frac{64}{3}H_{0,0,1,1} - \frac{256}{3}H_{0,1,1,1}\Big]$$

$$+L_M\Big[\frac{256\zeta_2z^2}{3} - \frac{128}{27}(14z^2+127z+64)H_0 + (z+1)\Big[-\frac{64}{9}H_0^3 - \frac{928}{9}H_0^2$$

$$+\frac{256}{3}H_{0,1}H_0 - \frac{256}{3}\zeta_2H_0 - \frac{256}{3}H_{0,0,1} + \frac{128}{3}H_{0,1,1} + \frac{128\zeta_3}{3}\Big] + \frac{64(z-1)(616z^2+667z+94)}{81z}$$

$$+\frac{128(z-1)(4z^2-26z+13)H_1}{27z} + \frac{(z-1)(4z^2+7z+4)\big[-\frac{32}{9}H_1^2 - \frac{128}{9}H_0H_1\big]}{z}$$

$$-\frac{128(2z^3-3z^2+3z+4)H_{0,1}}{9z}\Big]\Big]\Big]$$

$$+C_F N_F T_F^2 \left[\left[\frac{64}{9}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{27z} \right] L_Q^3 \right.$$

$$+ \left[-\frac{64}{3}(z+1)H_0^2 + \frac{64}{9}(4z^2-11z-8)H_0 + \frac{32(z-1)(10z^2+33z-2)}{9z} \right] L_Q^2$$

$$+ \left[\frac{\left(\frac{256}{9}H_{0,-1} - \frac{256}{9}H_{-1}H_0\right)(z+1)^3}{z} + \left[\frac{64}{3}H_0^3 - \frac{64}{3}H_{0,1,1} + \frac{64\zeta_3}{3} \right](z+1) \right.$$

$$- \frac{32}{9}(12z^2-59z-29)H_0^2 + \frac{16(z-1)(4z^2+7z+4)H_1^2}{9z} - \frac{64(z-1)(194z^2+683z+68)}{81z}$$

$$- \frac{64}{9}(2z^3+23z^2+8z+4)\frac{\zeta_2}{z} - \frac{64}{27}(79z^2-88z-190)H_0 - \frac{32(z-1)(38z^2+47z+20)H_1}{27z}$$

$$\left. + \frac{64}{9}(2z^2+11z+8)H_{0,1} \right] L_Q + \frac{16}{27}(8z^2+15z+3)H_0^3 + \left[\frac{32(z-1)(4z^2+7z+4)}{27z} \right.$$

$$\left. - \frac{64}{9}(z+1)H_0 \right] L_M^3 - \frac{32}{27}(56z^2+33z+21)H_0^2 - \frac{64(z-1)(1156z^2-203z+328)}{243z}$$

$$- \frac{16}{27}(z-1)(74z^2-43z+20)\frac{\zeta_2}{z} - \frac{32}{27}(100z^3+183z^2+33z-4)\frac{\zeta_3}{z}$$

$$+ \frac{32}{81}(800z^2-57z+111)H_0 + \frac{(z-1)(28z^2+z+10)\left[\frac{128}{27}H_0H_1 - \frac{128}{27}H_{0,1}\right]}{z}$$

$$- \frac{128(2z^3+6z^2+3z+2)H_0H_{0,1}}{9z} + \frac{64(12z^3+27z^2+9z+4)H_{0,0,1}}{9z}$$

$$+ \frac{32}{9}(6z^2+4z-5)H_0\zeta_2 + \frac{(z-1)(4z^2+7z+4)\left[-\frac{32}{9}H_1H_0^2 - \frac{16}{9}H_1\zeta_2\right]}{z}$$

$$+ L_M^2 \left[\frac{32}{3}(z-1)(2z-5) - \frac{32}{9}(4z^2-7z-13)H_0 + \frac{32(z-1)(4z^2+7z+4)H_1}{9z} \right.$$

$$\left. + (z+1)\left[\frac{32}{3}H_0^2 - \frac{64}{3}H_{0,1} + \frac{64\zeta_2}{3} \right] \right] + (z+1)\left[-\frac{8}{9}H_0^4 + \frac{64}{3}H_{0,1}H_0^2 - \frac{128}{3}H_{0,0,1}H_0 \right.$$

$$+ \frac{832}{9}\zeta_3 H_0 - \frac{32\zeta_2^2}{3} + \left[\frac{32}{3}H_{0,1} - \frac{32}{3}H_0^2 \right]\zeta_2 \right] + L_M \left[\frac{32}{9}(4z^2-7z-13)H_0^2 \right.$$

$$+ \frac{64}{27}(z^2+2z-58)H_0 + \frac{128(z-1)(25z^2+94z+34)}{81z} + \frac{32(z-1)(74z^2-43z+20)H_1}{27z}$$

$$+ \frac{(z-1)(4z^2+7z+4)\left[\frac{16}{9}H_1^2 - \frac{64}{9}H_0H_1\right]}{z} - \frac{64(2z^3+z^2-2z+4)H_{0,1}}{9z}$$

$$\left. + \frac{64}{9}(6z^2+4z-5)\zeta_2 + (z+1)\left[-\frac{64}{9}H_0^3 + \frac{128}{3}H_{0,1}H_0 - \frac{128}{3}\zeta_2 H_0 - \frac{128}{3}H_{0,0,1} \right.$$

$$\left. \left. \left. - \frac{64}{3}H_{0,1,1} + 64\zeta_3 \right] \right] \right] + C_F C_A T_F \left[-\frac{2}{9}(4z-17)H_0^4 - \frac{4}{9}(36z^2+47z+36)H_0^3 \right.$$

$$+ \frac{4}{27}(2132z^2-681z+855)H_0^2 + \frac{8(z-1)(122z^2-19z+113)H_1H_0^2}{9z}$$

$$- \frac{8(19z^2+19z+8)H_{0,1}H_0^2}{3z} - 32(2z-1)H_{0,0,1}H_0^2 - \frac{8}{3}(10z-17)\zeta_2 H_0^2 - \frac{16}{3}(14z-13)\zeta_3 H_0^2$$

$$- \frac{16(z-1)(19z^2+16z+10)H_1^2 H_0}{9z} + \frac{16}{5}(38z-5)\zeta_2^2 H_0 + \frac{16}{9}(13z^2-215z-4)\frac{\zeta_3}{z}H_0$$

$$-\frac{4\left(54204z^3+9339z^2+17082z+2624\right)H_0}{81z}-\frac{4}{9}\left(474z^3+121z^2+295z+80\right)\frac{\zeta_2}{z}H_0$$

$$-\frac{32(z-1)\left(769z^2-62z+337\right)H_1H_0}{27z}-\frac{32\left(19z^3-24z^2-6z+10\right)H_{0,-1}H_0}{9z}$$

$$+\frac{16\left(18z^3+127z^2+6z+51\right)H_{0,1}H_0}{3z}+\frac{64\left(10z^2+z+4\right)H_{0,0,1}H_0}{3z}+64(5z-2)H_{0,0,0,1}H_0$$

$$+\frac{8(z-1)(2z+1)(14z+1)H_1^3}{27z}+\frac{4(z-1)\left(328z^2+313z+67\right)H_1^2}{27z}$$

$$+\frac{4(z-1)\left(75516z^2-7654z+21765\right)}{81z}+\frac{8}{27}\left(1841z^3-1719z^2+1230z-515\right)\frac{\zeta_2}{z}$$

$$+\frac{8}{15}\left(20z^3+340z^2-137z+152\right)\frac{\zeta_2^2}{z}+\frac{8}{9}\left(768z^3+1158z^2+687z+76\right)\frac{\zeta_3}{z}$$

$$+\frac{4(z-1)\left(2500z^2+2755z+1771\right)H_1}{81z}+\frac{4}{9}(z-1)\left(154z^2+163z+46\right)\frac{\zeta_2}{z}H_1$$

$$+\frac{(z+1)\left(182z^2-122z+47\right)\left[\frac{32}{27}H_{-1}H_0-\frac{32}{27}H_{0,-1}\right]}{z}$$

$$+\frac{8\left(2964z^3-3393z^2+1368z-1348\right)H_{0,1}}{27z}+\frac{32\left(19z^3-51z^2-6z+10\right)H_{0,0,-1}}{9z}$$

$$-\frac{16\left(230z^3+621z^2+168z+193\right)H_{0,0,1}}{9z}+\frac{8\left(56z^3-105z^2-66z-40\right)H_{0,1,1}}{9z}$$

$$-\frac{16\left(21z^2-15z+8\right)H_{0,0,0,1}}{z}-\frac{16\left(20z^3+18z^2-15z-20\right)H_{0,0,1,1}}{3z}$$

$$-\frac{32(z-1)(z+2)(2z+1)H_{0,1,1,1}}{3z}-128(4z-1)H_{0,0,0,0,1}+L_M^3\left[\frac{16}{3}(2z-1)H_0^2\right.$$

$$+\frac{16\left(8z^2+11z+4\right)H_0}{9z}-\frac{8(z-1)\left(44z^2-z+44\right)}{9z}-\frac{16(z-1)\left(4z^2+7z+4\right)H_1}{9z}$$

$$+(z+1)\left[\frac{32}{3}H_{0,1}-\frac{32\zeta_2}{3}\right]\right]-\frac{8}{3}(23z+14)H_{0,1}\zeta_2+\frac{(z+1)\left(19z^2-16z+10\right)}{z}\left[-\frac{32}{9}H_0H_{-1}^2\right.$$

$$+\left(\frac{16}{9}H_0^2+\frac{64}{9}H_{0,-1}\right)H_{-1}-\frac{32}{9}\zeta_2H_{-1}-\frac{64}{9}H_{0,-1,-1}\right]+L_Q^3\left[-\frac{16}{3}(2z-1)H_0^2\right.$$

$$-\frac{16\left(8z^2+11z+4\right)H_0}{9z}+\frac{8(z-1)\left(44z^2-z+44\right)}{9z}+\frac{16(z-1)\left(4z^2+7z+4\right)H_1}{9z}$$

$$+(z+1)\left[\frac{32\zeta_2}{3}-\frac{32}{3}H_{0,1}\right]\right]+\frac{8}{3}(43z+37)\zeta_2\zeta_3+L_Q^2\left[\frac{16}{3}(4z-3)H_0^3\right.$$

$$-\frac{32\left(101z^3-53z^2+82z+13\right)H_0}{9z}-32(2z+1)\zeta_2H_0+\frac{8(z-1)\left(181z^2-413z+901\right)}{27z}$$

$$-\frac{16(z-1)\left(62z^2-7z+44\right)H_1}{9z}+\frac{(z-1)\left(4z^2+7z+4\right)\left[-\frac{16}{3}H_1^2-\frac{32}{3}H_0H_1\right]}{z}$$

$$+\frac{(z+1)\left(4z^2-7z+4\right)\left[\frac{32}{3}H_{-1}H_0-\frac{32}{3}H_{0,-1}\right]}{z}+16(2z-1)H_{0,1}+(z-1)\left[64H_0H_{0,-1}\right.$$

$$\left.-128H_{0,0,-1}\right]-96H_{0,0,1}+(z+1)\left[\frac{224}{3}H_0^2+64H_{0,1}H_0+64H_{0,1,1}\right]+\frac{16}{3}\left(8z^2-6z-3\right)\zeta_2$$

$$+32(z-2)\zeta_3\right]+L_M^2\left[\frac{8}{3}(16z-17)H_0^2+\frac{8\left(272z^3+103z^2+139z+40\right)H_0}{9z}\right.$$

$$+\frac{16(z-1)(4z^2+7z+4)H_1H_0}{3z}-\frac{8(z-1)(1864z^2-485z+694)}{27z}$$

$$-\frac{16}{3}(4z^3+17z^2+11z+4)\frac{\zeta_2}{z}-\frac{8(z-1)(104z^2+119z+32)H_1}{9z}$$

$$+\frac{(z+1)(4z^2-7z+4)\left[\frac{32}{3}H_{0,-1}-\frac{32}{3}H_{-1}H_0\right]}{z}+\frac{32}{3}(10z+7)H_{0,1}+(z+1)\Big[32H_{0,0,1}$$

$$-32H_0H_{0,1}\Big]+(z-1)\Big[-\frac{32}{3}H_0^3-64H_{0,-1}H_0-32\zeta_2H_0+128H_{0,0,-1}\Big]-64(2z-1)\zeta_3\Big]$$

$$+\frac{(z+1)(4z^2-7z+4)}{z}\Big[\frac{32}{9}H_0H_{-1}^3+\Big[-\frac{8}{3}H_0^2-\frac{32}{3}H_{0,-1}\Big]H_{-1}^2+\Big[-\frac{8}{9}H_0^3+(\frac{32}{3}H_{0,-1}$$

$$-\frac{32}{3}H_{0,1})H_0+\frac{64}{3}H_{0,-1,-1}-\frac{32}{3}H_{0,0,-1}+\frac{64}{3}H_{0,0,1}\Big]H_{-1}-16\zeta_3H_{-1}+\frac{8}{3}H_0^2H_{0,-1}$$

$$+H_0\Big[-\frac{32}{3}H_{0,-1,-1}+\frac{32}{3}H_{0,-1,1}-\frac{16}{3}H_{0,0,-1}+\frac{32}{3}H_{0,1,-1}\Big]-\frac{64}{3}H_{0,-1,-1,-1}-\frac{32}{3}H_{0,-1,0,1}$$

$$+\frac{32}{3}H_{0,0,-1,-1}-\frac{64}{3}H_{0,0,-1,1}+\frac{16}{3}H_{0,0,0,-1}-\frac{64}{3}H_{0,0,1,-1}+\Big[\frac{16}{3}H_{-1}^2+8H_0H_{-1}-8H_{0,-1}\Big]\zeta_2\Big]$$

$$+L_Q\Big[-\frac{4}{3}(12z-11)H_0^4-\frac{8}{3}(41z+34)H_0^3+\frac{4}{9}(1508z^2-1040z+1297)H_0^2$$

$$-\frac{24(z+1)(4z^2-5z+4)H_{-1}H_0^2}{z}+\frac{4(z-1)(56z^2+125z+56)H_1H_0^2}{3z}+8(19z-3)\zeta_2H_0^2$$

$$-\frac{16(466z^3+1961z^2+4007z+284)H_0}{27z}-\frac{8}{3}(40z^3-93z^2-21z-16)\frac{\zeta_2}{z}H_0$$

$$+\frac{32(z-1)(76z^2+19z+31)H_1H_0}{9z}-\frac{32(2z^3+3z^2+24z-22)H_{0,-1}H_0}{3z}$$

$$-\frac{8(31z^2-19z-8)H_{0,1}H_0}{z}+32(11z-7)H_{0,0,-1}H_0+32(7z+13)H_{0,0,1}H_0$$

$$+32(14z+15)\zeta_3H_0+\frac{4(z-1)(286z^2+25z+106)H_1^2}{9z}+\frac{8}{5}(41z+35)\zeta_2^2$$

$$+\frac{8(z-1)(3335z^2+2808z+4604)}{27z}-\frac{8}{9}(846z^3-371z^2+901z-264)\frac{\zeta_2}{z}$$

$$-\frac{16}{3}(76z^3+35z^2+14z-48)\frac{\zeta_3}{z}+16(z+1)(8z^2-5z+8)\frac{\zeta_2}{z}H_{-1}$$

$$-\frac{8(z-1)(968z^2+1301z+2210)H_1}{27z}-\frac{64}{3}(z-1)(5z^2+11z+5)\frac{\zeta_2}{z}H_1$$

$$+\frac{(z-1)(4z^2+7z+4)\left[\frac{40}{9}H_1^3+\frac{40}{3}H_0H_1^2\right]}{z}+\frac{(z+1)(73z^2+38z+118)}{z}\Big[\frac{32}{9}H_{-1}H_0$$

$$-\frac{32}{9}H_{0,-1}\Big]+\frac{8(542z^3+301z^2+853z+332)H_{0,1}}{9z}+\frac{16(44z^3+3z^2+87z-52)H_{0,0,-1}}{3z}$$

$$-\frac{16(8z^3-15z^2+33z-4)H_{0,0,1}}{3z}+\frac{(z+1)(z^2-z+1)}{z}\Big[-128H_{-1}H_{0,1}+128H_{0,-1,1}$$

$$+128H_{0,1,-1}\Big]+\frac{16}{3}(8z^2-19z+14)H_{0,1,1}-384zH_{0,0,0,-1}-128(2z+3)H_{0,0,0,1}$$

$$-32(z-5)H_{0,0,1,1}+(z-1)\Big[-112H_{0,-1}H_0^2-64H_{0,-1,-1}H_0+32H_{0,-1}^2-128H_{0,-1,0,1}$$

$$+96H_{0,-1}\zeta_2\Big]+(z+1)\Big[48H_0H_{-1}^2-96H_{0,-1}H_{-1}-136H_0^2H_{0,1}+96H_{0,-1,-1}-160H_0H_{0,1,1}$$

$$-160H_{0,1,1,1}+192H_{0,1}\zeta_2\Big]\Big]+L_M\Big[\frac{4}{3}(4z-5)H_0^4-\frac{8}{9}(35z-46)H_0^3$$

$$-\frac{4}{9}\left(606z^2 - 346z + 377\right)H_0^2 + \frac{32(z+1)\left(z^2 - z + 1\right)H_{-1}H_0^2}{z}$$

$$-\frac{4(z-1)\left(8z^2 + 17z + 8\right)H_1 H_0^2}{z} + \frac{16\left(3729z^3 + 2093z^2 + 2330z + 224\right)H_0}{27z}$$

$$+\frac{16(z-1)\left(203z^2 + 47z + 140\right)H_1 H_0}{9z} + \frac{16\left(4z^3 - 13z^2 + 3z - 4\right)H_{0,-1}H_0}{z}$$

$$+\frac{8\left(16z^3 - 41z^2 - 77z - 40\right)H_{0,1}H_0}{3z} - 32(3z+1)H_{0,0,-1}H_0 - 32(11z+5)H_{0,0,1}H_0$$

$$+\frac{8}{3}\left(8z^2 - 13z + 59\right)\zeta_2 H_0 - 32(16z+9)\zeta_3 H_0 - \frac{4(z-1)\left(20z^2 + 21z + 2\right)H_1^2}{3z}$$

$$-\frac{8}{5}(209z + 87)\zeta_2^2 - \frac{8(z-1)\left(11542z^2 + 399z + 3892\right)}{27z} - \frac{8}{9}\left(132z^3 + 313z^2 - 473z + 168\right)\frac{\zeta_2}{z}$$

$$+\frac{16}{3}\left(52z^3 - 78z^2 - 33z - 48\right)\frac{\zeta_3}{z} - \frac{8(z-1)\left(210z^2 - 559z - 186\right)H_1}{9z}$$

$$+\frac{(z+1)\left(125z^2 - 20z + 62\right)\left[\frac{32}{9}H_{0,-1} - \frac{32}{9}H_{-1}H_0\right]}{z} - \frac{8\left(274z^3 - 205z^2 + 659z - 200\right)H_{0,1}}{9z}$$

$$+\frac{(z+1)\left(4z^2 - 7z + 4\right)\left[\frac{32}{3}H_0 H_{-1}^2 - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1}\right]}{z}$$

$$-\frac{32(z-2)\left(6z^2 - z + 1\right)H_{0,0,-1}}{z} - \frac{32\left(4z^3 - 11z^2 - 17z - 14\right)H_{0,0,1}}{3z}$$

$$+\frac{(z+1)\left(2z^2 + z + 2\right)\left[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1}\right]}{z}$$

$$+\frac{16\left(8z^3 + 23z^2 + 5z - 8\right)H_{0,1,1}}{3z} + 96(z+3)H_{0,0,0,-1} + 32(23z+5)H_{0,0,0,1}$$

$$+\frac{(z-1)\left(4z^2 + 7z + 4\right)}{z}\left[\frac{8}{9}H_1^3 + 8H_0 H_1^2 - \frac{32}{3}H_{0,1}H_1 + \frac{16}{3}\zeta_2 H_1\right]$$

$$+(z-1)\left[32H_{0,-1}H_0^2 + 128H_{0,-1,-1}H_0 - 64H_{0,-1}^2 + \left(8H_0^2 + 64H_{0,-1}\right)\zeta_2\right]$$

$$+(z+1)\left[56H_{0,1}H_0^2 - 96H_{0,1,1}H_0 + 32H_{0,1}^2 + 32H_{0,0,1,1} - 32H_{0,1,1,1}\right.$$

$$\left.+\left[-96H_{-1} - 32H_{0,1}\right]\zeta_2\right] + \frac{(z-1)\left(4z^2 + 7z + 4\right)}{z}\left[\frac{2}{9}H_1^4 + \frac{4}{3}H_0^2 H_1^2\right.$$

$$+\left[\frac{80}{3}H_{0,0,1} + \frac{16}{3}H_{0,1,1}\right]H_1 - \frac{80}{9}\zeta_3 H_1 - \frac{16}{3}H_{0,1}^2 + H_0\left[-\frac{16}{9}H_1^3 - 16H_{0,1}H_1\right.$$

$$\left.+\frac{80}{3}H_{0,1,1}\right] + \left[\frac{4}{3}H_1^2 - \frac{8}{3}H_0 H_1\right]\zeta_2\right] + (z-1)\left[\frac{4}{15}H_0^5 - \frac{16}{3}H_{0,-1}H_0^3 + \left[32H_{0,0,-1}\right.\right.$$

$$\left.-32H_{0,-1,-1}\right]H_0^2 + \left[32H_{0,-1}^2 + 128H_{0,-1,-1,-1} - 64H_{0,-1,0,1} - 96H_{0,0,0,-1}\right]H_0$$

$$+H_{0,-1}\left[-128H_{0,-1,-1} - 64H_{0,0,-1} + 128H_{0,0,1}\right] + 256H_{0,-1,0,-1,-1} + 512H_{0,0,-1,-1,-1}$$

$$+64H_{0,0,-1,0,-1} - 128H_{0,0,-1,0,1} + 192H_{0,0,0,-1,-1} - 384H_{0,0,0,-1,1} + 128H_{0,0,0,0,-1}$$

$$-384H_{0,0,0,1,-1} - 128H_{0,0,1,0,-1} + \left[8H_0^3 + 48H_{0,-1}H_0 + 64H_{0,-1,-1} - 96H_{0,0,-1}\right]\zeta_2$$

$$-96H_{0,-1}\zeta_3\right] + (z+1)\left[-16H_{0,1,1}H_0^2 + \left[48H_{0,1}^2 - 128H_{0,0,1,1} + 64H_{0,1,1,1}\right]H_0\right.$$

$$-160H_{0,1}H_{0,0,1} + 864H_{0,0,0,1,1} + 288H_{0,0,1,0,1} - 128H_{0,0,1,1,1} - 32H_{0,1,0,1,1} - 32H_{0,1,1,1,1}$$

$$\left.+\left[16H_0 H_{0,1} - 16H_{0,0,1} - 16H_{0,1,1}\right]\zeta_2 + \frac{160}{3}H_{0,1}\zeta_3\right] + 80(7z-3)\zeta_5\right]$$

$$+ a_{Qq}^{\mathsf{PS},(3)} + \tilde{C}_{2,q}^{\mathsf{PS},(3)}(N_F + 1) \Bigg\} \ . \tag{597}$$

The Wilson coefficient $H_{g,2}^{\mathsf{S}}$ is given by :

$$H_{g,2}^{\mathsf{S}} =$$

$$a_s T_F \Bigg\{ -\gamma_{qg}^0 L_Q - 4\big(8z^2 - 8z + 1\big) + \gamma_{qg}^0 \Big[H_0 + H_1\Big] + \gamma_{qg}^0 L_M \Bigg\}$$

$$+ a_s^2 \Bigg\{ T_F^2 \Bigg[\frac{4}{3}\gamma_{qg}^0 L_M^2 - \frac{4}{3}\gamma_{qg}^0 L_Q L_M + \Big[\gamma_{qg}^0 \Big[\frac{4H_0}{3} + \frac{4H_1}{3}\Big] - \frac{16}{3}\big(8z^2 - 8z + 1\big)\Big] L_M \Bigg]$$

$$+ C_A T_F \Bigg[\frac{16}{3}(3z + 1)H_0^3 - 2z(57z - 92)H_0^2 + 4\big(8z^2 + 6z + 3\big)H_{-1}H_0^2 - 4\big(4z^2 - 6z + 3\big)H_1 H_0^2$$

$$- \frac{4}{9}\big(1445z^2 - 747z - 219\big)H_0 - \frac{4\big(199z^3 - 168z^2 - 3z - 16\big)H_1 H_0}{3z} - 8\big(4z^2 + 6z + 3\big)H_{0,-1}H_0$$

$$+ 8(14z + 5)H_{0,1}H_0 + 16\big(2z^2 - 8z - 1\big)\zeta_2 H_0 - \frac{2\big(107z^3 - 96z^2 + 9z - 8\big)H_1^2}{3z}$$

$$+ \Bigg[\frac{4(z - 1)\big(31z^2 + 7z + 4\big)}{3z} - 8(4z + 1)H_0 - 2\gamma_{qg}^0 H_1 \Bigg] L_M^2 - \frac{2\big(439z^3 - 130z^2 - 233z - 40\big)}{9z}$$

$$+ \frac{4}{3}\big(219z^3 - 204z^2 + 12z - 8\big)\frac{\zeta_2}{z} - \frac{4\big(749z^3 - 645z^2 - 84z + 52\big)H_1}{9z}$$

$$+ L_Q^2 \Bigg[-\frac{4(z - 1)\big(31z^2 + 7z + 4\big)}{3z} + 8(4z + 1)H_0 + 2\gamma_{qg}^0 H_1 \Bigg] + \frac{\big(13z^3 + 3z^2 - 9z - 2\big)}{z}\Bigg[\frac{16}{3}H_{-1}H_0$$

$$- \frac{16}{3}H_{0,-1}\Bigg] - \frac{4\big(20z^3 - 48z^2 + 15z + 16\big)H_{0,1}}{3z} + \gamma_{qg}^0 \Big[4H_0 H_1^2 - 4H_1 H_{0,1}\Big] + z^2\Big[-16H_0 H_{-1}^2$$

$$+ 32H_{0,-1}H_{-1} - 32H_{0,-1,-1}\Big] + 24(2z + 1)H_{0,0,-1} - 8(18z + 5)H_{0,0,1} + \big(2z^2 + 2z + 1\big)\Big[16H_{-1}H_{0,1}$$

$$- 16H_{0,-1,1} - 16H_{0,1,-1}\Big] - 32(z - 3)zH_{0,1,1} - 16\big(3z^2 + 2z + 1\big)H_{-1}\zeta_2 + 16(z - 1)^2 H_1 \zeta_2$$

$$+ L_M \Bigg[8(2z + 1)H_0^2 - \frac{8}{3}\big(44z^2 + 24z + 3\big)H_0 - 2\gamma_{qg}^0 H_1^2 + \frac{8\big(218z^3 - 225z^2 + 18z - 20\big)}{9z}$$

$$+ 32(z - 1)zH_1 + \big(2z^2 + 2z + 1\big)\Big[16H_{-1}H_0 - 16H_{0,-1}\Big] + 32z\zeta_2 \Bigg] + L_Q \Bigg[-16(3z + 1)H_0^2$$

$$+ 8z(25z - 24)H_0 + \frac{4\big(407z^3 - 276z^2 - 165z + 52\big)}{9z} + \frac{8\big(67z^3 - 60z^2 + 3z - 4\big)H_1}{3z} + \gamma_{qg}^0 \Big[-2H_1^2$$

$$- 4H_0 H_1\Big] + \big(2z^2 + 2z + 1\big)\Big[16H_{0,-1} - 16H_{-1}H_0\Big] - 16(4z + 1)H_{0,1} - 32(z - 2)z\zeta_2 \Bigg]$$

$$+ 8\big(6z^2 + 2z + 5\big)\zeta_3 \Bigg] + C_F T_F \Bigg[\Big[2(4z - 1) - 4\big(4z^2 - 2z + 1\big)H_0 + 2\gamma_{qg}^0 H_1\Big] L_Q^2$$

$$+ \Big[8\big(4z^2 - 2z + 1\big)H_0^2 + 8\big(10z^2 - 6z + 1\big)H_0 + 4\big(4z^2 - 17z + 9\big) + 4(2z - 1)(10z - 7)H_1$$

$$+ \gamma_{qg}^0 \Big[-4H_1^2 - 8H_0 H_1\Big] + 8(2z - 1)H_{0,1} + \Big[-4(4z - 1) + 8\big(4z^2 - 2z + 1\big)H_0 - 4\gamma_{qg}^0 H_1\Big] L_M$$

$$- 8\big(8z^2 - 6z + 3\big)\zeta_2 \Big] L_Q - \frac{4}{15}\big(72z^3 + 195z^2 - 10z + 15\big)H_0^2 - 2\big(42z^2 - 44z + 11\big)H_1^2$$

$$+\left[2(4z-1)-4\left(4z^2-2z+1\right)H_0+2\gamma_{qg}^0H_1\right]L_M^2+\frac{4\left(246z^3+51z^2-226z+4\right)}{15z}$$

$$-\frac{8\left(252z^3-51z^2+89z+2\right)H_0}{15z}-16z^2H_0^2H_1-4\left[24z^2-33z+7\right]H_1$$

$$-8\left(13z^2-14z+6\right)H_0H_1+\frac{\left(36z^5+40z^3+90z^2+1\right)\left[\frac{16}{15}H_{-1}H_0-\frac{16}{15}H_{0,-1}\right]}{z^2}$$

$$+(z-1)^2H_0\left[32H_{0,-1}-32H_{0,1}\right]-8\left(8z^2-6z-3\right)H_{0,1}+\gamma_{qg}^0\left[2H_1^3+4H_0H_1^2+4H_{0,1}H_1\right]$$

$$-32\left(3z^2-2z+3\right)H_{0,0,-1}-32(2z-1)H_{0,0,1}+8\left(4z^2-6z+3\right)H_{0,1,1}$$

$$+\frac{8}{15}\left(72z^3+315z^2-220z+45\right)\zeta_2+32\left(3z^2-2z+1\right)H_0\zeta_2+L_M\left[-8\left(4z^2-2z+1\right)H_0^2\right.$$

$$-8\left(10z^2-6z+1\right)H_0-4\left(4z^2-17z+9\right)-4(2z-1)(10z-7)H_1+\gamma_{qg}^0\left[4H_1^2+8H_0H_1\right]$$

$$-8(2z-1)H_{0,1}+8\left(8z^2-6z+3\right)\zeta_2\bigg]+(z+1)^2\left[-32H_0H_{-1}^2+\left[16H_0^2+64H_{0,-1}\right]H_{-1}\right.$$

$$\left.-32\zeta_2H_{-1}-64H_{0,-1,-1}\right]+\left(4z^2-2z+1\right)\left[16H_1\zeta_2-\frac{8}{3}H_0^3\right]+8\left(20z^2+2z+7\right)\zeta_3\bigg]\bigg\}$$

$$+a_s^3\left\{T_F^3\left[\frac{16}{9}\gamma_{qg}^0L_M^3-\frac{16}{9}\gamma_{qg}^0L_QL_M^2+\left[\gamma_{qg}^0\left[\frac{16H_0}{9}+\frac{16H_1}{9}\right]-\frac{64}{9}\left(8z^2-8z+1\right)\right]L_M^2-\frac{16\gamma_{qg}^0\zeta_3}{9}\right]\right.$$

$$+C_AT_F^2\left[-\frac{8}{9}(2z+1)H_0^4+\frac{16}{27}\left(23z^2-6z+3\right)H_0^3-\frac{8}{27}\left(822z^2+592z+229\right)H_0^2\right.$$

$$-\frac{16(z-1)\left(65z^2+17z+8\right)H_1H_0^2}{9z}+\frac{64}{3}(4z+1)H_{0,1}H_0^2-\frac{8}{3}(2z+9)\zeta_2H_0^2+\frac{64}{3}(z-1)zH_1^2H_0$$

$$+\frac{8}{81}\left(16303z^2-4390z+2000\right)H_0+\frac{16\left(1128z^3-1147z^2+146z-80\right)H_1H_0}{27z}$$

$$-\frac{32\left(23z^3+99z^2+15z+8\right)H_{0,1}H_0}{9z}-\frac{128}{3}(6z+1)H_{0,0,1}H_0+\frac{8}{9}\left(272z^2+360z+3\right)\zeta_2H_0$$

$$+\frac{64}{9}(41z+14)\zeta_3H_0-\frac{16}{9}(z-1)(5z+1)H_1^3+L_M^3\left[\frac{112(z-1)\left(31z^2+7z+4\right)}{27z}-\frac{56}{9}\gamma_{qg}^0H_1\right.$$

$$\left.-\frac{224}{9}(4z+1)H_0\right]+\frac{8}{27}\left(206z^2-143z+67\right)H_1^2-\frac{4\left(30335z^3-36798z^2+17367z-13244\right)}{243z}$$

$$-\frac{32}{15}(77z+2)\zeta_2^2-\frac{4}{9}\left(776z^3-1628z^2+59z-128\right)\frac{\zeta_2}{z}-\frac{16}{27}\left(1255z^3+150z^2+108z-28\right)\frac{\zeta_3}{z}$$

$$+\frac{8\left(6819z^3-6878z^2+106z-628\right)H_1}{81z}+L_Q^3\left[-\frac{16(z-1)\left(31z^2+7z+4\right)}{27z}\right.$$

$$\left.+\frac{32}{9}(4z+1)H_0+\frac{8}{9}\gamma_{qg}^0H_1\right]-\frac{32\left(672z^3-257z^2+76z-40\right)H_{0,1}}{27z}$$

$$+\frac{16\left(222z^3+244z^2+61z+16\right)H_{0,0,1}}{9z}-\frac{32}{3}z(5z-1)H_{0,1,1}+\frac{32}{3}(22z+1)H_{0,0,0,1}$$

$$-\frac{160}{9}\left(5z^2-5z+1\right)H_1\zeta_2+L_M^2\left[16H_0^2-\frac{16}{9}\left(127z^2+232z+25\right)H_0\right.$$

$$+\frac{8\left(2612z^3-2514z^2-33z-200\right)}{27z}+\frac{16\left(135z^3-128z^2+13z-4\right)H_1}{9z}+\gamma_{qg}^0\left[-\frac{20}{3}H_1^2\right.$$

$$-\frac{8}{3}H_0H_1\Big] + \big(2z^2+2z+1\big)\Big[32H_{-1}H_0 - 32H_{0,-1}\Big] - \frac{32}{3}(4z+1)H_{0,1} - \frac{64}{3}(z-6)z\zeta_2\Big]$$

$$+L_Q^2\Big[-\frac{16}{3}(8z+1)H_0^2 + \frac{16}{9}\big(49z^2-136z-13\big)H_0 + \frac{8\big(1048z^3-894z^2-87z-40\big)}{27z}$$

$$+\frac{16\big(87z^3-80z^2+13z-4\big)H_1}{9z} + \gamma_{qg}^0\Big[-\frac{4}{3}H_1^2 - \frac{8}{3}H_0H_1\Big]$$

$$+\big(2z^2+2z+1\big)\Big[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0\Big] - \frac{32}{3}(4z+1)H_{0,1} + L_M\Big[-\frac{16(z-1)\big(31z^2+7z+4\big)}{9z}$$

$$+\frac{32}{3}(4z+1)H_0 + \frac{8}{3}\gamma_{qg}^0 H_1\Big] - \frac{64}{3}(z-2)z\zeta_2\Big] + z(z+1)\Big[\frac{128}{3}H_0H_{-1}^2$$

$$+\Big[-\frac{64}{3}H_0^2 - 128H_0 - \frac{256}{3}H_{0,-1}\Big]H_{-1} + \frac{128}{3}\zeta_2 H_{-1} + \frac{128}{3}H_0H_{0,-1} + 128H_{0,-1}$$

$$+\frac{256}{3}H_{0,-1,-1} - \frac{128}{3}H_{0,0,-1}\Big] + L_Q\Big[\frac{448}{9}zH_0^3 - \frac{16}{9}\big(63z^2-296z-10\big)H_0^2$$

$$+\frac{32}{3}\big(5z^2+4z+2\big)H_{-1}H_0^2 - \frac{32}{3}\big(3z^2-4z+2\big)H_1H_0^2 - \frac{64}{27}\big(325z^2-832z-31\big)H_0$$

$$-\frac{64}{3}\big(3z^2+4z+2\big)H_{0,-1}H_0 + \frac{64}{3}\big(z^2+2z+2\big)H_{0,1}H_0 + \frac{128}{3}(z-5)z\zeta_2 H_0$$

$$+\Big[-\frac{16(z-1)\big(31z^2+7z+4\big)}{9z} + \frac{32}{3}(4z+1)H_0 + \frac{8}{3}\gamma_{qg}^0 H_1\Big]L_M^2$$

$$-\frac{8\big(25657z^3-23556z^2-969z-412\big)}{81z} + \frac{32}{9}\big(105z^3-184z^2+3z-4\big)\frac{\zeta_2}{z}$$

$$-\frac{32\big(1032z^3-964z^2+65z-20\big)H_1}{27z} + \frac{\big(87z^3-80z^2+13z-4\big)\Big[-\frac{16}{9}H_1^2 - \frac{32}{9}H_0H_1\Big]}{z}$$

$$+\frac{\big(17z^3+7z^2-4z-2\big)\Big[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1}\Big]}{z} - \frac{64\big(9z^3-59z^2-5z+2\big)H_{0,1}}{9z}$$

$$+\gamma_{qg}^0\Big[\frac{8}{9}H_1^3 + \frac{16}{3}H_0H_1^2 - \frac{32}{3}H_{0,1}H_1\Big] + (2z+1)\Big[\frac{32}{3}H_0H_{-1}^2 - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1}\Big]$$

$$+\frac{64}{3}\big(z^2+4z+2\big)H_{0,0,-1} - \frac{64}{3}\big(z^2-2z+2\big)H_{0,0,1} + \big(2z^2+2z+1\big)\Big[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1}$$

$$-\frac{64}{3}H_{0,1,-1}\Big] - \frac{128}{3}\big(3z^2-5z+1\big)H_{0,1,1} - \frac{32}{3}\big(4z^2+2z+1\big)H_{-1}\zeta_2 - \frac{32}{3}(2z-1)H_1\zeta_2$$

$$+L_M\Big[-\frac{32}{3}(4z+3)H_0^2 + \frac{32}{9}\big(101z^2-8z+13\big)H_0 - \frac{32\big(8z^3-14z^2+11z-8\big)}{3z}$$

$$+\frac{32\big(47z^3-40z^2-7z-4\big)H_1}{9z} + \gamma_{qg}^0\Big[-\frac{8}{3}H_1^2 - \frac{16}{3}H_0H_1\Big] + \big(2z^2+2z+1\big)\Big[\frac{64}{3}H_{0,-1}$$

$$-\frac{64}{3}H_{-1}H_0\Big] - \frac{64}{3}(4z+1)H_{0,1} - \frac{128}{3}(z-2)z\zeta_2\Big] + \frac{32}{3}\big(6z^2-22z+3\big)\zeta_3\Big]$$

$$+L_M\Big[-\frac{64}{9}(z-2)H_0^3 - \frac{32}{9}\big(54z^2+10z+5\big)H_0^2 + \frac{32}{3}\big(3z^2+2z+1\big)H_{-1}H_0^2$$

$$-\frac{32}{3}(z-1)^2 H_1 H_0^2 - \frac{16}{27}\big(1680z^2+1744z-269\big)H_0 - \frac{128\big(28z^3-22z^2-4z-3\big)H_1H_0}{9z}$$

$$-\frac{64}{3}(z+1)^2 H_{0,-1}H_0 - \frac{64}{3}\big(z^2-12z-3\big)H_{0,1}H_0 + \frac{128}{3}\big(z^2-3z-1\big)\zeta_2 H_0$$

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$$-\frac{64\left(5z^3-4z^2-z-1\right)H_1^2}{9z}+\frac{8\left(20863z^3-20616z^2+159z-172\right)}{81z}$$

$$+\frac{32}{9}\left(114z^3-20z^2+9z-4\right)\frac{\zeta_2}{z}+\frac{16\left(476z^3-548z^2+253z-144\right)H_1}{27z}$$

$$+\gamma_{qg}^0\left[\frac{16}{3}H_0H_1^2-\frac{8}{9}H_1^3\right]+\frac{\left(9z^3-z^2-14z-2\right)\left[\frac{64}{9}H_{-1}H_0-\frac{64}{9}H_{0,-1}\right]}{z}$$

$$-\frac{32\left(2z^3+70z^2+25z+12\right)H_{0,1}}{9z}+\left(4z^2+2z+1\right)\left[-\frac{32}{3}H_0H_{-1}^2+\frac{64}{3}H_{0,-1}H_{-1}\right.$$

$$-\frac{64}{3}H_{0,-1,-1}-\frac{64}{3}\left(z^2-2z-1\right)H_{0,0,-1}+\frac{64}{3}\left(z^2-20z-3\right)H_{0,0,1}+\left(2z^2+2z+1\right)\left[\frac{64}{3}H_{-1}H_{0,1}\right.$$

$$\left.-\frac{64}{3}H_{0,-1,1}-\frac{64}{3}H_{0,1,-1}\right]+\frac{128}{3}\left(z^2+z+1\right)H_{0,1,1}-\frac{32}{3}\left(8z^2+6z+3\right)H_{-1}\zeta_2$$

$$\left.+\frac{32}{3}\left(4z^2-6z+3\right)H_1\zeta_2+\frac{32}{3}\left(6z^2+26z+7\right)\zeta_3\right]+\left(2z^2+2z+1\right)\left[-\frac{128}{9}H_0H_{-1}^3\right.$$

$$+\left[\frac{32}{3}H_0^2+\frac{128}{3}H_{0,-1}\right]H_{-1}^2+\left[\frac{32}{9}H_0^3+\left[\frac{128}{3}H_{0,1}-\frac{128}{3}H_{0,-1}\right]H_0-\frac{256}{3}H_{0,-1,-1}+\frac{128}{3}H_{0,0,-1}\right.$$

$$\left.-\frac{256}{3}H_{0,0,1}\right]H_{-1}+64\zeta_3H_{-1}-\frac{32}{3}H_0^2H_{0,-1}+H_0\left[\frac{128}{3}H_{0,-1,-1}-\frac{128}{3}H_{0,-1,1}+\frac{64}{3}H_{0,0,-1}\right.$$

$$\left.-\frac{128}{3}H_{0,1,-1}\right]+\frac{256}{3}H_{0,-1,-1,-1}+\frac{128}{3}H_{0,-1,0,1}-\frac{128}{3}H_{0,0,-1,-1}+\frac{256}{3}H_{0,0,-1,1}-\frac{64}{3}H_{0,0,0,-1}$$

$$+\frac{256}{3}H_{0,0,1,-1}+\left[-\frac{64}{3}H_{-1}^2-\frac{80}{3}H_0H_{-1}+\frac{80}{3}H_{0,-1}\right]\zeta_2\right]+\gamma_{qg}^0\left[\frac{2}{9}H_1^4+\frac{4}{3}H_0^2H_1^2+\frac{10}{3}\zeta_2H_1^2\right.$$

$$+\left[\frac{80}{3}H_{0,0,1}+\frac{16}{3}H_{0,1,1}\right]H_1-\frac{40}{9}\zeta_3H_1-\frac{16}{3}H_{0,1}^2+H_0\left[-\frac{16}{9}H_1^3-16H_{0,1}H_1+\frac{80}{3}H_{0,1,1}\right]$$

$$\left.\left.-\frac{80}{3}H_{0,0,1,1}-\frac{16}{3}H_{0,1,1,1}\right]\right]+C_AN_FT_F^2\left[-\frac{4}{9}(2z+1)H_0^4+\frac{8}{27}\left(23z^2+12z+3\right)H_0^3\right.$$

$$-\frac{8}{27}\left(292z^2+39z+42\right)H_0^2-\frac{8(z-1)\left(65z^2+17z+8\right)H_1H_0^2}{9z}+\frac{32}{3}(4z+1)H_{0,1}H_0^2$$

$$+\frac{32}{3}(z-1)zH_1^2H_0+\frac{16}{81}\left(3392z^2+645z+111\right)H_0+\frac{32(z-1)\left(254z^2-7z+20\right)H_1H_0}{27z}$$

$$-\frac{16\left(23z^3+96z^2+15z+8\right)H_{0,1}H_0}{9z}+\frac{8}{9}\left(62z^2-16z-7\right)\zeta_2H_0+\frac{32}{9}(28z+13)\zeta_3H_0$$

$$-\frac{8}{9}(z-1)(5z+1)H_1^3+\left[\frac{16(z-1)\left(31z^2+7z+4\right)}{27z}-\frac{32}{9}(4z+1)H_0-\frac{8}{9}\gamma_{qg}^0H_1\right]L_M^3$$

$$-\frac{8}{3}\left(4z^2-6z+1\right)H_1^2-\frac{32\left(5854z^3-6219z^2+531z-328\right)}{243z}+\frac{4}{9}\left(19z^3-50z^2+2z-4\right)\frac{\zeta_2}{z}$$

$$-\frac{16}{27}\left(550z^3+228z^2+33z-4\right)\frac{\zeta_3}{z}+\frac{32}{3}\left(2z^2-2z-1\right)H_1+L_Q^3\left[-\frac{16(z-1)\left(31z^2+7z+4\right)}{27z}\right.$$

$$\left.+\frac{32}{9}(4z+1)H_0+\frac{8}{9}\gamma_{qg}^0H_1\right]-\frac{32\left(290z^3-261z^2+27z-20\right)H_{0,1}}{27z}$$

$$+\frac{16\left(111z^3+144z^2+21z+8\right)H_{0,0,1}}{9z}-\frac{16}{3}z(5z-4)H_{0,1,1}-\frac{16}{9}\left(2z^2-2z-5\right)H_1\zeta_2$$

$$+L_Q^2\left[-\frac{16}{3}(8z+1)H_0^2+\frac{16}{9}\left(49z^2-136z-13\right)H_0+\frac{8\left(1048z^3-894z^2-87z-40\right)}{27z}\right.$$

$$+\frac{16\left(87z^3-80z^2+13z-4\right)H_1}{9z}+\gamma_{qg}^0\left[-\frac{4}{3}H_1^2-\frac{8}{3}H_0H_1\right]+\left(2z^2+2z+1\right)\left[\frac{32}{3}H_{0,-1}\right.$$

$$-\frac{32}{3}H_{-1}H_0\Big] - \frac{32}{3}(4z+1)H_{0,1} - \frac{64}{3}(z-2)z\zeta_2\Big] + z(z+1)\Big[\frac{64}{3}H_0H_{-1}^2 + \Big[-\frac{32}{3}H_0^2 - 64H_0$$

$$-\frac{128}{3}H_{0,-1}\Big]H_{-1} + \frac{64}{3}\zeta_2 H_{-1} + \frac{64}{3}H_0 H_{0,-1} + 64H_{0,-1} + \frac{128}{3}H_{0,-1,-1} - \frac{64}{3}H_{0,0,-1}\Big]$$

$$+(6z+1)\Big[-\frac{8}{3}\zeta_2 H_0^2 - \frac{64}{3}H_{0,0,1}H_0\Big] + z\Big[128H_{0,0,0,1} - \frac{1216\zeta_2^2}{15}\Big] + L_M^2\Big[-\frac{4}{3}\gamma_{qg}^0 H_1^2$$

$$-\frac{32}{9}(4z^2-4z+5)H_1 - \frac{8(205z^3-168z^2+42z-52)}{27z} - \frac{32}{9}(9z^2-20z-5)H_0$$

$$+(2z^2+2z+1)\Big[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\Big] + z\Big[\frac{64}{3}H_0^2 + \frac{64\zeta_2}{3}\Big]\Big] + L_M\Big[-\frac{16}{9}(10z-1)H_0^3$$

$$+\frac{8}{9}(7z^2-66z-13)H_0^2 + \frac{16}{27}(58z^2-269z-2)H_0 - \frac{16(z-1)(65z^2+17z+8)H_1 H_0}{9z}$$

$$-\frac{32}{3}(2z^2-10z-1)H_{0,1}H_0 - \frac{8}{9}(13z^2-16z+23)H_1^2 + \frac{8(592z^3-268z^2-119z-4)}{27z}$$

$$+\frac{16}{27}(64z^2-64z+29)H_1 + (4z^2+4z+5)\Big[\frac{64}{9}H_{0,-1} - \frac{64}{9}H_{-1}H_0\Big]$$

$$+\frac{16(68z^3-54z^2-9z-8)H_{0,1}}{9z} + \frac{32}{3}(2z^2-18z-3)H_{0,0,1} - \frac{16}{9}z(3z+10)\zeta_2$$

$$+(2z^2+2z+1)\Big[-\frac{32}{3}H_0 H_{-1}^2 + \big(\frac{64}{3}H_{0,-1} - \frac{16}{3}H_0^2\big)H_{-1} - \frac{32}{3}\zeta_2 H_{-1} + \frac{32}{3}H_0 H_{0,-1} - \frac{64}{3}H_{0,-1,-1}$$

$$-\frac{32}{3}H_{0,0,-1}\Big] + \gamma_{qg}^0\Big[-\frac{8}{9}H_1^3 - \frac{4}{3}H_0^2 H_1 + \frac{16}{3}H_{0,1}H_1 - \frac{8}{3}\zeta_2 H_1 - \frac{32}{3}H_{0,1,1}\Big] + \frac{128}{3}(5z+1)\zeta_3\Big]$$

$$+L_Q\Big[\frac{16}{9}(26z+1)H_0^3 - \frac{8}{9}(126z^2-542z-39)H_0^2 + \frac{32}{3}(5z^2+4z+2)H_{-1}H_0^2$$

$$-\frac{32}{3}(3z^2-4z+2)H_1 H_0^2 - \frac{16}{27}(1520z^2-2822z-89)H_0 - \frac{64}{3}(3z^2+4z+2)H_{0,-1}H_0$$

$$+\frac{64}{3}(z^2+2z+2)H_{0,1}H_0 + \frac{128}{3}(z-5)z\zeta_2 H_0 - \frac{8(2306z^3-2056z^2-181z+16)}{9z}$$

$$+\frac{32}{9}(105z^3-181z^2+3z-4)\frac{\zeta_2}{z} - \frac{16(1934z^3-1807z^2+83z-40)H_1}{27z}$$

$$+\frac{(87z^3-80z^2+13z-4)\big[-\frac{16}{9}H_1^2 - \frac{32}{9}H_0 H_1\big]}{z} + \frac{(17z^3+7z^2-4z-2)\big[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1}\big]}{z}$$

$$-\frac{32(18z^3-115z^2-10z+4)H_{0,1}}{9z} + \gamma_{qg}^0\Big[\frac{8}{9}H_1^3 + \frac{16}{3}H_0 H_1^2 - \frac{32}{3}H_{0,1}H_1\Big] + (2z+1)\Big[\frac{32}{3}H_0 H_{-1}^2$$

$$-\frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1}\Big] + \frac{64}{3}(z^2+4z+2)H_{0,0,-1} - \frac{64}{3}(z^2-2z+2)H_{0,0,1}$$

$$+(2z^2+2z+1)\Big[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1}\Big] - \frac{128}{3}(3z^2-5z+1)H_{0,1,1}$$

$$-\frac{32}{3}(4z^2+2z+1)H_{-1}\zeta_2 - \frac{32}{3}(2z-1)H_1\zeta_2 + \frac{32}{3}(6z^2-22z+3)\zeta_3\Big]$$

$$+(2z^2+2z+1)\Big[-\frac{64}{9}H_0 H_{-1}^3 + \Big[\frac{16}{3}H_0^2 + \frac{64}{3}H_{0,-1}\Big]H_{-1}^2 + \Big[\frac{16}{9}H_0^3 + \Big[\frac{64}{3}H_{0,1} - \frac{64}{3}H_{0,-1}\Big]H_0$$

$$-\frac{128}{3}H_{0,-1,-1} + \frac{64}{3}H_{0,0,-1} - \frac{128}{3}H_{0,0,1}\Big]H_{-1} + 32\zeta_3 H_{-1} - \frac{16}{3}H_0^2 H_{0,-1} + H_0\Big[\frac{64}{3}H_{0,-1,-1}$$

$$-\frac{64}{3}H_{0,-1,1} + \frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,1,-1}\Big] + \frac{128}{3}H_{0,-1,-1,-1} + \frac{64}{3}H_{0,-1,0,1} - \frac{64}{3}H_{0,0,-1,-1}$$

$$+\frac{128}{3}H_{0,0,-1,1}-\frac{32}{3}H_{0,0,0,-1}+\frac{128}{3}H_{0,0,1,-1}+\left[-\frac{32}{3}H_{-1}^2-\frac{32}{3}H_0H_{-1}+\frac{32}{3}H_{0,-1}\right]\zeta_2\right]+\gamma_{qg}^0\left[\frac{1}{9}H_1^4\right.$$

$$+\frac{2}{3}H_0^2H_1^2+\frac{4}{3}\zeta_2H_1^2+\left[\frac{40}{3}H_{0,0,1}+\frac{8}{3}H_{0,1,1}\right]H_1-\frac{40}{9}\zeta_3H_1-\frac{8}{3}H_{0,1}^2+H_0\left[-\frac{8}{9}H_1^3-8H_{0,1}H_1\right.$$

$$\left.+\frac{40}{3}H_{0,1,1}\right]-\frac{40}{3}H_{0,0,1,1}-\frac{8}{3}H_{0,1,1,1}\right]\right]+C_A^2T_F\left[\frac{1}{9}\left(18z^2-26z+23\right)H_0^4+\frac{2}{9}\left(225z^2-12z-35\right)H_0^3\right.$$

$$-\frac{4\left(60z^3+46z^2+5z+8\right)H_{-1}H_0^3}{9z}+\frac{8(z-1)\left(65z^2+17z+8\right)H_1H_0^3}{9z}-\frac{32}{3}(4z+1)H_{0,1}H_0^3$$

$$+\frac{8}{3}(8z-5)\zeta_2H_0^3-\frac{4\left(120z^3+106z^2+5z+8\right)H_{-1}^2H_0^2}{3z}+\frac{2\left(190z^3-142z^2-13z-24\right)H_1^2H_0^2}{3z}$$

$$+\frac{2}{27}\left(5248z^2-6573z+738\right)H_0^2+\frac{8(z+1)\left(359z^2-32z+20\right)H_{-1}H_0^2}{9z}$$

$$+\frac{2(z-1)\left(1619z^2-169z+308\right)H_1H_0^2}{9z}+\frac{4\left(96z^3+70z^2+5z+8\right)H_{0,-1}H_0^2}{3z}$$

$$+\frac{8}{3}\left(32z^2-62z+7\right)H_{0,1}H_0^2-16\left(2z^2-6z+3\right)H_{0,0,-1}H_0^2-64(z-1)H_{0,0,1}H_0^2$$

$$+96(z-3)zH_{0,1,1}H_0^2+\frac{2}{3}\left(38z^2-6z+47\right)\zeta_2H_0^2-\frac{32}{3}(31z-5)\zeta_3H_0^2$$

$$-\frac{8\left(93z^3-82z^2+8z-8\right)H_1^3H_0}{9z}-\frac{8\left(241z^3-229z^2+11z-20\right)H_1^2H_0}{3z}+32\left(z^2+9z+3\right)H_{0,1}^2H_0$$

$$+\frac{16}{5}(124z-19)\zeta_2^2H_0-\frac{4\left(225962z^3+100242z^2+16377z+2624\right)H_0}{81z}$$

$$-\frac{2}{9}\left(6474z^3+1864z^2+1165z+160\right)\frac{\zeta_2}{z}H_0+\frac{8}{9}\left(210z^3+340z^2-269z-8\right)\frac{\zeta_3}{z}H_0$$

$$+\frac{8}{3}\left(67z^3+46z^2+2z+12\right)\frac{\zeta_2}{z}H_{-1}H_0-\frac{8\left(16040z^3-16335z^2+1575z-1388\right)H_1H_0}{27z}$$

$$-\frac{8}{3}(z-1)\left(137z^2+41z+20\right)\frac{\zeta_2}{z}H_1H_0-\frac{16\left(323z^3+111z^2-12z+20\right)H_{0,-1}H_0}{9z}$$

$$+\frac{4\left(1501z^3+5892z^2+93z+468\right)H_{0,1}H_0}{9z}-\frac{8\left(406z^3-330z^2-3z-40\right)H_1H_{0,1}H_0}{3z}$$

$$-\frac{16\left(108z^3+82z^2+5z+8\right)H_{0,-1,-1}H_0}{3z}-\frac{8\left(132z^3+94z^2+5z+8\right)H_{0,0,-1}H_0}{3z}$$

$$-\frac{16\left(129z^3-210z^2+28z-8\right)H_{0,0,1}H_0}{3z}+\frac{8\left(768z^3-398z^2+19z-56\right)H_{0,1,1}H_0}{3z}$$

$$-128\left(4z^2+3\right)H_{0,-1,-1,-1}H_0+128(7z-1)H_{0,0,0,1}H_0-32\left(6z^2+14z+9\right)H_{0,0,1,1}H_0$$

$$-32\left(6z^2-14z+1\right)H_{0,1,1,1}H_0+8\left(2z^2+38z+11\right)H_{0,1}\zeta_2H_0+\frac{\left(204z^3-166z^2-19z-8\right)H_1^4}{9z}$$

$$+\frac{2\left(1321z^3-1440z^2+15z-4\right)H_1^3}{27z}+\frac{8}{3}\left(132z^3+118z^2+5z+8\right)\frac{\zeta_2}{z}H_{-1}^2$$

$$+\frac{2\left(3536z^3-3108z^2+39z-134\right)H_1^2}{27z}+\frac{4}{3}\left(102z^3-94z^2+11z-8\right)\frac{\zeta_2}{z}H_1^2+32z^2H_{0,-1}^2$$

$$-\frac{8\left(161z^3-130z^2-4z-16\right)H_{0,1}^2}{3z}+\frac{4\left(1158802z^3-1178838z^2+83079z-66607\right)}{243z}$$

$$+\frac{2}{27}\left(57548z^3-10209z^2+4590z-2176\right)\frac{\zeta_2}{z}+\frac{16}{15}\left(86z^3+1673z^2-90z+76\right)\frac{\zeta_2^2}{z}$$

$$+\frac{4}{27}\left(16364z^3+25128z^2+3537z+448\right)\frac{\zeta_3}{z}-\frac{16}{9}(z+1)\left(413z^2-32z+20\right)\frac{\zeta_2}{z}H_{-1}$$

$$-8\left(112z^3+98z^2+5z+8\right)\frac{\zeta_3}{z}H_{-1}+\frac{4\left(18046z^3-18101z^2+1286z-1465\right)H_1}{27z}$$

$$+\frac{4}{9}\big(3111z^3 - 3052z^2 + 116z - 242\big)\frac{\zeta_2}{z}H_1 - \frac{8}{9}\big(874z^3 - 698z^2 - 41z - 80\big)\frac{\zeta_3}{z}H_1$$

$$+\frac{(z+1)\big(2443z^2 - 244z + 94\big)\Big[\frac{16}{27}H_{-1}H_0 - \frac{16}{27}H_{0,-1}\Big]}{z} - \frac{8}{3}\big(103z^3 + 70z^2 + 2z + 12\big)\frac{\zeta_2}{z}H_{0,-1}$$

$$+\frac{8\big(9594z^3 - 17343z^2 + 1125z - 1388\big)H_{0,1}}{27z} - \frac{8}{3}\big(124z^3 + 258z^2 + 33z + 16\big)\frac{\zeta_2}{z}H_{0,1}$$

$$+\frac{(z+1)\big(305z^2 - 32z + 20\big)\Big[-\frac{16}{9}H_0H_{-1}^2 + \frac{32}{9}H_{0,-1}H_{-1} - \frac{32}{9}H_{0,-1,-1}\Big]}{z}$$

$$+\frac{16\big(287z^3 - 105z^2 - 12z + 20\big)H_{0,0,-1}}{9z} - \frac{4\big(3525z^3 + 9840z^2 + 495z + 628\big)H_{0,0,1}}{9z}$$

$$+\frac{8\big(728z^3 - 590z^2 - 11z - 72\big)H_1H_{0,0,1}}{3z} - 192\big(z^2 + 5z + 2\big)H_{0,1}H_{0,0,1}$$

$$+\frac{\big(84z^3 + 70z^2 + 5z + 8\big)\Big[H_0\Big[\frac{16}{3}H_{0,-1,1} + \frac{16}{3}H_{0,1,-1}\Big] - \frac{16}{3}H_{-1}H_0H_{0,1}\Big]}{z}$$

$$+\frac{4\big(3425z^3 - 3312z^2 + 48z - 240\big)H_{0,1,1}}{9z} + \frac{8\big(84z^3 - 70z^2 + 5z - 8\big)H_1H_{0,1,1}}{3z}$$

$$+\frac{\big(108z^3 + 94z^2 + 5z + 8\big)\Big[\frac{16}{9}H_0H_{-1}^3 - \frac{16}{3}H_{0,-1}H_{-1}^2 + \frac{32}{3}H_{0,-1,-1}H_{-1} - \frac{32}{3}H_{0,-1,-1,-1}\Big]}{z}$$

$$+\frac{\big(96z^3 + 82z^2 + 5z + 8\big)\Big[\frac{16}{3}H_{-1}H_0H_{0,-1} - \frac{16}{3}H_{0,-1,0,1}\Big]}{z} + (4z - 1)\Big[\frac{4}{15}H_0^5 - 64H_0H_{0,-1,0,1}\Big]$$

$$+\frac{\big(72z^3 + 58z^2 + 5z + 8\big)\Big[\frac{16}{3}H_{0,0,-1,-1} - \frac{16}{3}H_{-1}H_{0,0,-1}\Big]}{z} + \frac{8\big(168z^3 + 118z^2 + 5z + 8\big)H_{0,0,0,-1}}{3z}$$

$$+\frac{16\big(203z^3 - 408z^2 + 69z - 16\big)H_{0,0,0,1}}{3z} + \frac{\big(90z^3 + 76z^2 + 5z + 8\big)}{z}\Big[\frac{32}{3}H_{-1}H_{0,0,1} - \frac{32}{3}H_{0,0,-1,1}$$

$$-\frac{32}{3}H_{0,0,1,-1}\Big] - \frac{8\big(932z^3 - 266z^2 + 31z - 56\big)H_{0,0,1,1}}{3z} + z(z+1)\Big[-64H_{0,1}H_{-1}^2$$

$$+\Big[192H_{0,1} + 128H_{0,-1,1} + 128H_{0,1,-1}\Big]H_{-1} - 192H_{0,-1,1} - 192H_{0,1,-1} - 128H_{0,-1,-1,1}$$

$$-128H_{0,-1,1,-1} - 128H_{0,1,-1,-1}\Big] - \frac{8\big(30z^3 - 10z^2 - 7z - 8\big)H_{0,1,1,1}}{3z} + \big(z^2 - z + 1\big)\Big[\frac{32}{3}H_{0,-1}H_0^3$$

$$-64H_{0,-1}^2H_0 + H_{0,-1}\Big[256H_{0,-1,-1} + 128H_{0,0,-1}\Big] - 512H_{0,-1,0,-1,-1} - 1024H_{0,0,-1,-1,-1}$$

$$-128H_{0,0,-1,0,-1}\Big] - 32\big(2z^2 - 22z + 7\big)H_{0,0,0,-1,-1} - 32(54z - 1)H_{0,0,0,0,1}$$

$$+\big(z^2 - 5z + 2\big)\Big[64H_0H_{0,0,0,-1} + 256H_{0,0,0,-1,1} + 256H_{0,0,0,1,-1}\Big] + 64\big(13z^2 + 83z + 32\big)H_{0,0,0,1,1}$$

$$+\big(2z^2 - 14z + 5\big)\Big[-32H_{0,-1}H_{0,0,1} + 32H_{0,0,-1,0,1} - 32H_{0,0,0,0,-1} + 32H_{0,0,1,0,-1}\Big]$$

$$+64\big(5z^2 + 29z + 11\big)H_{0,0,1,0,1} - 128\big(12z^2 - 8z + 7\big)H_{0,0,1,1,1} - 64\big(9z^2 - 7z + 5\big)H_{0,1,0,1,1}$$

$$-64\big(z^2 + z + 1\big)H_{0,1,1,1,1} + \big(6z^2 - 10z + 7\big)\big(24H_{0,0,-1} - 16H_{0,-1,-1}\big)\zeta_2$$

$$-8\big(2z^2 + 46z + 7\big)H_{0,0,1}\zeta_2 - 16\big(30z^2 - 22z + 17\big)H_{0,1,1}\zeta_2 + L_M^3\Big[\frac{16}{3}(8z - 1)H_0^2$$

$$-\frac{8\big(18z^3 - 152z^2 - 11z - 8\big)H_0}{9z} - \frac{4(z - 1)\big(1883z^2 - 97z + 272\big)}{27z} - \frac{8\big(146z^3 - 118z^2 - z - 16\big)H_1}{9z}$$

$$+\gamma_{qg}^0\Big[\frac{8}{3}H_1^2 + \frac{8}{3}H_0H_1\Big] + \frac{32}{3}\big(2z^2 + 6z + 3\big)H_{0,1} - \frac{64}{3}(4z + 1)\zeta_2\Big] + L_Q^3\Big[-\frac{16}{3}(8z - 1)H_0^2$$

$$+\frac{8\left(18z^3-152z^2-11z-8\right)H_0}{9z}+\frac{4(z-1)\left(1883z^2-97z+272\right)}{27z}+\frac{8\left(146z^3-118z^2-z-16\right)H_1}{9z}$$

$$+\gamma_{qg}^0\left[-\frac{8}{3}H_1^2-\frac{8}{3}H_0H_1\right]-\frac{32}{3}\left(2z^2+6z+3\right)H_{0,1}+\frac{64}{3}(4z+1)\zeta_2\right]+\left(6z^2-2z+5\right)\left[16H_0^2H_{0,-1,-1}\right.$$

$$\left.-24H_0H_{0,-1}\zeta_2\right]+32\left(4z^2-8z+5\right)H_{0,-1}\zeta_3+\frac{160}{3}\left(2z^2+6z+3\right)H_{0,1}\zeta_3$$

$$+\frac{8}{3}(278z+83)\zeta_2\zeta_3+L_Q^2\left[\frac{16}{3}(16z-3)H_0^3-\frac{4}{3}\left(54z^2-436z-23\right)H_0^2\right.$$

$$-\frac{4\left(5183z^3-3512z^2+1009z+104\right)H_0}{9z}-\frac{8\left(280z^3-238z^2+5z-24\right)H_1H_0}{3z}$$

$$-16\left(6z^2-10z+7\right)H_{0,-1}H_0+16\left(2z^2+22z+7\right)H_{0,1}H_0-32(14z+3)\zeta_2H_0$$

$$-\frac{4\left(352z^3-310z^2+17z-24\right)H_1^2}{3z}-\frac{2\left(6639z^3-7318z^2-509z+1260\right)}{9z}$$

$$-\frac{4\left(4549z^3-4204z^2+59z-168\right)H_1}{9z}+\frac{\left(146z^3+118z^2-z+16\right)\left[\frac{8}{3}H_{-1}H_0-\frac{8}{3}H_{0,-1}\right]}{z}$$

$$+\frac{8\left(28z^3+272z^2+23z-8\right)H_{0,1}}{3z}+16\left(6z^2-26z+11\right)H_{0,0,-1}+16\left(2z^2-6z-5\right)H_{0,0,1}$$

$$+64\left(z^2+5z+2\right)H_{0,1,1}+\frac{16}{3}\left(126z^2-196z-9\right)\zeta_2+\left(2z^2+2z+1\right)\left[-32H_0H_{-1}^2\right.$$

$$+\left[24H_0^2+64H_{0,-1}+32H_{0,1}\right]H_{-1}-64\zeta_2H_{-1}-64H_{0,-1,-1}-32H_{0,-1,1}$$

$$\left.-32H_{0,1,-1}\right]+\gamma_{qg}^0\left[4H_1^3+12H_0H_1^2+6H_0^2H_1-16\zeta_2H_1\right]+32(z-3)\zeta_3\right]$$

$$+L_M^2\left[-\frac{32}{3}(4z-1)H_0^3+\frac{16}{3}\left(9z^2+z-3\right)H_0^2+\frac{8\left(1335z^3+710z^2+266z+40\right)H_0}{9z}\right.$$

$$+\frac{16(z-1)\left(19z^2+7z+4\right)H_1H_0}{3z}+16\left(6z^2-10z+7\right)H_{0,-1}H_0-16\left(2z^2+6z+3\right)H_{0,1}H_0$$

$$-64(z-1)\zeta_2H_0-\frac{4\left(132z^3-118z^2+5z-8\right)H_1^2}{3z}-\frac{8\left(2695z^3-2661z^2+192z-244\right)}{9z}$$

$$-\frac{16}{3}\left(44z^3+83z^2-3z+4\right)\frac{\zeta_2}{z}-\frac{8\left(617z^3-604z^2-70z-28\right)H_1}{9z}$$

$$+\frac{\left(146z^3+118z^2-z+16\right)\left[\frac{8}{3}H_{0,-1}-\frac{8}{3}H_{-1}H_0\right]}{z}+\frac{16}{3}z(25z+36)H_{0,1}$$

$$-16\left(6z^2-26z+11\right)H_{0,0,-1}+32(4z+1)H_{0,1,1}+\left(2z^2+2z+1\right)\left[32H_0H_{-1}^2+\left[-24H_0^2\right.\right.$$

$$\left.-64H_{0,-1}-32H_{0,1}\right]H_{-1}+64\zeta_2H_{-1}+64H_{0,-1,-1}+32H_{0,-1,1}+16H_{0,0,1}+32H_{0,1,-1}\right]$$

$$+\gamma_{qg}^0\left[4H_1^3-2H_0^2H_1-8\zeta_2H_1\right]-32(17z-2)\zeta_3\right]+\left(2z^2+2z+1\right)\left[-16H_0H_{-1}^4+\left[\frac{64}{3}H_0^2\right.\right.$$

$$\left.+64H_{0,-1}+\frac{64}{3}H_{0,1}\right]H_{-1}^3+\left[-\frac{8}{3}H_0^3+\left[32H_{0,1}-64H_{0,-1}\right]H_0-192H_{0,-1,-1}-64H_{0,-1,1}\right.$$

$$\left.-96H_{0,0,1}-64H_{0,1,-1}\right]H_{-1}^2-\frac{376}{5}\zeta_2^2H_{-1}+\left[-\frac{4}{3}H_0^4+\left[-16H_{0,-1}-32H_{0,1}\right]H_0^2\right.$$

$$+\left[128H_{0,-1,-1}-64H_{0,-1,1}+96H_{0,0,-1}+64H_{0,0,1}-64H_{0,1,-1}-64H_{0,1,1}\right]H_0$$

$$+384H_{0,-1,-1,-1}+128H_{0,-1,-1,1}+128H_{0,-1,0,1}+128H_{0,-1,1,-1}+192H_{0,0,-1,1}-160H_{0,0,0,-1}$$

$$\left.-32H_{0,0,0,1}+192H_{0,0,1,-1}+128H_{0,0,1,1}+128H_{0,1,-1,-1}\right]H_{-1}+H_0^2\left[32H_{0,-1,1}+32H_{0,1,-1}\right]$$

$$+H_0\Big[64H_{0,-1,-1,1}+64H_{0,-1,1,-1}+64H_{0,-1,1,1}-96H_{0,0,-1,-1}-64H_{0,0,-1,1}-64H_{0,0,1,-1}$$

$$+64H_{0,1,-1,-1}+64H_{0,1,-1,1}+64H_{0,1,1,-1}\Big]-384H_{0,-1,-1,-1,-1}$$

$$-128H_{0,-1,-1,-1,1}-128H_{0,-1,-1,0,1}-128H_{0,-1,-1,1,-1}-128H_{0,-1,0,-1,1}-128H_{0,-1,0,1,-1}$$

$$-64H_{0,-1,0,1,1}-128H_{0,-1,1,-1,-1}-192H_{0,0,-1,-1,1}-192H_{0,0,-1,1,-1}$$

$$-128H_{0,0,-1,1,1}-192H_{0,0,1,-1,-1}-128H_{0,0,1,-1,1}-128H_{0,0,1,1,-1}-128H_{0,1,-1,-1,-1}$$

$$+\Big[-\frac{160}{3}H_{-1}^3+40H_0H_{-1}^2+\Big[36H_0^2-16H_{0,-1}+96H_{0,1}\Big]H_{-1}-96H_{0,-1,1}-96H_{0,1,-1}\Big]\zeta_2$$

$$+\Big[160H_{-1}^2-64H_{-1}H_0\Big]\zeta_3+\gamma_{qg}^0\Big[-\frac{1}{3}H_1^5-\frac{2}{3}H_0^3H_1^2+\Big[-44H_{0,0,1}-4H_{0,1,1}\Big]H_1^2$$

$$+46\zeta_2^2H_1+8H_0^2H_{0,1}H_1+\Big[20H_{0,1}^2-16H_{0,0,0,1}+24H_{0,0,1,1}\Big]H_1+H_0\Big[\frac{5}{3}H_1^4+24H_{0,1}H_1^2$$

$$+\Big[-8H_{0,0,1}-56H_{0,1,1}\Big]H_1\Big]-40H_{0,1}H_{0,1,1}+\Big[-\frac{8}{3}H_1^3+2H_0H_1^2+H_0^2H_1-32H_{0,1}H_1\Big]\zeta_2$$

$$+\Big[\frac{52}{3}H_1^2+\frac{40}{3}H_0H_1\Big]\zeta_3\Big]+L_M\Big[\frac{4}{3}(19z-5)H_0^4+\frac{4\big(136z^3+160z^2+59z+24\big)H_{-1}H_0^2}{3z}$$

$$-\frac{4}{9}\big(54z^2+40z-37\big)H_0^3-\frac{2}{9}\big(4303z^2+216z+1303\big)H_0^2-\frac{4\big(18z^3+59z^2-58z-8\big)H_1H_0^2}{3z}$$

$$+16\big(z^2+8z-1\big)H_{0,-1}H_0^2+4\big(36z^2-2z+23\big)H_{0,1}H_0^2+\frac{40(z-1)\big(31z^2+7z+4\big)H_1^2H_0}{3z}$$

$$+4\big(24z^2-22z-1\big)\zeta_2H_0^2+\frac{8\big(30972z^3+13409z^2+3023z+448\big)H_0}{27z}$$

$$+\frac{4(z-1)\big(1583z^2-5z+192\big)H_1H_0}{3z}+\frac{8\big(128z^3-346z^2-5z-24\big)H_{0,-1}H_0}{3z}$$

$$+\frac{8\big(40z^3-329z^2-86z-24\big)H_{0,1}H_0}{3z}-64\big(5z^2-4z+5\big)H_{0,-1,-1}H_0$$

$$-32\big(3z^2+16z+3\big)H_{0,0,-1}H_0-32(z+1)(13z+7)H_{0,0,1}H_0$$

$$-32\big(12z^2+8z+11\big)H_{0,1,1}H_0+8\big(30z^2-31z+13\big)\zeta_2H_0-96(22z+3)\zeta_3H_0$$

$$-\frac{8}{9}\big(103z^2-94z+2\big)H_1^3-\frac{2\big(2023z^3-2032z^2-247z-84\big)H_1^2}{9z}$$

$$+32\big(3z^2-6z+4\big)H_{0,-1}^2+64\big(3z^2-z+2\big)H_{0,1}^2-\frac{4}{5}\big(112z^2+1758z+245\big)\zeta_2^2$$

$$-\frac{4\big(290560z^3-295527z^2+31944z-24160\big)}{81z}-\frac{4}{9}\big(1333z^3+2860z^2-762z+336\big)\frac{\zeta_2}{z}$$

$$+\frac{32}{3}\big(180z^3-343z^2-14z-24\big)\frac{\zeta_3}{z}-\frac{4\big(9884z^3-13409z^2+3625z+492\big)H_1}{27z}$$

$$-\frac{8}{3}\big(54z^3-76z^2+41z-8\big)\frac{\zeta_2}{z}H_1+\frac{\big(541z^3+476z^2-50z+62\big)\Big[\frac{32}{9}H_{0,-1}-\frac{32}{9}H_{-1}H_0\Big]}{z}$$

$$-\frac{4\big(3416z^3-3816z^2+1353z-416\big)H_{0,1}}{9z}-\frac{16\big(40z^3-26z^2-17z-8\big)H_1H_{0,1}}{3z}$$

$$+\frac{\big(176z^3+112z^2-37z+16\big)\Big[\frac{8}{3}H_0H_{-1}^2-\frac{16}{3}H_{0,-1}H_{-1}+\frac{16}{3}H_{0,-1,-1}\Big]}{z}$$

$$-\frac{8\big(392z^3-532z^2+49z-24\big)H_{0,0,-1}}{3z}-\frac{8\big(152z^3-624z^2-75z-40\big)H_{0,0,1}}{3z}$$

$$+\frac{(z+1)\big(5z^2+4z+2\big)\Big[\frac{64}{3}H_{-1}H_{0,1}-\frac{64}{3}H_{0,-1,1}-\frac{64}{3}H_{0,1,-1}\Big]}{z}+\frac{32}{3}\big(8z^2+49z-11\big)H_{0,1,1}$$

$$-64\big(2z^2-4z+1\big)H_{0,-1,0,1}+16\big(10z^2+46z+23\big)H_{0,0,0,-1}+16\big(22z^2+126z+13\big)H_{0,0,0,1}$$

$$+32(12z^2 + 4z + 9)H_{0,0,1,1} + \frac{8}{3}(136z^2 + 40z - 85)H_{-1}\zeta_2 - 16(16z^2 - 2z + 13)H_{0,-1}\zeta_2$$

$$-16(20z^2 - 26z + 9)H_{0,1}\zeta_2 + (2z^2 + 2z + 1)\left[-\frac{160}{3}H_0 H_{-1}^3 + \left[8H_0^2 + 160H_{0,-1}\right]H_{-1}^2\right.$$

$$+\left[\frac{8}{3}H_0^3 + 64H_{0,-1}H_0 - 320H_{0,-1,-1} - 160H_{0,0,-1} + 32H_{0,0,1} - 64H_{0,1,1}\right]H_{-1}$$

$$+192\zeta_3 H_{-1} + 320H_{0,-1,-1,-1} + 64H_{0,-1,1,1} + 160H_{0,0,-1,-1} - 32H_{0,0,-1,1} - 32H_{0,0,1,-1}$$

$$+64H_{0,1,-1,1} + 64H_{0,1,1,-1} + \left[64H_{-1}H_0 - 80H_{-1}^2\right]\zeta_2\right] + \gamma_{qg}^0\left[2H_1^4 - 4H_0^2 H_1^2 + 8H_{0,1}H_1^2\right.$$

$$+\frac{10}{3}H_0^3 H_1 + \left[48H_{0,0,-1} + 64H_{0,0,1}\right]H_1 - 136\zeta_3 H_1 - 24H_{0,-1}H_{0,1} + H_0\left[-\frac{20}{3}H_1^3\right.$$

$$+\left[-24H_{0,-1} - 16H_{0,1}\right]H_1 + 24H_{0,-1,1} + 24H_{0,1,-1}\right] - 8H_{0,1,1,1} + \left[-8H_1^2 - 40H_0 H_1\right]\zeta_2\right]\right]$$

$$+L_Q\left[-\frac{4}{3}(51z - 11)H_0^4 + \frac{4}{9}(198z^2 - 1648z - 83)H_0^3 - \frac{8}{3}(22z^2 + 18z + 9)H_{-1}H_0^3\right.$$

$$+\frac{8}{3}(22z^2 - 26z + 13)H_1 H_0^3 + 8(16z^2 + 10z + 5)H_{-1}^2 H_0^2 + 16(7z^2 - 8z + 4)H_1^2 H_0^2$$

$$+\frac{2}{9}(18688z^2 - 14584z + 3849)H_0^2 - \frac{8(389z^3 + 317z^2 + 7z + 28)H_{-1}H_0^2}{3z}$$

$$+\frac{4(822z^3 - 631z^2 - 61z - 72)H_1 H_0^2}{3z} + 16(3z^2 - 22z + 9)H_{0,-1}H_0^2$$

$$-4(36z^2 + 134z + 63)H_{0,1}H_0^2 - 4(24z^2 - 230z - 9)\zeta_2 H_0^2$$

$$+\frac{8(477z^3 - 428z^2 + 37z - 28)H_1^2 H_0}{3z} + \frac{4(37209z^3 - 68257z^2 - 7837z - 1136)H_0}{27z}$$

$$-\frac{8}{3}(708z^3 - 1801z^2 - 21z - 16)\frac{\zeta_2}{z}H_0 + \frac{16(3243z^3 - 3023z^2 + 10z - 58)H_1 H_0}{9z}$$

$$+\frac{16(129z^3 + 194z^2 - 50z + 36)H_{0,-1}H_0}{3z} - 64(7z^2 + 6z + 3)H_{-1}H_{0,-1}H_0$$

$$-\frac{8(42z^3 + 833z^2 - 49z - 40)H_{0,1}H_0}{3z} - 64(z-1)^2 H_1 H_{0,1}H_0 + 128(5z^2 + 2z + 3)H_{0,-1,-1}H_0$$

$$+32(5z^2 + 44z - 7)H_{0,0,-1}H_0 + 32(11z^2 + 24z + 19)H_{0,0,1}H_0 + (10z^2 - 2z + 5)\left[32H_{0,-1,1}\right.$$

$$\left.+32H_{0,1,-1}\right]H_0 + 32(2z^2 - 32z - 5)H_{0,1,1}H_0 + 32(14z^2 + 10z + 5)H_{-1}\zeta_2 H_0$$

$$-128(5z^2 - 6z + 3)H_1\zeta_2 H_0 + 32(64z + 17)\zeta_3 H_0 + \frac{8(424z^3 - 382z^2 + 29z - 24)H_1^3}{9z}$$

$$+\frac{8(3661z^3 - 3443z^2 + 79z - 71)H_1^2}{9z} - 32(3z^2 - 2z + 3)H_{0,-1}^2 - 32(3z^2 - 4z + 2)H_{0,1}^2$$

$$+\frac{4}{5}(352z^2 - 18z + 157)\zeta_2^2 + \frac{2(503837z^3 - 529368z^2 + 89781z - 57320)}{81z}$$

$$-\frac{16}{9}(4600z^3 - 3718z^2 + 405z - 136)\frac{\zeta_2}{z} - \frac{8}{3}(1252z^3 - 1106z^2 + 15z - 96)\frac{\zeta_3}{z}$$

$$+\frac{8}{3}(536z^3 + 424z^2 - 7z + 24)\frac{\zeta_2}{z}H_{-1} + \frac{4(52691z^3 - 54734z^2 + 469z + 3996)H_1}{27z}$$

$$-\frac{8}{3}(766z^3 - 676z^2 + 11z - 64)\frac{\zeta_2}{z}H_1 + \frac{(272z^3 + 257z^2 - 128z - 240)\left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right]}{z}$$

$$+\frac{16(1357z^3 - 952z^2 + 395z + 162)H_{0,1}}{9z} + \frac{(100z^3 + 92z^2 + 19z - 8)}{z}\left[\frac{8}{3}H_0 H_{-1}^2 - \frac{16}{3}H_{0,-1}H_{-1}\right.$$

$$+\frac{16}{3}H_{0,-1,-1}\Big] + \gamma_{qg}^0\Big[-2H_1^4 + \Big[\frac{88}{3}H_{0,1} - 48H_{0,0,-1}\Big]H_1 + H_0\Big[24H_1H_{0,-1} - \frac{28}{3}H_1^3\Big] + 24H_{0,-1}H_{0,1}\Big]$$

$$+\frac{16(131z^3 - 71z^2 + 107z - 44)H_{0,0,-1}}{3z} - \frac{16(15z^3 - 125z^2 + 29z + 4)H_{0,0,1}}{3z}$$

$$+64(4z^2 - 6z + 3)H_1H_{0,0,1} + \frac{(218z^3 + 166z^2 - 13z + 16)}{z}\Big[-\frac{16}{3}H_{-1}H_{0,1} + \frac{16}{3}H_{0,-1,1}$$

$$+\frac{16}{3}H_{0,1,-1}\Big] + \frac{16(60z^3 - 488z^2 + z + 8)H_{0,1,1}}{3z} + (2z^2 + 6z + 3)\Big[\frac{32}{3}H_0H_{-1}^3 - 32H_{0,-1}H_{-1}^2$$

$$+64H_{0,-1,-1}H_{-1} - 64H_{0,-1,-1,-1}\Big] + 64(5z^2 - 8z + 5)H_{0,-1,0,1} + (12z^2 + 14z + 7)\Big[32H_{-1}H_{0,0,-1}$$

$$-32H_{0,0,-1,-1}\Big] - 16(26z^2 + 114z + 3)H_{0,0,0,-1} - 16(22z^2 + 50z + 37)H_{0,0,0,1}$$

$$+(8z^2 + 6z + 3)\Big[-32H_{-1}H_{0,0,1} + 32H_{0,0,-1,1} + 32H_{0,0,1,-1}\Big] - 32(6z^2 - 8z - 1)H_{0,0,1,1}$$

$$+(3z^2 + 2z + 1)\Big[64H_{0,1}H_{-1}^2 + \Big[-128H_{0,-1,1} - 128H_{0,1,-1}\Big]H_{-1} + 128H_{0,-1,-1,1} + 128H_{0,-1,1,-1}$$

$$+128H_{0,1,-1,-1}\Big] + (2z^2 + 2z + 1)\Big[H_{-1}\Big[-64H_0H_{0,1} - 64H_{0,1,1}\Big] + 64H_{0,-1,1,1} + 64H_{0,1,-1,1}$$

$$+64H_{0,1,1,-1}\Big] - 32(2z^2 + 22z + 7)H_{0,1,1,1} - 16(10z^2 + 2z + 1)H_{-1}^2\zeta_2 - 16(4z^2 - 14z + 9)H_{0,-1}\zeta_2$$

$$+16(24z^2 + 38z + 31)H_{0,1}\zeta_2 + 16(20z^2 + 6z + 3)H_{-1}\zeta_3 + (8z^2 - 10z + 5)\Big[-32\zeta_2H_1^2 - 112\zeta_3H_1\Big]\Big]$$

$$+272(8z - 1)\zeta_5\Big] + C_F^2T_F\Big[\frac{2}{15}(8z^2 - 2z + 1)H_0^5 + \frac{1}{3}(52z^2 - 16z - 3)H_0^4 - \frac{2}{3}(36z^2 + 2z + 5)H_0^3$$

$$+\frac{16}{3}(13z^2 - 17z + 3)H_1H_0^3 + 16(2z - 1)H_{0,1}H_0^3 - \frac{2}{3}(2z - 3)(4z - 1)\zeta_2H_0^3$$

$$+2(40z^2 - 56z + 15)H_1^2H_0^2 + (384z^2 - 196z + 63)H_0^2 - 4(18z^2 - 18z - 7)H_1H_0^2$$

$$-4(12z^2 - 60z + 17)H_{0,1}H_0^2 - 8(12z^2 + 10z - 5)H_{0,0,1}H_0^2 + 8(16z^2 - 6z + 3)H_{0,1,1}H_0^2$$

$$-2(52z^2 - 37z + 4)\zeta_2H_0^2 - \frac{16}{3}(22z^2 - 10z + 5)\zeta_3H_0^2 - \frac{4}{3}(4z - 3)(4z + 1)H_1^3H_0$$

$$-2(88z^2 - 86z - 7)H_1^2H_0 - 8(20z^2 - 14z + 7)H_{0,1}^2H_0 - \frac{8}{5}(284z^2 - 94z + 57)\zeta_2^2H_0$$

$$+4(136z^2 - 91z - 13)H_0 + 4(192z^2 - 156z + 25)H_1H_0 - 16(13z^2 + 7z + 3)H_{0,1}H_0$$

$$-8(64z - 25)H_{0,0,1}H_0 - 16(z - 1)(16z - 3)H_{0,1,1}H_0 + 32(12z^2 + 2z - 1)H_{0,0,0,1}H_0$$

$$+192(z - 1)^2H_{0,0,1,1}H_0 - 16(36z^2 - 38z + 19)H_{0,1,1,1}H_0 + 2(122z^2 + 38z + 29)\zeta_2H_0$$

$$-4(4z - 3)(16z - 9)H_1\zeta_2H_0 - 32z^2H_{0,-1}\zeta_2H_0 + 8(44z^2 - 58z + 29)H_{0,1}\zeta_2H_0$$

$$+8(4z - 3)(8z + 1)\zeta_3H_0 - \frac{1}{3}(72z^2 - 76z + 19)H_1^4 - \frac{4}{3}(62z^2 - 28z - 31)H_1^3$$

$$-78z^2 + 2(136z^2 - 129z + 74)H_1^2 - 4(12z^2 - 24z + 7)H_{0,1}^2 - \frac{8}{5}(176z^2 - 79z + 1)\zeta_2^2$$

$$-84z + 2(272z^2 - 313z + 39)H_1 - 2(112z^2 - 7z + 22)H_{0,1} + 2(104z^2 + 30z - 17)H_{0,0,1}$$

$$+64(5z^2 - 2z + 1)H_{0,1}H_{0,0,1} + (2z - 3)(2z - 1)\Big[48H_1H_{0,0,1} - 24H_0H_1H_{0,1}\Big]$$

$$-4(36z^2 + 102z - 37)H_{0,1,1} + 64(3z^2 - 4z + 2)H_{0,1}H_{0,1,1} + (4z - 1)\Big[8H_1^2H_{0,1} - 32H_1H_{0,1,1}\Big]$$

$$+8(24z^2 + 67z - 33)H_{0,0,0,1} + 8(32z^2 - 60z + 17)H_{0,0,1,1} - 32(2z - 1)(7z - 5)H_{0,1,1,1}$$

$$-8(48z^2 - 2z + 1)H_{0,0,0,0,1} - 16(92z^2 - 62z + 31)H_{0,0,0,1,1} - 32(20z^2 - 6z + 3)H_{0,0,1,0,1}$$

$$-16(36z^2 - 50z + 25)H_{0,0,1,1,1} - 96(4z^2 - 6z + 3)H_{0,1,0,1,1} - 16(40z^2 - 34z + 17)H_{0,1,1,1,1}$$

$$+16z(8z - 9)H_1^2\zeta_2 - \frac{1}{2}(232z^2 - 510z - 375)\zeta_2 + 4(61z^2 - 82z - 10)H_1\zeta_2$$

$$+(z+1)(3z+2)\Big[16H_{0,-1}-16H_{-1}H_0\Big]\zeta_2+4\big(128z^2-166z+47\big)H_{0,1}\zeta_2$$

$$+16\big(6z^2+2z+1\big)H_{0,0,-1}\zeta_2-8\big(20z^2-74z+37\big)H_{0,0,1}\zeta_2-16\big(2z^2-6z+3\big)H_{0,1,1}\zeta_2$$

$$+96\gamma_{qg}^0\log(2)\zeta_2+L_Q^3\bigg[\frac{64}{3}H_{0,1}z^2-8H_0z+\frac{4}{3}\big(8z^2-2z+1\big)H_0^2-\frac{2}{3}(2z-11)-\frac{16}{3}(4z-1)H_1$$

$$+\gamma_{qg}^0\Big[-\frac{8}{3}H_1^2-\frac{8}{3}H_0H_1\Big]-\frac{32}{3}\big(4z^2-2z+1\big)\zeta_2\bigg]+L_M^3\bigg[-\frac{64}{3}H_{0,1}z^2+8H_0z$$

$$-\frac{4}{3}\big(8z^2-2z+1\big)H_0^2+\frac{2}{3}(2z-11)+\frac{16}{3}(4z-1)H_1+\gamma_{qg}^0\Big[\frac{8}{3}H_1^2+\frac{8}{3}H_0H_1\Big]$$

$$+\frac{32}{3}\big(4z^2-2z+1\big)\zeta_2\bigg]+\big(2z^2+2z+1\big)\Big[16H_{-1}\zeta_2^2+\Big[16H_0H_{-1}^2+\big[-8H_0^2-32H_{0,-1}\big]H_{-1}$$

$$+32H_{0,-1,-1}\Big]\zeta_2\bigg]+\frac{4}{3}\big(252z^2-34z+13\big)\zeta_3+\frac{64}{3}\big(12z^2-7z-2\big)H_1\zeta_3$$

$$+\frac{128}{3}\big(5z^2-6z+3\big)H_{0,1}\zeta_3-\frac{16}{3}\big(20z^2+14z-1\big)\zeta_2\zeta_3+L_M^2\bigg[-\frac{4}{3}\big(24z^2+2z+3\big)H_0^3$$

$$-4z(32z+9)H_0^2-4\big(18z^2-6z+11\big)H_0-16\big(10z^2-18z+5\big)H_1H_0+64z^2H_{0,-1}H_0$$
$$-32\big(6z^2-2z+1\big)H_{0,1}H_0+8\big(40z^2-10z+9\big)\zeta_2H_0+104z^2-32(z-1)(5z-2)H_1^2$$
$$-92z-2\big(36z^2-126z+65\big)H_1+(z+1)(3z+2)\Big[32H_{-1}H_0-32H_{0,-1}\Big]-8\big(20z^2+5z-4\big)H_{0,1}$$
$$-32\big(6z^2+2z+1\big)H_{0,0,-1}+8\big(24z^2-14z+7\big)H_{0,0,1}-32\big(2z^2+2z-1\big)H_{0,1,1}$$
$$+8\big(40z^2-11z+6\big)\zeta_2+\big(2z^2+2z+1\big)\Big[-32H_0H_{-1}^2+\big[16H_0^2+64H_{0,-1}\big]H_{-1}$$

$$-32\zeta_2H_{-1}-64H_{0,-1,-1}\Big]+\gamma_{qg}^0\Big[8H_1^3+20H_0H_1^2+8H_0^2H_1-32\zeta_2H_1\Big]+8\big(32z^2+2z+7\big)\zeta_3-33\bigg]$$

$$+L_Q^2\bigg[-\frac{4}{3}\big(24z^2+2z+3\big)H_0^3-4z(32z+9)H_0^2-4\big(18z^2-6z+11\big)H_0-16\big(10z^2-18z+5\big)H_1H_0$$

$$+64z^2H_{0,-1}H_0-32\big(6z^2-2z+1\big)H_{0,1}H_0+8\big(40z^2-10z+9\big)\zeta_2H_0+104z^2$$
$$-32(z-1)(5z-2)H_1^2-92z-2\big(36z^2-126z+65\big)H_1+(z+1)(3z+2)\Big[32H_{-1}H_0-32H_{0,-1}\Big]$$
$$-8\big(20z^2+5z-4\big)H_{0,1}-32\big(6z^2+2z+1\big)H_{0,0,-1}+8\big(24z^2-14z+7\big)H_{0,0,1}$$

$$-32\big(2z^2+2z-1\big)H_{0,1,1}+8\big(40z^2-11z+6\big)\zeta_2+L_M\bigg[-64H_{0,1}z^2+24H_0z-4\big(8z^2-2z+1\big)H_0^2$$

$$+2(2z-11)+16(4z-1)H_1+\gamma_{qg}^0\Big[8H_1^2+8H_0H_1\Big]+32\big(4z^2-2z+1\big)\zeta_2\bigg]$$

$$+\big(2z^2+2z+1\big)\Big[-32H_0H_{-1}^2+\big[16H_0^2+64H_{0,-1}\big]H_{-1}-32\zeta_2H_{-1}-64H_{0,-1,-1}\Big]$$

$$+\gamma_{qg}^0\Big[8H_1^3+20H_0H_1^2+8H_0^2H_1-32\zeta_2H_1\Big]+8\big(32z^2+2z+7\big)\zeta_3-33\bigg]+\gamma_{qg}^0\Big[\frac{1}{3}H_1^5+\frac{16}{3}H_{0,1}H_1^3$$

$$-\frac{4}{3}H_0^3H_1^2+\big[-32H_{0,0,1}-16H_{0,1,1}\big]H_1^2-\frac{2}{3}H_0^4H_1+20H_{0,1}^2H_1+\frac{264}{5}\zeta_2^2H_1+12H_0^2H_{0,1}H_1$$

$$+H_0\Big[H_1^4+20H_{0,1}H_1^2+\big[-24H_{0,0,1}-56H_{0,1,1}\big]H_1\Big]+\Big[-\frac{40}{3}H_1^3-18H_0H_1^2$$

$$+4H_0^2H_1-24H_{0,1}H_1\Big]\zeta_2+\Big[\frac{4}{3}H_1^2+\frac{88}{3}H_0H_1\Big]\zeta_3\Big]+L_M\bigg[-\frac{2}{3}\big(40z^2+6z+5\big)H_0^4$$

$$-\frac{8}{3}z(82z+29)H_0^3 + \frac{2}{15}\big(576z^3 - 4620z^2 - 5510z - 585\big)H_0^2 + 128(z+1)(2z+3)H_{-1}H_0^2$$

$$-8\big(50z^2 - 40z - 3\big)H_1H_0^2 + 32\big(10z^2 - 2z + 3\big)H_{0,-1}H_0^2 - 16\big(8z^2 - 14z - 1\big)H_{0,1}H_0^2$$

$$+8\big(72z^2 - 22z + 15\big)\zeta_2 H_0^2 - 64\big(11z^2 - 13z + 4\big)H_1^2 H_0 - \frac{4\big(1464z^3 + 1593z^2 + 308z - 16\big)H_0}{15z}$$

$$-16\big(55z^2 - 95z + 29\big)H_1 H_0 - 64\big(7z^2 + 18z - 2\big)H_{0,-1}H_0 - 128\big(4z^2 + 1\big)H_{0,0,-1}H_0$$

$$-64\big(4z^2 + 10z + 3\big)H_{0,0,1}H_0 + 128\big(4z^2 - 6z + 3\big)H_{0,1,1}H_0 + 16\big(100z^2 - 31z + 9\big)\zeta_2 H_0$$

$$+32\big(44z^2 - 30z + 9\big)\zeta_3 H_0 - 48\big(7z^2 - 8z + 2\big)H_1^3 - 2\big(368z^2 - 546z + 155\big)H_1^2$$

$$-32(14z - 1)H_{0,-1}^2 - 16\big(8z^2 - 6z + 3\big)H_{0,1}^2 - \frac{8}{5}\big(192z^2 - 346z + 57\big)\zeta_2^2$$

$$+\frac{2\big(1092z^3 - 1893z^2 + 383z - 32\big)}{15z} - 2\big(272z^2 - 216z - 85\big)H_1 - 16\big(37z^2 - 6z - 13\big)H_{0,1}$$

$$+\frac{\big(72z^5 - 165z^4 - 1090z^3 - 915z^2 + 2\big)\big[\frac{32}{15}H_{0,-1} - \frac{32}{15}H_{-1}H_0\big]}{z^2} + 16(4z - 1)H_1 H_{0,1}$$

$$+(z+1)(z+14)\big[-32H_0 H_{-1}^2 + 64H_{0,-1}H_{-1} - 64H_{0,-1,-1}\big] + 128\big(3z^2 + 8z - 8\big)H_{0,0,-1}$$

$$+16\big(26z^2 - 77z + 26\big)H_{0,0,1} + (z+1)(3z+2)\big[64H_{-1}H_{0,1} - 64H_{0,-1,1} - 64H_{0,1,-1}\big]$$

$$-16\big(38z^2 - 3z - 15\big)H_{0,1,1} + \big(3z^2 - 4z + 2\big)\big[-128H_0 H_{0,-1,-1} - 128H_{0,-1,0,1}\big]$$

$$-64\big(10z^2 - 2z + 5\big)H_{0,0,0,-1} + 16\big(48z^2 + 34z + 27\big)H_{0,0,0,1} - 16\big(32z^2 - 34z + 17\big)H_{0,0,1,1}$$

$$-192(2z - 1)H_{0,1,1,1} - \frac{16}{15}\big(144z^3 - 1380z^2 - 665z - 240\big)\zeta_2 - 32(z+1)(7z+18)H_{-1}\zeta_2$$

$$+16\big(86z^2 - 78z + 5\big)H_1\zeta_2 - 32\big(12z^2 - 6z + 5\big)H_{0,-1}\zeta_2 - 64(2z+3)H_{0,1}\zeta_2$$

$$+16\big(128z^2 - 54z + 23\big)\zeta_3 + \big(2z^2 + 2z + 1\big)\Big[\frac{128}{3}H_0 H_{-1}^3 + \big[-112H_0^2 - 128H_{0,-1} - 64H_{0,1}\big]H_{-1}^2$$

$$+\Big[\frac{64}{3}H_0^3 + 192H_{0,-1}H_0 + 256H_{0,-1,-1} + 128H_{0,-1,1} + 64H_{0,0,-1} + 64H_{0,0,1} + 128H_{0,1,-1}\Big]H_{-1}$$

$$-224\zeta_3 H_{-1} - 256H_{0,-1,-1,-1} - 128H_{0,-1,-1,1} - 128H_{0,-1,1,-1} - 64H_{0,0,-1,-1} - 64H_{0,0,-1,1}$$

$$-64H_{0,0,1,-1} - 128H_{0,1,-1,-1} + \big[128H_{-1}^2 - 64H_{-1}H_0\big]\zeta_2\Big] + \gamma_{qg}^0\Big[8H_1^4 + 20H_0^2 H_1^2$$

$$+8H_{0,1}H_1^2 + 8H_0^3 H_1 + \big[128H_{0,0,-1} - 80H_{0,0,1}\big]H_1 - 120\zeta_3 H_1 - 64H_{0,-1}H_{0,1}$$

$$+H_0\Big[24H_1^3 + \big[64H_{0,1} - 64H_{0,-1}\big]H_1 + 76H_{0,1} + 64H_{0,-1,1} + 64H_{0,1,-1}\Big]$$

$$+\big[-64H_1^2 - 128H_0 H_1\big]\zeta_2\Big]\Big] + L_Q\Big[\frac{2}{3}\big(40z^2 + 6z + 5\big)H_0^4 + \frac{8}{3}z(82z+29)H_0^3$$

$$-\frac{2}{15}\big(576z^3 - 4620z^2 - 5510z - 585\big)H_0^2 - 128(z+1)(2z+3)H_{-1}H_0^2 + 8\big(50z^2 - 40z - 3\big)H_1 H_0^2$$

$$-32\big(10z^2 - 2z + 3\big)H_{0,-1}H_0^2 + 16\big(8z^2 - 14z - 1\big)H_{0,1}H_0^2 - 8\big(72z^2 - 22z + 15\big)\zeta_2 H_0^2$$

$$+64\big(11z^2 - 13z + 4\big)H_1^2 H_0 + \frac{4\big(1464z^3 + 1593z^2 + 308z - 16\big)H_0}{15z} + 16\big(55z^2 - 95z + 29\big)H_1 H_0$$

$$+64\big(7z^2 + 18z - 2\big)H_{0,-1}H_0 + 128\big(4z^2 + 1\big)H_{0,0,-1}H_0 + 64\big(4z^2 + 10z + 3\big)H_{0,0,1}H_0$$

$$-128\big(4z^2 - 6z + 3\big)H_{0,1,1}H_0 - 16\big(100z^2 - 31z + 9\big)\zeta_2 H_0 - 32\big(44z^2 - 30z + 9\big)\zeta_3 H_0$$

$$+48\big(7z^2 - 8z + 2\big)H_1^3 + 2\big(368z^2 - 546z + 155\big)H_1^2 + 32(14z - 1)H_{0,-1}^2 + 16\big(8z^2 - 6z + 3\big)H_{0,1}^2$$

$$+\frac{8}{5}\big(192z^2 - 346z + 57\big)\zeta_2^2 - \frac{2\big(1092z^3 - 1893z^2 + 383z - 32\big)}{15z} + 2\big(272z^2 - 216z - 85\big)H_1$$

$$+\frac{\big(72z^5 - 165z^4 - 1090z^3 - 915z^2 + 2\big)\big[\frac{32}{15}H_{-1}H_0 - \frac{32}{15}H_{0,-1}\big]}{z^2} + 16\big(37z^2 - 6z - 13\big)H_{0,1}$$

$$-16(4z-1)H_1H_{0,1} + (z+1)(z+14)\Big[32H_0H_{-1}^2 - 64H_{0,-1}H_{-1} + 64H_{0,-1,-1}\Big]$$

$$-128\big(3z^2+8z-8\big)H_{0,0,-1} - 16\big(26z^2-77z+26\big)H_{0,0,1} + (z+1)(3z+2)\Big[-64H_{-1}H_{0,1}$$

$$+64H_{0,-1,1} + 64H_{0,1,-1}\Big] + 16\big(38z^2-3z-15\big)H_{0,1,1} + \big(3z^2-4z+2\big)\Big[128H_0H_{0,-1,-1}$$

$$+128H_{0,-1,0,1}\Big]$$

$$+64\big(10z^2-2z+5\big)H_{0,0,0,-1} - 16\big(48z^2+34z+27\big)H_{0,0,0,1} + 16\big(32z^2-34z+17\big)H_{0,0,1,1}$$

$$+192(2z-1)H_{0,1,1,1} + \frac{16}{15}\big(144z^3-1380z^2-665z-240\big)\zeta_2 + 32(z+1)(7z+18)H_{-1}\zeta_2$$

$$-16\big(86z^2-78z+5\big)H_1\zeta_2 + 32\big(12z^2-6z+5\big)H_{0,-1}\zeta_2 + 64(2z+3)H_{0,1}\zeta_2$$

$$+L_M^2\Bigg[64H_{0,1}z^2 - 24H_0z + 4\big(8z^2-2z+1\big)H_0^2 - 2(2z-11) - 16(4z-1)H_1$$

$$+\gamma_{qg}^0\Big[-8H_1^2-8H_0H_1\Big] - 32\big(4z^2-2z+1\big)\zeta_2\Bigg] - 16\big(128z^2-54z+23\big)\zeta_3$$

$$+L_M\Bigg[\frac{8}{3}\big(24z^2+2z+3\big)H_0^3 + 8z(32z+9)H_0^2 + 8\big(18z^2-6z+11\big)H_0$$

$$+32\big(10z^2-18z+5\big)H_1H_0 - 128z^2H_{0,-1}H_0 + 64\big(6z^2-2z+1\big)H_{0,1}H_0 - 16\big(40z^2-10z+9\big)\zeta_2H_0$$

$$+64(z-1)(5z-2)H_1^2 - 2\big(104z^2-92z-33\big) + 4\big(36z^2-126z+65\big)H_1$$

$$+(z+1)(3z+2)\Big[64H_{0,-1}-64H_{-1}H_0\Big] + 16\big(20z^2+5z-4\big)H_{0,1} + 64\big(6z^2+2z+1\big)H_{0,0,-1}$$

$$-16\big(24z^2-14z+7\big)H_{0,0,1} + 64\big(2z^2+2z-1\big)H_{0,1,1} - 16\big(40z^2-11z+6\big)\zeta_2$$

$$+\big(2z^2+2z+1\big)\Big[64H_0H_{-1}^2 + \big[-32H_0^2-128H_{0,-1}\big]H_{-1} + 64\zeta_2H_{-1} + 128H_{0,-1,-1}\Big]$$

$$+\gamma_{qg}^0\Big[-16H_1^3-40H_0H_1^2-16H_0^2H_1+64\zeta_2H_1\Big] - 16\big(32z^2+2z+7\big)\zeta_3\Bigg]$$

$$+\big(2z^2+2z+1\big)\Bigg[-\frac{128}{3}H_0H_{-1}^3 + \Big[112H_0^2+128H_{0,-1}+64H_{0,1}\Big]H_{-1}^2 + \Big[-\frac{64}{3}H_0^3$$

$$-192H_{0,-1}H_0 - 256H_{0,-1,-1} - 128H_{0,-1,1} - 64H_{0,0,-1} - 64H_{0,0,1} - 128H_{0,1,-1}\Big]H_{-1}$$

$$+224\zeta_3H_{-1} + 256H_{0,-1,-1,-1} + 128H_{0,-1,-1,1} + 128H_{0,-1,1,-1} + 64H_{0,0,-1,-1} + 64H_{0,0,-1,1}$$

$$+64H_{0,0,1,-1} + 128H_{0,1,-1,-1} + \Big[64H_{-1}H_0-128H_{-1}^2\Big]\zeta_2 + \gamma_{qg}^0\Big[-8H_1^4-20H_0^2H_1^2-8H_{0,1}H_1^2$$

$$-8H_0^3H_1 + \Big[80H_{0,0,1}-128H_{0,0,-1}\Big]H_1 + 120\zeta_3H_1 + 64H_{0,-1}H_{0,1} + H_0\Big[-24H_1^3$$

$$+\Big[64H_{0,-1}-64H_{0,1}\Big]H_1 - 76H_{0,1} - 64H_{0,-1,1} - 64H_{0,1,-1}\Big] + \Big[64H_1^2+128H_0H_1\Big]\zeta_2\Big]\Bigg]$$

$$-8\big(56z^2-2z+1\big)\zeta_5 - 77\Bigg] + C_FT_F^2\Bigg[\frac{4}{9}\big(4z^2-20z+1\big)H_0^4$$

$$+\frac{16}{9}\big(10z^2-63z+4\big)H_0^3 - \frac{8}{3}\big(68z^2+75z-22\big)H_0^2 + \frac{16}{3}\big(10z^2-12z+1\big)H_1H_0^2$$

$$-\frac{64}{3}(z-1)^2H_{0,1}H_0^2 + \frac{4}{3}\big(32z^2+112z-59\big)\zeta_2H_0^2 + \frac{4}{3}\big(538z^2-1578z+21\big)H_0$$

$$-\frac{32}{3}\big(16z^2-27z+10\big)H_1H_0 - \frac{64}{3}\big(5z^2-4z+2\big)H_{0,1}H_0 - \frac{128}{3}\big(z^2+2z-1\big)H_{0,0,1}H_0$$

$$+\frac{4}{9}\big(24z^2+1124z-1063\big)\zeta_2H_0 - \frac{32}{9}\big(36z^2-74z+37\big)\zeta_3H_0 + \frac{32}{9}(z-1)(3z+1)H_1^3$$

$$+\left[-\frac{16}{3}(2z-1)H_0^2-\frac{16}{9}\left(24z^2-8z-5\right)H_0+\frac{16\left(62z^3-111z^2+75z-8\right)}{27z}\right.$$

$$\left.+\frac{32}{9}\gamma_{qg}^0 H_1\right]L_M^3+\frac{16}{3}\left(12z^2-17z+4\right)H_1^2-\frac{16}{15}\left(218z^2-194z+91\right)\zeta_2^2$$

$$+\frac{2\left(6994z^3-8610z^2+939z-208\right)}{9z}+\frac{2}{27}\left(1984z^3+7716z^2-12243z-256\right)\frac{\zeta_2}{z}$$

$$-\frac{16}{27}\left(170z^3-1128z^2+525z-8\right)\frac{\zeta_3}{z}+\frac{4\left(810z^3-1558z^2+793z+16\right)H_1}{3z}$$

$$+L_Q^3\left[\frac{16}{3}(2z-1)H_0^2+\frac{16}{9}(4z-11)H_0-\frac{16\left(62z^3-147z^2+84z-8\right)}{27z}+\frac{16}{9}\gamma_{qg}^0 H_1\right]$$

$$+\frac{16}{3}\left(20z^2-113z+140\right)H_{0,1}+\frac{16}{3}\left(20z^2-113z+56\right)H_{0,0,1}+\frac{64}{3}\left(3z^2-6z-1\right)H_{0,1,1}$$

$$+\frac{16}{3}\left(40z^2-28z+11\right)H_{0,0,0,1}+\frac{80}{9}\left(4z^2-4z-1\right)H_1\zeta_2+\gamma_{qg}^0\left[-\frac{2}{9}H_1^4+\frac{16}{3}H_{0,1}H_1^2\right.$$

$$-\frac{8}{9}H_0^3 H_1+\left[-\frac{64}{3}H_{0,0,1}-\frac{64}{3}H_{0,1,1}\right]H_1+\frac{160}{9}\zeta_3 H_1+\frac{32}{3}H_{0,1}^2+H_0\left[\frac{32}{3}H_1 H_{0,1}\right.$$

$$\left.-\frac{64}{3}H_{0,1,1}\right]+\frac{80}{3}H_{0,1,1,1}+\left[-\frac{26}{3}H_1^2-\frac{16}{3}H_0 H_1-12H_{0,1}\right]\zeta_2\right]+L_Q^2\left[-\frac{8}{3}\left(8z^2+56z-31\right)H_0^2\right.$$

$$+\frac{8}{9}\left(244z^2-236z+571\right)H_0+\frac{4\left(2156z^3-7632z^2+4977z+256\right)}{27z}$$

$$+\frac{16\left(130z^3-215z^2+112z-8\right)H_1}{9z}+\gamma_{qg}^0\left[-4H_1^2-\frac{32}{3}H_0 H_1\right]-\frac{16}{3}\left(12z^2-8z-5\right)H_{0,1}$$

$$+\left[-16(2z-1)H_0^2-\frac{16}{3}\left(8z^2-9\right)H_0+\frac{16(z-1)\left(62z^2-73z+8\right)}{9z}\right]L_M$$

$$-\frac{16}{3}\left(4z^2-8z+13\right)\zeta_2+(2z-1)\left[-16H_0^3+32\zeta_2 H_0-32H_{0,0,1}+32\zeta_3\right]\right]$$

$$+L_M^2\left[-\frac{8}{3}\left(40z^2+40z-23\right)H_0^2+\frac{8}{9}\left(4z^2-92z+547\right)H_0+\frac{4\left(1652z^3-6192z^2+4221z+256\right)}{27z}\right.$$

$$+\frac{16\left(10z^3-71z^2+70z-8\right)H_1}{9z}+\gamma_{qg}^0\left[\frac{20}{3}H_1^2+\frac{32}{3}H_0 H_1\right]-16\left(4z^2-3\right)H_{0,1}$$

$$+\frac{16}{3}\left(28z^2-16z-1\right)\zeta_2+(2z-1)\left[-16H_0^3+32\zeta_2 H_0-32H_{0,0,1}+32\zeta_3\right]\right]$$

$$+L_Q\left[\frac{64}{3}\left(2z^2+15z-5\right)H_0^3-\frac{16}{45}\left(72z^3+560z^2-2660z+2975\right)H_0^2\right.$$

$$-\frac{64}{3}\left(5z^2-4z+2\right)H_1 H_0^2-\frac{16\left(1444z^3-5632z^2+9213z+4\right)H_0}{45z}$$

$$-\frac{32\left(198z^3-283z^2+140z-8\right)H_1 H_0}{9z}+\frac{128}{3}(z-2)(3z-2)H_{0,-1}H_0$$

$$+\frac{32}{3}\left(16z^2-8z-5\right)H_{0,1}H_0-\frac{32}{3}\left(4z^2+84z-33\right)\zeta_2 H_0-\frac{16\left(168z^3-253z^2+131z-8\right)H_1^2}{9z}$$

$$+\left[-\frac{16\left(62z^3-123z^2+78z-8\right)}{9z}+16(2z-1)H_0^2+\frac{16}{3}\left(16z^2-4z-7\right)H_0-\frac{16}{3}\gamma_{qg}^0 H_1\right]L_M^2$$

$$-\frac{4\left(269954z^3-828996z^2+567861z-11744\right)}{405z}+\frac{16}{45}\left(144z^4+1600z^3-1400z^2+3045z-80\right)\frac{\zeta_2}{z}$$

$$-\frac{8\left(3080z^3-8448z^2+5247z+256\right)H_1}{27z}-\frac{128}{3}\left(7z^2-14z+9\right)H_{0,0,-1}$$

$$+\frac{\left(36z^5-155z^4+40z^3+225z^2-20z+1\right)\left[\frac{64}{45}H_{-1}H_0-\frac{64}{45}H_{0,-1}\right]}{z^2}$$

$$+\frac{16\left(76z^3-254z^2-329z-16\right)H_{0,1}}{9z}+\gamma_{qg}^0\left[\frac{8}{3}H_1^3+8H_0H_1^2+\frac{32}{3}H_{0,1}H_1\right]$$

$$-\frac{32}{3}\left(8z^2-60z+15\right)H_{0,0,1}+\frac{32}{3}\left(24z^2-20z+1\right)H_{0,1,1}+\frac{64}{3}\left(8z^2-6z+3\right)H_1\zeta_2$$

$$+(z+1)^2\left[-\frac{128}{3}H_0H_{-1}^2+\left[\frac{64}{3}H_0^2+\frac{256}{3}H_{0,-1}\right]H_{-1}-\frac{128}{3}\zeta_2H_{-1}-\frac{256}{3}H_{0,-1,-1}\right]$$

$$+L_M\left[16\left(8z^2+16z-9\right)H_0^2-\frac{16}{9}\left(124z^2-164z+559\right)H_0-\frac{8}{3}\gamma_{qg}^0H_1^2\right.$$

$$-\frac{8\left(1904z^3-6912z^2+4599z+256\right)}{27z}-\frac{32\left(70z^3-143z^2+91z-8\right)H_1}{9z}+\left(12z^2-4z-7\right)\left[\frac{32}{3}H_{0,1}\right.$$

$$\left.-\frac{32\zeta_2}{3}\right]+(2z-1)\left[32H_0^3-64\zeta_2H_0+64H_{0,0,1}-64\zeta_3\right]\right]+\frac{128}{3}\left(4z^2-21z+11\right)\zeta_3$$

$$+(2z-1)\left[16H_0^4-96\zeta_2H_0^2+64H_{0,0,1}H_0-128\zeta_3H_0-\frac{32\zeta_2^2}{5}+64H_{0,0,1,1}\right]\right]$$

$$+L_M\left[-\frac{64}{9}\left(10z^2+43z-14\right)H_0^3-\frac{16}{45}\left(72z^3-170z^2+2640z-2945\right)H_0^2\right.$$

$$+\frac{64}{3}\left(3z^2-4z+2\right)H_1H_0^2+\frac{32\left(173z^3-2669z^2+4406z-2\right)H_0}{45z}$$

$$+\frac{32\left(120z^3-199z^2+104z-8\right)H_1H_0}{9z}-\frac{128}{3}\left(z^2-4z+2\right)H_{0,-1}H_0-32\left(8z^2-8z+1\right)H_{0,1}H_0$$

$$+\frac{32}{3}\left(28z^2+68z-25\right)\zeta_2H_0+\frac{16\left(42z^3-121z^2+98z-8\right)H_1^2}{9z}$$

$$+\frac{4\left(274706z^3-812364z^2+549969z-11456\right)}{405z}+\frac{16}{45}\left(144z^4-340z^3+520z^2-2865z+80\right)\frac{\zeta_2}{z}$$

$$+\frac{8\left(2108z^3-7152z^2+4941z+256\right)H_1}{27z}+\gamma_{qg}^0\left[\frac{8}{3}H_1^3+\frac{8}{3}H_0H_1^2\right]$$

$$+\frac{\left(36z^5+155z^4+40z^3-45z^2+20z+1\right)\left[\frac{64}{45}H_{-1}H_0-\frac{64}{45}H_{0,-1}\right]}{z^2}$$

$$-\frac{16\left(172z^3-326z^2-365z-16\right)H_{0,1}}{9z}+\frac{128}{3}\left(z^2-10z+3\right)H_{0,0,-1}$$

$$+\frac{32}{3}\left(8z^2-76z+23\right)H_{0,0,1}-\frac{32}{3}\left(16z^2-8z-5\right)H_{0,1,1}+(z+1)^2\left[-\frac{128}{3}H_0H_{-1}^2\right.$$

$$+\left[\frac{64}{3}H_0^2+\frac{256}{3}H_{0,-1}\right]H_{-1}-\frac{128}{3}\zeta_2H_{-1}-\frac{256}{3}H_{0,-1,-1}\right]+\frac{64}{3}\left(12z^2+44z-15\right)\zeta_3$$

$$+(2z-1)\left[-16H_0^4-64H_{0,0,1}H_0+128\zeta_3H_0+\frac{32\zeta_2^2}{5}-64H_{0,0,1,1}+\left[96H_0^2+\frac{64H_1}{3}\right]\zeta_2\right]\right]$$

$$+(2z-1)\left[\frac{40}{3}\zeta_2H_0^3+\frac{64}{3}\zeta_3H_0^2+\frac{64}{5}\zeta_2^2H_0-32H_{0,0,0,0,1}+32\zeta_5\right]\right]$$

$$+C_FN_FT_F^2\left[-\frac{4}{9}(z-2)(6z+1)H_0^4+\frac{4}{27}\left(404z^2-54z+69\right)H_0^3-\frac{4}{27}\left(3112z^2-1329z-210\right)H_0^2\right.$$

$$-\frac{8\left(94z^3 - 234z^2 + 159z - 16\right)H_1 H_0^2}{9z} + \frac{16}{3}\left(6z^2 + 4z - 11\right)H_{0,1}H_0^2$$

$$-\frac{8}{3}\left(14z^2 - 5z - 11\right)\zeta_2 H_0^2 + \frac{8}{81}\left(27824z^2 + 4929z + 2631\right)H_0 - \frac{32}{3}\left(2z^2 + 28z - 17\right)H_{0,0,1}H_0$$

$$+\frac{16\left(1556z^3 - 1539z^2 + 72z - 80\right)H_1 H_0}{27z} - \frac{32\left(101z^3 + 222z^2 - 39z + 8\right)H_{0,1}H_0}{9z}$$

$$+\frac{8}{9}\left(240z^2 + 166z + 193\right)\zeta_2 H_0 + \frac{32}{9}\left(78z^2 - 133z - 28\right)\zeta_3 H_0 + \frac{16}{9}(z-1)(3z+1)H_1^3$$

$$+\left[-\frac{16}{3}(2z-1)H_0^2 - \frac{32}{9}\left(6z^2 - z - 4\right)H_0 + \frac{8\left(124z^3 - 258z^2 + 159z - 16\right)}{27z} + \frac{8}{9}\gamma_{qg}^0 H_1\right]L_M^3$$

$$+\frac{8}{3}\left(12z^2 - 17z + 4\right)H_1^2 - \frac{64}{15}\left(2z^2 + 58z - 11\right)\zeta_2^2 - \frac{4\left(221158z^3 - 226026z^2 + 17163z - 5248\right)}{243z}$$

$$-\frac{2}{27}\left(7280z^3 - 5646z^2 - 555z - 368\right)\frac{\zeta_2}{z} - \frac{8}{27}\left(3784z^3 + 2046z^2 + 1095z - 16\right)\frac{\zeta_3}{z}$$

$$+\frac{16}{3}\left(20z^2 - 17z - 1\right)H_1 + L_Q^3\left[\frac{16}{3}(2z-1)H_0^2 + \frac{16}{9}(4z-11)H_0 - \frac{16\left(62z^3 - 147z^2 + 84z - 8\right)}{27z}\right.$$

$$\left.+\frac{16}{9}\gamma_{qg}^0 H_1\right] - \frac{16\left(1448z^3 - 1341z^2 + 18z - 80\right)H_{0,1}}{27z} + \frac{16\left(498z^3 + 654z^2 + 3z + 16\right)H_{0,0,1}}{9z}$$

$$+\frac{32}{3}\left(3z^2 - 6z - 1\right)H_{0,1,1} - \frac{32}{3}\left(14z^2 - 74z + 19\right)H_{0,0,0,1} + L_M^2\left[\frac{32}{3}(2z-1)H_0^3\right.$$

$$+\frac{16}{3}(2z+3)(4z-3)H_0^2 - \frac{8}{9}\left(160z^2 + 146z + 305\right)H_0 + \frac{4\left(1000z^3 + 1356z^2 - 2247z - 208\right)}{27z}$$

$$+\frac{32}{9}\left(4z^2 - 4z + 5\right)H_1 + \gamma_{qg}^0\left[\frac{4}{3}H_1^2 + \frac{8}{3}H_{0,1} - \frac{8\zeta_2}{3}\right]\right] + \frac{16}{9}\left(2z^2 - 2z - 5\right)H_1\zeta_2 + \gamma_{qg}^0\left[-\frac{1}{9}H_1^4\right.$$

$$+\frac{8}{3}H_{0,1}H_1^2 - \frac{4}{9}H_0^3 H_1 + \left[-\frac{32}{3}H_{0,0,1} - \frac{32}{3}H_{0,1,1}\right]H_1 + \frac{88}{9}\zeta_3 H_1 + \frac{16}{3}H_{0,1}^2 + H_0\left[\frac{16}{3}H_1 H_{0,1}\right.$$

$$\left.-\frac{32}{3}H_{0,1,1}\right] + \frac{40}{3}H_{0,1,1,1} + \left[-4H_1^2 - \frac{4}{3}H_0 H_1 - \frac{20}{3}H_{0,1}\right]\zeta_2\right] + L_Q^2\left[-\frac{8}{3}\left(8z^2 + 56z - 31\right)H_0^2\right.$$

$$+\frac{8}{9}\left(244z^2 - 236z + 571\right)H_0 + \frac{4\left(2156z^3 - 7632z^2 + 4977z + 256\right)}{27z} + \gamma_{qg}^0\left[-4H_1^2 - \frac{32}{3}H_0 H_1\right]$$

$$+\frac{16\left(130z^3 - 215z^2 + 112z - 8\right)H_1}{9z} - \frac{16}{3}\left(12z^2 - 8z - 5\right)H_{0,1} + \left[-\frac{8}{3}(4z-1)\right.$$

$$+\frac{16}{3}\left(4z^2 - 2z + 1\right)H_0 - \frac{8}{3}\gamma_{qg}^0 H_1\right]L_M - \frac{16}{3}\left(4z^2 - 8z + 13\right)\zeta_2 + (2z-1)\left[-16H_0^3\right.$$

$$\left.\left.+32\zeta_2 H_0 - 32H_{0,0,1} + 32\zeta_3\right]\right] + L_Q\left[\frac{8}{3}\left(16z^2 + 108z - 35\right)H_0^3\right.$$

$$-\frac{8}{45}\left(144z^3 + 1120z^2 - 3745z + 5320\right)H_0^2 - \frac{16\left(1984z^3 - 5407z^2 + 7593z + 4\right)H_0}{45z}$$

$$-\frac{64}{3}\left(5z^2 - 4z + 2\right)H_1 H_0^2 - \frac{32\left(198z^3 - 283z^2 + 140z - 8\right)H_1 H_0}{9z} + \frac{128}{3}(z-2)(3z-2)H_{0,-1}H_0$$

$$+\frac{32}{3}\left(16z^2 - 8z - 5\right)H_{0,1}H_0 - \frac{32}{3}\left(4z^2 + 84z - 33\right)\zeta_2 H_0 - \frac{16\left(168z^3 - 253z^2 + 131z - 8\right)H_1^2}{9z}$$

$$-\frac{8\left(91102z^3 - 318513z^2 + 229863z - 6952\right)}{405z} + \frac{16}{45}\left(144z^4 + 1600z^3 - 1400z^2 + 3045z - 80\right)\frac{\zeta_2}{z}$$

$$
-\frac{8\left(3080z^3 - 8448z^2 + 5247z + 256\right)H_1}{27z} + \frac{16\left(76z^3 - 254z^2 - 329z - 16\right)H_{0,1}}{9z}
$$

$$
+\frac{\left(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1\right)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} + \gamma_{qg}^0\left[\frac{8}{3}H_1^3 + 8H_0H_1^2\right.
$$

$$
+\frac{32}{3}H_{0,1}H_1\right] - \frac{128}{3}\left(7z^2 - 14z + 9\right)H_{0,0,-1} - \frac{32}{3}\left(8z^2 - 60z + 15\right)H_{0,0,1}
$$

$$
+\frac{32}{3}\left(24z^2 - 20z + 1\right)H_{0,1,1} + \frac{64}{3}\left(8z^2 - 6z + 3\right)H_1\zeta_2 + (z+1)^2\left[-\frac{128}{3}H_0H_{-1}^2 + \left[\frac{64}{3}H_0^2\right.\right.
$$

$$
\left.+\frac{256}{3}H_{0,-1}\right]H_{-1} - \frac{128}{3}\zeta_2 H_{-1} - \frac{256}{3}H_{0,-1,-1}\right] + L_M\left[\frac{8}{9}\left(36z^2 + 82z - 61\right)\right.
$$

$$
-\frac{16}{9}\left(76z^2 - 38z + 25\right)H_0 - \frac{16}{9}\left(76z^2 - 88z + 41\right)H_1 + \gamma_{qg}^0\left[\frac{8}{3}H_1^2 + \frac{32}{3}H_0H_1\right]
$$

$$
+\frac{32}{3}\left(4z^2 - 6z + 3\right)H_{0,1} + \left(4z^2 - 2z + 1\right)\left[\frac{32\zeta_2}{3} - \frac{32}{3}H_0^2\right]\right] + \frac{128}{3}\left(4z^2 - 21z + 11\right)\zeta_3
$$

$$
+(2z-1)\left[\frac{44}{3}H_0^4 - 96\zeta_2 H_0^2 + 64H_{0,0,1}H_0 - 128\zeta_3 H_0 - \frac{32\zeta_2^2}{5} + 64H_{0,0,1,1}\right]\right]
$$

$$
+L_M\left[-\frac{8}{9}\left(16z^2 + 28z - 47\right)H_0^3 + \frac{8}{9}\left(422z^2 + 137z + 410\right)H_0^2 - \frac{80}{27}\left(176z^2 - 213z - 297\right)H_0\right.
$$

$$
+\frac{32\left(35z^3 + 29z^2 - 40z + 8\right)H_1 H_0}{9z} - \frac{32}{3}\left(4z^2 - 16z + 17\right)H_{0,1}H_0 - \frac{64}{3}\left(4z^2 - 2z + 1\right)\zeta_2 H_0
$$

$$
+\frac{8}{9}\left(86z^2 - 104z + 67\right)H_1^2 + \frac{8\left(3224z^3 - 19683z^2 + 16893z - 1064\right)}{81z} + \frac{8}{9}\left(152z^2 - 258z + 163\right)H_1
$$

$$
+\frac{16\left(16z^3 - 88z^2 + 131z - 16\right)H_{0,1}}{9z} - \frac{128}{3}\left(5z - 7\right)H_{0,0,1} - \frac{32}{3}\left(12z^2 - 14z + 7\right)H_{0,1,1}
$$

$$
-\frac{16}{9}\left(86z^2 - 30z + 51\right)\zeta_2 + \gamma_{qg}^0\left[-\frac{32}{3}H_1 H_0^2 - \frac{8}{3}H_1^2 H_0 - \frac{16}{3}H_1 H_{0,1} + \frac{32}{3}H_1\zeta_2\right]
$$

$$
+\frac{32}{3}\left(16z^2 + 2z - 19\right)\zeta_3 + (2z-1)\left[-\frac{20}{3}H_0^4 + 64H_{0,0,1}H_0 + 128\zeta_3 H_0 + \frac{384\zeta_2^2}{5} - 192H_{0,0,0,1}\right]\right]
$$

$$
+(2z-1)\left[-\frac{4}{15}H_0^5 - 8\zeta_2 H_0^3 + 32H_{0,0,1}H_0^2 + \frac{208}{3}\zeta_3 H_0^2 - 128H_{0,0,0,1}H_0 + 128H_{0,0,0,0,1} - 128\zeta_5\right]\right]
$$

$$
+C_F C_A T_F\left[-\frac{2}{15}H_0^5 + \frac{1}{18}\left(16z^2 + 178z + 13\right)H_0^4 - \frac{1}{27}\left(7624z^2 + 1752z + 1131\right)H_0^3\right.
$$

$$
+\frac{16}{3}(z+1)(3z+2)H_{-1}H_0^3 - \frac{4\left(404z^3 - 334z^2 - 25z - 40\right)H_1 H_0^3}{9z} + \frac{16}{3}\left(2z^2 + 16z + 7\right)H_{0,1}H_0^3
$$

$$
+\frac{8}{3}\left(4z^2 + 5z + 4\right)\zeta_2 H_0^3 - 2\left(20z^2 + 24z + 19\right)H_{-1}^2 H_0^2 - \frac{4\left(151z^3 - 138z^2 - 16\right)H_1^2 H_0^2}{3z}
$$

$$
+\frac{1}{9}\left(10622z^2 - 2987z - 346\right)H_0^2 - 4\left(4z^2 - 7\right)H_{-1}H_0^2 + \frac{2\left(124z^3 - 1392z^2 + 1269z - 40\right)H_1 H_0^2}{9z}
$$

$$
-12(4z+3)H_{0,-1}H_0^2 - \frac{8\left(13z^3 + 46z^2 - 23z + 20\right)H_{0,1}H_0^2}{3z} + 16\left(2z^2 - 28z - 9\right)H_{0,0,1}H_0^2
$$

$$
-16\left(14z^2 - 26z + 1\right)H_{0,1,1}H_0^2 - \frac{1}{3}\left(128z^2 - 106z - 49\right)\zeta_2 H_0^2 + \frac{16}{3}\left(6z^2 + 13z + 10\right)\zeta_3 H_0^2
$$

$$
-\frac{4\left(94z^3 - 48z^2 - 3z - 16\right)H_1^3 H_0}{9z} + \frac{4\left(728z^3 - 1068z^2 + 303z - 8\right)H_1^2 H_0}{9z}
$$

$$
+16\left(6z^2 - 10z - 1\right)H_{0,1}^2 H_0 - \frac{16}{5}\left(103z + 11\right)\zeta_2^2 H_0 - \frac{2}{81}\left(9514z^2 - 34311z + 17499\right)H_0
$$

$$+\frac{4\left(12769z^3-12504z^2-1401z+740\right)H_1H_0}{27z}+\frac{4}{3}\left(314z^3-242z^2-47z-48\right)\frac{\zeta_2}{z}H_1H_0$$

$$+8\left(2z^2+14z-3\right)H_{0,-1}H_0+8\left(4z^2+5\right)H_{-1}H_{0,-1}H_0-\frac{8\left(782z^3+387z^2+705z-20\right)H_{0,1}H_0}{9z}$$

$$+\frac{32\left(54z^3-56z^2+z-4\right)H_1H_{0,1}H_0}{3z}-8\left(4z^2-16z-11\right)H_{0,-1,-1}H_0-16\left(2z^2-6z-5\right)H_{0,0,-1}H_0$$

$$-\frac{8}{3}\left(448z^2-328z-49\right)H_{0,1,1}H_0+\frac{8\left(370z^3-88z^2-133z+40\right)H_{0,0,1}H_0}{3z}$$

$$-32\left(10z^2-34z-7\right)H_{0,0,0,1}H_0+16\left(8z^2-22z+1\right)H_{0,0,1,1}H_0+16\left(40z^2-22z+23\right)H_{0,1,1,1}H_0$$

$$+\frac{2}{9}\left(1180z^2-1409z-347\right)\zeta_2H_0-8\left(17z^2+27z+13\right)H_{-1}\zeta_2H_0+80(z+1)^2H_{0,-1}\zeta_2H_0$$

$$-8\left(26z^2+14z+29\right)H_{0,1}\zeta_2H_0-\frac{4}{9}\left(1828z^2+134z+131\right)\zeta_3H_0-\frac{2\left(12z^3+31z^2-56z-4\right)H_1^4}{9z}$$

$$-\frac{4\left(57z^3-94z^2+24z+4\right)H_1^3}{3z}-\frac{2\left(2369z^3-3096z^2+2241z+142\right)H_1^2}{27z}$$

$$-\frac{4}{3}\left(290z^3-306z^2+57z-8\right)\frac{\zeta_2}{z}H_1^2-64(z+1)H_{0,-1}^2+\frac{4\left(384z^3-256z^2+35z-16\right)H_{0,1}^2}{3z}$$

$$+\frac{4}{15}\left(2354z^2-3310z+101\right)\zeta_2^2-\frac{54588z^3-98666z^2-2299z+8388}{81z}-\frac{1}{18}\big(11000z^3+11030z^2$$

$$-1283z+1440\big)\frac{\zeta_2}{z}-\frac{4\left(30389z^3-44025z^2+11274z-365\right)H_1}{81z}-\frac{4}{9}\big(2752z^3-3646z^2+593z$$

$$-72\big)\frac{\zeta_2}{z}H_1+\frac{4}{9}\left(1088z^3-1036z^2+65z-64\right)\frac{\zeta_3}{z}H_1+\left(35z^2+40z+21\right)\Big[8H_{0,-1}-8H_{-1}H_0\Big]$$

$$+\frac{8\left(84z^3-70z^2+5z-8\right)H_1^2H_{0,1}}{3z}-\frac{4\left(9292z^3-15039z^2-1215z+740\right)H_{0,1}}{27z}$$

$$-\frac{4}{3}\left(150z^3-338z^2+169z-48\right)\frac{\zeta_2}{z}H_{0,1}+\frac{16}{3}(z-1)(15z+1)H_1H_{0,1}+\left(3z^2+z-5\right)\Big[8H_0H_{-1}^2$$

$$-16H_{0,-1}H_{-1}+16H_{0,-1,-1}\Big]-8(28z+1)H_{0,0,-1}+\frac{8\left(568z^3+2052z^2+801z-20\right)H_{0,0,1}}{9z}$$

$$-\frac{8\left(432z^3-436z^2+5z-32\right)H_1H_{0,0,1}}{3z}+48(6z+5)H_{0,1}H_{0,0,1}+\left(4z^2+8z+1\right)\Big[16H_{-1}H_0H_{0,1}$$

$$+H_0\Big[-16H_{0,-1,1}-16H_{0,1,-1}\Big]\Big]+(z-2)(z+1)\Big[-16H_{-1}H_{0,1}+16H_{0,-1,1}+16H_{0,1,-1}\Big]$$

$$-\frac{8\left(2599z^3-2553z^2+459z-8\right)H_{0,1,1}}{9z}-\frac{8\left(336z^3-292z^2+23z-32\right)H_1H_{0,1,1}}{3z}$$

$$-32\left(14z^2-6z+9\right)H_{0,1}H_{0,1,1}+\left(z^2+z+1\right)\Big[32H_0H_{-1}^3-96H_{0,-1}H_{-1}^2$$

$$+192H_{0,-1,-1}H_{-1}-192H_{0,-1,-1,-1}\Big]+8(2z-1)(2z+5)H_{0,-1,0,1}$$

$$+(2z+1)(2z+3)\Big[24H_{-1}H_{0,0,-1}-24H_{0,0,-1,-1}\Big]+\left(4z^2+2z+1\right)\Big[32H_0H_{0,0,-1,-1}$$

$$-8H_0^2H_{0,-1,-1}\Big]-8(20z+11)H_{0,0,0,-1}-\frac{8\left(590z^3-248z^2-233z+40\right)H_{0,0,0,1}}{3z}$$

$$+\left(4z^2+16z-3\right)\Big[-8H_{-1}H_{0,0,1}+8H_{0,0,-1,1}+8H_{0,0,1,-1}\Big]+\frac{8}{3}\left(512z^2-240z-183\right)H_{0,0,1,1}$$

$$+\left(8z^2+12z+7\right)\Big[-8H_{0,1}H_{-1}^2+\Big[16H_{0,-1,1}+16H_{0,1,-1}\Big]H_{-1}-16H_{0,-1,-1,1}-16H_{0,-1,1,-1}$$

$$-16H_{0,1,-1,-1}\Big]+z(z+1)\Big[64H_{-1}H_{0,1,1}-64H_{0,-1,1,1}-64H_{0,1,-1,1}-64H_{0,1,1,-1}\Big]$$

$$+64\left(5z^2+6z+3\right)H_{0,-1,-1,0,1}+\frac{16\left(251z^3-139z^2+17z-32\right)H_{0,1,1,1}}{3z}+z^2\Big[64H_0H_{0,-1,0,1}$$

$$+64H_{0,0,-1,0,1}\Big] + 32\big(20z^2 - 36z - 3\big)H_{0,0,0,0,1} + \big(16z^2 - 2z - 1\big)\Big[16H_{0,0,0,-1,1} + 16H_{0,0,0,1,-1}\Big]$$

$$-16\big(8z^2 + 86z + 75\big)H_{0,0,0,1,1} + \big(4z^2 - 2z - 1\big)\Big[-8H_{0,0,-1}H_0^2 + 16H_{0,0,0,-1}H_0 - 16H_{0,-1}H_{0,0,1}$$

$$-80H_{0,0,0,-1,-1} + 16H_{0,0,1,0,-1}\Big] - 16\big(8z^2 + 46z + 37\big)H_{0,0,1,0,1} + 16\big(120z^2 - 50z + 79\big)H_{0,0,1,1,1}$$

$$+16\big(56z^2 - 10z + 41\big)H_{0,1,0,1,1} + 16\big(32z^2 - 18z + 15\big)H_{0,1,1,1,1} + 8\big(14z^2 + 18z + 13\big)H_{-1}^2\zeta_2$$

$$+8\big(5z^2 - z - 9\big)H_{-1}\zeta_2 + 8(z + 1)(z + 14)H_{0,-1}\zeta_2 - 16\big(2z^2 + 6z + 3\big)H_{0,-1,-1}\zeta_2$$

$$+8\big(2z^2 - 14z - 7\big)H_{0,0,-1}\zeta_2 + 8\big(26z^2 + 50z + 47\big)H_{0,0,1}\zeta_2 + 64\big(8z^2 - 7z + 5\big)H_{0,1,1}\zeta_2$$

$$-48\gamma_{qg}^0\log(2)\zeta_2 + L_M^3\bigg[-\frac{16}{3}(4z + 1)H_0^2 + \frac{4}{9}\big(292z^2 - 94z + 11\big)H_0 + \frac{2}{3}\big(40z^2 - 20z - 31\big)$$

$$+\frac{8\big(146z^3 - 142z^2 + 5z - 16\big)H_1}{9z} + \gamma_{qg}^0\Big[-\frac{16}{3}H_1^2 - \frac{16}{3}H_0H_1\Big] - 32(2z + 1)H_{0,1}$$

$$-\frac{32}{3}\big(4z^2 - 10z - 1\big)\zeta_2\bigg] + L_Q^3\bigg[-\frac{8}{3}(4z + 1)H_0^2 + \frac{8}{9}\big(106z^2 - 40z + 11\big)H_0 + \frac{4}{3}\big(10z^2 - 16z - 5\big)$$

$$+\frac{8\big(106z^3 - 104z^2 + 19z - 8\big)H_1}{9z} + \gamma_{qg}^0\Big[-\frac{8}{3}H_1^2 - \frac{8}{3}H_0H_1\Big] - 16(2z + 1)H_{0,1}$$

$$-\frac{16}{3}\big(4z^2 - 10z - 1\big)\zeta_2\bigg] + \frac{2}{3}\big(1360z^2 - 4980z + 33\big)\zeta_3 - 4\big(52z^2 + 56z + 43\big)H_{-1}\zeta_3$$

$$-8\big(48z^2 - 34z + 33\big)H_{0,1}\zeta_3 + \frac{4}{3}\big(236z^2 - 638z - 125\big)\zeta_2\zeta_3$$

$$+L_Q^2\bigg[8(4z + 1)H_0^3 - \frac{8}{3}\big(141z^2 - 49z + 14\big)H_0^2 - \frac{2}{9}\big(2320z^2 - 1126z - 109\big)H_0$$

$$-\frac{8\big(248z^3 - 232z^2 + 53z - 16\big)H_1H_0}{3z} + 32\big(z^2 + 6z + 3\big)H_{0,1}H_0 + 16\big(8z^2 - 20z - 3\big)\zeta_2H_0$$

$$-\frac{8\big(131z^3 - 123z^2 + 30z - 8\big)H_1^2}{3z} - \frac{52z^3 - 536z^2 + 193z - 48}{3z}$$

$$-\frac{4\big(1052z^3 - 1078z^2 + 191z + 32\big)H_1}{9z} - \frac{8\big(66z^3 + 80z^2 - 31z + 16\big)H_{0,1}}{3z}$$

$$-32\big(z^2 + 7z + 2\big)H_{0,0,1} + 16(14z + 5)H_{0,1,1} + \bigg[\frac{22}{3}(4z - 1) - \frac{44}{3}\big(4z^2 - 2z + 1\big)H_0$$

$$+\frac{22}{3}\gamma_{qg}^0H_1\bigg]L_M + \frac{16}{3}\big(157z^2 - 85z + 11\big)\zeta_2 + \big(2z^2 + 2z + 1\big)\Big[-16H_0H_{0,-1} + 24H_{0,-1}$$

$$+H_{-1}\Big[8H_0^2 - 24H_0 + 32H_{0,1}\Big] - 32H_{0,-1,1} + 16H_{0,0,-1} - 32H_{0,1,-1} - 32H_{-1}\zeta_2\Big] + \gamma_{qg}^0\Big[4H_1^3$$

$$+12H_0H_1^2 + 10H_0^2H_1 - 24\zeta_2H_1\Big] + 64z^2\zeta_3\bigg] + L_M^2\bigg[-\frac{8}{3}(4z + 1)H_0^3 + 8\big(9z^2 - 11z - 1\big)H_0^2$$

$$+\frac{2}{9}\big(3584z^2 - 1850z - 455\big)H_0 + \frac{8\big(208z^3 - 192z^2 + 9z - 16\big)H_1H_0}{3z} + 32\big(z^2 - 8z - 2\big)H_{0,1}H_0$$

$$-16\big(8z^2 - 12z + 1\big)\zeta_2H_0 + \frac{8\big(157z^3 - 161z^2 + 22z - 8\big)H_1^2}{3z} + \frac{1556z^3 - 868z^2 - 477z - 48}{3z}$$

$$+\frac{8\big(1022z^3 - 1051z^2 - 28z - 64\big)H_1}{9z} - \frac{16\big(33z^3 + 35z^2 + 5z - 8\big)H_{0,1}}{3z} - 32\big(z^2 - 13z - 3\big)H_{0,0,1}$$

$$-16(10z + 7)H_{0,1,1} - \frac{8}{3}\big(142z^2 - 244z - 1\big)\zeta_2 + \big(2z^2 + 2z + 1\big)\Big[-16H_0H_{0,-1} + 24H_{0,-1}$$

$$+H_{-1}\Big[8H_0^2 - 24H_0 + 32H_{0,1}\Big] - 32H_{0,-1,1} + 16H_{0,0,-1} - 32H_{0,1,-1} - 32H_{-1}\zeta_2\Big] + \gamma_{qg}^0\Big[-12H_1^3$$

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$$-20H_0H_1^2 - 6H_0^2H_1 + 40\zeta_2H_1\Big] - 96\big(2z^2+1\big)\zeta_3\Big] + \big(2z^2+2z+1\big)\Big[8H_0H_{-1}^4 + \big(-8H_0^2$$

$$-32H_{0,-1}\big)H_{-1}^3 + \Big[-\frac{16}{3}H_0^3 + \Big[48H_{0,-1} - 64H_{0,1}\Big]H_0 + 96H_{0,-1,-1} - 48H_{0,0,-1} + 112H_{0,0,1}$$

$$-32H_{0,1,1}\Big]H_{-1}^2 + 72\zeta_2^2H_{-1} + \Big[\frac{8}{3}H_0^4 + \Big[16H_{0,-1} + 32H_{0,1}\Big]H_0^2 + \Big[-96H_{0,-1,-1}$$

$$+128H_{0,-1,1} - 64H_{0,0,-1} - 32H_{0,0,1} + 128H_{0,1,-1} + 64H_{0,1,1}\Big]H_0 - 192H_{0,-1,-1,-1}$$

$$-160H_{0,-1,0,1} + 64H_{0,-1,1,1} + 96H_{0,0,-1,-1} - 224H_{0,0,-1,1} + 160H_{0,0,0,-1} - 32H_{0,0,0,1} - 224H_{0,0,1,-1}$$

$$-96H_{0,0,1,1} + 64H_{0,1,-1,1} + 64H_{0,1,1,-1}\Big]H_{-1} + H_0^2\Big[-32H_{0,-1,1} - 32H_{0,1,-1}\Big] + H_0\Big[96H_{0,-1,-1,-1}$$

$$-128H_{0,-1,-1,1} - 128H_{0,-1,1,-1} - 64H_{0,-1,1,1} + 32H_{0,0,-1,1} + 32H_{0,0,1,-1} - 128H_{0,1,-1,-1}$$

$$-64H_{0,1,-1,1} - 64H_{0,1,1,-1}\Big] + 192H_{0,-1,-1,-1,-1} - 64H_{0,-1,-1,1,1} + 160H_{0,-1,0,-1,1}$$

$$+160H_{0,-1,0,1,-1}$$

$$+32H_{0,-1,0,1,1} - 64H_{0,-1,1,-1,1} - 64H_{0,-1,1,1,-1} - 96H_{0,0,-1,-1,-1} + 224H_{0,0,-1,-1,1}$$

$$+224H_{0,0,-1,1,-1}$$

$$+96H_{0,0,-1,1,1} + 224H_{0,0,1,-1,-1} + 96H_{0,0,1,-1,1} + 96H_{0,0,1,1,-1} - 64H_{0,1,-1,-1,1} - 64H_{0,1,-1,1,-1}$$

$$-64H_{0,1,1,-1,-1} + \Big[16H_{-1}^3 + 24H_0H_{-1}^2 + \Big[-60H_0^2 + 16H_{0,-1} - 96H_{0,1}\Big]H_{-1} + 96H_{0,-1,1}$$

$$+96H_{0,1,-1}\Big]\zeta_2 + \Big[-56H_{-1}^2 - 16H_0H_{-1}\Big]\zeta_3\Big] + \gamma_{qg}^0\Big[\frac{1}{3}H_1H_0^4 + 2H_1^2H_0^3$$

$$+\Big[\frac{2}{3}H_1^3 - 16H_1H_{0,1}\Big]H_0^2 + \Big[-\frac{5}{3}H_1^4 - 32H_{0,1}H_1^2 + \Big[24H_{0,0,1} + 88H_{0,1,1}\Big]H_1\Big]H_0$$

$$-\frac{434}{5}H_1\zeta_2^2 - \frac{16}{3}H_1^3H_{0,1} + H_1^2\Big[52H_{0,0,1} + 20H_{0,1,1}\Big] + H_1\Big[-28H_{0,1}^2 + 16H_{0,0,0,1} - 24H_{0,0,1,1}\Big]$$

$$+\Big[16H_1^3 + 20H_0H_1^2 - H_0^2H_1 + 32H_{0,1}H_1\Big]\zeta_2 + \Big[-\frac{14}{3}H_1^2 - \frac{116}{3}H_0H_1\Big]\zeta_3\Big]$$

$$+\big(2z+1\big)\Big[-\frac{16}{3}H_{0,-1}H_0^3 + 8H_{0,-1}^2H_0 - 48H_{0,-1}H_{0,0,-1} + 80H_{0,0,-1,0,-1} + 32H_{0,0,0,0,-1} + 8H_{0,-1}\zeta_3\Big]$$

$$+L_Q\Big[-\frac{4}{3}(18z+5)H_0^4 + \frac{8}{45}\big(144z^3 + 2305z^2 - 610z + 225\big)H_0^3 - \frac{16}{3}\big(18z^2 + 22z + 11\big)H_{-1}H_0^3$$

$$+\frac{16}{3}\big(6z^2 - 2z + 1\big)H_1H_0^3 + 8\big(18z^2 - 14z + 7\big)H_1^2H_0^2 + 8\big(4z^2 - 6z + 9\big)H_{0,-1}H_0^2$$

$$+\frac{4\big(2016z^4 + 16274z^3 - 10977z^2 - 602z + 24\big)H_0^2}{45z} + \frac{8\big(303z^3 - 364z^2 + 161z - 16\big)H_1H_0^2}{3z}$$

$$-\frac{4\big(288z^5 + 1140z^4 + 1000z^3 + 295z^2 + 80z + 8\big)H_{-1}H_0^2}{15z^2} - 24\big(4z^2 - 18z - 5\big)\zeta_2H_0^2$$

$$+\frac{8\big(398z^3 - 366z^2 + 69z - 32\big)H_1^2H_0}{3z} + \frac{4\big(242z^3 + 11758z^2 + 3445z + 64\big)H_0}{45z}$$

$$+\frac{8\big(1834z^3 - 1520z^2 + 223z + 64\big)H_1H_0}{9z} - \frac{8\big(1260z^4 - 300z^3 - 1655z^2 - 80z - 8\big)H_{0,-1}H_0}{15z^2}$$

$$+\frac{8\big(518z^3 + 172z^2 - 257z + 32\big)H_{0,1}H_0}{3z} + 32\big(2z^2 - 6z + 3\big)H_1H_{0,1}H_0$$

$$+32\big(12z^2 + 38z - 5\big)H_{0,-1,-1}H_0 + 64\big(11z^2 + 12z + 8\big)H_{0,0,-1}H_0 - 32\big(18z^2 + 14z + 19\big)H_{0,0,1}H_0$$

$$+\big(2z^2 - 10z + 1\big)\Big[-64H_{0,-1,1} - 64H_{0,1,-1}\Big]H_0 - 32\big(2z^2 + 22z + 13\big)H_{0,1,1}H_0$$

$$-\frac{8}{5}\big(192z^3 + 2020z^2 - 320z + 85\big)\zeta_2H_0 + 32\big(34z^2 + 42z + 21\big)H_{-1}\zeta_2H_0 + 256(z-1)^2H_1\zeta_2H_0$$

$$-32\big(4z^2 + 58z + 3\big)\zeta_3H_0 + \frac{8\big(93z^3 - 82z^2 + 26z - 8\big)H_1^3}{3z} + \frac{8\big(518z^3 - 346z^2 + 65z + 32\big)H_1^2}{9z}$$

$$-16\left(4z^2 + 26z - 11\right)H_{0,-1}^2 + 32\left(5z^2 - 8z + 4\right)H_{0,1}^2 + \frac{8}{5}\left(188z^2 - 234z + 39\right)\zeta_2^2$$

$$-\frac{2\left(67432z^3 - 39737z^2 - 18123z - 2512\right)}{45z} - \frac{4}{45}\left(4032z^4 + 29968z^3 - 22628z^2 + 1731z + 96\right)\frac{\zeta_2}{z}$$

$$-\frac{8}{3}\left(144z^4 - 6z^3 + 646z^2 + 49z + 32\right)\frac{\zeta_3}{z} - \frac{2\left(32084z^3 - 22856z^2 - 10349z - 544\right)H_1}{45z}$$

$$+\frac{16}{15}\left(216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6\right)\frac{\zeta_2}{z^2}H_{-1}$$

$$-\frac{16}{15}\left(72z^5 + 1610z^4 - 1585z^3 + 495z^2 - 120z - 2\right)\frac{\zeta_2}{z^2}H_1$$

$$+\frac{\left(1008z^5 - 558z^4 - 6071z^3 - 4851z^2 + 12z + 28\right)}{z^2}\left[\frac{16}{45}H_{0,-1} - \frac{16}{45}H_{-1}H_0\right]$$

$$+\frac{4\left(11628z^3 + 16856z^2 - 499z - 592\right)H_{0,1}}{45z} + \frac{16}{3}\left(44z^2 - 32z + 19\right)H_1 H_{0,1}$$

$$+\frac{\left(72z^5 + 600z^4 + 405z^3 - 55z^2 + 40z + 2\right)\left[\frac{16}{15}H_0 H_{-1}^2 - \frac{32}{15}H_{0,-1}H_{-1} + \frac{32}{15}H_{0,-1,-1}\right]}{z^2}$$

$$+\frac{8\left(288z^5 + 3660z^4 + 400z^3 - 3015z^2 - 80z - 8\right)H_{0,0,-1}}{15z^2} + \gamma_{qg}^0\left[-24H_{0,1}H_0^2 + \left[-\frac{40}{3}H_1^3\right.\right.$$

$$\left.\left. -48H_{0,-1}H_1\right]H_0 + 8H_1^2 H_{0,1} - 48H_{0,-1}H_{0,1} + 96H_1 H_{0,0,-1}\right] - 128\left(4z^2 - 6z + 3\right)H_1 H_{0,0,1}$$

$$+\frac{8\left(288z^4 - 2150z^3 + 260z^2 + 1215z - 160\right)H_{0,0,1}}{15z} - 128\left(5z^2 - 6z + 3\right)H_{0,-1,0,1}$$

$$+\frac{\left(144z^5 + 300z^4 + 670z^3 + 615z^2 + 4\right)\left[-\frac{16}{15}H_{-1}H_{0,1} + \frac{16}{15}H_{0,-1,1} + \frac{16}{15}H_{0,1,-1}\right]}{z^2}$$

$$+\frac{8\left(34z^3 + 304z^2 - 107z + 64\right)H_{0,1,1}}{3z} + \left(4z^2 + 6z + 3\right)\left[-\frac{128}{3}H_0 H_{-1}^3 + \left[48H_0^2 + 128H_{0,-1}\right]H_{-1}^2\right.$$

$$\left. + \left[-64H_0 H_{0,-1} - 256H_{0,-1,-1} - 64H_{0,0,-1}\right]H_{-1} + 256H_{0,-1,-1,-1} + 64H_{0,0,-1,-1}\right]$$

$$-16\left(108z^2 + 82z + 101\right)H_{0,0,0,-1} + 16\left(36z^2 + 98z + 61\right)H_{0,0,0,1} + (z+1)^2\left[-128H_{-1}H_{0,0,1}\right.$$

$$\left. +128H_{0,0,-1,1} + 128H_{0,0,1,-1}\right] + 64\left(3z^2 + 13z + 6\right)H_{0,0,1,1} + \left(3z^2 + 4z + 2\right)\left[128H_{0,1}H_{-1}^2\right.$$

$$\left. + \left[-256H_{0,-1,1} - 256H_{0,1,-1}\right]H_{-1} + 256H_{0,-1,-1,1} + 256H_{0,-1,1,-1} + 256H_{0,1,-1,-1}\right]$$

$$+\left(2z^2 + 2z + 1\right)\left[H_{-1}\left[-128H_0 H_{0,1} - 128H_{0,1,1}\right] + 128H_{0,-1,1,1} + 128H_{0,1,-1,1} + 128H_{0,1,1,-1}\right]$$

$$+128\left(z^2 - 7z - 1\right)H_{0,1,1,1} - 64\left(10z^2 + 14z + 7\right)H_{-1}^2\zeta_2 - 64(2z-1)H_1^2\zeta_2$$

$$-16\left(8z^2 + 26z + 13\right)H_{0,-1}\zeta_2 - 16\left(28z^2 - 74z + 13\right)H_{0,1}\zeta_2 + L_M^2\left[8(4z+1)H_0^2\right.$$

$$\left. -\frac{16}{3}z(31z - 9)H_0 - \frac{8}{3}(z-1)(15z + 13) - \frac{8\left(62z^3 - 60z^2 - 3z - 8\right)H_1}{3z} + \gamma_{qg}^0\left[8H_1^2 + 8H_0 H_1\right]\right.$$

$$\left. +48(2z+1)H_{0,1} + 16\left(4z^2 - 10z - 1\right)\zeta_2\right] + 16\left(74z^2 + 102z + 51\right)H_{-1}\zeta_3$$

$$+112\left(10z^2 - 14z + 7\right)H_1\zeta_3 + L_M\left[-\frac{16}{3}(4z+1)H_0^3 + \frac{8}{3}\left(114z^2 - 16z + 17\right)H_0^2\right.$$

$$\left. -\frac{4}{9}\left(632z^2 - 362z - 173\right)H_0 + \frac{32}{3}\left(10z^2 - 10z + 11\right)H_1 H_0 + 64(2z+1)\zeta_2 H_0\right.$$

$$\left. -\frac{16}{3}\left(13z^2 - 19z - 4\right)H_1^2 - \frac{2}{3}\left(752z^2 - 166z - 335\right) - \frac{4\left(992z^3 - 1024z^2 - 247z - 160\right)H_1}{9z}\right.$$

$$+8\left(44z^2+50z-7\right)H_{0,1}+32\left(2z^2-6z-1\right)H_{0,0,1}-32(2z-1)H_{0,1,1}-\frac{8}{3}\left(172z^2+74z+23\right)\zeta_2$$

$$+\left(2z^2+2z+1\right)\left[32H_0H_{0,-1}-48H_{0,-1}+H_{-1}\left[-16H_0^2+48H_0-64H_{0,1}\right]+64H_{0,-1,1}\right.$$

$$\left.-32H_{0,0,-1}+64H_{0,1,-1}+64H_{-1}\zeta_2\right]+\gamma_{qg}^0\left[8H_1^3-4H_0^2H_1-16\zeta_2H_1+H_0\left[8H_1^2+8H_{0,1}\right]\right]\right]$$

$$+32\left(4z^2+3\right)\zeta_3\Big]\Big]+L_M\left[\frac{4}{3}(10z+3)H_0^4-\frac{8}{9}\left(293z^2-56z+34\right)H_0^3\right.$$

$$-\frac{4}{45}\left(432z^3+8165z^2-5075z+385\right)H_0^2+4\left(8z^2-16z-27\right)H_{-1}H_0^2$$

$$-\frac{8\left(159z^3-232z^2+113z-8\right)H_1H_0^2}{3z}-8\left(12z^2+10z+9\right)H_{0,-1}H_0^2-32\left(4z^2-2z+1\right)H_{0,1}H_0^2$$

$$-8\left(4z^2+14z+11\right)\zeta_2H_0^2-\frac{16\left(13z^3-5z^2-5z-8\right)H_1^2H_0}{3z}$$

$$+\frac{4\left(27226z^3-15808z^2-3393z-24\right)H_0}{45z}+\frac{8\left(461z^3-676z^2-73z-184\right)H_1H_0}{9z}$$

$$+8\left(16z^2+52z-61\right)H_{0,-1}H_0-\frac{8\left(218z^3+572z^2-193z+16\right)H_{0,1}H_0}{3z}$$

$$-32\left(8z^2+14z-1\right)H_{0,-1,-1}H_0-64\left(5z^2+2\right)H_{0,0,-1}H_0+32\left(18z^2+6z+11\right)H_{0,0,1}H_0$$

$$+\left(2z^2-10z+1\right)\left[64H_{0,-1,1}+64H_{0,1,-1}\right]H_0-64\left(3z^2-10z-1\right)H_{0,1,1}H_0$$

$$+\frac{8}{3}\left(552z^2+328z+7\right)\zeta_2H_0-64\left(2z^2-29z-4\right)\zeta_3H_0+8\left(39z^2-40z+4\right)H_1^3$$

$$+\frac{2\left(5614z^3-6188z^2-77z-272\right)H_1^2}{9z}+16\left(4z^2+10z-3\right)H_{0,-1}^2-96\left(z^2+1\right)H_{0,1}^2$$

$$-\frac{8}{5}\left(12z^2-162z-79\right)\zeta_2^2+\frac{2\left(53302z^3-22043z^2-25297z-552\right)}{45z}$$

$$-\frac{16}{3}\left(501z^3-490z^2+26z-24\right)\frac{\zeta_3}{z}+\frac{2\left(14956z^3-14194z^2-1243z-344\right)H_1}{9z}$$

$$-\frac{16}{3}\left(32z^3-z^2-49z+8\right)\frac{\zeta_2}{z}H_1+\frac{\left(72z^5+15z^4-1105z^3-1215z^2+2\right)\left[\frac{16}{15}H_{-1}H_0-\frac{16}{15}H_{0,-1}\right]}{z^2}$$

$$+\frac{4\left(252z^3-3316z^2+65z+368\right)H_{0,1}}{9z}+\frac{32\left(20z^3-19z^2-7z-4\right)H_1H_{0,1}}{3z}$$

$$+(z+1)(2z-5)\left[-48H_0H_{-1}^2+96H_{0,-1}H_{-1}-96H_{0,-1,-1}\right]-8\left(40z^2+88z-149\right)H_{0,0,-1}$$

$$+\frac{8\left(202z^3+460z^2-167z+16\right)H_{0,0,1}}{3z}+\left(20z^2+34z+17\right)\left[16H_{-1}H_{0,1}-16H_{0,-1,1}-16H_{0,1,-1}\right]$$

$$-\frac{16}{3}\left(53z^2+209z-10\right)H_{0,1,1}+16\left(68z^2+2z+37\right)H_{0,0,0,-1}-16\left(52z^2+42z+41\right)H_{0,0,0,1}$$

$$+32\left(6z^2-10z+1\right)H_{0,0,1,1}-32\left(8z^2-14z+7\right)H_{0,1,1,1}+\frac{4}{45}\left(864z^3-5870z^2+10080z+405\right)\zeta_2$$

$$-16\left(26z^2+25z+2\right)H_{-1}\zeta_2+16\left(16z^2+26z+13\right)H_{0,-1}\zeta_2+16\left(20z^2-30z+11\right)H_{0,1}\zeta_2$$

$$+\left(2z^2+2z+1\right)\left[\frac{128}{3}H_0H_{-1}^3+\left[-48H_0^2-128H_{0,-1}-128H_{0,1}\right]H_{-1}^2+\left[\frac{112}{3}H_0^3\right.\right.$$

$$+\left[64H_{0,-1}+128H_{0,1}\right]H_0+256H_{0,-1,-1}+256H_{0,-1,1}+64H_{0,0,-1}+256H_{0,1,-1}+128H_{0,1,1}\right]H_{-1}$$

$$-368\zeta_3H_{-1}-256H_{0,-1,-1,-1}-256H_{0,-1,-1,1}-256H_{0,-1,1,-1}-128H_{0,-1,1,1}-64H_{0,0,-1,-1}$$

$$-256H_{0,1,-1,-1}-128H_{0,1,-1,1}-128H_{0,1,1,-1}+\left[192H_{-1}^2-416H_{-1}H_0\right]\zeta_2\right]+\gamma_{qg}^0\left[-8H_1^4+6H_0^2H_1^2\right.$$

$$-16H_{0,1}H_1^2+\frac{4}{3}H_0^3H_1+\left[-96H_{0,0,-1}-16H_{0,0,1}\right]H_1+196\zeta_3H_1+H_0\left[H_1\left[48H_{0,-1}-24H_{0,1}\right]\right.$$

$$-\frac{16}{3}H_1^3 + 48H_{0,-1}H_{0,1} - 64H_{0,-1,0,1} + \left[48H_1^2 + 80H_0H_1\right]\zeta_2\Big]\Big] - 4\big(212z^2 - 174z + 1\big)\zeta_5\Big]$$

$$+a_{Qg}^{(3)} + \tilde{C}_{2,g}^{\mathsf{S},(3)}(N_F + 1)\Bigg\} \ . \tag{598}$$

Numerical implementations of the harmonic polylogarithms were given in Refs. [66].

C The Longitudinal Wilson Coefficients in z-space

The Wilson coefficients for the longitudinal structure function $F_L(x, Q^2)$ in the asymptotic region in z-space are presented in the following. $L_{q,L}^{\mathsf{PS}}$ and $L_{g,L}^{\mathsf{S}}$ read :

$$L_{q,L}^{\mathsf{PS}} =$$

$$a_s^3\Bigg\{ C_F N_F T_F^2\Bigg[L_M\Bigg[z\Big[\frac{256}{3}H_{0,1} - \frac{256\zeta_2}{3}\Big] - \frac{256(z-1)\big(2z^2 + 2z - 1\big)H_1}{9z}$$

$$-\frac{256}{9}z(2z+11)H_0 + \frac{256(z-1)\big(19z^2 + 16z - 5\big)}{27z}\Bigg] + z(2z+11)\Big[\frac{128}{9}H_{0,1} - \frac{128\zeta_2}{9}\Big]$$

$$+z\Big[\frac{128\zeta_3}{3} - \frac{128}{3}H_{0,1,1}\Big] + L_Q^2\Big[\frac{128(z-1)\big(2z^2 + 2z - 1\big)}{9z} - \frac{128}{3}zH_0\Big]$$

$$+L_Q\Big[-\frac{256}{9}\big(4z^2 - 8z - 3\big)H_0 + \frac{256}{3}zH_0^2 - \frac{256(z-1)\big(3z^2 + 6z - 2\big)}{9z}\Big]$$

$$+\Big[\frac{128(z-1)\big(2z^2 + 2z - 1\big)}{9z} - \frac{128}{3}zH_0\Big]L_M^2 + \frac{64(z-1)\big(2z^2 + 2z - 1\big)H_1^2}{9z}$$

$$-\frac{128(z-1)\big(19z^2 + 16z - 5\big)H_1}{27z} - \frac{128}{27}z(19z+67)H_0 + \frac{256(z-1)\big(55z^2 + 43z - 14\big)}{81z}\Big]$$

$$+N_F\hat{\tilde{C}}_{L,q}^{\mathsf{PS},(3)}(N_F)\Bigg\} \ , \tag{599}$$

$$L_{g,L}^{\mathsf{S}} =$$

$$-\frac{64}{3}a_s^2 N_F T_F^2(z-1)zL_M + a_s^3\Bigg\{-N_F T_F^3\frac{256}{9}(z-1)zL_M^2$$

$$+C_A T_F^2 N_F\Bigg[\Big[\frac{64(z-1)\big(17z^2 + 2z - 1\big)}{9z} - \frac{256}{3}zH_0 + \frac{128}{3}(z-1)zH_1\Big]L_Q^2$$

$$+\Big[-\frac{64(z-1)\big(461z^2 + 11z - 25\big)}{27z} - \frac{128}{9}z(26z - 59)H_0$$

$$-\frac{128(z-1)\big(39z^2 + 2z - 1\big)H_1}{9z} + (z-1)z\Big[-\frac{128}{3}H_1^2 - \frac{256}{3}H_0H_1\Big]$$

$$+z(z+1)\Big[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1}\Big] + z\Big[\frac{512}{3}H_0^2 + \frac{512}{3}H_{0,1}\Big]$$

$$+\Big[\frac{128(z-1)\big(17z^2 + 2z - 1\big)}{9z} - \frac{512}{3}zH_0 + \frac{256}{3}(z-1)zH_1\Big]L_M$$

$$+\frac{256}{3}(z-2)z\zeta_2\bigg]L_Q-\frac{32}{9}(28z-3)H_0^2+\bigg[\frac{64(z-1)\left(17z^2+2z-1\right)}{9z}-\frac{256}{3}zH_0$$

$$+\frac{128}{3}(z-1)zH_1\bigg]L_M^2+\frac{32(z-1)\left(2714z^2-106z-139\right)}{81z}-\frac{64}{27}\left(110z^2+277z-33\right)H_0$$

$$+\frac{4160}{27}(z-1)zH_1+z\bigg[-\frac{64}{9}H_0^3-\frac{64}{3}H_{0,1}+\frac{64\zeta_2}{3}\bigg]+L_M\bigg[\frac{64(z-1)\left(68z^2+z-7\right)}{9z}$$

$$-\frac{128}{9}(4z-1)(13z+6)H_0-\frac{128(z-1)\left(19z^2+2z-1\right)H_1}{9z}+(z-1)z\bigg[-\frac{128}{3}H_1^2$$

$$-\frac{256}{3}H_0H_1\bigg]+z(z+1)\bigg[\frac{256}{3}H_{-1}H_0-\frac{256}{3}H_{0,-1}\bigg]+z\bigg[\frac{256}{3}H_0^2$$

$$+\frac{512}{3}H_{0,1}\bigg]+\frac{256}{3}(z-2)z\zeta_2\bigg]\bigg]+C_FT_F^2N_F\bigg[-\frac{16}{3}zH_0^4-\frac{32}{3}(7z-1)H_0^3-32(19z-3)H_0^2$$

$$-64\left(6z^2+7z-8\right)H_0+\bigg[-64zH_0^2-\frac{64}{3}\left(4z^2+3z-3\right)H_0$$

$$+\frac{128(z-1)\left(17z^2-10z-1\right)}{9z}\bigg]L_M^2+\frac{16(z-1)\left(343z^2-242z+4\right)}{3z}$$

$$+L_Q^2\bigg[-64zH_0^2-\frac{64}{3}(z+1)(4z-3)H_0+\frac{64(z-1)\left(28z^2-23z-2\right)}{9z}\bigg]$$

$$+L_Q\bigg[-\frac{128(25z+2)H_1(z-1)^2}{9z}-\frac{32\left(2474z^2-4897z+44\right)(z-1)}{135z}$$

$$-\frac{64}{45}\left(12z^3-180z^2-265z+90\right)H_0^2-\frac{64\left(354z^3-397z^2+388z+4\right)H_0}{45z}$$

$$+\frac{(z+1)\left(6z^4-6z^3+z^2-z+1\right)\left(\frac{256}{45}H_{-1}H_0-\frac{256}{45}H_{0,-1}\right)}{z^2}+\frac{128}{3}(z+1)(4z-3)H_{0,1}$$

$$+\bigg[128zH_0^2+\frac{128}{3}(2z-1)(2z+3)H_0-\frac{64(z-1)\left(74z^2-37z-4\right)}{9z}\bigg]L_M$$

$$+\frac{128}{45}\left(12z^3-60z^2-25z+45\right)\zeta_2+z\bigg[128H_0^3-256\zeta_2H_0+256H_{0,0,1}-256\zeta_3\bigg]\bigg]$$

$$+L_M\bigg[-\frac{64}{45}\left(12z^3+180z^2+335z-90\right)H_0^2+\frac{64\left(456z^3-708z^2+347z-4\right)H_0}{45z}$$

$$+\frac{64(z-1)\left(2368z^2-2189z-2\right)}{135z}+\frac{64(z-1)\left(80z^2-37z-4\right)H_1}{9z}$$

$$+\frac{(z+1)\left(6z^4-6z^3+z^2-z+1\right)\left[\frac{256}{45}H_{-1}H_0-\frac{256}{45}H_{0,-1}\right]}{z^2}-\frac{128}{3}(2z-1)(2z+3)H_{0,1}$$

$$+\frac{128}{45}\left(12z^3+60z^2+50z-45\right)\zeta_2+z\bigg[-128H_0^3+256\zeta_2H_0-256H_{0,0,1}+256\zeta_3\bigg]\bigg]\bigg]$$

$$+N_F\hat{\tilde{C}}_{L,g}^{\mathsf{S},(3)}(N_F)\bigg\}\,. \tag{600}$$

The flavor non-singlet Wilson coefficient is given by :

$$L_{q,L}^{\mathsf{NS}}=$$

$$a_s^2 C_F T_F \left\{ \frac{16 L_Q z}{3} + \left[-\frac{32}{3} H_0 - \frac{16 H_1}{3} \right] z - \frac{8}{9}(25z - 6) \right\}$$

$$+ a_s^3 \left\{ C_F^2 T_F \left[\left[8(z+2) + z\left[-16H_0 - 32H_1 \right] \right] L_Q^2 + \left[\frac{16}{15} z\left(24z^2 - 5 \right) H_0^2 + \frac{80}{9}(5z-6) H_1 \right. \right. \right.$$

$$+ \frac{16\left(144z^3 - 227z^2 - 72z - 96 \right) H_0}{45z} + \frac{32\left(72z^3 - 223z^2 - 77z + 48 \right)}{45z}$$

$$+ \frac{\left(3z^5 - 5z^3 - 10z^2 - 2 \right)\left[\frac{256}{15} H_{0,-1} - \frac{256}{15} H_{-1} H_0 \right]}{z^2} - \frac{32}{5} z\left(8z^2 + 5 \right) \zeta_2 + z\left[-\frac{256}{3} H_0 H_{-1}^2 \right.$$

$$+ \left[\frac{128}{3} H_0^2 + \frac{512}{3} H_{0,-1} \right] H_{-1} + \frac{128}{3} H_1^2 - \frac{128}{3} H_0^2 H_1 + 32 H_{0,1} + H_0\left[\frac{256 H_1}{3} - \frac{256}{3} H_{0,-1} \right.$$

$$\left. + \frac{256}{3} H_{0,1} \right] - \frac{512}{3} H_{0,-1,-1} + \frac{256}{3} H_{0,0,-1} - \frac{256}{3} H_{0,0,1} + \left[\frac{256 H_1}{3} - \frac{256}{3} H_{-1} \right] \zeta_2 + \frac{512 \zeta_3}{3} \right] \Bigg] L_Q$$

$$+ \frac{8}{9}(z+3) H_0^2 + \left[\frac{8(z+2)}{3} + z\left[-\frac{16}{3} H_0 - \frac{32 H_1}{3} \right] \right] L_M^2 - \frac{2}{27}(653z - 872) + \frac{16}{27}(11z + 42) H_0$$

$$+ z\left[-\frac{8}{9} H_0^3 - \frac{16}{3} H_1 H_0^2 + \left[\frac{32}{3} H_{0,1} - \frac{160 H_1}{9} \right] H_0 - \frac{896 H_1}{27} + \frac{160}{9} H_{0,1} - \frac{32}{3} H_{0,0,1} \right]$$

$$+ \left[-\frac{8}{9}(53z - 56) + \frac{32}{9}(z+3) H_0 + z\left[-\frac{16}{3} H_0^2 - \frac{64}{3} H_1 H_0 - \frac{320 H_1}{9} + \frac{64}{3} H_{0,1} \right] \right] L_M \Bigg]$$

$$+ \left[\left[\frac{64 z L_Q^2}{9} + \left[z\left[-\frac{256}{9} H_0 - \frac{128 H_1}{9} \right] - \frac{64}{27}(25z - 6) \right] L_Q + N_F \left[\frac{128 z L_Q^2}{9} + \left[z\left[-\frac{512}{9} H_0 \right. \right. \right. \right. \right.$$

$$\left. \left. \left. - \frac{256 H_1}{9} \right] - \frac{128}{27}(25z - 6) \right] L_Q \right] \right] T_F^2 + C_A \left[L_Q \left[-\frac{16\left(216z^3 - 3329z^2 + 624z + 144 \right)}{135z} \right. \right.$$

$$- \frac{64\left(18z^3 - 149z^2 + 6z - 12 \right) H_0}{45z} + \frac{\left(3z^5 - 5z^3 - 10z^2 - 2 \right)\left[\frac{128}{15} H_{-1} H_0 - \frac{128}{15} H_{0,-1} \right]}{z^2}$$

$$+ z\left(3z^2 - 5 \right)\left[\frac{128 \zeta_2}{15} - \frac{64}{15} H_0^2 \right] + z\left[\frac{128}{3} H_0 H_{-1}^2 + \left[-\frac{64}{3} H_0^2 - \frac{256}{3} H_{0,-1} \right] H_{-1} + \frac{64}{3} H_0^2 H_1 \right.$$

$$+ \frac{1088 H_1}{9} + H_0\left[\frac{128}{3} H_{0,-1} - \frac{128}{3} H_{0,1} \right] + \frac{256}{3} H_{0,-1,-1} - \frac{128}{3} H_{0,0,-1} + \frac{128}{3} H_{0,0,1} + \left[\frac{128}{3} H_{-1} \right.$$

$$\left. \left. \left. - \frac{128 H_1}{3} \right] \zeta_2 - \frac{256 \zeta_3}{3} \right] \right] - \frac{352 L_Q^2 z}{9} \right] T_F \right] C_F + \hat{C}_{L,q}^{\mathsf{NS},(3)}(N_F) \right\} . \tag{601}$$

The pure-singlet Wilson coefficient $H_{q,L}^{\mathsf{PS}}$ reads :

$$H_{q,L}^{\mathsf{PS}} =$$

$$a_s^2 C_F T_F \left\{ \frac{32(z-1)\left(10z^2 - 2z + 1 \right)}{9z} - 32(z+1)(2z-1) H_0 - \frac{32(z-1)\left(2z^2 + 2z - 1 \right) H_1}{3z} \right.$$

$$\left. + L_Q \left[\frac{32(z-1)\left(2z^2 + 2z - 1 \right)}{3z} - 32z H_0 \right] + z\left[32 H_0^2 + 32 H_{0,1} - 32 \zeta_2 \right] \right\}$$

$$+ a_s^3 \left\{ C_F^2 T_F \left[-\frac{8}{3}(5z+2) H_0^3 - \frac{8}{3}\left(8z^2 + 3 \right) H_0^2 + \frac{16}{9}\left(160z^2 + 93z - 39 \right) H_0 \right. \right.$$

$$+ \left[16z H_0^2 - 16(z+2) H_0 - \frac{16(z-1)\left(4z^2 - 11z - 2 \right)}{3z} \right] L_M^2 - \frac{32(z-1)\left(440z^2 - 91z - 28 \right)}{27z}$$

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$$+\frac{(z-1)(4z^2-11z-2)\left[\frac{32}{3}H_0H_1-\frac{32}{3}H_{0,1}\right]}{z}+\left[-\frac{32}{3}zH_0^3+16(5z+2)H_0^2\right.$$

$$+\frac{32}{3}(8z^2+18z+3)H_0-\frac{32(z-1)(80z^2+17z-10)}{9z}\right]L_M+L_Q^2\left[-\frac{64}{3}(2z^2-3)H_0\right.$$

$$+\frac{32(z-1)(2z^2-9z-1)}{3z}-\frac{64(z-1)(2z^2+2z-1)H_1}{3z}+z\left[64H_{0,1}-64\zeta_2\right]\right]$$

$$+(z+2)\left[32H_0H_{0,1}-64H_{0,0,1}+64\zeta_3\right]+z\left[\frac{4}{3}H_0^4-64H_{0,0,1}H_0-128\zeta_3H_0-\frac{384\zeta_2^2}{5}\right.$$

$$+192H_{0,0,0,1}\right]+L_Q\left[-\frac{32(z-1)(86z^2-33z+6)}{45z}-\frac{32(56z^3-813z^2+142z+16)H_0}{45z}\right.$$

$$+\frac{64(z-1)(4z^2+55z-5)H_1}{9z}+\frac{(z-1)(2z^2+2z-1)\left[\frac{64}{3}H_1^2+\frac{128}{3}H_0H_1\right]}{z}$$

$$+\frac{(z+1)(6z^4-6z^3+11z^2+4z-4)\left[\frac{128}{45}H_{0,-1}-\frac{128}{45}H_{-1}H_0\right]}{z^2}$$

$$+\frac{128(4z^3-6z^2-3z-1)H_{0,1}}{3z}+(6z^3+90z^2-85z-90)\left[\frac{64}{45}H_0^2-\frac{128\zeta_2}{45}\right]$$

$$+z\left[-\frac{64}{9}H_0^3+\left[\frac{128}{3}H_{0,-1}-128H_{0,1}\right]H_0+\frac{896}{3}\zeta_2H_0-\frac{256}{3}H_{0,0,-1}-128H_{0,1,1}+192\zeta_3\right]\right]\right]$$

$$+C_FT_F^2N_F\left[\left[\frac{128(z-1)(2z^2+2z-1)}{9z}-\frac{128}{3}zH_0\right]L_Q^2+\left[\frac{256}{3}zH_0^2\right.\right.$$

$$-\frac{256}{9}(4z^2-8z-3)H_0-\frac{256(z-1)(3z^2+6z-2)}{9z}\right]L_Q\right]$$

$$+C_FT_F^2\left[\left[\frac{128(z-1)(2z^2+2z-1)}{9z}-\frac{128}{3}zH_0\right]L_Q^2+\left[\frac{256}{3}zH_0^2-\frac{256}{9}(4z^2-8z-3)H_0\right.\right.$$

$$-\frac{256(z-1)(3z^2+6z-2)}{9z}\right]L_Q+\frac{64(z-1)(2z^2+2z-1)H_1^2}{9z}+\left[\frac{128(z-1)(2z^2+2z-1)}{9z}\right.$$

$$-\frac{128}{3}zH_0\right]L_M^2+\frac{256(z-1)(55z^2+43z-14)}{81z}-\frac{128}{27}z(19z+67)H_0$$

$$-\frac{128(z-1)(19z^2+16z-5)H_1}{27z}+L_M\left[\frac{256(z-1)(19z^2+16z-5)}{27z}-\frac{256}{9}z(2z+11)H_0\right.$$

$$-\frac{256(z-1)(2z^2+2z-1)H_1}{9z}+z\left[\frac{256}{3}H_{0,1}-\frac{256\zeta_2}{3}\right]\right]+z(2z+11)\left[\frac{128}{9}H_{0,1}-\frac{128\zeta_2}{9}\right]$$

$$+z\left[\frac{128\zeta_3}{3}-\frac{128}{3}H_{0,1,1}\right]\right]+C_FC_AT_F\left[\left[-\frac{16(z-1)(46z^2-z-21)}{3z}+\frac{64(7z^2-3z-1)H_0}{3z}\right.\right.$$

$$-\frac{64(z-1)(2z^2+2z-1)H_1}{3z}+z\left[64H_0^2+64H_{0,1}-64\zeta_2\right]\right]L_Q^2+\left[-\frac{32}{3}(19z-12)H_0^2\right.$$

$$+\frac{32(422z^3-137z^2-114z+4)H_0}{9z}-\frac{32(z-1)(670z^2-245z+46)}{27z}$$

$$+\frac{32(z-1)(106z^2-23z-65)H_1}{9z}+\frac{(z-1)(2z^2+2z-1)\left[\frac{128}{3}H_1^2+\frac{256}{3}H_0H_1\right]}{z}$$

$$+\frac{(z+1)\left(2z^2-2z-1\right)\left[\frac{256}{3}H_{0,-1}-\frac{256}{3}H_{-1}H_0\right]}{z}-64(z-4)H_{0,1}-\frac{64}{3}z(8z-3)\zeta_2$$

$$+z\left[-\frac{256}{3}H_0^3+\left[-256H_{0,-1}-256H_{0,1}\right]H_0+256\zeta_2 H_0+512H_{0,0,-1}-256H_{0,1,1}\right.$$

$$\left.\left.\left.-128\zeta_3\right]\right]L_Q\right]+\tilde{C}_{L,q}^{\mathsf{PS},(3)}(N_F+1)\Bigg\}\ . \tag{602}$$

Finally, the gluonic Wilson coefficient is given by :

$$H_{g,L}^{\mathsf{S}}=$$

$$-16T_F(z-1)za_s+a_s^2\Bigg\{-\frac{64}{3}(z-1)zL_M T_F^2+C_A T_F\left[-\frac{32(z-1)\left(53z^2+2z-1\right)}{9z}\right.$$

$$-32\left(13z^2-8z-1\right)H_0-\frac{32(z-1)\left(29z^2+2z-1\right)H_1}{3z}+L_Q\left[\frac{32(z-1)\left(17z^2+2z-1\right)}{3z}\right.$$

$$\left.-128zH_0+64(z-1)zH_1\right]+(z-1)z\left[-32H_1^2-64H_0 H_1\right]+z(z+1)\left[64H_{-1}H_0-64H_{0,-1}\right]$$

$$+z\left[96H_0^2+128H_{0,1}\right]+64(z-2)z\zeta_2\Bigg]+C_F T_F\left[-\frac{64}{15}z\left(3z^2+5\right)H_0^2\right.$$

$$+\frac{16\left(36z^3-78z^2-13z-4\right)H_0}{15z}+\frac{32(z-1)\left(63z^2+6z-2\right)}{15z}$$

$$+L_Q\left[32zH_0-16(z-1)(2z+1)\right]+16(z-1)(4z+1)H_1$$

$$+\frac{(z+1)\left(6z^4-6z^3+z^2-z+1\right)}{z^2}\left[\frac{64}{15}H_{-1}H_0-\frac{64}{15}H_{0,-1}\right]-32zH_{0,1}$$

$$+\left[16(z-1)(2z+1)-32zH_0\right]L_M+\frac{32}{15}z\left(12z^2+5\right)\zeta_2\Bigg]\Bigg\}$$

$$+a_s^3\Bigg\{-T_F^3\frac{256}{9}(z-1)zL_M^2+C_A T_F^2\left[\left[\frac{64(z-1)\left(17z^2+2z-1\right)}{9z}-\frac{256}{3}zH_0\right.\right.$$

$$\left.+\frac{128}{3}(z-1)zH_1\right]L_Q^2+\left[-\frac{64(z-1)\left(461z^2+11z-25\right)}{27z}-\frac{128}{9}z(26z-59)H_0\right.$$

$$-\frac{128(z-1)\left(39z^2+2z-1\right)H_1}{9z}+(z-1)z\left[-\frac{128}{3}H_1^2-\frac{256}{3}H_0 H_1\right]$$

$$+z(z+1)\left[\frac{256}{3}H_{-1}H_0-\frac{256}{3}H_{0,-1}\right]+z\left[\frac{512}{3}H_0^2+\frac{512}{3}H_{0,1}\right]+\left[\frac{128(z-1)\left(17z^2+2z-1\right)}{9z}\right.$$

$$\left.-\frac{512}{3}zH_0+\frac{256}{3}(z-1)zH_1\right]L_M+\frac{256}{3}(z-2)z\zeta_2\Bigg]L_Q-\frac{32}{9}(28z-3)H_0^2$$

$$+\left[\frac{64(z-1)\left(17z^2+2z-1\right)}{9z}-\frac{256}{3}zH_0+\frac{128}{3}(z-1)zH_1\right]L_M^2$$

$$+\frac{32(z-1)\left(2714z^2-106z-139\right)}{81z}-\frac{64}{27}\left(110z^2+277z-33\right)H_0+\frac{4160}{27}(z-1)zH_1$$

$$+z\left[-\frac{64}{9}H_0^3-\frac{64}{3}H_{0,1}+\frac{64\zeta_2}{3}\right]+L_M\left[\frac{64(z-1)\left(68z^2+z-7\right)}{9z}-\frac{128}{9}(4z-1)(13z+6)H_0\right.$$

$$-\frac{128(z-1)\left(19z^2+2z-1\right)H_1}{9z} + (z-1)z\left[-\frac{128}{3}H_1^2 - \frac{256}{3}H_0H_1\right]$$

$$+z(z+1)\left[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1}\right] + z\left[\frac{256}{3}H_0^2 + \frac{512}{3}H_{0,1}\right] + \frac{256}{3}(z-2)z\zeta_2\bigg]\bigg]$$

$$+C_A T_F^2 N_F\Bigg[\left[\frac{64(z-1)\left(17z^2+2z-1\right)}{9z} - \frac{256}{3}zH_0 + \frac{128}{3}(z-1)zH_1\right]L_Q^2$$

$$+\left[-\frac{64(z-1)\left(461z^2+11z-25\right)}{27z} - \frac{128}{9}z(26z-59)H_0 - \frac{128(z-1)\left(39z^2+2z-1\right)H_1}{9z}\right.$$

$$+(z-1)z\left[-\frac{128}{3}H_1^2 - \frac{256}{3}H_0H_1\right] + z(z+1)\left[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1}\right] + z\left[\frac{512}{3}H_0^2 + \frac{512}{3}H_{0,1}\right]$$

$$+\frac{256}{3}(z-2)z\zeta_2\bigg]L_Q\Bigg]$$

$$+C_A^2 T_F\Bigg[\left[-\frac{16(z-1)\left(1033z^2-26z-65\right)}{9z} - \frac{64\left(6z^3-47z^2+3z+1\right)H_0}{3z}\right.$$

$$-\frac{32(z-1)\left(79z^2+8z-4\right)H_1}{3z} + (z-1)z\left[-128H_1^2 - 128H_0H_1\right] + 128z(z+3)H_{0,1}$$

$$+z\left[256H_0^2 - 512\zeta_2\right]\bigg]L_Q^2 + \left[\frac{64}{3}\left(18z^2-91z+6\right)H_0^2 + \frac{32\left(2713z^3-1405z^2-60z+4\right)H_0}{9z}\right.$$

$$+\frac{64(z-1)\left(137z^2+12z-6\right)H_1H_0}{3z} + 128z(3z-5)H_{0,-1}H_0 - 128z(z+11)H_{0,1}H_0$$

$$+\frac{32(z-1)\left(161z^2+12z-6\right)H_1^2}{3z} + \frac{32(z-1)\left(680z^2-60z-13\right)}{9z}$$

$$+\frac{32(z-1)\left(1919z^2+30z-93\right)H_1}{9z} + \frac{(z+1)\left(79z^2-8z-4\right)\left[\frac{64}{3}H_{0,-1} - \frac{64}{3}H_{-1}H_0\right]}{z}$$

$$-\frac{128\left(z^3+53z^2-6z+1\right)H_{0,1}}{3z} - 128z(3z-13)H_{0,0,-1} - 128(z-3)zH_{0,0,1}$$

$$-256z(z+5)H_{0,1,1} - \frac{64}{3}\left(135z^2-160z-6\right)\zeta_2 + z(z+1)\left[256H_0H_{-1}^2 + \left[-192H_0^2\right.\right.$$

$$\left.-512H_{0,-1} - 256H_{0,1}\right]H_{-1} + 512\zeta_2 H_{-1} + 512H_{0,-1,-1} + 256H_{0,-1,1} + 256H_{0,1,-1}\bigg]$$

$$+(z-1)z\left[128H_1^3 + 384H_0H_1^2 + 192H_0^2H_1 - 512\zeta_2 H_1\right] + z\left[-\frac{1024}{3}H_0^3 + 1792\zeta_2 H_0 - 128\zeta_3\right]\bigg]L_Q\Bigg]$$

$$+C_F^2 T_F\Bigg[-\frac{8}{3}\left(4z^2-4z-1\right)H_0^3 - 4\left(20z^2-11z-1\right)H_0^2 + 8\left(24z^2+37z-7\right)H_0$$

$$-16(z-1)(10z-1)H_1H_0 + 32\left(2z^2+5z-2\right)H_{0,1}H_0 + 48(z-1)zH_1^2 - 16(z-1)(20z+3)$$
$$+32(z-1)(6z-1)H_1 + 16\left(16z^2-24z+3\right)H_{0,1} - 32\left(2z^2+17z-3\right)H_{0,0,1}$$

$$+(z-1)(2z+1)\left[\frac{16}{3}H_1^3 - 16H_0^2H_1 + 32H_{0,1,1}\right] - 16(z-2)(6z-1)\zeta_2 + L_Q^2\bigg[32(2z+1)H_1(z-1)$$

$$+24(z-1) + 16(2z-1)(2z+1)H_0 + z\left[-16H_0^2 - 64H_{0,1} + 64\zeta_2\right]\bigg]\Bigg]$$

$$+L_M^2\bigg[32(2z+1)H_1(z-1) + 24(z-1) + 16(2z-1)(2z+1)H_0 + z\left[-16H_0^2\right.$$

$$-64H_{0,1}+64\zeta_2\Bigg]\Bigg]-32\big(2z^2-19z+1\big)\zeta_3+z\Bigg[\frac{4}{3}H_0^4+32H_{0,1}H_0^2-192H_{0,0,1}H_0+192\zeta_2H_0$$

$$-64\zeta_3H_0-\frac{608\zeta_2^2}{5}+384H_{0,0,0,1}-64H_{0,0,1,1}-64H_{0,1,1,1}\Bigg]+L_M\Bigg[\frac{32}{15}\big(24z^3+90z^2-95z-15\big)H_0^2$$

$$+\frac{32\big(78z^3+141z^2-34z+8\big)H_0}{15z}+128\big(2z^2-3z-1\big)H_{0,1}H_0-\frac{8(z-1)(6z+1)(153z-32)}{15z}$$

$$+16(z-1)(4z-3)H_1+\frac{(z+1)\big(12z^4+3z^3-73z^2-2z+2\big)\big[\frac{128}{15}H_{0,-1}-\frac{128}{15}H_{-1}H_0\big]}{z^2}$$

$$+32(6z+1)H_{0,1}-64\big(4z^2-5z-2\big)H_{0,0,1}-\frac{32}{15}\big(48z^3+120z^2-250z-45\big)\zeta_2$$

$$+(z+1)(2z-1)\Big[128H_0H_{-1}^2+\big[-64H_0^2-256H_{0,-1}\big]H_{-1}+128\zeta_2H_{-1}-128H_0H_{0,-1}$$

$$+256H_{0,-1,-1}+384H_{0,0,-1}\Big]+(z-1)(2z+1)\Big[-64H_1H_0^2+128H_1H_0+64H_1^2+128H_1\zeta_2\Big]$$

$$+z\Big[\frac{32}{3}H_0^3+\big[128H_{0,1}-128H_{0,-1}\big]H_0^2+\big[512H_{0,-1,-1}+512H_{0,0,-1}-512H_{0,0,1}\big]H_0$$

$$-256H_{0,-1}^2+\frac{768\zeta_2^2}{5}-256H_{0,1,1}-768H_{0,0,0,-1}+768H_{0,0,0,1}+\big[64H_0+256H_{0,-1}-256H_{0,1}\big]\zeta_2$$

$$+\big[-512H_0-576\big]\zeta_3\Big]\Bigg]+L_Q\Bigg[-\frac{32}{15}\big(24z^3+90z^2-95z-15\big)H_0^2$$

$$-\frac{32\big(78z^3+141z^2-34z+8\big)H_0}{15z}-128\big(2z^2-3z-1\big)H_{0,1}H_0+\frac{8(z-1)(6z+1)(153z-32)}{15z}$$

$$-16(z-1)(4z-3)H_1+\frac{(z+1)\big(12z^4+3z^3-73z^2-2z+2\big)\big[\frac{128}{15}H_{-1}H_0-\frac{128}{15}H_{0,-1}\big]}{z^2}$$

$$-32(6z+1)H_{0,1}+64\big(4z^2-5z-2\big)H_{0,0,1}+L_M\Bigg[-64(2z+1)H_1(z-1)-48(z-1)$$

$$-32(2z-1)(2z+1)H_0+z\big[32H_0^2+128H_{0,1}-128\zeta_2\big]\Bigg]+\frac{32}{15}\big(48z^3+120z^2-250z-45\big)\zeta_2$$

$$+(z+1)(2z-1)\Big[-128H_0H_{-1}^2+\big[64H_0^2+256H_{0,-1}\big]H_{-1}-128\zeta_2H_{-1}+128H_0H_{0,-1}$$

$$-256H_{0,-1,-1}-384H_{0,0,-1}\Big]+(z-1)(2z+1)\Big[64H_1H_0^2-128H_1H_0-64H_1^2-128H_1\zeta_2\Big]$$

$$+z\Big[-\frac{32}{3}H_0^3+\big[128H_{0,-1}-128H_{0,1}\big]H_0^2+\big[-512H_{0,-1,-1}-512H_{0,0,-1}+512H_{0,0,1}\big]H_0+256H_{0,-1}^2$$

$$-\frac{768\zeta_2^2}{5}+256H_{0,1,1}+768H_{0,0,0,-1}-768H_{0,0,0,1}+\big[-64H_0-256H_{0,-1}+256H_{0,1}\big]\zeta_2$$

$$+\big[512H_0+576\big]\zeta_3\Big]\Bigg]\Bigg]+\textcolor{blue}{C_FT_F^2}\Bigg[-\frac{16}{3}zH_0^4-\frac{32}{3}(7z-1)H_0^3$$

$$-32(19z-3)H_0^2-64\big(6z^2+7z-8\big)H_0+\Bigg[-64zH_0^2-\frac{64}{3}\big(4z^2+5z-3\big)H_0$$

$$+\frac{64(z-1)\big(40z^2-17z-2\big)}{9z}\Bigg]L_M^2+\frac{16(z-1)\big(343z^2-242z+4\big)}{3z}+L_Q^2\Bigg[-64zH_0^2$$

$$-\frac{64}{3}(z+1)(4z-3)H_0+\frac{64(z-1)\big(28z^2-23z-2\big)}{9z}\Bigg]+L_Q\Bigg[-\frac{128(25z+2)H_1(z-1)^2}{9z}$$

$$-\frac{32\left(2474z^2 - 4897z + 44\right)(z-1)}{135z} - \frac{64}{45}\left(12z^3 - 180z^2 - 265z + 90\right)H_0^2$$

$$-\frac{64\left(354z^3 - 397z^2 + 388z + 4\right)H_0}{45z} + \frac{(z+1)\left(6z^4 - 6z^3 + z^2 - z + 1\right)}{z^2}\left[\frac{256}{45}H_{-1}H_0\right.$$

$$\left. -\frac{256}{45}H_{0,-1}\right] + \frac{128}{3}(z+1)(4z-3)H_{0,1} + \left[128zH_0^2 + \frac{128}{3}\left(4z^2 + 3z - 3\right)H_0\right.$$

$$\left. -\frac{256(z-1)\left(17z^2 - 10z - 1\right)}{9z}\right]L_M + \frac{128}{45}\left(12z^3 - 60z^2 - 25z + 45\right)\zeta_2 + z\left[128H_0^3\right.$$

$$\left.\left. -256\zeta_2 H_0 + 256H_{0,0,1} - 256\zeta_3\right]\right] + L_M\left[-\frac{64}{45}\left(12z^3 + 180z^2 + 305z - 90\right)H_0^2\right.$$

$$+\frac{64\left(426z^3 - 553z^2 + 362z - 4\right)H_0}{45z} + \frac{32(z-1)\left(3716z^2 - 4753z - 4\right)}{135z}$$

$$+\frac{128(z-1)\left(37z^2 - 20z - 2\right)H_1}{9z} + \frac{(z+1)\left(6z^4 - 6z^3 + z^2 - z + 1\right)\left[\frac{256}{45}H_{-1}H_0 - \frac{256}{45}H_{0,-1}\right]}{z^2}$$

$$-\frac{128}{3}\left(4z^2 + 3z - 3\right)H_{0,1} + \frac{128}{45}\left(12z^3 + 60z^2 + 35z - 45\right)\zeta_2$$

$$\left.+z\left[-128H_0^3 + 256\zeta_2 H_0 - 256H_{0,0,1} + 256\zeta_3\right]\right]\right] + C_F N_F T_F^2\left[\left[-64zH_0^2\right.\right.$$

$$-\frac{64}{3}(z+1)(4z-3)H_0 + \frac{64(z-1)\left(28z^2 - 23z - 2\right)}{9z}\right]L_Q^2 + \left[-\frac{128(25z+2)H_1(z-1)^2}{9z}\right.$$

$$-\frac{32\left(2474z^2 - 4897z + 44\right)(z-1)}{135z} - \frac{64}{45}\left(12z^3 - 180z^2 - 265z + 90\right)H_0^2$$

$$-\frac{64\left(354z^3 - 397z^2 + 388z + 4\right)H_0}{45z} + \frac{(z+1)\left(6z^4 - 6z^3 + z^2 - z + 1\right)}{z^2}\left[\frac{256}{45}H_{-1}H_0\right.$$

$$\left.-\frac{256}{45}H_{0,-1}\right] + \frac{128}{3}(z+1)(4z-3)H_{0,1} + \left[\frac{64}{3}(z-1)(2z+1) - \frac{128}{3}zH_0\right]L_M$$

$$+\frac{128}{45}\left(12z^3 - 60z^2 - 25z + 45\right)\zeta_2 + z\left[128H_0^3 - 256\zeta_2 H_0 + 256H_{0,0,1} - 256\zeta_3\right]\right]L_Q$$

$$+L_M\left[-\frac{32}{9}(z-1)(68z+25) - \frac{64}{9}\left(6z^2 - 31z - 3\right)H_0 - \frac{64}{3}(z-1)(2z+1)H_1\right.$$

$$\left.\left.+z\left[\frac{128}{3}H_0^2 + \frac{128}{3}H_{0,1} - \frac{128\zeta_2}{3}\right]\right]\right] + C_A C_F T_F\left[-\frac{16}{3}(3z+1)H_0^3\right.$$

$$-\frac{8}{3}\left(11z^2 - 18z + 3\right)H_0^2 + \frac{16}{9}\left(772z^2 + 480z - 39\right)H_0 + \frac{16(z-1)\left(77z^2 - 25z - 4\right)H_1 H_0}{3z}$$

$$-32(7z-2)H_{0,1}H_0 - 8(z-1)(9z-1)H_1^2 - \frac{32(z-1)\left(2168z^2 - 91z - 28\right)}{27z}$$

$$-16(z-1)(16z-1)H_1 + (z-1)(2z+1)\left[16H_0 H_1^2 - \frac{16}{3}H_1^3\right]$$

$$+z(z+1)\left[192H_{0,-1} - 192H_{-1}H_0\right]$$

$$-\frac{16\left(68z^3 - 117z^2 + 21z + 4\right)H_{0,1}}{3z} - 32\left(2z^2 - 2z - 1\right)H_{0,1,1} + L_M^2\left[\right.$$

$$-\frac{16(z-1)\left(43z^2 - 11z - 2\right)}{3z} + 32(3z-1)H_0 - 32(z-1)(2z+1)H_1$$

$$+z\Big[64H_0^2+64H_{0,1}-64\zeta_2\Big]\Big]-16z(3z+17)\zeta_2+(z+1)(2z-1)\Big[32H_0H_{-1}^2$$

$$+\Big[-16H_0^2-64H_{0,-1}\Big]H_{-1}+32\zeta_2H_{-1}+32H_0H_{0,-1}+64H_{0,-1,-1}-32H_{0,0,-1}\Big]$$

$$+L_Q^2\Big[\frac{16(z-1)\big(65z^2-2\big)}{3z}-\frac{32}{3}(20z-3)H_0+32(z-1)(2z+1)H_1+z\Big[-64H_0^2$$

$$-64H_{0,1}+64\zeta_2\Big]\Big]+(15z-4)\Big[32H_{0,0,1}-32\zeta_3\Big]+z\Big[\frac{8}{3}H_0^4-32H_{0,-1}H_0^2+\Big[128H_{0,-1,-1}$$

$$+128H_{0,0,-1}-256H_{0,0,1}-64H_{0,1,1}\Big]H_0-448\zeta_3H_0-64H_{0,-1}^2-\frac{1472\zeta_2^2}{5}-192H_{0,0,0,-1}$$

$$+768H_{0,0,0,1}+128H_{0,0,1,1}+64H_{0,1,1,1}+\Big[64H_{0,-1}-32H_0\Big]\zeta_2\Big]+L_Q\Big[\frac{128}{45}z\big(6z^2+35\big)H_0^3$$

$$+\frac{64\big(84z^4-9z^3+272z^2-48z+6\big)H_0^2}{45z}-\frac{32(z+1)\big(24z^4+6z^3-11z^2-4z+4\big)H_{-1}H_0^2}{15z^2}$$

$$-32(z-1)(2z+1)H_1H_0^2-\frac{16\big(4668z^3-5233z^2-130z-64\big)H_0}{45z}-64(z-1)(4z+1)H_1H_0$$

$$-\frac{64\big(30z^4-35z^3-15z^2-4\big)H_{0,-1}H_0}{15z^2}+64(z+1)(2z-1)H_{0,1}H_0-\frac{256}{5}z\big(4z^2+5\big)\zeta_2H_0$$

$$-32(z-1)(6z+1)H_1^2-\frac{8(z-1)\big(11062z^2+1335z-168\big)}{45z}$$

$$-\frac{64}{45}\big(168z^4-108z^3+343z^2-6z+24\big)\frac{\zeta_2}{z}+\frac{64}{5}(z+1)\big(12z^4-2z^3-3z^2-2z+2\big)\frac{\zeta_2}{z^2}H_{-1}$$

$$-\frac{64(z-1)\big(371z^2+37z-14\big)H_1}{15z}-\frac{64}{15}(z-1)\big(12z^4-18z^3-13z^2+2z+2\big)\frac{\zeta_2}{z^2}H_1$$

$$+\frac{(z+1)\big(42z^4-69z^3-35z^2-4z+7\big)\Big[\frac{256}{45}H_{0,-1}-\frac{256}{45}H_{-1}H_0\Big]}{z^2}$$

$$+\frac{64\big(24z^3+208z^2-17z+4\big)H_{0,1}}{15z}+\frac{(z+1)\big(12z^4+18z^3-13z^2-2z+2\big)}{z^2}\Big[\frac{64}{15}H_0H_{-1}^2$$

$$-\frac{128}{15}H_{0,-1}H_{-1}+\frac{128}{15}H_{0,-1,-1}\Big]+\frac{64\big(24z^5+90z^4-75z^3-45z^2-4\big)H_{0,0,-1}}{15z^2}$$

$$+\frac{64}{15}\big(24z^3-30z^2+55z+15\big)H_{0,0,1}+\frac{(z+1)\big(6z^4-6z^3+z^2-z+1\big)}{z^2}\Big[-\frac{256}{15}H_{-1}H_{0,1}$$

$$+\frac{256}{15}H_{0,-1,1}+\frac{256}{15}H_{0,1,-1}\Big]+\Big[\frac{352}{3}zH_0-\frac{176}{3}(z-1)(2z+1)\Big]L_M-\frac{256}{3}z\big(3z^2+2\big)\zeta_3$$

$$+z\Big[\Big[64H_{0,1}-64H_{0,-1}\Big]H_0^2+\Big[256H_{0,-1,-1}+256H_{0,0,-1}-256H_{0,0,1}\Big]H_0-256\zeta_3H_0$$

$$-128H_{0,-1}^2+\frac{384\zeta_2^2}{5}+128H_{0,1,1}-384H_{0,0,0,-1}+384H_{0,0,0,1}+\Big[128H_{0,-1}-128H_{0,1}\Big]\zeta_2\Big]\Big]$$

$$+L_M\Big[-\frac{32}{5}\big(4z^3+5z^2-10z-5\big)H_0^2+\frac{16\big(2226z^3-43z^2-63z-24\big)H_0}{45z}$$

$$-\frac{8(z-1)\big(3758z^2-1299z-152\big)}{45z}-\frac{16}{3}(z-1)(2z-23)H_1$$

$$+\frac{(z+1)\big(12z^4-27z^3-58z^2-2z+2\big)}{z^2}\Big[\frac{64}{15}H_{-1}H_0-\frac{64}{15}H_{0,-1}\Big]-64\big(6z^2-z-3\big)H_{0,0,-1}$$

$$+\frac{32}{15}z\big(24z^2-85\big)\zeta_2+(z+1)(2z-1)\Big[-64H_0H_{-1}^2+\Big[32H_0^2+128H_{0,-1}\Big]H_{-1}-64\zeta_2H_{-1}$$

$$-128 H_{0,-1,-1}\Big] + (z-1)(2z+1)\Big[32 H_1 H_0^2 + \Big[64 H_{0,-1} - 64 H_{0,1}\Big]H_0 - 32 H_1^2 + 64 H_{0,0,1}$$

$$-64 H_1 \zeta_2\Big] + z\Big[-\frac{128}{3}H_0^3 + \Big[64 H_{0,-1} - 64 H_{0,1}\Big]H_0^2 + \Big[-256 H_{0,-1,-1} - 256 H_{0,0,-1}$$

$$+256 H_{0,0,1}\Big]H_0 + 256 \zeta_3 H_0 + 128 H_{0,-1}^2 - \frac{384\zeta_2^2}{5} - \frac{544}{3}H_{0,1} + 128 H_{0,1,1} + 384 H_{0,0,0,-1}$$

$$-384 H_{0,0,0,1} + \Big[128 H_{0,1} - 128 H_{0,-1}\Big]\zeta_2\Big]\Big]\Big] + \tilde{C}_{L,g}^{\mathsf{S},(3)}(N_F+1)\Bigg\} \ . \tag{603}$$

D The OMEs in z-Space

In the following, we present the massive operator matrix elements in z-space. They are given by :

$$A_{qq,Q}^{\mathsf{PS}} =$$

$$a_s^3\Bigg\{ C_F N_F T_F^2 \Bigg[L_M^2\Bigg[(z+1)\Big[\frac{64}{3}H_{0,1} - \frac{32}{3}H_0^2 - \frac{64\zeta_2}{3}\Big] + \frac{32}{9}\big(4z^2 - 7z - 13\big)H_0$$

$$-\frac{32(z-1)\big(4z^2 + 7z + 4\big)H_1}{9z} - \frac{32}{3}(z-1)(2z-5)\Bigg] + L_M\Bigg[\big(4z^2 - 7z - 13\big)\Big[\frac{64\zeta_2}{9} - \frac{64}{9}H_{0,1}\Big]$$

$$+(z+1)\Big[\frac{128}{3}H_{0,0,1} - \frac{128}{3}H_{0,1,1} - \frac{128}{3}H_0\zeta_2 - \frac{32}{9}H_0^3 - \frac{464}{9}H_0^2\Big] + \frac{32(z-1)\big(4z^2 + 7z + 4\big)H_1^2}{9z}$$

$$+\frac{128}{27}\big(19z^2 - 16z - 40\big)H_0 + \frac{64}{3}(z-1)(2z-5)H_1 - \frac{32(z-1)\big(80z^2 - 511z - 136\big)}{81z}\Bigg]$$

$$+\big(57z^2 - 131z - 203\big)\Big[\frac{64\zeta_2}{81} - \frac{64}{81}H_{0,1}\Big] + (z+1)\Big[\frac{1856}{27}H_{0,0,1} + \frac{128}{9}H_{0,0,0,1} - \frac{256}{9}H_{0,0,1,1}$$

$$+\frac{320}{9}H_{0,1,1,1} + H_0^2\Big[-\frac{64}{9}\zeta_2 - \frac{5312}{81}\Big] + H_0\Big[\frac{640\zeta_3}{9} - \frac{1856\zeta_2}{27}\Big] - \frac{8}{27}H_0^4 - \frac{464}{81}H_0^3$$

$$-\frac{256}{15}\zeta_2^2\Big] + \frac{64}{27}\big(6z^2 - 25z - 34\big)H_{0,1,1} + L_M^3\Big[\frac{32(z-1)\big(4z^2 + 7z + 4\big)}{27z} - \frac{64}{9}(z+1)H_0\Big]$$

$$-\frac{80(z-1)\big(4z^2 + 7z + 4\big)H_1^3}{81z} - \frac{16(z-1)\big(34z^2 - 227z - 20\big)H_1^2}{81z}$$

$$-\frac{64(z-1)\big(10z^2 + 213z + 64\big)H_1}{81z} + \frac{32}{243}\big(660z^2 - 1577z - 2351\big)H_0$$

$$-\frac{64\big(22z^3 + 16z^2 - 17z - 16\big)\zeta_3}{27z} - \frac{64(z-1)\big(139z^2 - 3701z - 752\big)}{729z}\Bigg]\Bigg\} \ , \tag{604}$$

$$A_{qg,Q} =$$

$$a_s^3\Bigg\{ C_A T_F^2 N_F\Bigg[-\frac{8}{27}\big(4z^2 + 16z - 5\big)H_0^4 + \frac{32}{81}z(14z + 29)H_0^3$$

$$-\frac{16}{81}\big(44z^2 + 243z - 56\big)H_0^3 + \frac{16}{81}z(200z + 347)H_0^2 - \frac{16}{81}\big(402z^2 + 1472z - 205\big)H_0^2$$

$$-\frac{64}{27}\big(7z^2 + 7z + 5\big)H_{-1}H_0^2 + \frac{8}{243}\big(5772z^2 - 27934z - 451\big)H_0 + z\Big[\frac{96}{9}\zeta_2$$

$$+\frac{11392}{81}\Big]H_0 + \frac{256}{9}(4z + 1)\zeta_3 H_0 - \frac{16}{9}\big(4z^2 - 7z - 1\big)H_1 H_0$$

$$+\frac{64}{27}\left(14z^2+11z+10\right)H_{0,-1}H_0-\frac{32}{27}\left(14z^2-17z+10\right)H_{0,1}H_0$$

$$+L_M^3\left[\frac{16(z-1)\left(31z^2+7z+4\right)}{27z}-\frac{32}{9}(4z+1)H_0-\frac{8}{9}\gamma_{qg}^0H_1\right]-\frac{16}{81}\left(218z^2-200z+85\right)H_1^2$$

$$-\frac{11392}{81}(z-1)z+\frac{4\left(173275z^3-157668z^2+21651z-17368\right)}{729z}+\frac{32}{81}\left(109z^2+47z+47\right)\zeta_2$$

$$-\frac{16}{81}\left(254z^2+137z+112\right)\zeta_2-\frac{32\left(145z^3-123z^2+3z-16\right)\zeta_3}{27z}$$

$$+z(z+1)\left[\frac{32}{27}H_0^4+\frac{0}{3}\zeta_2H_0^2\right]+\frac{32\left(1184z^3-1067z^2+487z+18\right)H_1}{243z}$$

$$+\left(200z^2+191z+112\right)\left[\frac{32}{81}H_{0,-1}-\frac{32}{81}H_{-1}H_0\right]$$

$$+L_M^2\left[-\frac{64}{3}zH_0^2+\frac{32}{9}\left(9z^2-20z-5\right)H_0+\frac{8\left(205z^3-168z^2+42z-52\right)}{27z}\right.$$

$$+\frac{32}{9}\left(4z^2-4z+5\right)H_1+\gamma_{qg}^0\left(\frac{4}{3}H_1^2-\frac{4\zeta_2}{3}\right)+\left(2z^2+2z+1\right)\left[-\frac{16}{3}\zeta_2-\frac{32}{3}H_{-1}H_0\right.$$

$$\left.+\frac{32}{3}H_{0,-1}\right]\left]+\left(7z^2-7z+5\right)\left[\frac{32}{81}H_1^3+\frac{32}{27}H_0H_1^2+\frac{32}{27}H_0^2H_1+\left[\frac{64\zeta_2}{27}-\frac{128}{27}H_{0,1}\right]H_1\right]\right.$$

$$+\frac{16}{27}\left(24z^2-134z+3\right)H_{0,1}+\left(7z^2+4z+5\right)\left[\frac{32}{9}\zeta_3-\frac{128}{27}H_{0,0,-1}\right]$$

$$+\frac{32}{27}\left(14z^2-35z+10\right)H_{0,0,1}+L_M\left[-\frac{16}{9}(10z-3)H_0^3-\frac{8}{9}\left(16z^2+128z-3\right)H_0^2\right.$$

$$+\frac{16}{27}\left(238z^2-646z+5\right)H_0-\frac{32}{9}\left(7z^2-7z+5\right)H_1^2+\frac{8\left(3791z^3-3318z^2+465z-344\right)}{81z}$$

$$+\frac{32}{9}\left(7z^2-z+5\right)\zeta_2+\frac{16}{27}\left(158z^2-149z+85\right)H_1+\left(7z^2+7z+5\right)\left[-\frac{32}{9}\zeta_2\right.$$

$$\left.-\frac{64}{9}H_{-1}H_0+\frac{64}{9}H_{0,-1}\right]-\frac{64}{3}zH_{0,1}+\left(2z^2+2z+1\right)\left[-\frac{32}{3}H_{-1}H_0^2\right.$$

$$\left.+\frac{64}{3}H_{0,-1}H_0+16\zeta_3-\frac{64}{3}H_{0,0,-1}\right]+\gamma_{qg}^0\left[-\frac{4}{9}H_1^3-\frac{4}{3}H_0^2H_1+\left[\frac{16}{3}H_{0,1}\right.\right.$$

$$\left.\left.-\frac{8\zeta_2}{3}\right]H_1+4\zeta_3+H_0\left[\frac{8}{3}H_{0,1}-\frac{4}{3}H_1^2\right]-\frac{8}{3}H_{0,0,1}-8H_{0,1,1}\right]\right]+\frac{32}{9}\left(14z^2-9z+10\right)H_{0,1,1}$$

$$+\left(2z^2+2z+1\right)\left[-\frac{64}{27}H_{-1}H_0^3+\frac{64}{9}H_{0,-1}H_0^2-\frac{128}{9}H_{0,0,-1}H_0-\frac{224}{45}\zeta_2^2\right.$$

$$\left.+\frac{128}{9}H_{0,0,0,-1}\right]+\gamma_{qg}^0\left[\frac{1}{27}H_1^4+\left[\frac{4\zeta_2}{9}-\frac{8}{9}H_{0,1}\right]H_1^2-\frac{4}{9}H_0^3H_1+\left[\frac{16\zeta_3}{3}+\frac{16}{9}H_{0,0,1}\right.\right.$$

$$\left.+\frac{16}{9}H_{0,1,1}\right]H_1-\frac{56}{45}\zeta_2^2-\frac{8}{9}H_{0,1}^2+\frac{8}{9}H_{0,1}\zeta_2+H_0^2\left[\frac{4}{3}H_{0,1}-\frac{2}{9}H_1^2\right]+H_0\left[\frac{4}{27}H_1^3-\frac{8}{9}\zeta_2H_1\right.$$

$$\left.\left.-\frac{8}{3}H_{0,0,1}+\frac{8}{9}H_{0,1,1}\right]+\frac{8}{3}H_{0,0,0,1}-\frac{8}{9}H_{0,0,1,1}-\frac{8}{9}H_{0,1,1,1}\right]\right]$$

$$+C_FT_F^2N_F\left[-\frac{4}{27}\left(28z^2+116z-67\right)H_0^4+\frac{4}{81}\left(288z^2-2582z+2341\right)H_0^3\right.$$

$$-\frac{4}{81}\left(2152z^2+5141z-13876\right)H_0^2+\frac{4}{243}\left(14880z^2+73472z\right.$$

$$\left.+154967\right)H_0+\frac{256}{9}\left(6z^2-z-4\right)\zeta_3H_0-\frac{448}{81}\left(z^2-z+2\right)H_1H_0+L_M^3\left[-\frac{16}{3}(2z-1)H_0^2\right.$$

$$
-\frac{32}{9}\left(6z^2 - z - 4\right)H_0 + \frac{8\left(124z^3 - 258z^2 + 159z - 16\right)}{27z} + \frac{8}{9}\gamma_{qg}^0 H_1\Bigg]
$$

$$
+\frac{8}{81}\left(364z^2 - 373z + 224\right)H_1^2 - \frac{179524z^3 + 2535258z^2 - 2713863z - 42688}{729z}
$$

$$
-\frac{448}{81}\left(13z^2 - 13z + 8\right)\zeta_2 - \frac{64\left(117z^3 - 251z^2 + 154z - 16\right)\zeta_3}{27z} + (2z - 1)\Bigg[\frac{128}{3}H_0^2\zeta_3
$$

$$
-\frac{16}{15}H_0^5\Bigg] - \frac{16}{243}\left(2188z^2 - 2278z + 1613\right)H_1 + L_M^2\Bigg[-\frac{32}{3}(2z - 1)H_0^3
$$

$$
-\frac{16}{3}(2z + 3)(4z - 3)H_0^2 + \frac{8}{9}\left(160z^2 + 146z + 305\right)H_0
$$

$$
-\frac{4\left(1000z^3 + 1356z^2 - 2247z - 208\right)}{27z} - \frac{32}{9}\left(4z^2 - 4z + 5\right)H_1
$$

$$
+\gamma_{qg}^0\Bigg[-\frac{4}{3}H_1^2 + \frac{8\zeta_2}{3} - \frac{8}{3}H_{0,1}\Bigg]\Bigg] + \left(16z^2 - 16z + 5\right)\Bigg[\frac{16}{27}H_1 H_0^2 - \frac{32}{27}H_{0,1}H_0 + \frac{32}{27}H_{0,0,1}\Bigg]
$$

$$
+\left(7z^2 - 7z + 5\right)\Bigg[-\frac{32}{81}H_1^3 + \frac{896}{81}H_{0,1} - \frac{64}{27}H_{0,1,1}\Bigg] + L_M\Bigg[-\frac{20}{3}(2z - 1)H_0^4
$$

$$
-\frac{16}{9}\left(14z^2 + 37z - 26\right)H_0^3 + \frac{4}{9}\left(136z^2 - 174z + 909\right)H_0^2 - \frac{8}{27}\left(32z^2 - 2393z - 4145\right)H_0
$$

$$
+\frac{64}{3}(z - 1)z H_1 H_0 - \frac{4\left(3556z^3 + 33342z^2 - 38175z + 800\right)}{81z} - \frac{16}{27}\left(140z^2 - 149z + 112\right)H_1
$$

$$
+\left(7z^2 - 7z + 5\right)\Bigg[\frac{32}{9}H_1^2 - \frac{64\zeta_2}{9}\Bigg] + \frac{64}{9}\left(4z^2 - 4z + 5\right)H_{0,1} + \gamma_{qg}^0\Bigg[\frac{4}{9}H_1^3 - \frac{4}{3}H_0^2 H_1 - \frac{8\zeta_3}{3}
$$

$$
+\frac{8}{3}H_0 H_{0,1} - \frac{8}{3}H_{0,0,1} + \frac{8}{3}H_{0,1,1}\Bigg]\Bigg] + \gamma_{qg}^0\Bigg[-\frac{1}{27}H_1^4 - \frac{8}{27}H_0^3 H_1 - \frac{64}{9}\zeta_3 H_1 + \frac{16}{45}\zeta_2^2
$$

$$
+\frac{8}{9}H_0^2 H_{0,1} - \frac{16}{9}H_0 H_{0,0,1} + \frac{16}{9}H_{0,0,0,1} - \frac{8}{9}H_{0,1,1,1}\Bigg]\Bigg]\Bigg\}\,, \tag{605}
$$

$$
A_{Qq}^{\mathsf{PS}} =
$$

$$
a_s^2\Bigg\{C_F T_F\Bigg[\frac{2}{3}\left(8z^2 + 15z + 3\right)H_0^2 - \frac{8}{9}\left(56z^2 + 33z + 21\right)H_0
$$

$$
+L_M^2\Bigg[\frac{4(z - 1)\left(4z^2 + 7z + 4\right)}{3z} - 8(z + 1)H_0\Bigg] + \frac{4(z - 1)\left(400z^2 + 121z + 112\right)}{27z}
$$

$$
+\frac{(z - 1)\left(4z^2 + 7z + 4\right)\left[\frac{8}{3}H_{0,1} - \frac{8}{3}H_0 H_1\right]}{z} + (z + 1)\Bigg[-\frac{4}{3}H_0^3 + 16H_{0,1}H_0 + 32\zeta_3 - 32H_{0,0,1}\Bigg]
$$

$$
+L_M\Bigg[8(z + 1)H_0^2 - \frac{8}{3}\left(8z^2 + 15z + 3\right)H_0 + \frac{16(z - 1)\left(28z^2 + z + 10\right)}{9z}\Bigg]\Bigg]\Bigg\}
$$

$$
+a_s^3\Bigg\{a_{Qq}^{\mathsf{PS},(3)} + C_F^2 T_F\Bigg[-\frac{2}{9}\left(4z^2 - 3z + 3\right)H_0^4 - \frac{2}{9}\left(40z^2 + 149z + 115\right)H_0^3
$$

$$
+\frac{2}{3}\left(160z^2 + 191z - 117\right)H_0^2 - \frac{8}{3}\left(4z^2 - 3z - 3\right)\zeta_2 H_0^2 - \frac{4(z - 1)\left(20z^2 + 41z - 4\right)H_1 H_0^2}{3z}
$$

$$
+\frac{8\left(4z^3 + 27z^2 + 3z - 4\right)H_{0,1}H_0^2}{3z} - \frac{4}{3}\left(400z^2 - 135z + 222\right)H_0 - \frac{2}{3}\left(80z^2 + 469z + 221\right)\zeta_2 H_0
$$

$$+\frac{16}{9}\left(44z^2+51z-18\right)\zeta_3 H_0 - \frac{8(z-1)\left(16z^2-43z+66\right)H_1 H_0}{3z}$$

$$+\frac{16(z-1)\left(10z^2+11z-2\right)H_{0,1}H_0}{3z} + \frac{16\left(4z^3-33z^2-15z+4\right)H_{0,0,1}H_0}{3z}$$

$$-\frac{8(z-1)\left(6z^2-z-6\right)H_1^3}{9z} + \frac{8}{15}\left(188z^2-27z-105\right)\zeta_2^2 - \frac{4(z-1)\left(24z^2-13z+17\right)H_1^2}{3z}$$

$$+\frac{64(z+1)^2(2z-1)H_{0,1}^2}{3z} + \frac{4(z-1)(2z+1)(352z+233)}{9z} + \frac{4\left(84z^3+79z^2+75z-60\right)\zeta_2}{3z}$$

$$-\frac{4(z-1)\left(40z^2+33z+4\right)H_1\zeta_2}{3z} - \frac{8\left(12z^3+15z^2+9z+8\right)H_{0,1}\zeta_2}{3z}$$

$$+\frac{4}{9}\left(72z^2-809z-145\right)\zeta_3 - \frac{4(z-1)\left(80z^2-181z-9\right)H_1}{3z} + L_M^3\left[\frac{16\left(4z^2+7z+4\right)H_1(z-1)}{9z}\right.$$

$$\left.+\frac{92(z-1)}{9} + \frac{16}{9}z(4z+3)H_0 + (z+1)\left[-\frac{8}{3}H_0^2 + \frac{32\zeta_2}{3} - \frac{32}{3}H_{0,1}\right]\right]$$

$$-\frac{8\left(8z^3+100z^2-85z+66\right)H_{0,1}}{3z} - \frac{32}{3}(z-1)H_1 H_{0,1} - \frac{8\left(20z^3-95z^2-43z+4\right)H_{0,0,1}}{3z}$$

$$+L_M^2\left[\frac{8}{3}\left(4z^2-9z-3\right)H_0^2 + \frac{8}{3}(z+1)(32z-31)H_0 + \frac{16(z-1)\left(4z^2+7z+4\right)H_1 H_0}{3z}\right.$$

$$+\frac{4(z-1)\left(32z^2+81z+12\right)}{3z} + 32(3z+2)\zeta_2 + \frac{8(z-1)\left(32z^2+35z+8\right)H_1}{3z}$$

$$\left.-\frac{16\left(4z^3+21z^2+9z-4\right)H_{0,1}}{3z} + (z+1)\left[-\frac{16}{3}H_0^3 + \left[32\zeta_2-32H_{0,1}\right]H_0 - 32\zeta_3 + 32H_{0,0,1}\right]\right]$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[-\frac{2}{9}H_1^4 + \left[\frac{16}{3}H_{0,1} - \frac{20\zeta_2}{3}\right]H_1^2 - \frac{8}{9}H_0^3 H_1\right.$$

$$\left.+\left[\frac{176\zeta_3}{9} - \frac{64}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1}\right]H_1 + H_0\left[H_1\left[\frac{32}{3}H_{0,1} - \frac{16\zeta_2}{3}\right] - \frac{64}{3}H_{0,1,1}\right]\right]$$

$$-\frac{8}{3}\left(12z^2-23z-22\right)H_{0,1,1} - \frac{16\left(20z^3-21z^2-33z+4\right)H_{0,0,0,1}}{3z} - \frac{32\left(3z^2+15z+8\right)H_{0,0,1,1}}{3z}$$

$$+\frac{16\left(20z^3+15z^2-27z-24\right)H_{0,1,1,1}}{3z} + L_M\left[-\frac{32}{3}\left(4z^2+z+1\right)H_0^3\right.$$

$$+\frac{2}{3}\left(136z^2-111z+213\right)H_0^2 + \frac{8}{27}\left(242z^2-3984z-633\right)H_0 + \frac{32}{3}(2z+3)(8z-3)\zeta_2 H_0$$

$$+\frac{8(z-1)\left(140z^2-127z+104\right)H_1 H_0}{9z} - 32\left(4z^2+3z+1\right)H_{0,1}H_0$$

$$+\frac{4(z-1)\left(28z^2+21z+4\right)H_1^2}{3z} + \frac{4(z-1)\left(3204z^2+1625z+180\right)}{27z} - \frac{16}{9}\left(74z^2+18z+297\right)\zeta_2$$

$$-\frac{16\left(12z^3-z^2+z-8\right)\zeta_3}{z} + \frac{16(z-1)\left(229z^2-1175z-239\right)H_1}{27z}$$

$$+\frac{8\left(8z^3+303z^2+363z+104\right)H_{0,1}}{9z} + \frac{(z-1)\left(4z^2+7z+4\right)\left[-\frac{8}{9}H_1^3 - \frac{16}{3}H_0 H_1^2 + \frac{32}{3}H_{0,1}H_1\right]}{z}$$

$$+\frac{32}{3}\left(8z^2+15\right)H_{0,0,1} - \frac{16\left(12z^3+27z^2+3z-8\right)H_{0,1,1}}{3z} + (z+1)\left[6H_0^4 - 96\zeta_2 H_0^2\right.$$

$$\left.\left.+\left[192\zeta_3+96H_{0,0,1}+64H_{0,1,1}\right]H_0 + \frac{288}{5}\zeta_2^2 - 32H_{0,1}^2 - 96H_{0,0,0,1} + 32H_{0,1,1,1}\right]\right]$$

$$+(z+1)\left[\frac{2}{15}H_0^5 + \left[4\zeta_2 + \frac{16}{3}H_{0,1}\right]H_0^3 + \left[-\frac{88}{3}\zeta_3 - 48H_{0,0,1}\right]H_0^2 + \left[-\frac{448}{5}\zeta_2^2 + 32H_{0,1}\zeta_2\right.\right.$$

$$
\begin{aligned}
&\left. -32H_{0,1}^2 + 160H_{0,0,0,1} + 128H_{0,0,1,1}\right]H_0 + 32H_{0,0,1}\zeta_2 + 80H_{0,1,1}\zeta_2 - \frac{80}{3}\zeta_2\zeta_3 + 160\zeta_5 \\
&+H_{0,1}\left[-\frac{352}{3}\zeta_3 + 128H_{0,0,1} - 64H_{0,1,1}\right] - 192H_{0,0,0,0,1} - 768H_{0,0,0,1,1} - 320H_{0,0,1,0,1} \\
&\left.\left. +416H_{0,0,1,1,1} + 192H_{0,1,0,1,1} + 32H_{0,1,1,1,1}\right]\right] + C_F T_F^2\left[\frac{16}{27}\left(8z^2 + 15z + 3\right)H_0^3\right. \\
&-\frac{32}{27}\left(56z^2 + 33z + 21\right)H_0^2 + \frac{32}{81}\left(1020z^2 + 697z + 607\right)H_0 + \frac{32}{9}\left(12z^2 + 37z + 19\right)\zeta_2 H_0 \\
&+\frac{128(z-1)\left(28z^2 + z + 10\right)H_1 H_0}{27z} - \frac{128\left(2z^3 + 6z^2 + 3z + 2\right)H_{0,1}H_0}{9z} \\
&+L_M^3\left[\frac{128(z-1)\left(4z^2 + 7z + 4\right)}{27z} - \frac{256}{9}(z+1)H_0\right] \\
&-\frac{16(z-1)\left(38z^2 + 47z + 20\right)H_1^2}{27z} - \frac{64(z-1)\left(1781z^2 + 539z + 656\right)}{243z} \\
&-\frac{32\left(56z^3 - 179z^2 - 95z - 40\right)\zeta_2}{27z} - \frac{64\left(62z^3 + 129z^2 + 36z - 8\right)\zeta_3}{27z} \\
&+\frac{128(z-1)\left(55z^2 + 64z + 28\right)H_1}{81z} + \frac{(z-1)\left(4z^2 + 7z + 4\right)\left[\frac{16}{27}H_1^3 - \frac{32}{9}H_0^2 H_1 + \frac{32}{9}\zeta_2 H_1\right]}{z} \\
&-\frac{64\left(75z^3 + 13z^2 + 61z - 20\right)H_{0,1}}{27z} + L_M^2\left[\frac{32(z-1)\left(94z^2 + 49z + 40\right)}{27z}\right. \\
&\left.-\frac{32}{9}\left(12z^2 + 37z + 19\right)H_0 - \frac{32(z-1)\left(4z^2 + 7z + 4\right)H_1}{9z} + (z+1)\left[\frac{32}{3}H_0^2 - \frac{64\zeta_2}{3} + \frac{64}{3}H_{0,1}\right]\right] \\
&+\frac{64\left(12z^3 + 27z^2 + 9z + 4\right)H_{0,0,1}}{9z} + L_M\left[\frac{256(z-1)\left(40z^2 + 79z + 31\right)}{81z}\right. \\
&-\frac{64}{27}\left(18z^2 + 65z + 101\right)H_0 + \left(4z^2 - 7z - 13\right)\left[\frac{32}{9}H_0^2 + \frac{64\zeta_2}{9}\right] + \frac{64}{3}(z-1)(2z-5)H_1 \\
&+\frac{(z-1)\left(4z^2 + 7z + 4\right)\left[\frac{32}{9}H_1^2 - \frac{64}{9}H_0 H_1\right]}{z} + \frac{128\left(5z^2 + 5z - 2\right)H_{0,1}}{9z} \\
&\left.+(z+1)\left[-\frac{64}{9}H_0^3 + \left[\frac{128}{3}H_{0,1} - \frac{128\zeta_2}{3}\right]H_0 + \frac{256\zeta_3}{3} - \frac{128}{3}H_{0,0,1} - \frac{128}{3}H_{0,1,1}\right]\right] \\
&+\frac{64}{9}\left(2z^2 + 11z + 8\right)H_{0,1,1} + (z+1)\left[-\frac{8}{9}H_0^4 + \left[\frac{64}{3}H_{0,1} - \frac{32\zeta_2}{3}\right]H_0^2\right. \\
&\left.\left.+\left[\frac{1024\zeta_3}{9} - \frac{128}{3}H_{0,0,1}\right]H_0 + \frac{448}{15}\zeta_2^2 - \frac{64}{3}H_{0,1}\zeta_2 - \frac{64}{3}H_{0,1,1,1}\right]\right] \\
&+C_F T_F^2 N_F\left[L_M^2\left[(z+1)\left[-\frac{64}{3}H_{0,1} + \frac{32}{3}H_0^2 + \frac{64\zeta_2}{3}\right] - \frac{32}{9}\left(4z^2 - 7z - 13\right)H_0\right.\right. \\
&\left.+\frac{32(z-1)\left(4z^2 + 7z + 4\right)H_1}{9z} + \frac{32}{3}(z-1)(2z-5)\right] + L_M\left[(z+1)\left[\frac{128}{3}H_0 H_{0,1}\right.\right. \\
&\left.\left.-\frac{128}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1} - \frac{128}{3}\zeta_2 H_0 - \frac{64}{9}H_0^3 + 64\zeta_3\right] - \frac{64\left(2z^3 + z^2 - 2z + 4\right)H_{0,1}}{9z}\right. \\
&+\frac{32}{9}\left(4z^2 - 7z - 13\right)H_0^2 + \frac{64}{27}\left(z^2 + 2z - 58\right)H_0 \\
&+\frac{32(z-1)\left(74z^2 - 43z + 20\right)H_1}{27z} + \frac{(z-1)\left(4z^2 + 7z + 4\right)\left[\frac{16}{9}H_1^2 - \frac{64}{9}H_0 H_1\right]}{z}
\end{aligned}
$$

$$+\frac{64}{9}\zeta_2\big(6z^2+4z-5\big)+\frac{128(z-1)\big(25z^2+94z+34\big)}{81z}\bigg]+(z+1)\bigg[\zeta_2\bigg[\frac{32}{3}H_{0,1}-\frac{32}{3}H_0^2\bigg]$$

$$+\frac{64}{3}H_0^2H_{0,1}-\frac{128}{3}H_0H_{0,0,1}+\frac{832}{9}\zeta_3H_0-\frac{8}{9}H_0^4-\frac{32\zeta_2^2}{3}\bigg]+\frac{(z-1)\big(28z^2+z+10\big)}{z}\bigg[$$

$$\frac{128}{27}H_0H_1-\frac{128}{27}H_{0,1}\bigg]-\frac{128\big(2z^3+6z^2+3z+2\big)H_0H_{0,1}}{9z}+\frac{64\big(12z^3+27z^2+9z+4\big)H_{0,0,1}}{9z}$$

$$-\frac{16}{27}(z-1)\big(74z^2-43z+20\big)\frac{\zeta_2}{z}-\frac{32}{27}\big(100z^3+183z^2+33z-4\big)\frac{\zeta_3}{z}$$

$$+L_M^3\bigg[\frac{32(z-1)\big(4z^2+7z+4\big)}{27z}-\frac{64}{9}(z+1)H_0\bigg]+\frac{32}{9}\zeta_2\big(6z^2+4z-5\big)H_0$$

$$+\frac{(z-1)\big(4z^2+7z+4\big)\big[-\frac{16}{9}\zeta_2H_1-\frac{32}{9}H_1H_0^2\big]}{z}+\frac{16}{27}\big(8z^2+15z+3\big)H_0^3$$

$$-\frac{32}{27}\big(56z^2+33z+21\big)H_0^2+\frac{32}{81}\big(800z^2-57z+111\big)H_0$$

$$-\frac{64(z-1)\big(1156z^2-203z+328\big)}{243z}\bigg]+C_AC_FT_F\bigg[-\frac{2}{9}(4z-17)H_0^4-\frac{4}{9}\big(36z^2+47z+36\big)H_0^3$$

$$-\frac{8}{3}(z+3)\zeta_2H_0^3+\frac{64}{3}z^2H_0^2+\frac{4}{27}\big(1988z^2-681z+855\big)H_0^2+8(z-1)(2z+1)\zeta_2H_0^2$$

$$-\frac{8}{3}(2z+5)(3z-4)\zeta_2H_0^2-\frac{16}{3}(20z-13)\zeta_3H_0^2+\frac{8(z-1)\big(122z^2-19z+113\big)H_1H_0^2}{9z}$$

$$-\frac{8\big(19z^2+19z+8\big)H_{0,1}H_0^2}{3z}+\frac{16}{5}(9z-4)\zeta_2^2H_0+\frac{16}{5}(29z-1)\zeta_2^2H_0$$

$$-\frac{16(z-1)\big(19z^2+16z+10\big)H_1^2H_0}{9z}-\frac{64}{9}\big(37z^2+6\big)H_0$$

$$-\frac{4\big(48876z^3+9339z^2+16218z+2624\big)H_0}{81z}-\frac{8}{9}\big(152z^2-39z+60\big)\zeta_2H_0$$

$$-\frac{4\big(170z^3+199z^2+175z+80\big)\zeta_2H_0}{9z}-\frac{16\big(24z^3-31z^2+215z+4\big)\zeta_3H_0}{9z}$$

$$-\frac{32(z-1)\big(733z^2-62z+301\big)H_1H_0}{27z}-\frac{32\big(19z^3-24z^2-6z+10\big)H_{0,-1}H_0}{9z}$$

$$+\frac{16\big(18z^3+119z^2-2z+51\big)H_{0,1}H_0}{3z}-\frac{32\big(4z^3-23z^2-2z-8\big)H_{0,0,1}H_0}{3z}$$

$$+\frac{8(z-1)(2z+1)(14z+1)H_1^3}{27z}-\frac{8\big(96z^3-427z^2+134z-148\big)\zeta_2^2}{15z}$$

$$+\frac{8\big(116z^3-87z^2-3z+4\big)\zeta_2^2}{15z}+\frac{4(z-1)\big(616z^2+313z+355\big)H_1^2}{27z}$$

$$+\frac{4(z-1)\big(75516z^2-7654z+23205\big)}{81z}-\frac{8}{9}\big(9z^2+185z+38\big)\zeta_2$$

$$+\frac{8\big(1868z^3-1164z^2+1344z-515\big)\zeta_2}{27z}+\frac{4(z-1)\big(154z^2+163z+46\big)H_1\zeta_2}{9z}$$

$$-\frac{8}{3}(23z+14)H_{0,1}\zeta_2-\frac{8\big(247z^3-9z^2+18z+50\big)\zeta_3}{9z}$$

$$+\frac{8\big(1015z^3+1149z^2+705z+126\big)\zeta_3}{9z}-\frac{256}{3}(z-2)\zeta_2\zeta_3+8(25z-9)\zeta_2\zeta_3$$

$$+8(3z+5)\zeta_5+8(67z-35)\zeta_5-\frac{64(z-1)\big(37z^2+16\big)H_1}{9z}$$

$$+\frac{4(z-1)\left(7828z^2+2755z+4075\right)H_1}{81z}+\frac{(z-1)\left(z^2+1\right)\left[-\frac{128}{3}H_1^2-\frac{128}{3}H_0H_1\right]}{z}$$

$$+\frac{(z+1)\left(182z^2-122z+47\right)\left[\frac{32}{27}H_{-1}H_0-\frac{32}{27}H_{0,-1}\right]}{z}+\frac{64\left(6z^3+19z^2+10z-6\right)H_{0,1}}{9z}$$

$$+\frac{8\left(2820z^3-3849z^2+1128z-1204\right)H_{0,1}}{27z}+L_M^3\left[\frac{16}{3}(2z-1)H_0^2+\frac{16\left(8z^2+11z+4\right)H_0}{9z}\right.$$

$$\left.-\frac{8(z-1)\left(44z^2-z+44\right)}{9z}-\frac{16(z-1)\left(4z^2+7z+4\right)H_1}{9z}+(z+1)\left[\frac{32}{3}H_{0,1}-\frac{32\zeta_2}{3}\right]\right]$$

$$+\frac{32\left(19z^3-51z^2-6z+10\right)H_{0,0,-1}}{9z}+\frac{64}{9}\left(19z^2-15z-6\right)H_{0,0,1}$$

$$-\frac{16\left(306z^3+561z^2+144z+193\right)H_{0,0,1}}{9z}+L_M^2\left[-\frac{8}{3}\left(4z^2-25z+23\right)H_0^2\right.$$

$$+\frac{8}{3}\left(4z^2-9z+6\right)H_0^2+\frac{16}{9}\left(13z^2-30z+24\right)H_0+\frac{8\left(246z^3+163z^2+91z+40\right)H_0}{9z}$$

$$+\frac{16(z-1)\left(4z^2+7z+4\right)H_1H_0}{3z}-\frac{8(z-1)\left(35z^2-82z+89\right)}{27z}$$

$$-\frac{8(z-1)\left(1829z^2-403z+605\right)}{27z}-\frac{8(z-1)\left(104z^2+119z+32\right)H_1}{9z}$$

$$+\frac{(z+1)\left(4z^2-7z+4\right)\left[-\frac{16}{3}\zeta_2-\frac{32}{3}H_{-1}H_0+\frac{32}{3}H_{0,-1}\right]}{z}+(10z+7)\left[\frac{32}{3}H_{0,1}-\frac{32\zeta_2}{3}\right]$$

$$+(z-1)\left[-\frac{32}{3}H_0^3+\left[-32\zeta_2-64H_{0,-1}\right]H_0-96\zeta_3+128H_{0,0,-1}\right]$$

$$+(z+1)\left[-32\zeta_3-32H_0H_{0,1}+32H_{0,0,1}\right]\right]+\frac{(z+1)\left(19z^2-16z+10\right)}{z}\left[-\frac{32}{9}H_0H_{-1}^2\right.$$

$$+\left[\frac{16}{9}H_0^2-\frac{32}{9}\zeta_2+\frac{64}{9}H_{0,-1}\right]H_{-1}-\frac{64}{9}H_{0,-1,-1}\right]+\frac{8\left(56z^3-201z^2-162z-40\right)H_{0,1,1}}{9z}$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[\frac{2}{9}H_1^4+\frac{4}{3}H_0^2H_1^2+\frac{4}{3}\zeta_2H_1^2+\left[-\frac{80}{9}\zeta_3+\frac{80}{3}H_{0,0,1}+\frac{16}{3}H_{0,1,1}\right]H_1\right.$$

$$\left.-\frac{16}{3}H_{0,1}^2+H_0\left[-\frac{16}{9}H_1^3+\left[-\frac{8}{3}\zeta_2-16H_{0,1}\right]H_1+\frac{80}{3}H_{0,1,1}\right]\right]$$

$$+\frac{16\left(32z^3-87z^2+45z-24\right)H_{0,0,0,1}}{3z}+(2z-1)\left[128H_0H_{0,0,0,1}-32H_0^2H_{0,0,1}\right]$$

$$-\frac{16\left(36z^3+6z^2-15z-20\right)H_{0,0,1,1}}{3z}+z(4z-3)\left[H_0\left[\frac{32}{3}\zeta_3+\frac{32}{3}H_{0,0,1}\right]\right.$$

$$\left.-\frac{128}{3}H_{0,0,0,1}+\frac{64}{3}H_{0,0,1,1}\right]+\frac{(z+1)\left(4z^2-7z+4\right)}{z}\left[\frac{32}{9}H_0H_{-1}^3\right.$$

$$+\left[-\frac{8}{3}H_0^2-\frac{-16}{3}\zeta_2-\frac{32}{3}H_{0,-1}\right]H_{-1}^2+\left[-\frac{8}{9}H_0^3+\left[\frac{24}{3}\zeta_2+\frac{32}{3}H_{0,-1}-\frac{32}{3}H_{0,1}\right]H_0\right.$$

$$\left.-16\zeta_3+\frac{64}{3}H_{0,-1,-1}-\frac{32}{3}H_{0,0,-1}+\frac{64}{3}H_{0,0,1}\right]H_{-1}-\frac{24}{3}H_{0,-1}\zeta_2+\frac{8}{3}H_0^2H_{0,-1}$$

$$+H_0\left[-\frac{32}{3}H_{0,-1,-1}+\frac{32}{3}H_{0,-1,1}-\frac{16}{3}H_{0,0,-1}+\frac{32}{3}H_{0,1,-1}\right]-\frac{64}{3}H_{0,-1,-1,-1}$$

$$-\frac{32}{3}H_{0,-1,0,1}+\frac{32}{3}H_{0,0,-1,-1}-\frac{64}{3}H_{0,0,-1,1}+\frac{16}{3}H_{0,0,0,-1}-\frac{64}{3}H_{0,0,1,-1}\right]$$

$$+L_M\left[\frac{4}{3}(4z-5)H_0^4-\frac{8}{9}(35z-46)H_0^3-\frac{4}{9}\left(606z^2-346z+377\right)H_0^2\right.$$

$$+\frac{8(z+1)\left(16z^2-19z+16\right)H_{-1}H_0^2}{3z}-\frac{4(z-1)\left(8z^2+17z+8\right)H_1H_0^2}{z}$$

$$+\frac{128}{3}z^2H_0+\frac{16\left(3657z^3+2093z^2+2330z+224\right)H_0}{27z}-\frac{16}{3}\left(16z^2-19z-25\right)\zeta_2H_0$$

$$+\frac{8}{3}\left(40z^2-51z+9\right)\zeta_2H_0+8(9z-25)\zeta_3H_0-8(73z+11)\zeta_3H_0$$

$$+\frac{16(z-1)\left(203z^2+47z+140\right)H_1H_0}{9z}+\frac{64\left(2z^3-9z^2+3z-4\right)H_{0,-1}H_0}{3z}$$

$$+\frac{8\left(16z^3-41z^2-77z-40\right)H_{0,1}H_0}{3z}-32(5z-1)H_{0,0,-1}H_0-32(11z+5)H_{0,0,1}H_0$$

$$-\frac{16}{5}(7z+27)\zeta_2^2-\frac{24}{5}(65z+11)\zeta_2^2-\frac{4(z-1)\left(20z^2+21z+2\right)H_1^2}{3z}$$

$$-\frac{8(z-1)\left(11542z^2+399z+4036\right)}{27z}-\frac{32(z+1)\left(53z^2-14z+26\right)\zeta_2}{9z}$$

$$+\frac{8\left(80z^3-157z^2+521z-64\right)\zeta_2}{9z}-\frac{16(z+1)\left(4z^2-z+4\right)H_{-1}\zeta_2}{z}$$

$$+\frac{4\left(44z^3-81z^2-213z-180\right)\zeta_3}{3z}+\frac{4\left(164z^3-231z^2+81z-12\right)\zeta_3}{3z}$$

$$+\frac{128(z-1)\left(z^2+1\right)H_1}{3z}-\frac{8(z-1)\left(258z^2-559z-138\right)H_1}{9z}$$

$$+\frac{(z+1)\left(19z^2-16z+10\right)}{z}\left[\frac{32}{9}H_{0,-1}-\frac{32}{9}H_{-1}H_0\right]$$

$$+\frac{(z+1)\left(53z^2-2z+26\right)}{z}\left[\frac{64}{9}H_{0,-1}-\frac{64}{9}H_{-1}H_0\right]$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[\frac{8}{9}H_1^3+8H_0H_1^2+\left[\frac{16\zeta_2}{3}-\frac{32}{3}H_{0,1}\right]H_1\right]$$

$$-\frac{8\left(274z^3-253z^2+611z-200\right)H_{0,1}}{9z}-\frac{16\left(32z^3-75z^2+21z-16\right)H_{0,0,-1}}{3z}$$

$$-\frac{32}{3}z(4z-3)H_{0,0,1}+\frac{32\left(8z^2+17z+14\right)H_{0,0,1}}{3z}$$

$$+\frac{(z+1)\left(4z^2-7z+4\right)}{z}\left[\frac{32}{3}H_0H_{-1}^2+\left[-\frac{8}{3}H_0^2+16\zeta_2-\frac{64}{3}H_{0,-1}\right.\right.$$

$$\left.-\frac{32}{3}H_{0,1}\right]H_{-1}+\frac{16}{3}H_0H_{0,-1}+\frac{64}{3}H_{0,-1,-1}+\frac{32}{3}H_{0,-1,1}-\frac{16}{3}H_{0,0,-1}+\frac{32}{3}H_{0,1,-1}\right]$$

$$+\frac{(z+1)\left(8z^2-5z+8\right)\left[\frac{32}{3}H_{-1}H_{0,1}-\frac{32}{3}H_{0,-1,1}-\frac{32}{3}H_{0,1,-1}\right]}{z}$$

$$+\frac{16\left(8z^3+23z^2+5z-8\right)H_{0,1,1}}{3z}+(z-1)\left[\left[8\zeta_2+32H_{0,-1}\right]H_0^2\right.$$

$$+\left[128H_{0,-1,-1}+64H_{0,0,-1}\right]H_0-64H_{0,-1}^2+64H_{0,-1}\zeta_2$$

$$\left.-96H_{0,0,0,-1}\right]-64zH_{0,0,0,1}+160(5z+1)H_{0,0,0,1}$$

$$+(z+1)\left[56H_{0,1}H_0^2-96H_{0,1,1}H_0+32H_{0,1}^2+H_{0,1}\left[-32\zeta_2-\frac{128}{3}\right]\right.$$

$$\left.+192H_{0,0,0,-1}+32H_{0,0,1,1}-32H_{0,1,1,1}\right]+\frac{128(z-1)}{3z}\Bigg]$$

$$-\frac{32(z-1)(z+2)(2z+1)H_{0,1,1,1}}{3z}-64(3z-2)H_{0,0,0,0,1}+32(23z+27)H_{0,0,0,1,1}$$

$$+ z\left[\frac{32}{3}\zeta_2 H_0^3 + 32\zeta_3 H_0^2 + 64H_{0,0,0,1}H_0 - 320H_{0,0,0,0,1} + 128H_{0,0,0,1,1}\right]$$

$$+ (z-1)\left[\frac{4}{15}H_0^5 - \frac{16}{3}H_{0,-1}H_0^3 + \left[32H_{0,0,-1} - 32H_{0,-1,-1}\right]H_0^2\right.$$

$$+ \left[32H_{0,-1}^2 + 16(-2+5)\zeta_2 H_{0,-1} + 128H_{0,-1,-1,-1} - 64H_{0,-1,0,1} - 96H_{0,0,0,-1}\right]H_0$$

$$+ 64H_{0,-1,-1}\zeta_2 - 96H_{0,0,-1}\zeta_2 + H_{0,-1}\left[48(-5+3)\zeta_3 - 128H_{0,-1,-1} - 64H_{0,0,-1} + 128H_{0,0,1}\right]$$

$$+ 256H_{0,-1,0,-1,-1} + 512H_{0,0,-1,-1,-1} + 64H_{0,0,-1,0,-1} - 128H_{0,0,-1,0,1} + 192H_{0,0,0,-1,-1}$$

$$\left. - 384H_{0,0,0,-1,1} + 128H_{0,0,0,0,-1} - 384H_{0,0,0,1,-1} - 128H_{0,0,1,0,-1}\right]$$

$$+ (z+1)\left[-16H_{0,1,1}H_0^2 + \left[48H_{0,1}^2 + \left[16\zeta_2 + \frac{128}{3}\right]H_{0,1} - 128H_{0,0,1,1} + 64H_{0,1,1,1}\right]H_0\right.$$

$$+ H_{0,1,1}\left[\frac{256}{3} - 16\zeta_2\right] - 16H_{0,0,1}\zeta_2 + H_{0,1}\left[\frac{160\zeta_3}{3} - 160H_{0,0,1}\right] + 288H_{0,0,1,0,1}$$

$$\left.\left.\left. - 128H_{0,0,1,1,1} - 32H_{0,1,0,1,1} - 32H_{0,1,1,1,1}\right] - \frac{640(z-1)}{9z}\right]\right\}, \tag{606}$$

$$A_{Qg} =$$

$$a_s\gamma_{qg}^0 T_F L_M + a_s^2\left\{\frac{4}{3}\gamma_{qg}^0 T_F^2 L_M^2\right.$$

$$+ C_A T_F\left[-\frac{4}{3}(2z+1)H_0^3 + \frac{2}{3}(23z^2 + 12z + 3)H_0^2 - 32z^2 H_0 - \frac{4}{9}(328z^2 + 129z + 42)H_0\right.$$

$$- \frac{4(z-1)(65z^2 + 17z + 8)H_1 H_0}{3z} + 16(4z+1)H_{0,1}H_0 + 2(z-1)(5z+1)H_1^2$$

$$+ L_M^2\left[\frac{4(z-1)(31z^2 + 7z + 4)}{3z} - 8(4z+1)H_0 - 2\gamma_{qg}^0 H_1\right] + \frac{4(1588z^3 - 1413z^2 - 9z - 112)}{27z}$$

$$- 4(z-12)z\zeta_2 - 2(26z^2 + 26z + 5)\zeta_3 + 2(26z^2 + 82z + 21)\zeta_3 - 32(z-1)zH_1$$

$$+ 4(12z^2 - 12z - 1)H_1 + L_M\left[8(2z+1)H_0^2 - \frac{8}{3}(44z^2 + 24z + 3)H_0 + 32(z-1)zH_1\right.$$

$$\left.+ \frac{8(218z^3 - 225z^2 + 18z - 20)}{9z} + \gamma_{qg}^0\left[2\zeta_2 - 2H_1^2\right] + (2z^2 + 2z + 1)\left[8\zeta_2 + 16H_{-1}H_0 - 16H_{0,-1}\right]\right]$$

$$+ \frac{4(68z^3 - 72z^2 - 9z - 8)H_{0,1}}{3z} + z\left[32H_{0,1} - 32\zeta_2\right] - 32(z^2 + 5z + 1)H_{0,0,1}$$

$$+ z(z+1)\left[16H_{-1}H_0 - 16H_{0,-1} + 32H_{0,0,1}\right] + (2z^2 + 2z + 1)\left[-8H_0 H_{-1}^2 + \left[4H_0^2 - 8\zeta_2\right.\right.$$

$$\left.\left.\left. + 16H_{0,-1}\right]H_{-1} - 8H_0 H_{0,-1} - 16H_{0,-1,-1} + 8H_{0,0,-1}\right] + \gamma_{qg}^0\left[-\frac{1}{3}H_1^3 + H_0 H_1^2 - 2H_{0,1,1}\right]\right]$$

$$+ C_F T_F\left[\frac{2}{3}(4z^2 - 2z + 1)H_0^3 + (20z^2 - 12z - 1)H_0^2 - 2(24z^2 + 9z + 8)H_0\right.$$

$$+ 4(10z^2 - 12z + 1)H_1 H_0 - 16(z-1)^2 H_{0,1}H_0 - 4(z-1)(3z+1)H_1^2$$

$$+ L_M^2\left[2(4z-1) - 4(4z^2 - 2z + 1)H_0 + 2\gamma_{qg}^0 H_1\right] + 2(40z^2 - 41z + 13) + 8(3z^2 - 6z - 1)\zeta_2$$

$$+ 8(2z^2 + 2z - 1)\zeta_3 - 4z(12z - 13)H_1 + L_M\left[-4(4z^2 - 2z + 1)H_0^2 - 4(8z^2 - 4z + 3)H_0\right.$$

$$-4\left(20z^2 - 29z + 14\right) - 32(z-1)zH_1 + \gamma_{qg}^0\left[2H_1^2 + 4H_0H_1 - 4\zeta_2\right]\bigg] - 4\left(16z^2 - 24z - 1\right)H_{0,1}$$

$$+8\left(2z^2 - 6z + 3\right)H_{0,0,1} + \gamma_{qg}^0\left[\frac{1}{3}H_1^3 - H_0^2H_1 + 2H_{0,1,1}\right]\bigg]\bigg\}$$

$$+a_s^3\bigg\{a_{Qg}^{(3)} + T_F^3\left[\frac{16}{9}\gamma_{qg}^0 L_M^3 - \frac{16}{9}\gamma_{qg}^0\zeta_3\right] + C_A T_F^2\left[-\frac{4}{9}(2z+3)H_0^4\right.$$

$$+\frac{8}{27}\left(46z^2 + 74z - 13\right)H_0^3 - \frac{8}{27}\left(458z^2 - 382z + 221\right)H_0^2 + \frac{8}{3}\left(16z^2 + 10z - 7\right)\zeta_2 H_0^2$$

$$-\frac{16(z-1)\left(65z^2 + 17z + 8\right)H_1 H_0^2}{9z} + \frac{64}{3}(4z+1)H_{0,1}H_0^2 + \frac{16}{81}\left(6612z^2 + 5083z + 346\right)H_0$$

$$+\frac{8}{9}\left(176z^2 + 248z + 41\right)\zeta_2 H_0 + \frac{32}{9}\left(24z^2 + 100z + 31\right)\zeta_3 H_0$$

$$+\frac{16\left(854z^3 - 882z^2 + 99z - 80\right)H_1 H_0}{27z} - \frac{32\left(23z^3 + 72z^2 + 15z + 8\right)H_{0,1}H_0}{9z}$$

$$+\frac{128}{3}\left(2z^2 - 4z - 1\right)H_{0,0,1}H_0 - \frac{16}{9}(z-1)(5z+1)H_1^3 + L_M^3\left[\frac{112(z-1)\left(31z^2 + 7z + 4\right)}{27z}\right.$$

$$\left.-\frac{224}{9}(4z+1)H_0 - \frac{56}{9}\gamma_{qg}^0 H_1\right] + \frac{8}{15}\left(222z^2 - 102z - 1\right)\zeta_2^2 - \frac{8}{15}\left(222z^2 + 222z - 1\right)\zeta_2^2$$

$$-\frac{8}{3}\left(38z^2 - 43z + 3\right)H_1^2 - \frac{8\left(82666z^3 - 87018z^2 + 8835z - 5788\right)}{243z} - \frac{160}{9}\left(5z^2 - 5z + 1\right)H_1\zeta_2$$

$$-\frac{4\left(896z^3 - 336z^2 + 43z - 128\right)\zeta_2}{9z} + \frac{32}{3}z(13z+5)\zeta_3 - \frac{16\left(1489z^3 + 408z^2 + 51z - 28\right)\zeta_3}{27z}$$

$$+z^2\left[H_0\left[\frac{256}{3}\zeta_2 + \frac{1024}{3}\right] - \frac{128}{3}H_0^2\right] - \frac{16\left(2330z^3 - 2321z^2 + 391z + 36\right)H_1}{81z}$$

$$+(z-1)z\left[\frac{256}{3}H_1^2 + \frac{1024}{3}H_1 + H_0\left[\frac{64}{3}H_1^2 + \frac{256}{3}H_1\right]\right]\bigg]$$

$$+L_M^2\bigg[\frac{64}{3}(z+1)H_0^2 - \frac{32}{9}\left(79z^2 + 68z + 11\right)H_0 + \frac{8\left(1769z^3 - 1788z^2 + 96z - 212\right)}{27z}$$

$$+\frac{32}{9}\left(28z^2 - 28z + 5\right)H_1 + \gamma_{qg}^0\left[4\zeta_2 - 4H_1^2\right] + \left(2z^2 + 2z + 1\right)\left[16\zeta_2 + 32H_{-1}H_0\right.$$

$$\left.-32H_{0,-1}\right]\bigg] - \frac{16\left(980z^3 - 1218z^2 + 117z - 80\right)H_{0,1}}{27z} - \frac{256}{3}(z-2)zH_{0,0,1}$$

$$+\frac{32\left(135z^3 + 96z^2 + 21z + 8\right)H_{0,0,1}}{9z} + L_M\bigg[-\frac{16}{9}(14z-1)H_0^3 + \frac{8}{9}\left(30z^2 - 104z + 9\right)H_0^2$$

$$-\frac{256}{3}z^2 H_0 - \frac{16}{27}\left(418z^2 + 904z + 79\right)H_0 - \frac{32(z-1)\left(65z^2 + 17z + 8\right)H_1 H_0}{9z}$$

$$-\frac{32}{3}\left(2z^2 - 18z - 3\right)H_{0,1}H_0 + \frac{16}{9}\left(z^2 + 2z - 13\right)H_1^2 + \frac{8\left(3285z^3 - 2894z^2 + 95z - 264\right)}{27z}$$

$$+\frac{32}{9}\left(4z^2 + 35z + 5\right)\zeta_2 - \frac{32}{9}\left(7z^2 + 31z + 5\right)\zeta_2 - \frac{32}{3}\left(10z^2 + 10z + 1\right)\zeta_3$$

$$+\frac{32}{3}\left(10z^2 + 44z + 9\right)\zeta_3 - \frac{256}{3}(z-1)zH_1 + \frac{16}{27}\left(374z^2 - 365z + 67\right)H_1$$

$$+\left(7z^2 + 7z + 5\right)\left[\frac{64}{9}H_{0,-1} - \frac{64}{9}H_{-1}H_0\right] + \frac{256}{3}zH_{0,1} + \frac{32\left(68z^3 - 78z^2 - 9z - 8\right)H_{0,1}}{9z}$$

$$-\frac{32}{3}\left(6z^2 + 42z + 7\right)H_{0,0,1} + z(z+1)\left[\frac{128}{3}H_{-1}H_0 - \frac{128}{3}H_{0,-1} + \frac{256}{3}H_{0,0,1}\right]$$

$$+\left(2z^2+2z+1\right)\left[-\frac{64}{3}H_0H_{-1}^2+\left[-\frac{64}{3}\zeta_2+\frac{128}{3}H_{0,-1}\right]H_{-1}-\frac{128}{3}H_{0,-1,-1}\right]$$

$$+\gamma_{qg}^0\left[-\frac{4}{3}H_1^3+\frac{4}{3}H_0H_1^2-\frac{4}{3}H_0^2H_1+\left[\frac{16}{3}H_{0,1}-\frac{8\zeta_2}{3}\right]H_1-\frac{40}{3}H_{0,1,1}\right]\right]-\frac{32}{3}z(5z-19)H_{0,1,1}$$

$$+z\left[\frac{512}{3}\zeta_2-\frac{256}{3}H_0H_{0,1}-\frac{512}{3}H_{0,1,1}\right]-\frac{256}{3}z(4z+1)H_{0,0,0,1}$$

$$+\frac{64}{3}\left(18z^2-2z+5\right)H_{0,0,1,1}+z(z+1)\left[\frac{128}{3}H_0H_{-1}^2+\left[-\frac{64}{3}H_0^2-128H_0-\frac{-128}{3}\zeta_2\right.\right.$$

$$\left.-\frac{256}{3}H_{0,-1}\right]H_{-1}-\frac{128}{3}H_0^2\zeta_2+128H_{0,-1}-\frac{256}{3}H_{0,1}+\frac{256}{3}H_{0,-1,-1}-\frac{128}{3}H_{0,0,-1}$$

$$+H_0\left[-\frac{256}{3}\zeta_3+\frac{128}{3}H_{0,-1}-\frac{256}{3}H_{0,0,1}\right]+\frac{1024}{3}H_{0,0,0,1}-\frac{512}{3}H_{0,0,1,1}\right]$$

$$+\left(2z^2+2z+1\right)\left[-\frac{128}{9}H_0H_{-1}^3+\left[\frac{32}{3}H_0^2-\frac{64}{3}\zeta_2+\frac{128}{3}H_{0,-1}\right]H_{-1}^2$$

$$+\left[\frac{32}{9}H_0^3+\left[-\frac{80}{3}\zeta_2-\frac{128}{3}H_{0,-1}+\frac{128}{3}H_{0,1}\right]H_0+64\zeta_3-\frac{256}{3}H_{0,-1,-1}+\frac{128}{3}H_{0,0,-1}\right.$$

$$\left.-\frac{256}{3}H_{0,0,1}\right]H_{-1}+\frac{80}{3}H_{0,-1}\zeta_2-\frac{32}{3}H_0^2H_{0,-1}+H_0\left[\frac{128}{3}H_{0,-1,-1}-\frac{128}{3}H_{0,-1,1}+\frac{64}{3}H_{0,0,-1}\right.$$

$$-\frac{128}{3}H_{0,1,-1}\right]+\frac{256}{3}H_{0,-1,-1,-1}+\frac{128}{3}H_{0,-1,0,1}-\frac{128}{3}H_{0,0,-1,-1}+\frac{256}{3}H_{0,0,-1,1}-\frac{64}{3}H_{0,0,0,-1}$$

$$+\frac{256}{3}H_{0,0,1,-1}\right]+\gamma_{qg}^0\left[\frac{2}{9}H_1^4+\frac{4}{3}H_0^2H_1^2+\frac{10}{3}\zeta_2H_1^2+\left[-\frac{40}{9}\zeta_3+\frac{80}{3}H_{0,0,1}+\frac{16}{3}H_{0,1,1}\right]H_1\right.$$

$$-\frac{16}{3}H_{0,1}^2+H_0\left[-\frac{16}{9}H_1^3-16H_{0,1}H_1+\frac{80}{3}H_{0,1,1}\right]-\frac{16}{3}H_{0,1,1,1}\right]\right]$$

$$+C_AT_F^2N_F\left[-\frac{4}{9}(2z+1)H_0^4+\frac{8}{27}\left(23z^2+12z+3\right)H_0^3-\frac{8}{27}\left(292z^2+39z+42\right)H_0^2\right.$$

$$-\frac{8(z-1)\left(65z^2+17z+8\right)H_1H_0^2}{9z}+\frac{32}{3}(4z+1)H_{0,1}H_0^2+\frac{32}{3}(z-1)zH_1^2H_0$$

$$+\frac{16}{81}\left(3392z^2+645z+111\right)H_0+\frac{32(z-1)\left(254z^2-7z+20\right)H_1H_0}{27z}$$

$$-\frac{16\left(23z^3+96z^2+15z+8\right)H_{0,1}H_0}{9z}+\frac{8}{9}\left(62z^2-16z-7\right)\zeta_2H_0+\frac{32}{9}(28z+13)\zeta_3H_0$$

$$-\frac{8}{9}(z-1)(5z+1)H_1^3+L_M^3\left[\frac{16(z-1)\left(31z^2+7z+4\right)}{27z}-\frac{32}{9}(4z+1)H_0\right.$$

$$\left.-\frac{8}{9}\gamma_{qg}^0H_1\right]-\frac{8}{3}\left(4z^2-6z+1\right)H_1^2-\frac{32\left(5854z^3-6219z^2+531z-328\right)}{243z}$$

$$+\frac{4}{9}\left(19z^3-50z^2+2z-4\right)\frac{\zeta_2}{z}-\frac{16}{27}\left(550z^3+228z^2+33z-4\right)\frac{\zeta_3}{z}$$

$$+\frac{32}{3}\left(2z^2-2z-1\right)H_1-\frac{32\left(290z^3-261z^2+27z-20\right)H_{0,1}}{27z}-\frac{16}{3}z(5z-4)H_{0,1,1}$$

$$+\frac{16\left(111z^3+144z^2+21z+8\right)H_{0,0,1}}{9z}-\frac{16}{9}\left(2z^2-2z-5\right)H_1\zeta_2+z(z+1)\left[\frac{64}{3}H_0H_{-1}^2\right.$$

$$+\left[-\frac{32}{3}H_0^2-64H_0-\frac{128}{3}H_{0,-1}\right]H_{-1}+\frac{64}{3}\zeta_2H_{-1}+\frac{64}{3}H_0H_{0,-1}+64H_{0,-1}+\frac{128}{3}H_{0,-1,-1}$$

$$\left.-\frac{64}{3}H_{0,0,-1}\right]+(6z+1)\left[-\frac{8}{3}\zeta_2H_0^2-\frac{64}{3}H_{0,0,1}H_0\right]+z\left[128H_{0,0,0,1}-\frac{1216\zeta_2^2}{15}\right]$$

$$+L_M^2\left[-\frac{4}{3}\gamma_{qg}^0H_1^2-\frac{32}{9}\left(4z^2-4z+5\right)H_1-\frac{8\left(205z^3-168z^2+42z-52\right)}{27z}\right.$$

$$
-\frac{32}{9}\left(9z^2 - 20z - 5\right)H_0 + \left(2z^2 + 2z + 1\right)\left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\right] + z\left[\frac{64}{3}H_0^2 + \frac{64\zeta_2}{3}\right]\Bigg]
$$

$$
+L_M\Bigg[-\frac{16}{9}\left(10z - 1\right)H_0^3 + \frac{8}{9}\left(7z^2 - 66z - 13\right)H_0^2 + \frac{16}{27}\left(58z^2 - 269z - 2\right)H_0
$$

$$
-\frac{16(z-1)\left(65z^2 + 17z + 8\right)H_1 H_0}{9z} - \frac{32}{3}\left(2z^2 - 10z - 1\right)H_{0,1}H_0 - \frac{8}{9}\left(13z^2 - 16z + 23\right)H_1^2
$$

$$
+\frac{8\left(592z^3 - 268z^2 - 119z - 4\right)}{27z} + \frac{16}{27}\left(64z^2 - 64z + 29\right)H_1 + \left(4z^2 + 4z + 5\right)\left[\frac{64}{9}H_{0,-1}\right.
$$

$$
\left.-\frac{64}{9}H_{-1}H_0\right] + \frac{16\left(68z^3 - 54z^2 - 9z - 8\right)H_{0,1}}{9z} + \frac{32}{3}\left(2z^2 - 18z - 3\right)H_{0,0,1} - \frac{16}{9}z(3z + 10)\zeta_2
$$

$$
+\left(2z^2 + 2z + 1\right)\left[-\frac{32}{3}H_0 H_{-1}^2 + \left[\frac{64}{3}H_{0,-1} - \frac{16}{3}H_0^2\right]H_{-1} - \frac{32}{3}\zeta_2 H_{-1} + \frac{32}{3}H_0 H_{0,-1}\right.
$$

$$
\left.-\frac{64}{3}H_{0,-1,-1} - \frac{32}{3}H_{0,0,-1}\right] + \gamma_{qg}^0\left[-\frac{8}{9}H_1^3 - \frac{4}{3}H_0^2 H_1 + \frac{16}{3}H_{0,1}H_1 - \frac{8}{3}\zeta_2 H_1 - \frac{32}{3}H_{0,1,1}\right]
$$

$$
+\frac{128}{3}(5z + 1)\zeta_3\Bigg] + \left(2z^2 + 2z + 1\right)\left[-\frac{64}{9}H_0 H_{-1}^3 + \left[\frac{16}{3}H_0^2 + \frac{64}{3}H_{0,-1}\right]H_{-1}^2 + \left[\frac{16}{9}H_0^3\right.\right.
$$

$$
+\left[\frac{64}{3}H_{0,1} - \frac{64}{3}H_{0,-1}\right]H_0 - \frac{128}{3}H_{0,-1,-1} + \frac{64}{3}H_{0,0,-1} - \frac{128}{3}H_{0,0,1}\right]H_{-1} + 32\zeta_3 H_{-1}
$$

$$
-\frac{16}{3}H_0^2 H_{0,-1} + H_0\left[\frac{64}{3}H_{0,-1,-1} - \frac{64}{3}H_{0,-1,1} + \frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,1,-1}\right] + \frac{128}{3}H_{0,-1,-1,-1}
$$

$$
+\frac{64}{3}H_{0,-1,0,1} - \frac{64}{3}H_{0,0,-1,-1} + \frac{128}{3}H_{0,0,-1,1} - \frac{32}{3}H_{0,0,0,-1} + \frac{128}{3}H_{0,0,1,-1} + \left[-\frac{32}{3}H_{-1}^2\right.
$$

$$
\left.\left.-\frac{32}{3}H_0 H_{-1} + \frac{32}{3}H_{0,-1}\right]\zeta_2\right] + \gamma_{qg}^0\left[\frac{1}{9}H_1^4 + \frac{2}{3}H_0^2 H_1^2 + \frac{4}{3}\zeta_2 H_1^2 + \left[\frac{40}{3}H_{0,0,1} + \frac{8}{3}H_{0,1,1}\right]H_1\right.
$$

$$
\left.-\frac{40}{9}\zeta_3 H_1 - \frac{8}{3}H_{0,1}^2 + H_0\left[-\frac{8}{9}H_1^3 - 8H_{0,1}H_1 + \frac{40}{3}H_{0,1,1}\right] - \frac{40}{3}H_{0,0,1,1} - \frac{8}{3}H_{0,1,1,1}\right]\Bigg]
$$

$$
+C_A^2 T_F\Bigg[\frac{1}{9}\left(18z^2 - 26z + 23\right)H_0^4 + \frac{2}{9}\left(225z^2 - 12z - 35\right)H_0^3 - \frac{8}{3}(z - 15)z\zeta_2 H_0^3
$$

$$
+\frac{8}{3}\left(z^2 - 7z - 5\right)\zeta_2 H_0^3 - \frac{4\left(60z^3 + 46z^2 + 5z + 8\right)H_{-1}H_0^3}{9z} + \frac{8(z-1)\left(65z^2 + 17z + 8\right)H_1 H_0^3}{9z}
$$

$$
-\frac{32}{3}(4z + 1)H_{0,1}H_0^3 - \frac{4\left(120z^3 + 106z^2 + 5z + 8\right)H_{-1}^2 H_0^2}{3z} + \frac{2\left(190z^3 - 142z^2 - 13z - 24\right)H_1^2 H_0^2}{3z}
$$

$$
+\frac{2}{27}\left(5248z^2 - 6573z + 738\right)H_0^2 + \frac{8}{3}\left(37z^2 + 46z - 3\right)\zeta_2 H_0^2
$$

$$
-\frac{2}{3}\left(110z^2 + 190z - 59\right)\zeta_2 H_0^2 - \frac{32}{3}(43z - 5)\zeta_3 H_0^2 + \frac{8(z+1)\left(359z^2 - 32z + 20\right)H_{-1}H_0^2}{9z}
$$

$$
+\frac{2(z-1)\left(1907z^2 - 169z + 308\right)H_1 H_0^2}{9z} + \frac{4\left(96z^3 + 70z^2 + 5z + 8\right)H_{0,-1}H_0^2}{3z}
$$

$$
+\frac{8}{3}\left(32z^2 - 86z + 7\right)H_{0,1}H_0^2 + 16\left(6z^2 - 2z + 5\right)H_{0,-1,-1}H_0^2 - 64\left(z^2 + 2z - 1\right)H_{0,0,1}H_0^2
$$

$$
+96(z - 3)zH_{0,1,1}H_0^2 - \frac{8\left(93z^3 - 82z^2 + 8z - 8\right)H_1^3 H_0}{9z} + \frac{32}{5}(2z + 3)(10z - 3)\zeta_2^2 H_0
$$

$$
-\frac{16}{5}\left(40z^2 - 76z + 1\right)\zeta_2^2 H_0 - \frac{8\left(169z^3 - 157z^2 + 11z - 20\right)H_1^2 H_0}{3z} + 32\left(z^2 + 9z + 3\right)H_{0,1}^2 H_0
$$

$$
-\frac{32}{9}\left(701z^2 + 12\right)H_0 - \frac{4\left(175490z^3 + 100242z^2 + 15513z + 2624\right)H_0}{81z}
$$

$$
-\frac{4}{9}\left(2359z^2 + 462z + 120\right)\zeta_2 H_0 - \frac{2\left(1756z^3 + 940z^2 + 925z + 160\right)\zeta_2 H_0}{9z}
$$

$$-\frac{8\left(180z^3+166z^2+5z+8\right)H_{-1}\zeta_2 H_0}{3z}+\frac{8\left(247z^3+212z^2+7z+20\right)H_{-1}\zeta_2 H_0}{3z}$$

$$-\frac{8(z-1)\left(137z^2+41z+20\right)H_1\zeta_2 H_0}{3z}-8\left(18z^2-22z+19\right)H_{0,-1}\zeta_2 H_0$$

$$+8\left(2z^2+38z+11\right)H_{0,1}\zeta_2 H_0+\frac{8}{3}z(51z+62)\zeta_3 H_0$$

$$+\frac{8\left(57z^3+154z^2-269z-8\right)\zeta_3 H_0}{9z}-\frac{8\left(11936z^3-12231z^2+1431z-1244\right)H_1 H_0}{27z}$$

$$-\frac{16\left(323z^3+111z^2-12z+20\right)H_{0,-1}H_0}{9z}+\frac{64}{3}\left(6z^2+33z+2\right)H_{0,1}H_0$$

$$+\frac{4\left(1213z^3+4308z^2-3z+468\right)H_{0,1}H_0}{9z}-\frac{8\left(406z^3-330z^2-3z-40\right)H_1 H_{0,1}H_0}{3z}$$

$$-\frac{16\left(108z^3+82z^2+5z+8\right)H_{0,-1,-1}H_0}{3z}-\frac{8\left(132z^3+94z^2+5z+8\right)H_{0,0,-1}H_0}{3z}$$

$$+\frac{32}{3}z(54z+11)H_{0,0,1}H_0-\frac{16\left(237z^3-188z^2+28z-8\right)H_{0,0,1}H_0}{3z}$$

$$+\frac{8\left(768z^3-542z^2+19z-56\right)H_{0,1,1}H_0}{3z}+64\left(z^2-5z+2\right)H_{0,0,0,-1}H_0$$

$$-256z(2z+1)H_{0,0,0,1}H_0+128\left(4z^2+9z-1\right)H_{0,0,0,1}H_0-32\left(18z^2+26z+9\right)H_{0,0,1,1}H_0$$

$$-32\left(6z^2-14z+1\right)H_{0,1,1,1}H_0+\frac{\left(204z^3-166z^2-19z-8\right)H_1^4}{9z}+\frac{2\left(3049z^3-3168z^2+15z-4\right)H_1^3}{27z}$$

$$-\frac{2\left(3846z^3-10226z^2+675z-592\right)\zeta_2^2}{15z}+\frac{2\left(4534z^3+3158z^2-45z+16\right)\zeta_2^2}{15z}$$

$$+\frac{2\left(19952z^3-19524z^2+615z-710\right)H_1^2}{27z}-\frac{8\left(161z^3-130z^2-4z-16\right)H_{0,1}^2}{3z}$$

$$+\frac{4\left(1158802z^3-1178838z^2+87399z-70927\right)}{243z}+\frac{4\left(102z^3-94z^2+11z-8\right)H_1^2\zeta_2}{3z}$$

$$+\frac{2}{9}\left(5643z^2-5900z-143\right)\zeta_2+\frac{2\left(40619z^3+7491z^2+5019z-2176\right)\zeta_2}{27z}$$

$$+\frac{4\left(2247z^3-2188z^2+116z-242\right)H_1\zeta_2}{9z}+\frac{8\left(72z^3+94z^2+5z+8\right)H_{0,-1}\zeta_2}{3z}$$

$$-\frac{8\left(175z^3+164z^2+7z+20\right)H_{0,-1}\zeta_2}{3z}-\frac{8\left(124z^3+114z^2+33z+16\right)H_{0,1}\zeta_2}{3z}$$

$$+16\left(6z^2-26z+11\right)H_{0,-1,-1}\zeta_2+8\left(18z^2-62z+29\right)H_{0,0,-1}\zeta_2$$

$$+8\left(46z^2+2z-7\right)H_{0,0,1}\zeta_2-16\left(30z^2-22z+17\right)H_{0,1,1}\zeta_2$$

$$-\frac{4\left(3551z^3+1551z^2+36z+100\right)\zeta_3}{9z}+\frac{4\left(27017z^3+29781z^2+3645z+748\right)\zeta_3}{27z}$$

$$-4\left(66z^2-134z+19\right)\zeta_2\zeta_3+\frac{4}{3}\left(198z^2+154z+223\right)\zeta_2\zeta_3$$

$$+\frac{4\left(304z^3+262z^2+15z+24\right)H_{-1}\zeta_3}{z}-\frac{4\left(528z^3+458z^2+25z+40\right)H_{-1}\zeta_3}{z}$$

$$-\frac{8\left(874z^3-698z^2-41z-80\right)H_1\zeta_3}{9z}+32\left(13z^2-17z+14\right)H_{0,-1}\zeta_3$$

$$+\frac{160}{3}\left(2z^2+6z+3\right)H_{0,1}\zeta_3+8(12z+5)\zeta_5+104(20z-3)\zeta_5$$

$$+(4z-1)\left[\frac{4}{15}H_0^5-32H_{0,-1}\zeta_2 H_0+64H_{0,0,-1}\zeta_2\right]-\frac{32(z-1)\left(701z^2+32\right)H_1}{9z}$$

$$+\frac{4\big(34870z^3-34925z^2+2054z-2233\big)H_1}{27z}+\frac{(z-1)\big(57z^2+2\big)\Big[-\frac{64}{3}H_1^2-\frac{64}{3}H_0H_1\Big]}{z}$$

$$+(z-1)z\Big[-128H_1^3-192H_0H_1^2-64H_0^2H_1+384\zeta_2H_1\Big]$$

$$+\frac{(z+1)\big(2443z^2-244z+94\big)\Big[\frac{16}{27}H_{-1}H_0-\frac{16}{27}H_{0,-1}\Big]}{z}+\frac{32\big(17z^2+20z-12\big)H_{0,1}}{9z}$$

$$+\frac{8\big(9594z^3-17547z^2+885z-1244\big)H_{0,1}}{27z}+\frac{8\big(84z^3-70z^2+5z-8\big)H_1H_{0,1,1}}{3z}$$

$$+L_M^3\Bigg[\frac{16}{3}(8z-1)H_0^2-\frac{8\big(18z^3-152z^2-11z-8\big)H_0}{9z}-\frac{4(z-1)\big(1883z^2-97z+272\big)}{27z}$$

$$-\frac{64}{3}(4z+1)\zeta_2-\frac{8\big(146z^3-118z^2-z-16\big)H_1}{9z}+\gamma_{qg}^0\Big[\frac{8}{3}H_1^2+\frac{8}{3}H_0H_1\Big]+\frac{32}{3}\big(2z^2+6z+3\big)H_{0,1}\Bigg]$$

$$+\frac{16\big(287z^3-105z^2-12z+20\big)H_{0,0,-1}}{9z}+\frac{32}{9}\big(251z^2-375z-12\big)H_{0,0,1}$$

$$-\frac{4\big(5533z^3+6840z^2+399z+628\big)H_{0,0,1}}{9z}+\frac{8\big(728z^3-590z^2-11z-72\big)H_1H_{0,0,1}}{3z}$$

$$+\frac{(z+1)\big(305z^2-32z+20\big)}{z}\Big[-\frac{16}{9}H_0H_{-1}^2+\Big[-\frac{16}{3}\zeta_2+\frac{32}{9}H_{0,-1}+\frac{32}{9}H_{0,1}\Big]H_{-1}$$

$$-\frac{32}{9}H_{0,-1,-1}-\frac{32}{9}H_{0,-1,1}-\frac{32}{9}H_{0,1,-1}\Big]-192\big(z^2+5z+2\big)H_{0,1}H_{0,0,1}$$

$$+\frac{(z+1)\big(251z^2-32z+20\big)}{z}\Big[H_{-1}\Big[\frac{32}{9}\zeta_2-\frac{32}{9}H_{0,1}\Big]+\frac{32}{9}H_{0,-1,1}+\frac{32}{9}H_{0,1,-1}\Big]$$

$$+\frac{128}{3}\big(9z^2+30z+2\big)H_{0,1,1}+\frac{4\big(2561z^3-6192z^2-144z-240\big)H_{0,1,1}}{9z}$$

$$+L_M^2\Bigg[\frac{16}{3}z(z+1)H_0^3-\frac{16}{3}\big(z^2+9z-2\big)H_0^3-\frac{8}{3}\big(13z^2-8z+12\big)H_0^2$$

$$+\frac{8}{3}\big(31z^2-6z+6\big)H_0^2+\frac{8}{9}\big(197z^2-42z+48\big)H_0+\frac{16\big(569z^3+376z^2+109z+20\big)H_0}{9z}$$

$$+32(2z+1)\zeta_2H_0-32(4z-1)\zeta_2H_0+\frac{16(z-1)\big(19z^2+7z+4\big)H_1H_0}{3z}$$

$$+16\big(6z^2-10z+7\big)H_{0,-1}H_0-16\big(2z^2+6z+3\big)H_{0,1}H_0-\frac{4\big(132z^3-118z^2+5z-8\big)H_1^2}{3z}$$

$$-\frac{4(z-1)\big(1231z^2-191z+178\big)}{27z}-\frac{4\big(14939z^3-14544z^2+783z-1286\big)}{27z}$$

$$-\frac{4}{3}\big(30z^2+214z-11\big)\zeta_2+8\big(2z^2-46z+13\big)\zeta_3$$

$$-8\big(2z^2+22z+5\big)\zeta_3-\frac{8\big(617z^3-604z^2-70z-28\big)H_1}{9z}+\gamma_{qg}^0\Big[4H_1^3-2H_0^2H_1-8\zeta_2H_1\Big]$$

$$+\frac{\big(146z^3+118z^2-z+16\big)}{z}\Big[-\frac{4}{3}\zeta_2-\frac{8}{3}H_{-1}H_0+\frac{8}{3}H_{0,-1}\Big]+\frac{16}{3}z(25z+36)H_{0,1}$$

$$-16\big(6z^2-26z+11\big)H_{0,0,-1}+\big(2z^2+2z+1\big)\Big[32H_0H_{-1}^2+\Big[-24H_0^2$$

$$+64\zeta_2-64H_{0,-1}-32H_{0,1}\Big]H_{-1}+64H_{0,-1,-1}+32H_{0,-1,1}$$

$$+16H_{0,0,1}+32H_{0,1,-1}\Big]+32(4z+1)H_{0,1,1}\Bigg]-64z(3z+2)H_{0,-1,0,1}$$

$$
-\frac{16\left(60z^3+58z^2+5z+8\right)H_{0,-1,0,1}}{3z}+\frac{8\left(168z^3+118z^2+5z+8\right)H_{0,0,0,-1}}{3z}
$$

$$
-\frac{64}{3}z(93z+46)H_{0,0,0,1}+\frac{16\left(575z^3-224z^2+69z-16\right)H_{0,0,0,1}}{3z}
$$

$$
+\frac{\left(72z^3+58z^2+5z+8\right)}{z}\left[H_{-1}\left[\frac{16}{3}H_{0,0,1}-\frac{16}{3}H_{0,0,-1}\right]+\frac{16}{3}H_{0,0,-1,-1}\right.
$$

$$
\left.-\frac{16}{3}H_{0,0,-1,1}-\frac{16}{3}H_{0,0,1,-1}\right]+\frac{64}{3}z(54z-1)H_{0,0,1,1}-\frac{8\left(1364z^3-274z^2+31z-56\right)H_{0,0,1,1}}{3z}
$$

$$
+\gamma_{qg}^0\left[-\frac{1}{3}H_1^5-\frac{8}{3}\zeta_2H_1^3-\frac{2}{3}H_0^3H_1^2+\left[\frac{52\zeta_3}{3}-44H_{0,0,1}-4H_{0,1,1}\right]H_1^2+H_0^2\left[\zeta_2+8H_{0,1}\right]H_1\right.
$$

$$
+\left[46\zeta_2^2-32H_{0,1}\zeta_2+20H_{0,1}^2-16H_{0,0,0,1}+24H_{0,0,1,1}\right]H_1+H_0\left[\frac{5}{3}H_1^4+\left[2\zeta_2+24H_{0,1}\right]H_1^2\right.
$$

$$
\left.\left.+\left(\frac{40\zeta_3}{3}-8H_{0,0,1}-56H_{0,1,1}\right)H_1\right]-40H_{0,1}H_{0,1,1}\right]
$$

$$
+\frac{\left(96z^3+82z^2+5z+8\right)}{z}\left[\left[\frac{16}{3}H_{0,1}-\frac{16}{3}\zeta_2\right]H_{-1}^2+\left[\frac{16}{3}H_0H_{0,-1}-\frac{32}{3}H_{0,-1,1}\right.\right.
$$

$$
\left.\left.-\frac{32}{3}H_{0,1,-1}\right]H_{-1}+\frac{32}{3}H_{0,-1,-1,1}+\frac{32}{3}H_{0,-1,1,-1}+\frac{32}{3}H_{0,1,-1,-1}\right]
$$

$$
+\frac{\left(108z^3+94z^2+5z+8\right)}{z}\left[\frac{16}{9}H_0H_{-1}^3+\left[8\zeta_2-\frac{16}{3}H_{0,-1}-\frac{16}{3}H_{0,1}\right]H_{-1}^2+\left[-\frac{16}{3}H_0H_{0,1}\right.\right.
$$

$$
\left.+\frac{32}{3}H_{0,-1,-1}+\frac{32}{3}H_{0,-1,1}+\frac{16}{3}H_{0,0,1}+\frac{32}{3}H_{0,1,-1}\right]H_{-1}
$$

$$
+H_0\left[\frac{16}{3}H_{0,-1,1}+\frac{16}{3}H_{0,1,-1}\right]-\frac{32}{3}H_{0,-1,-1,-1}-\frac{32}{3}H_{0,-1,-1,1}-\frac{32}{3}H_{0,-1,1,-1}
$$

$$
\left.-\frac{16}{3}H_{0,0,-1,1}-\frac{16}{3}H_{0,0,1,-1}-\frac{32}{3}H_{0,1,-1,-1}\right]+L_M\left[\frac{4}{3}(19z-5)H_0^4-\frac{4}{9}\left(54z^2\right.\right.
$$

$$
+40z-37)H_0^3-\frac{2}{9}\left(4303z^2+216z+1303\right)H_0^2+4(34z-9)\zeta_2H_0^2+32\left(3z^2-7z+1\right)
$$

$$
\times\zeta_2H_0^2-\frac{4\left(108z^3+94z^2+5z+8\right)H_{-1}H_0^2}{3z}+\frac{8\left(122z^3+127z^2+32z+16\right)H_{-1}H_0^2}{3z}
$$

$$
-\frac{4\left(18z^3+59z^2-58z-8\right)H_1H_0^2}{3z}-16\left(z^2-10z+3\right)H_{0,-1}H_0^2
$$

$$
+4\left(36z^2-2z+23\right)H_{0,1}H_0^2+\frac{40(z-1)\left(31z^2+7z+4\right)H_1^2H_0}{3z}
$$

$$
+\frac{8\left(28596z^3+13409z^2+3023z+448\right)H_0}{27z}+\frac{4(z-1)\left(1487z^2-5z+192\right)H_1H_0}{3z}
$$

$$
-\frac{32}{3}\left(57z^2+38z-3\right)\zeta_2H_0+\frac{8}{3}\left(318z^2+59z+27\right)\zeta_2H_0+8\left(4z^2+24z-21\right)\zeta_3H_0
$$

$$
-8\left(4z^2+288z+15\right)\zeta_3H_0+\frac{16\left(28z^3-208z^2-5z-16\right)H_{0,-1}H_0}{3z}
$$

$$
+\frac{8\left(72z^3+70z^2+5z+8\right)H_{0,-1}H_0}{3z}+\frac{8\left(40z^3-281z^2-86z-24\right)H_{0,1}H_0}{3z}
$$

$$
-32\left(8z^2-2z+7\right)H_{0,-1,-1}H_0+\left(2z^2-6z+3\right)\left[-32H_{0,-1,-1}-32H_{0,0,-1}\right]H_0
$$

$$
-32(z+1)(9z+7)H_{0,0,1}H_0+\left(2z^2-10z+1\right)\left[-32H_{0,-1,1}-32H_{0,1,-1}\right]H_0
$$

$$
-32\left(12z^2+8z+11\right)H_{0,1,1}H_0-\frac{8}{9}\left(103z^2-94z+2\right)H_1^3+\frac{8}{5}\left(24z^2-696z-71\right)\zeta_2^2
$$

$$
-\frac{4}{5}\left(160z^2+366z+103\right)\zeta_2^2-\frac{2\left(2599z^3-2608z^2-247z-84\right)H_1^2}{9z}-32z(z+22)H_{0,0,-1}H_0
$$

$$+32\left(z^2 - 4z + 2\right)H_{0,-1}^2 + 64\left(3z^2 - z + 2\right)H_{0,1}^2 - \frac{4\left(290560z^3 - 295527z^2 + 32808z - 25024\right)}{81z}$$

$$+\frac{4\left(617z^3 - 2690z^2 + 490z - 128\right)\zeta_2}{9z} - \frac{8\left(975z^3 + 85z^2 - 136z + 104\right)\zeta_2}{9z}$$

$$-\frac{16\left(76z^3 + 103z^2 + 50z + 12\right)H_{-1}\zeta_2}{3z} - \frac{8\left(54z^3 - 76z^2 + 41z - 8\right)H_1\zeta_2}{3z}$$

$$-16(6z - 1)H_{0,-1}\zeta_2 - 32\left(8z^2 - 4z + 7\right)H_{0,-1}\zeta_2 - 16\left(20z^2 - 26z + 9\right)H_{0,1}\zeta_2$$

$$+\frac{2\left(640z^3 - 4874z^2 - 543z - 360\right)\zeta_3}{3z} + \frac{2\left(2240z^3 - 614z^2 + 319z - 24\right)\zeta_3}{3z}$$

$$+\frac{64(z - 1)\left(33z^2 + 2\right)H_1}{3z} - \frac{4\left(14636z^3 - 18161z^2 + 3913z + 204\right)H_1}{27z}$$

$$+(z - 1)z\left[128H_1^2 + 128H_0H_1\right] + \frac{(z + 1)\left(251z^2 - 32z + 20\right)}{z}\left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right]$$

$$+\frac{\left(831z^3 + 733z^2 - 88z + 104\right)}{z}\left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right] + \left(z^2 - z + 1\right)\left[32H_{0,-1}H_0^2\right.$$

$$+64H_{0,-1}^2\bigg] - \frac{64}{3}(21z + 2)H_{0,1} - \frac{4\left(3416z^3 - 4824z^2 + 1257z - 416\right)H_{0,1}}{9z}$$

$$-\frac{16\left(40z^3 - 26z^2 - 17z - 8\right)H_1H_{0,1}}{3z} + \frac{(z + 1)\left(40z^2 - 25z + 4\right)}{z}\left[\frac{16}{3}H_0H_{-1}^2 - \frac{32}{3}H_{0,-1}H_{-1}\right.$$

$$+\frac{32}{3}H_{0,-1,-1}\bigg] - \frac{8\left(36z^3 + 46z^2 + 5z + 8\right)H_{0,0,-1}}{3z} - \frac{16\left(178z^3 - 289z^2 + 22z - 16\right)H_{0,0,-1}}{3z}$$

$$-\frac{32}{3}z(42z + 11)H_{0,0,1} + \frac{8\left(16z^3 + 668z^2 + 75z + 40\right)H_{0,0,1}}{3z}$$

$$+\frac{\left(96z^3 + 82z^2 + 5z + 8\right)}{z}\left[\frac{8}{3}H_0H_{-1}^2 + \left[8\zeta_2 - \frac{16}{3}H_{0,-1} - \frac{16}{3}H_{0,1}\right]H_{-1} + \frac{16}{3}H_{0,-1,-1}\right.$$

$$+\frac{16}{3}H_{0,-1,1} + \frac{16}{3}H_{0,1,-1}\bigg] + \frac{\left(116z^3 + 118z^2 + 29z + 16\right)}{z}\left[\frac{16}{3}H_{-1}H_{0,1} - \frac{16}{3}H_{0,-1,1} - \frac{16}{3}H_{0,1,-1}\right]$$

$$+\frac{32}{3}\left(8z^2 + 73z - 11\right)H_{0,1,1} + z\left[-128H_0H_{0,1} - 256H_{0,1,1}\right] + z^2\left[704H_0\right.$$

$$-128H_{0,-1,0,1}\bigg] + 64(4z - 1)H_{0,-1,0,1} + 64\left(z^2 - 5z + 2\right)H_{0,0,0,-1}$$

$$+48\left(2z^2 + 22z + 5\right)H_{0,0,0,-1} + 128z(3z + 1)H_{0,0,0,1} - 16\left(2z^2 - 118z - 13\right)H_{0,0,0,1}$$

$$+32\left(20z^2 + 12z + 9\right)H_{0,0,1,1} + z(z + 1)\left[-128H_0H_{0,0,1} - 256H_{0,0,1,1}\right]$$

$$+\left(2z^2 + 2z + 1\right)\left[-\frac{160}{3}H_0H_{-1}^3 + \left[8H_0^2 - 80\zeta_2 + 160H_{0,-1}\right]H_{-1}^2 + \left[\frac{8}{3}H_0^3 + \left[64\zeta_2 + 64H_{0,-1}\right]H_0\right.\right.$$

$$+192\zeta_3 - 320H_{0,-1,-1} - 160H_{0,0,-1} + 32H_{0,0,1} - 64H_{0,1,1}\bigg]H_{-1} + H_0\left[-64H_{0,-1,1} - 64H_{0,1,-1}\right]$$

$$+320H_{0,-1,-1,-1} + 64H_{0,-1,1,1} + 160H_{0,0,-1,-1} - 32H_{0,0,-1,1} - 32H_{0,0,1,-1}$$

$$+64H_{0,1,-1,1} + 64H_{0,1,1,-1}\bigg] + \gamma_{qg}^0\left[2H_1^4 - 4H_0^2H_1^2 + \left[8H_{0,1} - 8\zeta_2\right]H_1^2\right.$$

$$+\frac{10}{3}H_0^3H_1 + \left[-136\zeta_3 + 48H_{0,0,-1} + 64H_{0,0,1}\right]H_1 + H_0\left[H_1\left[-40\zeta_2 - 24H_{0,-1} - 16H_{0,1}\right]\right.$$

$$-\frac{20}{3}H_1^3\bigg] - 24H_{0,-1}H_{0,1} - 8H_{0,1,1,1}\bigg] + \frac{128(z - 1)}{3z}\bigg] - \frac{8\left(30z^3 + 278z^2 - 7z - 8\right)H_{0,1,1,1}}{3z}$$

$$+z\left[\left[128\zeta_3 + 64H_{0,1}\right]H_0^2 + 384H_{0,1,1}H_0 - 384H_{0,1}\zeta_2\right.$$

$$+768H_{0,1,1,1}\bigg] + \left(2z^2 - 6z + 3\right)\left[-16H_{0,0,-1}H_0^2 - 96H_{0,-1,-1}\zeta_2 + 64H_{0,-1,0,1}\right]$$

$$+\left(z^2 - z + 1\right)\left[\frac{32}{3}H_{0,-1}H_0^3 + \left[128H_{0,-1,0,1} - 64H_{0,-1}^2\right]H_0 + H_{0,-1}\left[-288\zeta_3 + 256H_{0,-1,-1}\right.\right.$$

$$+128H_{0,0,-1}\Big] + 256H_{0,-1,-1,0,1} - 512H_{0,-1,0,-1,-1} - 256H_{0,-1,0,1,1} - 1024H_{0,0,-1,-1,-1}$$

$$-128H_{0,0,-1,0,-1}\Big] - 32\big(2z^2 + 18z - 3\big)H_{0,0,-1,0,1} - 32\big(2z^2 - 22z + 7\big)H_{0,0,0,-1,-1}$$

$$-32\big(2z^2 - 14z + 5\big)H_{0,0,0,0,-1} - 32\big(40z^2 + 54z - 1\big)H_{0,0,0,0,1} + z^2\Big[32H_{0,-1}^2 + 1280H_{0,0,0,0,1}\Big]$$

$$+\big(4z^2 + 16z - 1\big)\Big[-64H_{0,0,0,-1,1} - 64H_{0,0,0,1,-1}\Big] + \big(8z^2 - 4z + 7\big)\Big[64H_{0,0,0,-1,1}$$

$$+64H_{0,0,0,1,-1}\Big] - 128z(11z + 7)H_{0,0,0,1,1} + 64\big(35z^2 + 97z + 32\big)H_{0,0,0,1,1}$$

$$+\big(6z^2 + 14z + 1\big)\Big[32H_{0,-1}H_{0,0,1} - 32H_{0,0,1,0,-1}\Big] + \big(4z^2 + 3\big)\Big[-64H_{0,-1}H_{0,0,1}$$

$$-128H_0H_{0,-1,-1,-1} - 128H_{0,-1,-1,0,1} + 64H_{0,0,1,0,-1}\Big] + 64\big(5z^2 + 29z + 11\big)H_{0,0,1,0,1}$$

$$-128\big(18z^2 - 2z + 7\big)H_{0,0,1,1,1} + z(z + 1)\Big[64H_{0,0,1}H_0^2 + 128H_{-1}H_{0,1}H_0 + \big(-128H_{0,-1,1}$$

$$-128H_{0,1,-1} + 384H_{0,0,1,1}\big)H_0 - 384H_{0,0,1}\zeta_2 + 768H_{0,0,1,1,1}\Big]$$

$$-64\big(9z^2 - 7z + 5\big)H_{0,1,0,1,1} + \big(2z^2 + 2z + 1\big)\Big[-16H_0H_{-1}^4 + \Big[\frac{64}{3}H_0^2 - \frac{160}{3}\zeta_2$$

$$+64H_{0,-1} - \frac{-64}{3}H_{0,1}\Big]H_{-1}^3 + \Big[-\frac{8}{3}H_0^3 + \Big[40\zeta_2 - 64H_{0,-1}$$

$$+32H_{0,1}\Big]H_0 + 160\zeta_3 - 192H_{0,-1,-1} - 64H_{0,-1,1} - 96H_{0,0,1} - 64H_{0,1,-1}\Big]H_{-1}^2 + \Big[-\frac{4}{3}H_0^4$$

$$+\Big[36\zeta_2 - 16H_{0,-1} - 32H_{0,1}\Big]H_0^2 + \Big[-64\zeta_3 + 128H_{0,-1,-1} - 64H_{0,-1,1} + 96H_{0,0,-1}$$

$$+64H_{0,0,1} - 64H_{0,1,-1} - 64H_{0,1,1}\Big]H_0 - \frac{376}{5}\zeta_2^2 - 16H_{0,-1}\zeta_2 + 96H_{0,1}\zeta_2 + 384H_{0,-1,-1,-1}$$

$$+128H_{0,-1,-1,1} + 128H_{0,-1,0,1} + 128H_{0,-1,1,-1} + 192H_{0,0,-1,1} - 160H_{0,0,0,-1}$$

$$-32H_{0,0,0,1} + 192H_{0,0,1,-1} + 128H_{0,0,1,1} + 128H_{0,1,-1,-1}\Big]H_{-1} - 96H_{0,-1,1}\zeta_2 - 96H_{0,1,-1}\zeta_2$$

$$+H_0^2\Big[32H_{0,-1,1} + 32H_{0,1,-1}\Big] + H_0\Big[64H_{0,-1,-1,1} - 64H_{0,-1,0,1}$$

$$+64H_{0,-1,1,-1} + 64H_{0,-1,1,1} - 96H_{0,0,-1,-1} - 64H_{0,0,-1,1}$$

$$-64H_{0,0,1,-1} + 64H_{0,1,-1,-1} + 64H_{0,1,-1,1} + 64H_{0,1,1,-1}\Big] - 384H_{0,-1,-1,-1,-1}$$

$$-128H_{0,-1,-1,-1,1} - 128H_{0,-1,-1,1,-1} - 128H_{0,-1,0,-1,1} - 128H_{0,-1,0,1,-1} - 128H_{0,-1,1,-1,-1}$$

$$-192H_{0,0,-1,-1,1} + 64H_{0,0,-1,0,1} - 192H_{0,0,-1,1,-1} - 128H_{0,0,-1,1,1} - 192H_{0,0,1,-1,-1}$$

$$-128H_{0,0,1,-1,1} - 128H_{0,0,1,1,-1} - 128H_{0,1,-1,-1,-1}\Big] - 64\big(z^2 + z + 1\big)H_{0,1,1,1,1} - \frac{640(z - 1)}{9z}\Big]$$

$$+C_F^2T_F\Big[\frac{1}{3}\big(20z^2 - 7z - 1\big)H_0^4 - \frac{1}{3}\big(32z^2 + 2z - 13\big)H_0^3 + \frac{16}{3}z(z + 1)\zeta_2H_0^3$$

$$+\frac{4}{3}\big(20z^2 - 28z + 3\big)H_1H_0^3 + (2z - 1)\Big[\frac{2\zeta_2}{3} + 8H_{0,1}\Big]H_0^3 + 4\big(10z^2 - 12z + 1\big)H_1^2H_0^2$$

$$+3z(176z - 67)H_0^2 + 8z(3z + 4)\zeta_2H_0^2 - 2\big(24z^2 - 19z - 4\big)\zeta_2H_0^2$$

$$-\frac{4}{3}\big(112z^2 - 34z + 17\big)\zeta_3H_0^2 - 2\big(16z^2 - 46z - 1\big)H_1H_0^2 + 4(6z - 7)H_{0,1}H_0^2$$

$$-8\big(12z^2 + 2z - 1\big)H_{0,0,1}H_0^2 + 32\big(5z^2 - 4z + 2\big)H_{0,1,1}H_0^2$$

$$-8\big(4z^2 + 2z + 1\big)\zeta_2^2H_0 - \frac{48}{5}\big(44z^2 - 14z + 7\big)\zeta_2^2H_0 - 16(z - 1)(2z - 3)H_1^2H_0$$

$$-32\big(4z^2 - 2z + 1\big)H_{0,1}^2H_0 - 2\big(160z^2 + 105z + 59\big)H_0 + 64z\zeta_2H_0 - 4z(11z - 4)\zeta_2H_0$$

$$-4\big(24z^2 - 20z + 1\big)H_1\zeta_2H_0 + 8\big(36z^2 - 38z + 19\big)H_{0,1}\zeta_2H_0 - 8\big(32z^2 - 19z + 1\big)\zeta_3H_0$$

$$+4\big(264z^2 - 233z - 6\big)H_1H_0 - 8(14z + 11)H_{0,1}H_0 - 8\big(20z^2 - 14z - 9\big)H_{0,0,1}H_0$$

$$+32\big(8z^2 - 18z - 1\big)H_{0,1,1}H_0 + z^2\Big[64H_{0,0,1,1} - 32H_{0,-1}\zeta_2\Big]H_0 + \big(16z^2 - 12z - 5\big)H_1^4$$

$$+\frac{8}{3}\big(24z^2 - 40z + 7\big)H_1^3 + 562z^2 - \frac{8}{5}\big(255z^2 - 140z + 13\big)\zeta_2^2 + 2\big(152z^2 - 137z + 36\big)H_1^2$$

$$-8\big(10z^2 - 16z + 1\big)H_{0,1}^2 - 718z + 4\big(4z^2 - 20z + 7\big)H_1^2\zeta_2 - \frac{1}{2}\big(264z^2 + 1078z - 311\big)\zeta_2$$

$$-4\big(11z^2 + 36z - 21\big)H_1\zeta_2 + 4\big(32z^2 + 22z + 23\big)H_{0,1}\zeta_2 + 16\big(6z^2 + 2z + 1\big)H_{0,0,-1}\zeta_2$$

$$-40\big(4z^2 - 10z + 5\big)H_{0,0,1}\zeta_2 - 32\big(5z^2 - 8z + 4\big)H_{0,1,1}\zeta_2 + (z + 1)(3z + 2)\Big[-8\zeta_2^2 - 16H_{-1}H_0\zeta_2$$

$$+16H_{0,-1}\zeta_2\Big] + \big(2z^2 + 2z + 1\big)\Big[16H_0\zeta_2 H_{-1}^2 + \Big[16\zeta_2^2 - 8H_0^2\zeta_2 - 32H_{0,-1}\zeta_2\Big]H_{-1} + 32H_{0,-1,-1}\zeta_2\Big]$$

$$-\frac{2}{3}\big(384z^2 - 862z + 277\big)\zeta_3 - 16\big(5z^2 + 2z + 1\big)\zeta_2\zeta_3 - 4\big(112z^2 - 271z - 24\big)H_{0,1}$$

$$+\frac{16}{3}\big(19z^2 - 8z + 4\big)\zeta_2\zeta_3 - \frac{32}{3}\big(24z^2 - 34z + 1\big)H_1\zeta_3 + \frac{32}{3}\big(14z^2 - 30z + 15\big)H_{0,1}\zeta_3$$

$$-4\big(80z^2 - 111z + 41\big)H_1 + (z - 1)(3z + 1)\Big[\frac{16}{3}H_0H_1^3 - 16\zeta_2\Big]$$

$$+L_M^3\Big[-\frac{64}{3}H_{0,1}z^2 + 8H_0z - \frac{4}{3}\big(8z^2 - 2z + 1\big)H_0^2 + \frac{2}{3}(2z - 11) + \frac{32}{3}\big(4z^2 - 2z + 1\big)\zeta_2$$

$$+\frac{16}{3}(4z - 1)H_1 + \gamma_{qg}^0\Big[\frac{8}{3}H_1^2 + \frac{8}{3}H_0H_1\Big]\Big] + 4(4z + 39)H_{0,0,1} + \big(5z^2 - 8z + 1\big)\Big[64H_1H_{0,0,1}$$

$$-32H_0H_1H_{0,1}\Big] + \big(6z^2 + 2z - 1\big)\Big[32H_{0,1}H_{0,0,1} - 64\zeta_5\Big] + 8\big(56z^2 - 95z + 29\big)H_{0,1,1}$$

$$+32\big(4z^2 - 6z + 3\big)H_{0,1}H_{0,1,1} + L_M^2\Big[-\frac{32}{3}z(z + 1)H_0^3 - 16z(3z + 4)H_0^2 - 4\big(8z^2 - 2z + 1\big)H_0^2$$

$$-128zH_0 - 2\big(100z^2 - 90z + 27\big)H_0 + 16\big(4z^2 + 2z + 1\big)\zeta_2H_0 - 16\big(4z^2 - 8z + 1\big)H_1H_0$$
$$+64z^2H_{0,-1}H_0 + 8z^2 - 4\big(16z^2 - 20z + 1\big)H_1^2 + 20z + 32(z - 1)(3z + 1)$$
$$+16z(5z - 2)\zeta_2 + 32\big(5z^2 + 2z + 1\big)\zeta_3 - 8\big(25z^2 - 37z + 17\big)H_1$$
$$+\gamma_{qg}^0\Big[4H_1^3 + 12H_0H_1^2 + 4H_0^2H_1 - 16\zeta_2H_1 - 4\zeta_3\Big] + (z + 1)(3z + 2)\Big[16\zeta_2$$

$$+32H_{-1}H_0 - 32H_{0,-1}\Big] + \big(4z^2 - 2z + 1\big)\Big[H_0\Big[32\zeta_2 - 32H_{0,1}\Big] - \frac{8}{3}H_0^3\Big]$$

$$-16\big(4z^2 + z + 1\big)H_{0,1} + \big(2z^2 + 2z + 1\big)\Big[-32H_0H_{-1}^2 + \Big[16H_0^2 - 32\zeta_2$$

$$+64H_{0,-1}\Big]H_{-1} - 64H_{0,-1,-1}\Big] - 32\big(6z^2 + 2z + 1\big)H_{0,0,-1} + 16\big(8z^2 - 6z + 3\big)H_{0,0,1}$$

$$-16(2z - 1)H_{0,1,1} + 5\Big] + (4z - 1)\Big[8H_1^2H_{0,1} - 32H_1H_{0,1,1}\Big] + 16\big(30z^2 - 22z - 5\big)H_{0,0,0,1}$$

$$+\big(8z^2 - 2z + 1\big)\Big[\frac{1}{15}H_0^5 + 48H_{0,0,0,1}H_0\Big] - 32\big(8z^2 - 23z - 2\big)H_{0,0,1,1} + 8\big(36z^2 - 44z$$

$$-17\big)H_{0,1,1,1} + L_M\Big[-\frac{1}{3}\big(24z^2 + 2z + 3\big)H_0^4 - \frac{4}{3}\big(40z^2 + 23z + 4\big)H_0^3 - 4\big(112z^2$$

$$+32z + 25\big)H_0^2 + 16(2z - 1)(4z - 1)\zeta_2H_0^2 + 32\big(2z^2 + 1\big)\zeta_2H_0^2 + 32(z + 1)(5z + 4)H_{-1}H_0^2$$
$$-4(16z - 11)H_1H_0^2 + 16\big(6z^2 + 2z + 3\big)H_{0,-1}H_0^2 - 8\big(4z^2 - 6z - 1\big)H_{0,1}H_0^2$$
$$-8\big(6z^2 + 2z - 7\big)H_1^2H_0 - 4\big(94z^2 + 79z + 67\big)H_0 + 16\big(8z^2 + 2z + 11\big)\zeta_2H_0$$
$$+16\big(10z^2 - 18z - 7\big)\zeta_2H_0 + 32\big(8z^2 - 8z - 1\big)\zeta_3H_0 + 16\big(12z^2 - 6z + 7\big)\zeta_3H_0$$
$$-4\big(184z^2 - 214z + 63\big)H_1H_0 - 32\big(2z^2 + 16z - 3\big)H_{0,-1}H_0 - 8\big(12z^2 - 46z + 17\big)H_{0,1}H_0$$

$$-128H_{0,-1,-1}H_0 - 16\big(8z^2 + 2z + 7\big)H_{0,0,1}H_0 + 32\big(14z^2 - 10z + 5\big)H_{0,1,1}H_0$$

$$-16(z-1)(5z+1)H_1^3 + \frac{16}{5}\big(12z^2 - 30z + 11\big)\zeta_2^2 - \frac{8}{5}\big(156z^2 - 154z + 85\big)\zeta_2^2$$

$$-4\big(109z^2 - 128z + 29\big)H_1^2 - 32\big(2z^2 + 2z - 1\big)H_{0,-1}^2 - 32\big(4z^2 - 2z + 1\big)H_{0,1}^2$$

$$+\frac{1}{2}\big(-324z^2 + 1134z - 959\big) + 8\big(99z^2 - 56z - 14\big)\zeta_2 + 16\big(10z^2 - 2z - 7\big)H_1\zeta_2$$

$$-64H_{0,-1}\zeta_2 - 64H_{0,1}\zeta_2 + 16\big(11z^2 - 17z + 23\big)\zeta_3$$

$$+16\big(49z^2 - 16z - 20\big)\zeta_3 - 8\big(47z^2 - 20z - 29\big)H_1 + \big(5z^2 + 23z + 20\big)\Big[16\zeta_2$$

$$+32H_{-1}H_0 - 32H_{0,-1}\Big] - 4\big(34z^2 + 10z - 11\big)H_{0,1} - 64(z-1)(3z-2)H_1H_{0,1}$$

$$+(z+1)^2\Big[-128H_0H_{-1}^2 + \Big[256H_{0,-1} - 128\zeta_2\Big]H_{-1} - 256H_{0,-1,-1}\Big]$$

$$-64\big(3z^2 - 7z + 7\big)H_{0,0,-1} + \big(2z^2 + 2z + 1\big)\Big[-32H_{-1}^2H_0^2 - 96H_{0,0,-1}H_0$$

$$+H_{-1}\Big[\frac{16}{3}H_0^3 + 64H_{0,-1}H_0\Big]\Big] - 24\big(4z^2 + 12z - 5\big)H_{0,0,1} + 16(z+10)H_{0,1,1}$$

$$-32\big(2z^2 - 10z + 1\big)H_{0,0,0,-1} + 64\big(5z^2 - 2z + 4\big)H_{0,0,0,1} - 16\big(16z^2 - 10z + 5\big)H_{0,0,1,1}$$

$$+\gamma_{qg}^0\Big[2H_1^4 + 6H_0^2H_1^2 + \Big[16H_{0,1} - 24\zeta_2\Big]H_1^2 + \frac{8}{3}H_0^3H_1 + \Big[-56\zeta_3 + 32H_{0,0,-1} - 24H_{0,0,1}$$

$$-24H_{0,1,1}\Big]H_1 - 16H_{0,-1}H_{0,1} + H_0\Big[\frac{8}{3}H_1^3 + \big(-48\zeta_2 - 16H_{0,-1} + 24H_{0,1}\big)H_1 + 16H_{0,-1,1}$$

$$+16H_{0,1,-1}\Big] + 16H_{0,-1,0,1} + 8H_{0,1,1,1}\Big] + \gamma_{qg}^0\Big[-\frac{1}{3}H_1^5 + \Big[\frac{16}{3}H_{0,1} - 8\zeta_2\Big]H_1^3 - \frac{2}{3}H_0^3H_1^2$$

$$+\Big[\frac{28\zeta_3}{3} - 24H_{0,0,1} - 16H_{0,1,1}\Big]H_1^2 - \frac{1}{3}H_0^4H_1 + 12H_0^2H_{0,1}H_1 + \Big[\frac{264}{5}\zeta_2^2 - 16H_{0,1}\zeta_2$$

$$+16H_{0,1}^2\Big]H_1 + 96\ln(2)\zeta_2 + H_0\Big[-\frac{1}{3}H_1^4 + \Big[16H_{0,1} - 18\zeta_2\Big]H_1^2 + \Big[\frac{112\zeta_3}{3} - 24H_{0,0,1}$$

$$-48H_{0,1,1}\Big]H_1 + 40H_{0,1,1,1}\Big]\Big] - 16\big(24z^2 - 14z + 7\big)H_{0,0,0,0,1} - 32\big(26z^2 + 10z - 5\big)H_{0,0,0,1,1}$$

$$-96\big(4z^2 + 2z - 1\big)H_{0,0,1,0,1} - 16\big(28z^2 - 54z + 27\big)H_{0,0,1,1,1} - 32\big(8z^2 - 14z + 7\big)H_{0,1,0,1,1}$$

$$-16\big(16z^2 - 18z + 9\big)H_{0,1,1,1,1} + 133\Big] + C_F T_F^2\Big[\frac{2}{9}\big(8z^2 + 32z - 13\big)H_0^4 + \frac{8}{9}\big(20z^2 + 93z$$

$$-43\big)H_0^3 - \frac{16}{3}\big(16z^2 - 135z + 58\big)H_0^2 + \frac{4}{3}\big(32z^2 + 40z - 29\big)\zeta_2H_0^2 + \frac{16}{3}\big(10z^2 - 12z + 1\big)H_1H_0^2$$

$$-\frac{64}{3}(z-1)^2H_{0,1}H_0^2 + \frac{64}{3}\big(29z^2 + 9z - 69\big)H_0 + \frac{4}{9}\big(24z^2 - 136z - 559\big)\zeta_2H_0$$

$$-\frac{16}{9}\big(72z^2 - 40z + 29\big)\zeta_3H_0 - \frac{32}{3}\big(16z^2 - 27z + 10\big)H_1H_0 - \frac{64}{3}\big(5z^2 - 4z + 2\big)H_{0,1}H_0$$

$$-\frac{128}{3}\big(z^2 + 2z - 1\big)H_{0,0,1}H_0 + \frac{32}{9}(z-1)(3z+1)H_1^3 + L_M^3\Big[-\frac{16}{3}(2z-1)H_0^2$$

$$-\frac{16}{9}\big(24z^2 - 8z - 5\big)H_0 + \frac{16\big(62z^3 - 111z^2 + 75z - 8\big)}{27z} + \frac{32}{9}\gamma_{qg}^0H_1\Big]$$

$$-\frac{16}{15}\big(218z^2 - 122z + 61\big)\zeta_2^2 + \frac{16}{3}\big(12z^2 - 17z + 4\big)H_1^2 - \frac{2\big(9794z^3 - 20814z^2 + 12015z + 160\big)}{9z}$$

$$-\frac{2\big(608z^3 - 6636z^2 + 4467z + 256\big)\zeta_2}{27z} + \frac{80}{9}\big(4z^2 - 4z - 1\big)H_1\zeta_2 - \frac{16\big(170z^3 - 183z^2 + 147z - 8\big)\zeta_3}{27z}$$

$$+(2z-1)\Big[\frac{4}{15}H_0^5 + 8\zeta_2H_0^3 + \frac{16}{3}\zeta_3H_0^2\Big] + \frac{32}{3}\big(20z^2 - 17z - 1\big)H_1 + \frac{32}{3}\big(28z^2 - 49z + 16\big)H_{0,1}$$

$$
+L_M^2\left[-\frac{32}{3}(2z-1)H_0^3-\frac{64}{3}(z+1)(3z-2)H_0^2+\frac{8}{9}\left(144z^2+166z+325\right)H_0\right.
$$

$$
-\frac{4\left(2296z^3-24z^2-1677z-208\right)}{27z}-\frac{32}{9}\left(8z^2-8z-5\right)H_1+\gamma_{qg}^0\left[4H_1^2+\frac{16}{3}H_0H_1-8\zeta_2\right.
$$

$$
\left.\left.+\frac{8}{3}H_{0,1}\right]\right]+\frac{32}{3}\left(10z^2-4z+7\right)H_{0,0,1}+\frac{64}{3}\left(3z^2-6z-1\right)H_{0,1,1}
$$

$$
+L_M\left[-\frac{20}{3}(2z-1)H_0^4-\frac{8}{9}\left(16z^2+80z-55\right)H_0^3+\frac{4}{9}\left(296z^2-260z+931\right)H_0^2\right.
$$

$$
-\frac{8}{27}\left(240z^2-2203z-4228\right)H_0+\frac{64}{9}\left(23z^2-26z+4\right)H_1H_0-\frac{64}{3}\left(4z^2-6z+3\right)H_{0,1}H_0
$$

$$
-\frac{32}{9}\left(2z^2+z-8\right)H_1^2+\frac{2\left(6892z^3-85110z^2+85137z-1600\right)}{81z}+\frac{64}{9}\left(2z^2-11z-8\right)\zeta_2
$$

$$
+\frac{32}{3}\left(6z^2+2z-1\right)\zeta_3-\frac{16}{27}\left(244z^2-271z+56\right)H_1-\frac{64}{9}\left(25z^2-37z-4\right)H_{0,1}
$$

$$
\left.+\frac{256}{3}(z-1)^2H_{0,0,1}+\gamma_{qg}^0\left[\frac{4}{3}H_1^3-\frac{16}{3}H_0^2H_1+8H_{0,1,1}\right]\right]+\frac{64}{3}\left(10z^2+2z-1\right)H_{0,0,0,1}
$$

$$
+\gamma_{qg}^0\left[-\frac{2}{9}H_1^4+\left[\frac{16}{3}H_{0,1}-\frac{26\zeta_2}{3}\right]H_1^2-\frac{8}{9}H_0^3H_1+\left[\frac{160\zeta_3}{9}-\frac{64}{3}H_{0,0,1}-\frac{64}{3}H_{0,1,1}\right]H_1\right.
$$

$$
\left.+\frac{32}{3}H_{0,1}^2-12H_{0,1}\zeta_2+H_0\left[H_1\left[\frac{32}{3}H_{0,1}-\frac{16\zeta_2}{3}\right]-\frac{64}{3}H_{0,1,1}\right]+\frac{80}{3}H_{0,1,1,1}\right]
$$

$$
\textcolor{blue}{+C_FT_F^2N_F}\left[-\frac{4}{9}(z-2)(6z+1)H_0^4+\frac{4}{27}\left(404z^2-54z+69\right)H_0^3\right.
$$

$$
-\frac{4}{27}\left(3112z^2-1329z-210\right)H_0^2-\frac{8\left(94z^3-234z^2+159z-16\right)H_1H_0^2}{9z}
$$

$$
+\frac{16}{3}\left(6z^2+4z-11\right)H_{0,1}H_0^2-\frac{8}{3}\left(14z^2-5z-11\right)\zeta_2H_0^2
$$

$$
+\frac{8}{81}\left(27824z^2+4929z+2631\right)H_0+\frac{16\left(1556z^3-1539z^2+72z-80\right)H_1H_0}{27z}
$$

$$
-\frac{32\left(101z^3+222z^2-39z+8\right)H_{0,1}H_0}{9z}-\frac{32}{3}\left(2z^2+28z-17\right)H_{0,0,1}H_0
$$

$$
+\frac{8}{9}\left(240z^2+166z+193\right)\zeta_2H_0+\frac{32}{9}\left(78z^2-133z-28\right)\zeta_3H_0+\frac{16}{9}(z-1)(3z+1)H_1^3
$$

$$
+L_M^3\left[-\frac{16}{3}(2z-1)H_0^2-\frac{32}{9}\left(6z^2-z-4\right)H_0+\frac{8\left(124z^3-258z^2+159z-16\right)}{27z}+\frac{8}{9}\gamma_{qg}^0H_1\right]
$$

$$
+\frac{8}{3}\left(12z^2-17z+4\right)H_1^2-\frac{64}{15}\left(2z^2+58z-11\right)\zeta_2^2-\frac{2}{27}\left(7280z^3-5646z^2-555z-368\right)\frac{\zeta_2}{z}
$$

$$
-\frac{4\left(221158z^3-226026z^2+17163z-5248\right)}{243z}-\frac{8}{27}\left(3784z^3+2046z^2+1095z-16\right)\frac{\zeta_3}{z}
$$

$$
+\frac{16}{3}\left(20z^2-17z-1\right)H_1-\frac{16\left(1448z^3-1341z^2+18z-80\right)H_{0,1}}{27z}
$$

$$
+\frac{16\left(498z^3+654z^2+3z+16\right)H_{0,0,1}}{9z}+\frac{32}{3}\left(3z^2-6z-1\right)H_{0,1,1}
$$

$$
-\frac{32}{3}\left(14z^2-74z+19\right)H_{0,0,0,1}+L_M^2\left[\frac{32}{3}(2z-1)H_0^3+\frac{16}{3}(2z+3)(4z-3)H_0^2\right.
$$

$$
-\frac{8}{9}\left(160z^2+146z+305\right)H_0+\frac{4\left(1000z^3+1356z^2-2247z-208\right)}{27z}+\frac{32}{9}\left(4z^2-4z+5\right)H_1
$$

$$+\gamma_{qg}^0\left[\frac{4}{3}H_1^2+\frac{8}{3}H_{0,1}-\frac{8\zeta_2}{3}\right]\right]+\frac{16}{9}\left(2z^2-2z-5\right)H_1\zeta_2+\gamma_{qg}^0\left[-\frac{1}{9}H_1^4+\frac{8}{3}H_{0,1}H_1^2\right.$$

$$-\frac{4}{9}H_0^3H_1+\left[-\frac{32}{3}H_{0,0,1}-\frac{32}{3}H_{0,1,1}\right]H_1+\frac{88}{9}\zeta_3H_1+\frac{16}{3}H_{0,1}^2+H_0\left[\frac{16}{3}H_1H_{0,1}-\frac{32}{3}H_{0,1,1}\right]$$

$$+\frac{40}{3}H_{0,1,1,1}+\left[-4H_1^2-\frac{4}{3}H_0H_1-\frac{20}{3}H_{0,1}\right]\zeta_2\right]+L_M\left[-\frac{16}{3}\left(6z^2+3z-7\right)H_0^3\right.$$

$$+\frac{8}{9}\left(290z^2+194z+377\right)H_0^2-\frac{16}{27}\left(1244z^2-1226z-1319\right)H_0+\frac{32}{3}\left(2z^2+8z-13\right)H_{0,1}H_0$$

$$-\frac{16\left(62z^3-202z^2+149z-16\right)H_1H_0}{9z}+\frac{2\left(20960z^3-77478z^2+62367z-4256\right)}{81z}$$

$$+\frac{16}{9}\left(5z^2-8z+13\right)H_1^2-\frac{32}{27}\left(68z^2-77z+28\right)H_1+\frac{16\left(72z^3-194z^2+175z-16\right)H_{0,1}}{9z}$$

$$-\frac{32}{3}\left(10z^2+10z-23\right)H_{0,0,1}+\gamma_{qg}^0\left[\frac{8}{9}H_1^3-4H_0^2H_1+\frac{16}{3}H_{0,1,1}\right]-\frac{32}{9}\left(5z^2+4z+13\right)\zeta_2$$

$$+\frac{64}{3}\left(10z^2-9\right)\zeta_3+(2z-1)\left[-\frac{20}{3}H_0^4+64H_{0,0,1}H_0+128\zeta_3H_0+\frac{384\zeta_2^2}{5}-192H_{0,0,0,1}\right]\right]$$

$$+(2z-1)\left[-\frac{4}{15}H_0^5-8\zeta_2H_0^3+32H_{0,0,1}H_0^2+\frac{208}{3}\zeta_3H_0^2-128H_{0,0,0,1}H_0+128H_{0,0,0,0,1}-128\zeta_5\right]\right]$$

$$+C_AC_FT_F\left[-\frac{1}{18}\left(76z^2-178z+11\right)H_0^4-\frac{1}{27}\left(4496z^2+1656z+993\right)H_0^3-\frac{8}{3}z(9z+5)\zeta_2H_0^3\right.$$

$$+\frac{8}{3}\left(13z^2+8z+3\right)\zeta_2H_0^3-\frac{4}{3}\left(8z^2+4z-1\right)H_{-1}H_0^3-\frac{4\left(274z^3-238z^2-7z-24\right)H_1H_0^3}{9z}$$

$$+\frac{16}{3}\left(2z^2+8z+5\right)H_{0,1}H_0^3+4\left(12z^2+16z+1\right)H_{-1}^2H_0^2-\frac{2\left(166z^3-156z^2-9z-16\right)H_1^2H_0^2}{3z}$$

$$+\frac{1}{27}\left(36352z^2+3459z-48\right)H_0^2+4z(13z+4)\zeta_2H_0^2-\frac{1}{3}\left(76z^2+110z-1\right)\zeta_2H_0^2$$

$$-32z(2z+1)\zeta_3H_0^2+\frac{8}{3}\left(36z^2+2z+11\right)\zeta_3H_0^2-8\left(5z^2+7z-4\right)H_{-1}H_0^2$$

$$+\frac{2\left(952z^3-1740z^2+969z-184\right)H_1H_0^2}{9z}-4(4z+1)H_{0,-1}H_0^2$$

$$-\frac{4\left(32z^3+122z^2-25z+24\right)H_{0,1}H_0^2}{3z}-32(z+1)^2H_{0,-1,-1}H_0^2+96\left(z^2-2z-1\right)H_{0,0,1}H_0^2$$

$$-8\left(36z^2-46z+11\right)H_{0,1,1}H_0^2-\frac{8}{3}(z+2)(3z-2)H_1^3H_0-\frac{32}{5}\left(4z^2-8z-1\right)\zeta_2^2H_0+\frac{48}{5}\left(12z^2\right.$$

$$\left.-49z-3\right)\zeta_2^2H_0+\frac{8\left(229z^3-297z^2+63z-40\right)H_1^2H_0}{9z}+8(2z-5)(8z+1)H_{0,1}^2H_0$$

$$-\frac{2}{81}\left(122248z^2-67419z+15057\right)H_0-96z(z+2)\zeta_2H_0-\frac{2}{9}\left(724z^2+1757z+629\right)\zeta_2H_0$$

$$-8z(13z+19)H_{-1}\zeta_2H_0+8\left(32z^2+36z+1\right)H_{-1}\zeta_2H_0+\frac{4\left(138z^3-122z^2-23z-16\right)H_1\zeta_2H_0}{3z}$$

$$+16\left(z^2+4z+2\right)H_{0,-1}\zeta_2H_0-8\left(26z^2-18z+21\right)H_{0,1}\zeta_2H_0-\frac{4}{9}\left(736z^2+1826z+149\right)\zeta_3H_0$$

$$+16z(13z+9)\zeta_3H_0-\frac{4\left(888z^3-695z^2+380z-396\right)H_1H_0}{9z}-16\left(7z^2-z+4\right)H_{0,-1}H_0$$

$$+\frac{32\left(9z^3+231z^2-99z+23\right)H_{0,1}H_0}{9z}+\frac{8\left(382z^3-368z^2+7z-32\right)H_1H_{0,1}H_0}{3z}$$

$$-64z(2z-3)H_{0,0,1}H_0+\frac{16\left(189z^3-8z^2-35z+12\right)H_{0,0,1}H_0}{3z}$$

$$-\frac{8\left(792z^3 - 700z^2 + 5z - 48\right)H_{0,1,1}H_0}{3z} + 16\left(8z^2 + 2z + 1\right)H_{0,0,0,-1}H_0$$

$$+64z(6z+7)H_{0,0,0,1}H_0 - 64\left(11z^2 - 3z - 2\right)H_{0,0,0,1}H_0 + 32(2z+1)(8z+9)H_{0,0,1,1}H_0$$

$$+32\left(20z^2 - 18z + 9\right)H_{0,1,1,1}H_0 - \frac{2\left(174z^3 - 137z^2 - 32z - 4\right)H_1^4}{9z}$$

$$-\frac{4\left(221z^3 - 262z^2 + 13z + 4\right)H_1^3}{3z} - \frac{2}{5}\left(26z^2 - 102z - 23\right)\zeta_2^2 + \frac{2}{15}\left(7754z^2 - 9554z + 31\right)\zeta_2^2$$

$$-\frac{4\left(990z^3 - 1029z^2 + 151z - 17\right)H_1^2}{3z} + \frac{8\left(275z^3 - 200z^2 + 19z - 16\right)H_{0,1}^2}{3z}$$

$$-\frac{96828z^3 - 132146z^2 - 16891z + 8388}{81z} - \frac{8\left(47z^3 - 45z^2 + 6z - 4\right)H_1^2\zeta_2}{z}$$

$$-8\left(141z^2 + 46z + 9\right)\zeta_2 + \frac{\left(5072z^3 - 69594z^2 + 5289z - 4320\right)\zeta_2}{54z}$$

$$-\frac{4\left(2302z^3 - 2680z^2 + 293z - 216\right)H_1\zeta_2}{9z} - \frac{4\left(126z^3 - 218z^2 + 121z - 16\right)H_{0,1}\zeta_2}{3z}$$

$$+16\left(6z^2 - 2z - 1\right)H_{0,-1,-1}\zeta_2 + 8\left(10z^2 - 2z - 1\right)H_{0,0,-1}\zeta_2$$

$$-8\left(22z^2 + 46z - 35\right)H_{0,0,1}\zeta_2 + 16\left(48z^2 - 26z + 31\right)H_{0,1,1}\zeta_2$$

$$+\left(4z^2 + 2z + 1\right)\left[16H_0H_{0,-1}\zeta_2 - 32H_{0,0,-1}\zeta_2\right] - 4\left(59z^2 + 65z + 20\right)\zeta_3$$

$$+\frac{2}{3}\left(3526z^2 - 3590z + 323\right)\zeta_3 - 8\left(7z^2 - 12z + 2\right)\zeta_2\zeta_3 + \frac{4}{3}\left(134z^2 - 374z - 77\right)\zeta_2\zeta_3$$

$$+12\left(36z^2 + 56z + 5\right)H_{-1}\zeta_3 - 4\left(52z^2 + 88z + 9\right)H_{-1}\zeta_3 + \frac{8\left(1666z^3 - 1454z^2 - 17z - 128\right)H_1\zeta_3}{9z}$$

$$+8\left(8z^2 - 22z - 11\right)H_{0,-1}\zeta_3 - 32\left(12z^2 + 6z + 13\right)H_{0,1}\zeta_3 - 4\left(12z^2 + 6z - 5\right)\zeta_5$$

$$-4\left(28z^2 - 34z - 17\right)\zeta_5 - \frac{4\left(76964z^3 - 85257z^2 + 9162z - 2381\right)H_1}{81z}$$

$$+(z+1)(2z+5)\left[16H_{0,-1} - 16H_{-1}H_0\right] + \frac{8\left(84z^3 - 70z^2 + 5z - 8\right)H_1^2H_{0,1}}{3z}$$

$$+32z(16z+23)H_{0,1} + \frac{4\left(2084z^3 + 5661z^2 + 1260z - 1188\right)H_{0,1}}{27z} + \frac{16}{3}(z-1)(15z+1)H_1H_{0,1}$$

$$+L_M^3\left[-\frac{16}{3}(4z+1)H_0^2 + \frac{4}{9}\left(292z^2 - 94z + 11\right)H_0 + \frac{2}{3}\left(40z^2 - 20z - 31\right)\right.$$

$$-\frac{32}{3}\left(4z^2 - 10z - 1\right)\zeta_2 + \frac{8\left(146z^3 - 142z^2 + 5z - 16\right)H_1}{9z} + \gamma_{qg}^0\left[-\frac{16}{3}H_1^2 - \frac{16}{3}H_0H_1\right]$$

$$\left.-32(2z+1)H_{0,1}\right] + 16\left(19z^2 + 5z + 4\right)H_{0,0,-1} - 32(z-17)zH_{0,0,1}$$

$$-\frac{4\left(2012z^3 + 1014z^2 - 807z + 184\right)H_{0,0,1}}{9z} - \frac{8\left(764z^3 - 724z^2 + 11z - 64\right)H_1H_{0,0,1}}{3z}$$

$$-16\left(4z^2 - 50z - 23\right)H_{0,1}H_{0,0,1} + \left(z^2 - z + 4\right)\left[-16H_0H_{-1}^2 + \left[-48\zeta_2 + 32H_{0,-1} + 32H_{0,1}\right]H_{-1}\right.$$

$$\left.-32H_{0,-1,-1} - 32H_{0,-1,1} - 32H_{0,1,-1}\right] + \left(7z^2 + 5z + 4\right)\left[H_{-1}\left[32\zeta_2 - 32H_{0,1}\right]\right.$$

$$\left.+32H_{0,-1,1} + 32H_{0,1,-1}\right] - \frac{8\left(2105z^3 - 3075z^2 + 216z - 80\right)H_{0,1,1}}{9z}$$

$$-\frac{8\left(336z^3 - 292z^2 + 23z - 32\right)H_1H_{0,1,1}}{3z} - 32\left(18z^2 - 10z + 11\right)H_{0,1}H_{0,1,1}$$

$$+L_M^2\left[-\frac{8}{3}\left(13z^2 + 21z + 6\right)H_0^2 - 8z(5z-6)H_0 + \frac{2}{9}\left(2468z^2 - 3050z - 587\right)H_0\right.$$

$$-16\left(4z^2-6z+1\right)\zeta_2 H_0 + \frac{16(z-1)\left(55z^2+7z+4\right)H_1 H_0}{3z} + 16\left(2z^2-10z-1\right)H_{0,1}H_0$$

$$+\frac{8\left(90z^3-89z^2+4z-4\right)H_1^2}{3z} - 4(z-1)(3z-1) + \frac{2380z^3-1636z^2-533z-48}{3z}$$

$$-\frac{4}{3}\left(70z^2-278z+31\right)\zeta_2 - 8\left(18z^2-2z+9\right)\zeta_3 + \frac{8\left(842z^3-937z^2+83z-28\right)H_1}{9z}$$

$$+24z(z+1)H_0^2 + \gamma_{qg}^0\left[-8H_1^3-12H_0H_1^2-2H_0^2H_1+24\zeta_2 H_1\right] - \frac{16\left(33z^3+26z^2-13z-4\right)H_{0,1}}{3z}$$

$$-32\left(z^2-8z-1\right)H_{0,0,1} + \left(2z^2+2z+1\right)\left[-12\zeta_2+8\zeta_3+24H_{0,-1}-16H_0H_{0,-1}\right.$$

$$+H_{-1}\left[8H_0^2-24H_0-32\zeta_2+32H_{0,1}\right] - 32H_{0,-1,1}+16H_{0,0,-1}-32H_{0,1,-1}\bigg] - 48(2z+1)H_{0,1,1}\bigg]$$

$$-8\left(16z^2+20z+1\right)H_{0,0,0,-1} - \frac{8\left(730z^3-260z^2-137z+24\right)H_{0,0,0,1}}{3z} + (z-1)z\Big[128H_1^3$$

$$+512H_1^2+64H_0^2H_1+\left[-384\zeta_2-224\right]H_1-24H_{0,-1}\zeta_2+H_0\left[192H_1^2+512H_1\right]+256H_{0,0,0,1}\Big]$$

$$+(2z-1)(2z+1)\Big[H_{-1}\left[16H_{0,0,1}-16H_{0,0,-1}\right]-16H_{0,-1,0,1}+16H_{0,0,-1,-1}$$

$$-16H_{0,0,-1,1}-16H_{0,0,1,-1}\Big] + \gamma_{qg}^0\Big[\frac{2}{3}H_1^5+\left[\frac{32\zeta_2}{3}-\frac{16}{3}H_{0,1}\right]H_1^3$$

$$+\frac{4}{3}H_0^3H_1^2+\left[-\frac{80}{3}\zeta_3+68H_{0,0,1}+20H_{0,1,1}\right]H_1^2+\frac{1}{3}H_0^4H_1+H_0^2\left[-\zeta_2-20H_{0,1}\right]H_1$$

$$+\left[-\frac{494}{5}\zeta_2^2+48H_{0,1}\zeta_2-36H_{0,1}^2+16H_{0,0,0,1}-24H_{0,0,1,1}\right]H_1-48\ln(2)\zeta_2$$

$$+H_0\left[-\frac{4}{3}H_1^4+\left[16\zeta_2-40H_{0,1}\right]H_1^2+\left[-\frac{152}{3}\zeta_3+32H_{0,0,1}+104H_{0,1,1}\right]H_1\right]\Big]$$

$$-128z(2z-5)H_{0,0,1,1} + \frac{8\left(740z^3-546z^2-93z-48\right)H_{0,0,1,1}}{3z}$$

$$+\left(4z^2+8z+1\right)\Big[\left(16\zeta_2-16H_{0,1}\right)H_{-1}^2+\left[-16H_0H_{0,-1}+32H_{0,-1,1}+32H_{0,1,-1}\right]H_{-1}$$

$$+16H_0H_{0,-1,-1}-32H_{0,-1,-1,1}-32H_{0,-1,1,-1}-32H_{0,1,-1,-1}\Big] + \left(8z^2+12z+1\right)\Big[-\frac{16}{3}H_0H_{-1}^3$$

$$+\left(-24\zeta_2+16H_{0,-1}+16H_{0,1}\right)H_{-1}^2+\Big[16H_0H_{0,1}-32H_{0,-1,-1}-32H_{0,-1,1}-16H_{0,0,1}$$

$$-32H_{0,1,-1}\Big]H_{-1}-8H_{0,-1}\zeta_2+H_0\Big[-16H_{0,-1,1}+8H_{0,0,-1}-16H_{0,1,-1}\Big]$$

$$+32H_{0,-1,-1,-1}+32H_{0,-1,-1,1}+32H_{0,-1,1,-1}+16H_{0,0,-1,1}+16H_{0,0,1,-1}+32H_{0,1,-1,-1}\Big]$$

$$+L_M\Big[\frac{4}{3}(5z+2)H_0^4-\frac{2}{3}\left(236z^2-2z+11\right)H_0^3-\frac{2}{9}\left(1578z^2-1883z-353\right)H_0^2$$

$$+16\left(2z^2-4z-3\right)\zeta_2 H_0^2-8\left(8z^2+2z+3\right)\zeta_2 H_0^2-8\left(6z^2+8z+5\right)H_{-1}H_0^2$$

$$+4\left(8z^2+12z+1\right)H_{-1}H_0^2-\frac{4\left(274z^3-360z^2+111z-16\right)H_1H_0^2}{3z}$$

$$+8\left(2z^2-10z-3\right)H_{0,-1}H_0^2-16\left(7z^2-10z+2\right)H_{0,1}H_0^2-\frac{8\left(97z^3-99z^2+6z-16\right)H_1^2H_0}{3z}$$

$$+\frac{4}{27}\left(4322z^2-10579z+200\right)H_0-16\left(5z^2+13\right)\zeta_2 H_0+8\left(128z^2+124z+29\right)\zeta_2 H_0$$

$$-4\left(52z^2+10z+33\right)\zeta_3 H_0+4\left(52z^2+370z+97\right)\zeta_3 H_0$$

$$+\frac{4\left(450z^3-812z^2+415z-208\right)H_1H_0}{9z}-8(4z+1)H_{0,-1}H_0-16\left(4z^2-16z+21\right)H_{0,-1}H_0$$

$$-\frac{8\left(108z^3+394z^2-101z+16\right)H_{0,1}H_0}{3z}-64\left(6z^2+4z+1\right)H_{0,-1,-1}H_0$$

$$-16\left(26z^2-2z+11\right)H_{0,0,-1}H_0+16\left(18z^2-22z+15\right)H_{0,0,1}H_0$$

$$+\left(2z^2-10z+1\right)\left[32H_{0,-1,1}+32H_{0,1,-1}\right]H_0-16\left(8z^2-30z-9\right)H_{0,1,1}H_0$$

$$+\frac{8}{9}\left(193z^2-166z-16\right)H_1^3-\frac{12}{5}\left(88z^2+6z+75\right)\zeta_2^2+\frac{12}{5}\left(88z^2+106z+113\right)\zeta_2^2$$

$$+\frac{4\left(2365z^3-2510z^2+121z-56\right)H_1^2}{9z}+16\left(6z^2+2z-1\right)H_{0,-1}^2-32\left(2z^2+3\right)H_{0,1}^2$$

$$+\frac{122748z^3-97910z^2-30697z-1440}{54z}+8\left(50z^2-19z-37\right)\zeta_2$$

$$-\frac{8}{9}\left(1207z^2-1190z-233\right)\zeta_2-16\left(5z^2+6z+4\right)H_{-1}\zeta_2$$

$$+\frac{8\left(52z^3-78z^2+45z-16\right)H_1\zeta_2}{3z}+32(8z+1)H_{0,-1}\zeta_2$$

$$+16\left(4z^2+10z+5\right)H_{0,-1}\zeta_2+32\left(6z^2-16z+3\right)H_{0,1}\zeta_2-2\left(44z^2-184z+235\right)\zeta_3$$

$$-\frac{2\left(3428z^3-3120z^2-825z-192\right)\zeta_3}{3z}+\frac{8\left(8065z^3-9010z^2+1094z+102\right)H_1}{27z}$$

$$+(z-1)z\left[-128H_1^2-128H_0H_1\right]+(z+1)(5z+2)\left[16H_{-1}H_0-16H_{0,-1}\right]$$

$$+(z+1)(18z-37)\left[16H_{-1}H_0-16H_{0,-1}\right]+(2z+1)\left[-8H_{0,-1}H_0^2-16H_{0,-1}^2\right]$$

$$+\frac{4\left(1640z^3-1658z^2-215z+208\right)H_{0,1}}{9z}+\frac{16\left(80z^3-90z^2+9z-8\right)H_1H_{0,1}}{3z}$$

$$+(z+1)(5z+9)\left[16H_0H_{-1}^2-32H_{0,-1}H_{-1}+32H_{0,-1,-1}\right]$$

$$+z^2\left[H_0\left[-64H_{0,-1,-1}-64H_{0,0,-1}\right]-64H_0^2\right]-8\left(8z^2+4z-1\right)H_{0,0,-1}$$

$$+16\left(14z^2-24z+47\right)H_{0,0,-1}+\frac{32(17z-4)\left(2z^2+2z-1\right)H_{0,0,1}}{3z}$$

$$+\left(4z^2+8z+1\right)\left[-8H_0H_{-1}^2+\left[-24\zeta_2+16H_{0,-1}+16H_{0,1}\right]H_{-1}-16H_{0,-1,-1}-16H_{0,-1,1}\right.$$

$$\left.-16H_{0,1,-1}\right]+\left(10z^2+20z+13\right)\left[16H_{-1}H_{0,1}-16H_{0,-1,1}-16H_{0,1,-1}\right]$$

$$-\frac{8}{3}\left(72z^2+290z+11\right)H_{0,1,1}+z\left[128H_0H_{0,1}-192H_{0,0,1}+256H_{0,1,1}\right]$$

$$+\gamma_{qg}^0\left[-4H_1^4+6H_0^2H_1^2+\left[32\zeta_2-24H_{0,1}\right]H_1^2-\frac{2}{3}H_0^3H_1+\left[168\zeta_3-48H_{0,0,-1}-40H_{0,0,1}\right.\right.$$

$$\left.+24H_{0,1,1}\right]H_1+H_0\left[4H_1^3+\left[64\zeta_2+24H_{0,-1}-8H_{0,1}\right]H_1\right]+24H_{0,-1}H_{0,1}\right]$$

$$-32(6z-1)H_{0,-1,0,1}+32\left(8z^2-6z+3\right)H_{0,-1,0,1}+16\left(8z^2+2z+1\right)H_{0,0,0,-1}$$

$$+16\left(50z^2+2z+31\right)H_{0,0,0,-1}-64z(4z+5)H_{0,0,0,1}-16\left(18z^2-18z+25\right)H_{0,0,0,1}$$

$$-32\left(10z^2+14z+5\right)H_{0,0,1,1}+z(z+1)\left[-128H_{0,1}+128H_0H_{0,0,1}+256H_{0,0,1,1}\right]$$

$$+\left(2z^2+2z+1\right)\left[\frac{160}{3}H_0H_{-1}^3+\left(-72H_0^2+208\zeta_2-160H_{0,-1}-128H_{0,1}\right)H_{-1}^2+\left[\frac{104}{3}H_0^3\right.\right.$$

$$+\left[-352\zeta_2+128H_{0,-1}+64H_{0,1}\right]H_0-384\zeta_3+320H_{0,-1,-1}+256H_{0,-1,1}+32H_{0,0,-1}+96H_{0,0,1}$$

$$+256H_{0,1,-1}+64H_{0,1,1}\Big]H_{-1}-320H_{0,-1,-1,-1}-256H_{0,-1,-1,1}-256H_{0,-1,1,-1}-64H_{0,-1,1,1}$$

$$\left.-32H_{0,0,-1,-1}-96H_{0,0,-1,1}-96H_{0,0,1,-1}-256H_{0,1,-1,-1}-64H_{0,1,-1,1}-64H_{0,1,1,-1}\right]\Bigg]$$

$$+\frac{8\left(282z^3+160z^2+25z-48\right)H_{0,1,1,1}}{3z}+z\Big[\frac{4}{15}H_0^5-64H_{0,1}H_0^2+\Big[-128H_{0,1}-384H_{0,1,1}\Big]H_0$$

$$+384H_{0,1}\zeta_2-768H_{0,1,1,1}\Big]+z^2\Big[-\frac{64}{3}H_0^3+\Big[256-32H_{0,0,-1}\Big]H_0^2-224H_0-192H_{0,-1,-1}\zeta_2$$

$$+128H_{0,-1,0,1,1}\Big]+(2z+1)\Big[-\frac{8}{3}H_{0,-1}H_0^3+\Big[16H_{0,-1}^2-32H_{0,-1,0,1}\Big]H_0$$

$$+H_{0,-1}\Big[72\zeta_3-64H_{0,-1,-1}-32H_{0,0,-1}\Big]-64H_{0,-1,-1,0,1}+128H_{0,-1,0,-1,-1}$$

$$+64H_{0,-1,0,1,1}+256H_{0,0,-1,-1,-1}+32H_{0,0,-1,0,-1}\Big]+32\left(10z^2+6z+3\right)H_{0,0,-1,0,1}$$

$$-64\left(5z^2+2z+1\right)H_{0,0,0,-1,-1}-32\left(6z^2+2z+1\right)H_{0,0,0,0,-1}-320z(2z+3)H_{0,0,0,0,1}$$

$$+16\left(80z^2+22z-1\right)H_{0,0,0,0,1}+\left(20z^2+14z+7\right)\Big[32H_{0,0,0,-1,1}+32H_{0,0,0,1,-1}\Big]$$

$$+\left(4z^2+10z+5\right)\Big[-32H_{0,0,0,-1,1}-32H_{0,0,0,1,-1}\Big]+128z(9z+10)H_{0,0,0,1,1}$$

$$-16\left(60z^2+374z+129\right)H_{0,0,0,1,1}+\left(5z^2+4z+2\right)\Big[64H_{0,0,1,0,-1}-64H_{0,-1}H_{0,0,1}\Big]$$

$$+\left(4z^2+6z+3\right)\Big[32H_{0,-1}H_{0,0,1}+64H_0H_{0,-1,-1,-1}+64H_{0,-1,-1,0,1}-32H_{0,0,1,0,-1}\Big]$$

$$-16(110z+53)H_{0,0,1,0,1}+32\left(104z^2-8z+55\right)H_{0,0,1,1,1}+z(z+1)\Big[-64H_{0,0,1}H_0^2$$

$$-128H_{-1}H_{0,1}H_0+\Big[128H_{0,-1,1}+128H_{0,1,-1}-384H_{0,0,1,1}\Big]H_0+384H_{0,0,1}\zeta_2$$

$$-128H_{0,1,1}+128H_{0,-1,0,1}-768H_{0,0,1,1,1}\Big]+16\left(72z^2-26z+49\right)H_{0,1,0,1,1}$$

$$+\left(2z^2+2z+1\right)\Big[16H_0H_{-1}^4+\Big[-\frac{64}{3}H_0^2-\frac{-160}{3}\zeta_2-64H_{0,-1}$$

$$-\frac{64}{3}H_{0,1}\Big]H_{-1}^3+\Big[\frac{8}{3}H_0^3+\Big[-56\zeta_2+64H_{0,-1}-32H_{0,1}\Big]H_0-160\zeta_3+192H_{0,-1,-1}$$

$$+64H_{0,-1,1}+96H_{0,0,1}+64H_{0,1,-1}\Big]H_{-1}^2+\Big[\frac{4}{3}H_0^4+\Big[-28\zeta_2+16H_{0,-1}+32H_{0,1}\Big]H_0^2$$

$$+\Big[64\zeta_3-128-128H_{0,-1,-1}+64H_{0,-1,1}-96H_{0,0,-1}-64H_{0,0,1}+64H_{0,1,-1}+64H_{0,1,1}\Big]H_0$$

$$+\frac{296}{5}\zeta_2^2+48H_{0,-1}\zeta_2-96H_{0,1}\zeta_2-384H_{0,-1,-1,-1}-128H_{0,-1,-1,1}-128H_{0,-1,0,1}-128H_{0,-1,1,-1}$$

$$-192H_{0,0,-1,1}+160H_{0,0,0,-1}+32H_{0,0,0,1}-192H_{0,0,1,-1}-128H_{0,0,1,1}-128H_{0,1,-1,-1}\Big]H_{-1}$$

$$+96H_{0,-1,1}\zeta_2+96H_{0,1,-1}\zeta_2+128H_{0,-1}+H_0^2\Big[-32H_{0,-1,1}-32H_{0,1,-1}\Big]+H_0\Big[-64H_{0,-1,-1,1}$$

$$+64H_{0,-1,0,1}-64H_{0,-1,1,-1}-64H_{0,-1,1,1}+96H_{0,0,-1,-1}+64H_{0,0,-1,1}+64H_{0,0,1,-1}-64H_{0,1,-1,-1}$$

$$-64H_{0,1,-1,1}-64H_{0,1,1,-1}\Big]+384H_{0,-1,-1,-1,-1}+128H_{0,-1,-1,-1,1}$$

$$+128H_{0,-1,-1,1,-1}+128H_{0,-1,0,-1,1}+128H_{0,-1,0,1,-1}+128H_{0,-1,1,-1,-1}+192H_{0,0,-1,-1,1}$$

$$-64H_{0,0,-1,0,1}+192H_{0,0,-1,1,-1}+128H_{0,0,-1,1,1}+192H_{0,0,1,-1,-1}$$

$$+128H_{0,0,1,-1,1}+128H_{0,0,1,1,-1}+128H_{0,1,-1,-1,-1}\Big]+16\left(20z^2-14z+13\right)H_{0,1,1,1,1}\Big]\Big\}\,. \tag{607}$$

The OME $A_{gg,Q}(z)$ as a diagonal element in the singlet-gluon matrix has distribution-valued (+, $\delta(1-z)$) and regular (reg) contributions :

$$A_{gg,Q}(z)=\left[A_{gg,Q,+}(z)\right]_++A_{gg,Q,\mathrm{reg}}(z)+C_{gg,Q}\delta(1-z), \tag{608}$$

with

$$\int_0^1 dz f(z)\left[A_{gg,Q}(z)\right]_+ \;=\; \int_0^1 dz[f(z)-f(1)]A_{gg,Q,+}(z) \tag{609}$$

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$$\int_0^1 dz\, C_{gg,Q}\delta(1-z) \;=\; C_{gg,Q}\;. \tag{610}$$

The different parts are given by :

$A_{gg,Q,+} =$

$$a_s^2 \frac{1}{z-1}\left\{C_A T_F\left[-\frac{8}{3}L_M^2 - \frac{80}{9}L_M - \frac{224}{27}\right]\right\}$$

$$+a_s^3 \frac{1}{z-1}\left\{C_A^2 T_F\left[L_M\left[\left[H_0\left[-\frac{64}{3}H_{0,-1} + \frac{32}{3}H_{0,1} - \frac{640}{9}H_1 - \frac{16}{3}\right] + \frac{128}{3}H_{0,0,-1} - \frac{64}{3}H_{0,0,1}\right.\right.\right.\right.$$

$$+\left[-\frac{16}{3}H_1 - \frac{160}{9}\right]H_0^2 + \frac{320\zeta_2}{9} - \frac{256\zeta_3}{3} - \frac{1240}{81}\right] + L_M^2\left[\left[-\frac{16}{3}H_0^2 - \frac{64}{3}H_1 H_0 + \frac{32\zeta_2}{3} - \frac{184}{9}\right]\right]$$

$$+\left[\zeta_2\left[\frac{8}{3}H_0^2 + \frac{32}{3}H_1 H_0 + \frac{16}{27}\right] - \frac{88 H_0}{9} - \frac{16\zeta_2^2}{3} - \frac{176\zeta_3}{27} - \frac{22672}{243}\right] + L_M^3 \frac{176}{27}\right]$$

$$+C_A C_F T_F\left[L_M\left[64\zeta_3 - \frac{40}{3}\right] - 8L_M^2 + \left[-40\zeta_2 - \frac{466}{9}\right]\right]$$

$$+C_A T_F^2\left[\left[\frac{16 H_0}{3} + \frac{560\zeta_2}{27} + \frac{224\zeta_3}{27} + \frac{5248}{81}\right] - L_M^3\frac{224}{27} - L_M^2\frac{640}{27} - L_M\frac{320}{9}\right]$$

$$+C_A T_F^2 N_F\left[\left[\frac{32 H_0}{9} + \frac{160\zeta_2}{27} + \frac{64\zeta_3}{27} + \frac{10496}{243}\right] - L_M^3\frac{64}{27} - L_M\frac{2176}{81}\right] + a_{gg,Q,(+)}^{(3)}\right\}, \tag{611}$$

$C_{gg,Q} =$

$$\frac{4}{3}a_s T_F L_M + a_s^2\left\{C_A T_F\left[\frac{16}{3}L_M + \frac{10}{9}\right] + C_F T_F\left[4L_M - 15\right] + \frac{16}{9}T_F^2 L_M^2\right\}$$

$$+a_s^3\left\{C_A^2 T_F\left[\left[\frac{16\zeta_3}{3} - \frac{2}{3}\right]L_M^2 + \left[\frac{16\zeta_2^2}{3} + \frac{160\zeta_3}{9} + \frac{277}{9}\right]L_M + \zeta_2\left(4 - \frac{8\zeta_3}{3}\right) - \frac{616}{27}\right]\right.$$

$$+C_F C_A T_F\left\{-\frac{22}{3}L_M^2 + \frac{736}{9}L_M + \frac{20\zeta_2}{3} + 16\zeta_3 - \frac{1045}{6} - 64\zeta_2\log(2)\right\}$$

$$+C_F T_F^2 N_F\left\{28\zeta_2 + \frac{118}{3} - \frac{268}{9}L_M\right\} + C_F T_F^2\left[\frac{40}{3}L_M^2 - \frac{584}{9}L_M + \frac{782}{9} - \frac{40\zeta_2}{3}\right]$$

$$+C_A T_F^2 N_F\left[\frac{224}{27} - \frac{4\zeta_2}{3} - \frac{44}{3}L_M\right] + C_A T_F^2\left[\frac{56}{3}L_M^2 - 2L_M - \frac{44\zeta_2}{3} - \frac{8}{27}\right]$$

$$+C_F^2 T_F\left[-2L_M + -80\zeta_2 - 32\zeta_3 - 39 + 128\zeta_2\log(2)\right] + T_F^3\left[\frac{64}{27}L_M^3 - \frac{64\zeta_3}{27}\right]$$

$$+a_{gg,Q,\delta}^{(3)}\bigg\}, \tag{612}$$

and

$A_{gg,Q,\mathrm{reg}} =$

$$a_s^2\left\{C_A T_F\left[\frac{4}{3}(z+1)H_0^2 + \frac{4}{9}(22z+13)H_0 - \frac{8(z^3 - z^2 + 2z - 1)L_M^2}{3z}\right.\right.$$

$$
-\frac{4\left(175z^3 - 137z^2 + 157z - 139\right)}{27z} + \frac{4}{3}zH_1 + L_M\left[\frac{16}{3}(z+1)H_0 - \frac{8\left(23z^3 - 19z^2 + 29z - 23\right)}{9z}\right]\right]\right]
$$

$$
+C_F T_F\left[\frac{4}{3}(z+1)H_0^3 + 2(5z+3)H_0^2 + 16(3z+2)H_0 + L_M^2\left[8(z+1)H_0 - \frac{4(z-1)\left(4z^2 + 7z + 4\right)}{3z}\right]\right.
$$

$$
\left.-\frac{8(z-1)\left(3z^2 + 9z - 1\right)}{z} + L_M\left[8(z+1)H_0^2 + 8(5z+3)H_0 - \frac{16(z-1)\left(5z^2 + 11z - 1\right)}{3z}\right]\right]\right\}
$$

$$
+a_s^3\left\{C_A^2 T_F\left[\frac{88}{27}(z+1)H_0^3 + \frac{44}{27}(22z+13)H_0^2 + \frac{8}{3}\left(2z^3 - 2z^2 - 4z - 1\right)\frac{\zeta_2}{z+1}H_0^2\right.\right.
$$

$$
+\frac{88}{81}(161z+62)H_0 + \frac{8}{9}\left(44z^2 + 11z + 47\right)\zeta_2 H_0 + \frac{176\left(z^3 - z^2 + 2z - 1\right)L_M^3}{27z} - \frac{44}{9}zH_1^2
$$

$$
+L_M^2\left[\frac{\left(z^2 + z + 1\right)^2}{z(z+1)}\left[\frac{64}{3}H_{-1}H_0 - \frac{64}{3}H_{0,-1}\right] - \frac{16\left(2z^3 - 2z^2 - 4z - 1\right)H_0^2}{3(z+1)}\right.
$$

$$
+\frac{8\left(208z^3 - 273z^2 + 204z - 208\right)}{27z} + \frac{32}{3}\left(2z^3 + 2z^2 + 4z + 3\right)\frac{\zeta_2}{z+1} - \frac{64}{9}\left(11z^2 + 9\right)H_0
$$

$$
\left.-\frac{64\left(z^3 - z^2 + 2z - 1\right)H_0 H_1}{3z}\right] - \frac{8\left(12283z^3 - 9665z^2 + 8143z - 7927\right)}{243z}
$$

$$
-\frac{4}{27}\left(416z^3 - 437z^2 + 433z - 416\right)\frac{\zeta_2}{z} - \frac{16}{3}\left(2z^3 + 2z^2 + 4z + 3\right)\frac{\zeta_2^2}{z+1} + \frac{176\left(4z^2 + 3z - 3\right)H_1}{27z}
$$

$$
+\left(z^2 + z + 1\right)^2\frac{\zeta_2}{z(z+1)}\left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0\right] + \frac{\left(z^3 - z^2 + 2z - 1\right)}{z}\left[\frac{32}{3}H_0 H_1\zeta_2 - \frac{176\zeta_3}{27}\right]
$$

$$
+L_M\left[\frac{(z+1)^3}{z}\left[-\frac{64}{3}H_{-1}H_{0,1} + \frac{64}{3}H_{0,-1,1} + \frac{64}{3}H_{0,1,-1}\right] + \frac{8\left(4z^2 - 31z + 4\right)H_{-1}H_0^2(z+1)}{3z}\right.
$$

$$
-\frac{16}{3}\left(2z^2 - 35z + 2\right)\frac{\zeta_2}{z}H_{-1}(z+1) + \frac{\left(2z^2 - 9z + 2\right)(z+1)}{z}\left[-16H_0 H_{-1}^2 + 32H_{0,-1}H_{-1}\right.
$$

$$
\left.-32H_{0,-1,-1}\right] + \left[-24H_{0,1}H_0^2 + 96H_{0,0,1}H_0 - \frac{1832}{9}H_{0,1} + 192H_{0,0,0,-1} - 192H_{0,0,0,1}\right.
$$

$$
\left.+96H_{0,1}\zeta_2\right](z+1) + \frac{8}{9}(19z - 12)H_0^3 - \frac{24}{5}(13z + 23)\zeta_2^2 + \frac{4\left(13194z^3 - 13763z^2 + 12661z - 12402\right)}{81z}
$$

$$
+\frac{8}{9}\left(132z^4 - 137z^3 + 156z^2 + 481z + 96\right)\frac{\zeta_2}{z(z+1)} - \frac{32}{3}\left(8z^3 + 7z^2 + 31z - 11\right)\frac{\zeta_3}{z}
$$

$$
-\frac{4\left(3918z^3 - 1495z^2 + 4022z + 568\right)H_0}{27z} - \frac{8}{3}\left(8z^4 + 43z^3 + 34z^2 - 5z - 8\right)\frac{\zeta_2}{z(z+1)}H_0
$$

$$
+\frac{4}{3}(25z - 29)H_0^2 H_1 + \frac{4\left(410z^3 + 1965z^2 - 2031z - 410\right)H_1}{27z} - 16(z-1)\left(2z^2 + 9z + 2\right)\frac{\zeta_2}{z}H_1
$$

$$
-\frac{32\left(33z^3 - 29z^2 + 49z - 33\right)H_0 H_1}{9z} + \frac{64}{3}\left(z^2 + 9z - 7\right)H_0 H_{0,-1} - \frac{8}{3}(17z - 37)H_0 H_{0,1}
$$

$$
-\frac{16\left(12z^3 + 45z^2 - 83z + 4\right)H_{0,0,-1}}{3z} - 32(5z - 1)H_0 H_{0,0,-1} + \frac{16}{3}\left(4z^2 + 31z - 23\right)H_{0,0,1}
$$

$$
+8(5z + 3)H_0^2\zeta_2 + (z-1)\left[48H_{0,-1}H_0^2 - 192H_{0,-1,-1}H_0 + 96H_{0,-1}^2 - 96H_{0,-1}\zeta_2\right]
$$

$$
+z\left(192H_0\zeta_3 - \frac{4}{3}H_0^4\right) - \frac{4\left(124z^3 - 459z^2 - 613z - 70\right)H_0^2}{9(z+1)}
$$

$$
\left.+\frac{\left(114z^4 - 77z^3 - 342z^2 - 77z + 114\right)}{z(z+1)}\left[\frac{16}{9}H_{-1}H_0 - \frac{16}{9}H_{0,-1}\right]\right]\right]
$$

$$
+C_A T_F^2 \left[-L_M^3 \frac{224(z^3 - z^2 + 2z - 1)}{27z} + L_M^2 \left[\frac{128}{9}(z+1)H_0 - \frac{64(23z^3 - 19z^2 + 29z - 23)}{27z} \right] \right.
$$

$$
+ L_M \left[-\frac{8(814z^3 - 633z^2 + 777z - 598)}{81z} - \frac{8}{27}(52z^2 - 89z - 2)H_0 - \frac{8(52z^3 - 51z^2 + 33z - 52)H_1}{27z} \right.
$$

$$
\left. + (z+1)\left[\frac{80}{9}H_0^2 + \frac{128}{9}H_{0,1} - \frac{128\zeta_2}{9} \right] \right] - \frac{8}{9}(22z + 13)H_0^2 + \frac{8}{3}zH_1^2 - \frac{16}{27}(161z + 62)H_0
$$

$$
+ \frac{16(1187z^3 - 949z^2 + 881z - 791)}{81z} + \frac{56}{27}(23z^3 - 19z^2 + 29z - 23)\frac{\zeta_2}{z} + \frac{224}{27}(z^3 - z^2 + 2z - 1)\frac{\zeta_3}{z}
$$

$$
- \frac{32(4z^2 + 3z - 3)H_1}{9z} + (z+1)\left[-\frac{16}{9}H_0^3 - \frac{112}{9}\zeta_2 H_0 \right] \right] + C_A N_F T_F^2 \left[-\frac{64(z^3 - z^2 + 2z - 1)L_M^3}{27z} \right.
$$

$$
+ L_M \left[-\frac{16(114z^3 - 85z^2 + 149z - 42)}{81z} - \frac{16}{27}(52z^2 - z + 50)H_0 - \frac{16(52z^3 - 39z^2 + 33z - 52)H_1}{27z} \right.
$$

$$
\left. + (z+1)\left[\frac{32}{3}H_0^2 + \frac{256}{9}H_{0,1} - \frac{256\zeta_2}{9} \right] \right] - \frac{16}{27}(22z + 13)H_0^2 + \frac{32(1187z^3 - 949z^2 + 881z - 791)}{243z}
$$

$$
+ \frac{16}{9}zH_1^2 + \frac{16}{27}(23z^3 - 19z^2 + 29z - 23)\frac{\zeta_2}{z} + \frac{64}{27}(z^3 - z^2 + 2z - 1)\frac{\zeta_3}{z} - \frac{32}{81}(161z + 62)H_0
$$

$$
\left. - \frac{64(4z^2 + 3z - 3)H_1}{27z} + (z+1)\left[-\frac{32}{27}H_0^3 - \frac{32}{9}\zeta_2 H_0 \right] \right]
$$

$$
+ C_F^2 T_F \left[\frac{2}{9}(4z^2 - 3z + 3)H_0^4 + \frac{2}{9}(40z^2 + 149z + 115)H_0^3 - \frac{2}{3}(64z^2 + 23z - 69)H_0^2 \right.
$$

$$
+ \frac{4(z-1)(20z^2 + 41z - 4)H_1 H_0^2}{3z} - \frac{8(4z^3 + 27z^2 + 3z - 4)H_{0,1}H_0^2}{3z} + \frac{8}{3}(4z^2 - 3z - 3)\zeta_2 H_0^2
$$

$$
+ \frac{4}{3}(80z^2 + 33z + 246)H_0 - \frac{8(z-1)(32z^2 + 127z - 18)H_1 H_0}{3z} - \frac{16(10z^3 - 35z^2 - 49z + 2)H_{0,1}H_0}{3z}
$$

$$
- \frac{16(4z^3 - 33z^2 - 15z + 4)H_{0,0,1}H_0}{3z} + \frac{2}{3}(80z^2 + 469z + 221)\zeta_2 H_0 - \frac{16}{9}(44z^2 + 51z - 18)\zeta_3 H_0
$$

$$
+ \frac{8(z-1)(6z^2 - z - 6)H_1^3}{9z} + \frac{4(z-1)(24z^2 - 13z + 17)H_1^2}{3z} - \frac{64(z+1)^2(2z-1)H_{0,1}^2}{3z}
$$

$$
- \frac{8}{15}(188z^2 - 27z - 105)\zeta_2^2 - \frac{4(z-1)(56z^2 + 418z - 5)}{3z} - \frac{4}{3}(84z^3 + 79z^2 + 75z - 60)\frac{\zeta_2}{z}
$$

$$
+ \frac{4(z-1)(80z^2 - 181z - 9)H_1}{3z} + \frac{4}{3}(z-1)(40z^2 + 33z + 4)\frac{\zeta_2}{z}H_1
$$

$$
+ \frac{8(56z^3 + 136z^2 - 121z + 18)H_{0,1}}{3z} + \frac{8}{3}(12z^3 + 15z^2 + 9z + 8)\frac{\zeta_2}{z}H_{0,1} + \frac{32}{3}(z-1)H_1 H_{0,1}
$$

$$
+ \frac{8(20z^3 - 239z^2 - 187z + 4)H_{0,0,1}}{3z} + \frac{8}{3}(12z^2 - 23z - 22)H_{0,1,1}
$$

$$
+ \frac{16(20z^3 - 21z^2 - 33z + 4)H_{0,0,0,1}}{3z} + \frac{32(3z^2 + 15z + 8)H_{0,0,1,1}}{3z}
$$

$$
- \frac{16(20z^3 + 15z^2 - 27z - 24)H_{0,1,1,1}}{3z} + L_M^3 \left[-\frac{16(4z^2 + 7z + 4)H_1(z-1)}{9z} - \frac{92(z-1)}{9} \right.
$$

$$
\left. - \frac{16}{9}z(4z+3)H_0 + (z+1)\left[\frac{8}{3}H_0^2 + \frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3} \right] \right] - \frac{4}{9}(72z^2 - 1673z - 1009)\zeta_3
$$

$$+L_M\left[\frac{16}{3}(5z+2)H_0^3-\frac{2}{3}\left(72z^2-599z-207\right)H_0^2-\frac{8}{3}\left(78z^2-256z-135\right)H_0\right.$$

$$-\frac{8(z-1)\left(36z^2+25z+24\right)H_1H_0}{3z}+\frac{128(6z+1)H_{0,1}H_0}{3z}-256zH_{0,0,-1}H_0$$

$$-32(8z+3)\zeta_2H_0+64(z-3)\zeta_3H_0-\frac{4(z-1)\left(28z^2+21z+4\right)H_1^2}{3z}-\frac{32}{5}(13z+23)\zeta_2^2$$

$$-\frac{4(z-1)\left(268z^2+377z-68\right)}{3z}+\frac{16}{3}\left(36z^3-51z^2+3z-8\right)\frac{\zeta_3}{z}-\frac{16(z-1)\left(39z^2-45z-13\right)H_1}{3z}$$

$$+\frac{8\left(8z^3-23z^2-43z-24\right)H_{0,1}}{3z}+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[\frac{8}{9}H_1^3+\frac{16}{3}H_0H_1^2-\frac{32}{3}H_{0,1}H_1\right]$$

$$-\frac{128\left(3z^3+3z^2-9z-1\right)H_{0,0,-1}}{3z}-\frac{32\left(4z^3-12z^2+27z+4\right)H_{0,0,1}}{3z}$$

$$+\frac{16\left(12z^3+27z^2+3z-8\right)H_{0,1,1}}{3z}+\frac{16}{3}\left(14z^2-128z+21\right)\zeta_2$$

$$+\frac{(z+1)\left(z^2-4z+1\right)}{z}\left[-\frac{128}{3}H_0H_{-1}^2+\left(\frac{64}{3}H_0^2+\frac{256}{3}H_{0,-1}\right)H_{-1}-\frac{128}{3}\zeta_2H_{-1}-\frac{256}{3}H_{0,-1,-1}\right]$$

$$+\frac{(z-1)\left(z^2+4z+1\right)}{z}\left[\frac{64}{3}H_1H_0^2+\frac{128}{3}H_{0,-1}H_0-\frac{128}{3}H_1\zeta_2\right]+(z-1)\left[64H_{0,-1}H_0^2\right.$$

$$-256H_{0,-1,-1}H_0+128H_{0,-1}^2-128H_{0,-1}\zeta_2\Big]+(z+1)\Big[2H_0^4-64H_{0,1}H_0^2-\frac{2144}{3}H_{-1}H_0$$

$$+\left[288H_{0,0,1}-64H_{0,1,1}\right]H_0+32H_{0,1}^2+\frac{2144}{3}H_{0,-1}+384H_{0,0,0,-1}-416H_{0,0,0,1}-32H_{0,1,1,1}$$

$$+\left[128H_{0,1}-32H_0^2\right]\zeta_2\Big]+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[\frac{2}{9}H_1^4-\frac{16}{3}H_{0,1}H_1^2+\frac{8}{9}H_0^3H_1\right.$$

$$+\left[\frac{64}{3}H_{0,0,1}+\frac{64}{3}H_{0,1,1}\right]H_1-\frac{176}{9}\zeta_3H_1+H_0\left[\frac{64}{3}H_{0,1,1}-\frac{32}{3}H_1H_{0,1}\right]+\left[\frac{20}{3}H_1^2+\frac{16}{3}H_0H_1\right]\zeta_2\right]$$

$$+L_M^2\left[-\frac{8}{3}\left(4z^2-9z-3\right)H_0^2-\frac{8}{3}(z+1)(32z-31)H_0-\frac{16(z-1)\left(4z^2+7z+4\right)H_1H_0}{3z}\right.$$

$$-\frac{4(z-1)\left(32z^2+81z+12\right)}{3z}-\frac{8(z-1)\left(32z^2+35z+8\right)H_1}{3z}+\frac{16\left(4z^3+21z^2+9z-4\right)H_{0,1}}{3z}$$

$$-32(3z+2)\zeta_2+(z+1)\left[\frac{16}{3}H_0^3+32H_{0,1}H_0-32\zeta_2H_0-32H_{0,0,1}+32\zeta_3\right]\right]+(z+1)\left[-\frac{2}{15}H_0^5\right.$$

$$-\frac{16}{3}H_{0,1}H_0^3+48H_{0,0,1}H_0^2+\frac{448}{5}\zeta_2^2H_0+\left[32H_{0,1}^2-160H_{0,0,0,1}-128H_{0,0,1,1}\right]H_0$$

$$+H_{0,1}\left[64H_{0,1,1}-128H_{0,0,1}\right]+192H_{0,0,0,0,1}+768H_{0,0,0,1,1}+320H_{0,0,1,0,1}-416H_{0,0,1,1,1}$$

$$-192H_{0,1,0,1,1}-32H_{0,1,1,1,1}+\left[\frac{88}{3}H_0^2+\frac{352}{3}H_{0,1}\right]\zeta_3+\zeta_2\left[-4H_0^3-32H_{0,1}H_0\right.$$

$$-32H_{0,0,1}-80H_{0,1,1}+\frac{80\zeta_3}{3}\Big]-160\zeta_5\Big]\right]$$

$$+C_FT_F^2\left[-\frac{8}{3}(5z+3)H_0^3-32(3z+2)H_0^2-64(7z+5)H_0+\frac{8}{9}\left(8z^2-61z-31\right)\zeta_2H_0\right.$$

$$+L_M^3\left[\frac{160}{9}(z+1)H_0-\frac{80(z-1)\left(4z^2+7z+4\right)}{27z}\right]+\frac{32(z-1)\left(19z^2+64z-5\right)}{3z}$$

$$+\frac{16}{27}(z-1)\left(67z^2+184z-41\right)\frac{\zeta_2}{z}+L_M^2\left[-\frac{32(z-1)\left(22z^2+85z-32\right)}{27z}\right.$$

$$-\frac{64}{9}\left(2z^2-4z-1\right)H_0-\frac{32(z-1)\left(4z^2+7z+4\right)H_1}{9z}+(z+1)\left[\frac{64}{3}H_0^2+\frac{64}{3}H_{0,1}-\frac{64\zeta_2}{3}\right]\Big]$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[\frac{16}{9}H_1\zeta_2+\frac{80\zeta_3}{27}\right]+(z+1)\left[-\frac{4}{3}H_0^4-\frac{160}{9}\zeta_3H_0+\frac{32\zeta_2^2}{3}\right.$$

$$+\left[-\frac{56}{3}H_0^2-\frac{32}{3}H_{0,1}\right]\zeta_2\Big]+L_M\left[-\frac{8}{9}\left(16z^2+13z+19\right)H_0^2-\frac{64}{27}\left(11z^2-13z-76\right)H_0\right.$$

$$-\frac{32(z-1)\left(73z^2+631z+73\right)}{81z}-\frac{32(z-1)\left(22z^2+85z-32\right)H_1}{27z}$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[-\frac{16}{9}H_1^2-\frac{64}{9}H_0H_1\right]+\frac{64\left(2z^3+7z^2-2z-4\right)H_{0,1}}{9z}$$

$$+\frac{64}{9}\left(2z^2-4z-1\right)\zeta_2+(z+1)\left[\frac{80}{9}H_0^3+\frac{128}{3}H_{0,1}H_0-\frac{128}{3}\zeta_2H_0-\frac{128}{3}H_{0,0,1}+\frac{64}{3}H_{0,1,1}+\frac{64\zeta_3}{3}\right]\Big]\Big]$$

$$+C_FN_FT_F^2\left[-\frac{16}{9}(5z+3)H_0^3-\frac{64}{3}(3z+2)H_0^2-\frac{64}{81}\left(110z^2+755z+518\right)H_0\right.$$

$$-\frac{16}{9}\left(4z^2+37z+25\right)\zeta_2H_0+L_M^3\left[\frac{128}{9}(z+1)H_0-\frac{64(z-1)\left(4z^2+7z+4\right)}{27z}\right]$$

$$+\frac{16(z-1)\left(38z^2+47z+20\right)H_1^2}{27z}+\frac{64(z-1)\left(1138z^2+2470z+193\right)}{243z}$$

$$-\frac{16}{27}\left(8z^3+223z^2+271z+14\right)\frac{\zeta_2}{z}+\frac{64}{27}\left(10z^3+36z^2+21z-4\right)\frac{\zeta_3}{z}$$

$$-\frac{128(z-1)\left(55z^2+64z+28\right)H_1}{81z}+\frac{64}{27}\left(19z^2+67z+43\right)H_{0,1}-\frac{64}{9}\left(2z^2+11z+8\right)H_{0,1,1}$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[-\frac{16}{27}H_1^3-\frac{16}{9}\zeta_2H_1\right]+(z+1)\left[-\frac{8}{9}H_0^4-\frac{128}{9}\zeta_3H_0-\frac{96\zeta_2^2}{5}+\frac{64}{3}H_{0,1,1,1}\right.$$

$$+\left[\frac{32}{3}H_{0,1}-\frac{16}{3}H_0^2\right]\zeta_2\Big]+L_M\left[-\frac{16}{9}\left(16z^2+43z+37\right)H_0^2-\frac{64}{9}\left(z^2+5z-41\right)H_0\right.$$

$$-\frac{64(z-1)\left(7z^2+91z+61\right)}{27z}-\frac{32(z-1)\left(2z^2+41z-28\right)H_1}{9z}+\frac{64\left(2z^3+3z^2-12z-8\right)H_{0,1}}{9z}$$

$$+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[-\frac{16}{3}H_1^2-\frac{128}{9}H_0H_1\right]+\frac{64}{3}\left(2z^2+z+2\right)\zeta_2$$

$$+(z+1)\left[\frac{32}{3}H_0^3+\frac{256}{3}H_{0,1}H_0-\frac{256}{3}\zeta_2H_0-\frac{256}{3}H_{0,0,1}+64H_{0,1,1}+\frac{64\zeta_3}{3}\right]\Big]\Big]$$

$$+C_AC_FT_F\left[\frac{2}{9}(26z+5)H_0^4+\frac{4}{27}\left(20z^2+141z+174\right)H_0^3-\frac{4}{9}\left(300z^2-901z-169\right)H_0^2\right.$$

$$-\frac{8(z-1)\left(26z^2-32z+23\right)H_1H_0^2}{3z}-\frac{8\left(3z^2+3z-8\right)H_{0,1}H_0^2}{3z}+32(2z-1)H_{0,0,1}H_0^2$$

$$+\frac{8}{3}(32z+5)\zeta_2H_0^2+\frac{16}{3}(14z-13)\zeta_3H_0^2+\frac{16(z-1)\left(19z^2+16z+10\right)H_1^2H_0}{9z}-\frac{16}{5}(38z-5)\zeta_2^2H_0$$

$$+\frac{4\left(36604z^3+28737z^2+27060z+2624\right)H_0}{81z}+\frac{4}{9}\left(298z^3+121z^2+427z+80\right)\frac{\zeta_2}{z}H_0$$

$$-\frac{16}{9}\left(145z^2-83z-4\right)\frac{\zeta_3}{z}H_0+\frac{32(z-1)\left(461z^2-73z+227\right)H_1H_0}{27z}$$

$$+\frac{32\left(19z^3-24z^2-6z+10\right)H_{0,-1}H_0}{9z}-\frac{16\left(10z^3+249z^2-48z+109\right)H_{0,1}H_0}{9z}$$

$$-\frac{32\left(9z^2-9z+8\right)H_{0,0,1}H_0}{3z}-64(5z-2)H_{0,0,0,1}H_0-\frac{8(z-1)(2z+1)(14z+1)H_1^3}{27z}$$

$$-\frac{4(z-1)\left(328z^2+313z+67\right)H_1^2}{27z}-\frac{2\left(417275z^3-288519z^2-19632z-102833\right)}{243z}$$

$$-\frac{16}{27}\left(845z^3-432z^2+255z-182\right)\frac{\zeta_2}{z}-\frac{8}{15}\left(20z^3+340z^2-137z+152\right)\frac{\zeta_2^2}{z}$$

$$-\frac{8}{9}\left(416z^3+498z^2+555z+76\right)\frac{\zeta_3}{z}-\frac{4\left(2500z^3-69z^2-984z-1771\right)H_1}{81z}-\frac{4}{9}$$

$$(z-1)\left(154z^2+163z+46\right)\frac{\zeta_2}{z}H_1+\frac{(z+1)\left(182z^2-122z+47\right)}{z}\left[\frac{32}{27}H_{0,-1}-\frac{32}{27}H_{-1}H_0\right]$$

$$-\frac{8\left(1732z^3-2205z^2+972z-908\right)H_{0,1}}{27z}-\frac{32\left(19z^3-51z^2-6z+10\right)H_{0,0,-1}}{9z}$$

$$+\frac{16\left(98z^3+324z^2+69z+149\right)H_{0,0,1}}{9z}-\frac{8\left(56z^3-105z^2-66z-40\right)H_{0,1,1}}{9z}$$

$$+\frac{16\left(21z^2-15z+8\right)H_{0,0,0,1}}{z}+\frac{16\left(20z^3+18z^2-15z-20\right)H_{0,0,1,1}}{3z}$$

$$+\frac{32(z-1)(z+2)(2z+1)H_{0,1,1,1}}{3z}+128(4z-1)H_{0,0,0,0,1}+\frac{8}{3}(23z+14)H_{0,1}\zeta_2$$

$$+\frac{(z+1)\left(19z^2-16z+10\right)}{z}\left[\frac{32}{9}H_0H_{-1}^2+\left[-\frac{16}{9}H_0^2-\frac{64}{9}H_{0,-1}\right]H_{-1}+\frac{32}{9}\zeta_2H_{-1}+\frac{64}{9}H_{0,-1,-1}\right]$$

$$+L_M^3\left[-\frac{16}{3}(2z-1)H_0^2-\frac{16\left(8z^2+11z+4\right)H_0}{9z}+\frac{8(z-1)\left(44z^2-z+44\right)}{9z}\right.$$

$$\left.+\frac{16(z-1)\left(4z^2+7z+4\right)H_1}{9z}+(z+1)\left[\frac{32\zeta_2}{3}-\frac{32}{3}H_{0,1}\right]\right]-\frac{8}{3}(43z+37)\zeta_2\zeta_3$$

$$+L_M^2\left[-\frac{8}{3}(38z+5)H_0^2-\frac{8\left(184z^3+103z^2+205z+40\right)H_0}{9z}\right.$$

$$-\frac{16(z-1)\left(4z^2+7z+4\right)H_1H_0}{3z}+\frac{8\left(517z^3-444z^2+45z-127\right)}{9z}+\frac{16}{3}\left(4z^3+17z^2+11z+4\right)\frac{\zeta_2}{z}$$

$$+\frac{8(z-1)\left(104z^2+119z+32\right)H_1}{9z}+\frac{(z+1)\left(4z^2-7z+4\right)\left[\frac{32}{3}H_{-1}H_0-\frac{32}{3}H_{0,-1}\right]}{z}$$

$$-\frac{32}{3}(10z+7)H_{0,1}+(z+1)\left[32H_0H_{0,1}-32H_{0,0,1}\right]+(z-1)\left[\frac{32}{3}H_0^3+64H_{0,-1}H_0+32\zeta_2H_0\right.$$

$$\left.-128H_{0,0,-1}\right]+64(2z-1)\zeta_3\right]+\frac{(z+1)\left(4z^2-7z+4\right)}{z}\left[-\frac{32}{9}H_0H_{-1}^3+\left[\frac{8}{3}H_0^2+\frac{32}{3}H_{0,-1}\right]H_{-1}^2\right.$$

$$+\left[\frac{8}{9}H_0^3+\left[\frac{32}{3}H_{0,1}-\frac{32}{3}H_{0,-1}\right]H_0-\frac{64}{3}H_{0,-1,-1}+\frac{32}{3}H_{0,0,-1}-\frac{64}{3}H_{0,0,1}\right]H_{-1}+16\zeta_3H_{-1}$$

$$-\frac{8}{3}H_0^2H_{0,-1}+H_0\left[\frac{32}{3}H_{0,-1,-1}-\frac{32}{3}H_{0,-1,1}+\frac{16}{3}H_{0,0,-1}-\frac{32}{3}H_{0,1,-1}\right]+\frac{64}{3}H_{0,-1,-1,-1}$$

$$+\frac{32}{3}H_{0,-1,0,1}-\frac{32}{3}H_{0,0,-1,-1}+\frac{64}{3}H_{0,0,-1,1}-\frac{16}{3}H_{0,0,0,-1}+\frac{64}{3}H_{0,0,1,-1}+\left[-\frac{16}{3}H_{-1}^2-8H_0H_{-1}\right.$$

$$\left.\left.+8H_{0,-1}\right]\zeta_2\right]+L_M\left[\frac{4}{3}(11z-7)H_0^4+\frac{16}{9}(37z-23)H_0^3-\frac{4}{9}\left(234z^2+1487z+1031\right)H_0^2\right.$$

$$+\frac{8(z+1)\left(8z^2+43z+8\right)H_{-1}H_0^2}{3z}-\frac{8(z-1)\left(16z^2+55z+16\right)H_1H_0^2}{3z}$$

$$-\frac{8\left(2384z^3+4795z^2+2230z-120\right)H_0}{27z}+\frac{64}{3}\left(3z^3+16z^2-2z-2\right)\frac{\zeta_2}{z}H_0$$

$$
\begin{aligned}
&-\frac{16(z-1)\left(115z^2-17z+52\right)H_1H_0}{9z}-\frac{16\left(32z^3+9z^2-27z+16\right)H_{0,-1}H_0}{3z}\\
&+\frac{16}{3}\left(12z^2+35z-40\right)H_{0,1}H_0+64(3z+1)H_{0,0,-1}H_0-192(z+2)H_{0,0,1}H_0-32(2z+3)\zeta_3 H_0\\
&+\frac{4(z-1)\left(20z^2+21z+2\right)H_1^2}{3z}+\frac{16}{5}(89z+73)\zeta_2^2+\frac{8\left(2021z^3-1164z^2-1313z+441\right)}{9z}\\
&+\frac{16}{9}\left(30z^3+539z^2-334z-56\right)\frac{\zeta_2}{z}-\frac{16}{3}\left(4z^3-154z^2-49z+16\right)\frac{\zeta_3}{z}\\
&+\frac{16}{3}(z+1)\left(4z^2-79z+4\right)\frac{\zeta_2}{z}H_{-1}+\frac{16(z-1)\left(340z^2-1007z-254\right)H_1}{27z}\\
&+\frac{32}{3}(z-1)\left(8z^2+35z+8\right)\frac{\zeta_2}{z}H_1+\frac{(z+1)\left(29z^2+259z-34\right)}{z}\left[\frac{32}{9}H_{-1}H_0-\frac{32}{9}H_{0,-1}\right]\\
&+\frac{16\left(85z^3-95z^2+403z-64\right)H_{0,1}}{9z}+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[-\frac{8}{9}H_1^3-8H_0H_1^2+\frac{32}{3}H_{0,1}H_1\right]\\
&+\frac{(z+1)\left(4z^2-21z+4\right)}{z}\left[16H_0H_{-1}^2-32H_{0,-1}H_{-1}+32H_{0,-1,-1}\right]\\
&+\frac{16\left(56z^3-33z^2-105z+24\right)H_{0,0,-1}}{3z}-\frac{16\left(20z^3+53z^2-49z+16\right)H_{0,0,1}}{3z}\\
&+\frac{(z+1)^3}{z}\left[\frac{128}{3}H_{-1}H_{0,1}-\frac{128}{3}H_{0,-1,1}-\frac{128}{3}H_{0,1,-1}\right]-\frac{16\left(8z^3+23z^2+5z-8\right)H_{0,1,1}}{3z}\\
&-96(3z+5)H_{0,0,0,-1}+96(z+7)H_{0,0,0,1}+(z-1)\left[-48H_{0,-1}H_0^2+448H_{0,-1,-1}H_0-224H_{0,-1}^2\right.\\
&\left.+224H_{0,-1}\zeta_2\right]+(z+1)\left[112H_{0,1}H_0^2+96H_{0,1,1}H_0-32H_{0,1}^2-32H_{0,0,1,1}+32H_{0,1,1,1}+\left[-96H_0^2\right.\right.\\
&\left.\left.-256H_{0,1}\right]\zeta_2\right]+\frac{(z-1)\left(4z^2+7z+4\right)}{z}\left[-\frac{2}{9}H_1^4-\frac{4}{3}H_0^2H_1^2+\left[-\frac{80}{3}H_{0,0,1}-\frac{16}{3}H_{0,1,1}\right]H_1+\frac{80}{9}\zeta_3 H_1\right.\\
&+\frac{16}{3}H_{0,1}^2+H_0\left[\frac{16}{9}H_1^3+16H_{0,1}H_1-\frac{80}{3}H_{0,1,1}\right]+\left[\frac{8}{3}H_0H_1-\frac{4}{3}H_1^2\right]\zeta_2\right]+(z-1)\left[-\frac{4}{15}H_0^5\right.\\
&+\frac{16}{3}H_{0,-1}H_0^3+\left[32H_{0,-1,-1}-32H_{0,0,-1}\right]H_0^2+\left[-32H_{0,-1}^2-128H_{0,-1,-1,-1}+64H_{0,-1,0,1}\right.\\
&\left.+96H_{0,0,0,-1}\right]H_0+H_{0,-1}\left[128H_{0,-1,-1}+64H_{0,0,-1}-128H_{0,0,1}\right]-256H_{0,-1,0,-1,-1}\\
&-512H_{0,0,-1,-1,-1}-64H_{0,0,-1,0,-1}+128H_{0,0,-1,0,1}-192H_{0,0,0,-1,-1}+384H_{0,0,0,-1,1}\\
&-128H_{0,0,0,0,-1}+384H_{0,0,0,1,-1}+128H_{0,0,1,0,-1}+\left[-8H_0^3-48H_{0,-1}H_0-64H_{0,-1,-1}\right.\\
&\left.\left.+96H_{0,0,-1}\right]\zeta_2+96H_{0,-1}\zeta_3\right]+(z+1)\left[16H_{0,1,1}H_0^2+\left[-48H_{0,1}^2+128H_{0,0,1,1}-64H_{0,1,1,1}\right]H_0\right.\\
&+160H_{0,1}H_{0,0,1}-864H_{0,0,0,1,1}-288H_{0,0,1,0,1}+128H_{0,0,1,1,1}+32H_{0,1,0,1,1}+32H_{0,1,1,1,1}\\
&\left.+\left[-16H_0H_{0,1}+16H_{0,0,1}+16H_{0,1,1}\right]\zeta_2-\frac{160}{3}H_{0,1}\zeta_3\right]-80(7z-3)\zeta_5+a_{gg,Q,\mathrm{reg}}^{(3)}\Bigg\}.
\end{aligned}
\tag{613}
$$

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