MAST90053 EXPERIMENTAL MATHEMATICS

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2014, SEMESTER 1, WEEK 3

Bifurcation points of the logistic map

The logistic map is the iteration

$$x_{n+1} = f(x_n) = rx_n(1 - x_n).$$

When $1 < r < 3$ this map is regular and iterates to the fixed point $x^\dagger$ determined by

$$f(x^\dagger) = x^\dagger.$$

The fixed point is thus given by $x^\dagger = (r-1)/r$. This fixed point is stable when the Jacobian is less than one, i.e.,

$$|f'(x^\dagger)| < 1 \quad \Leftrightarrow \quad 1 < r < 3.$$

At $r = 3$ the fixed point $x^\dagger$ becomes unstable and the map jumps back and forth between two values of x. This is the onset of a period doubling bifurcation. The fixed points of the double iterated map are solution of the polynomial equation

$$f_2(x) \equiv f(f(x)) = x$$

or

$$x(rx - r + 1)\left(r^2x^2 - r^2x - rx + r + 1\right) = 0$$

The fixed point $x^\dagger$ is also a fixed point (albeith unstable) of the double iterated map and indeed appears as a root. You can check that the fixed points of $f_2(x)$ are stable for $3 < r < 1 + \sqrt{6}$.

The quickest way to determine the bifurcation points involves computing the discriminat of the fixed point equation. The discriminant of a polynomial p with leading coefficient one and roots x_i is the product

$$D(p) = \prod_{i<j} (x_i - x_j)^2.$$

Therefore $D(p) = 0$ if and only if the polynomial p has at least a double root. This happens exactly at the bifurcation points of the logistic map, where a single fixed point splits into two new fixed points. The discriminant can be computed as the determinant of a matrix whose entries are associated with the coefficients of p and, in particular, one does not need to know the roots of p in advance. `Mathematica` can compute the discriminant of a polynomial with the command `Discriminant`.

Factoring out the fixed point of the single iterate, we can compute the discriminant of

$$\frac{f_2(x) - x}{f(x) - x} = r^2x^2 - r^2x - rx + r + 1$$

which is

$$(r-3)r^2(r+1),$$

showing the bifurcation point at $r = 3$. The onset of higher order bifurcation points can be computed in a similar way, but the degree of the polynomial that has r as a root increases rapidly.

EXERCISE 1. Compute the values of r denoting the onset of a 3-cycle and 4-cycle.

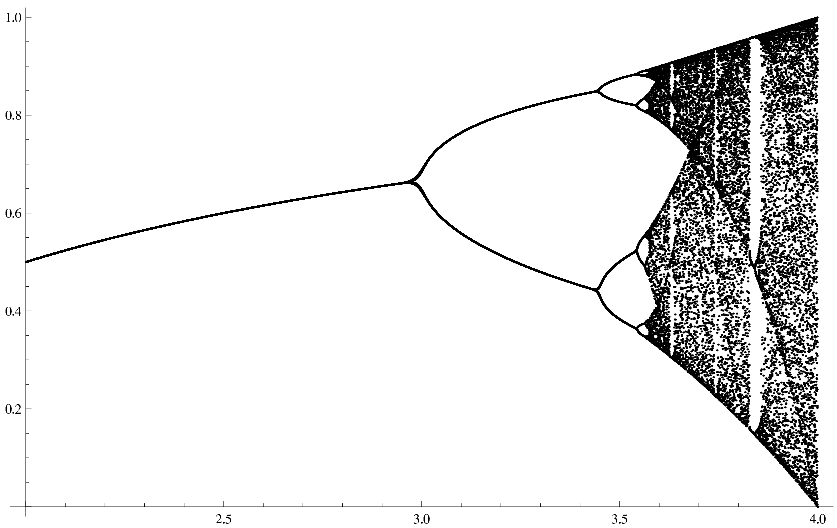

Figure 1: Bifurcation diagram of the logistic map

The bifurcation diagram of the logistic map shows the values of x to which the iteration converges as a function of r. This diagram is generated by the following `Mathematica` code:

```
Clear[x,r,n]
ListPlot[Table[
  {r,#} & /@
   RecurrenceTable[
      {x[n+1]==r x[n](1-x[n]),x[1]==0.5},
      x,{n,100,140}
    ],
  {r,2,4,0.001}],
 PlotStyle -> PointSize[Tiny], ImageSize -> Large
 ]
```

It is clear that for $r < 3$ there is only one fixed point. For $3 < r \lesssim 3.45$ there is a period-2 orbit and after that point a period-4 orbit becomes stable. Computing the onset of the period-8 orbit is more difficult, and in the next section we shall use the PSLQ algorithm to achieve that.

Computing r_8

Bailey et al. (2006) sketched a techinque to compute the onset of the period-8 orbit up to very high precision. Let $f_8(x)$ be the eigth-times iterate of $f(x)$and let $g_8(x) = f_8(x) - x$. Given some initial r slightly less than the expected bifurcation point, a "comb" of function values $g_8(x)$ at n evenly spaced x values (with spacing h_x) is computed near the limit of the iteration $x_{n+1} = f_8(x_n)$. The comb is constructed such that it has $n/2$ positive function values of $g_8(x)$, followed by $n/2$ negative function values. Then r is incremented by some small amount h_r and the comb is reevaluated. This process is repeated until two sign changes are observed among the n function values. This means that a bifurcation occurred just prior to the current value of r. The value of r is then restored to its previous value and the values of h_r and h_x are reduced to make the "comb" finer. This process is continued, moving the value of r and its associated comb back and forth near the bifurcation point with progressively smaller values of h_r.

In this way we can compute the third bifurcation point r_8 to a very high precision in just a few seconds. There is a `Mathematica` notebook `LogisticMap.nb` available on LMS that does just that.

EXERCISE 2. Compute the bifurcation point r_8 to high enough precision to be able to verify, using PSLQ, that is it algebraic. Verify that it is the smallest positive real root of the following polynomial of degree 12.

$$\begin{aligned} 4913 + 2108r^2 - 604r^3 - 977r^4 + 8r^5 + 44r^6 + 392r^7 - 193r^8 \\ - 40r^9 + 48r^{10} - 12r^{11} + r^{12}, \end{aligned}$$

Two important numbers associated to iterated functions are Feigenbaum's constants α and δ Feigenbaum (1979). These numbers are defined by

$$\delta = \lim_{n\to\infty} \frac{B_n - B_{n-1}}{B_{n+1} - B_n},$$

where B_n is the nth bifurcation point r_{2^n}, and

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}},$$

where d_n is the value of the x-value nearest to 0 in the nth cycle. The values of these constants are believed to be universal, i.e. independent of the details of the iteration as long as the iteration has a quadratic maximum. The values of α and δ have been computed to very high accuracy, but their nature is unknown. Their approximate values are given by

$$\begin{aligned} \alpha &= 2.502907875095892822283902873218\ldots, \\ \delta &= 4.669201609102990671853203820466\ldots. \end{aligned}$$

The decimal expansion of the constants α and δ can be found on the *The On-Line Encyclopedia of Integer Sequences* as the sequences A006891 and A006890 respectively.

References

Bailey, David H, Jonathan M Borwein, Vishaal Kapoor, and Eric W Weisstein. 2006. "Ten Problems in Experimental Mathematics." *American Mathematical Monthly* 113 (6). Mathematical Association of America: 481–509.

Feigenbaum, Mitchell J. 1979. "The Universal Metric Properties of Nonlinear Transformations." *Journal of Statistical Physics* 21 (6). Springer: 669–706.