HASKELL SPACEFLIGHT WORKSHOP

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Introduction

This workshop consists of these published notes and a set of problems contained in the associated GitHub Haskell project.

How much of the notes it will be necessary to read depends on the background of an individual participant. Those who are already familiar with numerical methods might be able to skip most of the notes and simply work on the problems directly. However, we have tried to make the notes fairly comprehensive for newcomers, so that people unfamiliar with numerical methods at least have a starting point.

2

ODE Integration and Initial Value Problems

The models we use for a spacecraft depend upon a set of variables that represent its state at an instant in time. These state variables typically include:

- Position
- Velocity
- Mass

They may be scalar quantities or vectors, as appropriate to the problem.

Our simulations are all examples of "Initial Value Problems". In an initial value problem, we know the starting state of the spacecraft, and we have a set of first-order ordinary differential equations (ODEs), which describe how its state evolves with time. We will integrate these ODEs to predict the state at future times. Using this approach, we can compute the time history of state variables that are critial to mission or maneuver planning. For example, we might find the trajectory of a spacecraft (its position as a function of time), and check whether it places the spacecraft in a desired orbit.

The motion of a spacecraft depends on multiple forces that might be acting on it. For example:

- Gravity
- Atmospheric drag
- Rocket thrust

Thrust from a rocket engine may be controlled, and control inputs can be modeled easily in our system. Testing the behavior of a control system, particularly under conditions of real-world variations, is a modern practical use of the methods we cover (eg. [1, 2]).

2.1 1D Euler's Method

We can write a set of coupled, first-order ODEs as:

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = f(t, \mathbf{x}) \tag{2.1}$$

Here, $\mathbf{x}$ is the state vector, t is time, and f is some function. In Euler's method, we approximate a step forward in time by adding the product of the gradient, $\dot{\mathbf{x}}$, and the time step, h, to the current state, $\mathbf{x}$:

$$\mathbf{x}(t+h) \approx \mathbf{x}(t) + \dot{\mathbf{x}}\, h \tag{2.2}$$

$$\approx \mathbf{x}(t) + f\left(t, \mathbf{x}(t)\right)\, h \tag{2.3}$$

2.1.1 Radioactive Decay

We will begin implementing Euler's method with a 1D state, specialized to `Double`, using the process of radioactive decay as an example. Radioactive decay has an analytical solution, thus providing a ground truth against which the numerical result can be compared. It only involves a single state variable, N, which can be represented as a `Double`. Specializing to `Double` gives us a simple starting point that is close to the approach used in many other programming languages.

In radioactive decay, the rate of decay, $\dot{N}$, is proportional to the number of moles of radioactive particles that remain at any instant in time, N:

$$\dot{N} = -\lambda N \tag{2.4}$$

where λ is called the decay constant. This equation can be solved by knowing in advance that an exponential function happens to fit exactly the expected equation:

$$N = N_0 \exp\left(-\lambda t\right) \tag{2.5}$$

So that:

$$\dot{N} = -\lambda \left(N_0 \exp\left(-\lambda t\right)\right) \tag{2.6}$$

$$= -\lambda N \tag{2.7}$$

as required. Conventionally, λ is specified in terms of the half-life of an isotope, $t_{(1/2)}$:

$$\text{at } t = 0,\ N = N_0 \tag{2.8}$$

$$\text{at } t = t_{(1/2)},\ N = \frac{N_0}{2} \tag{2.9}$$

thus:

$$\frac{N_0}{2} = N_0 \exp\left(-\lambda t_{(1/2)}\right) \tag{2.10}$$

$$\ln\left(\tfrac{1}{2}\right) = -\lambda t_{(1/2)} \tag{2.11}$$

$$\lambda = \frac{\ln 2}{t_{(1/2)}} \tag{2.12}$$

As an example, consider the isotope Plutonium-238 (^{238}Pu), which has been used in radioisotope thermoelectric generators (RTGs) for spacecraft such as the Voyager 1 and 2 probes. This isotope has a half-life of approximately 87.7 years.

Problem 1: Euler integration specialized to `Double`.

In the file `ODE.hs`,

- implement `eulerStepDouble`, which takes a single step of Euler integration
- implement `integrateEulerDouble`, which takes multiple steps

In `ODEExamples.hs`,

- run `plotEulerDoubleExpDecay Screen`, to view a plot of Euler integration applied to the radioactive decay example

Figure 2.1 shows the result of applying Euler integration to the radioactive decay example. In this figure, it is evident that when smaller time-steps are taken, the Euler method more closely approximates the analytical solution. This is usually the case practically with numerical integration, although there is a limit beyond which smaller time steps will begin to diverge from the correct solution due to accruing floating-point errors. We will see later that raising the polynomial order of the integration approximation can improve accuracy with greater computational efficiency than taking smaller time steps.

2.2 Euler's Method for an `AffineSpace` State

We will now generalize Euler's method using the abstractions available in the `vector-space` package. The necessary constraints are captured in Listing 1. Don't panic if this seems a lot to take in, since we'll see a few concrete examples.

The first concept we introduce is the difference between an `AffineSpace` and its associated `VectorSpace`. In the present context, points in the `AffineSpace` are points belonging to the state space of the problem, of type `state` (eg. position, velocity, etc). Vectors of the associated `VectorSpace`, of type `diff`, represent deltas or differences between the points (eg. an offset of position, a delta in velocity, etc). We can add a vector to a point to obtain a new point, but we don't sum points directly. Similarly, we can multiply a vector by a scalar, but we cannot multiply a point by a scalar. Most widely-used frameworks for numerical integration do not make this distinction.

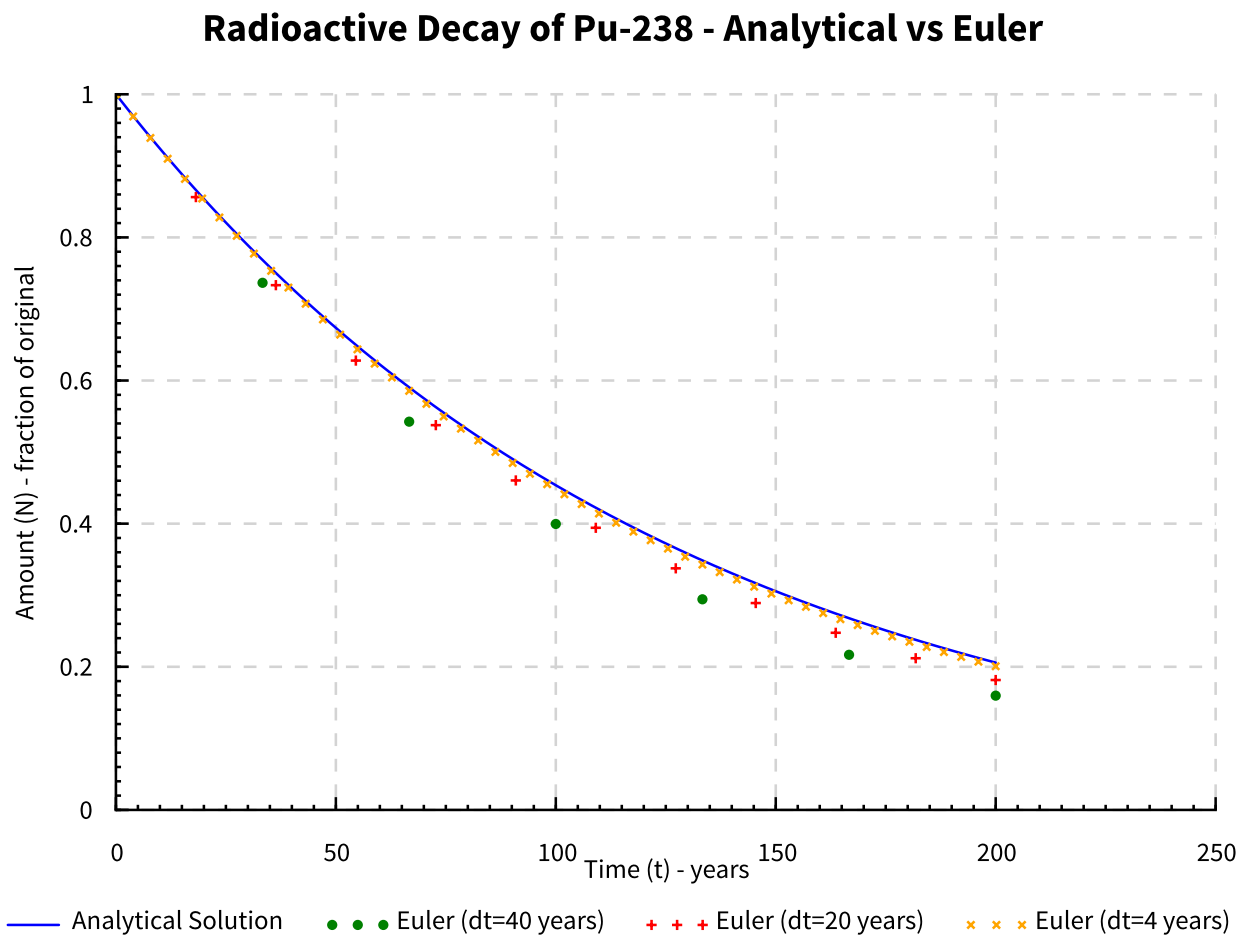

Figure 2.1: Comparison of Euler integration with the analytical result for radioactive decay of the isotope ^{238}Pu. The solid line shows the analytical solution while the points demonstrate the Euler approximation for different time steps.

```
eulerStep
  :: ( AffineSpace state
     , diff ~ Diff state, VectorSpace diff
     , HasBasis time, HasTrie (Basis time)
     , s ~ Scalar diff, s ~ Scalar time )
  => time                                -- ^ Step size @dt@
  -> ((time, state) -> time :-* diff)    -- ^ Gradient function @f (x, t)@
  -> (time, state)                       -- ^ Before the step @(t, x)@
  -> (time, state)                       -- ^ After the step @(t, x)@
```

Listing 1: Constraints for Euler's method generalized by `vector-space`.

Next is the concept of a linear map representing the derivative: `time :-* diff`. In our case, where we deal with finite differences, this constructor can be considered analogous to a (linear) function of type `time -> diff`. What this function represents is the delta, `diff`, which arises from taking a time-step, h, of type `time`. A very simple illustration of the linear map is shown in Listing 2, with a version for vectors shown in Listing 3. The `linear` function assumes that the function it has been provided is linear, and it memoizes the values of that function along each basis vector of the vector space.

```
> :set -XFlexibleContexts
> import Data.LinearMap ((:-*), linear, lapply)
> f = (*) 5 :: Double -> Double
> lm = linear f
> :t lm
lm :: Double :-* Double
> :force lm
lm = Data.LinearMap.LMap (Just 5.0)
> lapply lm 1.0
5.0
> lapply lm 2.0
10.0
```

Listing 2: A scalar linear map. Once the map has been defined (by the `linear` function), the `lapply` function multiplies the input vector (a `Double`) by the memoized value along the unit basis vector.

```
> :set -XFlexibleContexts
> import Data.LinearMap ((:-*), linear, lapply)
> :{
| f :: (Double, Double) -> (Double, Double, Double)
| f (x, y) = (2*x + y, 3*x - y, 4*y)
| :}
> lm = linear f
> :force lm
lm = Data.LinearMap.LMap
     (Just
       (Data.MemoTrie.EitherTrie ((,,) 2.0 3.0 0.0) ((,,) 1.0 -1.0 4.0)))
> lapply lm (5, 6)
(16.0,9.0,24.0)
```

Listing 3: A vector linear map, using tuples for vectors. This provides a better view of the memoisation that is occurring under the hood. The construction of a matrix-like representation (but with automatic dimension checking) is evident.

Finally, we need to describe the operations that can be used on instances of `AffineSpace` and its associated `VectorSpace`:

`lapply m h` applies linear map `m` to vector `h`

`p .+^ v` adds vector `v` to point `p`
`a ^+^ b` adds vector `a` to vector `b`

These operations are sufficient to implement the generalized form of Euler's method.

2.2.1 Simple Harmonic Motion

To motivate the generalized form of Euler's method, let's consider simple harmonic motion (SHM), with two scalar state variables:

- Position, r
- Velocity, v

The data type `State` in `ODEExamples.hs` describes this state, which is the `AffineSpace` of the problem. It introduces statically type-checked units from the `units` package for length and velocity. The data type `DState` is the correspondng `VectorSpace` of the problem, representing deltas in the state.

In SHM, a linear spring force, F, is proportional to the position, r, with a spring constant, k:

$$F = -kr \tag{2.13}$$

This is combined with the equations of motion of a point mass to produce the following governing ODE:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -kr/m \end{bmatrix} \tag{2.14}$$

where v is the velocity and m is the mass. If the initial conditions of the problem at $t = 0$ are $r = r_0$ and $v = 0$ then the analytical solution is:

$$\mathbf{x}(t) = \begin{bmatrix} r_0 \cos(\omega t) \\ -\omega r_0 \sin(\omega t) \end{bmatrix} \tag{2.15}$$

in which ω is the angular velocity, given by:

$$\omega = \sqrt{\frac{k}{m}} \tag{2.16}$$

This analytical solution can be differentiated twice manually to confirm that it satisfies the governing ODE, and values substituted to confirm that it satisfies the initial conditions.

Problem 2: Generalized Euler integration.

In the file `ODE.hs`,

- implement `eulerStep`
- implement `integrate` and `integrateWithDiff`

In `ODEExamples.hs`,

- run `plotEulerSHM Screen`, to view a plot of the position state variable, computed by Euler integration of SHM equations

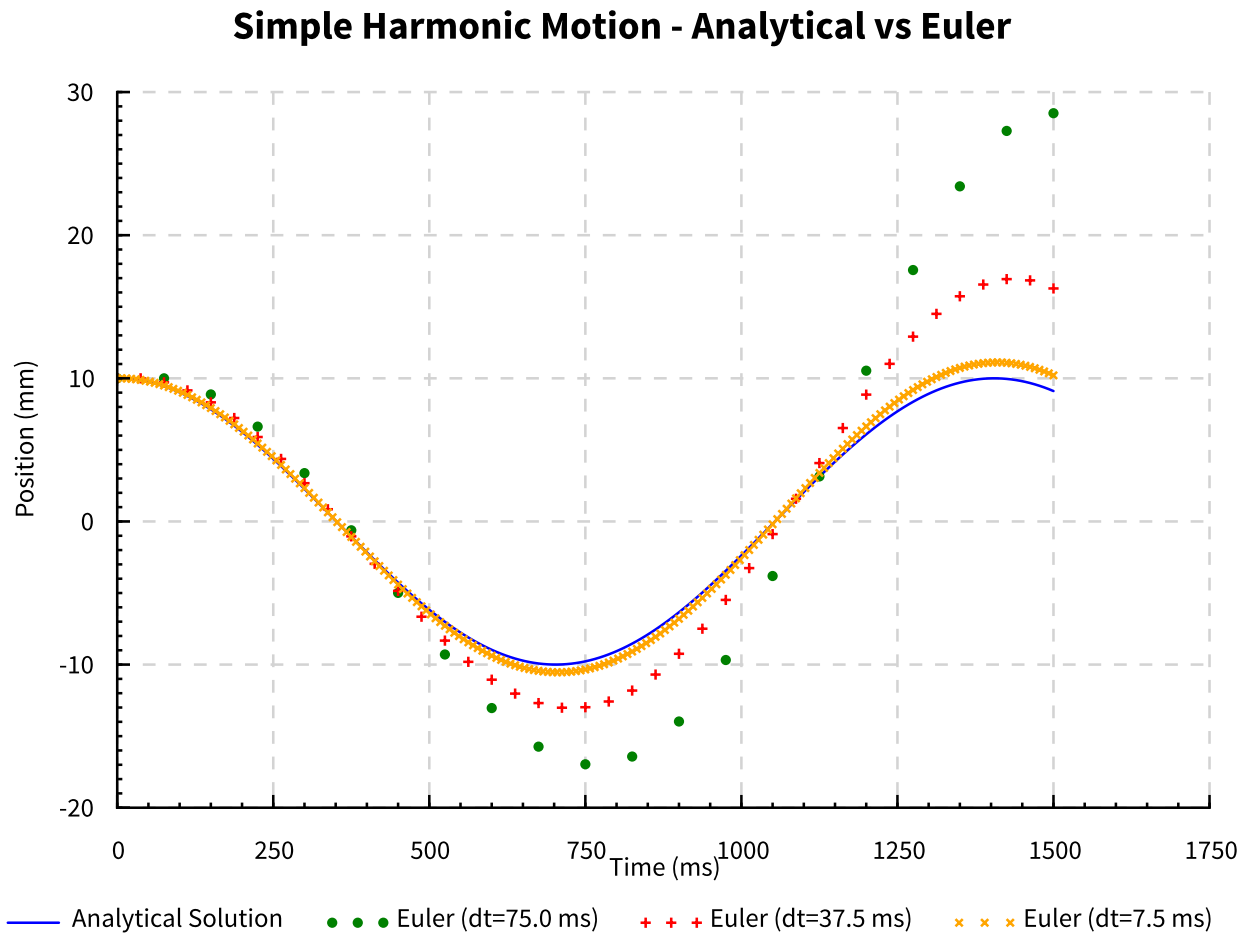

Figure 2.2: Comparison of Euler integration with the analytical result for the position variable of a simple harmonic oscillator. The solid line shows the analytical solution while the points demonstrate the Euler approximations for different time steps.

Figure 2.2 shows the position variable of a simple harmonic oscillator example. Once again, the smaller the time step, the more closely the result tracks the analytical solution.

2.3 4th-Order Runge-Kutta Integration

We have used Euler integration so far because it introduced the concepts we rely on from `vector-space`. However, there is a more common default for numerical integration of ODEs: the 4th-Order Runge-Kutta method (RK4). RK4 is the default in Matlab and SciPy (in those packages it is typically used with an embedded 5th-order approximation for step-size control; something we won't implement here). RK4 is usually a much better choice than the Euler method in terms of the accuracy/performance tradeoff.

RK4 is definitely not the final word though! The popular Numerical Recipes textbook recommends an 8th-order method (Dopr853) for general production use in non-stiff systems [3]. Specialized integrators may also be used for particular problems. The Apollo Guidance Computer used Nyström's Method to perform integration for efficiency and because of the dominant effect of a single, central gravitational force in most situations [4].[1] Long-duration astrodynamics problems, such as those concerning solar-system formation, or the behaviour of orbits over thousands of years, may have to use symplectic integrators to achieve reasonable accuracy (eg. [5]). We don't investigate these methods here because of time limitations, and because RK4 is both easy to implement and entirely sufficient for the examples.

We will only supply the equations for RK4 here and refer readers elsewhere (eg. [3, 6]) for a complete derivation:

$$\mathbf{k_1} = h\, f(t, \mathbf{x}) \tag{2.17}$$
$$\mathbf{k_2} = h\, f(t + \tfrac{1}{2}h, \mathbf{x} + \tfrac{1}{2}\mathbf{k_1}) \tag{2.18}$$
$$\mathbf{k_3} = h\, f(t + \tfrac{1}{2}h, \mathbf{x} + \tfrac{1}{2}\mathbf{k_2}) \tag{2.19}$$
$$\mathbf{k_4} = h\, f(t + h, \mathbf{x} + \mathbf{k_3}) \tag{2.20}$$
$$\mathbf{x}(t + h) \approx \mathbf{x} + \tfrac{1}{6}\mathbf{k_1} + \tfrac{1}{3}\mathbf{k_2} + \tfrac{1}{3}\mathbf{k_3} + \tfrac{1}{6}\mathbf{k_4} \tag{2.21}$$

The vectors $\mathbf{k_1} \ldots \mathbf{k_4}$ can be treated as stages of the computation, and are good candidates for let-floating. However, be aware that these equations do not directly represent the Haskell code. Instead, when implementing them, some care must be taken to consider what components are computed by `lapply`, what operations are adding vectors, and what operations are offsetting a point by a vector. Determining these are left as part of the exercise.

[1] The Draper Lab Apollo documents do refer to RK4 though, as "The usual fourth-order Runge-Kutta integration". Apparently it has a long history as the "go to" approach!

Problem 3: Runge-Kutta Integration.

In the file `ODE.hs`,

- implement `rk4Step`

In `ODEExamples.hs`,

- run `plotSHMComparison Screen`, to view a plot of Euler vs RK4 for the SHM example, using the same number of function evaluations

Figure 2.3 shows the comparison of Euler's method and RK4, for the same number of function evaluations. It is clear that RK4 much more closely approximates the analytical result.

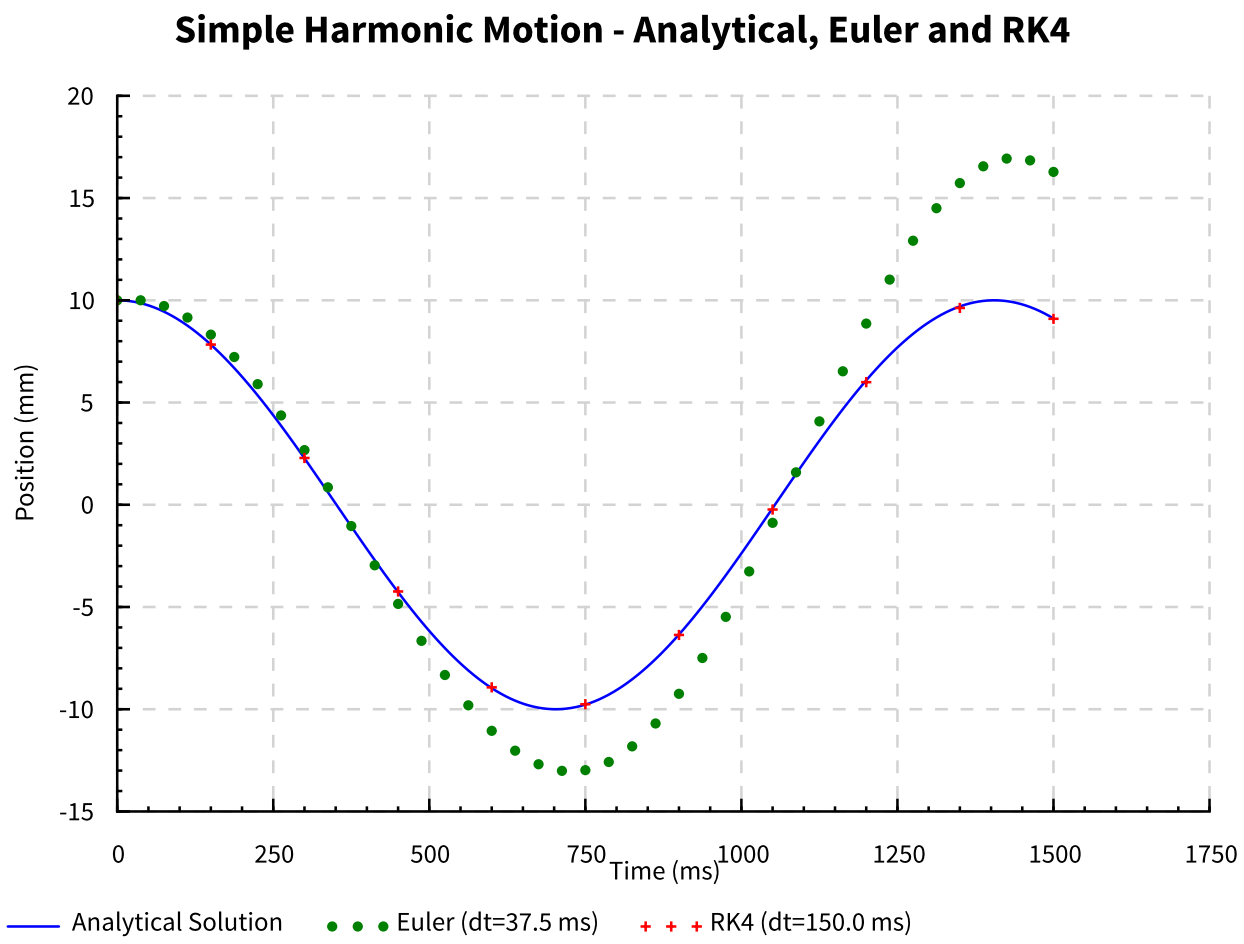

Figure 2.3: Comparison of Euler and RK4 integrations, for the same number of function evaluations, with the analytical result of a simple harmonic oscillator.

2.4 Simulating Apollo Lunar Ascent

Now that we have a working integrator, it's possible to look at an example of a somewhat realistic simulation: the algorithm used for Lunar Ascent Guidance during the Apollo missions.

This algorithm is described in summary by a NASA technical report [7] and in much greater detail by an MIT Draper Lab document [4], which contains sufficient detail to implement the algorithm.

In our version, we have made some minor simplifications to suit this workshop, none of which are substantial changes to the core algorithm or guidance concept. The main changes and approximations are:

- We projected the problem into 2D, by removing the parameter specifying the distance from the Command Service Module (CSM) orbital plane, and removing the associated control parameters.
- We removed the thrust filter computations and associated pre-launch initializations, since we don't model the behaviour and noise characteristics of the inertial guidance unit. Instead, we used the nominal initial value for the thrust, based on stored values for the Ascent Propulsion System (APS) exhaust velocity and mass flow rate.
- We did not allow the Reaction Control System (RCS) rockets to substitute for the APS, and we did not consider modeling abort scenarios.
- The Average-G routine is substituted with a simpler version.
- Window Pointing Direction (WDP) is unused and not specified (it always points toward the center of the moon in the original version anyway).
- The Digital Autopilot (not technically part of the Ascent Guidance itself) is not modelled in full, and is instead substituted by a computed constant angular acceleration.
- Our simulation does not take into account additional thrust produced by +X firing of the RCS for attitude control. Due to the regular angular acceleration, and thus regular RCS burns during ascent, this may result in our slightly underestimating the net thrust.

The approach we use for simulation involves checking-in with our version of the Apollo Guidance Computer (AGC) every 2 s of simulation time, matching the polling that was used in the original AGC. The ascent guidance returns a commanded thrust angle and an optional engine shutoff time. We take the commanded thrust angle and compute an angular acceleration that will allow the ascent stage to point toward the commanded angle at the end of the 2 s period (approximating the behaviour of the original Digital Autopilot, which achieved the same thing by firing the RCS rockets). We then integrate the equations of motion forward for 2 s. This swapping between calls to the AGC and forward integration for 2 s continues until we reach the commanded engine shutoff time, at which point we stop the engines. Following the burn phase, we take the final state and integrate the equations of motion for a further 10 000 s, to numerically compute the coasting trajectory.

A unique feature of this simulation (we believe) is that it incorporates statically-checked units, from the `units` package. Whether or not that was a worthwhile exercise is debatable

(the overhead and extra code complexity is somewhat overwhelming), but we do have static confirmation that almost all aspects of the algorithm use consistent units.

The ascent guidance itself has the following parameters as a target (see the `AscentTarget` type in `LunarAscent/Types.hs`):

- target velocity
- target radius

The desired insertion orbit of the ascent stage was computed in advance by Mission Control. That orbit determined a final radius and velocity that should be reached at the end of the burn (there is a pair of radius and velocity corresponding to each unique orbit). In our simulation, we did not perform any of those calculations, but instead used the target parameters for a "quick, early takeoff", which were programmed by default into the AGC, and would typically be overridden by the astronauts under nominal conditions [4]. The phase of the orbit relative to the CSM would be set approximately, by knowing the nominal ascent duration and timing the takeoff accordingly [4]. Achieving a suitable orbital phase was done to reduce the fuel used during later rendezvous with the CSM. Rendezvous with the CSM was an entirely different manouvre performed after the ascent, and not part of the ascent guidance itself [4].

The inner workings of the guidance algorithm are somewhat outside the scope of this workshop, and in fact are not derived in the Draper Lab document [4]. However, we can roughly summarise the approach based upon careful reading of the algorithm and reference to other texts, by core members of the Draper Lab, which deal with similar control of powered ascent [6]. Given the ascent target parameters, the guidance begins each polling loop by computing a velocity-to-be-gained. This is the difference between the target velocity and the current velocity, corrected for any velocity losses during the burn from gravitational acceleration. The gravitational velocity losses were found by analytically integrating the acceleration due to gravity. That also required an approximation of the remaining burn time (time-to-go estimate), as the limit of the integration, which was obtained from a Taylor Series expansion of the well-known Tsiolkovsky Rocket Equation, using the velocity-to-be-gained as the delta-V. Finally, a slightly-modified version of "cross-product" steering was used to choose the commanded thrust direction, so that the velocity-to-be-gained would fall to zero at the end of the burn, thus matching the target velocity, and the target radius would also be achieved. This modified cross-product steering was implemented using a core "linear guidance concept" [4], combined with tweaks such as prioritising radial thrust and performing bang-bang directional control of radial acceleration if the target radius was reached early. This guidance algorithm seems quite likely to have been based on earlier ballistic missile guidance, such as that used for the Minuteman ICBMs, given the Draper Lab's prior involvement with those projects [8]. It seems likely that earlier practical tests of those missile systems lent some confidence. However, it is naturally extremely difficult to track down any details of even the decomissioned parts of large missile systems for final confirmation.[2]

[2] I tried. Hopefully I'm not on any (more) watch lists now. - J. Merritt.

Symbols

f	A function.
F	Total force, scalar (N).
h	Time step (s).
k	Spring constant (N/m).
m	Mass (kg).
N	Number of moles of a substance.
N_0	Number of moles of a substance at $t = 0$.
r	Position, scalar (m).
r_0	Position, scalar, at $t = 0$ (m).
t	Time (s).
$t_{(1/2)}$	Radioactive half life (s).
v	Velocity, scalar (m/s).
$\mathbf{x}$	System state vector.
λ	Radioactive decay constant (1/s).
ω	Angular frequency (1/s).

References

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