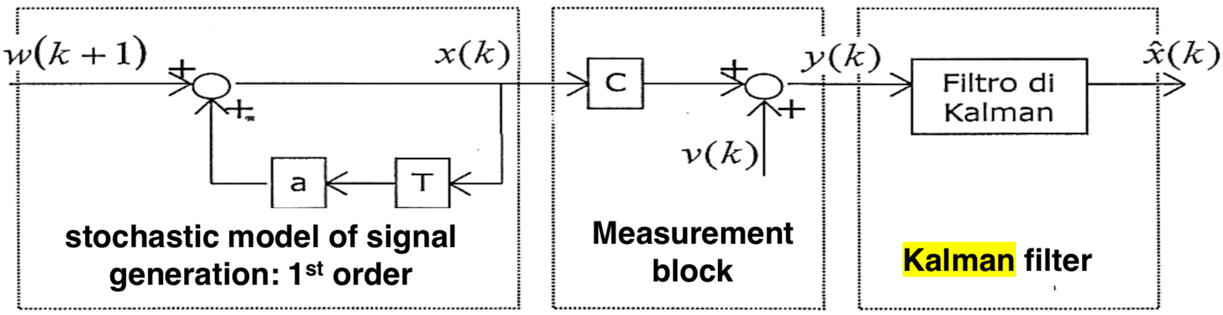
Kalman filter approach

We have three main blocks:

(Non dovrebbe essere w(k-1)l'ingresso qui sotto?)



- ***Signal Generation Model***

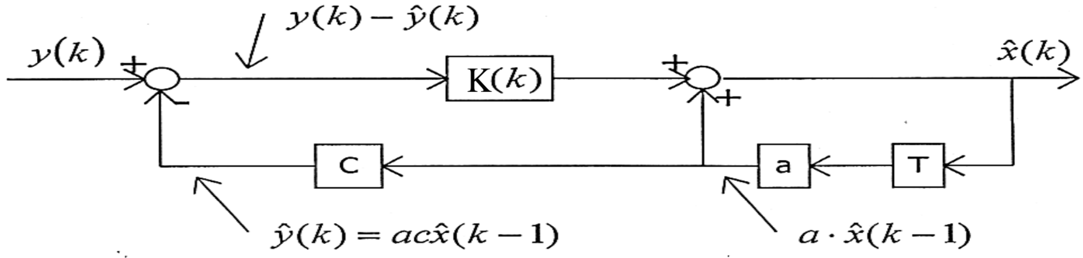
  It's a *stochastic* model (this means that the model is fed by a *white noise*) which generates the *signal* in which I'm interested, for this first block we can consider the $ARMA$ models family.

- ***Measure Block***

  In this block C is the *transformation matrix* used to map the *state vector parameters* (i.e. $\mathbf{x(k)}$) into the *measurement* domain. We then suppose that this new *measurement* vector is affected by a *white noise* $v(k)$.

- ***Kalman Filter*** :

  The signal $\mathbf{y(k)}$ then enters the *Kalman Filter* which can be represented schematically in the following way:

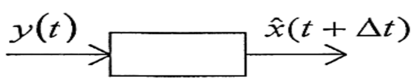


Note: both Wiener and Kalman:

$\Delta t = 0 \quad \Rightarrow \quad$ filtering

$\Delta t > 0 \quad \Rightarrow \quad$ predictor

$\Delta t < 0 \quad \Rightarrow \quad$ interpolator



In summary, we have the following equations:

Processes

$$
\begin{aligned}
\mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) + \mathbf{W}(k-1) \\
\mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k)
\end{aligned} \tag{1}
$$

Estimators

$$
\hat{\mathbf{x}}(k) = \mathbf{A}\hat{\mathbf{x}}(k-1) + \mathbf{K}(k)\underbrace{(\mathbf{y}(k) - \mathbf{C}\mathbf{A}\hat{\mathbf{x}}(k-1))}_{\text{Innovation!}} \tag{2}
$$

It's important to observe and comment the equation above in the following way:

The new estimate $\hat{\mathbf{x}}(k)$ is equal to the old estimate $\hat{\mathbf{x}}(k-1)$ multiplied by the state transition matrix $\mathbf{A}$ plus a weighting factor $\mathbf{K}(k)$ (Kalman Gain) times the innovation introduced by a measurement $\mathbf{y}(k)$. We can see that in the innovation expression the matrix $\mathbf{C}$ just represents the transformation matrix used to map the state vector parameters into the measurement domain (Of course you have to compare $\mathbf{y}(k)$ and $\mathbf{A}\hat{\mathbf{x}}(k-1)$ in the same domain!)

If you see how the Kalman Gain is calculated (see equations below) it's trivial to understand that in corrispondence of high uncertainty in the measurement ($\mathbf{R}(k)$ is high) the Kalman Gain $\mathbf{K}(k)$ becomes tiny $\rightarrow$ low trust in the measurement!

We define:

$$
\mathbf{P_1}(k) = \mathbf{A}\mathbf{P}(k-1)\mathbf{A}^T + \mathbf{Q}(k) \tag{3}
$$

$$
\mathbf{P}(k) = \mathbf{P_1}(k) - \mathbf{K}(k)\mathbf{C}(k)\mathbf{P_1}(k) \tag{4}
$$

$$
\mathbf{K}(k) = \mathbf{P_1}(k)\mathbf{C}^T\left(\mathbf{C}\mathbf{P_1}(k)\mathbf{C}^T + \mathbf{R}(k)\right)^{-1} \tag{5}
$$

Where

$$
\begin{aligned}
\mathbf{R}(k) &= \text{Measurement Noise} = \mathbb{E}\left[\mathbf{v}(k)\mathbf{v}^T(k)\right] \\
\mathbf{Q}(k) &= \text{Process Noise} = \mathbb{E}\left[\mathbf{W}(k)\mathbf{W}^T(k)\right] \\
\mathbf{P}(k) &= \text{Error Covariance Matrix} = \mathbb{E}\left[\left(\mathbf{x}(k) - \hat{\mathbf{x}}(k)\right)\left(\mathbf{x}(k) - \hat{\mathbf{x}}(k)\right)^T\right]
\end{aligned} \tag{6}
$$

How do we practically implement the *Kalman* filter?

- First, we have to define the physical model of the signal we are studying, this means that we have to specify the $\mathbf{A}$ and $\mathbf{C}$ matrices.
- Then we have to obtain the statistics of the stochastic processes involved, i.e. the matrices $\mathbf{Q}$ and $\mathbf{R}$.
- Then we have to initialize the state vector $\hat{\mathbf{x}}(0)$.
- Finally we have to test the efficiency of our model, we do that by testing the whiteness of the prediction error (if the prediction error is a *white noise* then we can claim that our model has extracted all the possible information from the signal we're observing) , to do so we use the Anderson test which checks the sample covariance function estimate or the sample spectrum estimate.

  A *white noise* is characterized by the following *sample covariance* and *spectrum*: