

LECTURE NOTES OF

MODEL IDENTIFICATION AND DATA ANALYSIS

PART 2

From Professor Sergio Savaresi's lectures

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Lecture Notes of Model Identification and Data Analysis - Part 2

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The L^AT_EX source code is available at

<https://github.com/andreabosisio/mida2>

The original repository is available at

<https://github.com/polimi-cheatsheet/MIDA2>

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Compiled with ♡

Please notify errors or changes through an email or a pull request.

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Part I

MIDA 2

Chapter 0

Introduction

0.1 General topics of MIDA course

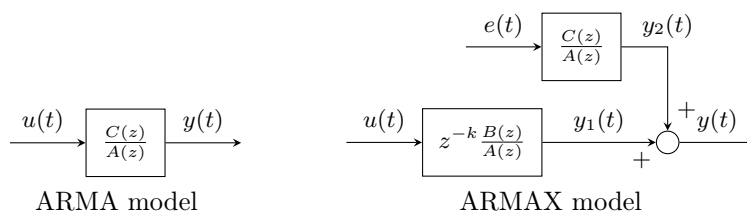
- Collect digitally data from real systems
- Build Black-Box (BB) (or Gray-Box (GB)) models from data, with emphasis on
 - Dynamic systems
 - Control/automation-oriented applications
- Purpose of modelling (area of machine learning focusing on “control”)
 - Prediction
 - Software-sensing
 - Modelling for control design

0.1.1 Super summary of MIDA1

The focus is on *Time Series* (output-only systems) and *input/output* (I/O) systems.

Models used in MIDA1:

- Auto Regressive Moving Average (ARMA) models for time series
- Auto Regressive Moving Average with Exogenous Input (ARMAX) models for I/O systems



We've used a *Black-Box model identification* of the model, indicated as $\mathcal{M}(\vartheta)$ where ϑ is the

parameter vector, which contains the coefficients of $A(z)$, $B(z)$, $C(z)$.

In particular we've seen the **Prediction Error Minimization (PEM)** method, a *parametric approach* based on the minimization of the *performance index* defined as:

DEFINITION 0.1 — PEM cost function. $J_N(\vartheta) = \frac{1}{N} \sum_{t=1}^N (y(t) - \hat{y}(t|t-1, \vartheta))^2$

which is the variance of the *prediction error* made by the model. The optimal ϑ is $\hat{\vartheta}_N = \arg \min_{\vartheta} J_N(\vartheta)$

0.1.2 MIDA 2

The focus is on *Dynamical System* with a special eye on control (or feedback) application (more close to real applications than time series). We can divide the course in three main blocks: *Advanced Model Identification* methods, *Software-Sensing (SW)* and *Minimum Variance Control*.

In particular:

Chapter 1 non-parametric (direct/constructive) Black-Box (BB) identification of I/O systems using State-Space models

Chapter 2 parametric identification for BB I/O systems, with a frequency-domain approach

Chapter 3 Kalman-filter for SW-sensing using feedback on White-Box (WB) models

Chapter 4 BB methods for SW-sensing without feedback

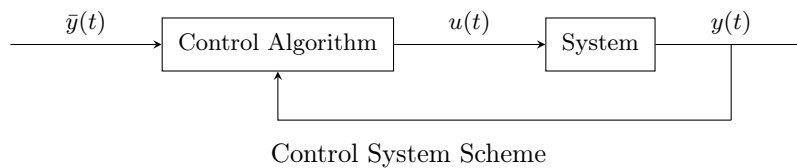
Chapter 5 GB system identification using Kalman-filter and using *simulation-error methods* (S.E.M.)

Chapter 6 Minimum-Variance Control (M.V.C.), design of optimal feedback controllers using the theory background of the MIDA course

Appendix Recursive (online) implementation of algorithms for system identification

0.2 Motivation example for the course: ABS (Anti-Lock Braking System)

ABS is an example of a control system since it follows this general scheme:

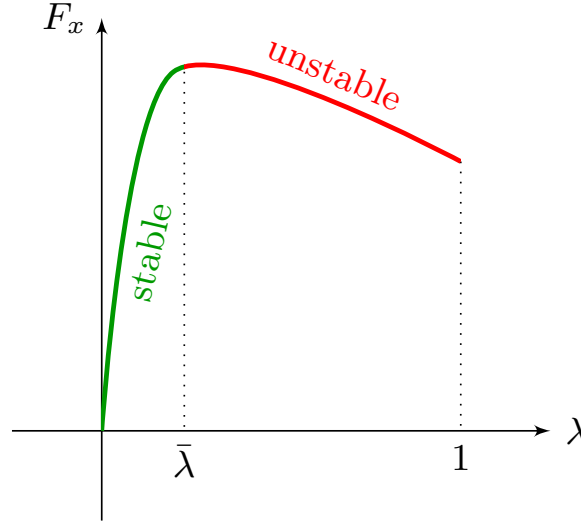


where $\bar{y}(t)$ is the reference value of $y(t)$.

We define the *slip* of the wheel as $\lambda(t) = \frac{v(t) - \omega(t)r}{v(t)}$, where $v(t)$ is the horizontal velocity of the car, $\omega(t)$ is the angular velocity of the wheel and r is the radius of the wheel.

During a brake $0 \leq \lambda(t) \leq 1$ (from free rolling wheel (i.e. $v(t) = \omega(t)r$) to locked wheel (i.e. $\omega(t) = 0$)).

The curve of F_x , the braking force, is as showed below:



Relation between λ and the braking force.

In the case of ABS, $u(t) = x(t)$ which is the voltage of the electric braking motor (control variable), $y(t) = \lambda(t)$ (controlled variable) and $\bar{y}(t) = \bar{\lambda}(t)$, which is the maximum braking point that we want to reach during an emergency brake.

The problem can be divided into subproblems:

- Model of the system
- SW-estimation of λ since v is **not** directly measurable, so λ cannot be computed
- Design of the ABS control algorithm

Because of that measurement problem we have to build a Black-Box model from data.

Why Black-Box modelling? The control variable x (the voltage to the actuator) controls a complex systems from the actuator to λ . The system can be seen as a chain of components:

- Current dynamics and electric motor
- Position dynamics of the actuator
- Dynamics of the hydraulic circuit of the braking system
- Tire dynamics
- Wheel rotational dynamics

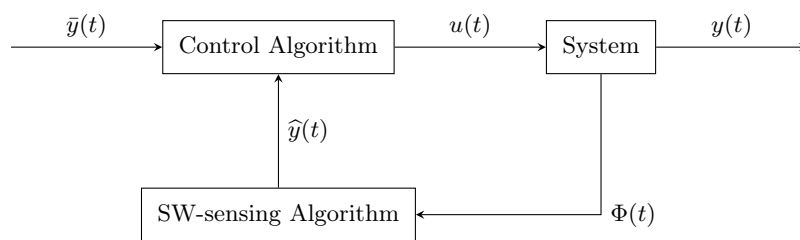
- Vehicle full dynamics

It's simply deducible that such system is really difficult to model.

RECALL — White-Box (WB) vs. Black-Box (BB).

- WB modelling: write the physical equations from *first principles*.
- BB modelling: experiment \rightarrow collect data \rightarrow build model. Using only I/O measured data we can *learn* a mathematical model of the I/O behavior of the system.

In order to estimate the variable $v(t)$ (and so also $\lambda(t)$) we use an SW-sensing algorithm which takes as inputs other measurable variables (e.g. angular velocities of all the wheels). Thus, the scheme of a general control system become something like this:

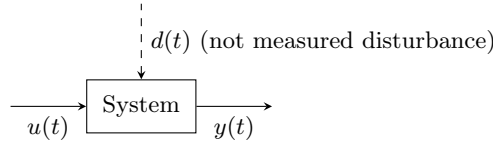


Control System Scheme with SW-sensing

where $\Phi(t)$ are the available (measurable) variables of the system and $\hat{y}(t)$ is the estimation of the SW-sensing algorithm of $y(t)$

Chapter 1

Black-Box non-parametric identification of I/O systems using State-Space models in time domain



RECALL — General path of a Parametric Identification Methods.

1. Collect data: inputs: $\{u(1), u(2), \dots, u(N)\}$, outputs: $\{y(1), y(2), \dots, y(N)\}$
2. Select **a-priori** a class/family of **parametric models**: $\mathcal{M}(\vartheta)$
3. Select **a-priori** a performance index $J(\vartheta)$ (which gives an order to the quality of the models)
4. Optimization step (minimize $J(\vartheta)$ w.r.t ϑ): $\hat{\vartheta}_N = \arg \min_{\vartheta} J(\vartheta) \rightarrow$ optimal model $\mathcal{M}(\hat{\vartheta}_N)$ characterized by the **optimal parameters** $\hat{\vartheta}_N$

Note: $J(\vartheta) : \mathbb{R}^{n_{\vartheta}} \rightarrow \mathbb{R}^+$ (where n_{ϑ} is the order of the model).

Note: $\mathcal{M}(\vartheta_1)$ can be sorted, that is $\mathcal{M}(\vartheta)$ is better than $\mathcal{M}(\vartheta_2)$ if $J(\vartheta_1) < J(\vartheta_2)$.

In this chapter we are presenting a totally different system identification approach, the **non-parametric** one, which means:

- No a-priori model-class selection
- No performance index definition
- No optimization task

Before entering in the system identification algorithm we need to recall the three main mathematical representation of **Discrete-time Dynamic Linear Systems** which are characterized by their internal variables called *states* and indicated with $x(t)$.

1.1 Representations

1.1.1 Representation #1: State-Space (SS)

$$\mathcal{S} : \begin{cases} x(t+1) = Fx(t) + Gu(t) & \text{state equations} \\ y(t+1) = Hx(t) + Du(t) & \text{output equations} \end{cases}$$

where F , G , H and D are matrices defined as follows:

$$F = \begin{bmatrix} n \times n \\ \text{state matrix} \end{bmatrix} \quad G = \begin{bmatrix} n \times 1 \\ \text{input} \\ \text{matrix} \end{bmatrix}$$

$$H = [1 \times n \quad \text{output matrix}] \quad D = [1 \times 1 \quad \text{i/o matrix}]$$

In this case we have 1 input and 1 output (i.e. *SISO* system), but those *difference equations* can be extended for multiple inputs and outputs systems. Usually $D = 0$ since we can say that the majority of real systems have this property.

DEFINITION 1.1 — Strictly-proper system. A system is called *strictly-proper* when the output of the system doesn't directly depend on the input (i.e. $D = 0$).

OBSERVATION. n is the *order* of the system.

EXAMPLE — SISO system of order $n = 2$.

$$\mathcal{S} : \begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + 2u(t) \\ x_2(t+1) = x_1(t) + 2x_2(t) + u(t) \\ y(t) = \frac{1}{4}x_1(t) + \frac{1}{2}x_2(t) \end{cases}$$

In this case $n = 2$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, one input $u(t)$ and one output $y(t)$.

$$F = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ H = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad D = 0$$

REMARK — State-Space representation is not unique. $F_1 = TFT^{-1}$, $G_1 = TG$, $H_1 = HT^{-1}$, $D_1 = D$ for any invertible $(n \times n)$ matrix T . The system $\{F, G, H, D\}$ is equivalent to $\{F_1, G_1, H_1, D_1\}$.

1.1.2 Representation #2: Transfer Function (TF)

$$W(z) = \frac{B(z)}{A(z)} z^{-k} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_p z^{-p}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}} z^{-k}$$

$$\mathcal{S} : y(t) = W(z)u(t)$$

The Transfer Function (TF) $W(z)$ is a rational function of z : it's a *digital filter*.

RECALL — z **operator**. The linear z operator is such that $z^{-1}[x(t)] = x(t-1)$ and $z[x(t)] = x(t+1)$.

From now on the " $[\cdot]$ " will be **omitted** for simplicity.

EXAMPLE.

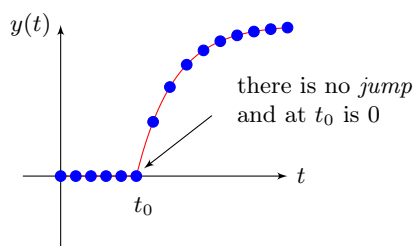
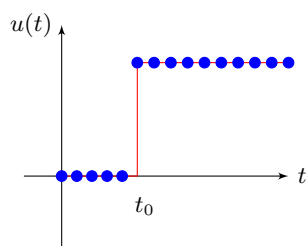
$$\begin{aligned} \mathcal{S} : y(t) &= \underbrace{\left[\frac{1 + \frac{1}{2}z^{-1}}{2 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2}} z^{-1} \right]}_{W(z)} u(t) \\ 2y(t) + \frac{1}{3}y(t-1) + \frac{1}{4}y(t-2) &= u(t-1) + \frac{1}{2}u(t-2) \\ y(t) &= \underbrace{-\frac{1}{6}y(t-1) - \frac{1}{8}y(t-2)}_{\text{recursive part}} + \underbrace{\frac{1}{2}u(t-1) + \frac{1}{4}u(t-2)}_{\text{past inputs}} \end{aligned}$$

REMARK — **IIR and FIR filters.**

$W(z) = \frac{z^{-1}}{1 + \frac{1}{3}z^{-1}}$ is an IIR (*Infinite Impulse Response*) filter since it has the recursive part because of the presence of *poles* in $W(z)$.

$W(z) = z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}$ is a FIR (*Finite Impulse Response*) filter since it depends only on a *finite* sequence of past inputs.

REMARK — **Strictly proper systems.** Notice that for strictly proper systems the delay of $W(z)$ is $k \geq 1$, or, equivalently, the order of the numerator of $W(z)$ is strictly smaller than the order of the denominator of $W(z)$.

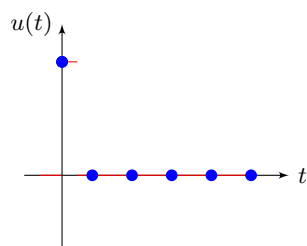


1.1.3 Representation #3: Convolution of the input with the Impulse Response (IR)

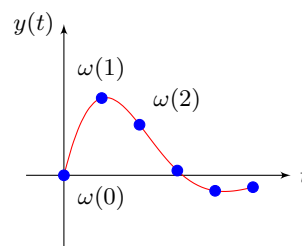
The third way to represent a system is through the *convolution* of the input with the *Impulse Response (IR)*.

DEFINITION 1.2 — Impulse Response. In the discrete time domain, the Impulse Response of a filter $W(z)$ is $y(t) = W(z)u(t)$ where the input $\omega(t)$ is an impulse (i.e. $u(t) = 0$ everywhere except from $t = 0$ where $u(t) = 1$).

$$\omega(t) = \{\omega(0), \omega(1), \omega(2), \dots\}$$



Impulse in input



Impulse Response (IR) in output

Note if the system is strictly proper then $\omega(0) = 0$.

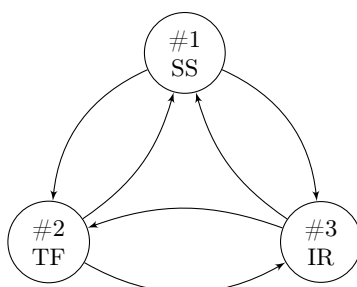
It can be proven that the input-output relationship from $u(t)$ to $y(t)$ can be written as

$$y(t) = \omega(0)u(t) + \omega(1)u(t-1) + \omega(2)u(t-2) + \dots = \sum_{k=0}^{\infty} \omega(k)u(t-k)$$

From this, it is clear that $y(t)$ is the *convolution* of the IR with the input signal.

1.2 Transformations between representations

It is possible to translate each representation into another one. Therefore, there are six possible transformations between the three representations.



Transformations between representations

1.2.1 State-Space to Transfer Function

Consider a strictly proper system with the following State-Space (SS) representation:

$$\mathcal{S} : \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Du(t) \end{cases} \Rightarrow \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

From the system we get

$$\begin{aligned} zx(t) &= Fx(t) + Gu(t) \Rightarrow x(t) = (zI - F)^{-1}Gu(t) \\ &\Rightarrow y(t) = H(zI - F)^{-1}G \cdot u(t) \end{aligned}$$

Thus, the Transfer Function is

$$\text{SS} \rightarrow \text{TF}: \quad W(z) = H(zI - F)^{-1}G \quad (1.1)$$

EXAMPLE. Consider the following SISO system of order $n = 2$:

$$F = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad H = [1 \quad 0] \quad D = 0$$

Using the transformation 1.1:

$$\begin{aligned} W(z) &= [1 \quad 0] \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1 \quad 0] \begin{bmatrix} z-1 & 0 \\ -\frac{1}{2} & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [1 \quad 0] \frac{1}{(z-1)(z-2)} \begin{bmatrix} z-2 & 0 \\ \frac{1}{2} & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(z-1)(z-2)} [z-2 \quad 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{\cancel{z-2}}{(z-1)\cancel{(z-2)}} = \frac{1}{z-1} = \frac{1}{1-z^{-1}} z^{-1} \end{aligned}$$

Notice that, due to **cancellation of singularities**, in this case we only have one pole, but the system is of order two; this comes from the fact that part of the system is *non observable*.

Alternatively, it could be noted that $\{F, G, H, D\}$ corresponds to the following system:

$$\mathcal{S} : \begin{cases} x_1(t+1) = x_1(t) + u(t) \\ x_2(t+1) = \frac{1}{2}x_1(t) + 2x_2(t) + u(t) \\ y(t) = x_1(t) \end{cases} \Rightarrow \begin{cases} zx_1(t) = x_1(t) + u(t) \\ zx_2(t) = \frac{1}{2}x_1(t) + 2x_2(t) + u(t) \\ y(t) = x_1(t) \end{cases}$$

From the first equation we have that $x_1(t) = \frac{1}{z-1}u(t)$ and substituting it to the third one we obtain the same result: $y(t) = x_1(t) = \frac{1}{z-1}u(t) \Rightarrow W(z) = \frac{1}{1-z^{-1}} z^{-1}$

1.2.2 Transfer Function to State-Space

This conversion is not very used in practice and it is called the *realization* of a Transfer Function into a State-Space system.

Issue: the State-Space representation is not unique! Thus from a single Transfer Function we can get infinite different equivalent State-Space models.

1.2.2.1 Control realization

We assume that the system is **strictly proper** and that the denominator is **monic** (i.e. $a_0 = 1$).

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

The formula for the control realization of $W(z)$ is

$$\text{TF} \rightarrow \text{SS:} \quad F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad H = [b_{n-1} \quad b_{n-2} \quad \cdots \quad b_0] \quad D = 0 \quad (1.2)$$

EXAMPLE. Consider the following Transfer Function:

$$W(z) = \frac{2z^2 + \frac{1}{2}z + \frac{1}{4}}{z^3 + \frac{1}{4}z^2 + \frac{1}{3}z + \frac{1}{5}}$$

Applying the transformation 1.2, the control realization is:

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{5} & -\frac{1}{3} & -\frac{1}{4} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad H = [\frac{1}{4} \quad \frac{1}{2} \quad 2] \quad D = 0$$

1.2.3 Transfer Function to Impulse Response

To get the IR from a Transfer Function $W(z)$ is sufficient to make the ∞ -long division between the numerator and denominator of $W(z)$.

$$\text{TF} \rightarrow \text{IR:} \quad \omega(t) = \begin{cases} e_t & \text{if } t \geq 1 \\ 0 & \text{if } t = 0 \text{ (and } W(z) \text{ is strictly-proper)} \end{cases} \quad (1.3)$$

where e_t are the coefficients of

$$E(z) = e_1 z^{-1} + e_2 z^{-2} + \cdots = \sum_{t=1}^{\infty} e_t z^{-t}$$

which is the *remainder* of the ∞ -long division.

EXAMPLE. Consider the following Transfer Function:

$$W(z) = \frac{1}{z - \frac{1}{2}} = \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}} \stackrel{\infty\text{-long div.}}{=} 0z^{-0} + 1z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3} + \cdots$$

Applying the transformation 1.3 the IR is $\omega(0) = 0$, $\omega(1) = 1$, $\omega(2) = \frac{1}{2}$, $\omega(3) = \frac{1}{4}$, \dots . Thus $\omega(0) = 0$ and $\omega(t) = \frac{1}{2^{t-1}} \quad \forall t \geq 1$

In this case there is also a quicker way

$$y(t) = \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}u(t) = \left(z^{-1} \frac{1}{1 - \frac{1}{2}z^{-1}}\right)u(t)$$

Remembering that for *geometric series* we have

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \text{ if } |a| < 1$$

We can rewrite $y(t)$ as follows

$$y(t) = \left(z^{-1} \sum_{k=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^k\right)u(t) = \left(0 + 1z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3} + \dots\right)u(t)$$

1.2.4 Impulse Response to Transfer Function

DEFINITION 1.3. Given a discrete-time signal $s(t)$ such that $\forall t < 0 : s(t) = 0$, its *Z-Transform* is defined as

$$\mathcal{Z}(s(t)) = \sum_{t=0}^{\infty} s(t)z^{-t}$$

Given this, it can be proven that

$$\text{TF} \rightarrow \text{IR}: \quad W(z) = \mathcal{Z}(\omega(t)) = \sum_{t=0}^{\infty} \omega(t)z^{-t} \quad (1.4)$$

This means that the **Transfer Function of a system** is the \mathcal{Z} -transform of a special signal, $\omega(t)$, that is the Impulse Response of the system.

REMARK. This formula cannot be used in practice to transform an IR representation to a TF representation. This is because we need infinite points of the Impulse Response, and it must be available *noise-free*. Thus, this transformation is only theoretical.

1.2.5 State-Space to Impulse Response

Consider the following State-Space model, with initial conditions $x(0) = 0$ and $y(0) = 0$

$$\mathcal{S} : \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

"Running the simulation of the system", we have that

$$\begin{aligned} x(1) &= Fx(0) + Gu(0) = Gu(0) \\ y(1) &= Hx(1) = HGu(0) \\ &\downarrow \\ x(2) &= Fx(1) + Gu(1) = FGu(0) + Gu(1) \\ y(2) &= Hx(2) = HFGu(0) + HGu(1) \end{aligned}$$

$$\begin{aligned}
& \Downarrow \\
& x(3) = Fx(2) + Gu(2) = F^2Gu(0) + FG u(1) + Gu(2) \\
& y(3) = Hx(3) = HF^2Gu(0) + HFG u(1) + HGu(2) \\
& \vdots
\end{aligned}$$

This can be generalized to

$$y(t) = 0u(t) + HGu(t-1) + HFGu(t-2) + HF^2Gu(t-3) + \dots$$

Recalling that

$$y(t) = \omega(0)u(t) + \omega(1)u(t-1) + \omega(2)u(t-2) + \omega(3)u(t-3) + \dots$$

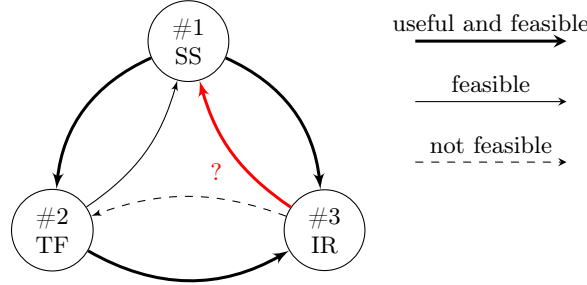
we deduce that $\omega(0) = 0$, $\omega(1) = HG$, $\omega(2) = HFG$, $\omega(3) = HF^2G$, ...

Thus, the Impulse Response is

$$\text{SS} \rightarrow \text{IR}: \quad \omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases} \quad (1.5)$$

1.2.6 Summary of transformations

Notice that the IR representation is very easy to obtain experimentally, since we only need to measure the system response to the impulse signal. However, given the IR representation, it is difficult to get to the other representations, since the transformation from IR to TF is only theoretical. Moving from the IR to the SS representation is the key task of the *Subspace-based State-Space System Identification*, also known as *4SID method*.



Transformations between representations in practice

Before moving to the topic of *4SID method*, a recall of the *observability* and *controllability* properties for a linear dynamic system is needed.

1.3 Fundamental concepts of Observability and Controllability

$$\mathcal{S}: \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) \end{cases}$$

DEFINITION 1.4 — Fully Observable. The system is fully observable (from the output) if and only if the **observability matrix** is full rank:

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \quad \text{rank}(O) = n$$

where n is the order of the system.

OBSERVATION — Observability. Observability is a property of the system: by observing the output $y(t)$ it's possible to observe the state $x(t)$.

Note: It refers only to state and output (i.e. F and H).

DEFINITION 1.5 — Fully Controllable. The system is fully controllable (from the input) if and only if the **controllability matrix** is full rank:

$$R = [G \quad FG \quad \dots \quad F^{n-1}G] \quad \text{rank}(R) = n$$

where, again, n is the order of the system.

R is also called *reachability* matrix.

OBSERVATION — Controllability. Controllability is a property of the system: by driving (or move) the input $u(t)$ it's possible to control the state $x(t)$.

Note: It refers only to input and state (i.e. F and G).

Note: Controllability and Reachability are synonyms.

EXAMPLE — SISO system of order $n = 2$.

$$\mathcal{S} : \begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + u(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) \\ y(t) = \frac{1}{4}x_1(t) \end{cases} \quad F = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{8} & 0 \end{bmatrix} \quad \text{rank}(O) = 1 < n = 2 \quad \implies \quad \text{not fully observable}$$

$$R = [G \quad FG] = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{3} \end{bmatrix} \quad \text{rank}(R) = 1 < n = 2 \quad \implies \quad \text{not fully controllable}$$

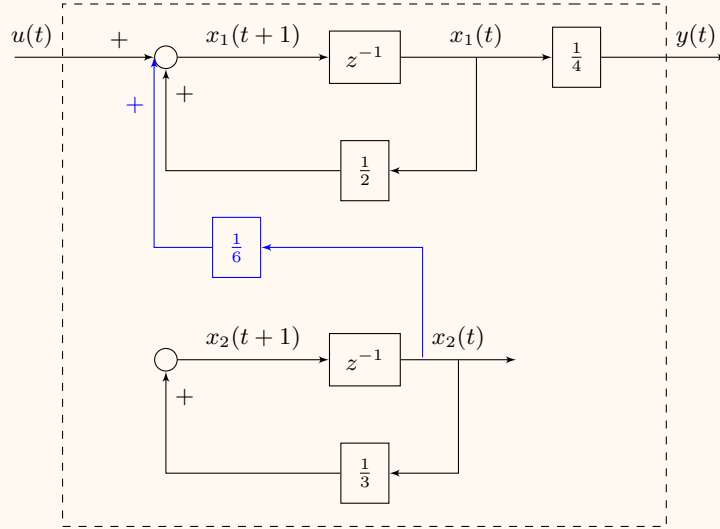
If \mathcal{S} become

$$\mathcal{S}' : \begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + u(t) + \frac{1}{6}x_2(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) \\ y(t) = \frac{1}{4}x_1(t) \end{cases} \quad \text{only } F \text{ change and become } F = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix}$$

And

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{24} \end{bmatrix} \quad \text{rank}(O) = 2 = n \implies \text{fully observable}$$

Observability and Controllability can also be checked graphically by building the block scheme



From this block scheme it can be seen that x_1 is *directly* observable and reachable; x_2 become only *indirectly* observable through $x_1(t)$ with the introduction of the blue block that correspond to the term $\frac{1}{6}x_2(t)$ in \mathcal{S}' .

EXAMPLE — SISO system of order $n = 2$.

$$\mathcal{S} : \begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) + u(t) \\ y(t) = \frac{1}{4}x_1(t) \end{cases} \quad F = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{8} & 0 \end{bmatrix} \quad \text{rank}(O) = 1 < n = 2 \implies \text{not fully observable}$$

$$R = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{3} \end{bmatrix} \quad \text{rank}(R) = 1 < n = 2 \implies \text{not fully controllable}$$

If \mathcal{S} become

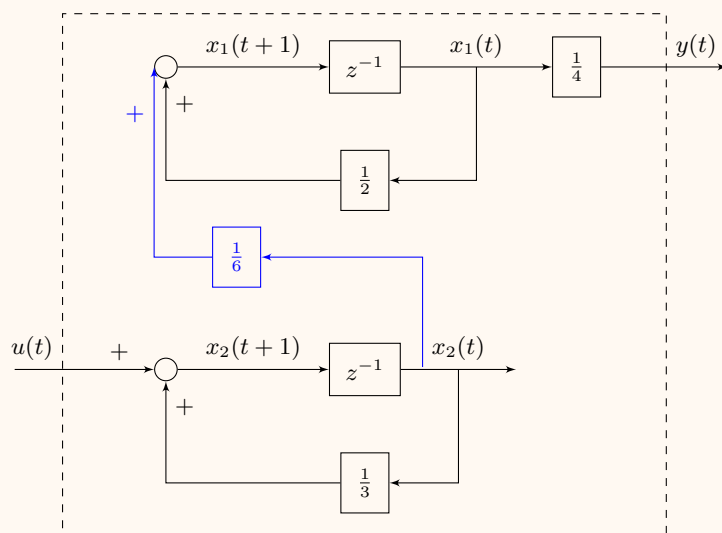
$$\mathcal{S}' : \begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + \frac{1}{6}x_2(t) \\ x_2(t+1) = \frac{1}{3}x_2(t) + u(t) \\ y(t) = \frac{1}{4}x_1(t) \end{cases} \quad \text{only } F \text{ change and become } F = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix}$$

And

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{24} \end{bmatrix} \quad \text{rank}(O) = 2 = n = 2 \quad \Rightarrow \quad \text{fully observable}$$

$$R = [G \quad FG] = \begin{bmatrix} 0 & \frac{1}{6} \\ 1 & \frac{1}{3} \end{bmatrix} \quad \text{rank}(R) = 2 = n = 2 \quad \Rightarrow \quad \text{fully controllable}$$

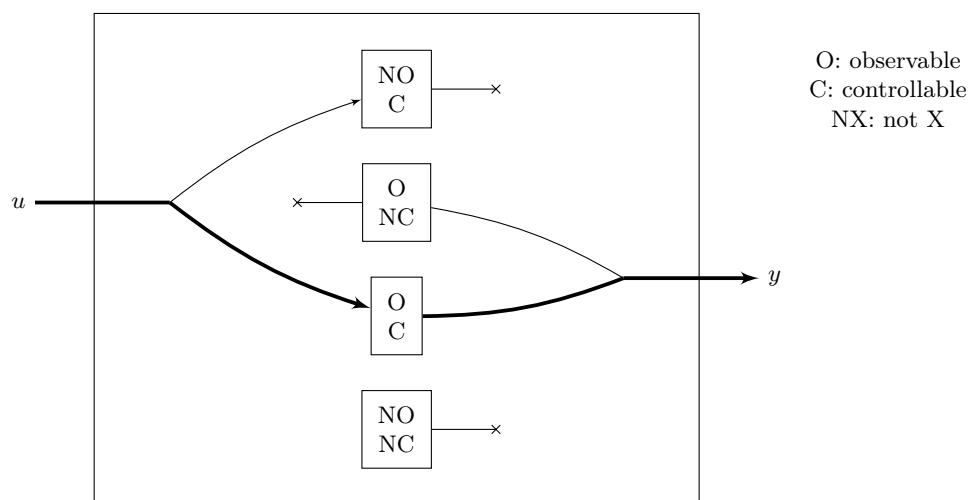
Again, this properties can be also check using the block scheme



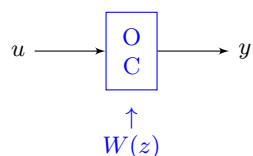
It can be noticed that $x_1(t)$ is *directly* observable but not reachable and $x_2(t)$ is *directly* controllable. In \mathcal{S}' , with the introduction of the same blue block, the system become fully observable and controllable; in particular, $x_2(t)$ become also *indirectly* observable and $x_1(t)$ become also *indirectly* controllable.

REMARK — 4 sub-systems.

Any State-Space system can be **internally** seen as 4 sub-systems as follows:



Which **externally** is equivalent to a systems like this:



Hence, this graphically proves the fact that the I/O representation of a system through the Transfer Function can only represent the *observable* and *controllable* part of the system; all the other sub-systems remain **hidden** with this representation. For this reason, the SS representation is "more complete" than the TF one when the system is not fully observable or not fully controllable.

Another final definition is needed before presenting the 4SID algorithm.

DEFINITION 1.6 — Hankel matrix of order n. Starting from $IR = \{\omega(1), \omega(2), \dots, \omega(N)\}$ we can build the Hankel matrix of order n .

$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) & \cdots & \omega(n) \\ \omega(2) & \omega(3) & \omega(4) & \cdots & \omega(n+1) \\ \omega(3) & \omega(4) & \omega(5) & \cdots & \omega(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega(n) & \omega(n+1) & \omega(n+2) & \cdots & \omega(2n-1) \end{bmatrix}$$

Note: it is a square matrix of size $n \times n$.

Note: we need IR up to time $2n - 1$.

Note: it starts from $\omega(1)$ and not from $\omega(0)$ (since, for strictly-proper systems, $\omega(0) = 0$).

Note: the anti-diagonals all have the same element repeated.

We know that $\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ HF^{t-1}G & \text{if } t > 0 \end{cases}$, therefore H_n can be rewritten as

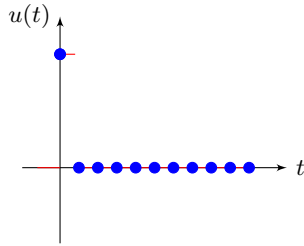
$$H_n = \begin{bmatrix} HG & HFG & HF^2G & \cdots & HF^{n-1}G \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ HF^{n-1}G & \cdots & \cdots & \cdots & HF^{2n-2}G \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \cdot [G \quad FG \quad \cdots \quad F^{n-1}G] = O \cdot R$$

$$\Rightarrow H_n = O \cdot R$$

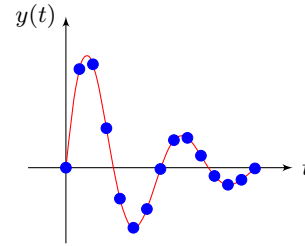
where O is the observability matrix and R is the reachability matrix.

1.4 Subspace-based State-Space System Identification (4SID)

The original 4SID method starts from the measurement of the system output in a very simple experiment, that is the *impulse experiment*.



Impulse in input



IR in output

The fundamental idea of this experiment is that it is very simple to measure the output of system given an impulse signal as the input. Since the goal of the method is to identify an SS model $\{\hat{F}, \hat{G}, \hat{H}\}$ starting from the $IR = \{\omega(0), \omega(1), \omega(2), \dots\}$, it can be said that it is a Black-Box system identification method.

We will see the solution of the problem with two different assumptions:

Basic problem IR measurement is assumed to be **noise-free**. Easier and not realistic problem. Described in 1.5.

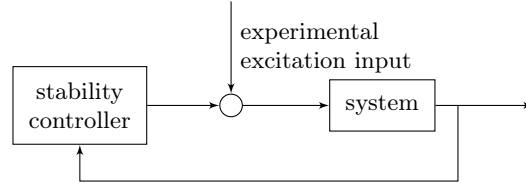
Real problem IR is measured with noise

$$\tilde{\omega}(t) = \omega(t) + \eta(t) \quad t = 0, 1, \dots, N$$

where $\tilde{\omega}(t)$ is the measured noisy IR, $\omega(t)$ is the “true” noise-free IR and $\eta(t)$ is the measurement noise (e.g. White Noise (WN)). Described in 1.6.

REMARK. We will see in detail only the original version of 4SID, that is that the experiment is an impulse-experiment. However 4SID can be extended to any generic input signal $\{u(1), u(2), \dots, u(N)\}$ that is sufficiently exciting.

REMARK — Unstable System. In case of an unstable system the measurements must be collected in a closed-loop experiment. Indeed, if the experiment was open-loop, the experiment would be unfeasible.



1.5 4SID procedure (noise-free)

Step 1 Build the Hankel matrix in increasing order and each time compute the rank of the matrix.

$$H_1 = [\omega(1)] \quad H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} \quad H_3 = \dots \quad \dots \quad H_n = \dots$$

Suppose that $\text{rank}(H_i) = i \quad \forall i \in \{1, \dots, n\}$ and $\text{rank}(H_{n+1}) = n$.

If this happens, it means that we found the first Hankel matrix which is not full rank and it also means that we have estimated (found) the order of the system.

Step 2 Take H_{n+1} (**recall:** $\text{rank}(H_{n+1}) = n$) and factorize it in two rectangular matrices of size $(n+1) \times n$ and $n \times (n+1)$.

$$H_{n+1} = \begin{bmatrix} \text{extended} \\ \text{observability} \\ \text{matrix:} \\ O_{n+1} \end{bmatrix} \cdot \begin{bmatrix} \text{extended} \\ \text{controllability} \\ \text{matrix:} \\ R_{n+1} \end{bmatrix}$$

where $O_{n+1} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^n \end{bmatrix}$ of size $(n+1) \times n$ and $R_{n+1} = \begin{bmatrix} G & FG & \dots & F^n G \end{bmatrix}$ of size $n \times (n+1)$.

Step 3 H, F, G estimation.

Using O_{n+1} and R_{n+1} we can easily find:

$$\begin{aligned} \hat{G} &= R_{n+1}(:, 1) \quad (\text{first column of } R_{n+1}) \\ \hat{H} &= O_{n+1}(1, :) \quad (\text{first row of } O_{n+1}) \end{aligned}$$

To estimate \hat{F} consider O_{n+1} (or, similarly, R_{n+1}):
define O_1 as O_{n+1} without the last row, and O_2 as O_{n+1} without the first row, that is

$$O_1 = O_{n+1}(1:n; :) \quad O_2 = O_{n+1}(2:n+1; :)$$

Note: O_1 and O_2 are $n \times n$ matrices. This is called *shift-invariance* property.

Note: O_1 is invertible since it's an observability matrix and the subsystem is fully observable.

Moreover $O_2 = O_1 F$, so $\hat{F} = O_1^{-1} O_2$.

In conclusion in a simple and *constructive* (i.e NON parametric) way we have estimated a State-Space model of the system starting from measured *noise-free* IR, using only $2n + 1$ samples of IR.

In practice $n \ll N$, where, again, n is the order of the system and N the number of sample in the dataset.

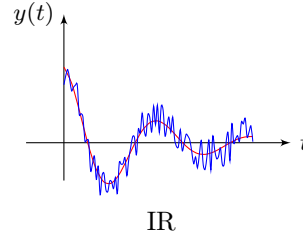
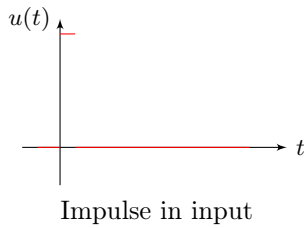
Conclusion

$$\{\hat{H}, \hat{G}, \hat{F}\} = \{O_{n+1}(1; :), R_{n+1}(:, 1), O_1^{-1} O_2\}$$

REMARK. If the measurement is noisy all this process is **useless**. Also, if n is unknown, Step 1 could never stop.

1.6 4SID procedure (with Noise)

In reality, the measurements of IR are noisy: $\tilde{\omega}(t) = \omega(t) + \eta(t)$. Therefore, the **real problem** is the estimation of the system taking into account the noise; indeed, the available dataset will be $IR = \tilde{\omega}(t) = \{\tilde{\omega}(0), \tilde{\omega}(1), \dots, \tilde{\omega}(N)\}$.



Step 1 Build the Hankel matrix from data using “one-shot” all the available N data points.

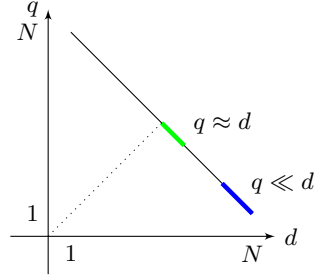
$$\tilde{H}_{qd} = \begin{bmatrix} \tilde{\omega}(1) & \tilde{\omega}(2) & \cdots & \tilde{\omega}(d) \\ \tilde{\omega}(2) & \tilde{\omega}(3) & \cdots & \tilde{\omega}(d+1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\omega}(q) & \tilde{\omega}(q+1) & \cdots & \tilde{\omega}(q+d-1) \end{bmatrix}$$

\tilde{H}_{qd} is a $q \times d$ matrix.

Note that, as before, we start from $\tilde{\omega}(1)$ since we are assuming $\tilde{\omega}(0) = 0$.

Note that $q + d - 1$ must equal to N so that we use all the dataset.

REMARK — Choice of q and d . Hypothesis: $q < d$ so that $q + d - 1 = N \Rightarrow q = N + 1 - d$.



If $q \approx d$ the method has better accuracy but high computational effort.

If $q \ll d$ it's computationally less intensive but has the worst accuracy.

Rule of thumb: If $q > \frac{d}{2}$ we get to a good enough result.

Step 2 *Singular Value Decomposition (SVD)* of \tilde{H}_{qd}

$$\underbrace{\tilde{H}_{qd}}_{q \times d} = \underbrace{\tilde{U}}_{q \times q} \underbrace{\tilde{S}}_{q \times d} \underbrace{\tilde{V}^T}_{d \times d}$$

where \tilde{U} and \tilde{V} are *square* and *unitary* matrices and \tilde{S} is a *rectangular diagonal* matrix such that:

$$\tilde{S} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix}$$

where $\sigma_1, \sigma_2, \dots, \sigma_q$ are the *singular values* of \tilde{H}_{qd} . Those are real, positive numbers, sorted in decreasing order (i.e. $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_q$).

DEFINITION 1.7 — Unitary matrix. A matrix M is *unitary* if:

- $\det M = 1$ (so it's invertible)
- $M^{-1} = M^T$

OBSERVATION. The singular values of a rectangular matrix are a *sort of eigenvalues* of a square matrix.

SVD is a *sort of diagonalization* of a rectangular matrix.

RECALL. For a square matrix, $\text{eig}(A) = \text{roots}(\det(A - \lambda I))$. If M is rectangular, $SV(M) = \sqrt{\text{eig}(MM^T)}$ (for non zero eigenvalues).

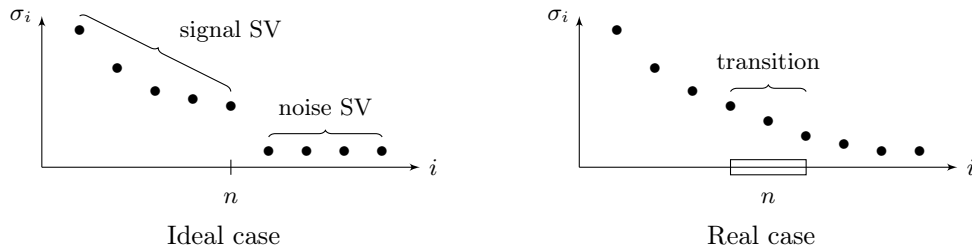
REMARK — How to compute SVD. The optimal numerical computation is not trivial. Theoretical method for SVD computation is to make 2 diagonalization steps:

$$\underbrace{\tilde{H}_{qd}\tilde{H}_{qd}^T}_{q \times q} = \tilde{U}\tilde{S}\tilde{S}^T\tilde{U}^T$$

$$\underbrace{\tilde{H}_{qd}^T\tilde{H}_{qd}}_{d \times d} = \tilde{V}\tilde{S}^T\tilde{S}\tilde{V}^T$$

Instead, use `svd(M)` in Matlab.

Step 3 Plot the singular values and separate (cut-off) the system from noise.



In the ideal case there is a perfect/clear separation between the signal and the noise S.V. (a jump). The index of the jump is n , that is the order of the system.

In the real case there is no clear distinction between signal and noise singular values, since the order of the system can assume values in an interval. With some empirical test we can select a good compromise between complexity, precision and overfitting (see *cross-validation*).

After the decision on the value of n we split \tilde{H}_{qd} in \tilde{U} , \tilde{S} and \tilde{V}^T :

$$\tilde{H}_{qd} = \left[\begin{array}{c} \overbrace{\tilde{U}}^{q \times n} \\ \tilde{U} \end{array} \right] \left[\begin{array}{c} \overbrace{\tilde{S}}^{n \times n} \\ \tilde{S} \end{array} \right] \left[\begin{array}{c} \overbrace{\tilde{V}^T}^{n \times d} \\ \tilde{V}^T \end{array} \right]$$

$$\tilde{H}_{qd} = \underbrace{\hat{U}\hat{S}\hat{V}^T}_{\hat{H}_{qd}} + H_{res,qd} \quad \text{rank}(\tilde{H}_{qd}) = q \quad \text{rank}(\hat{H}_{qd}) = n \quad \text{rank}(H_{res,qd}) = q$$

where \hat{H}_{qd} is the "signal part" and $H_{res,qd}$ is the "noise (or residual) part" of \tilde{H}_{qd} .

Note: \hat{S} is now a square diagonal matrix of $\sigma_1, \dots, \sigma_q$.

Note: from \tilde{H}_{qd} to \hat{H}_{qd} the rank is hugely reduced.

Note: \hat{H}_{qd} is the "cleaned" Hankel matrix.

Step 4 Estimation of \hat{F} , \hat{G} and \hat{H} using the cleaned matrix \hat{H}_{qd}

$$\hat{H}_{qd} = \hat{U} \hat{S} \hat{V}^T = \hat{U} \hat{S}^{\frac{1}{2}} \hat{S}^{\frac{1}{2}} \hat{V}^T$$

where $\hat{S}^{\frac{1}{2}}$ is the square diagonal matrix with elements the square roots of the elements of \hat{S} , that is $\hat{S}^{\frac{1}{2}} = \text{diag}(\sqrt{\sigma_1}, \dots, \sqrt{\sigma_q})$.

Defining $\hat{O} = \hat{U} \hat{S}^{\frac{1}{2}}$ and $\hat{R} = \hat{S}^{\frac{1}{2}} \hat{V}^T$, \hat{H}_{qd} become

$$\hat{H}_{qd} = \underbrace{\hat{O}}_{q \times n} \underbrace{\hat{R}}_{n \times d}$$

We can view \hat{O} as the *extended observability matrix* and the \hat{R} the *extended reachability matrix* of the system.

Now it is possible to estimate \hat{H} with the first row of \hat{O} and \hat{G} with the first column of \hat{R} , similarly to the Step 3 of the noise-free case.

$$\begin{aligned} \hat{G} &= \hat{R}(:, 1) \quad (\text{first column of } \hat{R}) \\ \hat{H} &= \hat{O}(1, :) \quad (\text{first row of } \hat{O}) \end{aligned}$$

What about the estimation of \hat{F} ? Let's try to proceed again similarly to what we have done at the Step 3 of the noise-free case. Consider for example \hat{O} and define \hat{O}_1 as \hat{O} without the last row, and \hat{O}_2 as \hat{O} without the first row.

Using the *shift-invariance* property we have that $\hat{O}_1 \hat{F} = \hat{O}_2$, but \hat{O}_1 is not a square matrix so it's NOT invertible. To avoid this problem we can use the approximate *least-squares* solution.

RECALL — Solution of Linear Systems. Consider a generic system $Ax = B$ with dimension $(h \times n) \cdot (n \times 1) = (h \times 1)$. We have 3 different cases:

1. $h < n$. We have less equations than variables: the system is *under determined* and we have infinite solutions.
2. $h = n$. We have one and only one solution if A is invertible.
3. $h > n$. We have more equations than variables: the system is *over determined* and it's impossible (no solutions).

REMARK — Least-Squares Method. In case we have more equations than variable (i.e. $h > n$) we can use an approximate solution using the least-squares method, which for a generic system is as follows:

$$\begin{aligned} Ax &= B \\ \Downarrow \\ A^T Ax &= A^T B \implies \hat{X} = \underbrace{(A^T A)^{-1} A^T}_{A^+} B \end{aligned}$$

A^+ is called *pseudo-inverse*, which is “surrogate inverse” when A is rectangular.

Applying the least-square method to $\hat{O}_1 \hat{F} = \hat{O}_2$ we have

$$\hat{O}_1 \hat{F} = \hat{O}_2 \implies \hat{O}_1^T \hat{O}_1 \hat{F} = \hat{O}_1^T \hat{O}_2 \implies \hat{F} = \left(\hat{O}_1^T \hat{O}_1 \right)^{-1} \hat{O}_1^T \hat{O}_2$$

Conclusion Starting from a noisy IR $\{\tilde{\omega}(1), \tilde{\omega}(2), \dots, \tilde{\omega}(N)\}$ we have estimated a model $\{\hat{F}, \hat{G}, \hat{H}\}$ in a non-parametric and constructive way.

$$\{\hat{H}, \hat{G}, \hat{F}\} = \left\{ \hat{O}(1; :), \hat{R}(:, 1), \left(\hat{O}_1^T \hat{O}_1 \right)^{-1} \hat{O}_1^T \hat{O}_2 \right\}$$

OBSERVATION. This method can be extended also to the case where the input signal is generic (i.e. not an impulse).

REMARK — Optimality of 4SID. The method is *optimal* in the sense that it makes the best possible rank reduction of \tilde{H}_{qd} , that is from q to n .

However, notice that in general there are infinite ways to make a rank reduction.

REMARK — Rank Reduction. Our goal is to obtain the desired rank reduction by discarding the minimum amount of information contained in the original matrix. SVD makes exactly this: $\tilde{H}_{res,qd}$ is the minimum possible (in the sense of the *Frobenius norm*).

$$\left| \tilde{H}_{res,qd} \right|_F = \sqrt{\sum_{ij} \left(\tilde{H}_{res,qd}^{(ij)} \right)^2}$$

EXAMPLE — Rank Reduction.

$$\underbrace{\begin{bmatrix} 2 & 5 & 3 & 6 & 5 \\ 5 & 3 & 6 & 5 & 7 \\ 3 & 6 & 5 & 7 & 1 \end{bmatrix}}_{\text{rank}=3} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{rank}=2} + \underbrace{\begin{bmatrix} 1 & 5 & 3 & 6 & 5 \\ 5 & 2 & 6 & 5 & 7 \\ 3 & 6 & 5 & 7 & 1 \end{bmatrix}}_{\text{rank}=3}$$

It's not the optimal rank reduction matrix, because it factors out a matrix with lower rank but a lot of information of the original matrix is lost.

REMARK. 4SID is a constructive method that can be implemented in a fully-automatic way, except for these steps:

- q and d selection (not critical)
- choice of n (typically supervised by the designer). It can be made automatic using a cross-validation method.

REMARK. SVD was an historical turning point in machine learning algorithms because it allows:

- very efficient compression of information.
- very efficient separation of *important* information from noise.
- order reduction of a model.

Notice that those are 3 different prospective of the same general problem.

EXAMPLE — similar to an exam exercise. Consider the following S.S. model

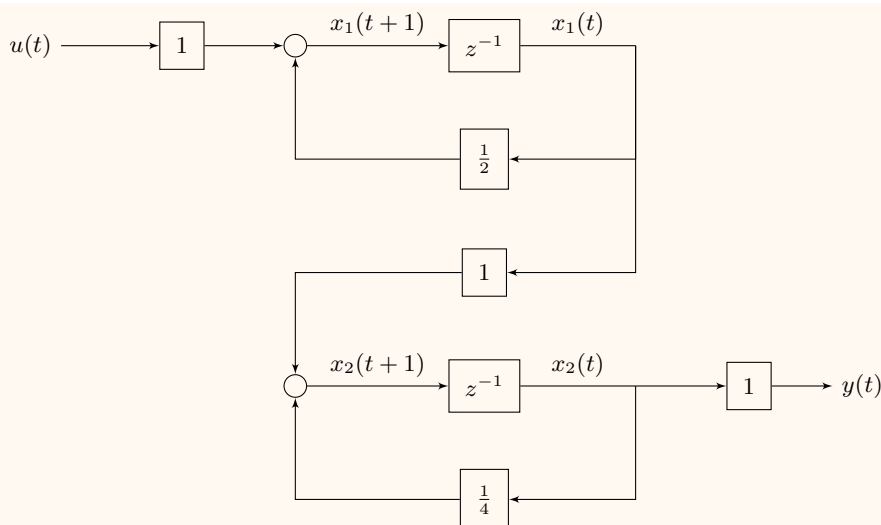
$$F = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad H = [0 \quad 1] \quad D = 0$$

Note Given the size of F the system has grade $n = 2$ and it is strictly-proper single-input single-output system. **Note** Since F is triangular, the eigenvalues are on the diagonal and $\text{eig}(F) = \{\frac{1}{2}, \frac{1}{4}\}$ and they have both absolute value less than one, thus the system is *asymptotically stable*.

Question 1 Write the time domain equations of the system in the state space representation.

$$\mathcal{S} : \begin{cases} x_1(t+1) &= \frac{1}{2}x_1(t) + u(t) \\ x_2(t+1) &= x_1(t) + \frac{1}{4}x_2(t) \\ y(t) &= x_2(t) \end{cases}$$

Question 2 Write the block scheme of the SS representation of the system.



By visual inspection *seems* that the system is fully observable and fully controllable.

Question 3 Make a formal verification that the system is fully observable and controllable.

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{4} \end{bmatrix} \quad \text{rank}(O) = 2 = n$$

$$R = [G \quad FG] = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \quad \text{rank}(R) = 2 = n$$

This confirms our visual inspection.

Let's compute the extended $(n + 1 = 2 + 1)$ O_3 and R_3 :

$$F^2 = FF = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & \frac{1}{16} \end{bmatrix}$$

$$O_3 = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{16} \end{bmatrix}$$

$$R_3 = [G \quad FG \quad F^2G] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \end{bmatrix}$$

Question 4 Compute the transfer function representation.

First method: direct manipulation of SS equations

$$\mathcal{S} : \begin{cases} x_1(t+1) &= \frac{1}{2}x_1(t) + u(t) \\ x_2(t+1) &= x_1(t) + \frac{1}{4}x_2(t) \\ y(t) &= x_2(t) \end{cases}$$

$$\begin{aligned}
zx_1(t+1) - \frac{1}{2}x_1(t) &= u(t) &\implies x_1(t) &= \frac{1}{z - \frac{1}{2}}u(t) \\
zx_2(t) - \frac{1}{4}x_2(t) &= \frac{1}{z - \frac{1}{2}}u(t) &\implies x_2(t) &= \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})}u(t) \\
y(t) &= \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})}u(t) \\
W(z) &= \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})}
\end{aligned}$$

There are 2 poles: $z = \frac{1}{4}$ and $z = \frac{1}{2}$. Since the system is fully observable and controllable the poles correspond to the eigenvalues of F .

Second method: use the formula

$$W(z) = H(zI - F)^{-1}G = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z - \frac{1}{2} & 0 \\ -1 & z - \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})}$$

Question 5 Write I/O time-domain representation

$$y(t) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}}u(t) = \frac{z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}u(t) = \frac{3}{4}y(t-1) - \frac{1}{8}y(t-2) + u(t-2)$$

Question 6 Compute the first 6 values (including $\omega(0)$) of IR.

We decide to compute it from the TF $\frac{z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$. Performing the 5-step long division of $W(z)$ the result is $z^{-2} + \frac{3}{4}z^{-3} + \frac{7}{16}z^{-4} + \frac{15}{64}z^{-5}$.

$$\begin{array}{lll}
\omega(0) = 0 & \omega(1) = 0 & \omega(2) = 1 \\
\omega(3) = \frac{3}{4} & \omega(4) = \frac{7}{16} & \omega(5) = \frac{15}{64}
\end{array}$$

Note Also $\omega(1) = \omega(0) = 0$: this means that the delay $k = 2$, not just 1.

Question 7 Build the Hankel matrix and stop when the rank is not full (noise-free case).

$$\begin{aligned}
H_1 &= \begin{bmatrix} 0 \end{bmatrix} & \text{rank}(H_1) &= 1 \\
H_2 &= \begin{bmatrix} 0 & 1 \\ 1 & \frac{3}{4} \end{bmatrix} & \text{rank}(H_2) &= 2 \\
H_3 &= \begin{bmatrix} 0 & 1 & \frac{3}{4} \\ 1 & \frac{3}{4} & \frac{7}{16} \\ \frac{3}{4} & \frac{7}{16} & \frac{15}{64} \end{bmatrix} & \text{rank}(H_3) &= 2 \neq 3
\end{aligned}$$

H_3 is not full rank so the order of the system is 2 (as we already know).

It can be proved that $O_3 R_3 = H_3$.

Chapter 2

Parametric Black-Box system identification of I/O system using a frequency domain approach

So far we have seen:

- In MIDA1 parametric black-box identification of I/O systems (ARMAX) and time series (ARMA)
- In Chapter 1 non-parametric black-box identification of I/O systems (4SID)

The **frequency domain approach** is a black-box and **parametric**, and it's very used in practice since it's very robust and reliable.

Since it's parametric it uses the 4 usual steps:

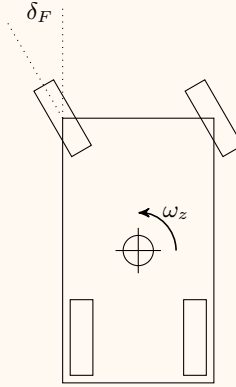
1. Experiment design and data pre-processing. A special type of experiment and data pre-processing is needed.
2. Selection of parametric model class ($\mathcal{M}(\vartheta)$)
3. Definition of a performance index ($J(\vartheta)$). A new special performance index is needed.
4. Optimization ($\hat{\vartheta} = \arg \min_{\vartheta} J(\vartheta)$)

1. and 2. are the special steps of this chapter.

The general intuitive idea of the method is:

- Make a set of “single sinusoid” (“single-tune”) excitation experiments
- From each experiment estimate a single point of the frequency response of the system
- Fit the estimated and modeled frequency response to obtain the optimal model

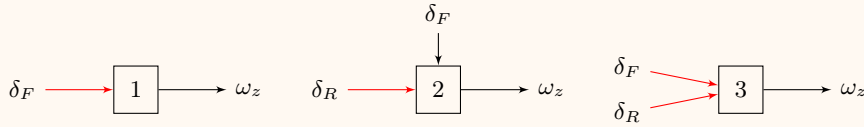
EXAMPLE — Car steer dynamics. Let's picture a top-view of a car that is steering.



where δ_F is the *steer angle* (input) and ω_z is the *rotational speed* around the vertical axis z .

This kind of dynamics relationship is very important for stability control systems design (ESG/ESP) and for autonomous cars.

There are 3 possible situations:

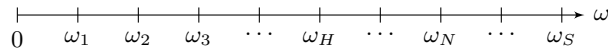


1. The control variable is the *front steer* δ_F . This is the case of autonomous cars
2. The human driver controls δ_F which is a measurable disturbance while the system controls the *rear steer* δ_R
3. Both δ_R and δ_F are control variables: application high performance autonomous car

Imagine to make an experiment where you make a sinusoid of the steer and you get the rotational speed as output.

2.1 Steps of the system identification with frequency domain method

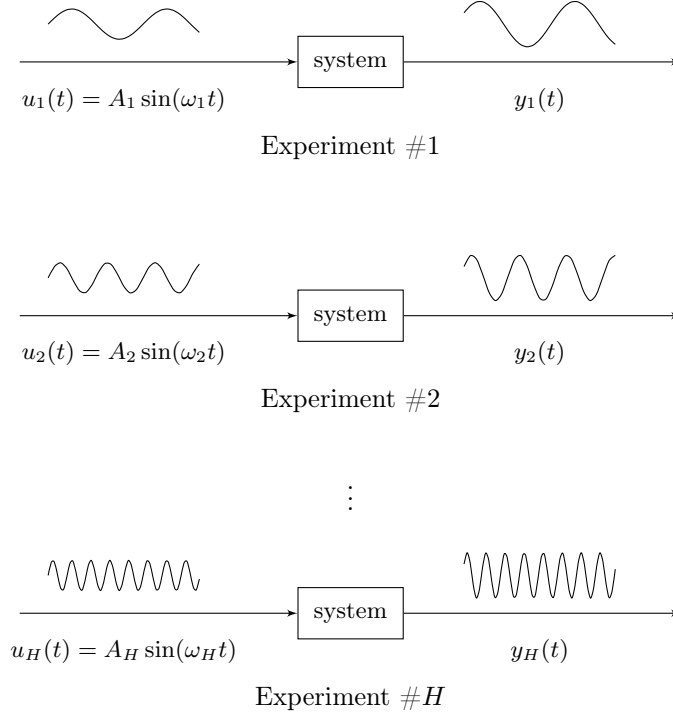
Step 1 In the experiment design step we first have to select a set of *excitation frequencies*.



where ω_S is the *sampling frequency*, $\omega_N = \frac{1}{2}\omega_S$ is the *Nyquist frequency* (which is the maximum frequency of a digital system) and ω_H is the maximum explored frequency. We have $\{\omega_1, \omega_2, \dots, \omega_H\}$ usually evenly spaced ($\Delta\omega$ is constant). ω_H must be selected according

to the bandwidth of the control system.

We make H independent experiments.



REMARK. The amplitudes A_1, A_2, \dots, A_H can be equal (constant) or, more frequently in practice, they decrease as the frequency increases to fulfill the power constraint on the input.

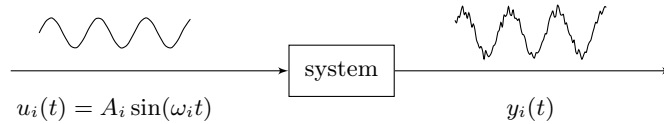
Indeed, if $\delta(t)$ is the steering angle (moved by an actuator), the requested steer torque is proportional to δ : $T(t) = K\delta(t)$. Therefore the steer power is proportional to $T(t)\dot{\delta}(t) = K\delta(t)\dot{\delta}(t)$. If $\delta(t) = A_i \sin(\omega_i t)$ the steering power is $K A_i \sin(\omega_i t) \omega_i A_i \cos(\omega_i t)$ which is proportional to $K A_i^2 \omega_i$.

If we have a limit to this power, this power should be constant during the H experiments, thus

$$K A_i^2 \omega_i = \text{const} \quad \implies \quad A_i = \sqrt{\frac{\text{const}}{K \omega_i}}$$

and this means that the amplitude must be inversely proportional to the frequency.

Focusing on the i -th experiment

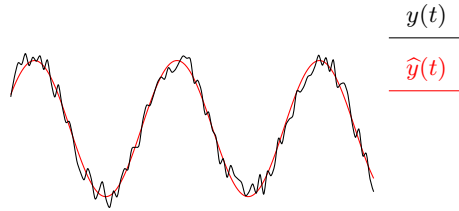


RECALL — Frequency Response Theorem for LTI Systems. If the system is LTI (linear time-invariant), the frequency response theorem says that if the input is a sine input of frequency ω_i the output must be a sine with frequency ω_i but with different amplitude or phase.

However $y_i(t)$ in real applications is not a perfect sinusoid because of those non-ideal behaviours:

- Noise on output measurements
- Noise on the system not directly related to measurements (e.g. roughness of the road).
- (Small) non-linear effects (that we neglect, since we will use LTI local approximations of the system)

In pre-processing of I/O data we want to extract from $y_i(t)$ a perfect sinusoid of frequency ω_i . We force the assumption that the system is LTI, so the output must be a pure sine wave of frequency ω_i (all the remaining signal is noise).



The model of the output signal is

$$\hat{y}_i = B_i \sin(\omega_i t + \phi_i) = a_i \sin(\omega_i t) + b_i \cos(\omega_i t)$$

There are 2 unknowns: B_i and ϕ_i (or a_i and b_i). It's better to use the model with a_i and b_i since it's *linear* in those parameters.

The unknown parameters are a_i and b_i and we can find them by parametric identification.

$$\{\hat{a}_i, \hat{b}_i\} = \arg \min_{\{a_i, b_i\}} J_N(a_i, b_i)$$

$$J_N(a_i, b_i) = \frac{1}{N} \sum_{t=1}^N (y_i(t) - \hat{y}_i(t))^2 = \frac{1}{N} \sum_{t=1}^N \left(\underbrace{y_i(t)}_{\text{measurement}} - \underbrace{a_i \sin(\omega_i t) + b_i \cos(\omega_i t)}_{\text{model output}} \right)^2$$

Since the model is linear in a_i and b_i , J_N , which is the *sample variance of the modelling error*, is a *quadratic function* of a_i and b_i . Thus, we can solve the problem explicitly.

$$\begin{aligned} \frac{\partial J_N}{\partial a_i} &= \frac{2}{N} \sum_{t=1}^N (-\sin(\omega_i t))(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)) = 0 \\ \frac{\partial J_N}{\partial b_i} &= \frac{2}{N} \sum_{t=1}^N (-\cos(\omega_i t))(y_i(t) - a_i \sin(\omega_i t) - b_i \cos(\omega_i t)) = 0 \end{aligned}$$

This is a 2×2 linear system which can be written in matrix form

$$\begin{bmatrix} \sum_{t=1}^N \sin(\omega_i t)^2 & \sum_{t=1}^N \sin(\omega_i t) \cos(\omega_i t) \\ \sum_{t=1}^N \sin(\omega_i t) \cos(\omega_i t) & \sum_{t=1}^N \cos(\omega_i t)^2 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^N y_i(t) \sin(\omega_i t) \\ \sum_{t=1}^N y_i(t) \cos(\omega_i t) \end{bmatrix}$$

At this point we prefer to go back to a *sin-only* form (B_i, ϕ_i) of the estimated sinusoid $\hat{y}_i(t)$:

$$\hat{B}_i \sin(\omega_i t + \phi_i) = \hat{B}_i \sin(\omega_i t) \cos(\hat{\phi}_i) + \hat{B}_i \cos(\omega_i t) \sin(\hat{\phi}_i) \stackrel{!}{=} \hat{a}_i \sin(\omega_i t) + \hat{b}_i \cos(\omega_i t)$$

\Downarrow

$$\begin{cases} \hat{B}_i \cos(\hat{\phi}_i) = \hat{a}_i \\ \hat{B}_i \sin(\hat{\phi}_i) = \hat{b}_i \end{cases} \quad (\star)$$

\Downarrow

$$\begin{aligned} \frac{\hat{b}_i}{\hat{a}_i} &= \frac{\sin \hat{\phi}_i}{\cos \hat{\phi}_i} = \tan(\hat{\phi}_i) & \hat{\phi}_i &= \arctan \left(\frac{\hat{b}_i}{\hat{a}_i} \right) \\ \hat{B}_i &= \frac{\frac{\hat{a}_i}{\cos \hat{\phi}_i} + \frac{\hat{b}_i}{\sin \hat{\phi}_i}}{2} & \text{average of the the 2 equation in } (\star) \end{aligned}$$

Therefore, we have now found a bidirectional mapping $\{\hat{a}_i, \hat{b}_i\} \iff \{\hat{B}_i, \hat{\phi}_i\}$.

Repeating the experiment and pre-processing for the H experiments we can compute

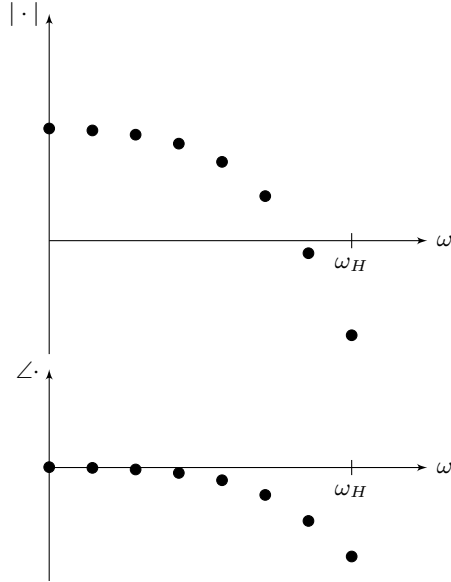
$$\begin{aligned} \{\hat{B}_1, \hat{\phi}_1\} &\mapsto \frac{\hat{B}_1}{A_1} e^{j\hat{\phi}_1} \\ \{\hat{B}_2, \hat{\phi}_2\} &\mapsto \frac{\hat{B}_2}{A_2} e^{j\hat{\phi}_2} \\ &\vdots \\ \{\hat{B}_H, \hat{\phi}_H\} &\mapsto \frac{\hat{B}_H}{A_H} e^{j\hat{\phi}_H} \end{aligned}$$

where each $\frac{\hat{B}_i}{A_i} e^{j\hat{\phi}_i}$ is a complex number that is the estimated point at frequency ω_i of the frequency response of the transfer function $W(z)$ from the input $u(t)$ to the output $y(t)$ of the system.

RECALL — Frequency Response Theorem for LTI systems. If for a LTI system characterized by TF $W(z)$ the input is $A_i \sin(\omega_i t)$ and the output is $\hat{B}_i \sin(\omega_i t + \hat{\phi}_i)$, then $W(z = e^{j\omega_i}) = \frac{\hat{B}_i}{A_i} e^{j\hat{\phi}_i}$ is the corresponding frequency response of the system at that frequency (ω_i).

In particular, $\frac{\hat{B}_i}{A_i}$ is the ratio between the output and input amplitude and $\hat{\phi}_i$ is the phase shift.

We can now plot these H point in a classical *Bode plot* exploiting that $|W(e^{j\omega_i})| = \frac{\hat{B}_i}{A_i}$ and that $\angle W(e^{j\omega_i}) = \hat{\phi}_i$.



At the end of *step 1* we have a frequency-domain dataset (H values) representing H estimated points of the frequency response of the system.

Step 2 Parametric model class (TF) selection

$$\mathcal{M}(\vartheta) : W(z; \vartheta) = \frac{b_0 + b_1 z^{-1} + \dots + b_p z^{-p}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} z^{-1} \quad \vartheta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_0 \\ \vdots \\ b_p \end{bmatrix}$$

OBSERVATION — Model order selection. In this case the order is composed by 2 parameters n and p : use cross-validation approach (or visual fitting in the Bode diagram) for finding the best choice of these parameters.

Step 3 New performance index: variance of the error in frequency domain.

$$J_H(\vartheta) = \frac{1}{H} \sum_{i=1}^H \left| W(e^{j\omega_i}; \vartheta) - \frac{\hat{B}_i}{A_i} e^{j\hat{\phi}_i} \right|^2$$

where $W(e^{j\omega_i}; \vartheta)$ is the modeled frequency response and $\frac{\hat{B}_i}{A_i} e^{j\hat{\phi}_i}$ is the measured frequency response.

Step 4 Optimization

$$\hat{\vartheta}_H = \arg \min_{\vartheta} J_H(\vartheta)$$

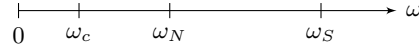
Usually $J_H(\vartheta)$ is a *non-quadratic* and *non-convex function*; Thus, iterative optimization methods are needed.

Conclusion We have obtained the estimated model which is a TF

$$\mathcal{M}(\hat{\vartheta}_H) : W(z; \hat{\vartheta}_H)$$

REMARK — Frequency bandwidth selection $\omega_H = ?$. Theoretically the standard best solution should be H points distributed uniformly from 0 to ω_N (Nyquist freq.).

In practice it's better to concentrate the experimental effort in a smaller and more focused bandwidth.

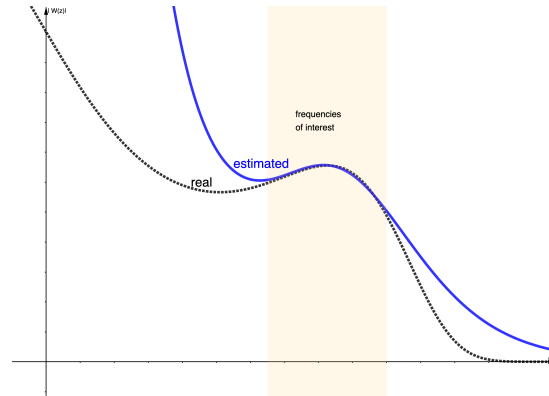


where ω_c is the *expected cut-off frequency* of the closed system.

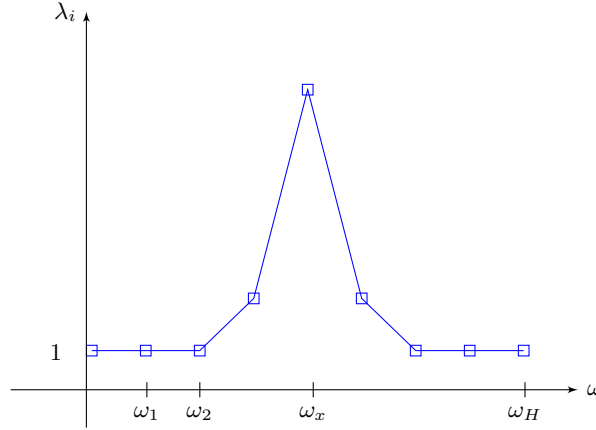
Rule of thumb: $\omega_H \approx 3\omega_c$

Example The Electronic Stability Control (ESC) system has an expected bandwidth of $\omega_c \approx 4\text{Hz}$, so $\omega_H \approx 12\text{Hz}$.

REMARK — Emphasis on special frequency range. In some cases, between ω_1 and ω_H , we want to be more accurate in system identification in some frequency regions (typically around cut-off-frequency or around resonances).



We can manage this selected focus on estimation precision using non-uniform weights λ_i (different weights for different frequencies).



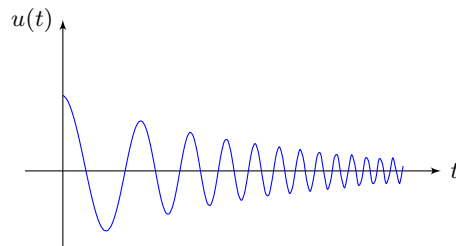
where ω_x is the frequency of interest.

The performance index can be redefined:

$$\tilde{J}_H(\vartheta) = \frac{1}{H} \sum_{i=1}^H \lambda_i \left(W(e^{j\omega_i}; \vartheta) - \frac{\hat{B}_i}{A_i} e^{j\hat{\phi}_i} \right)^2$$

Another *trick*: more dense ω_i spacing in the frequency region of special interest (not really used).

REMARK — Single experiment. Sometimes the set of H independent single-sinusoid experiments can be replaced by a long single “sine-sweep” experiment, that is to measure the frequency response of the system when the input is sinusoid that starts with frequency ω_1 and amplitude A_1 and ends with frequency ω_H and amplitude A_H .



Slowing-varying sinusoid with increasing frequency and decreasing amplitude.

The output will be a single long signal $y(t)$.

We can cut a-posteriori the signal into H pieces, and then back to the standard procedure or we can directly compute an estimation of $\widehat{W}(e^{j\omega})$ as a ration of the output and input spectra

$$\widehat{W}(e^{j\omega}) = \frac{\widehat{\Gamma}_y(e^{j\omega})}{\widehat{\Gamma}_u(e^{j\omega})}$$

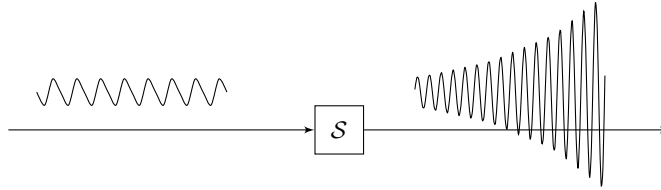
We can fit the estimated $\widehat{W}(e^{j\omega})$ with the model frequency response $W(e^{j\omega}; \vartheta)$ in the performance index.

$$\frac{1}{H} \sum_{i=1}^H |W(e^{j\omega_i}; \vartheta) - \widehat{W}(e^{j\omega_i})|^2$$

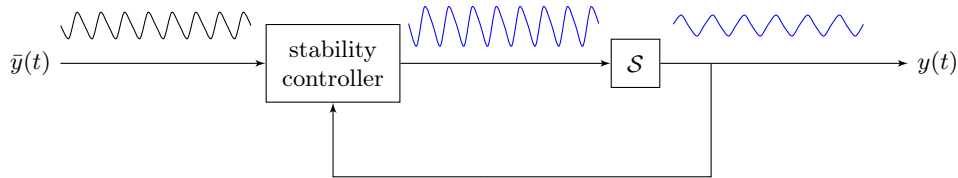
where, similarly, $W(e^{j\omega_i}; \vartheta)$ is the modeled frequency response and $\widehat{W}(e^{j\omega_i})$ is the measured frequency response.

This experiment is quicker but has usually a lower signal-to-noise-ration.

REMARK — Experiment on unstable system. What happen if the system is *open-loop* unstable? We have to stop the experiment because of the instability of the output (which of course diverges).



We can avoid this problem by performing the experiment in *closed-loop* and adding a *stability controller*.



where the blue sinusoids are the signal reflecting the dynamic of the system and, measuring them, they will compose the dataset I/O of the system.

Note that the model identified using that dataset will be a model of an unstable system.

This is just a trick to collect data from an unstable system.

2.2 Comparison between time domain and frequency domain parametric methods

Frequency domains

Pro Robust and very reliable because each experiment has a big signal-to-noise ratio (since we are forcing all the signal energy on a single clean sinusoid).

Pro Intuitive since it is easy to understand.

Pro Consistent with many control-design methods that work in the frequency domain.

Cons More demanding in terms of design of the experiment.

Cons Provides no noise model (unlike the PEM method of ARMAX system identification)

Note that F.D. and T.D. methods should provide approximately the same result if done correctly.

Chapter 3

Software (virtual) Sensing model-based in feedback method using Kalman Filter framework

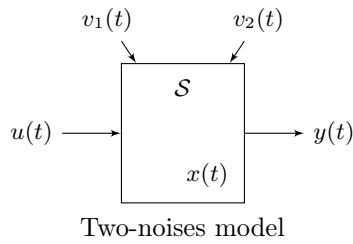
In MIDA1 we have mostly used I/O Transfer Function (TF) representations:

$$y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \quad e(t) \sim \text{WN}$$

Kalman filter theory is fully based on State-Space (SS) representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) + v_1(t) & v_1 \sim \text{WN} \\ y(t) = Hx(t) + Du(t) + v_2(t) & v_2 \sim \text{WN} \end{cases}$$

where we assume that the system model is given (typically obtained in a white-box approach): it's not a system identification problem! Indeed we are interested in the internal state of the system.

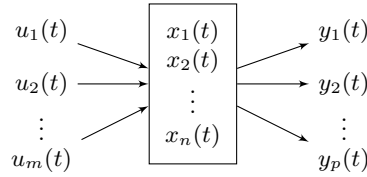


3.1 Motivation and Goals of Kalman Filter (KF)

Given a model description $\{F, G, H\}$ and noises variances (not a system identification technique), with Kalman Filter (KF) theory we can address the following problems:

- Find the k -steps ahead prediction of output: $\hat{y}(t+k|t)$ (already solved in MIDA1 with ARMAX).
- Find the k -steps ahead prediction of state: $\hat{x}(t+k|t)$.
- Find the filter of the state at present time: $\hat{x}(t|t)$. In practice it's a *software-sensing* problem, which is the most important problem solved by Kalman filter (reason of why it is named Kalman *filter*).
- Gray box system identification (see ??). We have a recorded data-set and the model structure with some unknown parameters.

Dynamical systems have this layout:



MIMO system with m inputs (actuators/control variables), p outputs (sensors) and n states.

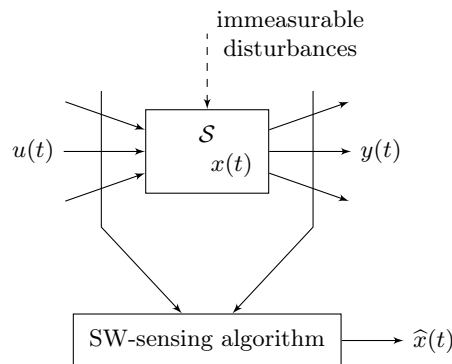
Key problem usually $p \ll n$, that is, physical sensors are much less than system states, because of:

- cost
- cables, power supply, installation
- maintenance (faults, degradation)
- existence of such sensors

That's why usually not all the states are measured (available) but it is very useful to have full "measurement" of states because:

- design of control algorithms (state feedback design)
- monitoring of the system (fault detection, predictive maintenance, ...)

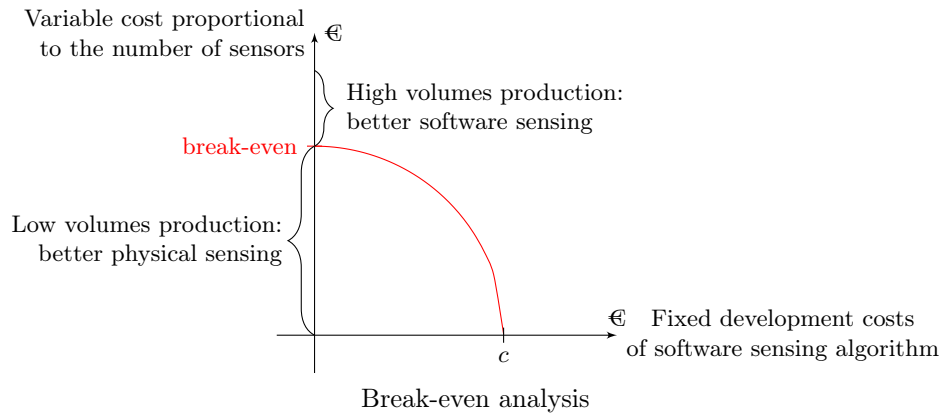
This problem can be solved with *software sensing* (*SW-sensing*) (also called virtual sensing) algorithms:



where $\hat{x}(t)$ is an estimation (or SW-sensing) of the internal state $x(t)$.

Designer Dilemma When using software sensing and when physical sensing?

- In some cases there is no option (not feasible installation of a physical sensor)
- In most cases both options are viable: variable vs fixed cost



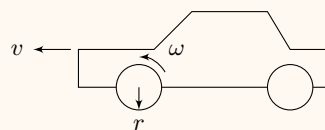
Given c , the fixed cost of development of the software sensor, it is possible to compute the break-even point. The break-even point coincides with the number of products such that above that number it is more convenient to use software sensing, while below that number it is more convenient to buy and install physical sensors per each product.

Anyway, in some cases we might use both physical and software sensing for redundancy (e.g. in very safety-critical or mission-critical applications).

Key questions for software sensing

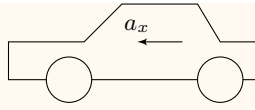
- Is software-sensing feasible? Test is the observability of the states from measured outputs (through physical sensors).
- If feasible, check if the level of noise (error) of the estimated variable is acceptable.

EXAMPLE — Slip estimation for ABS/traction control. Now we go back to the ABS example (described in 0.2) where we realized that SW-sensing is needed for the estimation of v , the horizontal velocity of the car, since it cannot be measured with a physical sensor.



Indeed, measure of v cannot be done with an optical sensor or a GPS: they both have a problem of availability (not guaranteed). Physical sensing is not an option for industrial production.

Intuitive solution: install a longitudinal accelerometer (a_x) and integrate.



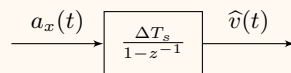
$$\hat{v} = \int a_x(t) dt \quad \Longleftrightarrow \quad a_x(t) \rightarrow 1/s \rightarrow \hat{v}(t)$$

In discrete time domain: discretization using approximation of derivative (Eulero forward method)

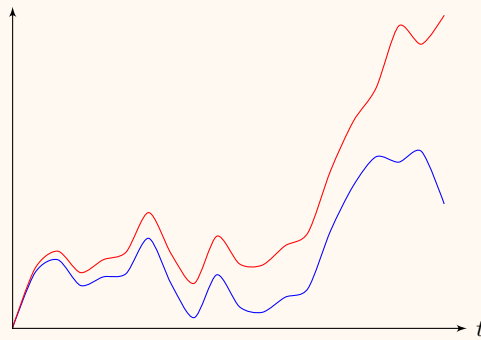
$$\frac{d}{dt} v(t) = a_x(t) \quad \frac{dv(t)}{dt} \approx \frac{v(t+1) - v(t)}{\Delta T_s} = a_x(t)$$

Where ΔT_s is the sampling interval (e.g. 10ms).

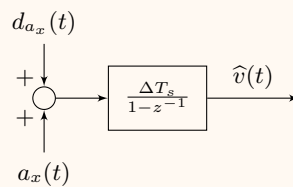
$$\hat{v}(t) = \hat{v}(t-1) + \Delta T_s a_x(t-1)$$



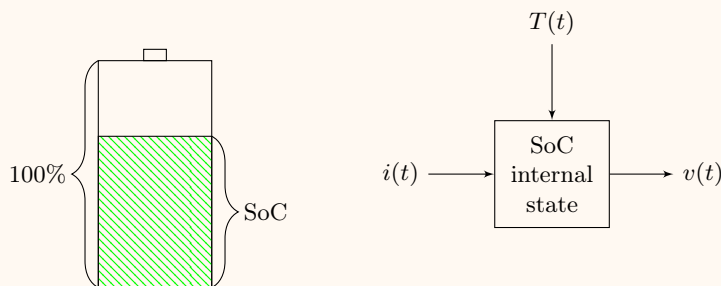
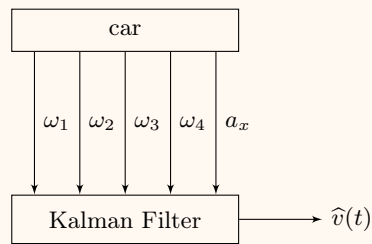
Unfortunately the measured signal is not $a_x(t)$ but $a_x(t) + d_{a_x}(t)$.



Integrating noise generates a *drift*. Integrator is not an asymptotic stable system.



Solution Use a Kalman Filter.



EXAMPLE — State of charge estimation of a battery.

$$\text{SoC}(t) = 1 - \frac{\int i(t)dt}{I} \quad 0 \leq \text{SoC} \leq 1$$

Where I is the total amount of *current* that can be extracted by the user of the battery. This solution is not feasible since it integrates the noise on $i(t)$.

3.2 Kalman Filter on Basic Systems

- No external inputs (~~$Gu(t)$~~): time series
- Linear systems
- Time invariant systems

The basic solution of this basic system is the 1-step prediction. After that, we will make the extensions to more general systems:

- k-step prediction
- filter $\hat{x}(t|t)$
- time-varying systems
- systems with exogenous inputs (presence of $Gu(t)$)
- non-linear systems (Extended Kalman Filter (EKF))

3.2.1 Detailed description of Basic System

The basic system we initially consider is a MIMO system with n states, (m inputs) and p outputs.

$$\mathcal{S} : \begin{cases} x(t+1) = Fx(t) + \cancel{Gu(t)} + v_1(t) & \text{state equation} \\ y(t) = Hx(t) + \cancel{Du(t)} + v_2(t) & \text{output equation} \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \left(u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \right) \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

State Noise $v_1(t)$ is a vector white-noise.

$$v_1(t) \sim \text{WN}(0, V_1) \quad v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \\ \vdots \\ v_{1n}(t) \end{bmatrix}$$

and it is called *state noise* or *model noise*. It is used to model immeasurable noises affecting the system and small modelling errors.

Properties of $v_1(t)$:

1. $\mathbb{E}[v_1(t)] = \vec{0}$
2. $\mathbb{E}[v_1(t) \cdot v_1^T(t)] = V_1$, where V_1 is an $n \times n$ covariance matrix (square, symmetric and semi-definite positive by definition)
3. $\mathbb{E}[v_1(t) \cdot v_1^T(t - \tau)] = 0 \quad \forall t \quad \forall \tau \neq 0$ (*whiteness* property)

Output Noise $v_2(t)$ is a vector white-noise.

$$v_2(t) \sim \text{WN}(0, V_2) \quad v_2(t) = \begin{bmatrix} v_{21}(t) \\ v_{22}(t) \\ \vdots \\ v_{2p}(t) \end{bmatrix}$$

and it is called *output noise* or *sensor noise*. It is the noise affecting the output sensor measurements.

Properties of $v_2(t)$:

1. $\mathbb{E}[v_2(t)] = \vec{0}$
2. $\mathbb{E}[v_2(t) \cdot v_2^T(t)] = V_2$, where V_2 is a $p \times p$ covariance matrix (square, symmetric and semi-definite positive¹ by definition)

¹We make the **assumption that V_2 is definite positive** (i.e. $V_2 > 0$) because we will need this property in the Riccati equation.

$$3. \mathbb{E}[v_2(t) \cdot v_2^T(t - \tau)] = 0 \quad \forall t \quad \forall \tau \neq 0 \text{ (whiteness property)}$$

Assumptions about the relationships between $v_1(t)$ and $v_2(t)$:

$$\mathbb{E}[v_1(t) \cdot v_2^T(t - \tau)] = \begin{cases} 0 & \text{if } \tau \neq 0 \\ V_{12} & \text{if } \tau = 0 \end{cases} \quad \clubsuit$$

where V_{12} is a cross-correlation matrix of size $n \times p$.

The system (\clubsuit) means that v_1 and v_2 can be correlated only at the same time, but, in practice, $V_{12} = 0$ is the most common assumption. Thus, in practice

$$\mathbb{E}[v_1(t) \cdot v_2^T(t - \tau)] = 0 \quad \forall t \forall \tau \quad (1^{st} \text{ assumption})$$

Since the system \mathcal{S} is dynamic we need to define the (in this case, probabilistic) initial conditions:

$$\mathbb{E}[x(1)] = \underbrace{X_0}_{n \times 1} \quad \mathbb{E}[(x(1) - X_0)(x(1) - X_0)^T] = \underbrace{P_0}_{n \times n} \geq 0$$

If the covariance matrix $P_0 = 0$ the initial state is perfectly known.

Finally we assume that the two noises $v_1(t)$ and $v_2(t)$ are uncorrelated with the initial state:

$$x(1) \perp v_1(t) \quad x(1) \perp v_2(t) \quad (2^{nd} \text{ assumption})$$

3.2.2 KF Basic Solution of the Basic System

$$\text{State eq.:} \quad \hat{x}(t+1|t) = F\hat{x}(t|t-1) + K(t)e(t) \quad (3.1)$$

$$\text{Output eq.:} \quad \hat{y}(t|t-1) = H\hat{x}(t|t-1) \quad (3.2)$$

$$\text{Prediction output error eq.:} \quad e(t) = y(t) - \hat{y}(t|t-1) \quad (3.3)$$

$$\text{Gain of the filter:} \quad K(t) = (FP(t)H^T + V_{12}) (HP(t)H^T + V_2)^{-1} \quad (3.4)$$

$$\text{DRE:} \quad P(t+1) = (FP(t)F^T + V_1) - (FP(t)H^T + V_{12}) (HP(t)H^T + V_2)^{-1} (FP(t)H^T + V_{12})^T \quad (3.5)$$

Since 3.1 and 3.5 are dynamical equations, two initial conditions are needed:

$$\text{Init. condition for 3.1:} \quad \hat{x}(1|0) = \mathbb{E}[x(1)] = X_0 \quad (3.6)$$

$$\text{Init. condition for 3.5:} \quad P(1) = \text{var}[x(1)] = P_0 \quad (3.7)$$

DEFINITION 3.1 — Difference Riccati Equation (DRE). The equation 3.5 is called *Difference Riccati Equation (DRE)* and it is a special type of non-linear matrix difference equation.

Note The Difference Riccati Equation (DRE) is an autonomous (i.e. there are no inputs), non-linear, discrete time, multi-variable system, described by a non-linear difference matrix equation

$$P(t+1) = f_{NL}(P(t)) \quad P(1) = P_0$$

REMARK — Structure of $K(t)$ and DRE. Notice that $K(t)$ and the DRE have a *block-structure* having this form:

$$AP(t)B^T + N \quad \text{where } N \text{ is a noise matrix}$$

There are 3 different types of blocks:

$$\begin{array}{lll} \text{state:} & FP(t)F^T + V_1 & (\text{since } F \text{ refers to 3.1}) \\ \text{output:} & HP(t)H^T + V_2 & (\text{since } H \text{ refers to 3.2}) \\ \text{mix:} & FP(t)H^T + V_{12} & \end{array}$$

Therefore

$$\begin{array}{ll} 3.4 \text{ become:} & K(t) \equiv (\text{mix})(\text{output})^{-1} \\ 3.5 \text{ become:} & P(t+1) \equiv (\text{state}) - (\text{mix})(\text{output})^{-1}(\text{mix})^T \end{array}$$

REMARK — Existence of the DRE. In order to guarantee the existence of the DRE for all time instant t , the only critical part is the inversion of the **output** block:

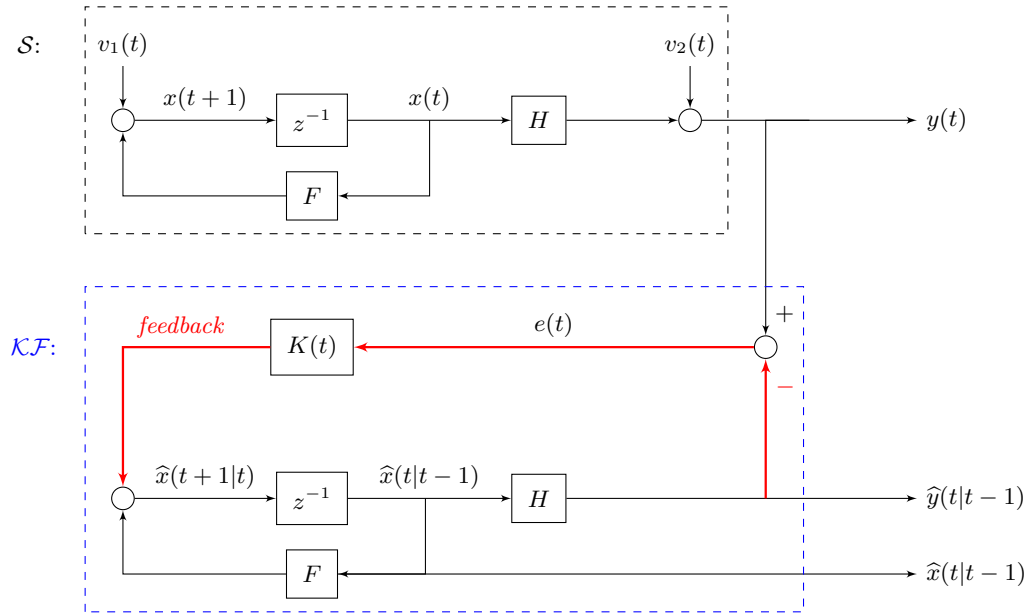
$$\underbrace{\underbrace{(HP(t)H^T + V_2)}_{\geq 0}}_{> 0}^{-1} \quad \text{thanks to } V_2 > 0, \text{ output is always invertible}$$

REMARK — Meaning of $P(t)$. The symmetric $n \times n$ matrix $P(t)$ has a very important meaning, indeed

$$\begin{aligned} P(t) &= \mathbb{E}[(x(t) - \hat{x}(t|t-1))(x(t) - \hat{x}(t|t-1))^T] = \text{Var}[x(t) - \hat{x}(t|t-1)] \\ &= \text{Var}[e_x(t)] \end{aligned}$$

Therefore, $P(t)$ is the covariance matrix of the 1-step prediction error of the state $x(t)$.

3.2.3 Block-scheme representation of the Kalman Filter



The idea behind Kalman Filter is simple and intuitive:

- we make a simulated replica (*digital twin*) of the system (without noises v_1 and v_2 which are not measurable)
- we compare the true measured output with the estimated/predicted output $\hat{y}(t|t-1)$
- we make corrections on the KF main equation, proportional (with gain $K(t)$) to the output error $e(t)$ in order to keep KF as close as possible to the system
- we extract the state estimation $\hat{x}(t|t-1)$ from the digital twin

OBSERVATION. Kalman Filter is a feedback system. Feedback here is not used for control, but for estimation.

This general structure was known since '30 (before Kalman Filter development) and was called *state observer*. Fundamental contribution of Kalman was to find the **optimal gain** $K(t)$. $K(t)$ is not a simple scalar gain but is a (maybe very large) $n \times p$ matrix.

The selection of gain matrix $K(t)$ is very critical:

- if $K(t)$ is *too small*: the estimation is not optimal because we are *under exploiting* the information in $y(t)$
- if $K(t)$ is *too big*: risk of over-exploiting $y(t)$ and we can get noise amplification, even risk of instability

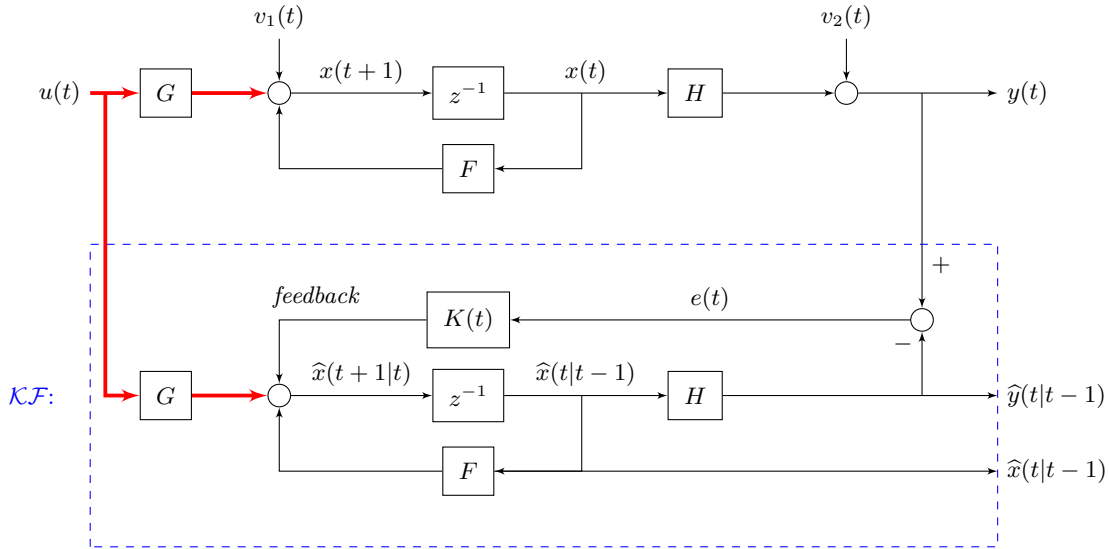
Design of a Kalman Filter does not require a *training dataset*, but a complete model of the system:

- F , G , H matrixes: usually obtained with a white-box physical modelling of the system
- V_1 , V_2 and V_{12} : V_2 is easily built from sensor specifications, while V_1 is much more

difficult to be designed (it's the most critical design parameter of KF). For what concerns V_{12} , we recall that in practice $V_{12} = 0$.

3.3 Extensions of Basic Problem of Basic System

3.3.1 Exogenous input



Notice that $K(t)$ remains the same because $P(t)$ is the covariance of the prediction error on $x(t)$ which remains the same because $Gu(t)$ doesn't introduce any additional noise or uncertainties to the system. That's because $Gu(t)$ is a totally known (deterministic) signal.

3.3.2 Multi-step Prediction

Assuming that $\hat{x}(t+1|t)$ is known from the basic solution, we can simply obtain a multi-step prediction as:

$$\begin{aligned}\hat{x}(t+2|t) &= F\hat{x}(t+1|t) \\ \hat{x}(t+3|t) &= F\hat{x}(t+2|t) = F^2\hat{x}(t+1|t) \\ &\vdots \\ \begin{cases} \hat{x}(t+k|t) = F^{k-1}\hat{x}(t+1|t) \\ \hat{y}(t+k|t) = H\hat{x}(t+k|t) \end{cases}\end{aligned}$$

3.3.3 Filter ($\hat{x}(t|t)$)

$$\hat{x}(t+1|t) = F\hat{x}(t|t) \implies \hat{x}(t|t) = F^{-1}\hat{x}(t+1|t)$$

This formula can be used only if F is invertible. If F is not invertible, the filter can be obtained with a specific *filter* formulation of KF

DEFINITION 3.2 — Kalman Filter in Filter Form. Reformulation of the equations described in 3.2.2.

State equation (3.5) and Gain equation (3.4) become

$$\begin{aligned}\hat{x}(t|t) &= F\hat{x}(t-1|t-1) + Gu(t-1) + K_0(t)e(t) \\ K_0(t) &= (P(t)H^T) (HP(t)H^T + V_2)^{-1}\end{aligned}$$

while the DRE equation (3.5) remains unchanged.

The initial condition for the new State equation is

$$\hat{x}(1|1) = X_0$$

REMARK. These equations are valid under the (legit) assumption $V_{12} = 0$.

OBSERVATION. Gain of KF in prediction form (eq. 3.4 assuming $V_{12} = 0$):

$$K(t) = (FP(t)H^T) (HP(t)H^T + V_2)^{-1}$$

Gain of KF in filter form:

$$K_0(t) = (P(t)H^T) (HP(t)H^T + V_2)^{-1}$$

Therefore, the only difference is the elimination of F .

3.3.4 Time-varying systems

$$F \mapsto F(t)$$

$$G \mapsto G(t)$$

$$H \mapsto H(t)$$

Therefore

$$\mathcal{S} : \begin{cases} x(t+1) = F(t)x(t) + G(t)u(t) + v_1(t) \\ y(t) = H(t)x(t) + v_2(t) \end{cases}$$

DEFINITION 3.3 — Linear Time Varying (LTV) system. The system \mathcal{S} is a *Linear Time Varying (LTV)* system, since its parameters (F, G, H) depends on the time instant t .

Kalman Filter equations remain of the same form as the ones described in 3.2.2: we just need to replace the parameter matrices (F, G, H) with the time-varying ones $(F(t), G(t), H(t))$.

3.3.5 Non-Linear Systems

This extensions is much more complicated. We will see Extended Kalman Filter (EKF) in section 3.4.2.

3.4 Asymptotic Solution of Kalman Filter

OBSERVATION. Even when the system \mathcal{S} is an LTI, the Kalman Filter is not itself an LTI system: it is an LTV system, since it depends on the gain $K(t)$ which is time-varying.

The fact that KF is an LTV system is the source of 2 problems:

- Checking the asymptotic stability of KF algorithm is very difficult, since the stability check of an LTV system is not simple as the stability check for LTI.
- Computational problem: $K(t)$ must be computed at each sampling time (e.g. every 5ms), including the inversion of $HP(t)H^T + V_2$ ($p \times p$ matrix) and the computation of $P(t)$ using the DRE.

RECALL — Asymptotic Stability of a system. If the system is:

- LTI: $x(t+1) = Fx(t) + Gu(t)$
the stability check considers only the sign of the eigenvalues of F .
- LTV: $x(t+1) = F(t)x(t) + G(t)u(t)$
even if all the eigenvalues of $F(t)$ are strictly inside the unit circle at any time, the system is not guaranteed to be asymptotically stable. In practice it is, if the time-variations are *slow* (e.g. aging).

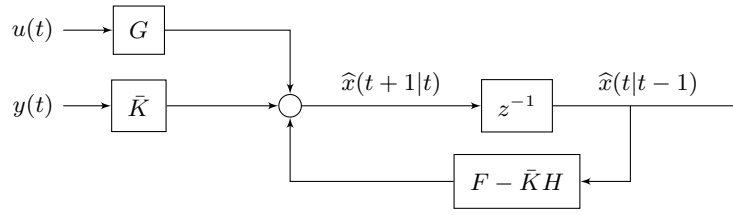
Because of those problems in real/practical applications the asymptotic version of KF is preferred.

Basic idea Since the dependency on t of $K(t)$ derives from the presence of $P(t)$, if $P(t)$ converges to a constant value \bar{P} (steady-state value of $P(t)$), then also $K(t)$ will converge to \bar{K} (steady-state value of $K(t)$).

Using \bar{K} the KF becomes an LTI system.

Let's analyze the asymptotic stability of the State equation (eq. 3.1) of the asymptotic KF when \bar{K} is used (assuming it exists).

$$\begin{aligned}
 \hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) + \bar{K}e(t) \\
 &= F\hat{x}(t|t-1) + Gu(t) + \bar{K}(y(t) - \hat{y}(t|t-1)) \\
 &= F\hat{x}(t|t-1) + Gu(t) + \bar{K}(y(t) - H\hat{x}(t|t-1)) \\
 &= \underbrace{(F - \bar{K}H)}_{\text{new state matrix}} \hat{x}(t|t-1) + Gu(t) + \bar{K}y(t)
 \end{aligned}$$



Asymptotic Kalman Filter with Exogenous input

Condition for Asymptotic Stability of KF If \bar{K} exists, the KF is asymptotically stable if and only if all the eigenvalues of $F - \bar{K}H$ are strictly inside the unit circle.

OBSERVATION. The stability of the system \mathcal{S} is related to matrix F , whereas the stability of KF is related to matrix $F - \bar{K}H$.

Therefore, KF can be asymptotically stable even if the system is unstable.

Existence of \bar{K} Starting from the equation 3.4, if exists $P(t)$ such that $P(t) = \bar{P}$ then $K(t)$ becomes

$$\bar{K} = (F\bar{P}H^T + V_{12}) (H\bar{P}H^T + V_2)^{-1}$$

Thus, \bar{K} exists if \bar{P} exists.

In order to check that, we need to check the converge properties of DRE

RECALL — Stability of a dynamical autonomous system. How to find the equilibrium points of a dynamical autonomous system?

Continuous time	Discrete time
$\dot{x} = f(x(t))$	$x(t+1) = f(x(t))$
equilibrium when $\dot{x} = 0$	equilibrium when $x(t+1) = x(t)$
\downarrow	\downarrow
$f(\bar{x}) = 0$	$f(\bar{x}) = \bar{x}$

DRE is an autonomous discrete time system, thus we impose $\bar{P} = f(\bar{P})$, where $f(\bar{P})$ is the DRE (eq. 3.5) evaluated in \bar{P} :

$$\text{ARE: } \bar{P} = (F\bar{P}F^T + V_1) - (F\bar{P}H^T + V_{12}) (H\bar{P}H^T + V_2)^{-1} (F\bar{P}H^T + V_{12})^T \quad (3.8)$$

DEFINITION 3.4 — Algebraic Riccati Equation (ARE). The equation 3.8 is a non-linear, matrix, static algebraic equation, known as *Algebraic Riccati Equation (ARE)*.

If a steady state \bar{P} solution of DRE (eq. 3.5) does exist, it must be a solution of A.R.E (eq. 3.8). There remains 3 questions:

1. **Existence:** does Algebraic Riccati Equation (ARE) have a semi-definite positive solution?
2. **Convergence:** if exists, does the DRE converges to \bar{P} ?
3. **Stability:** is the corresponding \bar{K} such that the KF is asymptotically stable?

To answer those questions we need two fundamental theorems (KF asymptotic theorems).

3.4.1 Asymptotic Kalman Filter Theorems

The two Asymptotic Kalman Filter Theorems that we are going to introduce provide *sufficient* conditions only.

THEOREM 3.5 — First Asymptotic KF Theorem. Assumptions: $V_{12} = 0$ and the system is asymptotically stable (i.e. all eigenvalues of F are strictly inside the unit circle). Then:

1. ARE has one and only one semi-definite positive solution: $\bar{P} \geq 0$.
2. DRE converges to \bar{P} , $\forall P_0 \geq 0$ (P_0 : initial semi-definite positive condition).
3. The corresponding \bar{K} is s.t. the KF is asymptotically stable.

RECALL — Observability and Controllability. Recall on Observability and Controllability needed for the introduction of the Second Asymptotic KF Theorem (3.6).

Observability of the state through the output The pair (F, H) is observable if and only if

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \quad \text{is full rank}$$

Controllability from the noise We are interested in controllability from $v_1(t)$ (and not from $u(t)$).

$$x(t+1) = Fx(t) + \cancel{Gu(t)} + v_1(t) \quad v_1(t) \sim WN(0, V_1)$$

It's always possible to factorize $V_1 = \Gamma \cdot \Gamma^T$ rewriting

$$x(t+1) = Fx(t) + \Gamma\omega(t) \quad \omega(t) \sim WN(0, I)$$

We can say that the state x is controllable/reachable from the input noise $v_1(t)$ if and only if:

$$R = [\Gamma \quad F\Gamma \quad \dots \quad F^{n-1}\Gamma] \quad \text{is full rank}$$

THEOREM 3.6 — Second Asymptotic KF Theorem. Assumptions: $V_{12} = 0$, (F, H) is observable and (F, Γ) is controllable. Then:

1. ARE has one and only one definite positive solution $\bar{P} > 0$.
2. DRE converges to \bar{P} , $\forall P_0 \geq 0$ (P_0 : initial semi-definite positive condition).
3. The corresponding \bar{K} is such that the KF is asymptotically stable.

OBSERVATION. The difference between theorems 3.5 and 3.6 is that the first one ensures that $\bar{P} \geq 0$ while the second one ensures that $\bar{P} > 0$.

These two theorems are very useful in practice because we can fully avoid the (very difficult) direct convergence analysis of DRE.

EXAMPLE.

$$S: \begin{cases} x(t+1) = \frac{1}{2}x(t) + v_1(t) & v_1 \sim WN(0, \frac{19}{20}) \\ y(t) = 2x(t) + v_2(t) & v_2 \sim WN(0, 1) \end{cases} \quad v_1 \perp v_2$$

Question Find (if exists) the steady state (asympt.) K.F. $\hat{x}(t+1|t)$ and $\hat{x}(t|t)$.

$$n = 1 \quad F = \frac{1}{2} \quad G = 0 \quad H = 2 \quad V_1 = \frac{19}{20} \quad V_2 = 1 \quad V_{12} = 0$$

Since $V_{12} = 0$ we can try to use the asymptotic theorems.

First step Compute the DRE

$$\begin{aligned} P(t+1) &= (FP(t)F^T + V_1) - (FP(t)H^T + V_{12}) (HP(t)H^T + V_2)^{-1} (FP(t)H^T + V_{12})^T \\ &= \frac{1}{4}P(t) + \frac{19}{20} - \frac{\left(\frac{1}{2}P(t)\right)^2}{4P(t) + 1} \\ &= \frac{\cancel{P(t)^2} + \frac{1}{4}P(t) + \frac{19}{5}P(t) + \frac{19}{20} - \cancel{P(t)^2}}{4P(t) + 1} \end{aligned}$$

Note The second order terms must cancel out.

$$P(t+1) = \frac{81P(t) + 19}{80P(t) + 20}$$

Second step Compute and solve the ARE

$$\bar{P} = \frac{81\bar{P} + 19}{80\bar{P} + 20} \implies 80\bar{P}^2 + 20\bar{P} - 81\bar{P} - 19 = 0$$

$$\bar{P}_1 = 1 \quad \bar{P}_2 = -\frac{19}{80} < 0$$

$\bar{P} = 1$ is the only definite positive solution of ARE

Question Does DRE converges to $\bar{P} = 1$, $\forall P_0 \geq 0$?

There are 2 methods for addressing this question:

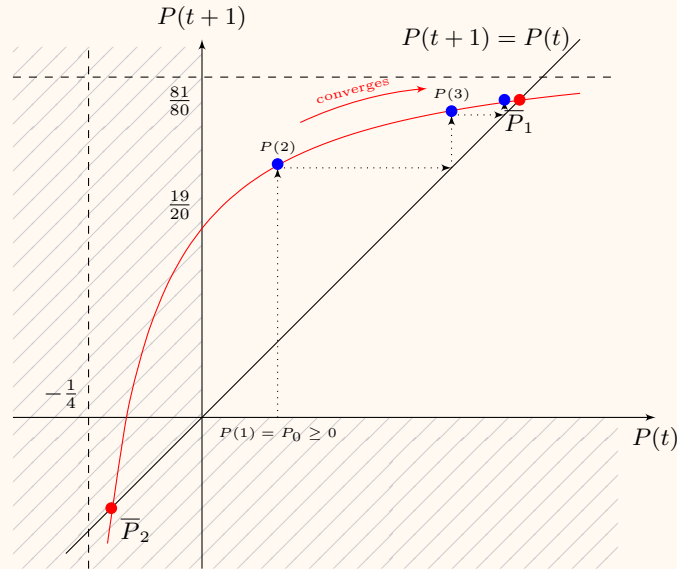
Direct analysis of DRE

- Using asymptotic theorems

First methods Direct analysis of DRE

$P(t+1) = f(P(t))$ we need to plot $f(\cdot)$ in the $P(t) - P(t+1)$ plane

$$\bar{P} = \frac{81\bar{P} + 19}{80\bar{P} + 20} \quad \begin{cases} \text{vertical asy. value} & P(t) = -\frac{20}{80} = -\frac{1}{4} \\ \text{horizontal asy. value} & P(t) = \frac{81}{80} \end{cases}$$



By direct analysis/inspection of DRE dynamics we can conclude that $\forall P_0 \geq 0$, DRE always converges to $\bar{P}_1 = 1$.

If $n = 1$ the direct inspection is feasible, but it's very difficult for $n \geq 2$.

Second method Use theorems

$$V_{12} = 0 \quad F = \frac{1}{2} \text{ (S is stable)} \quad \Rightarrow \quad \text{First theorem is fulfilled}$$

The observability matrix of $\{F, H\}$ is $O = [2]$ with full rank $O = 1$, the system is fully observable.

Controllability from noise $v_1(t)$

$$V_1 = \frac{19}{20} \quad \Gamma = \sqrt{\frac{19}{20}}$$

The controllability matrix of $\{F, \Gamma\}$ is $R = \left[\sqrt{\frac{19}{20}} \right]$ with full rank $R = 1$, the system is fully controllable from noise.

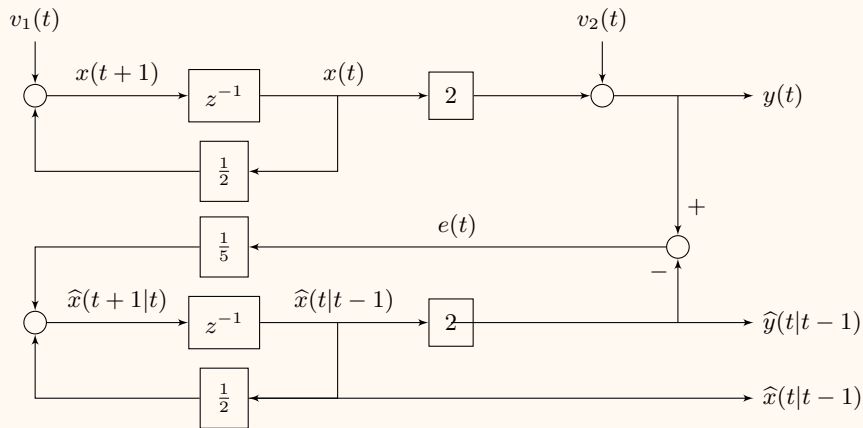
Both theorems are fulfilled, so ARE has one and only one solution $\bar{P} > 0$, DRE converges to \bar{P} , $\forall P_0 \geq 0$ and \bar{K} makes the K.F. asymptotically stable.

Third step Compute \bar{K}

$$\bar{K} = (F\bar{P}H^T + V_{12}) (H\bar{P}H^T + V_2)^{-1} = \left(\frac{1}{2} \cdot 1 \cdot 2 + 0 \right) (2 \cdot 1 \cdot 2 + 1)^{-1} = \frac{1}{5}$$

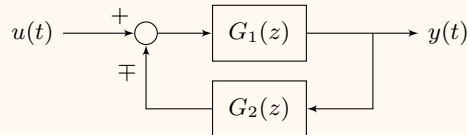
Double-check the asymptotical stability of K.F.

$$F - \bar{K}H = \frac{1}{2} - \frac{1}{5} \cdot 2 = \frac{5-4}{10} = \frac{1}{10}$$



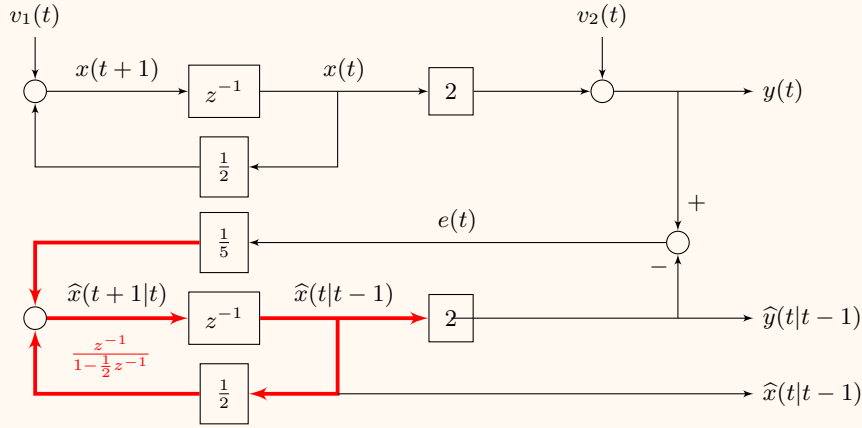
Question Find T.F. from $y(t)$ to $\hat{x}(t|t-1)$

Recall T.F. from block schemes of feedback systems.



$$y(t) = G_1(z) (u(t) \mp G_2(z)y(t)) \implies y(t) = \frac{G_1(z)}{1 \pm G_1(z)G_2(z)} u(t)$$

The K.F. is composed of 2 nested loops.



Predictor of state:

$$\hat{x}(t|t-1) = \frac{\frac{1}{5} \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}}{1 + \frac{1}{5} \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}} 2} y(t) = \frac{\frac{1}{5} z^{-1}}{1 - \frac{1}{10} z^{-1}} y(t)$$

Predictor of output:

$$\hat{y}(t|t-1) = H \hat{x}(t|t-1) = \frac{2}{5} \frac{1}{1 - \frac{1}{10} z^{-1}} y(t-1)$$

Filter of state:

$$\hat{x}(t|t) = F^{-1} \hat{x}(t+1|t) = \frac{2}{5} \frac{1}{1 - \frac{1}{10} z^{-1}} y(t)$$

REMARK — White noise. In the formulas of Kalman Filter there is a requirement that $v_1(t)$ and $v_2(t)$ must be white noises. In many practical applications this assumption can be too demanding.

We need a workaround to deal with practical applications where this assumption is not valid. The workaround is a simple trick called *state extension* and it consists on extending the model of S to incorporate the *noise dynamics*.

Let's see how it works with a numerical example.

Example Given a system \mathcal{S} of order $n = 1$

$$\mathcal{S} : \begin{cases} x(t+1) = ax(t) + \eta(t) & \eta(t) \text{ is not a white noise} \\ y(t) = bx(t) + v_2(t) & v_2(t) \sim \text{WN}(0, 1) \end{cases}$$

and a model of $\eta(t)$ (AR(1) stochastic model)

$$\eta(t) = \frac{1}{1 - cz^{-1}} e(t) \quad e(t) \sim \text{WN}(0, 1) \quad e \perp v_2$$

We cannot apply KF formula to this system, since $\eta(t)$ is not a WN, but we can proceed as follows

$$\begin{aligned} \eta(t) = c\eta(t-1) + e(t) &\xrightarrow{z} \eta(t+1) = c\eta(t) + e(t+1) \\ \eta(t+1) &= c\eta(t) + v(t) \end{aligned}$$

where $v(t) = e(t+1)$, $v \sim \text{WN}(0, 1)$ and $v \perp v_2$.

Trick Extension of the state vector.

$$\begin{aligned} x(t) &\mapsto x_1(t) \\ \eta(t) &\mapsto x_2(t) \end{aligned}$$

Now $n = 2$ and \mathcal{S} becomes \mathcal{S}'

$$\mathcal{S}' : \begin{cases} x_1(t+1) = ax_1(t) + x_2(t) \\ x_2(t+1) = cx_2(t) + v(t) \\ y(t) = bx_1(t) + v_2(t) \end{cases}$$

$$F = \begin{bmatrix} a & 1 \\ 0 & c \end{bmatrix} \quad H = \begin{bmatrix} b & 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 0 \\ v(t) \end{bmatrix} \quad V_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad v_2 \sim \text{WN}(0, 1) \quad V_{12} = 0$$

Now we can apply KF solution equations (3.2.2) to this system.

3.4.2 Extension to Non-Linear systems

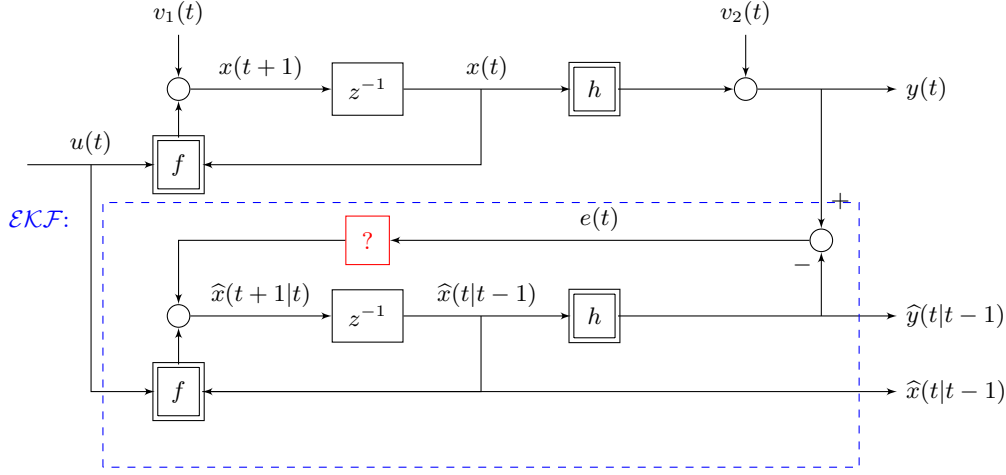
Coming back to extensions of the KF for the basic system (section 3.3), we consider a system with non-linear dynamics:

$$\mathcal{S} : \begin{cases} x(t+1) = f(x(t), u(t)) + v_1(t) \\ y(t) = h(x(t)) + v_2(t) \end{cases}$$

where f and h are non-linear functions of $x(t)$ and $u(t)$ (smoothness class C^1 or higher), for example

$$f(x(t), u(t)) = \frac{1}{2}x^5(t) + u^3(t), \quad h(x(t)) = e^{x(t)} + v_2(t)$$

How can we design a Kalman Filter in this case? We can follow the general idea of the *state observer*.



For the gain block of KF we have 2 different types of solutions:

1. The gain is a non-linear function of $e(t)$ (most natural and intuitive solution)
2. The gain is a linear time-varying function

The second solution is less intuitive but is the most effective: we can reuse most of the KF theory's formulas for the LTV domain.

In practice, Extended Kalman Filter (EKF) idea is to make a time-varying local linearization (approximation) of the non-linear time-invariant system at each sampling time.

The gain equation $K(t)$ in Extended Kalman Filter can be computed as:

$$K(t) = (F(t)P(t)H(t)^T + V_{12}) (H(t)P(t)H(t)^T + V_2)^{-1}$$

and $P(t)$ can be computed from the DRE:

$$P(t+1) = (F(t)P(t)F(t)^T + V_1) - (F(t)P(t)H(t)^T + V_{12}) (H(t)P(t)H(t)^T + V_2)^{-1} (F(t)P(t)H(t)^T + V_{12})^T$$

OBSERVATION. Equations of K and DRE are the usual formulas of KF (3.2.2) with the introduction of time-varying parameters (as said in 3.3.4).

Linearization of the system We linearize f and h by computing at each sampling time, respectively, $F(t)$ and $H(t)$ as:

$$F(t) = \left. \frac{\partial f(x(t), u(t))}{\partial x(t)} \right|_{x(t)=\hat{x}(t|t-1)}$$

$$H(t) = \left. \frac{\partial h(x(t))}{\partial x(t)} \right|_{x(t)=\hat{x}(t|t-1)}$$

Therefore, EKF is the time-varying solution of KF where $F(t)$ and $H(t)$ are local linearized matrices computed around the last available state prediction $\hat{x}(t|t-1)$.

Summary Procedure to implement EKF at time t :

1. take last available state prediction $\hat{x}(t|t-1)$
2. using $\hat{x}(t|t-1)$, linearize the system by computing $F(t)$ and $H(t)$
3. compute $K(t)$ and update the DRE
4. compute $\hat{x}(t+1|t)$

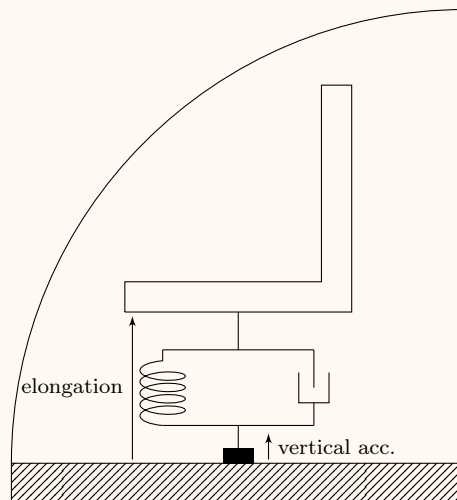
REMARK. Main issues of EKF (same of LTV KF):

- Very difficult (almost impossible) to have a theoretical guarantee of EKF stability (in practice extensive empirical testing is used).
- Computational load (at each time $F(t)$, $H(t)$, $K(t)$ and $P(t)$ must be computed at run-time).

However, EKF is largely used today with some limitations in:

- safety-critical applications
- mission-critical applications

EXAMPLE — KF full procedure. Suspended seat in the cabin of an off-highway vehicle (agriculture tractor, earth-moving machine, etc...).

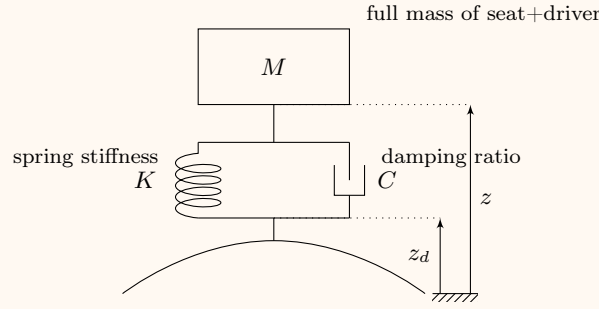


The physical sensors are:

- Vertical accelerometer placed at the basis of the cabin
- Elongation sensor between the basis of the cabin and the seat

Problem Estimation of the seat vertical speed.

Step 1: System Modeling Move from a pictorial representation to a schematic representation of it:



The sensor are:

- Acceleration \ddot{z}_d (+noise)
- Elongation $z - z_d$ (+noise)

Model of the system dynamics (physical white-box model in continuous time domain).

Core model equation (force balance in vertical direction):

$$\begin{aligned}
 M\ddot{z} &= \underbrace{-C \frac{d}{dt}(z - z_d)}_{\text{damping}} - \underbrace{K(z - z_d)}_{\text{spring}} + \cancel{\text{gravity}} = \cancel{Mg} \\
 &= -C(\dot{z} - \dot{z}_d) - K(z - z_d)
 \end{aligned}$$

Since we measure \ddot{z}_d the overall dimension of the system is 4. The vector of state variables is:

$$x(t) = \begin{bmatrix} z \\ \dot{z} \\ z_d \\ \dot{z}_d \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \quad u(t) = \ddot{z}_d \quad y(t) = z - z_d$$

We can write the full model in state space form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{C}{M}(x_2 - x_4) - \frac{K}{M}(x_1 - x_3) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \\ y = x_1 - x_3 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{K}{M}x_1 - \frac{C}{M}x_2 + \frac{K}{M}x_3 + \frac{C}{M}x_4 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u \\ y = x_1 - x_3 \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M} & -\frac{C}{M} & \frac{K}{M} & \frac{C}{M} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0 \quad -1 \quad 0]$$

Step 2: Discretization Next is discretization (we use digital systems), choice of sampling time Δ (for this application can be 5ms).

Using *Eulero-forward approximation* of time derivative $\dot{x}(t) \approx \frac{x(t+1) - x(t)}{\Delta}$.

Example for the first equation:

$$\frac{x_1(t+1) - x_1(t)}{\Delta} = x_2(t) \Rightarrow x_1(t+1) = x_1(t) + \Delta x_2(t)$$

Step 3: Add Noises In order to apply KF formulas we need to add noises v_1, v_2 to the model. The discretized system with noises is:

$$\mathcal{S} : \begin{cases} x_1(t+1) = x_1(t) + \Delta x_2(t) + v_{11}(t) \\ x_2(t+1) = -\frac{\Delta K}{M} x_1(t) + \left(-\frac{\Delta C}{M} + 1\right) x_2(t) + \frac{\Delta K}{M} x_3(t) + \frac{\Delta C}{M} x_4(t) + v_{12}(t) \\ x_3(t+1) = x_3(t) + \Delta x_4(t) + v_{13}(t) \\ x_4(t+1) = x_4(t) + \Delta u(t) + v_{14}(t) \\ y(t) = x_1(t) - x_3(t) + v_2(t) \end{cases}$$

This is normal state-space in discrete time:

$$\mathcal{S} : \begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + \mathcal{B}u(t) \end{cases}$$

$$F = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ -\frac{\Delta K}{M} & -\frac{\Delta C}{M} + 1 & \frac{\Delta K}{M} & \frac{\Delta C}{M} \\ 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta \end{bmatrix} \quad H = [1 \quad 0 \quad -1 \quad 0]$$

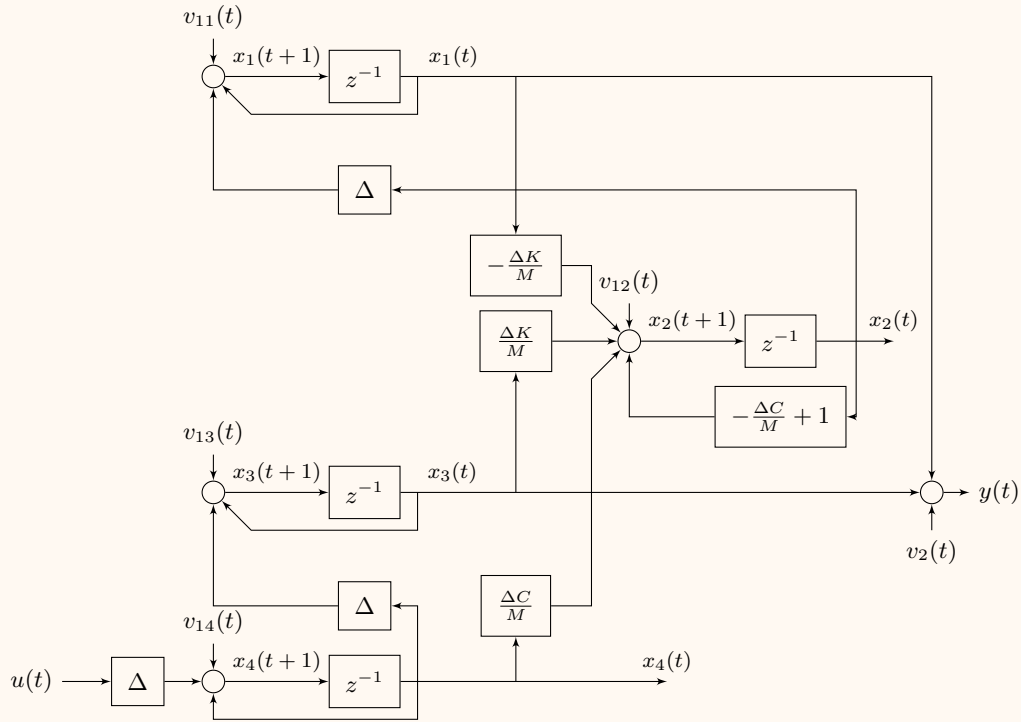
Noise on the state equations:

$$v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \\ v_{13}(t) \\ v_{14}(t) \end{bmatrix} \sim WN(0, V_1) \quad V_1 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & \lambda_1^2 \end{bmatrix} \quad v_2(t) \sim WN(0, V_2)$$

Assumptions:

- All white-noises
- All uncorrelated

V_2 can be estimated by datasheet of elongation sensor, λ_4 can be estimated by datasheet of accelerometer. We expect λ_1, λ_2 and λ_3 to be small, and we can use the simplifying assumption $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda^2$, estimated empirically.



We can compute

$$O = \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^3 \end{bmatrix} \quad R = [G \quad FG \quad F^2G \quad F^3G]$$

From visual inspection of the block schema we expect full observability from output and full controllability from input.

KF at this point can be applied. Theorem 3.5 and 3.6 are applicable, thus we can directly jump to ARE solution.