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Port-Hamiltonian flexible multibody dynamics

Andrea Brugnoli

[1]ISAE-SUPAERO, Toulouse

Outline

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Previous wok

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Disadvantages:

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Floating frame based approach

The floating frame approach relies on the hypothesis of small deformations: elastic motion is described w.r.t a reference that follows the large rigid motion.

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Advantages

- The most used paradigm in multibody dynamics;
- For control applications most references adopt this approach;
- Model reduction techniques are applicable.

Disadvantages:

- Effect due to geometric non-linearities are not considered: not suitable for large deformation (substructuring can be employed to describe large deformations).

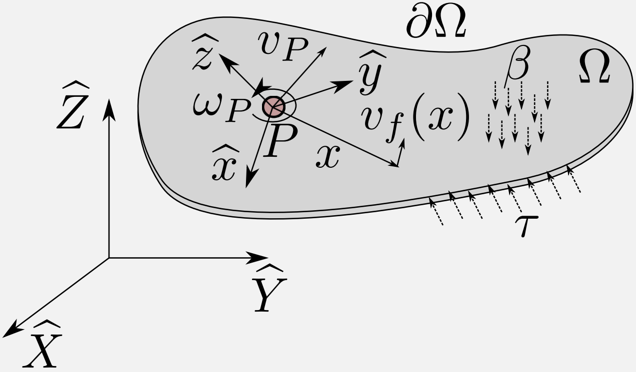

Figure: Thin floating body undergoing a surface traction $\boldsymbol{\tau}$ and body force density $\boldsymbol{\beta}$

Floating body kinematics

The velocity of a generic point is expressed by considering a small flexible displacement superimposed to the rigid motion

$$\boldsymbol{v} = \boldsymbol{v}_P + [\boldsymbol{\omega}_P]_\times (\boldsymbol{x} + \boldsymbol{u}_f) + \boldsymbol{v}_f.$$

This equation is expressed in the body reference frame $\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}$.

- $\boldsymbol{x}$ is the position vector of the current point;
- $\boldsymbol{v}_P$, $\boldsymbol{\omega}_P$ are the linear and angular velocities of point P;
- $\boldsymbol{v}_f := \dot{\boldsymbol{u}}_f$ is the time derivative of the deformation displacement $\boldsymbol{u}_f$ (computed in the body frame);
- The cross map $[\boldsymbol{a}]_\times$ denotes the skew-symmetric matrix associated to vector $\boldsymbol{a}$.

Equations of motion

The equations are obtained by application of the virtual work principle[3].
Linear momentum balance:

$$m(\dot{\boldsymbol{v}}_P + [\boldsymbol{\omega}_P]_\times \boldsymbol{v}_P) + [\boldsymbol{s}_u]_\times^\top \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho \ddot{\boldsymbol{u}}_f \, \mathrm{d}\Omega =$$
$$- [\boldsymbol{\omega}_P]_\times [\boldsymbol{\omega}_P]_\times \boldsymbol{s}_u - \int_\Omega 2\rho [\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f \, \mathrm{d}\Omega + \int_\Omega \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, \mathrm{d}\Gamma,$$

- ρ is the mass density;
- $m = \int_\Omega \rho \, \mathrm{d}\Omega$ the total mass;
- $\boldsymbol{s}_u = \int_\Omega \rho(\boldsymbol{x} + \boldsymbol{u}_f) \, \mathrm{d}\Omega$ the static moment.

[3] Bernd Simeon. *Computational flexible multibody dynamics*. Springer, 2013, Chapter 4.

Equations of motion

The equations are obtained by application of the virtual work principle[3].
Rearranged linear momentum balance:

$$m\dot{\boldsymbol{v}}_P + [\boldsymbol{s}_u]_\times^\top \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho \dot{\boldsymbol{v}}_f \, \mathrm{d}\Omega =$$

$$\left[m\boldsymbol{v}_P + [\boldsymbol{s}_u]_\times^\top \boldsymbol{\omega}_P + 2\int_\Omega \rho \boldsymbol{v}_f \, \mathrm{d}\Omega \right]_\times \boldsymbol{\omega}_P + \int_\Omega \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, \mathrm{d}\Gamma.$$

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Angular momentum balance:

$$[\boldsymbol{s}_u]_\times(\dot{\boldsymbol{v}}_P + [\boldsymbol{\omega}_P]_\times \boldsymbol{v}_P) + \boldsymbol{J}_u \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \ddot{\boldsymbol{u}}_f \, \mathrm{d}\Omega + [\boldsymbol{\omega}_P]_\times \boldsymbol{J}_u \boldsymbol{\omega}_P =$$
$$- \int_\Omega 2\rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times [\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f \, \mathrm{d}\Omega + \int_\Omega [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\tau} \, \mathrm{d}\Gamma,$$

$\boldsymbol{J}_u := \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times^\top [\boldsymbol{x} + \boldsymbol{u}_f]_\times \, \mathrm{d}\Omega$ is the inertia matrix.

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\begin{aligned}
&[\boldsymbol{s}_u]_\times \dot{\boldsymbol{v}}_P + \boldsymbol{J}_u \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \dot{\boldsymbol{v}}_f \, \mathrm{d}\Omega = \\
&\left[[\boldsymbol{s}_u]_\times^\top \boldsymbol{\omega}_P + 2 \int_\Omega \rho \boldsymbol{v}_f \, \mathrm{d}\Omega \right]_\times \boldsymbol{v}_P + \left[[\boldsymbol{s}_u]_\times \boldsymbol{v}_P + \boldsymbol{J}_u \boldsymbol{\omega}_P + 2 \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{v}_f \, \mathrm{d}\Omega \right]_\times \boldsymbol{\omega}_P + \\
&2 \int_\Omega \left[\rho \boldsymbol{v}_P + \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P \right]_\times \boldsymbol{v}_f \, \mathrm{d}\Omega + \int_\Omega [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\tau} \, \mathrm{d}\Gamma.
\end{aligned}
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Flexibility PDE:

$$\rho(\dot{\boldsymbol{v}}_P + [\boldsymbol{\omega}_P]_\times \boldsymbol{v}_P) + \rho([\dot{\boldsymbol{\omega}}_P]_\times + [\boldsymbol{\omega}_P]_\times[\boldsymbol{\omega}_P]_\times)(\boldsymbol{x} + \boldsymbol{u}_f) + \rho(2[\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f + \ddot{\boldsymbol{u}}_f) = \\ \mathrm{Div}\,\boldsymbol{\Sigma} + \boldsymbol{\beta},$$

together with boundary conditions

$$\begin{aligned} &\text{Neumann condition} \quad && \boldsymbol{\Sigma} \cdot \boldsymbol{n}|_{\Gamma_N} = \boldsymbol{\tau}|_{\Gamma_N}, \quad && \boldsymbol{n} \text{ is the outward normal}, \\ &\text{Dirichlet condition} \quad && \boldsymbol{u}_f|_{\Gamma_D} = \bar{\boldsymbol{u}}_f|_{\Gamma_D}, \end{aligned}$$

- $\boldsymbol{\Sigma}$ is the Cauchy stress tensor;
- The infinitesimal strain is given by $\boldsymbol{\varepsilon} = \mathrm{Grad}(\boldsymbol{u}_f) \quad \mathrm{Grad} = \frac{1}{2}[\nabla + \nabla^\top]$;
- To close the system, Hooke's law $\boldsymbol{\Sigma} = \boldsymbol{\mathcal{D}}\boldsymbol{\varepsilon}$, where $\boldsymbol{\mathcal{D}}$ is the stiffness tensor.

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$$\rho\dot{\boldsymbol{v}}_P + \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top\dot{\boldsymbol{\omega}}_P + \rho\dot{\boldsymbol{v}}_f = \left[\rho\boldsymbol{v}_P + \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top\boldsymbol{\omega}_P + 2\rho\boldsymbol{v}_f\right]_\times \boldsymbol{\omega}_P + \mathrm{Div}\,\boldsymbol{\Sigma} + \boldsymbol{\beta}.$$

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Energies and canonical momenta

Consider the total energy (Hamiltonian), given by the sum of kinetic and deformation energy:

$$\begin{aligned} H &= H_{\mathsf{kin}} + H_{\mathsf{def}}, \\ &= \frac{1}{2}\int_\Omega \left\{\rho ||\boldsymbol{v}_P + [\boldsymbol{\omega}_P]_\times (\boldsymbol{x} + \boldsymbol{u}_f) + \boldsymbol{v}_f||^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon}\right\} \, \mathrm{d}\Omega. \end{aligned}$$

The momenta (usually called energy variables in the pH framework) are then computed by derivation of the Hamiltonian:

$$\begin{aligned} \boldsymbol{p}_t &:= \frac{\partial H}{\partial \boldsymbol{v}_P} = m\boldsymbol{v}_P + [\boldsymbol{s}_u]_\times^\top \boldsymbol{\omega}_P + \int_\Omega \rho \boldsymbol{v}_f \, \mathrm{d}\Omega, \\ \boldsymbol{p}_r &:= \frac{\partial H}{\partial \boldsymbol{\omega}_P} = [\boldsymbol{s}_u]_\times \boldsymbol{v}_P + \boldsymbol{J}_u \boldsymbol{\omega}_P + \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{v}_f \, \mathrm{d}\Omega, \\ \boldsymbol{p}_f &:= \frac{\delta H}{\delta \boldsymbol{v}_f} = \rho \boldsymbol{v}_P + \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P + \rho \boldsymbol{v}_f, \\ \boldsymbol{\varepsilon} &:= \frac{\delta H}{\delta \boldsymbol{\Sigma}} = \boldsymbol{\mathcal{D}}^{-1} \boldsymbol{\Sigma}, \end{aligned}$$

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In matrix form

$$\begin{bmatrix} \boldsymbol{p}_t \\ \boldsymbol{p}_r \\ \boldsymbol{p}_f \\ \boldsymbol{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} m\boldsymbol{I}_{3\times 3} & [\boldsymbol{s}_u]_\times^\top & \mathcal{I}_\rho^\Omega & 0 \\ [\boldsymbol{s}_u]_\times & \boldsymbol{J}_u & \boldsymbol{\mathcal{I}}_{\rho x}^\Omega & 0 \\ (\mathcal{I}_\rho^\Omega)^* & (\boldsymbol{\mathcal{I}}_{\rho x}^\Omega)^* & \rho & 0 \\ 0 & 0 & 0 & \boldsymbol{\mathcal{D}}^{-1} \end{bmatrix}}_{\boldsymbol{\mathcal{M}}:\ \text{Mass operator}} \begin{bmatrix} \boldsymbol{v}_P \\ \boldsymbol{\omega}_P \\ \boldsymbol{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}, \qquad \begin{aligned} \mathcal{I}_\rho^\Omega &:= \int_\Omega \rho(\cdot) \, \mathrm{d}\Omega, \\ \boldsymbol{\mathcal{I}}_{\rho x}^\Omega &:= \int_\Omega \rho[\boldsymbol{x} + \boldsymbol{u}_f]_\times(\cdot). \end{aligned}$$

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The mass operator $\boldsymbol{\mathcal{M}}$ is a self-adjoint, positive operator. It holds

$$H_{\mathsf{kin}} + H_{\mathsf{def}} = \frac{1}{2}\langle \boldsymbol{e}_{\mathsf{kd}}, \, \boldsymbol{\mathcal{M}} \boldsymbol{e}_{\mathsf{kd}} \rangle, \qquad \boldsymbol{e}_{\mathsf{kd}} = [\boldsymbol{v}_P; \, \boldsymbol{\omega}_P; \, \boldsymbol{v}_f; \boldsymbol{\Sigma}]$$

Notice that the kinetic energy also depends on the flexible displacement

$$\frac{\delta H_{\mathsf{kin}}}{\delta \boldsymbol{u}_f} = [\boldsymbol{p}_f]_\times \boldsymbol{\omega}_{\boldsymbol{P}}.$$

This term is responsible for a coupling between the kinematic coordinates and the velocities.

Generalized coordinates are required for a complete formulation:

- ${}^{i}\boldsymbol{r}_P$ the position of point P in the inertial frame of reference;
- $\boldsymbol{R}$ the direction cosine matrix that transforms vectors from the body frame to the inertial frame (other attitude parametrizations are possible);
- $\boldsymbol{u}_f$ the flexible displacement;

The direction cosine matrix is converted into a vector by concatenating its rows

$$\boldsymbol{R}_{\mathsf{v}} = \mathrm{vec}(\boldsymbol{R}^\top) = [\boldsymbol{R}_x \ \boldsymbol{R}_y \ \boldsymbol{R}_z]^\top,$$

where $\boldsymbol{R}_x, \boldsymbol{R}_y, \boldsymbol{R}_z$ are the first, second and third row of matrix $\boldsymbol{R}$. Furthermore the corresponding cross map will be given by

$$[\boldsymbol{R}_{\mathsf{v}}]_\times = \begin{bmatrix} [\boldsymbol{R}_x]_\times \\ [\boldsymbol{R}_y]_\times \\ [\boldsymbol{R}_z]_\times \end{bmatrix}, \qquad [\boldsymbol{R}_{\mathsf{v}}]_\times : \mathbb{R}^9 \rightarrow \mathbb{R}^{9\times 3}.$$

The overall port-Hamiltonian formulation

$$
\underbrace{\left[\begin{array}{c:c} \boldsymbol{I} & 0 \\ \hdashline 0 & \boldsymbol{\mathcal{M}} \end{array}\right]}_{\boldsymbol{\mathcal{E}}} \frac{d}{dt} \underbrace{\begin{bmatrix} {}^{i}\mathbf{r}_P \\ \boldsymbol{R}_{\mathsf{v}} \\ \boldsymbol{u}_f \\ \hdashline \boldsymbol{v}_P \\ \boldsymbol{\omega}_P \\ \boldsymbol{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}}_{\boldsymbol{e}} = \underbrace{\left[\begin{array}{ccc:cccc} 0 & 0 & 0 & \boldsymbol{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\boldsymbol{R}_{\mathsf{v}}]_\times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I}_{3\times 3} & 0 \\ \hdashline -\boldsymbol{R}^\top & 0 & 0 & 0 & [\widetilde{\boldsymbol{p}}_t]_\times & 0 & 0 \\ 0 & -[\boldsymbol{R}_{\mathsf{v}}]_\times^\top & 0 & [\widetilde{\boldsymbol{p}}_t]_\times & [\widetilde{\boldsymbol{p}}_r]_\times & \boldsymbol{\mathcal{I}}_{p_f}^{\Omega} & 0 \\ 0 & 0 & -\boldsymbol{I}_{3\times 3} & 0 & -(\boldsymbol{\mathcal{I}}_{p_f}^{\Omega})^* & 0 & \mathrm{Div} \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Grad} & 0 \end{array}\right]}_{\boldsymbol{\mathcal{J}}} \underbrace{\begin{bmatrix} \partial_{\boldsymbol{r}_P} H \\ \partial_{\boldsymbol{R}_{\mathsf{v}}} H \\ \delta_{\boldsymbol{u}_f} H \\ \hdashline \boldsymbol{v}_P \\ \boldsymbol{\omega}_P \\ \boldsymbol{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}}_{\boldsymbol{z}}.
$$

Variables $\widetilde{\boldsymbol{p}}_t, \widetilde{\boldsymbol{p}}_r$ are defined as:

$$
\widetilde{\boldsymbol{p}}_t = \boldsymbol{p}_t + \int_{\Omega} \rho \boldsymbol{v}_f \, \mathrm{d}\Omega, \qquad \widetilde{\boldsymbol{p}}_r = \boldsymbol{p}_r + \int_{\Omega} \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{v}_f \, \mathrm{d}\Omega.
$$

The operator $\boldsymbol{\mathcal{I}}_{p_f}^{\Omega}$ is defined as: $\boldsymbol{\mathcal{I}}_{p_f}^{\Omega} := \int_{\Omega} \left\{ 2[\boldsymbol{p}_f]_\times + \rho [\boldsymbol{v}_f]_\times \right\} (\cdot) \, \mathrm{d}\Omega.$

This system fits into the framework detailed in[4] and extends it.

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}(\boldsymbol{e})\frac{\partial \boldsymbol{e}}{\partial t} &= \boldsymbol{\mathcal{J}}(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}) + \boldsymbol{\mathcal{B}}_d(\boldsymbol{e})\boldsymbol{u}_d + \boldsymbol{\mathcal{B}}_r(\boldsymbol{e})\boldsymbol{u}_\partial, \qquad \text{where } \boldsymbol{u}_d := \boldsymbol{\beta} \\
\boldsymbol{y}_d &= \boldsymbol{\mathcal{B}}_d^*(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}), \\
\boldsymbol{y}_r &= \boldsymbol{\mathcal{B}}_r^*(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}), \\
\boldsymbol{u}_\partial &= \boldsymbol{\mathcal{B}}_\partial \boldsymbol{z}(\boldsymbol{e}) = \boldsymbol{\Sigma}\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{\tau}|_{\partial\Omega}, \\
\boldsymbol{y}_\partial &= \boldsymbol{\mathcal{C}}_\partial \boldsymbol{z}(\boldsymbol{e}) = \boldsymbol{v}_f|_{\partial\Omega},
\end{aligned}
$$

Vector $\boldsymbol{y}_r$ represents the rigid body velocity at the boundary $\boldsymbol{y}_r = (\boldsymbol{v}_P + [\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P)|_{\partial\Omega}$, while $\boldsymbol{y}_d$ represents the velocity field in the domain $\boldsymbol{y}_d = (\boldsymbol{v}_P + [\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P + \boldsymbol{v}_f)|_{\Omega}$. Operator $\boldsymbol{\mathcal{E}}$ is positive self-adjoint, $\boldsymbol{\mathcal{J}}$ is formally skew-symmetric. The Hamiltonian satisfies

$$
\partial_{\boldsymbol{e}} H = \boldsymbol{\mathcal{E}}^* \boldsymbol{z}.
$$

[4] Volker Mehrmann and Riccardo Morandin. "Structure-preserving discretization for port-Hamiltonian descriptor systems". In: *Proceedings of the 59th IEEE Conference on Decision and Control*. 2019, pp. 6663–6868.

Energy balance

$$
\begin{aligned}
\dot{H}(\boldsymbol{e}) &= \langle \partial_{\boldsymbol{e}} H, \partial_t \boldsymbol{e} \rangle_{\mathscr{X}} = \langle \boldsymbol{\mathcal{E}}^* \boldsymbol{z}, \partial_t \boldsymbol{e} \rangle_{\mathscr{X}}, \\
&= \langle \boldsymbol{z}, \boldsymbol{\mathcal{E}} \partial_t \boldsymbol{e} \rangle_{\mathscr{X}}, \quad \text{Adjoint definition}, \\
&= \langle \boldsymbol{z}, \boldsymbol{\mathcal{J}} \boldsymbol{z} + \boldsymbol{\mathcal{B}}_d(\boldsymbol{e}) \boldsymbol{u}_d + \boldsymbol{\mathcal{B}}_r(\boldsymbol{e}) \boldsymbol{u}_\partial \rangle_{\mathscr{X}}, \\
&= \langle \boldsymbol{y}_\partial, \boldsymbol{u}_\partial \rangle_{\mathscr{L}^2(\partial\Omega, \mathbb{R}^3)} + \langle \boldsymbol{\mathcal{B}}_d^* \boldsymbol{z}, \boldsymbol{u}_d \rangle_{\mathscr{X}} + \langle \boldsymbol{\mathcal{B}}_r^* \boldsymbol{z}, \boldsymbol{u}_\partial \rangle_{\mathscr{X}}, \quad \text{I.B.P. on } \boldsymbol{\mathcal{J}}, \\
&= \langle \boldsymbol{y}_\partial + \boldsymbol{y}_r, \boldsymbol{u}_\partial \rangle_{\mathscr{L}^2(\partial\Omega, \mathbb{R}^3)} + \langle \boldsymbol{y}_d, \boldsymbol{u}_d \rangle_{\mathscr{L}^2(\Omega, \mathbb{R}^3)},
\end{aligned}
\tag{1}
$$

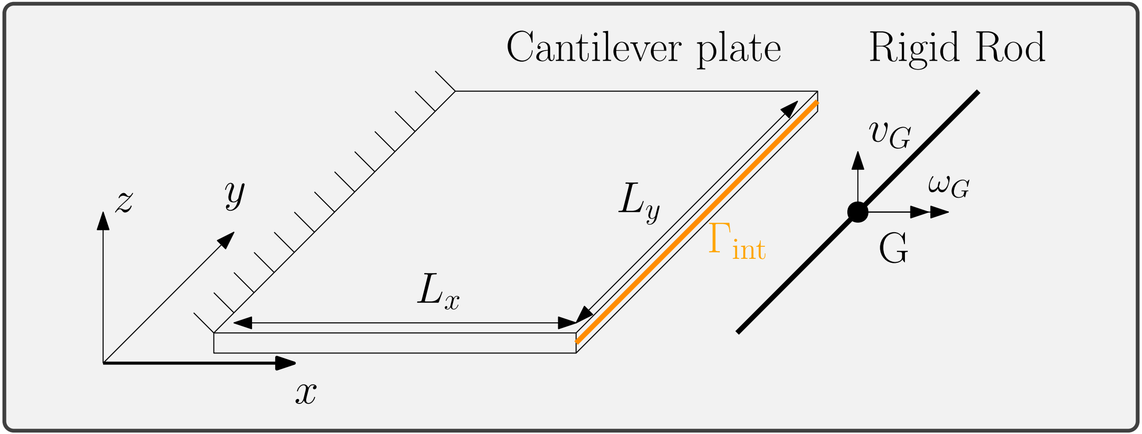
where the integration by parts (Stokes theorem) has been used

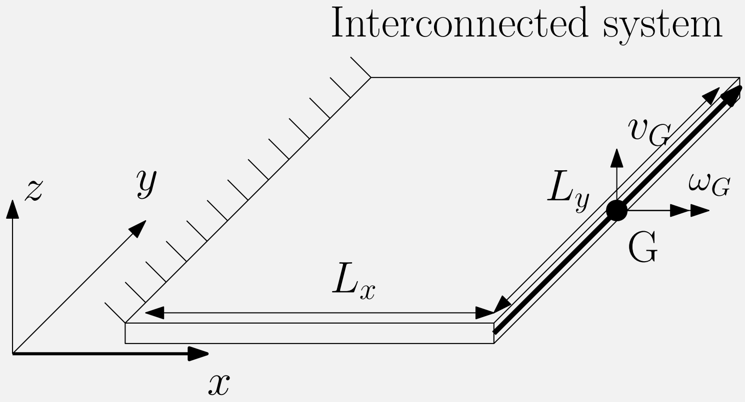
$$
\int_\Omega \boldsymbol{\Sigma} : \mathrm{Grad}(\boldsymbol{v}_f) \, \mathrm{d}\Omega + \int_\Omega \mathrm{Div}(\boldsymbol{\Sigma}) \cdot \boldsymbol{v}_f \, \mathrm{d}\Omega = \int_{\partial\Omega} (\boldsymbol{\Sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{v}_f \, \mathrm{d}\Gamma = \langle \boldsymbol{y}_\partial, \boldsymbol{u}_\partial \rangle_{\mathscr{L}^2(\partial\Omega)}. \tag{2}
$$

The power balance equals the power due to body force and surface traction

$$
\dot{H}(\boldsymbol{e}) = \int_{\partial\Omega} (\boldsymbol{\Sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{v} \, \mathrm{d}\Gamma + \int_\Omega \boldsymbol{u}_d \cdot \boldsymbol{v} \, \mathrm{d}\Omega, \quad \boldsymbol{v} := \boldsymbol{v}_P + [\boldsymbol{\omega}_P]_\times (\boldsymbol{x} + \boldsymbol{u}_f) + \boldsymbol{v}_f. \tag{3}
$$

Outline


Cantilever plate
Rigid Rod
z
y
x
L_x
L_y
Γ_{int}
v_G
ω_G
G


Interconnected system
z
y
x
L_x
L_y
v_G
ω_G
G

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod

$$
\text{dpH} \begin{cases} \dfrac{\partial x_1}{\partial t} = \mathcal{J} \dfrac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \dfrac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \dfrac{\delta H_1}{\delta x_1} \end{cases} \qquad \text{pH} \begin{cases} \dfrac{d\boldsymbol{x}_2}{dt} = J \dfrac{\partial H_2}{\partial \boldsymbol{x}_2} + B\boldsymbol{u}_2 \\ \boldsymbol{y}_2 = B^T \dfrac{\partial H_2}{\partial x_2} + D\boldsymbol{u}_2 \end{cases},
$$

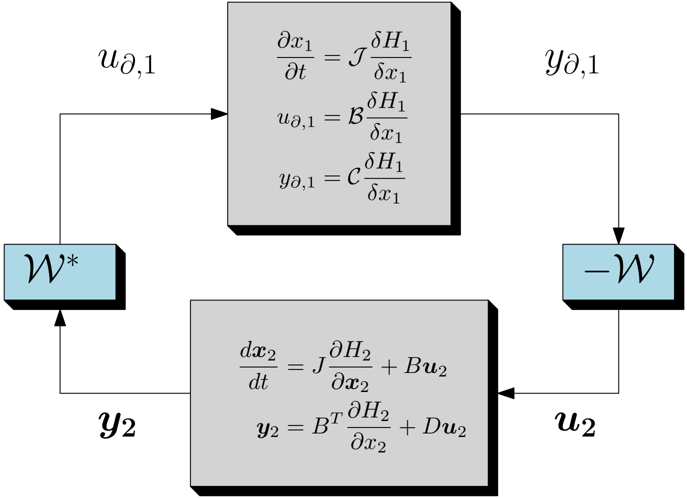
where $u_{\partial,1} \in \mathscr{U}$, $y_{\partial,1} \in \mathscr{Y} = \mathscr{U}'$ belong to some Hilbert spaces and $x_2 \in \mathbb{R}^n, u, y \in \mathbb{R}^m$.
The interconnection is power-preserving if

$$
\langle u_{\partial,1},\ y_{\partial,1} \rangle_{\mathscr{U} \times \mathscr{Y}} + \langle u_2,\ y_2 \rangle_{\mathbb{R}^m} = 0.
$$

This is achieved by introducing a compact operator $\mathcal{W} : \mathscr{Y} \to \mathbb{R}^m$

$$
u_2 = -\mathcal{W}\, y_{\partial,1}, \qquad u_{\partial,1} = \mathcal{W}^*\, y_2,
$$

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod



Boundary interconnection of the Kirchhoff plate

Plate $(\Omega = [0, L_x] \times [0, L_y])$

$$
\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix}
$$

$$
\begin{aligned}
u_{\partial,\text{pl}} &= \partial_t w(x = L_x, y), \\
y_{\partial,\text{pl}} &= \widetilde{q}_n(x = L_x, y).
\end{aligned}
$$

Rigid rod

$$
\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \boldsymbol{u}_{\text{rod}},
$$

$$
\boldsymbol{y}_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},
$$

Space $\mathscr{Y}$ is the space of square-integrable functions with support on $\Gamma_{\text{int}} = \{(x, y) |\ x = L_x, 0 \leq y \leq L_y\}$. The interconnection operator then provides the total force and torque acting on the rigid rod

$$
\mathcal{W} y_{\partial,\text{pl}} = - \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial,\text{pl}} \, \mathrm{d}s \\ \int_{\Gamma_{\text{int}}} (y - L_y/2) \, y_{\partial,\text{pl}} \, \mathrm{d}s \end{pmatrix}.
$$

The adjoint operator provides a rigid movement as the plate input at Γ_{int}

$$
\begin{aligned}
\langle \mathcal{W} y_{\partial,\text{pl}}, \ \boldsymbol{y}_{\text{rod}} \rangle_{\mathbb{R}^m} &= \langle y_{\partial,\text{pl}}, \ \mathcal{W}^* \boldsymbol{y}_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})}, \\
\mathcal{W}^* y_{\text{rod}} &= v_G + \omega_G \, (y - L_y/2) \, .
\end{aligned}
$$

Distributed load ($t_{\text{end}} = 10\,[\mathrm{ms}]$)

$$p = \begin{cases} 10^5 \left[y + 10\,(y - L_y/2)^2\right] [Pa], & \forall\, t < 2\,[\mathrm{ms}], \\ 0, & \forall\, t \geq 2\,[\mathrm{ms}]. \end{cases}$$

Play/Pause

Results

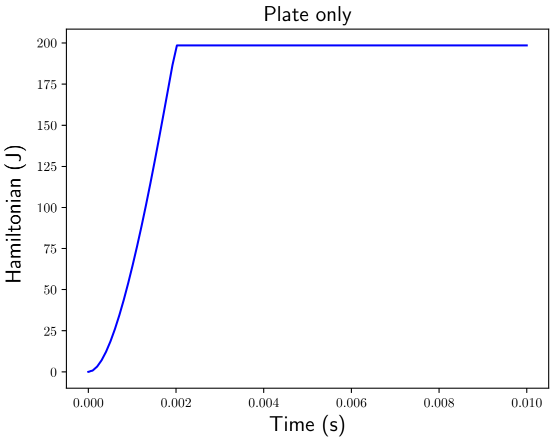


Plate only
Hamiltonian (J)
Time (s)
0
25
50
75
100
125
150
175
200
0.000
0.002
0.004
0.006
0.008
0.010

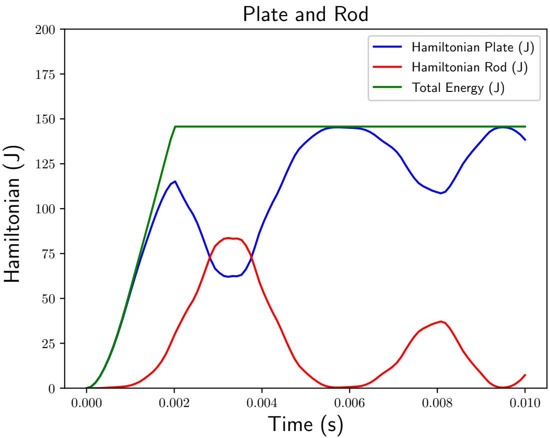


Plate and Rod
Hamiltonian Plate (J)
Hamiltonian Rod (J)
Total Energy (J)
Hamiltonian (J)
Time (s)
0
25
50
75
100
125
150
175
200
0.000
0.002
0.004
0.006
0.008
0.010

The following has been presented:

Thanks for your attention
Questions?

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