

# Interconnection of the Kirchhoff plate within the port-Hamiltonian framework

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- 1 The Kirchhoff plate as a port-Hamiltonian system
  - Classical formulation
  - Port-Hamiltonian formulation
- 2 Structure preserving discretization
  - The partitioned finite element method
  - Application to the Kirchhoff plate
- 3 Interconnection with rigid elements
- 4 Stabilization by boundary injection

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## General infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z, t) &= \mathcal{J} \frac{\delta H}{\delta x} + B u(z, t), \\ y(z, t) &= B^* \frac{\delta H}{\delta x}. \end{cases}$$

With boundary conditions

$$u_{\partial} = \mathcal{B} \frac{\delta H}{\delta x}, \quad y_{\partial} = \mathcal{C} \frac{\delta H}{\delta x}$$

Energy rate:  $\dot{H} = u_{\partial}^T y_{\partial} + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$

- $x$  energy variables,  $e = \delta_x H =:$  co-energy variables ;
- $\mathcal{J}$ : skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$ : boundary operator;
- $u, y, B$ : distributed input, output and control operator;

## Linear infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z, t) &= \mathcal{J}Qx + Bu(z, t), \\ y(z, t) &= B^*Qx. \end{cases}$$

With boundary conditions

$$u_{\partial} = \mathcal{B} \frac{\delta H}{\delta x}, \quad y_{\partial} = \mathcal{C} \frac{\delta H}{\delta x}$$

Energy rate:  $\dot{H} = u_{\partial}^T y_{\partial} + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$

- $x$  energy variables,  $e = \delta_x H = Qx$ : co-energy variables ( $Q$  symmetric positive operator);
- $\mathcal{J}$ : skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$ : boundary operator;
- $u, y, B$ : distributed input, output and control operator;

## Classical bilaplacian formulation

For an homogeneous isotropic material

$$\rho h \frac{\partial^2 w}{\partial t^2} + D \Delta^2 w = p, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2.$$

$\Delta^2 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$  is the bilaplacian operator

- $\rho$  [kg/m<sup>3</sup>] is the mass density;
- $h$  [m] is the plate thickness;
- $p$  [N/m<sup>2</sup>] is an external distributed force;
- $D$  [Pa m] is the bending stiffness;

## Bending moment formulation

$$\rho h \frac{\partial^2 w}{\partial t^2} + \operatorname{div} \operatorname{Div}(\mathbf{M}) = p, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2.$$

Where  $\mathbf{M} = \mathbb{D} \nabla^2 w \in \mathbb{R}_{\text{sym}}^{2 \times 2}$  is the bending moment tensor and  $\nabla^2 = \operatorname{Grad} \circ \operatorname{grad}$  the Hessian.

$$\operatorname{div} \operatorname{Div}(\mathbf{M}) = \partial_{xx} M_{11} + 2\partial_{xy} M_{12} + \partial_{yy} M_{22}$$

- $\rho$  [kg/m<sup>3</sup>] is the mass density;
- $h$  [m] is the plate thickness;
- $p$  [N/m<sup>2</sup>] is an external distributed force;
- $\mathbb{D}$  is the bending rigidity tensor (symmetric, positive). For an homogeneous isotropic material

$$\mathbb{D}\mathbf{A} = D \{ (1 - \nu)\mathbf{A} + \nu \operatorname{Tr}(\mathbf{A})\mathbf{I} \};$$

# Boundary conditions

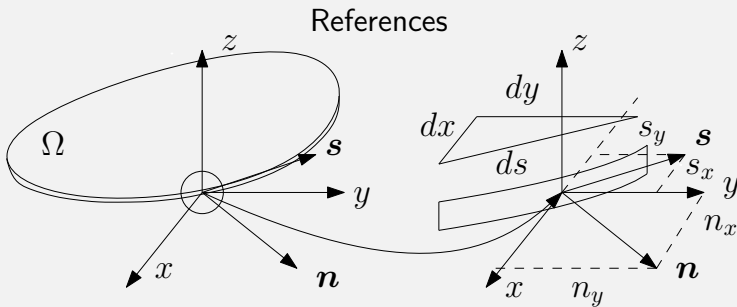
For the boundary variables consider the definitions

Flexural moment  $M_{nn} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{n})$ ,

Shear stress  $\mathbf{q} = \text{Div } \mathbf{M}$ ,

Torsional moment  $M_{ns} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{s})$ , Kirchhoff shear force  $\tilde{q}_n = -\mathbf{q} \cdot \mathbf{n} - \partial_s M_{ns}$ .

$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$  is the tensor contraction and  $\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^\top \in \mathbb{R}^{2 \times 2}$  is the dyadic product.





# Boundary conditions

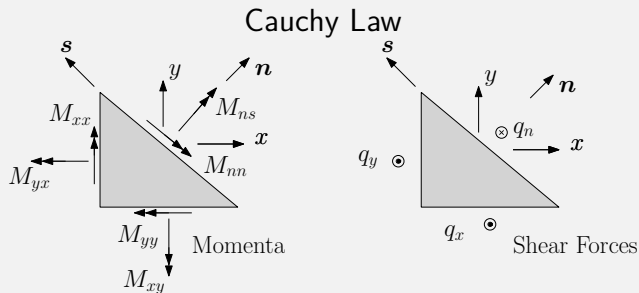
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Boundary conditions

$$\Gamma_f = \{\tilde{q}_n, M_{nn} \text{ known}\}$$

$$\Gamma_c = \{w, \frac{\partial w}{\partial n} \text{ known}\}$$

$$\Gamma_s = \{w, M_{nn} \text{ known}\}$$

$\Omega$

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The total energy of the system is given by the sum of kinetic and deformation energy

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho h (\partial_t w)^2 + \mathbb{D} \nabla^2 w : \nabla^2 w \right\} d\Omega$$

Consider the following choice for the energy variables

$$\alpha_1 := \rho h \partial_t w, \quad \text{Linear momentum}$$

$$\mathbf{A}_2 := \nabla^2 w, \quad \text{Curvature}$$

This leads to the following co-energy variables

$$e_1 := \frac{\delta H}{\delta \alpha_1} = \partial_t w, \quad \text{Vertical velocity}$$

$$\mathbf{E}_2 := \frac{\delta H}{\delta \mathbf{A}_2} = \mathbf{M}, \quad \text{Bending moment}$$

The system

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_1 \\ \mathbf{A}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_1 \\ \mathbf{E}_2 \end{pmatrix}, \quad \begin{pmatrix} e_1 \\ \mathbf{E}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} (\rho h)^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix}}_{\mathcal{Q}} \begin{pmatrix} \alpha_1 \\ \mathbf{A}_2 \end{pmatrix}$$

defines a Stokes-Dirac structure. The proof is readily obtained considering that the following holds

$$(\operatorname{div} \operatorname{Div})^* = \nabla^2$$

This means that the operator  $\mathcal{J}$  is formally skew-adjoint.

It is worth noticing that the boundary variables are defined by the power balance

$$\dot{H} = \int_{\partial\Omega} \{ \partial_t w \tilde{q}_n + \partial_n(\partial_t w) M_{nn} \} \, ds.$$

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# How to discretize pH systems?

## Infinite dimensional pHs

PDE:

$$\partial_t x(z, t) = \mathcal{J} \delta_x H + B u(z, t),$$

$$y(z, t) = B^* \delta_x H.$$

Boundary conditions:

$$u_\partial = \mathcal{B} \delta_x H, \quad y_\partial = \mathcal{C} \delta_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z, t) y(z, t) \, d\Omega$$

## Finite dimensional pHs

ODE:

$$\dot{x} = J \partial_x H + B_d u_d + B_\partial u_\partial,$$

$$y_d = B_d^T \partial_x H,$$

$$y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

## Available methods

- Spectral methods (Moulla 2012);
- Finite differences (Trenchant 2018);
- Finite elements based (Golo 2004, Kotyczka 2018, [Cardoso-Ribeiro 2018](#));

# The partitioned finite element method (PFEM)

General form of a linear pH system in co-energy variables

$$\mathcal{M} \frac{\partial e}{\partial t} = \mathcal{J}e, \quad \mathcal{M} = \mathcal{Q}^{-1}$$

## General procedure for PFEM

- 1 Put the system into weak form:

$$\left( v, \mathcal{M} \frac{\partial e}{\partial t} \right)_{\Omega} = (v, \mathcal{J}e)_{\Omega}.$$

- 2 Apply integration by part on a partition of  $\mathcal{J}$ :

$$(v, \mathcal{J}e)_{\Omega} \stackrel{i.b.p.}{=} j(v, e)_{\Omega} + b(v, u_{\partial})_{\partial\Omega},$$

so that  $j(v, e)_{\Omega}$  is a skew-symmetric bilinear form.

- 3 Discretization by Galerkin method (same basis function for test and co-energy variables)



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Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the **first line of the operator  $\mathcal{J}$**  is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega \\ &= \underbrace{\int_\Omega \left\{ -\nabla^2 v_1 : \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega}_{j_{\text{Hess}}(v, e)} + \underbrace{\int_{\partial\Omega} \{v_1 q_n + \partial_n v_1 M_{nn}\} ds}_{b_N(v, u_\partial)_{\partial\Omega}} \end{aligned}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the **second line of the operator  $\mathcal{J}$**  is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega \\ &= \underbrace{\int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \text{div Div } \mathbf{V}_2 e_1 \right\} d\Omega}_{j_{\text{div Div}}(v, e)} + \underbrace{\int_{\partial\Omega} \{ v_{q_n} \partial_t w + v_{m_n} \partial_n \partial_t w \} ds}_{b_D(v, u_\partial)_{\partial\Omega}}, \end{aligned}$$

where  $v_{q_n} = -(\text{Div } \mathbf{V}_2) \cdot \mathbf{n} - \partial_s v_{m_s}$ ,  $v_{m_s} = \mathbf{V}_2 : (\mathbf{n} \otimes \mathbf{s})$ ,  $v_{m_n} = \mathbf{V}_2 : (\mathbf{n} \otimes \mathbf{n})$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

The selection depends on the control variables. For **Neumann** control the first line is integrated by parts. For **Dirichlet control** the second.

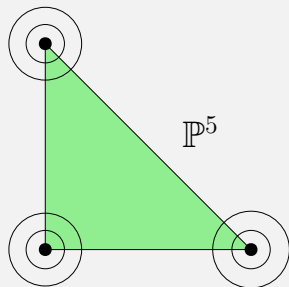
# Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

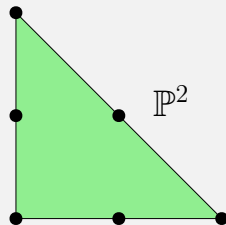
$$m(\mathbf{v}, \partial_t \mathbf{e}) = j_{\text{Hess}}(\mathbf{v}, \mathbf{e}) + b_N(\mathbf{v}, \mathbf{u}_\partial)_{\partial\Omega}$$

For both  $\mathbf{e}_1, \mathbf{E}_2$  the  $H^2$  conforming Bell elements are selected. For the boundary variables Lagrange polynomials of order two are selected.

Bell element



Lagrange polynomials



Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

$$m(\mathbf{v}, \partial_t \mathbf{e}) = j_{\text{Hess}}(\mathbf{v}, \mathbf{e}) + b_N(\mathbf{v}, \mathbf{u}_\partial)_{\partial\Omega}$$

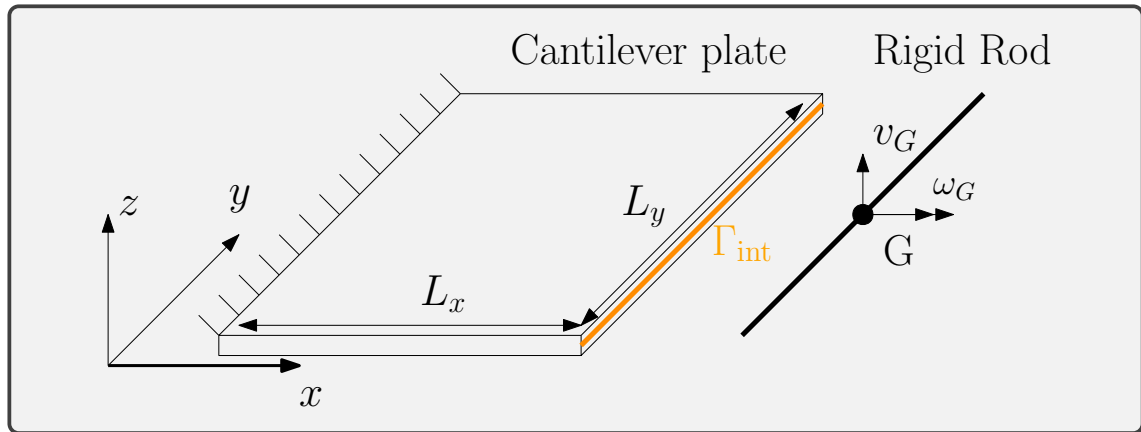
For both  $\mathbf{e}_1, \mathbf{E}_2$  the  $H^2$  conforming Bell elements are selected. For the boundary variables Lagrange polynomials of order two are selected. Dirichlet boundary conditions are enforced by Lagrange multipliers

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

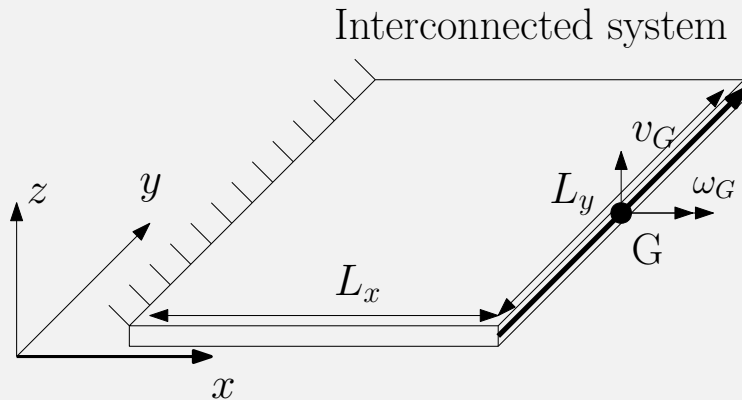
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# Rigid rod welded to a cantilever plate



# Rigid rod welded to a cantilever plate



# Boundary interconnection of the Kirchhoff plate

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod

$$\text{dpH} \begin{cases} \frac{\partial x_1}{\partial t} = \mathcal{J} \frac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \frac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \frac{\delta H_1}{\delta x_1} \end{cases} \quad \text{pH} \begin{cases} \frac{dx_2}{dt} = J \frac{\partial H_2}{\partial x_2} + B u_2 \\ y_2 = B^T \frac{\partial H_2}{\partial x_2} + D u_2 \end{cases},$$

where  $u_{\partial,1} \in \mathcal{U}$ ,  $y_{\partial,1} \in \mathcal{Y} = \mathcal{U}'$  belong to some Hilbert spaces and  $x_2 \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}^m$ .  
The interconnection is power-preserving if

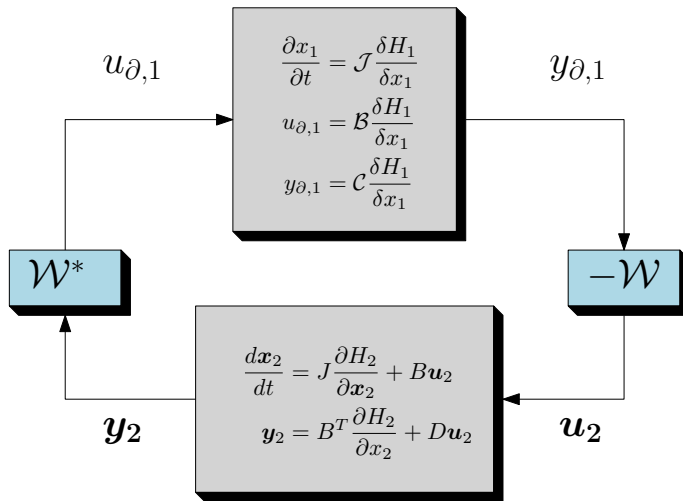
$$\langle u_{\partial,1}, y_{\partial,1} \rangle_{\mathcal{U} \times \mathcal{Y}} + \langle u_2, y_2 \rangle_{\mathbb{R}^m} = 0.$$

This is achieved by introducing a compact operator  $\mathcal{W} : \mathcal{Y} \rightarrow \mathbb{R}^m$

$$u_2 = -\mathcal{W} y_{\partial,1}, \quad u_{\partial,1} = \mathcal{W}^* y_2,$$

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# Boundary interconnection of the Kirchhoff plate

Plate ( $\Omega = [0, L_x] \times [0, L_y]$ )

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix}$$

$$u_{\partial, \text{pl}} = \partial_t w(x = L_x, y),$$

$$y_{\partial, \text{pl}} = \tilde{q}_n(x = L_x, y).$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \mathbf{u}_{\text{rod}},$$

$$\mathbf{y}_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space  $\mathcal{Y}$  is the space of square-integrable functions with support on  $\Gamma_{\text{int}} = \{(x, y) \mid x = L_x, 0 \leq y \leq L_y\}$ . The interconnection operator then provides the total force and torque acting on the rigid rod

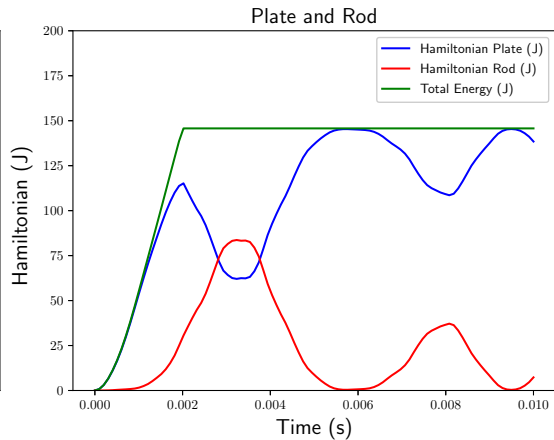
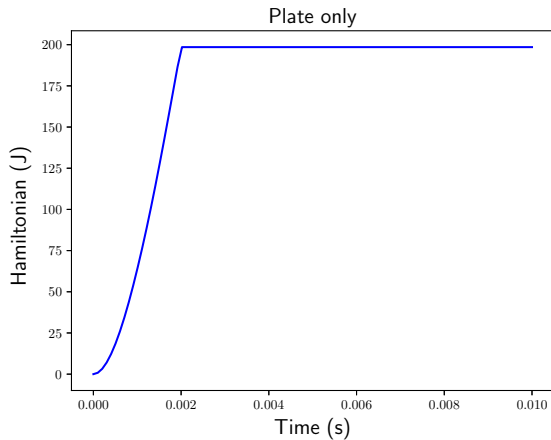
$$\mathcal{W}y_{\partial, \text{pl}} = - \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial, \text{pl}} \, ds \\ \int_{\Gamma_{\text{int}}} (y - L_y/2) y_{\partial, \text{pl}} \, ds \end{pmatrix}.$$

The adjoint operator provides a rigid movement as the plate input at  $\Gamma_{\text{int}}$

$$\langle \mathcal{W}y_{\partial, \text{pl}}, \mathbf{y}_{\text{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial, \text{pl}}, \mathcal{W}^* \mathbf{y}_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})},$$

$$\mathcal{W}^* \mathbf{y}_{\text{rod}} = v_G + \omega_G (y - L_y/2).$$

$$p = \begin{cases} \text{Distributed load } (t_{\text{end}} = 10 [\text{ms}]) \\ 10^5 \left[ y + 10 (y - L_y/2)^2 \right] [Pa], & \forall t < 2 [\text{ms}], \\ 0, & \forall t \geq 2 [\text{ms}]. \end{cases}$$

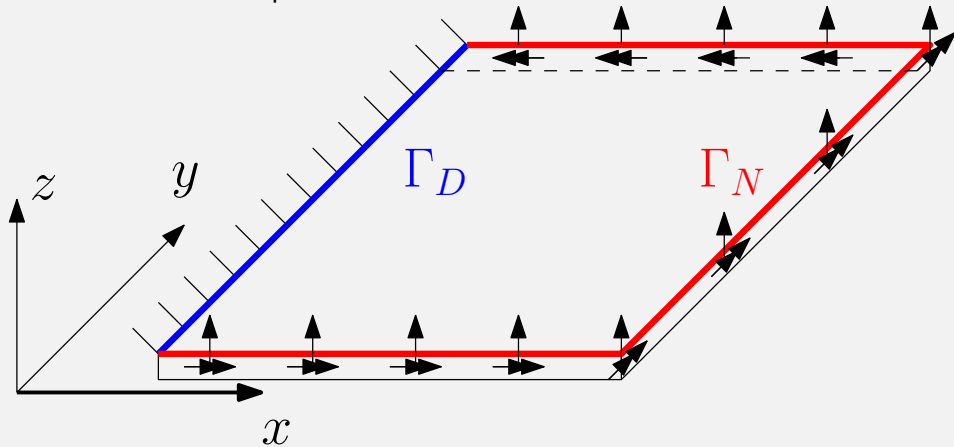


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# Boundary control of a cantilever plate

Cantilever plate at the left side and controlled elsewhere



Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} \quad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

subjected to the following boundary conditions

$$\begin{aligned} \partial_t w|_{\Gamma_D} &= 0, \\ \partial_x \partial_t w|_{\Gamma_D} &= 0, \\ M_{nn}|_{\Gamma_N} &= u_M, \\ \tilde{q}|_{\Gamma_N} &= u_F, \end{aligned} \quad \begin{aligned} \Gamma_D &= \{x = 0\} \\ \Gamma_N &= \{y = 0 \cup x = 1 \cup y = 1\} \end{aligned}$$

with initial conditions (compatible with the constraints):

$$w_t(x, y, 0) = x^2; \quad \mathbf{M}(x, y, 0) = 0.$$

Obtain a finite-dimensional uncontrolled system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

Apply the control law  $u = -Ky$ ,  $K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

with  $R = BKB^T \succeq 0$ .

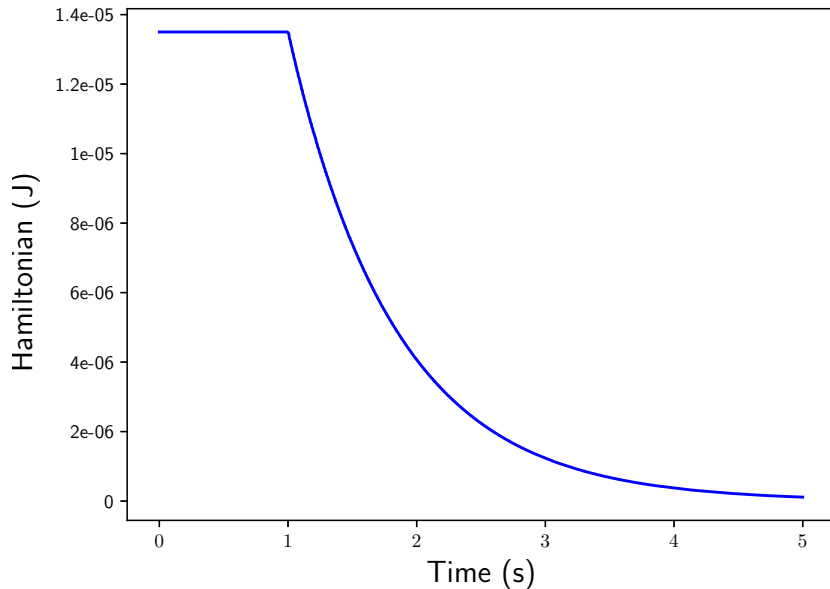
The Hamiltonian  $\dot{H} = -e^T R e \leq 0$  is a non increasing function and by La Salle principle the equilibrium point  $e = 0$  is asymptotically stable.

# Stabilization by boundary injection

Control parameter ( $t_{\text{end}} = 5[\text{s}]$ )

$$K = \begin{cases} 0, & \forall t < 1 [\text{s}], \\ 100, & \forall t \geq 1 [\text{s}]. \end{cases}$$

# Stabilization by boundary injection



The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic interconnections;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

- The Hellan-Herrmann-Johnson method<sup>1</sup>, but difficulties when dealing with inhomogeneous bcs;
- New discretization method capable that handles inhomogeneous bcs<sup>2</sup>

<sup>1</sup>H. Blum and R. Rannacher. "On mixed finite element methods in plate bending analysis". In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

<sup>2</sup>Katharina. Rafetseder and Walter. Zulehner. "A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach". In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

Thanks for your attention  
Questions?



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