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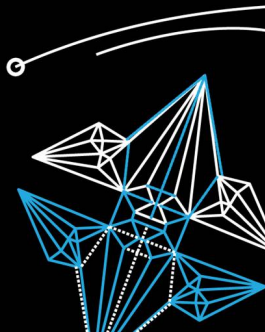
Portwings Internal Meeting

Challenges and outlook for the numerics

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What numerics for the portwings project?

Methods should preserve the continuous structure at the discrete level. Which structure?

- ❶ Cohomology: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla \times} V^2(\mathbb{R}^3) \xrightarrow{\nabla \cdot} V^3(\mathbb{R});$
- ❷ Variational structure $\delta \int I = 0$, (I Lagrangian density);
- ❸ Hamiltonian structure $\mathcal{F} = \{F, H\}, \{\cdot, \cdot\}$ Poisson brackets.
- ❹ ...

Recent developments:

- ▶ splitting of topological and metric operators (Bauer and Behrens 2018);
- ▶ Lie group structure and underlying variational formulation (Gawlik and Gay-Balmaz 2020);
- ▶ connection with algebraic topology, i.e. de Rham complex and more general Hilbert complexes, e.g. elasticity (Bochev and Hyman 2006; Arnold, Falk, and Winther 2006; Palha et al. 2014);

Principle behind split discretization

- ▶ Fluid equation written in covariant form (exterior calculus);
- ▶ Split Hamiltonian form $\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}$.
 - ▶ Topological bracket depending on d (exterior derivative) or ι_v (interior product).
 - ▶ Metric dependent \mathcal{H} , since it depends on $*$

Linear shallow water waves in Hamiltonian form

- ▶ $\mathcal{H} = \frac{1}{2} \int_{\Omega} \left\{ \bar{h} \|\mathbf{u}\|^2 + gh^2 \right\} d\Omega$, with $\frac{\delta \mathcal{H}}{\delta \mathbf{u}} = \bar{h} \mathbf{u}$, $\frac{\delta \mathcal{H}}{\delta h} = gh$, where g gravity acc. and \bar{h} equilibrium fluid height.
- ▶ $\{\mathcal{F}, \mathcal{G}\} = -\left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}}, \nabla \frac{\delta \mathcal{G}}{\delta h}\right) - \left(\frac{\delta \mathcal{F}}{\delta h}, \nabla \cdot \frac{\delta \mathcal{G}}{\delta \mathbf{u}}\right)$.

$$\mathcal{F} = \int \mathbf{u} d\Omega : \dot{\mathcal{F}} = -\left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}}, \nabla \frac{\delta \mathcal{H}}{\delta h}\right) \rightarrow \partial_t \mathbf{u} = -g \nabla h.$$

$$\mathcal{F} = \int h d\Omega : \dot{\mathcal{F}} = -\left(\frac{\delta \mathcal{F}}{\delta h}, \nabla \cdot \frac{\delta \mathcal{H}}{\delta \mathbf{u}}\right) \rightarrow \partial_t h = -\bar{h} \nabla \cdot \mathbf{u}.$$

Split and weak (or mixed) form

De Rham complex: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla \times} V^2(\mathbb{R}^3) \xrightarrow{\nabla \cdot} V^3(\mathbb{R})$.

Split form

$$\begin{array}{ccc} h \in V^0 & \xrightarrow{\nabla} & V^1 \ni \mathbf{u} \\ \tilde{h} = \tilde{*}h \downarrow & & \downarrow \mathbf{u} = \tilde{*}\mathbf{u} \\ \tilde{h} \in V^3 & \xleftarrow{\nabla \cdot} & V^2 \ni \tilde{\mathbf{u}} \end{array}$$

- ▶ Assume full $\tilde{*}$ in metric eqs.
- ▶ Both $\nabla, \nabla \cdot$ are imposed strongly.
- ▶ Both diff. eqs exact

Weak form (Mixed FE)

$$\begin{array}{ccc} h \in V^3 & \xrightarrow{\hat{\nabla}} & V^2 \ni \mathbf{u} \\ \tilde{h} = h \downarrow & & \downarrow \tilde{\mathbf{u}} = \mathbf{u} \\ \tilde{h} \in V^3 & \xleftarrow{\nabla \cdot} & V^2 \ni \tilde{\mathbf{u}} \end{array}$$

- ▶ Assume $\tilde{*} = \text{Id}$, i.e. $\tilde{\mathbf{u}} = \mathbf{u}, \tilde{h} = h$.
- ▶ Weak gradient $\hat{\nabla}$.
- ▶ Moment Eq. weak

Application of mixed finite elements

Infinite-dimensional pH system

PDE with boundary control:

$$\frac{\partial \alpha}{\partial t}(\mathbf{x}, t) = \mathcal{J} \delta_{\alpha} H.$$

Boundary conditions:

$$\mathbf{u}_{\partial} = \mathcal{B}_{\partial} \delta_{\alpha} H, \quad \mathbf{y}_{\partial} = \mathcal{C}_{\partial} \delta_{\alpha} H.$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} \mathbf{u}_{\partial} \cdot \mathbf{y}_{\partial} \, dS.$$

Structure-preserving discretization

Tools available

FEniCS: <https://fenicsproject.org/>.

Fluid Structure Interaction in Fenics: Bergersen et al. 2020.

Mesh morphing in FEniCS: <https://bitbucket.org/Epoxid/femorph/src/c7317791c8f00d70fe16d593344cb164a53cad9b/?at=dokken%2Frestructuring>

PyDec: <https://github.com/hirani/pydec>

Learning Python for scientific computing <https://faculty.math.illinois.edu/~hirani/cbmj/index.html>