The Euler-Bernoulli beam in differential form

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1 Classical formulation

The Euler-Bernoulli beam model consists of one PDE, describing the vertical displacement along the beam length:

$$\rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) = 0, \quad x \in \Omega = \{0, L\}, \tag{1}$$

where $w(x,t)$ is the transverse displacement of the beam. The coefficients $\rho(x), A(x)E(x)$ and $I(x)$ are the mass density, cross section, Young's modulus of elasticity and the moment of inertia of a cross section. The energy variables are then chosen as follows:

$$\alpha_w = \rho A(x)\frac{\partial w}{\partial t}(x,t), \quad \text{Linear Momentum}, \quad \alpha_\kappa = \frac{\partial^2 w}{\partial x^2}(x,t), \quad \text{Curvature}. \tag{2}$$

Those variables are collected in the vector $\boldsymbol{\alpha} = (\alpha_w,\, \alpha_\kappa)^T$, so that the Hamiltonian can be written as a quadratic functional in the energy variables:

$$H = \frac{1}{2}\int_\Omega \left\{\frac{1}{\rho A}\alpha_w^2 + EI\alpha_\kappa^2\right\} \,\mathrm{d}\Omega \tag{3}$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}(x,t), && \text{Vertical velocity}, \\ e_\kappa &:= \frac{\delta H}{\delta \alpha_\kappa} = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t), && \text{Flexural momentum}. \end{aligned} \tag{4}$$

The underlying interconnection structure is then found to be:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix}\begin{pmatrix} e_w \\ e_\kappa\end{pmatrix}. \tag{5}$$

The power flow gives access to the boundary variables:

$$
\begin{aligned}
\dot{H} &= \int_\Omega \{e_w \partial_t \alpha_w + e_\kappa \partial_t \alpha_\kappa\} \ \mathrm{d}\Omega, \\
&= \int_\Omega \{-e_w \partial_{xx} e_\kappa + e_\kappa \partial_{xx} e_w\} \ \mathrm{d}\Omega, \qquad \text{Integration by parts}, \\
&= \int_{\partial\Omega} \{-e_w \partial_x e_\kappa + e_\kappa \partial_x e_w\} \ \mathrm{d}s = \langle -e_w, \, \partial_x e_\kappa \rangle_{\partial\Omega} + \langle e_\kappa, \, \partial_x e_w \rangle_{\partial\Omega}
\end{aligned}
\tag{6}
$$

Since the system is of differential order two, two pairing appears, giving rise to four combination of uniform boundary causality

- First case $u_{\partial,1} = e_w, \quad u_{\partial,2} = \partial_x e_w, \quad y_{\partial,1} = -\partial_x e_\kappa, \quad y_{\partial,2} = e_\kappa$.
  This imposes the vertical $e_w := \partial_t w$ and angular velocity $\partial_x e_w := \partial_{xt} w$ as boundary inputs. If the inputs are null a clamped boundary condition is obtained.

- Second case $u_{\partial,1} = e_w, \quad u_{\partial,2} = e_\kappa, \quad y_{\partial,1} = -\partial_x e_\kappa, \quad y_{\partial,2} = \partial_x e_w$.
  This imposes the vertical velocity and flexural momentum $e_\kappa := EI \partial_{xx} w$ as boundary inputs. Zero inputs lead to a simply supported condition is found.

- Third case $u_{\partial,1} = -\partial_x e_\kappa, \quad u_{\partial,2} = e_\kappa, \quad y_{\partial,1} = e_w, \quad y_{\partial,2} = \partial_x e_w$.
  This imposes the shear force $\partial_x e_\kappa := \partial_x (EI \partial_{xx} w)$ and flexural momentum as boundary inputs. Null inputs correspond to a free condition.

- Forth case $u_{\partial,1} = -\partial_x e_\kappa, \quad u_{\partial,2} = \partial_x e_w, \quad y_{\partial,1} = e_w, \quad y_{\partial,2} = e_\kappa$.
  This imposes the shear force and angular velocity as boundary inputs.

2 Differential forms formulation

The co-energy now are 1-forms $e_w, e_\kappa \in \Lambda^1(\Omega)$ with the flows are 0-forms $f_w = \partial_t \alpha_w, f_\kappa = \partial_t \alpha_\kappa \in \Lambda^0(\Omega)$. To recast (5) using the exterior derivative, the Hodge star operator is needed.

$$
* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega). \tag{7}
$$

For one dimensional domain and using Euclidian coordinates this operator can be either used on 1-forms or 0-forms

$$
\begin{aligned}
* :&\Lambda^1(\Omega) \to \Lambda^0(\Omega), \\
&\mathrm{d}x \to 1
\end{aligned}
\tag{8}
$$

or

$$
\begin{aligned}
* :&\Lambda^0(\Omega) \to \Lambda^1(\Omega), \\
&1 \to \ \mathrm{d}x
\end{aligned}
\tag{9}
$$

Then the equivalent system reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w\\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix}0 & -\mathrm{d}{*}\mathrm{d}\\ \mathrm{d}{*}\mathrm{d} & 0\end{bmatrix}\begin{pmatrix}e_w\\ e_\kappa\end{pmatrix}. \tag{10}$$

Proof 1 *The operator* $\mathrm{d}{*}\,\mathrm{d} : \Lambda^0(\Omega) \to \Lambda^1(\Omega)$ *is a composition of operators that reads in Euclidean coordinates*

$$\begin{aligned}\mathrm{d}{*}\mathrm{d}e &= \mathrm{d}{*}(\frac{\partial e}{\partial x}\,\mathrm{d}x),\\ &= \mathrm{d}(\frac{\partial e}{\partial x}),\\ &= \frac{\partial^2 e}{\partial x^2}\,\mathrm{d}x,\end{aligned} \tag{11}$$

The Hamiltonian energy is then

$$H = \frac{1}{2}\int_\Omega e_w \wedge \alpha_w + e_\kappa \wedge \alpha_\kappa \tag{12}$$

The power rate then reads

$$\begin{aligned}\dot{H} &= \int_\Omega e_w \wedge \partial_t \alpha_w + e_\kappa \wedge \partial_t \alpha_\kappa,\\ &= \int_\Omega e_w \wedge \partial_t \alpha_w + e_\kappa \wedge \partial_t \alpha_\kappa,\\ &= \int_\Omega -e_w \wedge (\mathrm{d}{*}\mathrm{d}e_w) + e_\kappa \wedge (\mathrm{d}{*}\mathrm{d}e_\kappa)\end{aligned} \tag{13}$$