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Mixed finite elements for port-Hamiltonian models of von Kármán beams

7th IFAC Conference on Lagrangian and Hamiltonian method for non linear control

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11-13 October, 2021

Overview

1. Von-Kármán theory of thin beams in pH form
2. Numerical discretization
3. Numerical convergence study

Outline

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2. Numerical discretization
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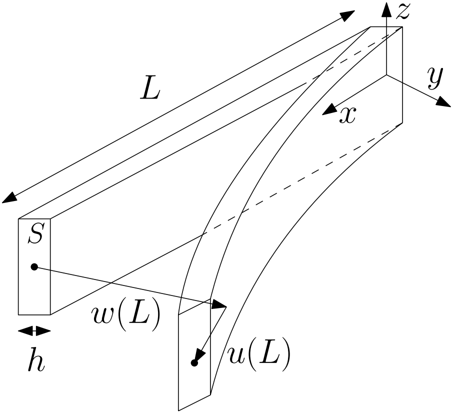
Linear vs Von-Kármán plate theory

Geoometrical non-linearities allow describing bifurcations (i.e. buckling).

The von-Kármán assumption

Basic geometric assumption

Out of plane deflection comparable compared to the thickness: $w/h = \mathcal{O}(1)$.



Aspect ratio: $\delta = h/L$.
The following terms are kept in the expansion:

$$w/L = \mathcal{O}(\delta),$$
$$u/L = \mathcal{O}(\delta^2),$$

Linear isotropic beams

The axial and bending behavior are uncoupled if $w/h \ll 1$:

Axial displacement (wave equation)

$$\rho A \partial_{tt} u = \partial_x n_{xx}, \qquad n_{xx} = EA\, \partial_x u.$$

Vertical displacement (Euler-Bernoulli equation)

$$\rho A \partial_{tt} w = -\partial_{xx} m_{xx}, \qquad m_{xx} = EI \partial_{xx} w.$$

For Von-Kármán beams the two are coupled.

Stresses and Strains in Von-Kármán beams

Decomposition strain field

$$\varepsilon_{xx} = \overbrace{\boxed{\frac{\partial u}{\partial x}} + \boxed{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2}}^{\varepsilon_{xx}^m} - z\overbrace{\boxed{\frac{\partial^2 w}{\partial x^2}}}^{\kappa_{xx}} = \varepsilon_{xx}^m - z\kappa_{xx}.$$

Linear axial def.

Non-linear axial def.

Linear bending def.

Membrane and bending stresses (isotropic material)

$$n_{xx} = \int_S E \, \mathrm{d}S \varepsilon_{xx}^m = EA\varepsilon_{xx}^m, \qquad \text{Axial stress resultant}$$

$$m_{xx} = -\int_S Ez^2 \, \mathrm{d}S \kappa_{xx} = EI\kappa_{xx}, \qquad \text{Bending stress resultant}$$

Port-Hamiltonian Von-Kármán beams

Dynamics

$$\rho A \partial_{tt} u = \partial_x \boxed{n_{xx}},$$
$$\rho A \partial_{tt} w = -\partial^2_{xx} m_{xx} + \partial_x (\boxed{n_{xx}} \partial_x w),$$

Energy and coenergy variables

Same selection as usual:

$$\alpha_u = \rho A \partial_t u, \qquad \alpha_\varepsilon = \varepsilon^m_{xx},$$
$$\alpha_w = \rho A \partial_t w, \qquad \alpha_\kappa = \kappa_{xx}.$$

Linear constitutive relation $\boldsymbol{e} = \mathcal{Q} \boldsymbol{\alpha}$ with

$$\mathcal{Q} = \mathrm{Diag}\left[\rho A,\ C_a,\ \rho A,\ C_b\right]^{-1}, \quad C_a = (EA)^{-1}, \quad C_b = (EI)^{-1},$$

where C_a, C_b are the axial and bending compliances.

The port-Hamiltonian realization

To close the system, variable w has to be accessible.

$$
\frac{\partial}{\partial t}\begin{pmatrix}\alpha_u\\ \alpha_\varepsilon\\ \alpha_w\\ \alpha_\kappa\\ w\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \partial_x & 0 & 0 & 0\\ \partial_x & 0 & \boxed{(\partial_x w)\,\partial_x} & 0 & 0\\ 0 & \boxed{\partial_x(\cdot\,\partial_x w)} & 0 & -\partial^2_{xx} & -1\\ 0 & 0 & \partial^2_{xx} & 0 & 0\\ 0 & 0 & 1 & & 0\end{bmatrix}}_{\mathcal{J}} \begin{pmatrix}e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\\ \delta_w H\end{pmatrix}.
$$

Proposition

The operator $\mathcal{J}$ is formally skew-adjoint.

The construction is analogous for plate problems[1].

[1] Andrea Brugnoli and Denis Matignon (2022). "A port-Hamiltonian formulation for the full von-Kàrmàn plate model". In: *10th European Nonlinear Dynamics Conference (ENOC)*.

Energy rate and boundary conditions

Proposition

The energy rate reads

$$\dot{H} = \langle e_u, e_\varepsilon \rangle_{\partial\Omega} + \langle e_w, e_\varepsilon \partial_x w - \partial_x e_\kappa \rangle_{\partial\Omega} + \langle \partial_x e_w, e_\kappa \rangle_{\partial\Omega}.$$

with $\Omega = [0, L]$ *and* $\langle \cdot, \cdot \rangle_\Omega$ *the* L^2 *inner product.*

Boundary conditions classification

BCs	Traction	Bending	
Dirichlet BCs.	$e_u\vert_0^L$	$e_w\vert_0^L$	$\partial_x e_w\vert_0^L$
Neumann BCs.	$e_\varepsilon\vert_0^L$	$e_\varepsilon \partial_x w - \partial_x e_\kappa\vert_0^L$	$e_\kappa\vert_0^L$

Same bcs. as in Puel and Tucsnak 1996 for global existence and uniqueness result.

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Pure coenergy formulation

Coenergy formulation for linear constitutive equations

If the $\mathcal{Q}$ operator is inverted:

$$\begin{pmatrix} \rho A \dot{e}_u \\ C_a \dot{e}_\varepsilon \\ \rho A \dot{e}_w \\ C_b \dot{e}_\kappa \\ \dot{w} \end{pmatrix} = \begin{bmatrix} 0 & \partial_x & 0 & 0 & 0 \\ \partial_x & 0 & \partial_x w \, \partial_x & 0 & 0 \\ 0 & \partial_x (\cdot \, \partial_x w) & 0 & -\partial_{xx}^2 & -1 \\ 0 & 0 & \partial_{xx}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_u \\ e_\varepsilon \\ e_w \\ e_\kappa \\ \delta_w H \end{pmatrix}.$$

In the sequel, the quantity $\delta_w H$ is removed as no displacement dependent potential (e.g. gravity) is considered

Weak formulation

Introducing test functions and integrating by parts the first, third and fourth, we get the weak formulation.

Weak formulation

Find $(e_u, e_w, e_\kappa, w) \in H^1(\Omega)$, $e_\varepsilon \in L^2(\Omega)$ such that

$$
\begin{aligned}
\langle \psi_u, \, \rho A \, \dot{e}_u \rangle_\Omega &= - \langle \partial_x \psi_u, \, e_\varepsilon \rangle_\Omega + \langle \psi_u, \, e_\varepsilon \rangle_{\partial\Omega} \, . \\
\langle \psi_\varepsilon, \, C_a \, \dot{e}_\varepsilon \rangle_\Omega &= \langle \psi_\varepsilon, \, \partial_x e_u \rangle_\Omega + \langle \psi_\varepsilon, \, \partial_x w \, \partial_x e_w \rangle_\Omega \, , \\
\langle \psi_w, \, \rho A \dot{e}_w \rangle_\Omega &= - \langle \partial_x \psi_w \partial_x w, \, e_\varepsilon \rangle_\Omega + \langle \partial_x \psi_w, \, \partial_x e_\kappa \rangle_\Omega \\
&\quad + \langle \psi_w, \, e_\varepsilon \partial_x w - \partial_x e_\kappa \rangle_{\partial\Omega} \, , \\
\langle \psi_\kappa, \, C_b \, \dot{e}_\kappa \rangle_\Omega &= - \langle \partial_x \psi_\kappa, \, \partial_x e_w \rangle_\Omega + \langle \psi_\kappa, \, \partial_x e_w \rangle_{\partial\Omega} \, , \\
\langle \psi, \, \dot{w} \rangle_\Omega &= \langle \psi, \, e_w \rangle_\Omega \, .
\end{aligned}
$$

holds $\forall (\psi_u, \psi_w, \psi_\kappa, \psi) \in H^1(\Omega)$, $\forall \psi_\varepsilon \in L^2(\Omega)$.

Weak formulation

Introducing test functions and integrating by parts the first, third and fourth, we get the weak formulation.

Weak formulation

Find $\boldsymbol{e} = (e_u, e_\varepsilon, e_w, e_\kappa) \in H^1 \times L^2 \times H^1 \times H^1$ such that

$$\begin{aligned} m(\boldsymbol{\psi}, \partial_t \boldsymbol{e}) &= j_w(\boldsymbol{\psi}, \boldsymbol{e}) + b(\boldsymbol{\psi})\mathbf{u}, \\ \partial_t w &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{e}, \\ \mathbf{y} &= b^\top(\boldsymbol{e}), \end{aligned}$$

$\forall \boldsymbol{\psi} \in H^1 \times L^2 \times H^1 \times H^1 := X$

- m is a symmetric, coercive, bilinear form;
- j_w is a skew-symmetric bilinear form modulated by w;
- $b : X \to \mathbb{R}^6$ vector-valued functional.

Mixed finite element construction[2]

Crucial concept: Hilbert complex $H^1 \xrightarrow{\partial_x} L^2$.

Key requirements for mixed Galerkin approximation

- The subspaces $H^1_h \subset H^1$, $L^2_h \subset L^2$ form a subcomplex $H^1_h \xrightarrow{\partial_x} L^2_h$ (i.e. $\partial_x H^1_h \subset L^2_h$).
- they admit bounded linear projections $\pi_h^{H^1} : H^1 \to H^1_h$ and $\pi_h^{L^2} : L^2 \to L^2_h$ which commute with ∂_x:
$$\partial_x \pi_h^{H^1} = \pi_h^{L^2} \partial_x.$$

Satisfied for $CG_k \xrightarrow{\partial_x} DG_{k-1}$

$$\mathrm{CG}_k = \{u \in H^1(\Omega) | \ \forall \text{edge in the mesh}, \ u|_{\text{edge}} \in P_k\},$$

$$\mathrm{DG}_{k-1} = \{u \in L^2(\Omega) | \ \forall \text{edge in the mesh}, \ u|_{\text{edge}} \in P_{k-1}\},$$

where P_k space of polynomials of degree k.

[2] Arnold, Falk, and Winther 2006.

Finite element choice and final system

For the proposed weak formulation, the FE spaces become

$$e_u^h \in \mathrm{CG}_{2k-1}, \qquad e_\varepsilon^h \in \mathrm{DG}_{2k-2}, \qquad (e_w^h,\ e_\kappa^h,\ w^h) \in CG_k, \quad k \geq 1.$$

Implications:

- Subcomplex property for the linear part: $\partial_x \mathrm{CG}_{2k-1} \subset \mathrm{DG}_{2k-2}$.
- The non linear part respects

$$\partial_x \mathrm{CG}_k \cdot \partial_x \mathrm{CG}_k \subset \mathrm{DG}_{2k-2}.$$

Finite dimensional system (Galerkin projection)

$$\begin{aligned}
\mathbf{M}\dot{\mathbf{e}} &= \mathbf{J}(\mathbf{w})\mathbf{e} + \mathbf{B}\mathbf{u}, \\
\dot{\mathbf{w}} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{e}, \\
\mathbf{y} &= \mathbf{B}^\top \mathbf{e}.
\end{aligned}$$

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Manufactured solution

The following manufactured solution is considered

$$u^{\text{ex}} = x^3[1-(x/L)^3]\sin(2\pi t), \qquad w^{\text{ex}} \quad = \sin(\pi x/L)\sin(2\pi t),$$

together with the boundary conditions

$$u|_0^L = 0, \quad w|_0^L = 0, \quad m_{xx}|_0^L = 0.$$

A Crank-Nicholson scheme is used for time integration.

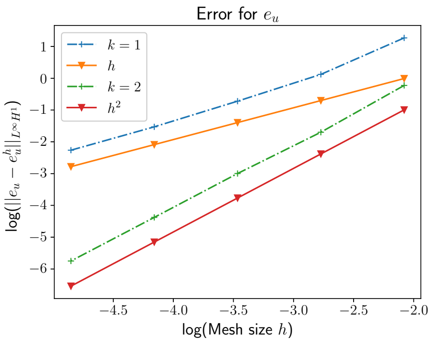
Convergence measure

The discrete time-space norm $L^\infty_{\Delta t}(\mathcal{X})(\mathcal{X} = H^1 \text{or } L^2)$ is used to measure convergence

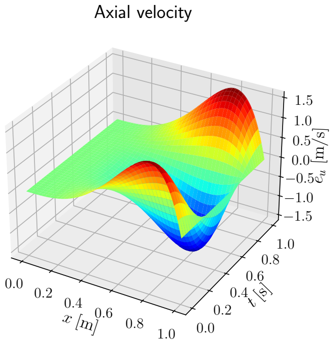
$$||\cdot||_{L^\infty(\mathcal{X})} \approx ||\cdot||_{L^\infty_{\Delta t}(\mathcal{X})} = \max_{t \in t_i} ||\cdot||_{\mathcal{X}},$$

where t_i are the discrete simulation instants.

Results

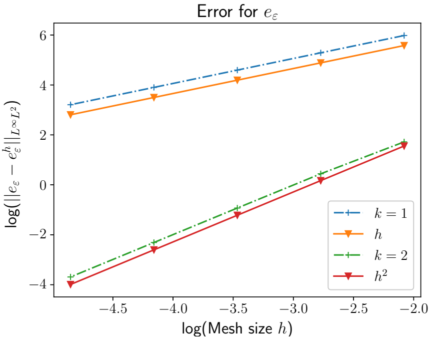


$L^\infty_{\Delta t}(H^1)$ error for e_u.

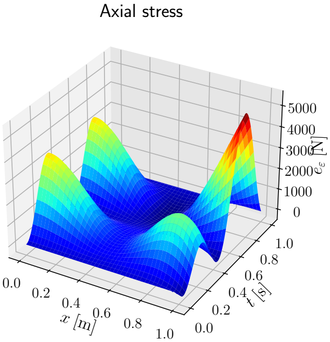


e_u^h ($h = 2^{-5}, k = 2$).

Results

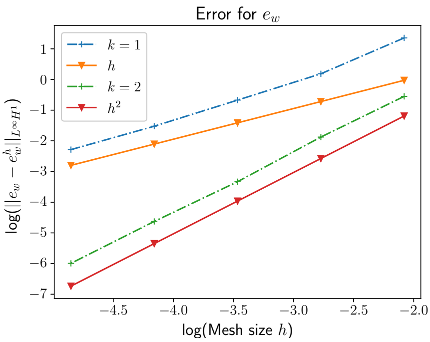


$L^\infty_{\Delta t}(L^2)$ error for e_ε.

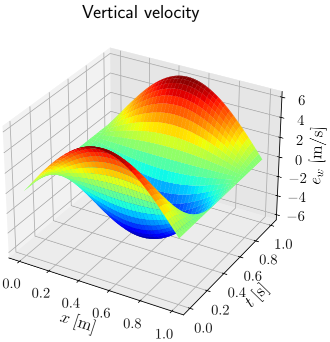


e^h_ε for $h = 2^{-5}, k = 2$.

Results

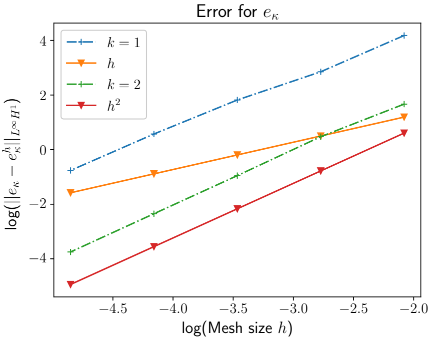


$L^\infty_{\Delta t}(H^1)$ error for e_w.

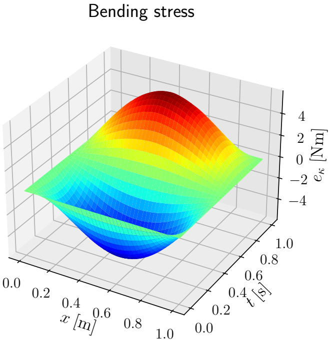


e_w^h for $h = 2^{-5}, k = 2$.

Results



$L^\infty_{\Delta t}(H^1)$ error for e_κ.



e^h_κ for $h = 2^{-5}, k = 2$.

Conclusion and Outlook

- First step into pH non linear mechanics. The geometrical non linearities belong to the interconnection operator.
- Natural extension for the 2D case (fancier FE).
- Can be used to study more complex phenomena. The discretization method guarantees energy conservation.

References I

Arnold, Douglas N., Richard S. Falk, and Ragnar Winther (2006). “Finite element exterior calculus, homological techniques, and applications”. In: *Acta Numerica* 15, 1–155.

Brugnoli, Andrea and Denis Matignon (2022). “A port-Hamiltonian formulation for the full von-Kàrmàn plate model”. In: *10th European Nonlinear Dynamics Conference (ENOC)*.

Puel, J.P. and M. Tucsnak (1996). “Global existence for the full von Kármán system”. In: *Applied Mathematics and Optimization* 34.2, pp. 139–160.

Port-Hamiltonian von-Kármán plates

$$
\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u \\ \boldsymbol{A}_\varepsilon \\ w \\ \alpha_w \\ \boldsymbol{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div}\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{Grad}\,\mathrm{grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix}\delta_{\boldsymbol{\alpha}_u} H \\ \delta_{\boldsymbol{A}_\varepsilon} H \\ \delta_w H \\ \delta_{\alpha_w} H \\ \delta_{\boldsymbol{A}_\kappa} H\end{pmatrix},
$$

where

$$
\mathcal{C}(w)(\boldsymbol{T}) = \operatorname{div}(\boldsymbol{T}\,\mathrm{grad}\, w),
$$

$$
\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\mathrm{grad}(\cdot) \otimes \mathrm{grad}(w) + \mathrm{grad}(w) \otimes \mathrm{grad}(\cdot)\right].
$$