

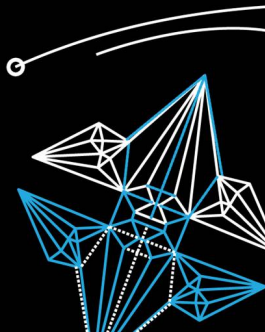
Portwings Internal Meeting

Challenges and outlook for the numerics

Andrea Brugnoli

Department of Robotics and Mechatronics,
University of Twente

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What numerics for the portwings project?

Methods should preserve the continuous structure at the discrete level. Which structure?

- ❶ Cohomology: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla \times} V^2(\mathbb{R}^3) \xrightarrow{\nabla \cdot} V^3(\mathbb{R});$
- ❷ Variational structure $\delta \int I = 0$, (I Lagrangian density);
- ❸ Hamiltonian structure $\mathcal{F} = \{F, H\}, \{\cdot, \cdot\}$ Poisson brackets.
- ❹ ...

Recent developments:

- ▶ splitting of topological and metric operators (Bauer and Behrens 2018);
- ▶ Lie group structure and underlying variational formulation (Gawlik and Gay-Balmaz 2020);
- ▶ connection with algebraic topology, i.e. de Rham complex and more general Hilbert complexes, e.g. elasticity (Bochev and Hyman 2006; Arnold, Falk, and Winther 2006; Palha et al. 2014);

Principle behind split discretization

- ▶ Fluid equation written in covariant form (exterior calculus);
- ▶ Split Hamiltonian form $\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}$.
 - ▶ Topological bracket depending on d (exterior derivative) or ι_v (interior product).
 - ▶ Metric dependent \mathcal{H} , since it depends on $*$

Linear shallow water waves in Hamiltonian form

- ▶ $\mathcal{H} = \frac{1}{2} \int_{\Omega} \left\{ \bar{h} \|\mathbf{u}\|^2 + gh^2 \right\} d\Omega$, with $\frac{\delta \mathcal{H}}{\delta \mathbf{u}} = \bar{h} \mathbf{u}$, $\frac{\delta \mathcal{H}}{\delta h} = gh$, where g gravity acc. and \bar{h} equilibrium fluid height.
- ▶ $\{\mathcal{F}, \mathcal{G}\} = -\left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}}, \nabla \frac{\delta \mathcal{G}}{\delta h}\right) - \left(\frac{\delta \mathcal{F}}{\delta h}, \nabla \cdot \frac{\delta \mathcal{G}}{\delta \mathbf{u}}\right)$.

$$\mathcal{F} = \int \mathbf{u} d\Omega : \dot{\mathcal{F}} = - \left(\frac{\delta \mathcal{F}}{\delta \mathbf{u}}, \nabla \frac{\delta \mathcal{H}}{\delta h} \right) \rightarrow \partial_t \mathbf{u} = -g \nabla h.$$

$$\mathcal{F} = \int h d\Omega : \dot{\mathcal{F}} = - \left(\frac{\delta \mathcal{F}}{\delta h}, \nabla \cdot \frac{\delta \mathcal{H}}{\delta \mathbf{u}} \right) \rightarrow \partial_t h = -\bar{h} \nabla \cdot \mathbf{u}.$$

Split and weak (or mixed) form

De Rham complex: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla \times} V^2(\mathbb{R}^3) \xrightarrow{\nabla \cdot} V^3(\mathbb{R})$.

Split form

$$\begin{array}{ccc} h \in V^0 & \xrightarrow{\nabla} & V^1 \ni \mathbf{u} \\ \tilde{h} = \tilde{*}h \downarrow & & \downarrow \mathbf{u} = \tilde{*}\mathbf{u} \\ \tilde{h} \in V^3 & \xleftarrow{\nabla \cdot} & V^2 \ni \tilde{\mathbf{u}} \end{array}$$

- Assume full $\tilde{*}$ in metric eqs.
- Both $\nabla, \nabla \cdot$ are imposed strongly.
- Both diff. eqs exact

Weak form (Mixed FE)

$$\begin{array}{ccc} h \in V^3 & \xrightarrow{\hat{\nabla}} & V^2 \ni \mathbf{u} \\ \tilde{h} = h \downarrow & & \downarrow \tilde{\mathbf{u}} = \mathbf{u} \\ \tilde{h} \in V^3 & \xleftarrow{\nabla \cdot} & V^2 \ni \tilde{\mathbf{u}} \end{array}$$

- Assume $\tilde{*} = \text{Id}$, i.e. $\tilde{\mathbf{u}} = \mathbf{u}, \tilde{h} = h$.
- Weak gradient $\hat{\nabla}$.
- Moment Eq. weak

Split decomposition in practice

- ❶ First projection of the strong form. For the 1D case:

$$\partial_t \mathbf{u}_e^{(1)} + g \mathbf{D}^{en} \mathbf{h}_n^{(0)} = \mathbf{0}, \quad \mathbf{u}_e^{(1)} : 1 \text{ form}, \quad \mathbf{h}_n^{(0)} : 0 \text{ form},$$

$$\partial_t \tilde{\mathbf{h}}_e^{(1)} + \bar{h} \mathbf{D}^{en} \tilde{\mathbf{u}}_n^{(0)} = \mathbf{0}, \quad \tilde{\mathbf{h}}_e^{(1)} : 1 \text{ form}, \quad \tilde{\mathbf{u}}_n^{(0)} : 0 \text{ form}.$$

\mathbf{D}^{en} metric free approximation of exterior derivative.

- ❷ Project the metric closure relations:

- High accuracy $CG_1^u - CG_1^h$ spaces:

$$\mathbf{M}^{nn} \tilde{\mathbf{u}}_n^{(0)} = \mathbf{P}^{ne} \mathbf{u}_e^{(1)}, \quad \mathbf{M}^{nn} \mathbf{h}_n^{(0)} = \mathbf{P}^{ne} \tilde{\mathbf{h}}_e^{(1)};$$

- Low accuracy $CG_0^u - CG_0^h$ spaces:

$$\mathbf{M}^{en} \tilde{\mathbf{u}}_n^{(0)} = \mathbf{I}^{ee} \mathbf{u}_e^{(1)}, \quad \mathbf{M}^{en} \mathbf{h}_n^{(0)} = \mathbf{I}^{ee} \tilde{\mathbf{h}}_e^{(1)};$$

- Medium accuracy $CG_1^u - CG_0^h$ spaces:

$$\mathbf{M}^{nn} \tilde{\mathbf{u}}_n^{(0)} = \mathbf{P}^{ne} \mathbf{u}_e^{(1)}, \quad \mathbf{M}^{en} \mathbf{h}_n^{(0)} = \mathbf{I}^{ee} \tilde{\mathbf{h}}_e^{(1)};$$

\mathbf{M}^{nn} , \mathbf{M}^{en} metric dependent, \mathbf{P}^{ne} metric free averaging op.

Weak (mixed) form

Weak formulation: find $\mathbf{u} \in V^2(\mathbb{R}^3)$, $h \in V^3(\mathbb{R})$

$$\begin{aligned}(\mathbf{v}_u, \partial_t \mathbf{u})_{L^2} &= -(\mathbf{v}_u, g \widehat{\nabla} h)_{L^2} = (\nabla \cdot \mathbf{v}_u, gh)_{L^2}, & \forall \mathbf{v}_u \in V^2, \\(v_h, \partial_t h)_{L^2} &= -(v_h, \bar{h} \nabla \cdot \mathbf{u})_{L^2}, & \forall v_h \in V^3.\end{aligned}$$

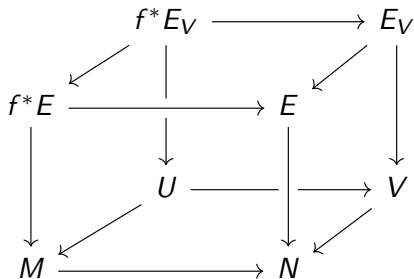
After projection

$$\begin{bmatrix} \mathbf{M}^{nn} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{ee} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{h}_e \end{pmatrix} = \begin{bmatrix} \mathbf{0} & g \mathbf{D}^{ne} \\ -\bar{h} \mathbf{D}^{en} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{h}_e \end{pmatrix}, \quad \mathbf{D}^{ne} = \mathbf{D}^{en \top}.$$

Another formulation with weak divergence: find $\mathbf{u} \in V^1(\mathbb{R}^3)$, $h \in V^0(\mathbb{R})$

$$\begin{aligned}(\mathbf{v}_u, \partial_t \mathbf{u})_{L^2} &= -(\mathbf{v}_u, g \nabla h)_{L^2}, & \forall \mathbf{v}_u \in V^1, \\(v_h, \partial_t h)_{L^2} &= -(v_h, \bar{h} \widehat{\nabla} \cdot \mathbf{u})_{L^2} = (\nabla v_h, \bar{h} \mathbf{u})_{L^2}, & \forall v_h \in V^0.\end{aligned}$$

Connection with algebraic topology



Tools available

Firedrake: <https://www.firedrakeproject.org/>.

FEniCS: <https://fenicsproject.org/>.

Fluid Structure Interaction in Fenics: Bergersen et al. 2020.

Mesh morphing in FEniCS: <https://bitbucket.org/Epoxid/femorph/src/c7317791c8f00d70fe16d593344cb164a53cad9b/?at=dokken%2Ffrestructuring>

PyDec: <https://github.com/hirani/pydec>

Learning Python for scientific computing <https://faculty.math.illinois.edu/~hirani/cbmgi/index.html>