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Port-Hamiltonian formulation and Symplectic discretization of Plate models

Part I : Mindlin model for thick plates

Part II : Kirchhoff model for thin plates

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Response to reviewers

We gratefully acknowledge each reviewer for the comments. We resume here the corrections apported to both papers.

Reviewer 1

The reviewer suggests to give a comment in the introduction part,in order to clarify that the paper addresses a different subject from the the papers of Symplectic approach for plate bending.

In the introduction of Part I we have added a remark upon the Symplectic elasticity advantages:
This approach offer insights on analytical solutions (see [4] for a closed solution of the eigenproblem of rectangular Reissner plates) and is of use whenever easy engineering solutions are sought after.
In this way the respective advantages of the two approaches are underlined. When it comes to obtain easy engineering solutions the Symplectic Elasticity paradigm is a powerful tool. The PH one on the contrary can be employed for complex applications involving large systems constructed in a modular way. The reader can therefore consult the Symplectic Elasticity references if its objective is the former.

Reviewer 2

Part I: Mindlin model for thick plates

Following remarks

- Introduction
  - page 1: formally skew-symmetric Hamiltonian differential operator is a pleonasm as a Hamiltonian operator should be skew-symmetric and obey the Jacobi identities [1].. Maybe you may just keep: Hamiltonian operator ? .
  - the authors might refer to for the relation between Lagrangian and Hamiltonian formu- lations including port variables and Dirac structures [2, 3]
- Reminder on port Hamiltonian systems
  - page 4: I do not like orthogonal complement which reminds of a metric structure but would rather prefer: isotropic and coisotropic.
  - page 6 Remark 1: Rather then inner product which reminds of a metric, I would write pairing (which just means the bilinearity)

we provide corresponding corrections in the manuscript. In the highlighted version the corresponding corrections are highlighted in blue.

For the remark "page 20 : The sentence Anyway, the Lagrange multipliers are defined only over the boundary. is cryptic." we have added a small period to better explain how the Lagrange multiplier are dealt with:

The Lagrange multipliers $\boldsymbol{\lambda}$ are discretized by using Lagrange polynomials defined over the boundary. The order of the Lagrange polynomials is the same as the one chosen for the co-energy variable. The corresponding finite element space is denoted by $H^1_r(\mathbb{P}_l, \partial\Omega)$.

Concerning the comment "page 20: Remark 4. The sentence This choice does not correspond to the optimal one given by $D(J)$. is not understandable. Please recall the equation where the operator J is defined ! Is it not H ? What means optimal : the projection space does not belong to the domain ?", with optimal we meant the fact that the finite elements have the minimum possible degree of regularity. Performing a numerical analysis of this PH system is still to be done and not easy. We simply wanted to suggest the fact that the for the numerical analysis the Arnold Winther element is more suited than Lagrange polynomials. We decide therefore to remove the remark and just leave a comment in the conclusion:
The Arnold-Winther element should be investigated as they provide a conforming approximation of space $H^{\mathrm{Div}}(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Unfortunately, this are not included inside FEniCS (or in any standard library).

Part II: Kirchhoff model for thin plates

We followed the suggestion
"page 6. After the sentence: This theorem states that, for smooth functions, higher order partial derivative commute., remove all detailed equations. This statement is enough as it corresponds to define the jet bundle over the displacements." and eliminate the superfluous comments.

Common remark

One comment was adressed to both papers:
"page 21: It is written: The symplectic Störmer-Verlet time integrator is employed, so that when no solicitation is applied to the system, the Hamiltonian is preserved. In theory the symplectic scheme should preserve the symplectic structure not the Hamiltonian which is expected not to be conserved by to oscillate around a mean value ? Furthermore how do you aply the scheme to a model which is defined with respect to a skew-symmetric matrix which is not in canonical coordinates?"

It is true that for closed systems the Hamiltonian is not perfectly conserved but only the symplectic structure. For this we modified the comment: The Störmer-Verlet time integrator is employed, so that the symplectic structure is preserved.
For what concerned how this integrator is employed in our particular case, this integrator (also known as leapfrog in the context of PDEs) does not require a skew-symmetric matrix in canonical coordinates (another application to non canonical skew-symmetric matrices can be found in [TRLGK18]). Given the system

$$\begin{aligned} M_p \dot{e}_p &= D e_q \\ M_q \dot{e}_q &= -D^T e_p \end{aligned}$$

the Leapfrog (Störmer-Verlet) scheme consist of three steps

$$\begin{aligned} M_p e_p^{n+1/2} &= M_p e_p^n + \frac{\Delta t}{2} D e_q^n \\ M_q e_q^{n+1} &= M_q e_q^n - \Delta t D^T e_p^{n+1/2} \\ M_p e_p^{n+1} &= M_p e_p^{n+1/2} + \frac{\Delta t}{2} D e_q^{n+1} \end{aligned}$$

Furthermore, the constrained case can be dealt with as well by applying the RATTLE scheme [HLW06, Chapter VII]. For the system

$$\begin{aligned} M_p \dot{e}_p &= D e_q + G\lambda \\ M_q \dot{e}_q &= -D^T e_p \\ 0 &= -G^T e_p \end{aligned}$$

the Leapfrog scheme is modified by inserting the projection matrix $P_\lambda = I - G\left(G^T M_p^{-1} G\right)^{-1} G^T M_p^{-1}$ in the dynamics of e_p (this corresponds to derive and solve for λ the constrain equation, given the actual

e_q variable)

$$
\begin{aligned}
M_p e_p^{n+1/2} &= M_p e_p^n + \frac{\Delta t}{2} P_\lambda D e_q^n \\
M_q e_q^{n+1} &= M_q e_q^n - \Delta t D^T e_p^{n+1/2} \\
M_p e_p^{n+1} &= M_p e_p^{n+1/2} + \frac{\Delta t}{2} P_\lambda D e_q^{n+1}
\end{aligned}
$$

Minor Modification for Part II

We also modify the definition of the flux variables in the augmented Dirac Structure considering dissipation of section 3.1.2 so that the pairing is analogous to the other pairing defined elsewhere. From 3.1.2. The augmented structure

$$
\begin{multline}
\mathcal{D}_r := \Big\{ (\boldsymbol{f}, \boldsymbol{f}_r) \in \mathcal{F}, \ (\boldsymbol{e}, \boldsymbol{e}_r) \in \mathcal{E}, \ \boldsymbol{z} \in \mathcal{Z} \mid \\
\boldsymbol{f} = -\frac{\partial \boldsymbol{\alpha}}{\partial t} = -J\boldsymbol{e} - G_R \boldsymbol{f}_r, \ \boldsymbol{f}_r = -S\boldsymbol{e}_r, \ \boldsymbol{e}_r = G_R^* \boldsymbol{e}, \ \boldsymbol{z} = B_\partial(\boldsymbol{e}) \Big\} \tag{1}
\end{multline}
$$

is a Stokes-Dirac structure with respect to the paring

$$
\begin{multline}
\ll (\boldsymbol{f}_1, \boldsymbol{f}_{r,1}, \boldsymbol{e}_1, \boldsymbol{e}_{r,1}, \boldsymbol{z}_1), (\boldsymbol{f}_2, \boldsymbol{f}_{r,2}, \boldsymbol{e}_2, \boldsymbol{e}_{r,2}, \boldsymbol{z}_2) \gg = \\
\int_\Omega \left[\boldsymbol{e}_1^T \boldsymbol{f}_2 + \boldsymbol{e}_2^T \boldsymbol{f}_1 + \boldsymbol{e}_{r,1}^T \boldsymbol{f}_{r,2} + \boldsymbol{e}_{r,2}^T \boldsymbol{f}_{r,1} \right] d\Omega + \int_{\partial\Omega} B_J(\boldsymbol{z}_1, \boldsymbol{z}_2) \, ds. \tag{2}
\end{multline}
$$

References

[HLW06] E. Hairer, C. Lubich, and G. Wanner. *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*, volume 31. Springer Science & Business Media, 2006.

[TRLGK18] Vincent Trenchant, Hector Ramírez, Yann Le Gorrec, and Paul Kotyczka. Finite differences on staggered grids preserving the port-Hamiltonian structure with application to an acoustic duct. *Journal of Computational Physics*, 373, 06 2018.