Tensorial Formulations for thin and thick plates
Weak Formulation and Discretization Procedure

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February 18, 2019

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The corresponding 1D models

Timoshenko beam

- Valid for thick beams
- Dimension of the PH model: 4
- Differential operator J of order 1

Euler-Bernoulli beam

- Valid for thin beams
- Dimension of the PH model: 2
- Differential operator J of order 2

$$\boldsymbol{\alpha} = [\rho v,\ I_\rho \omega_x,\ \frac{\partial \phi_x}{\partial x},\ \frac{\partial w}{\partial x} - \phi_x]^T$$

$$\boldsymbol{e} = [v,\ \omega_x,\ M_{xx},\ T_x]^T$$

$$J = \begin{pmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\alpha} = [\rho v,\ \frac{\partial^2 w}{\partial x^2}]^T$$

$$\boldsymbol{e} = [v,\ M_{xx}]^T$$

$$J = \begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

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Energy and co-energy variables

Linear momenta and curvature are taken as energy variables. Additional the shear strain are considered, leading to

$$\boldsymbol{\alpha} = \left(\rho h v,\ \rho \frac{h^3}{12}\omega_x,\ \rho \frac{h^3}{12}\omega_y,\ \kappa_{xx},\ \kappa_{yy},\ \kappa_{xy},\ \gamma_{xz},\ \gamma_{yz}\right)^T$$

where $v = \frac{\partial w}{\partial t},\ \omega_x = \frac{\partial \psi_x}{\partial t},\ \omega_y = \frac{\partial \psi_y}{\partial t}$. The Hamiltonian density is quadratic in the energy variables

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{D}_b & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{D}_s \end{bmatrix} \boldsymbol{\alpha}$$

The variational derivative provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v,\ \omega_x,\ \omega_y,\ M_{xx},\ M_{yy},\ M_{xy},\ Q_x,\ Q_y)^T$$

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 & 0 & 0 & 0 & \\ \frac{\partial}{\partial y} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial \boldsymbol{\alpha}}{\partial t} = J\mathbf{e}.$$

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Main References

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon.

Port-hamiltonian formulation and symplectic discretization of plate models. Part I : Mindlin model for thick plates.
arXiv preprint arXiv:1809.11131, 2018.
Under Review

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon.

Port-hamiltonian formulation and symplectic discretization of plate models. Part II : Kirchhoff model for thin plates.
arXiv preprint arXiv:1809.11136, 2018.
Under Review

A scalar-vector-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \qquad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now κ_{xy} now is half the value of the one in the vectorial case.
The curvatures tensor is the linear deformation tensor applied to the rotation vector $\boldsymbol{\theta} = (\psi_x, \psi_y)^T$

$$\mathbb{K} = \mathrm{Grad}(\boldsymbol{\theta}) = \frac{1}{2}\left(\nabla \otimes \boldsymbol{\theta} + \left(\nabla \otimes \boldsymbol{\theta}\right)^T\right),$$

where Grad is the symmetric gradient operator applied to a vector, which gives rise to a symmetric tensor.
The momenta are found by introducing a fourth order tensor $\mathbb{D}$, such that $\mathbb{M}_{ij} = \sum_{k,l} \mathbb{D}_{ijkl} \mathbb{K}_{kl}$, a linear relation between $\mathbb{M}$ and $\mathbb{K}$.

Energy Variables

The energy variables are now distinguished with respect to their different nature

$$
\begin{aligned}
\alpha_w &= \rho h \frac{\partial w}{\partial t}, \qquad & \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t}, \\
\mathbb{A}_\kappa &= \mathbb{K}, \qquad & \boldsymbol{\alpha}_{\epsilon_s} &= \boldsymbol{\epsilon}_s.
\end{aligned}
$$

The Hamiltonian is expressed as follows

$$
H = \int_\Omega \frac{1}{2} \left\{ \rho h \left(\frac{\partial w}{\partial t} \right)^2 + \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t} \cdot \frac{\partial \boldsymbol{\theta}}{\partial t} + \mathbb{M} : \mathbb{K} + \boldsymbol{Q} \cdot \boldsymbol{\epsilon}_s \right\} \mathrm{d}\Omega,
$$

where $\mathbb{M} : \mathbb{K}$ denotes the tensor contraction operation.
The momenta $\mathbb{M}$ depends linearly on $\mathbb{K}$, hence $\frac{1}{2}\mathbb{M} : \mathbb{K}$ is a quadratic form in $\mathbb{K}$.

Co-energy variables

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \qquad \boldsymbol{e}_\theta := \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t},$$

$$\mathbb{E}_\kappa := \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}, \qquad \boldsymbol{e}_{\epsilon_s} := \frac{\delta H}{\delta \boldsymbol{\epsilon}_s} = \boldsymbol{Q}.$$

where now the $\boldsymbol{\epsilon}_s$ and $\boldsymbol{Q}$ are the shear strain and stress respectively.

Proposition (see [1] for the proof)

The variational derivative of the Hamiltonian with respect to the curvatures tensor is the momenta tensor $\frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}$.

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Strong form and Interconnection structure

From div, the scalar divergence of a vector, we construct Div, the vector-valued divergence of a symmetric tensor, defined by

$$\boldsymbol{\varepsilon} = \mathrm{Div}(\mathbb{E}) \qquad \text{with } \boldsymbol{\varepsilon}_i = \mathrm{div}(\mathbb{E}_{ji}) = \sum_{j=1}^{n} \frac{\partial \mathbb{E}_{ji}}{\partial x_j}.$$

The port-Hamiltonian system is expressed as follows

$$\begin{cases} \dfrac{\partial \alpha_w}{\partial t} & = \mathrm{div}(\boldsymbol{e}_{\epsilon_s}), \\ \dfrac{\partial \boldsymbol{\alpha}_\theta}{\partial t} & = \mathrm{Div}(\mathbb{E}_\kappa) + \boldsymbol{e}_{\epsilon_s}, \\ \dfrac{\partial \mathbb{A}_\kappa}{\partial t} & = \mathrm{Grad}(\boldsymbol{e}_\theta), \\ \dfrac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} & = \mathrm{grad}(e_w) - \boldsymbol{e}_\theta \end{cases}.$$

If the variables are concatenated together, the formally skew-symmetric operator J can be highlighted

Strong form for the Mindlin plate

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \mathbb{A}_\kappa \\ \boldsymbol{\alpha}_{\epsilon_s} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ 0 & \mathrm{Grad} & 0 & 0 \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}}_{J} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \mathbb{E}_\kappa \\ \boldsymbol{e}_{\epsilon_s} \end{pmatrix},$$

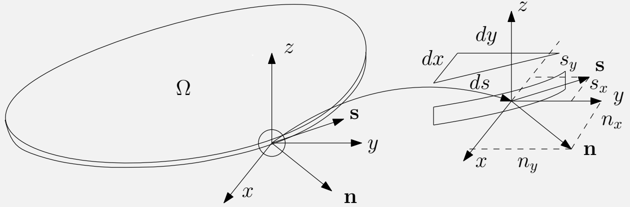
where all zeros are intended as nullifying operators from the space of input variables to the space of output variables.

Theorem (See [1] for additional details)

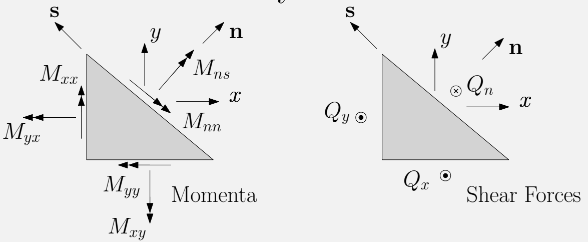
The adjoint of the tensor divergence Div *is* −Grad, *the opposite of the symmetric gradient.*

Boundary Variables

Conventions and references



Boundary variables



Energy flow

Again the boundary values can be found by evaluating the time derivative of the Hamiltonian

$$
\begin{aligned}
\dot{H} &= \int_{\Omega} \left\{ \frac{\partial \alpha_w}{\partial t} e_w + \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \cdot \boldsymbol{e}_\theta + \frac{\partial \mathbb{A}_\kappa}{\partial t} : \mathbb{E}_\kappa + \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} \cdot \boldsymbol{e}_{\epsilon_s} \right\} \, \mathrm{d}\Omega \\
&= \int_{\Omega} \left\{ \mathrm{div}(\boldsymbol{e}_{\epsilon_s}) e_w + [\mathrm{Div}(\mathbb{E}_\kappa) + \boldsymbol{e}_{\epsilon_s}] \cdot \boldsymbol{e}_\theta + \mathrm{Grad}(\boldsymbol{e}_\theta) : \mathbb{E}_\kappa + (\mathrm{grad}(e_w) - \boldsymbol{e}_\theta) \cdot \boldsymbol{e}_{\epsilon_s} \right\} \, \mathrm{d}\Omega \\
&= \int_{\partial\Omega} \left\{ \underbrace{(\boldsymbol{n} \cdot \boldsymbol{e}_{\epsilon_s})}_{Q_n} e_w + (\boldsymbol{n} \cdot \mathbb{E}_\kappa) \cdot \boldsymbol{e}_\theta \right\} \, \mathrm{d}\Omega \,, \\
&= \int_{\partial\Omega} \left\{ Q_n e_w + (\boldsymbol{n} \cdot \mathbb{E}_\kappa) \cdot (\omega_n \boldsymbol{n} + \omega_s \boldsymbol{s}) \right\} \, \mathrm{d}\Omega \,, \\
&= \int_{\partial\Omega} \left\{ Q_n e_w + \omega_n \underbrace{\boldsymbol{n}^T \mathbb{E}_\kappa \boldsymbol{n}}_{M_{nn}} + \omega_s \underbrace{\boldsymbol{s}^T \mathbb{E}_\kappa \boldsymbol{n}}_{M_{ns}} \right\} \, \mathrm{d}\Omega \,, \\
&= \int_{\partial\Omega} \left\{ Q_n e_w + \omega_n M_{nn} + \omega_s M_{ns} \right\} \, \mathrm{d}\Omega \,.
\end{aligned}
$$

Same result as in the vectorial case but with intrinsic operators.

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Weak Formulation

The first line is multiplied by v_w (multiplication by a scalar)

$$\int_\Omega v_w \frac{\partial \alpha_w}{\partial t} \, \mathrm{d}\Omega = \int_\Omega v_w \mathrm{div}(\boldsymbol{e}_{\epsilon_s}) \, \mathrm{d}\Omega,$$

the second and the fourth lines by $\boldsymbol{v}_\theta$, $\boldsymbol{v}_{\epsilon_s}$ (scalar product of $\mathbb{R}^2$)

$$\int_\Omega \boldsymbol{v}_\theta \cdot \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{v}_\theta \cdot (\mathrm{Div}(\mathbb{E}_\kappa) + \boldsymbol{e}_{\epsilon_s}) \, \mathrm{d}\Omega,$$
$$\int_\Omega \boldsymbol{v}_{\epsilon_s} \cdot \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{v}_{\epsilon_s} \cdot (\mathrm{grad}(e_w) - \boldsymbol{e}_\theta) \, \mathrm{d}\Omega,$$

the third one by $\mathbb{V}_\kappa$ (tensor contraction)

$$\int_\Omega \mathbb{V}_\kappa : \frac{\partial \mathbb{A}_\kappa}{\partial t} \, \mathrm{d}\Omega = \int_\Omega \mathbb{V}_\kappa : \mathrm{Grad}(\boldsymbol{e}_\theta) \, \mathrm{d}\Omega.$$

Choice I: Boundary control through forces and momenta

The first two line of the system in weak form are integrated by parts

$$\int_\Omega v_w \mathrm{div}(\boldsymbol{e}_{\epsilon_s}) \, \mathrm{d}\Omega = \int_{\partial\Omega} v_w \underbrace{\boldsymbol{n} \cdot \boldsymbol{e}_{\epsilon_s}}_{Q_n} \, \mathrm{d}s - \int_\Omega \mathrm{grad}(v_w) \cdot \boldsymbol{e}_{\epsilon_s} \, \mathrm{d}\Omega,$$

$$\int_\Omega \boldsymbol{v}_\theta \cdot (\mathrm{Div}(\mathbb{E}_\kappa) + \boldsymbol{e}_{\epsilon_s}) \, \mathrm{d}\Omega = \int_{\partial\Omega} \boldsymbol{v}_\theta \cdot (\boldsymbol{n} \cdot \mathbb{E}_\kappa) \, \mathrm{d}s - \int_\Omega \left\{ \mathrm{Grad}(\boldsymbol{v}_\theta) : \mathbb{E}_\kappa - \boldsymbol{v}_\theta \cdot \boldsymbol{e}_{\epsilon_s} \right\} \, \mathrm{d}\Omega.$$

The usual additional manipulation is performed on the boundary term containing the momenta, so that the proper boundary values arise

$$\int_{\partial\Omega} \boldsymbol{v}_\theta \cdot (\boldsymbol{n} \cdot \mathbb{E}_\kappa) \, \mathrm{d}s = \int_{\partial\Omega} \left\{ \underbrace{(\boldsymbol{v}_\theta \cdot \boldsymbol{n})}_{v_{\omega_n}} \boldsymbol{n} + \underbrace{(\boldsymbol{v}_\theta \cdot \boldsymbol{s})}_{v_{\omega_s}} \boldsymbol{s} \right\} \cdot (\boldsymbol{n} \cdot \mathbb{E}_\kappa) \, \mathrm{d}s$$

$$= \int_{\partial\Omega} \left\{ v_{\omega_n} M_{nn} + v_{\omega_s} M_{ns} \right\} \, \mathrm{d}s.$$

The final system exhibits as control inputs the boundary forces and momenta

Weak Form with forces and momenta as inputs

$$
\begin{cases}
\int_\Omega v_w \dfrac{\partial \alpha_w}{\partial t} \, \mathrm{d}\Omega & = - \int_\Omega \mathrm{grad}(v_w) \cdot \boldsymbol{e}_{\epsilon_s} \, \mathrm{d}\Omega + \int_{\partial\Omega} v_w \, Q_n \, \mathrm{d}s \\
\int_\Omega \boldsymbol{v}_\theta \cdot \dfrac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \, \mathrm{d}\Omega & = - \int_\Omega \left\{ \mathrm{Grad}(\boldsymbol{v}_\theta) : \mathbb{E}_\kappa - \boldsymbol{v}_\theta \cdot \boldsymbol{e}_{\epsilon_s} \right\} \, \mathrm{d}\Omega + \int_{\partial\Omega} \left\{ v_{\omega_n} M_{nn} + v_{\omega_s} M_{ns} \right\} \, \mathrm{d}s \\
\int_\Omega \mathbb{V}_\kappa : \dfrac{\partial \mathbb{A}_\kappa}{\partial t} \, \mathrm{d}\Omega & = \int_\Omega \mathbb{V}_\kappa : \mathrm{Grad}(\boldsymbol{e}_\theta) \, \mathrm{d}\Omega \\
\int_\Omega \boldsymbol{v}_{\epsilon_s} \cdot \dfrac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} \, \mathrm{d}\Omega & = \int_\Omega \boldsymbol{v}_{\epsilon_s} \cdot (\mathrm{grad}(e_w) - \boldsymbol{e}_\theta) \, \mathrm{d}\Omega
\end{cases}
.
$$

In this first case, the boundary controls $\boldsymbol{u}_\partial$ and the corresponding output $\boldsymbol{y}_\partial$ are

$$
\boldsymbol{u}_\partial = \begin{pmatrix} Q_n \\ M_{nn} \\ M_{ns} \end{pmatrix}_{\partial\Omega}, \qquad \boldsymbol{y}_\partial = \begin{pmatrix} e_w \\ \omega_n \\ \omega_s \end{pmatrix}_{\partial\Omega}.
$$

Choice II: Boundary control through kinematic variables

If instead the last two lines are integrated by parts

Weak Form with linear and rotational velocities as inputs

$$
\begin{cases}
\int_\Omega v_w \dfrac{\partial \alpha_w}{\partial t} \,\mathrm{d}\Omega &= \int_\Omega v_w \mathrm{div}(\boldsymbol{e}_{\epsilon_s}) \,\mathrm{d}\Omega \\
\int_\Omega \boldsymbol{v}_\theta \cdot \dfrac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \,\mathrm{d}\Omega &= \int_\Omega \boldsymbol{v}_\theta \cdot \left(\mathrm{Div}(\mathbb{E}_\kappa) + \boldsymbol{e}_{\epsilon_s}\right) \,\mathrm{d}\Omega \\
\int_\Omega \mathbb{V}_\kappa : \dfrac{\partial \mathbb{A}_\kappa}{\partial t} \,\mathrm{d}\Omega &= -\int_\Omega \mathrm{Div}(\mathbb{V}_\kappa) \cdot \boldsymbol{e}_\theta \,\mathrm{d}\Omega + \int_{\partial\Omega} \{ v_{M_{nn}} \omega_n + v_{M_{ns}} \omega_s \} \,\mathrm{d}s \\
\int_\Omega \boldsymbol{v}_{\epsilon_s} \cdot \dfrac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} \,\mathrm{d}\Omega &= -\int_\Omega \{ \mathrm{div}(\boldsymbol{v}_{\epsilon_s}) e_w + \boldsymbol{v}_{\epsilon_s} \cdot \boldsymbol{e}_\theta \} \,\mathrm{d}\Omega + \int_{\partial\Omega} v_{Q_n} e_w \,\mathrm{d}s
\end{cases}
$$

where $v_{M_{nn}} = \boldsymbol{n}^T \mathbb{V}_\kappa \boldsymbol{n}$, $v_{M_{ns}} = \boldsymbol{s}^T \mathbb{V}_\kappa \boldsymbol{n}$ and $v_{Q_n} = \boldsymbol{v}_{\epsilon_s} \cdot \boldsymbol{n}$. In this second case, the boundary controls $\boldsymbol{u}_\partial$ and corresponding output $\boldsymbol{y}_\partial$ are

$$
\boldsymbol{u}_\partial = \begin{pmatrix} e_w \\ \omega_n \\ \omega_s \end{pmatrix}_{\partial\Omega}, \qquad \boldsymbol{y}_\partial = \begin{pmatrix} Q_n \\ M_{nn} \\ M_{ns} \end{pmatrix}_{\partial\Omega}.
$$

Summary for the Mindlin Plate

Main points:

- variational derivative w.r.t. a tensor quantity;
- strong tensorial form;
- structure preserving weak formulation with different choices of the boundary inputs;
- only two choices for the boundary inputs were shown but others are possible.

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Energy and co-energy variables

This model is the 2D extension of the Bernoulli beam. It is logical to select as energy variable the linear momentum, together with the curvatures

$$\boldsymbol{\alpha} = (\mu v,\ \kappa_{xx},\ \kappa_{yy},\ \kappa_{xy})^T$$

where $v = \frac{\partial w}{\partial t}$. The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & \boldsymbol{D} \end{bmatrix} \boldsymbol{\alpha}$$

So the variational derivative of the total Hamiltonian $H = \int_\Omega \mathcal{H}\ d\Omega$ provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v,\ M_{xx},\ M_{yy},\ M_{xy})^T$$

Definition of J and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -\left(\frac{\partial^2}{\partial y\,\partial x} + \frac{\partial^2}{\partial x\,\partial y}\right) \\ \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y\,\partial x} + \frac{\partial^2}{\partial x\,\partial y} & 0 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial \boldsymbol{\alpha}}{\partial t} = J\mathbf{e}.$$

From the Schwarz theorem for C^2 functions the mixed derivative could be be expressed as $2\frac{\partial^2}{\partial x\,\partial y}$, instead of $\frac{\partial^2}{\partial y\,\partial x} + \frac{\partial^2}{\partial x\,\partial y}$. However, in this way the symmetry intrinsically present in $\gamma_{xy} = -z\left(\frac{\partial^2 w}{\partial y\,\partial x} + \frac{\partial^2 w}{\partial x\,\partial y}\right)$ would be lost. The mixed derivative is here split to reestablish the symmetric nature of curvatures and momenta (that are of tensorial nature).

Plan

A scalar-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \qquad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now κ_{xy} now is half the value of the one in the vectorial case. The curvatures tensor is the linear deformation tensor applied to the rotation vector $\boldsymbol{\theta} = \mathrm{grad}(w)$

$$\mathbb{K} = \mathrm{Grad}(\boldsymbol{\theta}) = \mathrm{Grad}(\mathrm{grad}(w)).$$

The momenta are found by introducing a fourth order tensor $\mathbb{D}$, such that $\mathbb{M}_{ij} = \mathbb{D}_{ijkl} \, \mathbb{K}_{kl}$

For what concerns the choice of the energy variables a scalar and a tensor variable are grouped together

$$\alpha_w = \mu \frac{\partial w}{\partial t} \qquad \mathbb{A}_\kappa = \mathbb{K}$$

The Hamiltonian energy is written as

$$H = \int_\Omega \left\{ \frac{1}{2} \mu \left(\frac{\partial w}{\partial t} \right)^2 + \frac{1}{2} \mathbb{M} : \mathbb{K} \right\} \, \mathrm{d}\Omega,$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \qquad \mathbb{E}_\kappa := \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}.$$

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Interconnection structure

The formally skew-symmetric operator J can be highlighted

Strong form for the Kirchhoff plate

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \mathbb{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\mathrm{div}\circ\mathrm{Div} \\ \mathrm{Grad}\circ\mathrm{grad} & 0 \end{bmatrix}}_{J}\begin{pmatrix} e_w \\ \mathbb{E}_\kappa\end{pmatrix}.$$

where all zeros are intended as nullifying operator from the space of input variables to the space of output variables.

Theorem (See [2] for additional details)

The adjoint of $\mathrm{div}\circ\mathrm{Div}$ *is* $\mathrm{Grad}\circ\mathrm{grad}$ *(i.e. the Hessian operator)*

Remark

The interconnection structure J now resembles that of the Bernoulli beam. The double divergence and the double gradient coincide, in dimension one, with the second derivative.

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Weak Formulation

The fist line is multiplied by v_w (scalar multiplication)

$$\int_\Omega v_w \frac{\partial \alpha_w}{\partial t} \ \mathrm{d}\Omega = \int_\Omega -v_w \, \mathrm{div}(\mathrm{Div}(\mathbb{E}_\kappa)) \ \mathrm{d}\Omega,$$

the second line by $\mathbb{V}_\kappa$ (tensor contraction)

$$\int_\Omega \mathbb{V}_\kappa : \frac{\partial \mathbb{A}_\kappa}{\partial t} \ \mathrm{d}\Omega = \int_\Omega \mathbb{V}_\kappa : \mathrm{Grad}(\mathrm{grad}(e_w)) \ \mathrm{d}\Omega.$$

Now depending on which line is integrated by parts different boundary control term can be selected.

Choice I: Boundary control through forces and momenta

The first line has to be integrated by parts twice

$$\int_\Omega -v_w \mathrm{div}(\mathrm{Div}(\mathbb{E}_\kappa)) \ \mathrm{d}\Omega = \int_{\partial\Omega} \underbrace{-\boldsymbol{n}\cdot\mathrm{Div}(\mathbb{E}_\kappa)}_{Q_n} v_w \ \mathrm{d}s + \int_\Omega \mathrm{grad}(v_w)\cdot\mathrm{Div}(\mathbb{E}_\kappa) \ \mathrm{d}\Omega$$

Applying again the integration by parts

$$\int_\Omega \mathrm{grad}(v_w)\cdot\mathrm{Div}(\mathbb{E}_\kappa) \ \mathrm{d}\Omega = \int_{\partial\Omega} \mathrm{grad}(v_w)\cdot(\boldsymbol{n}\cdot\mathbb{E}_\kappa) \ \mathrm{d}s - \int_\Omega \mathrm{Grad}(\mathrm{grad}(v_w)):\mathbb{E}_\kappa \ \mathrm{d}\Omega$$

The usual additional manipulation is performed on the boundary terms, so that the proper boundary values arise

$$\begin{aligned}
\int_{\partial\Omega} \mathrm{grad}(v_w)\cdot(\boldsymbol{n}\cdot\mathbb{E}_\kappa) \ \mathrm{d}s &= \int_{\partial\Omega} \left(\frac{\partial v_w}{\partial n}\boldsymbol{n} + \frac{\partial v_w}{\partial s}\boldsymbol{s}\right)\cdot(\boldsymbol{n}\cdot\mathbb{E}_\kappa) \ \mathrm{d}s \\
&= \int_{\partial\Omega} \left\{\frac{\partial v_w}{\partial n}\underbrace{\boldsymbol{n}^T\mathbb{E}_\kappa\boldsymbol{n}}_{M_{nn}} + \frac{\partial v_w}{\partial s}\underbrace{\boldsymbol{s}^T\mathbb{E}_\kappa\boldsymbol{n}}_{M_{ns}}\right\} \ \mathrm{d}s \\
&= \sum_{\Gamma_i\subset\partial\Omega} [M_{ns}v_w]_{\partial\Gamma_i} + \int_{\partial\Omega}\left\{\frac{\partial v_w}{\partial n}M_{nn} - v_w\frac{\partial M_{ns}}{\partial s}\right\} \ \mathrm{d}s
\end{aligned}$$

Defining the effective shear stress as $\widetilde{Q}_n = Q_n - \frac{\partial M_{ns}}{\partial s}$ If the boundary is regular the final expression becomes

$$\int_\Omega v_w \frac{\partial \alpha_w}{\partial t} \, \mathrm{d}\Omega = -\int_\Omega \mathrm{Grad}(\mathrm{grad}(v_w)) : \mathbb{E}_\kappa \, \mathrm{d}\Omega + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} + v_w \, \widetilde{Q}_n \right\} \, \mathrm{d}s.$$

Weak form with forces and momenta as inputs

$$\begin{cases} \int_\Omega v_w \frac{\partial \alpha_w}{\partial t} \, \mathrm{d}\Omega & = -\int_\Omega \mathrm{Grad}(\mathrm{grad}(v_w)) : \mathbb{E}_\kappa \, \mathrm{d}\Omega + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} + v_w \, \widetilde{Q}_n \right\} \, \mathrm{d}s, \\ \int_\Omega \mathbb{V}_\kappa : \frac{\partial \mathbb{A}_\kappa}{\partial t} \, \mathrm{d}\Omega & = \int_\Omega \mathbb{V}_\kappa : \mathrm{Grad}(\mathrm{grad}(e_w)) \, \mathrm{d}\Omega. \end{cases}$$

The control input $\boldsymbol{u}_\partial$ and the corresponding conjugate outputs $\boldsymbol{y}_\partial$ are

$$\boldsymbol{u}_\partial = \begin{pmatrix} \widetilde{Q}_n \\ M_{nn} \end{pmatrix}_{\partial\Omega}, \qquad \boldsymbol{y}_\partial = \begin{pmatrix} e_w \\ \frac{\partial e_w}{\partial n} \end{pmatrix}_{\partial\Omega}.$$

Choice II: Boundary control through kinematic variables

The same procedure can be performed on the second line of the system

Weak form with linear and angular velocities as inputs

$$\begin{cases} \displaystyle\int_\Omega v_w \frac{\partial \alpha_w}{\partial t} \, \mathrm{d}\Omega & = \displaystyle\int_\Omega -v_w \, \mathrm{div}(\mathrm{Div}(\mathbb{E}_\kappa)) \, \mathrm{d}\Omega, \\ \displaystyle\int_\Omega \mathbb{V}_\kappa : \frac{\partial \mathbb{A}_\kappa}{\partial t} \, \mathrm{d}\Omega & = \displaystyle\int_\Omega \mathrm{div}(\mathrm{Div}(\mathbb{V}_\kappa)) e_w \, d\Omega + \int_{\partial\Omega} \left\{ v_{M_{nn}} \frac{\partial e_w}{\partial n} + v_{\widetilde{Q}_n} e_w \right\} \mathrm{d}s. \end{cases}$$

where $v_{M_{nn}} = \boldsymbol{n}^T \mathbb{V}_\kappa \boldsymbol{n}$ and $v_{\widetilde{Q}_n} = -\mathrm{Div}(\mathbb{V}_\kappa) \cdot \boldsymbol{n} - \dfrac{\partial(\boldsymbol{s}^T \mathbb{V}_\kappa \boldsymbol{n})}{\partial s}$.
The control input $\boldsymbol{u}_\partial$ are now the kinematic boundary conditions, the corresponding conjugate outputs $\boldsymbol{y}_\partial$ are the dynamic boundary condition

$$\boldsymbol{u}_\partial = \begin{pmatrix} e_w \\ \dfrac{\partial e_w}{\partial n} \end{pmatrix}_{\partial\Omega}, \qquad \boldsymbol{y}_\partial = \begin{pmatrix} \widetilde{Q}_n \\ M_{nn} \end{pmatrix}_{\partial\Omega}.$$

Summary for the Kirchhoff Plate

Main points:

- new distributed port-Hamiltonian system involving second order differential operator in space dimension 2 for J;
- strong tensorial form where $\mathrm{div} \circ \mathrm{Div} = (\mathrm{Grad} \circ \mathrm{grad})^*$;
- structure preserving weak formulation with different choices of the boundary inputs;
- only two choices for the boundary variables were shown but others are possible;

Other contribution (not presented here):

- definition of the underlying Stokes-Dirac structure for the vectorial formulation of the Kirchhoff plate [2].

Plan

Main Reference

F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A structure-preserving partitioned finite element method for the 2d wave equation.
In *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 1–6, Valparaíso, CL, 2018.
http://oatao.univ-toulouse.fr/19965

General principles of the structure preserving discretization

The energy, co-energy and test functions of the *same* index are discretized by using the *same* bases:

$$
\begin{aligned}
\alpha_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x,y)\alpha_i^k(t), & e_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x,y) e_i^k(t), & v_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x,y) v_i^k \\
\alpha_i^{ap} &:= \boldsymbol{\phi}_i(x,y)^T \boldsymbol{\alpha}_i(t), & e_i^{ap} &:= \boldsymbol{\phi}_i(x,y)^T \boldsymbol{e}_i(t), & v_i^{ap} &:= \boldsymbol{\phi}_i(x,y)^T \boldsymbol{v}_i.
\end{aligned}
$$

The same procedure is applied for the boundary terms with a specific basis $\boldsymbol{\psi}$

$$
u_{\partial,i} \approx u_{\partial,i}^{ap} := \sum_{k=1}^{n_{\partial,i}} \psi_i^k(s) u_{\partial,i}^k(t) = \boldsymbol{\psi}_i(s)^T \boldsymbol{u}_{\partial,i}(t).
$$

Remark

The functions $\boldsymbol{\psi}_i(s)$ can be selected as the restriction of functions $\boldsymbol{\phi}$ over the boundary $\boldsymbol{\psi}(s) = \boldsymbol{\phi}(x(s), y(s))$ or in other ways.

Plan

Mindlin Plate: discretized operators

The formally skew-symmetric operator J is replaced with the following skew-symmetric matrix

$$J_d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -D_{x,71}^T & -D_{y,81}^T \\ 0 & 0 & 0 & -D_{x,42}^T & 0 & -D_{y,62}^T & D_{0,27} & 0 \\ 0 & 0 & 0 & 0 & -D_{y,53}^T & -D_{x,63}^T & 0 & D_{0,38} \\ 0 & D_{x,42} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{y,53} & 0 & 0 & 0 & 0 & 0 \\ 0 & D_{y,62} & D_{x,63} & 0 & 0 & 0 & 0 & 0 \\ D_{x,71} & -D_{0,27}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ D_{y,81} & 0 & -D_{0,38}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As a consequence of the integration by parts, a control input is included in the finite-dimensional system. Matrix B is defined by

$$B = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \\ 0_{\nu_4 \times n_{\partial,1}} & 0_{\nu_4 \times n_{\partial,2}} & 0_{\nu_4 \times n_{\partial,3}} \end{bmatrix}.$$

Final discretized system

The final system is written as

$$
\begin{aligned}
M\dot{\boldsymbol{\alpha}} &= J_d\,\boldsymbol{e} + B\,\boldsymbol{u}_\partial,\\
\boldsymbol{y}_\partial &= B^T\boldsymbol{e}
\end{aligned}
$$

where

- $M = \mathrm{Diag}[M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8]$ is a block diagonal matrix,
- $\boldsymbol{\alpha}$ is simply the concatenation of the $\boldsymbol{\alpha}_i$ (just like $\boldsymbol{e}$)
- $\boldsymbol{u}_\partial$ is the concatenation of the $\boldsymbol{u}_{\partial,i}$ ($\boldsymbol{u}_{\partial,1} = Q_n, \boldsymbol{u}_{\partial,2} = M_{nn}$ and $\boldsymbol{u}_{\partial,3} = M_{ns}$).

Plan

Kirchhoff Plate: discretized operators

The discretized system is written as

$$\begin{pmatrix} M_1 \dot{\boldsymbol{\alpha}}_1 \\ M_2 \dot{\boldsymbol{\alpha}}_2 \\ M_3 \dot{\boldsymbol{\alpha}}_3 \\ M_4 \dot{\boldsymbol{\alpha}}_4 \end{pmatrix} = \begin{bmatrix} 0 & -D_{xx}^T & -D_{yy}^T & -2D_{xy}^T \\ D_{xx} & 0 & 0 & 0 \\ D_{yy} & 0 & 0 & 0 \\ 2D_{xy} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_{\partial,1} \\ \boldsymbol{u}_{\partial,2} \end{pmatrix},$$

where M_i are square matrices (of size $N_i \times N_i$), D_{xx} is an $N_2 \times N_1$ matrix, D_{yy} is an $N_3 \times N_1$ matrix, D_{xy} is an $N_4 \times N_1$ matrix, B_1 is an $N_1 \times N_{\partial,1}$ matrix and finally B_2 is an $N_2 \times N_{\partial,2}$ matrix. The collocated output are defined as

$$\boldsymbol{y}_\partial = \begin{bmatrix} B_1^T & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix}.$$

Plan

Variational formulation using PFEM

The general variational form can be now stated:

- Energy variables belong to the mixed space $\alpha \in V_\alpha = V_{\alpha_1} \times \cdots \times V_{\alpha_n}$, where $\alpha_i \in V_{\alpha_i}$;
- Coenergy variables $e = \mathcal{Q}\alpha$ where $\mathcal{Q}$ is a coercive symmetrical operator $\mathcal{Q} : V_\alpha \to V_\alpha$;
- Test functions belong to the same space as α $v \in V_\alpha$

General structure preserving weak form with known inputs

If the boundary input u_∂ is known (i.e for the Mindlin plate free or clamped conditions) then

$$m\left(v, \frac{\partial \alpha}{\partial t}\right) = j(v, e) + b_{u_\partial}(v_{y_\partial}) \qquad \forall v \in V$$

where m is the mass bilinear, symmetric and coercive form, j is the interconnection bilinear and antisymmetric form and b is a linear functional.

Dealing with algebraic constraints

If the boundary input vector is partly unknown then Lagrange multipliers and corresponding test function have to be introduced. Variables $\lambda_{u_\partial}, v_{u_\partial} \in V_\lambda$ defined over Γ_λ, the boundary subset where the input are unknown and the conjugated output are set to zero (e.g. clamped plate with a free side).

General structure preserving weak form with unknown inputs

$$\begin{cases} m\left(v, \frac{\partial \alpha}{\partial t}\right) = j(v, e) + b(v_{y_\partial}, \lambda_{u_\partial}) \\ 0 = c(v_{u_\partial}, e_{y_\partial}) \end{cases} \qquad \forall v \in V, \ \forall v_{u_\partial} \in V_\lambda$$

where $v_{y_\partial} = \mathcal{B}_{y_\partial} v$ and $e_{y_\partial} = \mathcal{B}_{y_\partial} e$ ($\mathcal{B}$ is the boundary variables operator). Now b is a bilinear form on spaces $V_\alpha \times V_\lambda$ and c is defined over $V_\lambda \times V_\alpha$ and is obtained from b by swapping variables and applying $\mathcal{B}_{y_\partial}$ on e instead of v.

Numerical Study of the Mindlin Plate

This PH system can be discretized using a FE software. These numerical results were obtained using Fenics. To compare the eigenvalues publications [4, 5, 6] were used :

R. Durán, L. Hervella-Nieto, E. Liberman, and J. Solomin. Approximation of the vibration modes of a plate by Reissner-Mindlin equations.
Mathematics of Computation of the American Mathematical Society, 68(228):1447–1463, 1999

D.J. Dawe and O.L. Roufaeil. Rayleigh-Ritz vibration analysis of Mindlin plates.
Journal of Sound and Vibration, 69(3):345–359, 1980

H.C. Huang and E. Hinton. A nine node Lagrangian Mindlin plate element with enhanced shear interpolation.
Engineering Computations, 1(4):369–379, 1984

In the next tables these references will be denoted respectively by D-H, D-R, H-H. Our results are denoted by B-A.

Study case

Properties of the plate

- Square plate with $l_x = l_y = 1$;
- Young modulus $E = 70\, GPa$;
- Density $\rho = 2000\, kg/m^3$;
- Poisson modulus $\nu = 0.3$.

Boundary conditions considered

- All clamped CCCC ($w = 0$, $\omega_s = 0$, $\omega_n = 0$);
- Simply supported hard SSSS ($w = 0$, $\omega_s = 0$);
- Half clamped half simply supported SCSC;
- All clamped but one side free CCCF (F stands for free, i.e. $M_{nn} = 0, \; M_{ns} = 0, \; Q_n = 0,$).

Computational Complexity

A scalar viable discretized using n P^1 elements for each side gives rise to $(n+1) \times (n+1)$ dofs and one discretized using n P^2 elements contains $(2n+1) \times (2n+1)$ dofs. Hence since the system is of dimension 8:

- if 10 P^1 elements are used for each side the final system contains 968 states;
- if 10 P^2 elements are used for each side the final system contains 3528 states.

Benchmark variables

The variables are discretized by using Lagrange polynomials of order 1 (same order for each variable) or 2. Normalized eigenfrequency found in [5] are used as benchmark

$$\widehat{\omega}_{mn} = \omega^h_{mn} L \left(\frac{2(1+\nu)\rho}{E} \right)^{\frac{1}{2}}$$ Final result independent on E, ρ,

m and n being the numbers of half-waves occurring in the modes shapes in the x and y directions, respectively.
Colors used for comparing of the first 4 eigenvalues

- benchmark results;
- 1% error or less;
- up to 5% error;
- up to 15% error;
- spurious eigenvalue;

Eigenvalues for $t/L = 0.1$ using P^1

BCs	Mode	N : 10(B-A)	N : 10(D-H)	N : 20(B-A)	N : 20(D-H)	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	1.5999	1.5947	1.5917	1.5921	1.591	1.594
	$\widehat{\omega}_{21}$	3.0615	3.1181	3.0410	3.0595	3.039	3.046
	$\widehat{\omega}_{12}$	3.0615	3.1181	3.0410	3.0595	3.039	3.046
	$\widehat{\omega}_{22}$	4.3161	4.4477	4.2682	4.3106	4.263	4.285
SSSS	$\widehat{\omega}_{11}$	0.9324	0.9384	0.9324	0.9323	0.930	0.930
	$\widehat{\omega}_{21}$	2.2227	2.2893	2.2226	2.2366	2.219	2.219
	$\widehat{\omega}_{12}$	2.2227	2.2893	2.2226	2.2366	2.219	2.219
	$\widehat{\omega}_{22}$	3.4142	3.5657	3.3608	3.4450	3.405	3.406
SCSC	$\widehat{\omega}_{11}$	1.3111	1.3060	1.3013	1.3016	1.300	1.302
	$\widehat{\omega}_{21}$	2.4155	2.4664	2.3966	2.4120	2.394	2.398
	$\widehat{\omega}_{12}$	2.9082	2.9617	2.8871	2.9043	2.885	2.888
	$\widehat{\omega}_{22}$	3.8906	4.0126	3.8458	3.8830	3.839	3.852
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	1.0855	1.0812	1.0982	1.0848	1.081	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7636	1.7759	1.7461	1.7525	1.744	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.6696	2.7413	2.6575	2.6787	2.657	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.2248	3.3186	3.1997	3.2282	3.197	3.216

Eigenvalues for $t/L = 0.01$ using P^1

BCs	Mode	N : 10(B-A)	N : 10(D-H)	N : 20(B-A)	N : 20(D-H)	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	0.1967	.1754	.1765	.1754	.1754	.1754
	$\widehat{\omega}_{21}$	0.4030	.3668	.3604	.3599	.3574	.3576
	$\widehat{\omega}_{12}$	0.4030	.3668	.3604	.3599	.3574	.3576
	$\widehat{\omega}_{22}$	0.6431	.5487	.5358	.5323	.5264	.5274
SSSS	$\widehat{\omega}_{11}$	0.1706	.0972	.1128	.0965	.0963	.0963
	$\widehat{\omega}_{21}$	0.3576	.2486	.2660	.2426	.2406	.2406
	$\widehat{\omega}_{12}$	0.3576	.2486	.2660	.2426	.2406	.2406
	$\widehat{\omega}_{22}$	0.5803	.4035	.3865	.3893	.3847	.3848
SCSC	$\widehat{\omega}_{11}$	0.1864	.1417	.1487	.1413	.1411	.1411
	$\widehat{\omega}_{21}$	0.3649	.2748	.2829	.2688	.2668	.2668
	$\widehat{\omega}_{12}$	0.3987	.3474	.3485	.3402	.3377	.3377
	$\widehat{\omega}_{22}$	0.6075	.4814	.4933	.4657	.4604	.4608
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	0.1238	.1182	.1166	.1170	.1166	.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.2207	.1977	.1954	.1956	.1949	.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3204	.3193	.3078	.3109	.3080	.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.4144	.3874	.3751	.3771	.3736	.3740

Eigenvalues for $t/L = 0.1$ using P^2

BCs	Mode	$N = 5$ (B-A)	$N = 10$ (B-A)	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	1.5976	1.5914	1.591	1.594
	$\widehat{\omega}_{21}$	3.0584	3.0405	3.039	3.046
	$\widehat{\omega}_{12}$	3.0677	3.0405	3.039	3.046
	$\widehat{\omega}_{22}$	4.3109	4.2662	4.263	4.285
SSSS	$\widehat{\omega}_{11}$	0.9304	0.9302	0.930	0.930
	$\widehat{\omega}_{21}$	2.2223	2.2194	2.219	2.219
	$\widehat{\omega}_{12}$	2.2224	2.2194	2.219	2.219
	$\widehat{\omega}_{22}$	3.4128	3.4061	3.405	3.406
SCSC	$\widehat{\omega}_{11}$	1.3053	1.3004	1.300	1.302
	$\widehat{\omega}_{21}$	2.4040	2.3946	2.394	2.398
	$\widehat{\omega}_{12}$	2.9060	2.8858	2.885	2.888
	$\widehat{\omega}_{22}$	3.8721	3.8415	3.839	3.852
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	1.0845	1.0797	1.081	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7559	1.7425	1.744	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.6762	2.6547	2.657	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.2186	3.1954	3.197	3.216

Eigenvalues for $t/L = 0.01$ using P^2

BCs	Mode	$N = 5$ (B-A)	$N = 10$ (B-A)	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	0.1872	0.1762	0.1754	0.1754
	$\widehat{\omega}_{21}$	0.3725	0.3598	0.3574	0.3576
	$\widehat{\omega}_{12}$	0.4055	0.3598	0.3574	0.3576
	$\widehat{\omega}_{22}$	0.6043	0.5335	0.5264	0.5274
SSSS	$\widehat{\omega}_{11}$	0.0963	0.0963	0.0963	0.0963
	$\widehat{\omega}_{21}$	0.2422	0.2406	0.2406	0.2406
	$\widehat{\omega}_{12}$	0.2430	0.2406	0.2406	0.2406
	$\widehat{\omega}_{22}$	0.3874	0.3848	0.3847	0.3848
SCSC	$\widehat{\omega}_{11}$	0.1492	0.1418	0.1411	0.1411
	$\widehat{\omega}_{21}$	0.2827	0.2683	0.2668	0.2668
	$\widehat{\omega}_{12}$	0.3608	0.3394	0.3377	0.3377
	$\widehat{\omega}_{22}$	0.4940	0.4654	0.4604	0.4608
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	0.1197	0.1169	0.1166	0.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.2092	0.1960	0.1949	0.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3188	0.3089	0.3080	0.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.3938	0.3757	0.3736	0.3740

Temporal Simulation

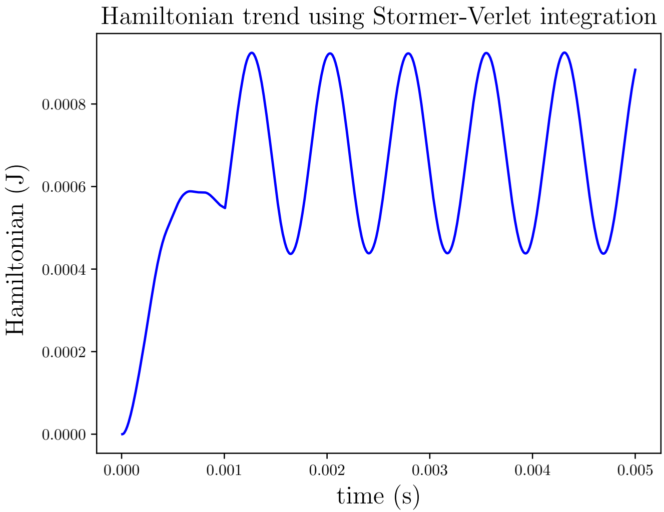
Consider a uniform specific load acting on a clamped plate at rest

$$\boldsymbol{f}(t) = \begin{cases} -1 \ m/s^2 \ \hat{\boldsymbol{z}} & t < 0.01 \ s \\ 0 & t > 0.01 \ s \end{cases}$$

Corresponding linear form: $l(v) = \int \rho h v_w f \; \mathrm{d}\Omega$.
Parameters:

- Total simulation time: $T = 0.05$
- Time step: $dt = 6.85\, 10^{-6} s$
- Integration in time: Stormer-Verlet

Simulation output $e_w = \frac{\partial w}{\partial t}$ (scale factor $\times 500$, 100 frame per second)


Hamiltonian trend using Stormer-Verlet integration
Hamiltonian (J)
0.0000
0.0002
0.0004
0.0006
0.0008
0.000
0.001
0.002
0.003
0.004
0.005
time (s)

[1] A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-hamiltonian formulation and symplectic discretization of plate models. Part I : Mindlin model for thick plates. *arXiv preprint arXiv:1809.11131*, 2018. Under Review.

[2] A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-hamiltonian formulation and symplectic discretization of plate models. Part II : Kirchhoff model for thin plates. *arXiv preprint arXiv:1809.11136*, 2018. Under Review.

[3] F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A structure-preserving partitioned finite element method for the 2d wave equation. In *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 1–6, Valparaíso, CL, 2018. `http://oatao.univ-toulouse.fr/19965`.

[4] R. Durán, L. Hervella-Nieto, E. Liberman, and J. Solomin. Approximation of the vibration modes of a plate by Reissner-Mindlin equations. *Mathematics of Computation of the American Mathematical Society*, 68(228):1447–1463, 1999.

[5] D.J. Dawe and O.L. Roufaeil. Rayleigh-Ritz vibration analysis of Mindlin plates. *Journal of Sound and Vibration*, 69(3):345–359, 1980.

[6] H.C. Huang and E. Hinton. A nine node Lagrangian Mindlin plate element with enhanced shear interpolation. *Engineering Computations*, 1(4):369–379, 1984.

[7] A. Macchelli, C. Melchiorri, and L. Bassi. Port-based modelling and control of the Mindlin plate. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 5989–5994, Dec 2005. `https://doi.org/10.1109/CDC.2005.1583120`.

[8] A. Macchelli, A. J. van der Schaft, and C. Melchiorri. *Port Hamiltonian formulation of infinite dimensional systems I. Modeling*, volume 4, pages 3762 – 3767. 01 2005. `https://doi.org/10.1109/CDC.2004.1429324`.

[9] M. Schöberl and K. Schlacher. Lagrangian and port-Hamiltonian formulation for distributed-parameter systems. In *8th Vienna International Conference on Mathematical Modelling*, volume 48, pages 610 – 615, 2015.
https://doi.org/10.1016/j.ifacol.2015.05.025.

[10] M. Schöberl and A. Siuka. Analysis and comparison of port-Hamiltonian formulations for field theories - demonstrated by means of the Mindlin plate. In *2013 European Control Conference (ECC)*, pages 548–553, July 2013.
https://doi.org/10.23919/ECC.2013.6669137.

[11] M. Grinfeld. *Mathematical Tools for Physicists*. 2nd edition, jan 2015.

[12] Alessandro Macchelli and Claudio Melchiorri. Modeling and control of the Timoshenko beam. The distributed port hamiltonian approach. *SIAM Journal on Control and Optimization*, 43(2):743–767, 2004.

Thank you for your attention. Questions?