Linear N-port modelling of thin plates

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SUPAERO

European Space Agency

Plan

1. TITOP model of the Kirchhoff plate
2. Modes of Kirchhoff plate element
3. Modelling of Sentinel Solar Panels

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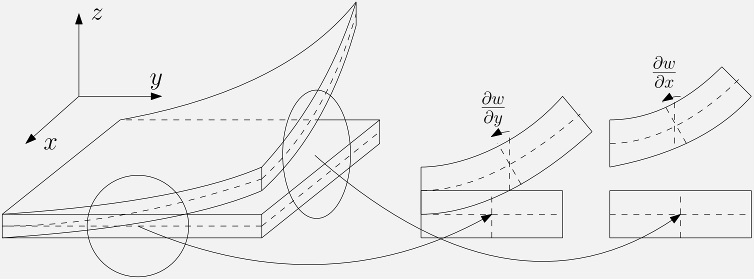
Model Hypothesis

Plane cross sections remains plane and normal to the middle-plane during deformations.



Consequently the displacement field assumes the following expression

$$u(x, y, z) = -z\frac{\partial w}{\partial x} \qquad v(x, y, z) = -z\frac{\partial w}{\partial y} \qquad w(x, y, z) = w(x, y)$$

Displacement Field approximation

General displacement field inside a quadrilateral element (Adini-Clough quadrilateral 1961):

$$
\begin{aligned}
w(x, y, t) &= P_w(x, y)^T \boldsymbol{a}(t) \\
\theta_x(x, y, t) &= \frac{\partial w}{\partial y} = P_{\theta_x}(x, y)^T \boldsymbol{a}(t) \\
\theta_y(x, y, t) &= -\frac{\partial w}{\partial x} = P_{\theta_y}(x, y)^T \boldsymbol{a}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
P_w(x, y) &= [1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2 y,\ xy^2,\ y^3,\ x^3 y,\ xy^3]^T \\
P_{\theta_x}(x, y) &= [0,\ 0,\ 1,\ 0,\ x,\ 2y,\ 0,\ x^2,\ 2xy,\ 3y^2,\ x^3,\ 3xy^2]^T \\
P_{\theta_y}(x, y) &= [0,\ -1,\ 0,\ -x,\ -y,\ 0,\ -3x^2,\ -2xy,\ y^2,\ 0,\ 3x^2 y,\ y^3]^T
\end{aligned}
$$

Rigid and Flexible components

The rigid and flexible part can be split by introducing a parent node (located at (x_P, y_P)) containing the rigid vertical displacement $(w_P(t))$ and the linearized rotation along x and y (θ_x, θ_y)

$$w(x, y, t) = w_P(t) + \widetilde{y}\theta_x - \widetilde{x}\theta_y + w_f(\widetilde{x}, \widetilde{y}, t)$$

where $\widetilde{x} = x - x_P$ and $\widetilde{y} = y - y_P$

The flexible part is expressed by considering that the constant and linear term of the $P_w(x, y)$ vector are included into the rigid part

$$w_f(x, y, t) = P_{w_f}(\widetilde{x}, \widetilde{y})^T \boldsymbol{a}_f(t)$$

where

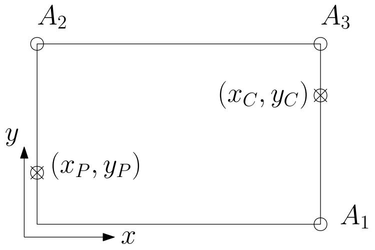
$$P_{w_f}(\widetilde{x}, \widetilde{y}) = [\widetilde{x}^2,\ \widetilde{x}\widetilde{y},\ \widetilde{y}^2,\ \widetilde{x}^3,\ \widetilde{x}^2\widetilde{y},\ \widetilde{x}\widetilde{y}^2,\ \widetilde{y}^3,\ \widetilde{x}^3\widetilde{y},\ \widetilde{x}\widetilde{y}^3]^T$$

Analogously for $\theta_{x,f}$ and $\theta_{y,f}$

Nodal Coordinates

It is now necessary to express $\boldsymbol{a}_f(t)$, in term of the nodal coordinates. The 9 components of this vector are found by expressing the flexible coordinates $w_f, \theta_{x,f}$ and $\theta_{y,f}$ at three different corner points. Defining

$$\begin{aligned} P_{A_i} &= \begin{bmatrix} P_{w_f}(\widetilde{x}_{A_i}, \widetilde{y}_{A_i}) & P_{\theta_{x,f}}(\widetilde{x}_{A_i}, \widetilde{y}_{A_i}) & P_{\theta_{y,f}}(\widetilde{x}_{A_i}, \widetilde{y}_{A_i}) \end{bmatrix} \\ \boldsymbol{q}_{A_i} &= \begin{bmatrix} w_f(\widetilde{x}_{A_i}, \widetilde{y}_{A_i}) & \theta_{x,f}(\widetilde{x}_{A_i}, \widetilde{y}_{A_i}) & \theta_{y,f}(\widetilde{x}_{A_i}, \widetilde{y}_{A_i}) \end{bmatrix}^T \end{aligned} \qquad \forall i = 1, \dots, 3$$



$$\boldsymbol{a}_f = \begin{bmatrix} P_{A_1}^T \\ P_{A_2}^T \\ P_{A_3}^T \end{bmatrix}^{-1} \begin{pmatrix} \boldsymbol{q}_{A_1} \\ \boldsymbol{q}_{A_2} \\ \boldsymbol{q}_{A_3} \end{pmatrix} = P_q^{-1} \boldsymbol{q}_f$$

For a better conditioning of matrix P_q nodes A_i $\forall i = 1, 2, 3$ need to be located as far as possible from (x_P, y_P).

Energies

The kinetic energy is given by

$$\mathcal{T} = \frac{1}{2}\int_0^{l_x}\int_0^{l_y} \begin{pmatrix} \dot{w} \\ \dot{\theta}_x \\ \dot{\theta}_y \end{pmatrix}^T \begin{bmatrix} \rho h & 0 & 0 \\ 0 & \frac{\rho h^3}{12} & 0 \\ 0 & 0 & \frac{\rho h^3}{12} \end{bmatrix} \begin{pmatrix} \dot{w} \\ \dot{\theta}_x \\ \dot{\theta}_y \end{pmatrix} dxdy$$

The elastic energy

$$\mathcal{K} = \frac{1}{2}\int_0^{l_x}\int_0^{l_y} \boldsymbol{\kappa}^T \boldsymbol{D}\boldsymbol{\kappa}\, dxdy$$

The curvatures and the flexural rigidity matrix are given by

$$\boldsymbol{\kappa} = \begin{pmatrix} -\frac{\partial^2 w_f}{\partial x^2} \\ -\frac{\partial^2 w_f}{\partial y^2} \\ -2\frac{\partial^2 w_f}{\partial x\, \partial y} \end{pmatrix} \qquad \boldsymbol{D} = \begin{bmatrix} D & D\nu & 0 \\ D\nu & D & 0 \\ 0 & 0 & \frac{D}{2}(1-\nu) \end{bmatrix}$$

where $D = \frac{Eh^3}{12(1-\nu)}$ is the flexural stiffness and ν is the Poisson ratio.

Mass matrix

The shape functions for the rigid part are

$$\begin{aligned}\boldsymbol{\phi}_{w,r} &= [1, \widetilde{y}, -\widetilde{x}]^T \\ \boldsymbol{\phi}_{\theta_x,r} &= [0, 1, 0]^T \\ \boldsymbol{\phi}_{\theta_y,r} &= [0, 0, 1]^T\end{aligned} \qquad \boldsymbol{\Phi}_r = \begin{bmatrix}\boldsymbol{\phi}_{w,r}^T \\ \boldsymbol{\phi}_{\theta_x,r}^T \\ \boldsymbol{\phi}_{\theta_x,r}^T\end{bmatrix}$$

For the flexible part

$$\begin{aligned}\boldsymbol{\phi}_{w,f} &= (P_{w,f}^T P_q^{-1})^T \\ \boldsymbol{\phi}_{\theta_x,f} &= (P_{\theta_x,f}^T P_q^{-1})^T \\ \boldsymbol{\phi}_{\theta_y,f} &= (P_{\theta_y,f}^T P_q^{-1})^T\end{aligned} \qquad \boldsymbol{\Phi}_f = \begin{bmatrix}\boldsymbol{\phi}_{w,f}^T \\ \boldsymbol{\phi}_{\theta_x,f}^T \\ \boldsymbol{\phi}_{\theta_x,f}^T\end{bmatrix}$$

The Mass matrix takes the form

$$\boldsymbol{M} = \begin{bmatrix}\boldsymbol{M}_{rr} & \boldsymbol{M}_{rf} \\ \boldsymbol{M}_{fr} & \boldsymbol{M}_{ff}\end{bmatrix} = \int_0^{l_x} \int_0^{l_y} \begin{bmatrix}\boldsymbol{\Phi}_r^T \\ \boldsymbol{\Phi}_f^T\end{bmatrix} \begin{bmatrix}\rho h & 0 & 0 \\ 0 & \frac{\rho h^3}{12} & 0 \\ 0 & 0 & \frac{\rho h^3}{12}\end{bmatrix} \begin{bmatrix}\boldsymbol{\Phi}_r & \boldsymbol{\Phi}_f\end{bmatrix} dxdy$$

Stiffness matrix

Shape function for the curvatures

$$\phi_{\kappa_{xx}} = \left(-\frac{\partial^2 P_{w,f}^T}{\partial x^2} P_q^{-1}\right)^T$$

$$\phi_{\kappa_{yy}} = \left(-\frac{\partial^2 P_{w,f}^T}{\partial y^2} P_q^{-1}\right)^T \qquad \boldsymbol{\Phi}_{\boldsymbol{\kappa}} = \begin{bmatrix} \phi_{\kappa_{xx}}^T \\ \phi_{\kappa_{yy}}^T \\ \phi_{\kappa_{xy}}^T \end{bmatrix}$$

$$\phi_{\kappa_{xy}} = \left(-2\frac{\partial^2 P_{w,f}^T}{\partial x\,\partial y} P_q^{-1}\right)^T$$

Stiffness matrix

$$\boldsymbol{K} = \int_0^{l_x}\int_0^{l_y} \boldsymbol{\Phi}_{\boldsymbol{\kappa}}^T \boldsymbol{D} \boldsymbol{\Phi}_{\boldsymbol{\kappa}} \, dxdy$$

Work of external forces

Calling F_P^z, T_P^x, T_P^y the force and torques that the element is exerting of the parent structure and F_C^z, T_C^x, T_C^y the external force applied at the child node C, the work can be evaluated as

$$W_e = \begin{pmatrix} w_P \\ \theta_{x,P} \\ \theta_{y,P} \\ \boldsymbol{q}_f \end{pmatrix}^T \begin{bmatrix} -\boldsymbol{I}_{3\times 3} & \boldsymbol{\tau}_{CP}^T \\ \boldsymbol{0}_{n\times 3} & \boldsymbol{\Phi}_f(\widetilde{x}_C, \widetilde{y}_C)^T \end{bmatrix} \begin{pmatrix} F_P^z \\ T_P^x \\ T_P^y \\ F_C^z \\ T_C^x \\ T_C^y \end{pmatrix}$$

The equations of motions ($\boldsymbol{q}_r = (w_P, \theta_x, \theta_y)^T$) read

$$\begin{bmatrix} \boldsymbol{M}_{rr} & \boldsymbol{M}_{rf} \\ \boldsymbol{M}_{fr} & \boldsymbol{M}_{ff} \end{bmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}}_r \\ \ddot{\boldsymbol{q}}_f \end{pmatrix} + \begin{bmatrix} \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 9} \\ \boldsymbol{0}_{9\times 3} & \boldsymbol{K} \end{bmatrix} \begin{pmatrix} \boldsymbol{q}_r \\ \boldsymbol{q}_f \end{pmatrix} = \begin{bmatrix} -\boldsymbol{I}_{3\times 3} & \boldsymbol{\tau}_{CP}^T \\ \boldsymbol{0}_{n\times 3} & \boldsymbol{\Phi}_f(\widetilde{x}_C, \widetilde{y}_C)^T \end{bmatrix} \begin{pmatrix} \boldsymbol{F}_P \\ \boldsymbol{F}_C \end{pmatrix}$$

Modal coordinates

The modal coordinates are introduced by solving the eigenvalue problem

$$\boldsymbol{M}_{ff}\,\boldsymbol{V}\boldsymbol{\Omega}^2 = \boldsymbol{K}\boldsymbol{V} \qquad \text{where } \boldsymbol{\Omega}^2 = \mathrm{Diag}(\omega_i^2)$$

Applying the transformation $\boldsymbol{q} = \boldsymbol{V}\boldsymbol{\eta}$ the system becomes

$$\begin{bmatrix} \boldsymbol{M}_{rr} & \boldsymbol{L}_P^T \\ \boldsymbol{L}_P & \boldsymbol{I}_{9\times 9} \end{bmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}}_r \\ \ddot{\boldsymbol{\eta}} \end{pmatrix} + \begin{bmatrix} \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 9} \\ \boldsymbol{0}_{9\times 3} & \boldsymbol{\Omega}^2 \end{bmatrix} \begin{pmatrix} \boldsymbol{q}_r \\ \boldsymbol{\eta} \end{pmatrix} = \begin{bmatrix} -\boldsymbol{I}_{3\times 3} & \boldsymbol{\tau}_{CP}^T \\ \boldsymbol{0}_{n\times 3} & \boldsymbol{\Phi}_{\eta,C}^T \end{bmatrix} \begin{pmatrix} \boldsymbol{F}_P \\ \boldsymbol{F}_C \end{pmatrix}$$

where $\boldsymbol{L}_P = \boldsymbol{V}^T \boldsymbol{M}_{fr}$, $\boldsymbol{\Phi}_\eta = \boldsymbol{\Phi}_f \boldsymbol{V}$ and $\boldsymbol{\Phi}_{\eta,C} = \boldsymbol{\Phi}_\eta(\widetilde{x} = x_C - x_P, \widetilde{y} = y_C - y_P)$. Furthermore

$$\boldsymbol{\tau}_{CP} = \begin{bmatrix} 1 & y_C - y_P & -(x_C - x_P) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Damping can be added by introducing matrix $\boldsymbol{\Xi} = \mathrm{Diag}(2\xi_i\omega_i)$. The acceleration at child node is written as

$$\ddot{\boldsymbol{x}}_C = \boldsymbol{\tau}_{CP}\ddot{\boldsymbol{x}}_P + \boldsymbol{\Phi}_{\eta,C}\boldsymbol{\eta}$$

where $\ddot{\boldsymbol{x}}_C = \begin{pmatrix} w_C & \theta_{x,C} & \theta_{y,C} \end{pmatrix}^T$ and $\ddot{\boldsymbol{x}}_P = \begin{pmatrix} w_C & \theta_{x,P} & \theta_{y,P} \end{pmatrix}^T$

2 ports model

The 2 ports model for the Kirchhoff plate considering as inputs the acceleration at the parent node and the reaction at the child node is the following (see [1] for a reference on the TITOP method). Given a generic body $\mathcal{L}_i$ the model reads

$$\begin{pmatrix} \dot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\eta}} \\ \hline \ddot{\boldsymbol{x}}_C \\ \boldsymbol{W}_P \end{pmatrix} = \underbrace{\left[\begin{array}{cc|cc} \mathbf{0}_{9\times 9} & \boldsymbol{I}_{9\times 9} & \mathbf{0}_{9\times 6} & \mathbf{0}_{9\times 6} \\ -\boldsymbol{\Omega}^2 & -\boldsymbol{\Xi} & \boldsymbol{\Phi}_{\eta,C}^T & -\boldsymbol{L}_P \\ \hline -\boldsymbol{\Phi}_{\eta,C}\boldsymbol{\Omega}^2 & -\boldsymbol{\Phi}_{\eta,C}\boldsymbol{\Xi} & \boldsymbol{\Phi}_{\eta,C}\boldsymbol{\Phi}_{\eta,C}^T & (\boldsymbol{\tau}_{CP} - \boldsymbol{\Phi}_{\eta,C}\boldsymbol{L}_P) \\ \boldsymbol{L}_P^T\boldsymbol{\Omega}^2 & \boldsymbol{L}_P^T\boldsymbol{\Xi} & (\boldsymbol{\tau}_{CP} - \boldsymbol{\Phi}_{\eta,C}\boldsymbol{L}_P)^T & \boldsymbol{L}_P^T\boldsymbol{L}_P - \boldsymbol{M}_{rr} \end{array}\right]}_{\mathcal{M}_{PC}^{\mathcal{L}_i}} \begin{pmatrix} \boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} \\ \hline \boldsymbol{W}_C \\ \ddot{\boldsymbol{x}}_P \end{pmatrix}$$

This model represent the case of a plate clamped at P and free at C. Thanks to the fact that the feedthrough matrix has consists of non null terms on the diagonal the inversion of the channel is possible. It is therefore possible to represent other boundary conditions by inversion of the channel.

Plan

Evaluation of modes

Test case for the representation of modes

Coefficient	Numerical value
Length along x: $l_x[m]$	4.143
Length along y: $l_y[m]$	2.200
Thickness: $t[m]$	0.04
Young modulus: $E[GPa]$	70
Poisson Modulus: $\nu[na]$	0.35
Parent node location along x: $x_P[m]$	0
Parent node location along y: $y_P[m]$	$l_y/2$
Child node location along x: $x_C[m]$	l_x
Child node location along y: $y_C[m]$	$l_y/2$

Modes I

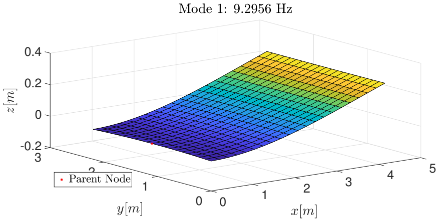


Figure: Reduced model

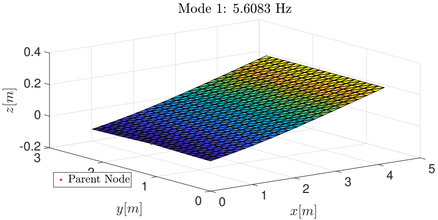


Figure: Model from [2] with 20×20 elements

Modes II

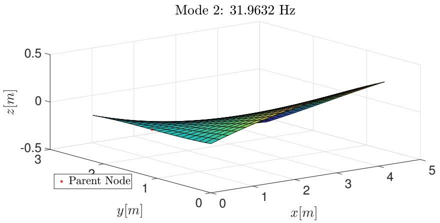


Figure: Reduced model

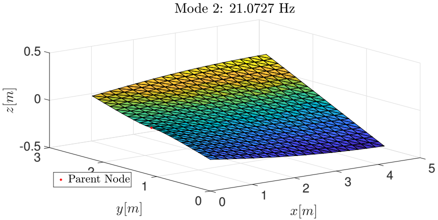


Figure: Model from [2] with 20×20 elements

Modes III

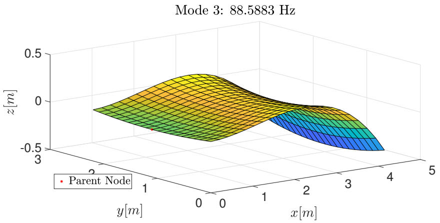


Figure: Reduced model

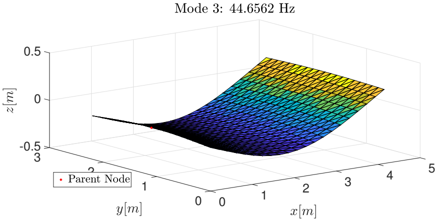


Figure: Model from [2] with 20×20 elements

Modes IV

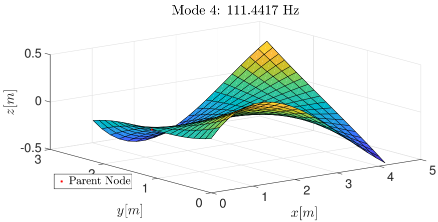


Figure: Reduced model

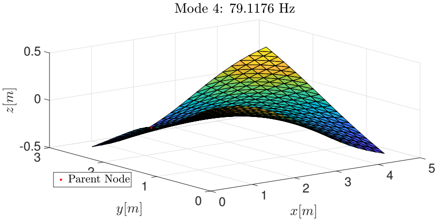


Figure: Model from [2] with 20×20 elements

Modes V

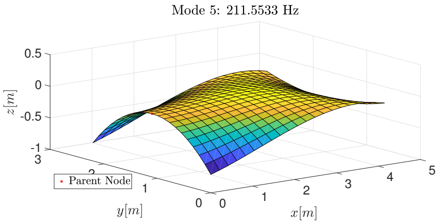


Figure: Reduced model

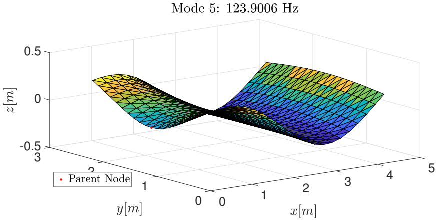


Figure: Model from [2] with 20×20 elements

Modes VI

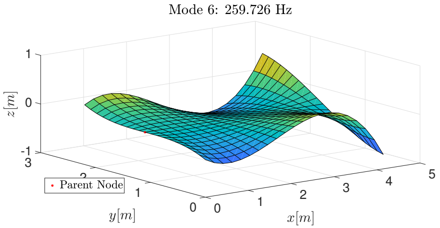


Figure: Reduced model

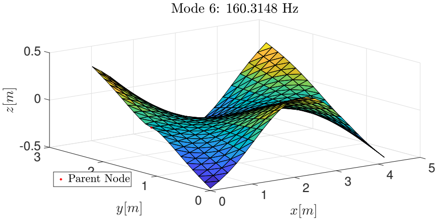


Figure: Model from [2] with 20×20 elements

Modes VII

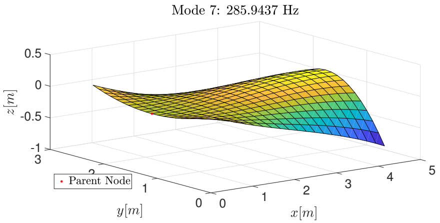


Figure: Reduced model

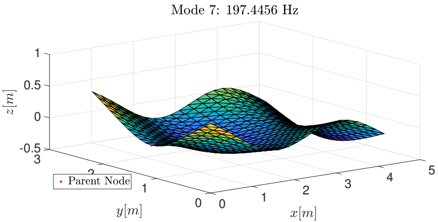


Figure: Model from [2] with 20×20 elements

Modes VIII

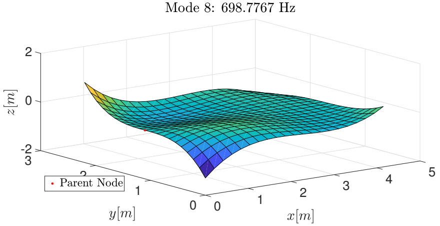


Figure: Reduced model

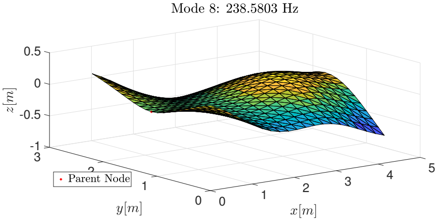


Figure: Model from [2] with 20×20 elements

Modes IX

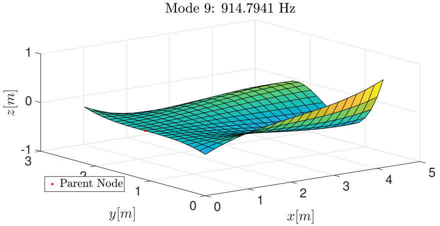


Figure: Reduced model

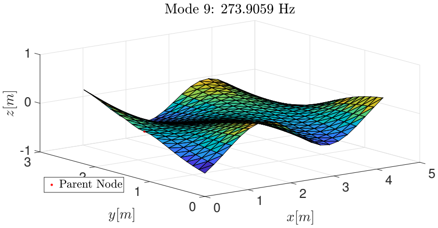


Figure: Model from [2] with 20×20 elements

Multiple children

Several children can be considered inside this model leading to an N ports model. If two children C_1, C_2 are considered

$$
\left[\begin{array}{cc|ccc}
\mathbf{0}_{9\times 9} & \boldsymbol{I}_{9\times 9} & \mathbf{0}_{9\times 6} & \mathbf{0}_{9\times 6} & \mathbf{0}_{9\times 6} \\
-\boldsymbol{\Omega}^2 & -\boldsymbol{\Xi} & \boldsymbol{\Phi}_{\eta,C_1}^T & \boldsymbol{\Phi}_{\eta,C_2}^T & -\boldsymbol{L}_P \\
\hline
-\boldsymbol{\Phi}_{\eta,C_1}\boldsymbol{\Omega}^2 & -\boldsymbol{\Phi}_{\eta,C_1}\boldsymbol{\Xi} & \boldsymbol{\Phi}_{\eta,C_1}\boldsymbol{\Phi}_{\eta,C_1}^T & \boldsymbol{\Phi}_{\eta,C_1}\boldsymbol{\Phi}_{\eta,C_2}^T & \left(\boldsymbol{\tau}_{C_1P} - \boldsymbol{\Phi}_{\eta,C_1}\boldsymbol{L}_P\right) \\
-\boldsymbol{\Phi}_{\eta,C_2}\boldsymbol{\Omega}^2 & -\boldsymbol{\Phi}_{\eta,C_2}\boldsymbol{\Xi} & \boldsymbol{\Phi}_{\eta,C_2}\boldsymbol{\Phi}_{\eta,C_1}^T & \boldsymbol{\Phi}_{\eta,C_2}\boldsymbol{\Phi}_{\eta,C_2}^T & \left(\boldsymbol{\tau}_{C_2P} - \boldsymbol{\Phi}_{\eta,C_2}\boldsymbol{L}_P\right) \\
\boldsymbol{L}_P^T\boldsymbol{\Omega}^2 & \boldsymbol{L}_P^T\boldsymbol{\Xi} & \left(\boldsymbol{\tau}_{C_1P} - \boldsymbol{\Phi}_{\eta,C_1}\boldsymbol{L}_P\right)^T & \left(\boldsymbol{\tau}_{C_2P} - \boldsymbol{\Phi}_{\eta,C_2}\boldsymbol{L}_P\right)^T & \boldsymbol{L}_P^T\boldsymbol{L}_P - \boldsymbol{M}_{rr}
\end{array}\right]
$$

The extension to a generic N-ports model is straightforward.

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Inversion of channels

The feedtrought matrix contains non null this term on the diagonal. This feature makes possible the inversion of whichever input port of the model. Given a generic vector of indexes $\boldsymbol{Y}$ the modes whose ports (indexed by $\boldsymbol{Y}$) are inverted will be denoted $\left[\mathcal{M}_{PC}^{\mathcal{L}_i}\right]^{-1\ \boldsymbol{Y}}$.
By inverting channels the construction of several interconnected panels is possible, like the Sentinel Satellite solar panels

Chain of plates: The Sentinel satellite case

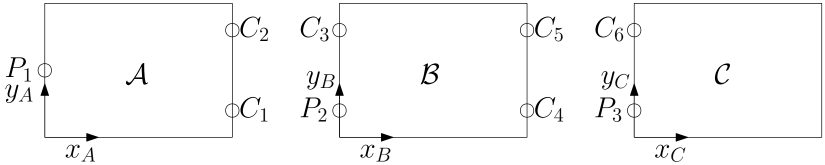


Figure: Schematic modelisation of the three panels

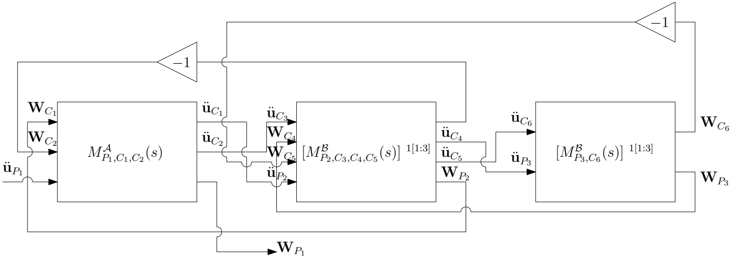
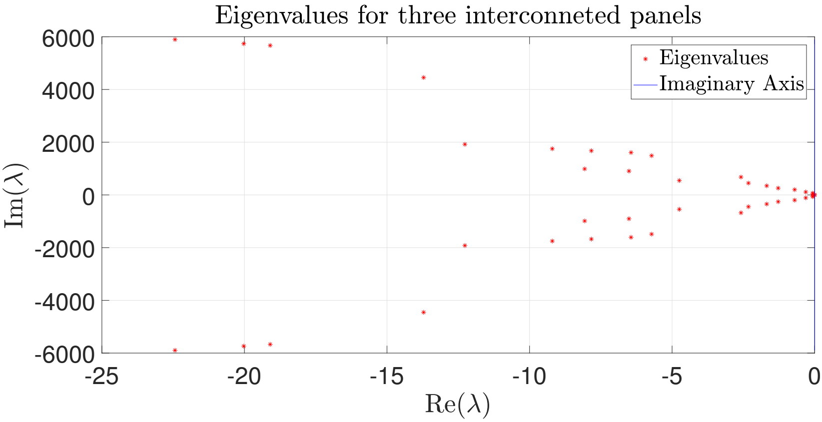


Figure: Equivalent Block Diagram

Eigenvalues for the complete model



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Journal of Sound and Vibration, vol. 415:pp. 128–146, February 2018.

Thank you for your attention. Questions?