December, the 13th, 2019

Interconnection of the Kirchhoff plate within the port-Hamiltonian framework

Andrea Brugnoli[1]

[1]ISAE-SUPAERO, Toulouse

Outline

1. The Kirchhoff plate as a port-Hamiltonian system
   - Classical formulation
   - Port-Hamiltonian formulation

2. Structure preserving discretization
   - The partitioned finite element method
   - Application to the Kirchhoff plate

3. Interconnection with rigid elements

4. Stabilization by boundary injection

Outline

1. The Kirchhoff plate as a port-Hamiltonian system
   - Classical formulation
   - Port-Hamiltonian formulation
2. Structure preserving discretization
3. Interconnection with rigid elements
4. Stabilization by boundary injection

The classical Kirchhoff model

Classical bilaplacian formulation

For an homogeneous isotropic material

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\Delta^2 w = p, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2.$$

$\Delta^2 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$ is the bilaplacian operator

- ρ $[\mathrm{kg/m^3}]$ is the mass density;
- h $[\mathrm{m}]$ is the plate thickness;
- p $[\mathrm{N/m^2}]$ is an external distributed force;
- D $[\mathrm{Pa\,m}]$ is the bending stiffness;

The classical Kirchhoff model

Bending moment formulation

$$\rho h \frac{\partial^2 w}{\partial t^2} + \operatorname{div} \operatorname{Div}(\boldsymbol{M}) = p, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2.$$

Where $\boldsymbol{M} = \mathbb{D} \nabla^2 w \in \mathbb{R}^{2\times 2}_{\text{sym}}$ is the bending moment tensor and $\nabla^2 = \operatorname{Grad} \circ \operatorname{grad}$ the Hessian.

$$\operatorname{div} \operatorname{Div}(\boldsymbol{M}) = \partial_{xx} M_{11} + 2\partial_{xy} M_{12} + \partial_{yy} M_{22}$$

- ρ $[\mathrm{kg/m^3}]$ is the mass density;
- h $[\mathrm{m}]$ is the plate thickness;
- p $[\mathrm{N/m^2}]$ is an external distributed force;
- $\mathbb{D}$ is the bending rigidity tensor (symmetric, positive). For an homogeneous isotropic material

$$\mathbb{D}\boldsymbol{A} = D\left\{(1-\nu)\boldsymbol{A} + \nu \operatorname{Tr}(\boldsymbol{A})\boldsymbol{I}\right\};$$

For the boundary variables consider the definitions

$$
\begin{aligned}
\text{Flexural moment} \quad & M_{nn} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}), \\
\text{Torsional moment} \quad & M_{ns} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{s}), \\
\text{Effective shear force} \quad & \widetilde{q}_n = -(\mathrm{Div}\, \boldsymbol{M}) \cdot \boldsymbol{n} - \partial_{\boldsymbol{s}} M_{ns},
\end{aligned}
$$

where $\boldsymbol{n}$, $\boldsymbol{s}$ are the normal and tangential versors along the boundary $\partial\Omega$.
$\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j} A_{ij} B_{ij}$ is the tensor contraction and $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top \in \mathbb{R}^{2\times 2}$ is the dyadic product between vectors.
Consider a partition of the boundary: $\partial\Omega = \Gamma_c \cup \Gamma_s \cup \Gamma_f$.

- Γ_c is the clamped part, i.e. w, $\partial_{\boldsymbol{n}} w$ known;
- Γ_s is the simply supported part, i.e. w, M_{nn} known;
- Γ_f is the free part, i.e. M_{nn}, q_n known;

Outline

1 The Kirchhoff plate as a port-Hamiltonian system
- Classical formulation
- Port-Hamiltonian formulation

2 Structure preserving discretization

3 Interconnection with rigid elements

4 Stabilization by boundary injection

The total energy of the system is given by the sum of kinetic and deformation energy

$$H = \frac{1}{2}\int_{\Omega} \left\{\rho \left(\partial_t w\right)^2 + \mathbb{D}\nabla^2 w : \nabla^2 w\right\} \, \mathrm{d}\Omega$$

Consider the following choice for the energy variables

$$\begin{aligned} \alpha_1 &:= \rho \partial_t w, \quad && \text{Linear momentum} \\ \boldsymbol{A}_2 &:= \nabla^2 w, \quad && \text{Curvature} \end{aligned}$$

This leads to the following co-energy variables

$$\begin{aligned} e_1 &:= \frac{\delta H}{\delta \alpha_1} = \partial_t w = (\rho h)^{-1} \alpha_1, \quad && \text{Velocity} \\ \boldsymbol{E}_2 &:= \frac{\delta H}{\delta \boldsymbol{A}_2} = \boldsymbol{M} = \mathbb{D} \boldsymbol{A}_2, \quad && \text{Bending moment} \end{aligned}$$

The system

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_1 \\ \boldsymbol{A}_2\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_1 \\ \boldsymbol{E}_2 \end{pmatrix}, \qquad \begin{pmatrix} e_1 \\ \boldsymbol{E}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} (\rho h)^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix}}_{\mathcal{Q}} \begin{pmatrix}\alpha_1 \\ \boldsymbol{A}_2\end{pmatrix}$$

with homogeneous boundary conditions

$$w|_{\Gamma_c} = \partial_{\boldsymbol{n}} w|_{\Gamma_c} = 0, \quad w|_{\Gamma_s} = M_{nn}|_{\Gamma_s} = 0, \quad q_n|_{\Gamma_f} = M_{nn}|_{\Gamma_f} = 0$$

defines a Stokes-Dirac structure.

Notice that $D(\mathcal{J}) = H^2(\Omega) \times H^{\operatorname{div}\operatorname{Div}}(\Omega)$:

$$H^2(\Omega) := \left\{ v \in L^2(\Omega) |\, \nabla^2 v \in L^2(\Omega,\, \mathbb{R}^{2\times 2}_{\text{sym}}) \right\},$$

$$H^{\operatorname{div}\operatorname{Div}}(\Omega) := \left\{ \boldsymbol{V} \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}}))|\, \operatorname{div}\operatorname{Div} \boldsymbol{V} \in L^2(\Omega) \right\}$$

Port-Hamiltonian formulation

Consider the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$,

$$\mathcal{F} = \mathcal{E} := L^2(\Omega) \times L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathsf{sym}}), \qquad (\partial_t \alpha_1, \ \partial_t \boldsymbol{A}_2) = \boldsymbol{f} \in \mathcal{F}, \quad (e_1, \ \boldsymbol{E}_2) = \boldsymbol{e} \in \mathcal{E}$$

It is necessary to show that the set

$$\mathcal{D}_{\mathcal{J}} := \{(\boldsymbol{f}, \boldsymbol{e}) \in \mathsf{Graph}(\mathcal{J}) | \ \boldsymbol{e} \in D(\mathcal{J})\} \subset \mathcal{B}$$

equals its orthogonal complement

$$\mathcal{D}_{\mathcal{J}}^{\perp} = \{b \in \mathcal{B} | \ \langle \boldsymbol{b}, \boldsymbol{b}' \rangle_{+} = 0, \ \forall \ \boldsymbol{b}' \in \mathcal{D}_{\mathcal{J}}\}$$

with respect to the canonical symmetrical pairing

$$\langle \boldsymbol{b}^1, \boldsymbol{b}^2 \rangle_{+} = \langle \boldsymbol{f}^1, \boldsymbol{e}^2 \rangle_{L^2} + \langle \boldsymbol{e}^1, \boldsymbol{f}^2 \rangle_{L^2}, \quad \boldsymbol{b}^i = (\boldsymbol{f}^i, \boldsymbol{e}^i) \in \mathcal{B}, \quad i = 1, 2$$

The proof is readily obtained considering that the following holds

$$(\mathrm{div}\,\mathrm{Div})^* = \nabla^2$$

This means that the operator $\mathcal{J}$ is formally skew-adjoint. By application of the Stokes theorem it is obtained $\mathcal{D}_{\mathcal{J}} = \mathcal{D}_{\mathcal{J}}^{\perp}$.

Inhomogeneous boundary conditions can be considered as well, but the definition of $\mathcal{D}_{\mathcal{J}}$ requires additional care.

It is worth noticing that the boundary variables are defined by the power balance

$$\dot{H} = \int_{\partial\Omega} \left\{ \partial_t w \, \widetilde{q}_n + \partial_{\boldsymbol{n}}(\partial_t w) \, M_{nn} \right\} \, \mathrm{d}s.$$

Outline

1 The Kirchhoff plate as a port-Hamiltonian system

2 Structure preserving discretization
- The partitioned finite element method
- Application to the Kirchhoff plate

3 Interconnection with rigid elements

4 Stabilization by boundary injection

How to discretize pH systems?

Infinite dimensional pHs

PDE:

$$\partial_t x(z,t) = \mathcal{J}\delta_x H + B u(z,t),$$
$$y(z,t) = B^* \delta_x H.$$

Boundary conditions:

$$u_\partial = \mathcal{B}\ \delta_x H, \quad y_\partial = \mathcal{C}\ \delta_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t) y(z,t)\ \mathrm{d}\Omega$$

Finite dimensional pHs

ODE:

$$\dot{x} = J\partial_x H + B_d u_d + B_\partial u_\partial,$$
$$y_d = B_d^T \partial_x H,$$
$$y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

Available methods

- Spectral methods (Moulla 2012);
- Finite differences (Trenchant 2018);
- Finite elements based (Golo 2004, Kotyczka 2018, Cardoso-Riberio 2019);

General idea of PFEM

General form of a linear pH system in co-energy variables

$$\mathcal{M}\frac{\partial e}{\partial t} = \mathcal{J} e, \qquad \mathcal{M} = \mathcal{Q}^{-1}$$

General procedure for PFEM

1. Put the system into weak form:
$$\left(v, \mathcal{M}\frac{\partial e}{\partial t}\right)_\Omega = (v, \mathcal{J} e)_\Omega .$$
2. Apply integration by part on a partition of $\mathcal{J}$:
$$(v, \mathcal{J} e)_\Omega \overbrace{=}^{i.b.p.} j(v, e)_\Omega + b(v, u_\partial)_{\partial\Omega},$$
so that $j(v, e)_\Omega$ is a skew-symmetric bilinear form.
3. Discretization by Galerkin method (same basis function for test and co-energy variables)

Outline

1 The Kirchhoff plate as a port-Hamiltonian system

2 Structure preserving discretization
- The partitioned finite element method
- Application to the Kirchhoff plate

3 Interconnection with rigid elements

4 Stabilization by boundary injection

For the Kirchhoff plate

$$\mathcal{M} = \mathsf{Diag}(\rho,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the first line of the operator is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e) &= \int_\Omega \left\{ -v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1 \right\} \, \mathrm{d}\Omega \\ &= \underbrace{\int_\Omega \left\{ -\nabla^2 v_1 : \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1 \right\} \, \mathrm{d}\Omega}_{j_{\mathsf{Hess}}(\boldsymbol{v}, \boldsymbol{e})} + \underbrace{\int_{\partial\Omega} \left\{ v_1 q_n + \partial_{\boldsymbol{n}} v_1 M_{nn} \right\} \, \mathrm{d}s}_{b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}} \end{aligned}$$

The Kirchhoff plate case

For the Kirchhoff plate

$$\mathcal{M} = \text{Diag}(\rho,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the second line of the operator is integrated by parts

$$\begin{aligned}
(v, \mathcal{J}e) &= \int_\Omega \left\{ -v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1 \right\} \, \mathrm{d}\Omega \\
&= \underbrace{\int_\Omega \left\{ -v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \operatorname{div}\operatorname{Div} \boldsymbol{V}_2 e_1 \right\} \, \mathrm{d}\Omega}_{j_{\nabla^2}(\boldsymbol{v},\boldsymbol{e})} + \underbrace{\int_{\partial\Omega} \left\{ v_1 q_n + \partial_{\boldsymbol{n}} v_1 M_{nn} \right\} \, \mathrm{d}s}_{b_N(\boldsymbol{v},\boldsymbol{u}_\partial)_{\partial\Omega}}
\end{aligned}$$

Application to the Kirchhoff plate

Outline

1 The Kirchhoff plate as a port-Hamiltonian system

2 Structure preserving discretization

3 Interconnection with rigid elements

4 Stabilization by boundary injection

Boundary interconnection of the Kirchhoff plate

The system is composed by a cantilever plate connected to a rigid rod. The interconnection is given by a compact operator.

$$
\text{dpH} \begin{cases} \dfrac{\partial x_1}{\partial t} = \mathcal{J} \dfrac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \dfrac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \dfrac{\delta H_1}{\delta x_1} \end{cases} \qquad \text{pH} \begin{cases} \dfrac{dx_2}{dt} = J \dfrac{\partial H_2}{\partial x_2} + B u_2 \\ y_2 = B^T \dfrac{\partial H_2}{\partial x_2} + D u_2 \end{cases},
$$

where $x_1 \in \mathscr{X}$, $u_{\partial,1} \in \mathscr{U}$, $y_{\partial,1} \in \mathscr{Y} = \mathscr{U}'$ belong to some Hilbert spaces (the prime denotes the topological dual of a space) and $x_2 \in \mathbb{R}^n, u, y \in \mathbb{R}^m$. The duality pairings for the boundary ports are denoted by

$$
\langle u_{\partial,1}, \; y_{\partial,1} \rangle_{\mathscr{U} \times \mathscr{Y}}, \qquad \langle u_2, \; y_2 \rangle_{\mathbb{R}^m}.
$$

For the interconnection, consider the compact operator $\mathcal{W} : \mathscr{Y} \to \mathbb{R}^m$ and the following power preserving interconnection

$$
u_2 = -\mathcal{W} \, y_{\partial,1}, \qquad u_{\partial,1} = \mathcal{W}^* \, y_2,
$$

Kirchhoff plate

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \operatorname{Grad} \operatorname{grad} & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix}$$

$$\begin{aligned} u_{\partial,\mathsf{pl}} &= \partial_t w(x = L_x, y), \\ y_{\partial,\mathsf{pl}} &= \widetilde{q}_n(x = L_x, y). \end{aligned}$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \boldsymbol{u}_{\mathsf{rod}},$$

$$\boldsymbol{y}_{\mathsf{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space $\mathscr{Y}$ is the space of square-integrable functions with support on $\Gamma_{\mathsf{int}} = \{(x, y) |\ x = L_x, 0 \leq y \leq L_y\}$. The compact interconnection operator then reads

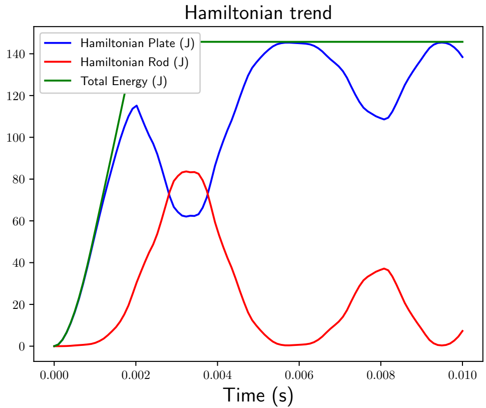
$$\mathcal{W} y_{\partial,\mathsf{pl}} = \begin{pmatrix} \int_{\Gamma_{\mathsf{int}}} y_{\partial,\mathsf{pl}} \, \mathrm{d}s \\ \int_{\Gamma_{\mathsf{int}}} (y - L_y/2) \, y_{\partial,\mathsf{pl}} \, \mathrm{d}s \end{pmatrix}.$$

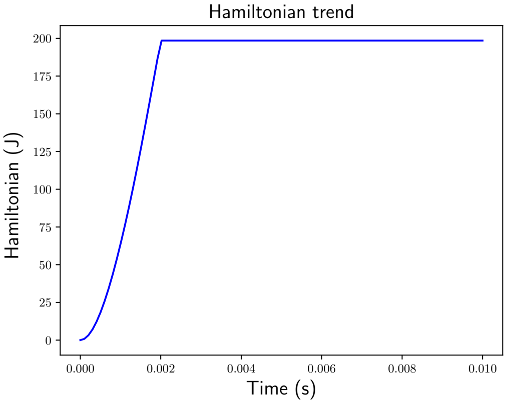
The adjoint operator is then obtained considering that $\boldsymbol{u}_{\mathsf{rod}} = \mathcal{W} y_{\partial,\mathsf{pl}}$ and that the inner product of $\mathbb{R}^m$ is easily converted to an inner product on the space $L^2(\Gamma_{\mathsf{int}})$

$$\langle \mathcal{W} y_{\partial,\mathsf{pl}}, \ \boldsymbol{y}_{\mathsf{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial,\mathsf{pl}}, \ \mathcal{W}^* \boldsymbol{y}_{\mathsf{rod}} \rangle_{L^2(\Gamma_{\mathsf{int}})},$$

$$\mathcal{W}^* y_{\mathsf{rod}} = v_G + \omega_G \, (y - L_y/2) \, .$$

Plate and rod


Hamiltonian trend
Hamiltonian Plate (J)
Hamiltonian Rod (J)
Total Energy (J)
0
20
40
60
80
100
120
140
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)


Hamiltonian trend
Hamiltonian (J)
0
25
50
75
100
125
150
175
200
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)

Outline

1 The Kirchhoff plate as a port-Hamiltonian system

2 Structure preserving discretization

3 Interconnection with rigid elements

4 Stabilization by boundary injection

Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} \qquad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

subjected to the following boundary conditions

$$\begin{aligned} \partial_t w|\Gamma_D &= 0, \\ \partial_x \partial_t w|\Gamma_D &= 0, \end{aligned} \qquad \Gamma_D = \{x = 0\}$$

$$\begin{aligned} \boldsymbol{M} : (n \otimes n)|\Gamma_N &= u_M, \\ \mathcal{D}\boldsymbol{M}|\Gamma_N := \widetilde{q}|\Gamma_N &= u_F, \end{aligned} \qquad \Gamma_N = \{x = 0,\ x = 1,\ y = 1\}$$

with initial conditions (compatible with the constraints):

$$w_t(x, y, 0) = x^2; \qquad \Sigma(x, y, 0) = 0.$$

Obtain a finite-dimensional uncontrolled system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

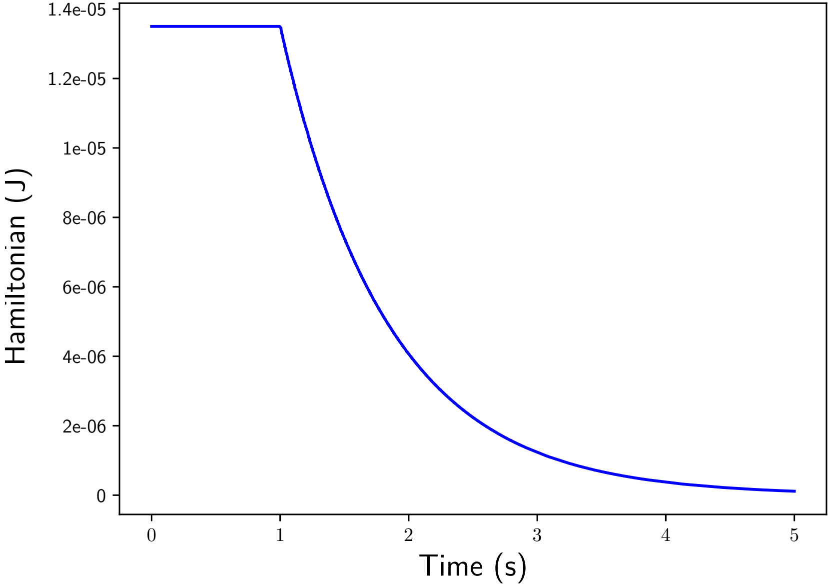
Apply the control law $u = -Ky, \ K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

with $R = BKB^T \succeq 0$.
The Hamiltonian $\dot{H} = -e^T Re \leq 0$ is a non increasing function and by La Salle principle the equilibrium point $e = 0$ is asymptotically stable.

Boundary stabilization of the Kirchhoff plate

Damping Kirchhoff Plate



Hamiltonian (J)
1.4e-05
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
0
1
2
3
4
5
Time (s)

Thank you

Questions

References I

Institut Supérieur de l'Aéronautique et de l'Espace

10 avenue Édouard Belin – BP 54032

31055 Toulouse Cedex 4 – France

Phone: +33 5 61 33 80 80

www.isae-supaero.fr

Institut Supérieur de l'Aéronautique et de l'Espace

SUPAERO