Partitioned Finite Element Method for port-Hamiltonian systems

– PFEM 4 pHs –

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Frauenchiemsee, Germany, 3rd March 2019.

PFEM 4 pHs? the team!

- Origin of the method:

  F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A structure-preserving Partitioned Finite Element Method for the 2D wave equation. In *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control* (LHMNLC), 6 pages, Valparaíso, Chile, 2018. IFAC-PapersOnLine, Vol. 51, Issue 3, 2018, pp. 119–124

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- Collaborators on the **PFEM 4 pHs** project since then:
  - Anass SERHANI, Ph.D. student
  - Andrea BRUGNOLI, Ph.D. student
  - Ghislain HAINE
  - Valérie POMMIER - BUDINGER
  - Daniel ALAZARD
  - Michel SALAÜN
  - Xavier VASSEUR

Overview

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General ideas on port-Hamiltonian systems (pHs)

1. strongly structured mathematical dynamical systems: both linear and non-linear, both finite-dimensional and infinite-dimensional,
2. based on physical grounds, allowing for different modelling levels,
3. all physics permitted: solid mechanics, structural mechanics, fluid mechanics, electromagnetism, electrical circuits, ...
4. comes along with specific numerical methods, which do preserve, at the discrete level, the structure of the continuous equations,
5. allows for open dynamical systems, with interacting ports,
6. modularity: interconnection of sub-systems, and... easy multiphysics modelling, e.g. Fluid-Structure Interaction,
7. physically-based strategy for control and stabilization,
8. extensions to dissipative dynamical systems are available.

Problem of power-preserving semi-discretization

Infinite-dimensional pHs

$$\dot{\boldsymbol{x}}(z,t) = \mathcal{J}\boldsymbol{e}(z,t)\,,$$

where $z \in \Omega$:
$\boldsymbol{x}(z,t)$ is the vector of energy variables;
$\boldsymbol{e}(z,t) := \delta_{\boldsymbol{x}} H(\boldsymbol{x})$ is the vector of co-energy variables.
With boundary control & observation:

$$u_\partial = \mathcal{B}\boldsymbol{e}\,, \quad y_\partial = \mathcal{C}\boldsymbol{e}$$

Power-balance:
$\dot{H} =< y_\partial, u_\partial >_{\partial\Omega}$

$\Longrightarrow$

Finite-dimensional approximation:

$$\dot{\boldsymbol{x}}(t) = J\boldsymbol{e}(t) + B\,\boldsymbol{u}_\partial\,,$$
$$\boldsymbol{y}_\partial = B^T\boldsymbol{e}(t)\,,$$

where:
$\boldsymbol{x}(t) \in \mathbb{R}^N$ is the vector of energy variables;
$\boldsymbol{e}(t) = \nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) \in \mathbb{R}^N$ is the vector of co-energy variables;

Power-balance:
$\dot{H}_d =< \boldsymbol{y}_\partial, \boldsymbol{u}_\partial >_{\mathbb{R}^{N_\partial}} := \boldsymbol{y}_\partial^T M_\partial\, \boldsymbol{u}_\partial$.

Our goals

- Develop a power-preserving method that works in a domain Ω with arbitrary geometry, in 2D and 3D, in a straightforward way;
- Use Finite Element Method: easy to code; many computational tools are already available (FreeFem++, FEniCS, etc.);
- Semi-discretization is structure-preserving: finite-dimensional system is automatically a port-Hamiltonian system;
- Taking into account the boundary control and observation proves straightforward.

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$\Longrightarrow$ Thus, PFEM provides a semi-discretization that proves structure-preserving: the finite-dimensional system is automatically a port-Hamiltonian system.

Overview

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Questions:

- What are the 2 inital data: $w(t=0,x) = w^0(x)$ and $\partial_t w(t=0,x) = w^1(x)$?
- What kind of boundary conditions go along with the PDE?

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$\Longrightarrow$ The geometry of physics has been lost... almost everywhere!

Our sample problem: 2D wave equation as pHs

The wave equation:

$$\dot{\alpha}_p(\boldsymbol{x},t) = -\mathrm{div}\, \boldsymbol{e_q}(\boldsymbol{x},t)\,,$$
$$\dot{\boldsymbol{\alpha}}_q(\boldsymbol{x},t) = -\,\mathbf{grad}\, e_p(\boldsymbol{x},t)\,,$$

where $\boldsymbol{x} \in \Omega$ is the position vector, $\alpha_p = \rho\, \partial_t w$ (*linear momentum*) and $\boldsymbol{\alpha}_q = \mathbf{grad}\, w$ (*strain*) are the energy variables.

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Hamiltonian and co-energy

$$H = \frac{1}{2} \int_\Omega \left(\frac{1}{\rho} \alpha_p^2 + \boldsymbol{\alpha}_q^\top \cdot \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q \right) \mathrm{d}\Omega\,,$$

and the co-energy variables are *velocity* and *stress*, computed as:

$$e_p = \frac{\delta H}{\delta \alpha_p} = \frac{1}{\rho} \alpha_p\,, \quad \boldsymbol{e}_q = \frac{\delta H}{\delta \boldsymbol{\alpha}_q} = \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q\,.$$

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Power balance

$$\dot{H} = \int_{\partial\Omega} y_\partial(s,t)\, u_\partial(s,t)\;\mathrm{d}s\,,$$

where the boundary input is: $u_\partial(s,t) := -\boldsymbol{n}\cdot\boldsymbol{e_q}(\boldsymbol{x}(s),t) \quad s \in \partial\Omega\,.$
and its power-conjugated boundary output is: $y_\partial(s,t) := e_p(\boldsymbol{x}(s),t) \quad s \in \partial\Omega\,.$

The partitioned weak-form (1/2)

Strong form:

$$
\begin{aligned}
\dot{\alpha}_p(x,y,t) &= -\mathrm{div}\,\boldsymbol{e_q}(x,y,t)\\
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Weak form:

Taking arbitrary test functions $v_p(x,y)$, and $\boldsymbol{v}_q(x,y)$:

$$\int_\Omega v_p\,\dot{\alpha}_p\,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega v_p\,\mathrm{div}\,\boldsymbol{e_q}\,\mathrm{d}x\,\mathrm{d}y\,,$$
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Applying Stokes'theorem* to the first equation, we get:

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* Recall: $\int_\Omega \mathrm{div}\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega}\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}s$, and apply it to $\boldsymbol{w} := v_p\,\boldsymbol{e}_q$.
Make use of Leibniz's rule $\mathrm{div}(v_p\,\boldsymbol{e}_q) = v_p\,\mathrm{div}\boldsymbol{e}_q + \mathbf{grad}\,v_p\cdot\boldsymbol{e}_q$.

The partitioned weak-form (2/2)

$$\int_\Omega v_p\,\dot{\alpha}_p\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \textbf{grad}\,v_p \cdot \boldsymbol{e}_q\,\mathrm{d}x\,\mathrm{d}y - \int_{\partial\Omega} v_p\,\underbrace{\boldsymbol{n}\cdot\boldsymbol{e}_q(x,y,t)}_{-u_\partial}\,\mathrm{d}s\,,$$

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The input of the system explicitely appears on the previous weak-form:

$$u_\partial(s,t) := -\boldsymbol{n} \cdot \boldsymbol{e_q}(\boldsymbol{x}(s),t)) \,.$$

It is the *normal component of the stress* applied to the structure;

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Note: Stokes' theorem applied to the second equation would have given the velocity as input, and the normal component of the stress as output, div instead of **grad** being involved in the integrals.

Structure-preserving Finite Element Method (1/5)

The energy, co-energy and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\alpha_p^{ap} := \sum_{k=1}^{N_p} \phi_p^k(x,y)\alpha_p^k(t) = \boldsymbol{\phi}_p^T \boldsymbol{\alpha}_p(t), \qquad e_p^{ap} := \sum_{k=1}^{N_p} \phi_p^k(x,y)e_p^k(t) = \boldsymbol{\phi}_p^T \boldsymbol{e}_p(t),$$

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with ϕ_p an $N_p \times 1$ matrix, and $\boldsymbol{\Phi}_q$ an $N_q \times 2$ matrix.

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with $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix, and $\boldsymbol{\Phi}_q$ an $N_q \times 2$ matrix.

The same procedure is applied for the boundary terms with a specific basis ψ, leading to $\boldsymbol{\psi}$ as $N_\partial \times 1$ matrix:

$$u_\partial \approx u_\partial^{ap} := \sum_{m=1}^{N_\partial} \psi^m(s)u_\partial^m(t) = \boldsymbol{\psi}(s)^T \boldsymbol{u}_\partial(t).$$

Remark: The functions $\psi(s)$ can be selected as the restriction of functions ϕ over the boundary $\psi(s) = \phi(x(s), y(s))$ or in other ways.

PFEM (2/5) (a): Structure

Selecting as test functions vs all possible basis functions, we get:

$$\underbrace{\int_\Omega \boldsymbol{\phi}_p \boldsymbol{\phi}_p^T \,\mathrm{d}x\,\mathrm{d}y}_{M_p} \,\dot{\boldsymbol{\alpha}}_p = \underbrace{\int_\Omega \mathbf{grad}(\boldsymbol{\phi}_p) \cdot \boldsymbol{\Phi}_q^T \,\mathrm{d}x\,\mathrm{d}y}_{D} \,\boldsymbol{e}_q - \underbrace{\int_{\partial\Omega} \boldsymbol{\phi}_p \boldsymbol{\psi}^T(s)\,\mathrm{d}s}_{B} \,\boldsymbol{u}_\partial(t)\,,$$

$$\underbrace{\int_\Omega \boldsymbol{\Phi}_q \boldsymbol{\Phi}_q^T \,\mathrm{d}x\,\mathrm{d}y}_{M_q} \,\dot{\boldsymbol{\alpha}}_q = -\underbrace{\int_\Omega \boldsymbol{\Phi}_2 \cdot \mathbf{grad}(\boldsymbol{\phi}_p)^T \,\mathrm{d}x\,\mathrm{d}y}_{D^T} \,\boldsymbol{e}_p\,.$$

The equations can be rewritten as:

$$\begin{aligned} M_p\,\dot{\boldsymbol{\alpha}}_p &= D\,\boldsymbol{e}_q + B\,\boldsymbol{u}_\partial(t)\,, \\ M_q\,\dot{\boldsymbol{\alpha}}_q &= -\,D^T \boldsymbol{e}_p\,, \end{aligned}$$

where M_p and M_q are square mass matrices (of size $N_p \times N_p$, and $N_q \times N_q$, respectively). With $D_{kl} = \int_\Omega \mathbf{grad}(\phi_p^k) \cdot \boldsymbol{\Phi}_q^l \,\mathrm{d}x\,\mathrm{d}y$, D is an $N_p \times N_q$ matrix. With $B_{km} = \int_{\partial\Omega} \phi_p^k\,\psi^m\,\mathrm{d}s$, B is an $N_p \times N_\partial$ matrix.

PFEM (3/5) (a): Structure

The discretized system is written as

$$\begin{bmatrix} M_p & 0 \\ 0 & M_q \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_p \\ \dot{\boldsymbol{\alpha}}_q \end{pmatrix} = \begin{bmatrix} 0 & D \\ -D^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_q \end{pmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} \boldsymbol{u}_\partial,$$

Next, defining the boundary mass matrix $M_\partial := \int_{\partial\Omega} \boldsymbol{\psi}\,\boldsymbol{\psi}^T \,\mathrm{d}s$ of size $N_\partial \times N_\partial$, the collocated output is defined by:

$$M_\partial\, \boldsymbol{y}_\partial = \begin{bmatrix} B_p^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_p \\ \boldsymbol{e}_q \end{pmatrix}.$$

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$\Longrightarrow$ In the language of <u>flows</u> ($\boldsymbol{f}_p = \dot{\boldsymbol{\alpha}}_p, \boldsymbol{f}_q = \dot{\boldsymbol{\alpha}}_q$) and <u>efforts</u> ($\boldsymbol{e}_p$, $\boldsymbol{e}_q$), the following **structural identity** can be easily recovered:

$$\begin{array}{rrcl} & \boldsymbol{f}_p^T \, M_p \, \boldsymbol{e}_p + \boldsymbol{f}_q^T \, M_q \, \boldsymbol{e}_q & = & \boldsymbol{y}_\partial^T \, M_\partial \, \boldsymbol{u}_\partial \,, \\ \text{or} & (\boldsymbol{f}_p, \boldsymbol{e}_p)_p + (\boldsymbol{f}_q, \boldsymbol{e}_q)_q & = & (\boldsymbol{y}_\partial, \boldsymbol{u}_\partial)_\partial \,, \end{array}$$

underlying a finite-dimensional Dirac structure.

PFEM (4/5) (b): Constitutive relations

1. We define the discretized Hamiltonian by direct substitution:

$$\begin{aligned} H_d(\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q) &:= H\left[\alpha_p(\boldsymbol{x},t) = \boldsymbol{\alpha}_p^T(t)\,\boldsymbol{\phi}_p(\boldsymbol{x})\,, \alpha_q(\boldsymbol{x},t) = \boldsymbol{\alpha}_q^T(t)\,\boldsymbol{\Phi}_q(\boldsymbol{x})\right]\,, \\ &= \frac{1}{2}\boldsymbol{\alpha}_p^T(t)\,M_{1/\rho}\,\boldsymbol{\alpha}_p(t) + \frac{1}{2}\boldsymbol{\alpha}_q^T(t)\,M_T\,\boldsymbol{\alpha}_q(t)\,, \end{aligned}$$

with the *spatially averaged* coefficient matrices:

$$M_{1/\rho} := \int_\Omega \boldsymbol{\phi}_p \frac{1}{\rho(x,y)} \boldsymbol{\phi}_p^T \,\mathrm{d}x\,\mathrm{d}y, \qquad M_T := \int_\Omega \boldsymbol{\Phi}_q \cdot \overline{\overline{T}}(x,y) \cdot \boldsymbol{\Phi}_q^T \,\mathrm{d}x\,\mathrm{d}y,$$

$$Q_{1/\rho} := M_p^{-1}\,M_{1/\rho}\,, \qquad Q_T := M_q^{-1}\,M_{1/\rho}\,.$$

$\Longrightarrow$ Hence, $2\,H_d(\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q) = (\boldsymbol{\alpha}_p, Q_{1/\rho}\,\boldsymbol{\alpha}_p)_p + (\boldsymbol{\alpha}_q, Q_T\,\boldsymbol{\alpha}_q)_q$.

PFEM (4/5) (b): Constitutive relations

1. We define the discretized Hamiltonian by direct substitution:

$$\begin{aligned} H_d(\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_q) &:= H\left[\alpha_p(\boldsymbol{x},t) = \boldsymbol{\alpha}_p^T(t)\,\boldsymbol{\phi}_p(\boldsymbol{x})\,, \alpha_q(\boldsymbol{x},t) = \boldsymbol{\alpha}_q^T(t)\,\boldsymbol{\Phi}_q(\boldsymbol{x})\right]\,, \\ &= \frac{1}{2}\boldsymbol{\alpha}_p^T(t)\,M_{1/\rho}\,\boldsymbol{\alpha}_p(t) + \frac{1}{2}\boldsymbol{\alpha}_q^T(t)\,M_T\,\boldsymbol{\alpha}_q(t)\,, \end{aligned}$$

with the *spatially averaged* coefficient matrices:

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2. The discrete effort variables are computed w.r.t. weighted scalar products:

$$\boldsymbol{e}_p := \nabla_{\boldsymbol{\alpha}_p} H_d = Q_{1/\rho}\,\boldsymbol{\alpha}_p \qquad \boldsymbol{e}_q := \nabla_{\boldsymbol{\alpha}_q} H_d = Q_T\,\boldsymbol{\alpha}_q\,.$$

PFEM (5/5) (b): Constitutive relations

The power-balance is preserved:

$$
\begin{aligned}
\dot{H}_d &= (\dot{\boldsymbol{\alpha}}_p, \boldsymbol{e}_p)_p + (\dot{\boldsymbol{\alpha}}_q, \boldsymbol{e}_q)_q \\
&= (\boldsymbol{u}_\partial, \boldsymbol{y}_\partial)_\partial \\
&= \boldsymbol{y}_\partial^T \, M_\partial \, \boldsymbol{u}_\partial \\
&= \boldsymbol{e}_p^T \, B \, \boldsymbol{u}_\partial \\
&= \boldsymbol{e}_p^T \left(\int_{\partial\Omega} \boldsymbol{\phi}_p \, \boldsymbol{\psi}^T(s) \, \mathrm{d}s \right) \boldsymbol{u}_\partial \, , \\
&= \int_{\partial\Omega} e_p^{ap}(\boldsymbol{x}(s), t) \, u_\partial^{ap}(s,t) \, \mathrm{d}s = (u_\partial^{ap}(.,t), y_\partial^{ap}(.,t))_{L^2(\partial\Omega)} \, .
\end{aligned}
$$

$\Longrightarrow$ this is a finite-dimensional pHs, according to e.g.

H. Egger, T. Kugler, B. Liljegren-Sailer, N. Marheineke and V. Mehrmann.

On structure preserving model reduction for damped wave propagation in transport networks. SIAM J. Sci. Comput., 40-1, A331-A365, 2018.

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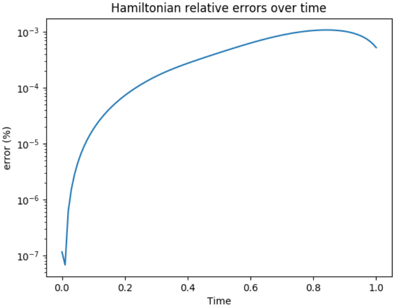

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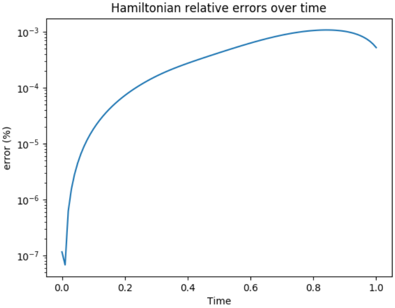
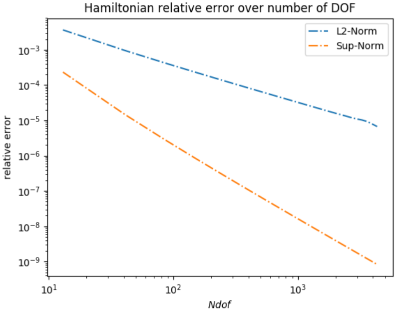


$$t \mapsto |H(t) - H_d(t)|/H(t)$$



$$N \mapsto \|H - H_d^N\|_{norm}/\|H\|_{norm}$$

Anisotropic, heterogeneous with Dirichlet boundary control

$$\rho(x,y) := x^2(2-x)+1, \qquad \overline{\overline{T}}(x,y) := \begin{pmatrix} x^2+1 & y \\ y & x+1 \end{pmatrix}$$

Anisotropic, heterogeneous with Impedance Boundary Condition (IBC)

$\rho(x,y) := x^2(2-x)+1,$ $\qquad \overline{\overline{T}}(x,y) := \begin{pmatrix} x^2+1 & y \\ y & x+1 \end{pmatrix}$ $\qquad Z_j$

Overview

The continuous model: from thermodynamics...

1. Physical quantities:
   - ρ the mass density,
   - u the internal energy density,
   - $\mathbf{J}$ the heat flux,
   - T the local temperature,
   - $\overline{\overline{\lambda}}$ the diffusivity **tensor**, symmetric positive definite,
   - C_V the heat capacity (at constant volume).

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4. Constitutive relations
   - Dulong-Petit's model: $u = C_V T$, with time-invariant heat capacity,
   - Fourier's law: $\mathbf{J} = -\overline{\overline{\lambda}} \cdot \mathbf{grad}(T)$

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1. Defining as *flows* and *efforts*:

$$\begin{aligned} f_u &:= \partial_t u, & e_u &:= \delta_u H = \frac{u}{C_V}, \\ \mathbf{f}_Q &:= -\,\textbf{grad}\left(\frac{u}{C_V}\right), & \mathbf{e}_Q &:= \mathbf{J}, \end{aligned}$$

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The continuous model: to sum up

- The *heat* equation, taking u as energy variable, leads to **the same structure as for the wave equation** when choosing as Hamiltonian the Lyapunov functional:

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- In terms of boundary control of PDEs: this corresponds to

$$\rho(\mathbf{x}) C_V(\mathbf{x}) \partial_t T(t,\mathbf{x}) - \mathrm{div}\left(\overline{\overline{\lambda}}(\mathbf{x}) \cdot \mathbf{grad}(T(t,\mathbf{x})) \right) = 0,$$

with <u>either</u> of the proposed boundary controls: *temperature* or *heat flux*.

Weak formulation (1/2): (a) Structure

Take Finite Element families like for the wave equation case, write:

$$\left\{ \begin{array}{ll} \int_\Omega \rho f_u \varphi & = -\int_\Omega \mathsf{div}(\mathbf{e}_Q)\varphi, \\ \int_\Omega \mathbf{f}_Q \cdot \boldsymbol{\varphi} & = -\int_\Omega \mathbf{grad}(e_u) \cdot \boldsymbol{\varphi}, \end{array} \right.$$

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Defining the dual observation $y_\partial := (e_u)_{|_{\partial\Omega}}$, the structure reads again:

$$\begin{pmatrix} M_\rho & 0 & 0 \\ 0 & \mathbf{M} & 0 \\ 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \underline{f_u} \\ \underline{\mathbf{f}_Q} \\ -\underline{y_\partial} \end{pmatrix} = \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e_u} \\ \underline{\mathbf{e}_Q} \\ \underline{v_\partial} \end{pmatrix},$$

with **the same matrices** as for the wave equation.

Weak formulation (2/2): (b) Constitutive relations

Write $u = C_V T$ and Fourier's law *in weak form*, using pHs variables:

$$\int_\Omega \rho C_V \, \partial_t T \, \varphi = \int_\Omega \rho f_u \, \varphi \quad \Rightarrow \quad M_{\rho C_V} \frac{d}{dt} \underline{T} = M_\rho \underline{f_u},$$

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The discrete Hamiltonian $H_d(t) := \frac{1}{2} \underline{T}^\top(t) \, M_{\rho C_V} \, \underline{T}(t)$ satisfies:

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Resolution

Using inversions and substitutions, **PFEM also provides ODEs**:

$$M_{\rho C_V} \frac{d}{dt} \underline{T}(t) = -D\mathbf{M}^{-1} \mathbf{\Lambda} \mathbf{M}^{-1} D^\top \underline{T}(t) + B \underline{v_\partial}(t),$$

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We thus have at least **3 ways** to numerically solve the heat PDE:

- the classical ODE-FEM approach + RK45 in time,
- the new DAE-PFEM approach + IDA (SUNDIALS) in time,
- the new ODE-PFEM approach + RK45 in time.

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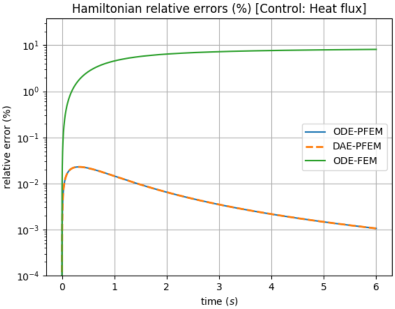

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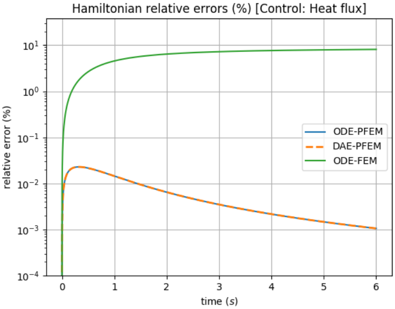

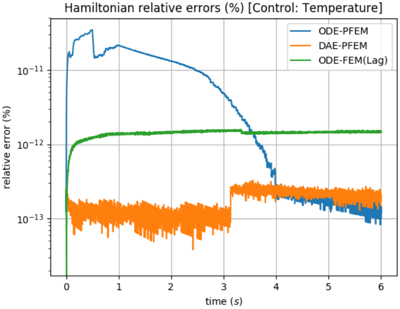

Anisotropic, heterogeneous & heat flux control

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The plate equation: a mathematician's viewpoint

Several scenarios...

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The plate equation: a mathematician's viewpoint

Several scenarios...

1. $\frac{1}{\kappa^2}\partial^2_{tt} w + \partial^4_{xxxx} w = 0$?
2. $\frac{1}{\kappa^2}\partial^2_{tt} w + \Delta^2 w = 0$?
3. $\rho\, \partial^2_{tt} w + \Delta(D \Delta w) = 0$?
4. $\rho\, \partial^2_{tt} w + \mathrm{div}\, \mathrm{Div}(\mathbb{D}\, \mathrm{Grad}(\mathbf{grad}\, w)) = 0$! (with $\mathbb{D}$ a fourth order tensor)

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- What kind of boundary conditions go along with the PDE?

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$\Longrightarrow$ The geometry of physics has been lost... almost everywhere!

Corresponding 1D models for beams

Timoshenko beam

- Valid for thick beams
- Dimension of the PH model: 4
- Differential operator $\mathcal{J}$ of order 1

$$\boldsymbol{\alpha} = [\rho v,\ I_\rho \omega_x,\ \partial_x \phi_x,\ \partial_x w - \phi_x]^T$$
$$\boldsymbol{e} = [v,\ \omega_x,\ M_{xx},\ T_x]^T$$
$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & \partial_x \\ 0 & 0 & \partial_x & 1 \\ 0 & \partial_x & 0 & 0 \\ \partial_x & -1 & 0 & 0 \end{pmatrix}$$

Euler-Bernoulli beam

- Valid for thin beams
- Dimension of the PH model: 2
- Differential operator $\mathcal{J}$ of order 2

$$\boldsymbol{\alpha} = [\rho v,\ \partial^2_{xx} w]^T$$
$$\boldsymbol{e} = [v,\ M_{xx}]^T$$
$$\mathcal{J} = \begin{pmatrix} 0 & -\partial^2_{xx} \\ \partial^2_{xx} & 0 \end{pmatrix}$$

Energy and co-energy variables (vector form)

This model is the 2D extension of the Bernoulli beam. It is logical to select as energy variables the *linear momentum*, together with the *curvatures*:

$$\boldsymbol{\alpha} = (\mu v,\ \kappa_{xx},\ \kappa_{yy},\ \kappa_{xy})^T$$

where $v = \partial_t w$. The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & \boldsymbol{D} \end{bmatrix} \boldsymbol{\alpha}\,.$$

So the variational derivative of the total Hamiltonian $H = \int_\Omega \mathcal{H}\ d\Omega$ provides as co-energy variables:

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v,\ M_{xx},\ M_{yy},\ M_{xy})^T$$

vertical velocity and *momenta*.

Definition of $\mathcal{J}$ & boundary variables (vector form)

The skew-adjoint operator relating energy and co-energy variables is found to be

$$\mathcal{J} = \begin{bmatrix} 0 & -\partial_{xx}^2 & -\partial_{yy}^2 & -\left(\partial_{yx}^2 + \partial_{xy}^2\right) \\ \partial_{xx}^2 & 0 & 0 & 0 \\ \partial_{yy}^2 & 0 & 0 & 0 \\ \partial_{yx}^2 + \partial_{xy}^2 & 0 & 0 & 0 \end{bmatrix}, \qquad \partial_t \boldsymbol{\alpha} = \mathcal{J} \mathbf{e}.$$

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From Schwarz theorem for C^2 functions the mixed derivative could be expressed as $2\,\partial_{xy}^2$, instead of $\partial_{yx}^2 + \partial_{xy}^2$. However, in this way the symmetry intrinsically present in $\gamma_{xy} = -z\left(\partial_{yx}^2 w + \partial_{xy}^2 w\right)$ would be lost. The mixed derivative is split here to reestablish the symmetric nature of curvatures and momenta, that are indeed of **tensorial** nature!

A scalar-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \qquad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now κ_{xy} now is half the value of the one in the vectorial case! The curvatures tensor is the linear deformation tensor applied to the rotation vector $\boldsymbol{\theta} = \textbf{grad}\, w$

$$\mathbb{K} = \mathrm{Grad}(\boldsymbol{\theta}) = \mathrm{Grad}(\textbf{grad}\, w).$$

The momenta are found by introducing a <u>fourth order tensor</u> $\mathbb{D}$, such that $\mathbb{M}_{ij} = \mathbb{D}_{ijkl}\, \mathbb{K}_{kl}$, i.e. $\mathbb{M} = \mathbb{D}\, \mathbb{K}$ for short.

For what concerns the choice of the energy variables a scalar and a tensor variable are grouped together

$$\alpha_w = \mu \partial_t w \qquad \mathbb{A}_\kappa = \mathbb{K}$$

The Hamiltonian energy is written as

$$H = \int_\Omega \left\{ \frac{1}{2} \mu \left(\partial_t w \right)^2 + \frac{1}{2} \mathbb{K} : \mathbb{D}\, \mathbb{K} \right\} \mathrm{d}\Omega,$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$e_w := \frac{\delta H}{\delta \alpha_w} = \partial_t w, \qquad \mathbb{E}_\kappa := \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}.$$

Interconnection structure

The formally skew-symmetric operator $\mathcal{J}$ can be highlighted

Strong form for the Kirchhoff plate

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \mathbb{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\mathrm{div}\circ\mathrm{Div} \\ \mathrm{Grad}\circ\mathbf{grad} & 0\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}e_w \\ \mathbb{E}_\kappa\end{pmatrix}.$$

Theorem

The adjoint of Div *is* $-\mathrm{Grad}$.
The adjoint of div $\circ\,\mathrm{Div}$ *is* $\mathrm{Grad}\circ\,$***grad*** *(i.e. the Hessian operator)*

Remark: The interconnection structure operator $\mathcal{J}$ now resembles that of the Euler-Bernoulli beam: both the double divergence and the double gradient do coincide, in dimension one, with the second derivative.

Kirchhoff Plate: discretized operators

The discretized system is written as

$$\begin{pmatrix} M_1 \dot{\boldsymbol{\alpha}}_1 \\ M_2 \dot{\boldsymbol{\alpha}}_2 \\ M_3 \dot{\boldsymbol{\alpha}}_3 \\ M_4 \dot{\boldsymbol{\alpha}}_4 \end{pmatrix} = \begin{bmatrix} 0 & -D_{xx}^T & -D_{yy}^T & -2D_{xy}^T \\ D_{xx} & 0 & 0 & 0 \\ D_{yy} & 0 & 0 & 0 \\ 2D_{xy} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_{\partial,1} \\ \boldsymbol{u}_{\partial,2} \end{pmatrix},$$

where M_i are square matrices (of size $N_i \times N_i$), D_{xx} is an $N_2 \times N_1$ matrix, D_{yy} is an $N_3 \times N_1$ matrix, D_{xy} is an $N_4 \times N_1$ matrix, B_1 is an $N_1 \times N_{\partial,1}$ matrix and finally B_2 is an $N_2 \times N_{\partial,2}$ matrix. The collocated output are defined as

$$\boldsymbol{y}_\partial = \begin{bmatrix} B_1^T & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_4 \end{pmatrix}.$$

Symmetric boundary excitation on Kirchhoff plate

Effect of gravity field on Kirchhoff plate

Distributed force on Kirchhoff plate

Distrib. force on Kirchhoff plate welded to a rod

Interconnected Kirchhoff plates

Boundary damping injection on Kirchhoff plate

Overview

SWE 2D in a disc, as a pHs

Energy variables: $\alpha_q = h$ *height*, and $\boldsymbol{\alpha}_p = \rho\,[u^r, u^\theta]^T$ *linear momentum*.

The Hamiltonian reads H

$$\begin{aligned} H &= \frac{1}{2}\int_{D_R} [\rho g\, h^2 + \rho h\,((u^r)^2 + (u^\theta)^2)]\; r\, \mathrm{d}r\, \mathrm{d}\theta\,, \\ &= \int_{D_R} [\frac{1}{2}\rho g\, \alpha_q^2 + \frac{1}{2\rho}\alpha_q|\boldsymbol{\alpha_p}|^2]\; r\, \mathrm{d}r\, \mathrm{d}\theta\;. \end{aligned}$$

The effort or co-energy variables can be computed as:

$e_q := \delta_q H = \rho g\, \alpha_q + \frac{1}{2\rho}|\boldsymbol{\alpha_p}|^2$ *hydrodynamic pressure*,

$\boldsymbol{e_p} := \delta_{\boldsymbol{p}} H = \frac{1}{\rho}\alpha_q\boldsymbol{\alpha_p} = h\,[u^r(t,r,\theta), u^\theta(t,r,\theta)]^T$ *volume flow*.

$$\begin{bmatrix} \dot{h} \\ \rho \begin{bmatrix} \dot{u^r} \\ \dot{u^\theta} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{div} \\ -\,\mathbf{grad} & 0 \end{bmatrix} \begin{bmatrix} \rho(gh + \frac{(u^r)^2+(u^\theta)^2}{2}) \\ h \begin{bmatrix} u^r \\ u^\theta \end{bmatrix} \end{bmatrix},$$

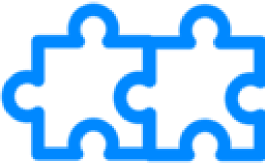
with boundary control $u_\partial(\theta,t) := -\boldsymbol{e_p}\cdot\boldsymbol{n} = -e_p^r(R,\theta,t)$ and collocated boundary observation $y_\partial(\theta,t) := e_q(R,\theta,t)$ at the boundary $\partial\Omega = C_R$.

Boundary feedback control of a circular water tank

Overview

SCRIMP software - Main objectives

- To provide the INFIDHEM partners with a **state-of-the-art**, comprehensive library for the numerical simulation of interconnected port-Hamiltonian systems (pHs).
- To provide well-documented tools to **make easier** the scientific cooperation within the INFIDHEM project.
- To develop a software that is **easy to learn and to use.**
- To develop a software that is **easy to modify or extend.**



Simulation and ContRol of Interconnected MultiPhysical systems (SCRIMP)

Programming language

- **Python** has been selected due to its **expressivity**, **ease of use and prototyping** and the **availability** of many well documented scientific libraries.

Interoperability

- **Python interoperability** offers the possibility to use any external library to define a port Hamiltonian subsystem in the finite- or infinite-dimensional case.
- **Examples**: simulations have been performed with PDE finite element discretizations based either on
  FEnICS (`https://fenicsproject.org/`), or
  Firedrake (`https://firedrakeproject.org/`).
- This seems especially important so as to tackle **coupled multi-physics** problems, where each subsystem may correspond to a different physical phenomenon.

SCRIMP software - Main features

Classes in SCRIMP are provided to

- Define specific elementary port-Hamiltonian systems in the finite dimensional setting.
- Define how to **interconnect** port-Hamiltonian subsystems to obtain the resulting system.
- Represent the algebraic dynamical system as a standard **pHS** or as a **pHDAE** system [Beattie et al, 2018].
- Use **state-of-the-art** numerical methods for time integration [Ongoing].
- Perform **structure-preserving** model reduction [Chaturantabut et al, 2016] and [Gugercin et al, 2012] [Ongoing].

Aim

S. Chaturantabut, C. Beattie, and S. Gugercin. *Structure-preserving model reduction for nonlinear Port-Hamiltonian systems*.
SIAM J. Sci. Comput., 38-5:B837–B865, 2016.

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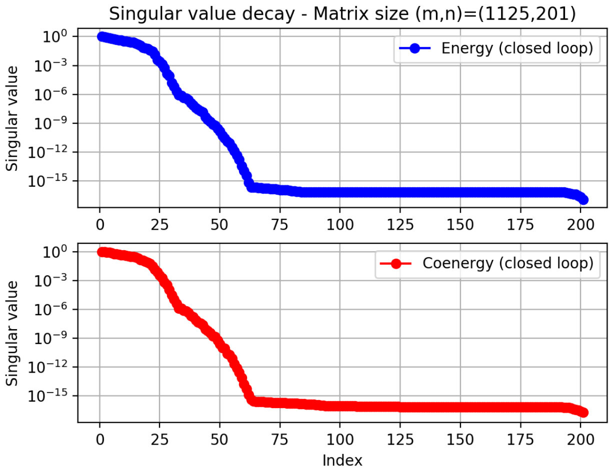
Very recent results: still need a lot of work!

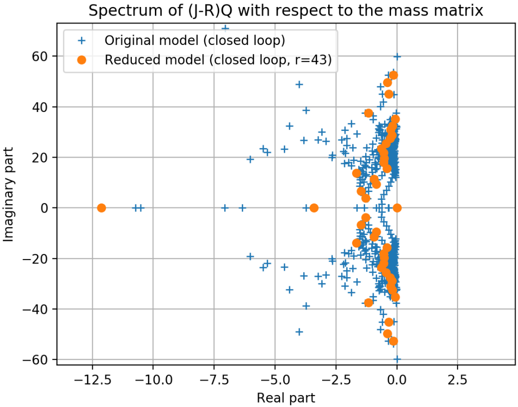
Wave with impedance boundary conditions

Step 1: SVD on snapshots of the energy and co-energy variables.



Wave with impedance boundary conditions

Step 2: choose a tolerance and construct 2 subspaces for $\underline{f}$ and $\underline{e}$.
From $N = 1125$ to $N_R = 43$ for $\mathrm{tol} = 10^{-8}$.
Comparison of the spectrum:



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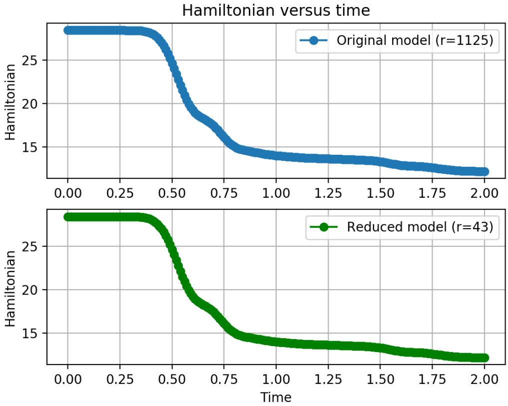
Comparison of the Hamiltonian:



Kirchhoff plate

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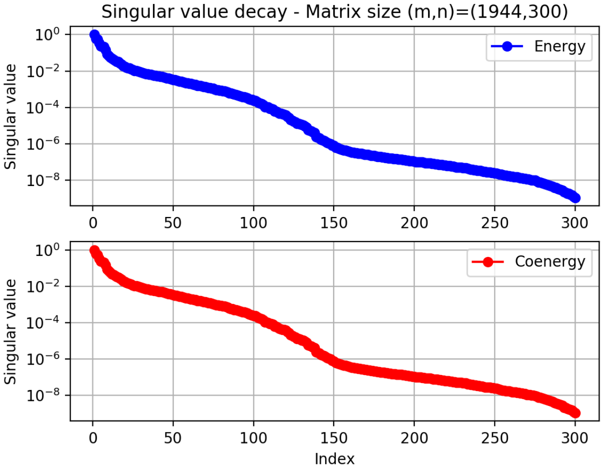
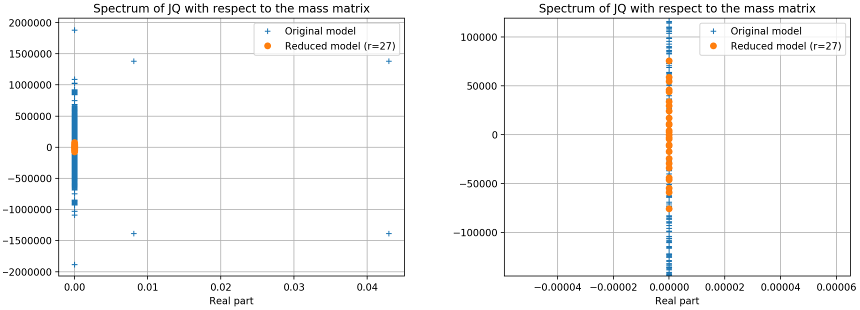


Kirchhoff plate

Step 2: choose a tolerance and construct 2 subspaces for $\underline{f}$ and $\underline{e}$.
From $N = 1944$ to $N_R = 27$ for $\mathrm{tol} = 10^{-2}$.
Comparison of the spectrum:



The “bad” eigenvalues should not exist here. The resolution of the eigenvalue problem should be investigated: a numerical artifact?

Overview

Conclusions

- The integration by parts of one of the weak-form equations naturally leads to skew-symmetric representation with the boundary input/output ports;
- 2D (or 3D) problems are straightforward to address;
- Interconnection structure and Constitutive relations are discretized separately;
- Same method can be used for other port-Hamiltonian systems (higher-order differential operators like Euler-Bernoulli beam, or Kirchhoff-Love plate equations, etc.);
- Other coordinate systems (as polar coordinates) can be used;
- Space varying coefficients can be easily taken into account;
- Nonlinear equations: non-quadratic Hamiltonian and non-linear interconnection structure;
- PFEM can be easily implemented using available Finite Element software allowing for complex geometries.

Onngoing work and open questions

- Other (mixed) choices of input/output are possible;
- Ongoing convergence analysis;
- Numerical methods for DAEs;
- Link with weak / strong formulation and differential forms;
- Multiphysics systems modelling: some useful 2D testcases (fluid-structure interaction (FSI), thermal-structure coupling, fluid-thermal coupling);
- Design and implementation of control laws.

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