

# Port-Hamiltonian flexible multibody dynamics

**Andrea Brugnoli**

<sup>1</sup>ISAE-SUPAERO, Toulouse

- 1 Previous work on multibody systems and the pH formalism
- 2 PH formulation of a floating body
  - Floating frame formulation
  - Energies and momenta
  - PH formulation
- 3 Discretization
- 4 Construction of multibody chain
  - General procedure for planar beams
  - The linear case

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Using Lie Algebra and differential forms a pH model of a flexible link has already been proposed<sup>1</sup>. This model can be embedded in a complex multibody system<sup>2</sup>.

Advantages:

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Drawbacks:

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The floating frame approach relies on the hypothesis of small deformations: elastic motion is described w.r.t a reference that follows the large rigid motion<sup>3</sup>.

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- The most used paradigm in multibody dynamics;



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## Advantages

- The most used paradigm in multibody dynamics;
- For control applications most references adopt this approach;
- Model reduction techniques are applicable.

## Drawbacks:

- Effect due to geometric non-linearities are not considered: not suitable for large deformations (substructuring can be employed to alleviate this).

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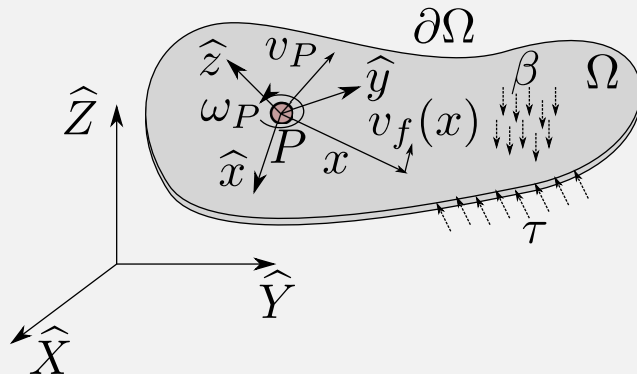


Figure: Thin floating body undergoing a surface traction  $\tau$  and body force density  $\beta$

The velocity of a generic point is expressed by considering a small flexible displacement superimposed to the rigid motion

$$\mathbf{v} = \mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times}(\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f.$$

This equation is expressed in the body reference frame  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ .

- $\mathbf{x}$  is the position vector of the current point;
- $\mathbf{v}_P, \boldsymbol{\omega}_P$  are the linear and angular velocities of point  $P$ ;
- $\mathbf{v}_f := \dot{\mathbf{u}}_f$  is the time derivative of the deformation displacement  $\mathbf{u}_f$  (computed in the body frame);
- The cross map  $[\mathbf{a}]_{\times}$  denotes the skew-symmetric matrix associated to vector  $\mathbf{a}$ .

The equations are obtained by application of the virtual work principle<sup>4</sup>.

## Linear momentum balance

$$m(\dot{\mathbf{v}}_P + [\boldsymbol{\omega}_P]_{\times} \mathbf{v}_P) + [\mathbf{s}_u]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho \ddot{\mathbf{u}}_f \, d\Omega = \\ - [\boldsymbol{\omega}_P]_{\times} [\boldsymbol{\omega}_P]_{\times} \mathbf{s}_u - \int_{\Omega} 2\rho [\boldsymbol{\omega}_P]_{\times} \dot{\mathbf{u}}_f \, d\Omega + \int_{\Omega} \boldsymbol{\beta} \, d\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, d\Gamma,$$

- $\rho$  is the mass density;
- $m = \int_{\Omega} \rho \, d\Omega$  the total mass;
- $\mathbf{s}_u = \int_{\Omega} \rho(\mathbf{x} + \mathbf{u}_f) \, d\Omega$  the static moment.

<sup>4</sup>Bernd Simeon. *Computational flexible multibody dynamics*. Springer, 2013, Chapter 4.

# Equations of motion

The equations are obtained by application of the virtual work principle<sup>4</sup>.

## Linear momentum balance

$$m\dot{\mathbf{v}}_P + [\mathbf{s}_u]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho \dot{\mathbf{v}}_f \, d\Omega =$$
$$\left[ m\mathbf{v}_P + [\mathbf{s}_u]_{\times}^{\top} \boldsymbol{\omega}_P + 2 \int_{\Omega} \rho \mathbf{v}_f \, d\Omega \right]_{\times} \boldsymbol{\omega}_P + \int_{\Omega} \boldsymbol{\beta} \, d\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, d\Gamma.$$

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## Angular momentum balance

$$\begin{aligned} [\mathbf{s}_u]_{\times} (\dot{\mathbf{v}}_P + [\boldsymbol{\omega}_P]_{\times} \mathbf{v}_P) + \mathbf{J}_u \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \ddot{\mathbf{u}}_f \, d\Omega + [\boldsymbol{\omega}_P]_{\times} \mathbf{J}_u \boldsymbol{\omega}_P = \\ - \int_{\Omega} 2\rho [\mathbf{x} + \mathbf{u}_f]_{\times} [\boldsymbol{\omega}_P]_{\times} \dot{\mathbf{u}}_f \, d\Omega + \int_{\Omega} [\mathbf{x} + \mathbf{u}_f]_{\times} \boldsymbol{\beta} \, d\Omega + \int_{\partial\Omega} [\mathbf{x} + \mathbf{u}_f]_{\times} \boldsymbol{\tau} \, d\Gamma, \end{aligned}$$

$\mathbf{J}_u := \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} [\mathbf{x} + \mathbf{u}_f]_{\times} \, d\Omega$  is the inertia matrix.

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## Flexibility PDE

$$\rho(\dot{\mathbf{v}}_P + [\boldsymbol{\omega}_P]_{\times} \mathbf{v}_P) + \rho([\dot{\boldsymbol{\omega}}_P]_{\times} + [\boldsymbol{\omega}_P]_{\times} [\boldsymbol{\omega}_P]_{\times})(\mathbf{x} + \mathbf{u}_f) + \rho(2[\boldsymbol{\omega}_P]_{\times} \dot{\mathbf{u}}_f + \ddot{\mathbf{u}}_f) = \text{Div } \boldsymbol{\Sigma} + \boldsymbol{\beta},$$

together with boundary conditions

Neumann condition  $\boldsymbol{\Sigma} \cdot \mathbf{n}|_{\Gamma_N} = \boldsymbol{\tau}|_{\Gamma_N}$ ,  $\mathbf{n}$  is the outward normal,

Dirichlet condition  $\mathbf{u}_f|_{\Gamma_D} = \bar{\mathbf{u}}_f|_{\Gamma_D}$ ,

- $\boldsymbol{\Sigma}$  is the Cauchy stress tensor;
- The infinitesimal strain is given by  $\boldsymbol{\varepsilon} = \text{Grad}(\mathbf{u}_f)$   $\text{Grad} = \frac{1}{2}[\nabla + \nabla^T]$ ;
- To close the system, Hooke's law  $\boldsymbol{\Sigma} = \boldsymbol{\mathcal{D}}\boldsymbol{\varepsilon}$ , where  $\boldsymbol{\mathcal{D}}$  is the stiffness tensor.

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## Flexibility PDE

$$\rho \dot{\mathbf{v}}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \rho \dot{\mathbf{v}}_f = \\ \left[ \rho \mathbf{v}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \boldsymbol{\omega}_P + 2\rho \mathbf{v}_f \right]_{\times} \boldsymbol{\omega}_P + \text{Div } \boldsymbol{\Sigma} + \boldsymbol{\beta}.$$

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# Energies and canonical momenta

Consider the total energy (Hamiltonian), given by the sum of kinetic and deformation energy:

$$H = H_{\text{kin}} + H_{\text{def}} = \frac{1}{2} \int_{\Omega} \left\{ \rho \| \mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times} (\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f \|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} d\Omega.$$

## Canonical momenta

$$\mathbf{p}_t := \frac{\partial H}{\partial \mathbf{v}_P} = m \mathbf{v}_P + [\mathbf{s}_u]_{\times}^{\top} \boldsymbol{\omega}_P + \int_{\Omega} \rho \mathbf{v}_f d\Omega,$$

$$\mathbf{p}_r := \frac{\partial H}{\partial \boldsymbol{\omega}_P} = [\mathbf{s}_u]_{\times} \mathbf{v}_P + \mathbf{J}_u \boldsymbol{\omega}_P + \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \mathbf{v}_f d\Omega,$$

$$\mathbf{p}_f := \frac{\delta H}{\delta \mathbf{v}_f} = \rho \mathbf{v}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \boldsymbol{\omega}_P + \rho \mathbf{v}_f,$$

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## Canonical momenta

$$\begin{bmatrix} \mathbf{p}_t \\ \mathbf{p}_r \\ \mathbf{p}_f \\ \boldsymbol{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} m\mathbf{I}_{3 \times 3} & [\mathbf{s}_u]_{\times}^{\top} & \mathcal{I}_{\rho}^{\Omega} & 0 \\ [\mathbf{s}_u]_{\times} & \mathbf{J}_u & \mathcal{I}_{\rho x}^{\Omega} & 0 \\ (\mathcal{I}_{\rho}^{\Omega})^* & (\mathcal{I}_{\rho x}^{\Omega})^* & \rho & 0 \\ 0 & 0 & 0 & \mathcal{D}^{-1} \end{bmatrix}}_{\mathcal{M}: \text{Mass operator}} \begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega}_P \\ \mathbf{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}, \quad \begin{aligned} \mathcal{I}_{\rho}^{\Omega} &:= \int_{\Omega} \rho(\cdot) d\Omega, \\ \mathcal{I}_{\rho x}^{\Omega} &:= \int_{\Omega} \rho[\mathbf{x} + \mathbf{u}_f]_{\times}(\cdot). \end{aligned}$$

The mass operator  $\mathcal{M}$  is a self-adjoint, positive operator. It holds

$$H_{\text{kin}} + H_{\text{def}} = \frac{1}{2} \langle \mathbf{e}_{\text{kd}}, \mathcal{M} \mathbf{e}_{\text{kd}} \rangle, \quad \mathbf{e}_{\text{kd}} = [\mathbf{v}_P; \boldsymbol{\omega}_P; \mathbf{v}_f; \boldsymbol{\Sigma}]$$

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Notice that the kinetic energy also depends on the flexible displacement

$$\frac{\delta H_{\text{kin}}}{\delta \mathbf{u}_f} = [\mathbf{p}_f]_{\times} \boldsymbol{\omega}_P.$$

This term is responsible for a coupling between the kinematic coordinates and the velocities.

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Generalized coordinates are required for a complete formulation:

- ${}^i\mathbf{r}_P$  the position of point  $P$  in the inertial frame of reference;
- $\mathbf{R}$  the direction cosine matrix (other attitude parametrizations are possible);
- $\mathbf{u}_f$  the flexible displacement;

The direction cosine matrix is converted into a vector by concatenating its rows

$$\mathbf{R}_v = \text{vec}(\mathbf{R}^\top) = [\mathbf{R}_1 \ \mathbf{R}_2 \ \mathbf{R}_3]^\top,$$

where  $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$  are the rows of matrix  $\mathbf{R}$ . Furthermore the corresponding cross map will be given by

$$[\mathbf{R}_v]_\times = \begin{bmatrix} [\mathbf{R}_1]_\times \\ [\mathbf{R}_2]_\times \\ [\mathbf{R}_3]_\times \end{bmatrix}, \quad [\mathbf{R}_v]_\times : \mathbb{R}^9 \rightarrow \mathbb{R}^{9 \times 3}.$$

The overall port-Hamiltonian formulation

$$\underbrace{\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathcal{M} \end{bmatrix}}_{\mathcal{E}} \frac{d}{dt} \underbrace{\begin{bmatrix} {}^i\mathbf{r}_P \\ \mathbf{R}_v \\ \mathbf{u}_f \\ \mathbf{v}_P \\ \boldsymbol{\omega}_P \\ \mathbf{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}}_e = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \mathbf{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\mathbf{R}_v]_{\times} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{3 \times 3} & 0 \\ -\mathbf{R}^{\top} & 0 & 0 & 0 & [\tilde{\mathbf{p}}_t]_{\times} & 0 & 0 \\ 0 & -[\mathbf{R}_v]_{\times}^{\top} & 0 & [\tilde{\mathbf{p}}_t]_{\times} & [\tilde{\mathbf{p}}_r]_{\times} & \mathcal{I}_{p_f}^{\Omega} & 0 \\ 0 & 0 & -\mathbf{I}_{3 \times 3} & 0 & -(\mathcal{I}_{p_f}^{\Omega})^* & 0 & \text{Div} \\ 0 & 0 & 0 & 0 & 0 & \text{Grad} & 0 \end{bmatrix}}_{\mathcal{J}} \underbrace{\begin{bmatrix} \partial_{\mathbf{r}_P} H \\ \partial_{\mathbf{R}_v} H \\ \delta_{\mathbf{u}_f} H \\ \mathbf{v}_P \\ \boldsymbol{\omega}_P \\ \mathbf{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}}_z.$$

Variables  $\tilde{\mathbf{p}}_t, \tilde{\mathbf{p}}_r$  are defined as:

$$\tilde{\mathbf{p}}_t = \mathbf{p}_t + \int_{\Omega} \rho \mathbf{v}_f \, d\Omega, \quad \tilde{\mathbf{p}}_r = \mathbf{p}_r + \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \mathbf{v}_f \, d\Omega.$$

The operator  $\mathcal{I}_{p_f}^{\Omega}$  is defined as:  $\mathcal{I}_{p_f}^{\Omega} := \int_{\Omega} \{2[\mathbf{p}_f]_{\times} + \rho[\mathbf{v}_f]_{\times}\} (\cdot) \, d\Omega.$

## Final pHDAE system

This system fits into the framework detailed in<sup>5</sup> and extends it.

$$\mathcal{E}(e) \frac{\partial e}{\partial t} = \mathcal{J}(e)z(e) + \mathcal{B}_d(e)u_d + \mathcal{B}_r(e)u_\partial, \quad \text{where } u_d := \beta$$

$$y_d = \mathcal{B}_d^*(e)z(e),$$

$$y_r = \mathcal{B}_r^*(e)z(e),$$

$$u_\partial = \mathcal{B}_\partial z(e) = \Sigma \cdot n|_{\partial\Omega} = \tau|_{\partial\Omega},$$

$$y_\partial = \mathcal{C}_\partial z(e) = v_f|_{\partial\Omega},$$

$$\text{with } y_r = (v_P + [x + u_f]^\top_\times \omega_P)|_{\partial\Omega}, \quad y_d = (v_P + [x + u_f]^\top_\times \omega_P + v_f)|_\Omega.$$

Operator  $\mathcal{E}$  is positive self-adjoint,  $\mathcal{J}$  is formally skew-symmetric. The Hamiltonian satisfies

$$\partial_e H = \mathcal{E}^* z.$$

<sup>5</sup>Volker Mehrmann and Riccardo Morandin. "Structure-preserving discretization for port-Hamiltonian descriptor systems". In: *Proceedings of the 59th IEEE Conference on Decision and Control*. 2019, pp. 6663–6668.



# Energy balance

State space  $\mathcal{X} = \mathbb{R}^3 \times \mathbb{R}^9 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathcal{L}^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}})$ .

$$\dot{H}(e) = \langle \partial_e H, \partial_t e \rangle_{\mathcal{X}},$$

# Energy balance

State space  $\mathcal{X} = \mathbb{R}^3 \times \mathbb{R}^9 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathcal{L}^2(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3})$ .

$$\begin{aligned}\dot{H}(e) &= \langle \partial_e H, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle \mathcal{E}^* z, \partial_t e \rangle_{\mathcal{X}},\end{aligned}$$

# Energy balance

State space  $\mathcal{X} = \mathbb{R}^3 \times \mathbb{R}^9 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathcal{L}^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}})$ .

$$\begin{aligned}\dot{H}(e) &= \langle \partial_e H, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle \mathcal{E}^* z, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle z, \mathcal{E} \partial_t e \rangle_{\mathcal{X}}, \quad \text{Adjoint definition,}\end{aligned}$$

# Energy balance

State space  $\mathcal{X} = \mathbb{R}^3 \times \mathbb{R}^9 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathcal{L}^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}})$ .

$$\begin{aligned}\dot{H}(e) &= \langle \partial_e H, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle \mathcal{E}^* z, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle z, \mathcal{E} \partial_t e \rangle_{\mathcal{X}}, \quad \text{Adjoint definition,} \\ &= \langle z, \mathcal{J}z + \mathcal{B}_d(e)u_d + \mathcal{B}_r(e)u_{\partial} \rangle_{\mathcal{X}},\end{aligned}$$

# Energy balance

State space  $\mathcal{X} = \mathbb{R}^3 \times \mathbb{R}^9 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathcal{L}^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}})$ .

$$\begin{aligned}\dot{H}(e) &= \langle \partial_e H, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle \mathcal{E}^* z, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle z, \mathcal{E} \partial_t e \rangle_{\mathcal{X}}, \quad \text{Adjoint definition,} \\ &= \langle z, \mathcal{J} z + \mathcal{B}_d(e) u_d + \mathcal{B}_r(e) u_{\partial} \rangle_{\mathcal{X}}, \\ &= \langle y_{\partial}, u_{\partial} \rangle_{\mathcal{L}^2(\partial\Omega, \mathbb{R}^3)} + \langle \mathcal{B}_d^* z, u_d \rangle_{\mathcal{X}} + \langle \mathcal{B}_r^* z, u_{\partial} \rangle_{\mathcal{X}}, \quad \text{I.B.P. on } \mathcal{J},\end{aligned}$$

where the integration by parts (Stokes theorem) has been used

$$\int_{\Omega} \Sigma : \text{Grad}(\mathbf{v}_f) \, d\Omega + \int_{\Omega} \text{Div}(\Sigma) \cdot \mathbf{v}_f \, d\Omega = \int_{\partial\Omega} (\Sigma \cdot \mathbf{n}) \cdot \mathbf{v}_f \, d\Gamma = \langle y_{\partial}, u_{\partial} \rangle_{\mathcal{L}^2(\partial\Omega)}.$$

# Energy balance

State space  $\mathcal{X} = \mathbb{R}^3 \times \mathbb{R}^9 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{L}^2(\Omega, \mathbb{R}^3) \times \mathcal{L}^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}})$ .

$$\begin{aligned}\dot{H}(e) &= \langle \partial_e H, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle \mathcal{E}^* z, \partial_t e \rangle_{\mathcal{X}}, \\ &= \langle z, \mathcal{J} z + \mathcal{B}_d(e) u_d + \mathcal{B}_r(e) u_{\partial} \rangle_{\mathcal{X}}, \quad \text{Adjoint definition,} \\ &= \langle y_{\partial}, u_{\partial} \rangle_{\mathcal{L}^2(\partial\Omega, \mathbb{R}^3)} + \langle \mathcal{B}_d^* z, u_d \rangle_{\mathcal{X}} + \langle \mathcal{B}_r^* z, u_{\partial} \rangle_{\mathcal{X}}, \quad \text{I.B.P. on } \mathcal{J}, \\ &= \langle y_{\partial} + y_r, u_{\partial} \rangle_{\mathcal{L}^2(\partial\Omega, \mathbb{R}^3)} + \langle y_d, u_d \rangle_{\mathcal{L}^2(\Omega, \mathbb{R}^3)},\end{aligned}$$

where the integration by parts (Stokes theorem) has been used

$$\int_{\Omega} \Sigma : \text{Grad}(v_f) \, d\Omega + \int_{\Omega} \text{Div}(\Sigma) \cdot v_f \, d\Omega = \int_{\partial\Omega} (\Sigma \cdot n) \cdot v_f \, d\Gamma = \langle y_{\partial}, u_{\partial} \rangle_{\mathcal{L}^2(\partial\Omega)}.$$

## Power balance

The power balance equals the power due to body force and surface traction

$$\begin{aligned}\dot{H}(e) &= \int_{\partial\Omega} \boldsymbol{\tau} \cdot \boldsymbol{v} \, d\Gamma + \int_{\Omega} \boldsymbol{\beta} \cdot \boldsymbol{v} \, d\Omega, \\ &= \int_{\partial\Omega} \boldsymbol{u}_d \cdot \boldsymbol{y}_d \, d\Gamma + \int_{\Omega} \boldsymbol{u}_{\partial} \cdot (\boldsymbol{y}_{\partial} + \boldsymbol{y}_r) \, d\Omega,\end{aligned}$$

where  $\boldsymbol{v} := \boldsymbol{v}_P + [\boldsymbol{\omega}_P]_{\times}(\boldsymbol{x} + \boldsymbol{u}_f) + \boldsymbol{v}_f$  is the velocity field with the body

# Some remarks

- Generic linear elastic model can be included.
- Conservative forces are easily accounted for by introducing an appropriate potential energy. The gravitational potential

$$H_{\text{pot}} = \int_{\Omega} \rho g {}^i r_z \, d\Omega = \int_{\Omega} \rho g \left[ {}^i r_{P,z} + \mathbf{R}_z(\mathbf{x} + \mathbf{u}_f) \right] d\Omega.$$

- Geometric stiffening could be considered by adding a potential energy associated to centrifugal forces or using a substructuring technique.
- If case of vanishing deformations  $\mathbf{u}_f \equiv 0$ , the Newton-Euler equations on the Euclidean group  $SE(3)$  are retrieved

$$\begin{bmatrix} {}^i \dot{\mathbf{r}}_P \\ \mathbf{R}_v \\ \dot{\mathbf{p}}_t \\ \dot{\mathbf{p}}_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & [\mathbf{R}_v]_{\times} \\ -\mathbf{R}^{\top} & 0 & 0 & [\mathbf{p}_t]_{\times} \\ 0 & -[\mathbf{R}_v]_{\times}^{\top} & [\mathbf{p}_t]_{\times} & [\mathbf{p}_r]_{\times} \end{bmatrix} \begin{bmatrix} \partial_{\mathbf{r}_P} H \\ \partial_{\mathbf{R}_v} H \\ \mathbf{v}_P \\ \boldsymbol{\omega}_P \end{bmatrix}, \quad \begin{bmatrix} \mathbf{p}_t \\ \mathbf{p}_r \end{bmatrix} = \begin{bmatrix} m\mathbf{I} & [\mathbf{s}]_{\times}^{\top} \\ [\mathbf{s}]_{\times} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega}_P \end{bmatrix}.$$



- 1 Previous work on multibody systems and the pH formalism
- 2 PH formulation of a floating body
- 3 Discretization**
- 4 Construction of multibody chain

The procedure boils down to three simple steps

- 1 The system is written in weak form;
- 2 An integration by parts is applied to highlight the appropriate boundary control;
- 3 A Galerkin method is employed to obtain a finite-dimensional system.

Test functions  $w$ , state  $e$  and effort functions  $z$  are discretized using the same bases

$$\mathbf{w}(\mathbf{x}, t) = \boldsymbol{\phi}(\mathbf{x})^\top \mathbf{w}(t), \quad \mathbf{e}(\mathbf{x}, t) = \boldsymbol{\phi}(\mathbf{x})^\top \mathbf{e}(t), \quad \mathbf{z}(\mathbf{x}, t) = \boldsymbol{\phi}(\mathbf{x})^\top \mathbf{z}(t),$$

## Finite-dimensional pHDAE system

After integration by parts of the Div operator

$$\mathbf{E}(\mathbf{e})\dot{\mathbf{e}} = \mathbf{J}(\mathbf{e})\mathbf{z}(\mathbf{e}) + \mathbf{B}_d(\mathbf{e})\mathbf{u}_d + \mathbf{B}_\partial(\mathbf{e})\mathbf{u}_\partial,$$

$$\mathbf{y}_d := \mathbf{M}_d \tilde{\mathbf{y}}_d = \mathbf{B}_d^\top \mathbf{z}(\mathbf{e}),$$

$$\mathbf{y}_\partial := \mathbf{M}_\partial \tilde{\mathbf{y}}_\partial = \mathbf{B}_\partial^\top \mathbf{z}(\mathbf{e}).$$

## Dirichlet conditions

The set  $\Gamma_D$  for the Dirichlet condition has to be non empty, otherwise the deformation field is allowed for rigid movement, leading to a singular mass matrix. Test and state shape functions must verify an homogeneous Dirichlet condition.

# Computation of the effort functions

The computation of vector  $\mathbf{z}$  is based on the discrete Hamiltonian gradient:

$$\frac{\partial H_d}{\partial \mathbf{e}} = \mathbf{E}^\top \mathbf{z}, \quad H_d = H_{d,\text{kin}} + H_{d,\text{def}} + H_{d,\text{pot}}.$$

The only term that requires additional care is  $z_u = \delta_{\mathbf{u}_f} H$ .

Flexible displacement contribution to the power balance

$$\dot{H}_u = \int_{\Omega} \frac{\partial \mathbf{u}_f}{\partial t} \cdot \mathbf{z}_u \, d\Omega = \int_{\Omega} \frac{\partial \mathbf{u}_f}{\partial t} \cdot \frac{\delta H}{\delta \mathbf{u}_f} \, d\Omega$$

Given that  $\mathbf{u}_f = \boldsymbol{\phi}_u^\top \mathbf{u}_f$ ,  $\mathbf{z}_u = \boldsymbol{\phi}_u^\top \mathbf{z}_u$ , the discrete Hamiltonian rate assumes the expressions

$$\dot{H}_{u,d}(\mathbf{u}_f) = \begin{cases} \dot{\mathbf{u}}_f^\top \mathbf{M}_u \mathbf{z}_u, \\ \dot{\mathbf{u}}_f^\top \frac{\partial H_d}{\partial \mathbf{u}_f}, \end{cases} \quad \Rightarrow \quad \mathbf{z}_u = \mathbf{M}_u^{-1} \frac{\partial H_d}{\partial \mathbf{u}_f}, \quad \text{where } \mathbf{M}_u = \int_{\Omega} \boldsymbol{\phi}_u \boldsymbol{\phi}_u^\top \, d\Omega$$

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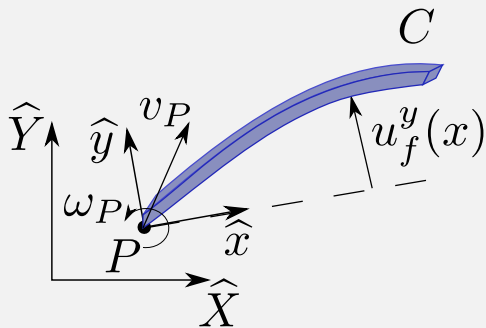


Figure: Floating beam.

## Beam discretized system

Neglecting the dependence on the deformation field in the mass matrix ( $\mathbf{M} = \text{const}$ )

$$\mathbf{E}\dot{\mathbf{e}} = \mathbf{J}(\mathbf{e})\mathbf{z}(\mathbf{e}) + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{B}^\top \mathbf{z},$$

with boundary variables

$$\mathbf{u} = [F_P^x, F_P^y, T_P^z, F_C^x, F_C^y, T_C^z]^\top,$$

$$\mathbf{y} = [v_P^x, v_P^y, \omega_P^z, v_C^x, v_C^y, \omega_C^z]^\top.$$

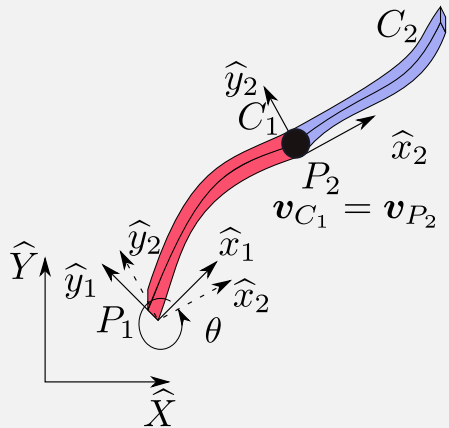


Figure: Two hinged beams.

The interconnection variables are

$$\mathbf{u}_1^{\text{int}} = [F_{C_1}^x, F_{C_1}^y]^\top := \mathbf{F}_{C_1},$$

$$\mathbf{u}_2^{\text{int}} = [F_{P_2}^x, F_{P_2}^y]^\top := \mathbf{F}_{P_2},$$

$$\mathbf{y}_1^{\text{int}} = [v_{C_1}^x, v_{C_1}^y]^\top := \mathbf{v}_{C_1},$$

$$\mathbf{y}_2^{\text{int}} = [v_{P_2}^x, v_{P_2}^y]^\top := \mathbf{v}_{P_2}.$$

## Hinged interconnected beams

The transformer interconnection

$$\mathbf{u}_1^{\text{int}} = -\mathbf{R}(\theta)\mathbf{u}_2^{\text{int}}, \quad \mathbf{y}_2^{\text{int}} = \mathbf{R}(\theta)^\top \mathbf{y}_1^{\text{int}},$$

where  $\mathbf{R}(\theta)$  is the relative rotation matrix, imposes the constraints on the velocity level and gives rise to a quasi-linear index 2 pHDAE.

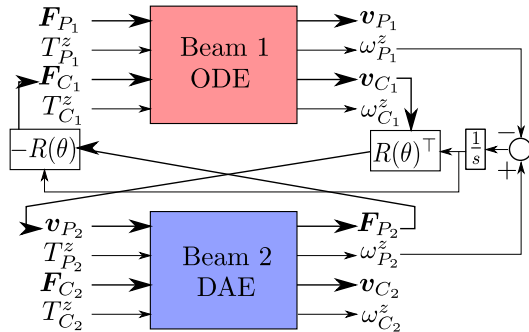
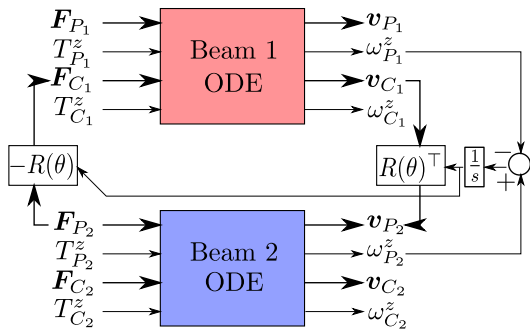
$$\begin{bmatrix} \mathbf{E}_1 & 0 & 0 \\ 0 & \mathbf{E}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1(\mathbf{e}_1) & 0 & -\mathbf{B}_1^{\text{int}}\mathbf{R} \\ 0 & \mathbf{J}_2(\mathbf{e}_2) & \mathbf{B}_2^{\text{int}} \\ \mathbf{R}^\top \mathbf{B}_1^{\text{int}\top} & -\mathbf{B}_2^{\text{int}\top} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{\partial 1}^{\text{ext}} & 0 \\ 0 & \mathbf{B}_{\partial 2}^{\text{ext}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\text{ext}} \\ \mathbf{u}_2^{\text{ext}} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{y}_1^{\text{ext}} \\ \mathbf{y}_2^{\text{ext}} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\partial 1}^{\text{ext}\top} & 0 & 0 \\ 0 & \mathbf{B}_{\partial 2}^{\text{ext}\top} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \lambda \end{bmatrix}.$$



# Equivalence of gyrator and transformer interconnection

The same result can be obtained by using a pHDAE system and a gyrator interconnection. It is sufficient to interchange the role of output and input of the second system  $\mathbf{u}_2^{\text{int}} \leftrightarrow \mathbf{y}_2^{\text{int}}$ .



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Hypothesis:

- 1 small angular velocities;
- 2 small relative configuration.

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf} & 0 \\ \mathbf{M}_{fr} & \mathbf{M}_{ff} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_r \\ \dot{\mathbf{p}}_f \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{G}_r^\top \\ 0 & \mathbf{J}_{ff} & \mathbf{G}_f^\top \\ -\mathbf{G}_r & -\mathbf{G}_f & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_r \\ \mathbf{p}_f \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_r \\ \mathbf{B}_f \\ 0 \end{bmatrix} \mathbf{u}.$$

with Hamiltonian  $H = \frac{1}{2} \mathbf{p}^\top \mathbf{M} \mathbf{p}$ . The modular construction of complex multi-body systems is then analogous to a sub-structuring technique<sup>6</sup>.

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<sup>6</sup>D. De Klerk, D. J. Rixen, and S. N. Voormeeren. "General Framework for Dynamic Substructuring: History, Review and Classification of Techniques". In: *AIAA Journal* 46.5 (2008), pp. 1169–1181. DOI: 10.2514/1.33274. URL: <https://doi.org/10.2514/1.33274>.

## Model reduction

Such system can be reduced using Linear model reduction methods directly in the DAE<sup>7</sup>.

Vector  $\mathbf{p}_f$  is projected on a meaningful subspace  $\mathbf{p}_f \approx \mathbf{V}_f^{\text{red}} \mathbf{p}_f^{\text{red}}$

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf}^{\text{red}} & 0 \\ \mathbf{M}_{fr}^{\text{red}} & \mathbf{M}_{ff}^{\text{red}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_r \\ \dot{\mathbf{p}}_f^{\text{red}} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{G}_r^{\top} \\ 0 & \mathbf{J}_{ff}^{\text{red}} & \mathbf{G}_f^{\text{red}\top} \\ -\mathbf{G}_r & -\mathbf{G}_f^{\text{red}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_r \\ \mathbf{p}_f^{\text{red}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{B}_r \\ \mathbf{B}_f^{\text{red}} \\ 0 \end{bmatrix} \mathbf{u},$$

<sup>7</sup>H. Egger et al. “On Structure-Preserving Model Reduction for Damped Wave Propagation in Transport Networks”. In: *SIAM Journal on Scientific Computing* 40.1 (2018), A331–A365. DOI: 10.1137/17M1125303.

## Index reduction

$$\begin{aligned} \mathbf{M}\dot{\mathbf{p}} &= \mathbf{J}\mathbf{p} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{B}\mathbf{u}, \\ \mathbf{0} &= \mathbf{G}\mathbf{p}, \end{aligned}$$

A null space matrix can be employed to eliminate the Lagrange multiplier and preserve the port-Hamiltonian structure.

$$\text{range}\{\mathbf{P}\} = \text{null}\{\mathbf{G}\}.$$

Then, the range of  $\mathbf{P}$  automatically satisfies the constraints. Considering the transformation  $\hat{\mathbf{p}} = \mathbf{P}\mathbf{p}$  and pre-multiplying the system by  $\mathbf{P}^\top$  an equivalent ODE is obtained

$$\widehat{\mathbf{M}} \dot{\hat{\mathbf{p}}} = \widehat{\mathbf{J}} \hat{\mathbf{p}} + \widehat{\mathbf{B}} \mathbf{u},$$

with  $\widehat{\mathbf{M}} = \mathbf{P}^\top \mathbf{M} \mathbf{P}$ ,  $\widehat{\mathbf{J}} = \mathbf{P}^\top \mathbf{J} \mathbf{P}$ ,  $\widehat{\mathbf{B}} = \mathbf{P}^\top \mathbf{B}$ .

Summarizing:

- port-Hamiltonian formulation of floating bodies;
- finite element discretization;
- interconnection of subcomponents;
- linearized case.

Some open questions

- stability and convergence of finite element;
- time discretization;
- non-linear model reduction of pHDAE;
- control strategies.

Additional information <https://arxiv.org/abs/2002.12816>

Thanks for your attention  
Questions?



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