Partitioned Finite Element Method for the Mindlin Plate as a Port-Hamiltonian system

Andrea Brugnoli

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Plan

1. PH formulation of the Mindlin plate
    - Mindlin-Reissner model for thick Plates
    - Port-Hamiltonian formulation
2. Structure preserving discretization
    - Boundary control through forces and momenta
    - Boundary control through kinematic variables
3. Discretization procedure
    - Finite-dimensional system
4. Numerical simulations

Plan

The Mindlin-Reissner model

The classical model is a system 3PDEs:

$$\begin{cases} \rho h \dfrac{\partial^2 w}{\partial t^2} & = \mathrm{div}(\boldsymbol{q}), \\ \rho \dfrac{h^3}{12} \dfrac{\partial^2 \boldsymbol{\theta}}{\partial t^2} & = \boldsymbol{q} + \mathrm{Div}(\boldsymbol{M}), \end{cases}$$

with the parameters and variables

- ρ the material density;
- h the plate thickness;
- the vertical displacement scalar field w;
- the cross section deflection vector field $\boldsymbol{\theta} = (\theta_x, \theta_y)$;
- the bending symmetric tensor field $\boldsymbol{M}$;
- shear stress vector field $\boldsymbol{q}$;

The divergence of a tensor field is a vector defined column-wise as

$$\mathrm{Div}(\boldsymbol{M}) := \left(\sum_{\alpha=1}^{2} \partial_{x_\alpha} m_{\alpha\beta} \right)_{\beta=1,\dots,2}.$$

Constitutive equations

For an homogeneous, isotropic material (Greek indexes equal 1,2)

$$\boldsymbol{M}_{\alpha\beta} = \boldsymbol{D}_{\alpha\beta\iota\lambda}\boldsymbol{K}_{\iota\lambda} \qquad \boldsymbol{q}_{\alpha} = \boldsymbol{C}_{\alpha\beta}\boldsymbol{\gamma}_{\beta}$$

The fourth and second order tensor $\boldsymbol{D}_{\alpha\beta\iota\lambda}$ (bending stiffness) and $\boldsymbol{C}_{\alpha\beta}$ (shear stiffness) are symmetric, positive definite.

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The variables

$$\boldsymbol{K} := \mathrm{Grad}(\boldsymbol{\theta}), \qquad \boldsymbol{\gamma} := \mathrm{grad}(w) - \boldsymbol{\theta}.$$

are the bending curvature and shear strain. The symmetric gradient of a vector field is defined as

$$\mathrm{Grad}(\boldsymbol{\theta}) := \frac{1}{2}\left(\nabla\boldsymbol{\theta} + \nabla^T\boldsymbol{\theta}\right).$$

Energies

The kinetic and potential energy density $\mathcal{K}$ and $\mathcal{U}$ read

$$\begin{aligned}\mathcal{K} &= \frac{1}{2}\left\{\rho h \left(\frac{\partial w}{\partial t}\right)^2 + \frac{\rho h^3}{12}\frac{\partial \boldsymbol{\theta}}{\partial t}\cdot\frac{\partial \boldsymbol{\theta}}{\partial t}\right\}, \\ \mathcal{U} &= \frac{1}{2}\left\{\boldsymbol{M} : \boldsymbol{K} + \boldsymbol{q}\cdot\boldsymbol{\gamma}\right\},\end{aligned}$$

where $\boldsymbol{M} : \boldsymbol{K} := \sum_{\alpha,\beta} m_{\alpha\beta}\,\kappa_{\alpha\beta}$ is the tensor contraction.
The Hamiltonian (total energy) is the sum of the two

$$H = \int_{\Omega} (\mathcal{K} + \mathcal{U})\ \mathrm{d}\Omega.$$

Energy, coenergy variables

The choice of the energy variables is the same as in[1]

$$
\begin{aligned}
\alpha_w &= \rho h \frac{\partial w}{\partial t}, \qquad & \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t}, \\
\boldsymbol{A}_\kappa &= \boldsymbol{K}, \qquad & \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}.
\end{aligned}
$$

[1] A. Macchelli, C. Melchiorri, and L. Bassi. "Port-based Modelling and Control of the Mindlin Plate". In: *Proceedings of the 44th IEEE Conference on Decision and Control*. 2005, pp. 5989–5994.

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$$
\boldsymbol{A}_\kappa = \boldsymbol{K}, \qquad \boldsymbol{\alpha}_\gamma = \boldsymbol{\gamma}.
$$

The coenergies are given by the Hamiltonian variational derivative

$$
e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \qquad \boldsymbol{e}_\theta := \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t},
$$

$$
\boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, \qquad \boldsymbol{e}_{\epsilon_s} := \frac{\delta H}{\delta \boldsymbol{\alpha}_{\boldsymbol{\gamma}}} = \boldsymbol{q}.
$$

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PH system

The port-Hamiltonian system is expressed as follows

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w\\ \boldsymbol{\alpha}_\theta\\ \boldsymbol{A}_\kappa\\ \boldsymbol{\alpha}_\gamma\end{pmatrix} = \underbrace{\begin{bmatrix}0 & 0 & 0 & \mathrm{div}\\ 0 & 0 & \mathrm{Div} & \boldsymbol{I}_{2\times 2}\\ 0 & \mathrm{Grad} & 0 & 0\\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0\end{bmatrix}}_{J}\begin{pmatrix}e_w\\ \boldsymbol{e}_\theta\\ \boldsymbol{E}_\kappa\\ \boldsymbol{e}_\gamma\end{pmatrix},$$

with J skew symmetric. Moreover

$$\begin{pmatrix}e_w\\ \boldsymbol{e}_\theta\\ \boldsymbol{E}_\kappa\\ \boldsymbol{e}_\gamma\end{pmatrix} = \underbrace{\begin{bmatrix}1/(\rho h) & 0 & 0 & 0\\ 0 & 12/(\rho h^3) & 0 & 0\\ 0 & 0 & \boldsymbol{D} & 0\\ 0 & 0 & 0 & \boldsymbol{C}\end{bmatrix}}_{Q}\begin{pmatrix}\alpha_w\\ \boldsymbol{\alpha}_\theta\\ \boldsymbol{A}_\kappa\\ \boldsymbol{\alpha}_\gamma\end{pmatrix},$$

with Q coercive.

Boundary variables

Taking the energy rate and applying of the Green theorem

$$\dot{H} = \int_{\partial\Omega} \left\{ w_t \, q_n + \omega_n \, m_{nn} + \omega_s \, m_{ns} \right\} \, \mathrm{d}s.$$

The dynamic boundary variable are defined as

$$\begin{aligned} \text{Shear Force} \qquad & q_n := \boldsymbol{e}_\gamma \cdot \boldsymbol{n}, \\ \text{Flexural momentum} \qquad & m_{nn} := \boldsymbol{E}_\kappa : (\boldsymbol{n} \otimes \boldsymbol{n}), \\ \text{Torsional momentum} \qquad & m_{ns} := \boldsymbol{E}_\kappa : (\boldsymbol{s} \otimes \boldsymbol{n}), \end{aligned}$$

where $\boldsymbol{u} \otimes \boldsymbol{v}$ denotes the outer product of vectors.
The corresponding power conjugated (and essential) boundary variables are

$$\begin{aligned} \text{Vertical velocity} \qquad & w_t := e_w, \\ \text{Flexural rotation} \qquad & \omega_n := \boldsymbol{e}_\theta \cdot \boldsymbol{n}, \\ \text{Torsional rotation} \qquad & \omega_s := \boldsymbol{e}_\theta \cdot \boldsymbol{s}. \end{aligned}$$

Plan

Main step to follow

The structure-preserving discretization consists of three steps:

1. write the system in weak form;
2. perform integrations by parts to get the chosen boundary control;
3. select the finite element spaces to achieve a finite-dimensional system.

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

$$J = \begin{bmatrix} 0 & 0 & 0 & \text{div} \\ 0 & 0 & \text{Div} & \boldsymbol{I}_{2\times 2} \\ 0 & \text{Grad} & 0 & 0 \\ \text{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

$$J_{\text{div}} := \begin{bmatrix} 0 & 0 & 0 & \text{div} \\ 0 & 0 & \text{Div} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

$$J_{\text{grad}} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \text{Grad} & 0 & 0 \\ \text{grad} & 0 & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

$$J_{\boldsymbol{I}} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_{2\times 2} \\ 0 & 0 & 0 & 0 \\ 0 & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

From these definitions, it holds

$$J_{\text{div}} = -J_{\text{grad}}^*,$$

where A^* is the formal adjoint of operator A.

Weak form

To simplify the notation, all test and unknown functions can be collected in one set variable

$$
\begin{aligned}
v &:= (v_w,\ \boldsymbol{v}_\theta,\ \boldsymbol{V}_\kappa,\ \boldsymbol{v}_\gamma),\\
\alpha &:= (\alpha_w,\ \boldsymbol{\alpha}_\theta,\ \boldsymbol{A}_\kappa,\ \boldsymbol{\alpha}_\gamma),\\
e &:= (e_w,\ \boldsymbol{e}_\theta,\ \boldsymbol{E}_\kappa,\ \boldsymbol{e}_\gamma),
\end{aligned}
$$

so that the previous system is rewritten compactly as

$$
\left(v, \frac{\partial \alpha}{\partial t}\right) = (v, Je).
$$

Gradient formulation

If the operator J_{div} is integrated by parts then

$$(v, Je) = j_{\mathrm{grad}}(v, e) + f_N(v). \tag{1}$$

The bilinear form

$$\begin{aligned} j_{\mathrm{grad}}(v, e) &= (J_{\mathrm{div}}^{*} v, e) + (v, J_{\mathrm{grad}}\, e) + (v, J_{\boldsymbol{I}}\, e), \\ &= (-J_{\mathrm{grad}}\, v, e) + (v, J_{\mathrm{grad}}\, e) + (v, J_{\boldsymbol{I}}\, e), \end{aligned}$$

is skew symmetric.

Gradient formulation

If the operator J_{div} is integrated by parts then

$$(v, Je) = j_{\text{grad}}(v, e) + f_N(v). \tag{1}$$

The functional

$$\begin{aligned} f_N(v) &= \int_{\partial\Omega} \{v_w\, q_n + v_{\omega_n} m_{nn} + v_{\omega_s} m_{ns}\} \, \mathrm{d}s, \\ &= \int_{\partial\Omega} v_\partial u_\partial \, \mathrm{d}s. \end{aligned}$$

express the boundary control $\boldsymbol{u}_\partial$ in terms of forces and momenta:

$$\boldsymbol{u}_\partial = \mathrm{Trace} \begin{pmatrix} q_n \\ m_{nn} \\ m_{ns} \end{pmatrix} \qquad \boldsymbol{y}_\partial = \mathrm{Trace} \begin{pmatrix} w_t \\ \omega_n \\ \omega_s \end{pmatrix}.$$

Divergence formulation

If the operator J_{grad} is integrated by parts then

$$(v, Je) = j_{\mathrm{div}}(v, e) + f_D(v), \tag{2}$$

The bilinear form

$$\begin{aligned} j_{\mathrm{div}}(v, e) &= (v, J_{\mathrm{div}}\, e) + (J_{\mathrm{grad}}^*\, v, e) + (v, J_{\boldsymbol{I}}\, e), \\ &= (v, J_{\mathrm{div}}\, e) + (-J_{\mathrm{div}}\, v, e) + (v, J_{\boldsymbol{I}}\, e), \end{aligned}$$

is skew symmetric.

Divergence formulation

If the operator J_{grad} is integrated by parts then

$$(v, Je) = j_{\text{div}}(v, e) + f_D(v), \tag{2}$$

The functional

$$\begin{aligned} f_D(v) &= \int_{\partial\Omega} \left\{ v_{q_n} w_t + v_{m_{nn}} \omega_n + v_{m_{ns}} \omega_s \right\} \, \mathrm{d}s, \\ &= \int_{\partial\Omega} v_\partial u_\partial \, \mathrm{d}s. \end{aligned}$$

expresses the boundary controls u_∂ in terms of linear and angular velocities:

$$u_\partial = \text{Trace} \begin{pmatrix} w_t \\ \omega_n \\ \omega_s \end{pmatrix} \qquad y_\partial = \text{Trace} \begin{pmatrix} q_n \\ m_{nn} \\ m_{ns} \end{pmatrix}_{\partial\Omega}.$$

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Homogeneous boundary conditions

Homogeneous boundary conditions:

- Clamped (C): $w_t = 0,\ \omega_n = 0,\ \omega_s = 0;$
- Simply supported hard (S): $w_t = 0,\ m_{nn} = 0,\ \omega_s = 0;$
- Free (F): $q_n = 0,\ m_{nn} = 0,\ m_{ns} = 0.$

The gradient formulation is adopted to discretize the system. This implies that

- variables in Blue are imposed weakly by setting $\boldsymbol{f}_N(v) = 0$
- variables in Red have to be imposed strongly (by select a functional space that incorporates those or by introducing Lagrange multipliers)

Galerkin method

Test and co-energy variables are discretized by a Galerkin Method, while energy variables are retrieve using the relation $\alpha = Q^{-1} e$. Replacing the approximated variables into the weak form

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{G}_D \\ -\boldsymbol{G}_D^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_N \\ 0 \end{pmatrix} \boldsymbol{u}_N$$

- $\boldsymbol{G}_D$ accounts for essential (Dirichlet) BCs;
- $\boldsymbol{B}_N$ account for inhomogeneous natural (Neumann) BCs;

Plan

BCs	Mode	$N = 10$	$N = 20$	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	1.5999	1.5917	1.591	1.594
	$\widehat{\omega}_{21}$	3.0615	3.0410	3.039	3.046
	$\widehat{\omega}_{12}$	3.0615	3.0410	3.039	3.046
	$\widehat{\omega}_{22}$	4.3161	4.2682	4.263	4.285
SSSS	$\widehat{\omega}_{11}$	0.9324	0.9324	0.930	0.930
	$\widehat{\omega}_{21}$	2.2227	2.2226	2.219	2.219
	$\widehat{\omega}_{12}$	2.2227	2.2226	2.219	2.219
	$\widehat{\omega}_{22}$	3.4142	3.3608	3.405	3.406
SCSC	$\widehat{\omega}_{11}$	1.3111	1.3013	1.300	1.302
	$\widehat{\omega}_{21}$	2.4155	2.3966	2.394	2.398
	$\widehat{\omega}_{12}$	2.9082	2.8871	2.885	2.888
	$\widehat{\omega}_{22}$	3.8906	3.8458	3.839	3.852
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	1.0855	1.0982	1.081	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7636	1.7461	1.744	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.6696	2.6575	2.657	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.2248	3.1997	3.197	3.216

Table: Eigenvalues for $h/L = 0.1$ using $\mathbb{P}_1$:
■ reference, ■ $\varepsilon < 2\%$.

Eigenvalues

BCs	Mode	$N = 5$	$N = 10$	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	1.5976	1.5914	1.591	1.594
	$\widehat{\omega}_{21}$	3.0584	3.0405	3.039	3.046
	$\widehat{\omega}_{12}$	3.0677	3.0405	3.039	3.046
	$\widehat{\omega}_{22}$	4.3109	4.2662	4.263	4.285
SSSS	$\widehat{\omega}_{11}$	0.9304	0.9302	0.930	0.930
	$\widehat{\omega}_{21}$	2.2223	2.2194	2.219	2.219
	$\widehat{\omega}_{12}$	2.2224	2.2194	2.219	2.219
	$\widehat{\omega}_{22}$	3.4128	3.4061	3.405	3.406
SCSC	$\widehat{\omega}_{11}$	1.3053	1.3004	1.300	1.302
	$\widehat{\omega}_{21}$	2.4040	2.3946	2.394	2.398
	$\widehat{\omega}_{12}$	2.9060	2.8858	2.885	2.888
	$\widehat{\omega}_{22}$	3.8721	3.8415	3.839	3.852
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	1.0845	1.0797	1.081	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7559	1.7425	1.744	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.6762	2.6547	2.657	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.2186	3.1954	3.197	3.216

Table: Eigenvalues for $h/L = 0.1$ using $\mathbb{P}_2$:
reference, $\varepsilon < 2\%$, $\varepsilon < 5\%$, $\varepsilon < 15\%$.

Eigenvalues

BCs	Mode	$N = 10$	$N = 20$	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	0.1967	0.1765	0.1754	0.1754
	$\widehat{\omega}_{21}$	0.4030	0.3604	0.3574	0.3576
	$\widehat{\omega}_{12}$	0.4030	0.3604	0.3574	0.3576
	$\widehat{\omega}_{22}$	0.6431	0.5358	0.5264	0.5274
SSSS	$\widehat{\omega}_{11}$	0.1706	0.1128	0.0963	0.0963
	$\widehat{\omega}_{21}$	0.3576	0.2660	0.2406	0.2406
	$\widehat{\omega}_{12}$	0.3576	0.2660	0.2406	0.2406
	$\widehat{\omega}_{22}$	0.5803	0.4442	0.3847	0.3848
SCSC	$\widehat{\omega}_{11}$	0.1864	0.1487	0.1411	0.1411
	$\widehat{\omega}_{21}$	0.3649	0.2829	0.2668	0.2668
	$\widehat{\omega}_{12}$	0.3987	0.3485	0.3377	0.3377
	$\widehat{\omega}_{22}$	0.6075	0.4933	0.4604	0.4608
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	0.1238	0.1166	0.1166	0.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.2207	0.1954	0.1949	0.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3204	0.3078	0.3080	0.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.4144	0.3751	0.3736	0.3740

Table: Eigenvalues for $h/L = 0.01$ using $\mathbb{P}_1$:

reference, $\varepsilon < 2\%$, $\varepsilon < 5\%$, $\varepsilon < 15\%$, $\varepsilon < 30\%$, $\varepsilon < 50\%$,

Eigenvalues

BCs	Mode	$N=5$	$N=10$	H-H	D-R
CCCC	$\widehat{\omega}_{11}$	0.1872	0.1762	0.1754	0.1754
	$\widehat{\omega}_{21}$	0.3725	0.3598	0.3574	0.3576
	$\widehat{\omega}_{12}$	0.4055	0.3598	0.3574	0.3576
	$\widehat{\omega}_{22}$	0.6043	0.5335	0.5264	0.5274
SSSS	$\widehat{\omega}_{11}$	0.0963	0.0963	0.0963	0.0963
	$\widehat{\omega}_{21}$	0.2422	0.2406	0.2406	0.2406
	$\widehat{\omega}_{12}$	0.2430	0.2406	0.2406	0.2406
	$\widehat{\omega}_{22}$	0.3874	0.3848	0.3847	0.3848
SCSC	$\widehat{\omega}_{11}$	0.1492	0.1418	0.1411	0.1411
	$\widehat{\omega}_{21}$	0.2827	0.2683	0.2668	0.2668
	$\widehat{\omega}_{12}$	0.3608	0.3394	0.3377	0.3377
	$\widehat{\omega}_{22}$	0.4940	0.4654	0.4604	0.4608
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	0.1197	0.1169	0.1166	0.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.2092	0.1960	0.1949	0.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3188	0.3089	0.3080	0.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.3938	0.3757	0.3736	0.3740

Table: Eigenvalues for $h/L = 0.01$ using $\mathbb{P}_2$:
reference, $\varepsilon < 2\%$, $\varepsilon < 5\%$, $\varepsilon < 15\%$.

Thank you for your attention. Questions?