Mixed finite elements for port-Hamiltonian von Kármán beams

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Abstract:

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1. INTRODUCTION

2. VON KÁRMÁN BEAMS

The classical von-Kármán beam model is presented in (Reddy, 2010, Chapter 4). Under the hypothesis of isotropic material, the extensional-bending stiffness is zero when the x-axis is taken along the geometric centroidal axis. With this assumption, the problem, defined on an open interval $\Omega = (0, L)$, takes the following form

$$\begin{aligned} \rho A \ddot{u} &= \partial_x n_{xx}, \\ \rho A \ddot{w} &= -\partial_{xx}^2 m_{xx} + \partial_x (n_{xx} \partial_x w), \end{aligned} \tag{1}$$

together with the stresses and strains expressions

$$\begin{aligned} n_{xx} &= EA\varepsilon_{xx}, \quad \varepsilon_{xx} = \partial_x u + 1/2(\partial_x w)^2, \\ m_{xx} &= EI\kappa_{xx}, \quad \kappa_{xx} = \partial_{xx}^2 w. \end{aligned} \tag{2}$$

Variable u is the horizontal displacement, w is the vertical displacement, n_{xx} is the axial stress resultant and m_{xx} is the bending stress resultant. The coefficients ρ, A, E, I are the mass density, the cross section, the Young module and the second moment of area.

The total energy of the model (Hamiltonian functional)

$$H = \frac{1}{2} \int_\Omega \left\{ \rho A (\dot{u}^2 + \dot{w}^2) + n_{xx}\varepsilon_{xx} + m_{xx}\kappa_{xx} \right\} \, \mathrm{d}\Omega, \tag{3}$$

consists of the kinetic energy and both membrane and bending deformation energies. This model proves conservative, see Bilbao et al. (2015). Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

3. THE EQUIVALENT PORT-HAMILTONIAN REALIZATION

To find a suitable port-Hamiltonian system, we first select a set of new energy variables to make the Hamiltonian functional quadratic

$$\alpha_u = \rho A \dot{u}, \quad \alpha_\varepsilon = \varepsilon_{xx}, \quad \alpha_w = \rho A \dot{w}, \quad \alpha_\kappa = \kappa_{xx}. \tag{4}$$

The energy is quadratic in these variables

$$H = \frac{1}{2} \int_\Omega \left\{ \frac{\alpha_u^2 + \alpha_w^2}{\rho A} + EA\varepsilon_{xx}^2 + EI\kappa_{xx}^2 \right\} \mathrm{d}\Omega. \tag{5}$$

By computing the variational derivative of the Hamiltonian, one obtains the so-called co-energy variables:

$$\begin{aligned} e_u &:= \delta_{\alpha_u} H = \dot{u}, \qquad e_\varepsilon := \delta_{\alpha_\varepsilon} H = n_{xx}, \\ e_w &:= \delta_{\alpha_w} H = \dot{w}, \qquad e_\kappa := \delta_{\alpha_\kappa} H = m_{xx}. \end{aligned} \tag{6}$$

Before stating the final formulation, consider the unbounded operator operator $\mathcal{C}(w)(\cdot) : L^2(\Omega) \rightarrow L^2(\Omega)$, that acts as follows

$$\mathcal{C}(w)(\cdot) = \partial_x (\cdot \, \partial_x w). \tag{7}$$

Proposition 1. The formal adjoint of the $\mathcal{C}(w)(\cdot)$ is given by

$$\mathcal{C}(w)^*(\cdot) = -\partial_x(\cdot)\partial_x(w). \tag{8}$$

Proof 1. Consider a smooth scalar fields with compact support $\psi \in C_0^\infty(\Omega)$ and $\xi \in C_0^\infty(\Omega)$. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation

$$\langle \psi, \, \mathcal{C}(w)(\xi) \rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)^*(\psi), \, \xi \rangle_{L^2(\Omega)} \,. \tag{9}$$

The proof follows from the computation

$$\begin{aligned} \langle \psi, \, \mathcal{C}(w)(\xi) \rangle_{L^2(\Omega)} &= \langle \psi, \, \partial_x (\xi \, \partial_x w) \rangle_{L^2(\Omega)} \,, \\ &= \langle -\partial_x \psi, \, \xi \partial_x w \rangle_{L^2(\Omega, \mathbb{R}^2)} \,, \\ &= \langle -\partial_x \psi \, \partial_x w, \, \xi \rangle_{L^2(\Omega)} \,. \end{aligned} \tag{10}$$

This means that

$$\mathcal{C}(w)^*(\cdot) = -\partial_x(\cdot)\partial_x w, \tag{11}$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_u \\ \alpha_\varepsilon \\ \alpha_w \\ \alpha_\kappa \end{pmatrix} = \begin{bmatrix} 0 & \partial_x & 0 & 0 \\ \partial_x & 0 & \partial_x w \, \partial_x & 0 \\ 0 & \partial_x (\cdot \, \partial_x w) & 0 & -\partial_{xx}^2 \\ 0 & 0 & \partial_{xx}^2 & 0 \end{bmatrix} \begin{pmatrix} e_u \\ e_\varepsilon \\ e_w \\ e_\kappa \end{pmatrix}, \tag{12}$$

The second line of system (12) represents the time derivative of the membrane strain tensor. To close the system, variable w has to be accessible. For this reason, its dynamics has to be included. The augmented system reads

$$
\frac{\partial}{\partial t}\begin{pmatrix}\alpha_u\\ \alpha_\varepsilon\\ \alpha_w\\ \alpha_\kappa\\ w\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \partial_x & 0 & 0 & 0\\ \partial_x & 0 & \partial_x w\,\partial_x & 0 & 0\\ 0 & \partial_x(\cdot\,\partial_x w) & 0 & -\partial^2_{xx} & -1\\ 0 & 0 & \partial^2_{xx} & 0 & 0\\ 0 & 0 & 1 & & 0\end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\\ \delta_w H\end{pmatrix}. \tag{13}
$$

The operator $\mathcal{J}$ is formally skew-adjoint. If only the kinetic and deformation energies are considered, it holds $\delta_w H = 0$. In general this terms allows accommodating other potentials, for example the gravitational one. Suitable boundary variables are then obtained considering the power balance

$$
\dot{H} = \langle e_u,\, e_\varepsilon\rangle_{\partial\Omega} + \langle e_w,\, e_\varepsilon\partial_x w - \partial_x e_\kappa\rangle_{\partial\Omega} + \langle \partial_x e_w,\, e_\kappa\rangle_{\partial\Omega}, \tag{14}
$$

The boundary conditions are consistent with the ones assumed in Puel and Tucsnak (1996) for deriving a global existence result for this model.

4. MIXED FINITE ELEMENT DISCRETIZATION

To perform the numerical discretization, the constitutive relations are first incorporated in the dynamics. The link between the energy variables (4) and the conergy variables (6) is given by the linear transformation

$$
\begin{pmatrix}\alpha_u\\ \alpha_\varepsilon\\ \alpha_w\\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix}\rho A & 0 & 0 & 0\\ 0 & C_a & 0 & 0\\ 0 & 0 & \rho A & 0\\ 0 & 0 & 0 & C_b\end{bmatrix}\begin{pmatrix} e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\end{pmatrix}, \tag{15}
$$

where $C_a = (EA)^{-1}$ and $C_b = (EI)^{-1}$ are the axial and bending compliance respectively. A pure coenergy formulation can then be employed once (15) is plugged into (13)

$$
\begin{pmatrix}\rho A\dot{e}_u\\ C_a\dot{e}_\varepsilon\\ \rho A\dot{e}_w\\ C_b\dot{e}_\kappa\\ \dot{w}\end{pmatrix} = \begin{bmatrix} 0 & \partial_x & 0 & 0 & 0\\ \partial_x & 0 & \partial_x w\,\partial_x & 0 & 0\\ 0 & \partial_x(\cdot\,\partial_x w) & 0 & -\partial^2_{xx} & -1\\ 0 & 0 & \partial^2_{xx} & 0 & 0\\ 0 & 0 & 1 & & 0\end{bmatrix} \begin{pmatrix} e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\\ \delta_w H\end{pmatrix}. \tag{16}
$$

To derive the discrete system, first (16) is put into weak form. To this aim the test functions $(\psi_u,\, \psi_\varepsilon,\, \psi_w,\, \psi_\kappa,\, \psi)$ are introduced. For sake of simplicity, no dependency between the displacements and the energy is considered, i.e. $\delta_w H = 0$:

$$
\begin{aligned}
\langle \psi_u,\, \rho A\,\dot{e}_u\rangle_\Omega &= \langle \psi_u,\, \partial_x e_\varepsilon\rangle_\Omega\,.\\
\langle \psi_\varepsilon,\, C_a\,\dot{e}_\varepsilon\rangle_\Omega &= \langle \psi_\varepsilon,\, \partial_x e_u\rangle_\Omega + \langle \psi_\varepsilon,\, \partial_x w\,\partial_x e_w\rangle_\Omega\,,\\
\langle \psi_w,\, \rho A\dot{e}_w\rangle_\Omega &= \langle \psi_w,\, \partial_x(e_\varepsilon\partial_x w)\rangle_\Omega - \left\langle \psi_w,\, \partial^2_{xx} e_\kappa\right\rangle_\Omega\,,\\
\langle \psi_\kappa,\, C_b\,\dot{e}_\kappa\rangle_\Omega &= \left\langle \psi_\kappa,\, \partial^2_{xx} e_w\right\rangle_\Omega\,,\\
\langle \psi,\, \dot{w}\rangle_\Omega &= \langle \psi,\, e_w\rangle_\Omega\,.
\end{aligned} \tag{17}
$$

Then the integration by parts is performed on the first line, the third and fourth line. This choice is such to retain the skew-symmetric structure at the discrete level and to lower the regularity requirement for the finite elements (Brugnoli, 2020, Chap. 8). The weak formulation then looks for $(e_u, e_w, e_\kappa, w) \in H^1(\Omega)$, $e_\varepsilon \in L^2(\Omega)$ such that the following system

$$
\begin{aligned}
\langle \psi_u,\, \rho A\,\dot{e}_u\rangle_\Omega &= -\langle \partial_x\psi_u,\, e_\varepsilon\rangle_\Omega + \langle \psi_u,\, e_\varepsilon\rangle_{\partial\Omega}\,.\\
\langle \psi_\varepsilon,\, C_a\,\dot{e}_\varepsilon\rangle_\Omega &= \langle \psi_\varepsilon,\, \partial_x e_u\rangle_\Omega + \langle \psi_\varepsilon,\, \partial_x w\,\partial_x e_w\rangle_\Omega\,,\\
\langle \psi_w,\, \rho A\dot{e}_w\rangle_\Omega &= -\langle \partial_x\psi_w\partial_x w,\, e_\varepsilon\rangle_\Omega + \langle \partial_x\psi_w,\, \partial_x e_\kappa\rangle_\Omega\\
&\quad + \langle \psi_w,\, e_\varepsilon\partial_x w - \partial_x e_\kappa\rangle_{\partial\Omega}\,,\\
\langle \psi_\kappa,\, C_b\,\dot{e}_\kappa\rangle_\Omega &= -\langle \partial_x\psi_\kappa,\, \partial_x e_w\rangle_\Omega + \langle \psi_\kappa,\, \partial_x e_w\rangle_{\partial\Omega}\,,\\
\langle \psi,\, \dot{w}\rangle_\Omega &= \langle \psi,\, e_w\rangle_\Omega\,.
\end{aligned} \tag{18}
$$

holds $\forall(\psi_u, \psi_w, \psi_\kappa, \psi) \in H^1(\Omega)$, $\forall\psi_\varepsilon \in L^2(\Omega)$ In this formulation, the boundary axial forces $e_\varepsilon|_0^L$, vertical forces $e_\varepsilon\partial_x w - \partial_x e_\kappa|_0^L$ and rotations $\partial_x e_w|_0^L$ are enforced weakly. To obtain the associated finite-dimensional system, the following Galerkin approximation is considered

$$
\begin{aligned}
e_u &= \sum_{i=1}^{n_u}\xi_u^i(x)e_u^i(t), & \psi_u &= \sum_{i=1}^{n_u}\xi_u^i(x)\psi_u^i,\\
e_\varepsilon &= \sum_{i=1}^{n_\varepsilon}\xi_\varepsilon^i(x)e_\varepsilon^i(t), & \psi_\varepsilon &= \sum_{i=1}^{n_\varepsilon}\xi_\varepsilon^i(x)\psi_\varepsilon^i,\\
e_w &= \sum_{i=1}^{n_w}\xi_w^i(x)e_w^i(t), & \psi_w &= \sum_{i=1}^{n_w}\xi_w^i(x)\psi_w^i,\\
e_\kappa &= \sum_{i=1}^{n_\kappa}\xi_\kappa^i(x)e_\kappa^i(t), & \psi_\kappa &= \sum_{i=1}^{n_\kappa}\xi_\kappa^i(x)\psi_\kappa^i,\\
w &= \sum_{i=1}^{n_w}\xi_w^i(x)w^i(t), & \psi &= \sum_{i=1}^{n_w}\xi_w^i(x)\psi^i,
\end{aligned} \tag{19}
$$

Notice that w, e_w have been discretized using the same test functions. Plugging (19) into (18), the following finite dimensional system is obtained

$$
\begin{pmatrix}\mathbf{M}_u\dot{\mathbf{e}}_u\\ \mathbf{M}_\varepsilon\dot{\mathbf{e}}_\varepsilon\\ \mathbf{M}_w\dot{\mathbf{e}}_w\\ \mathbf{M}_\kappa\dot{\mathbf{e}}_\kappa\\ \dot{\mathbf{w}}\end{pmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{D}_{\varepsilon u}^\top & \mathbf{0} & \mathbf{0}\\ \mathbf{D}_{\varepsilon u} & \mathbf{0} & \mathbf{D}_{\varepsilon w}(\mathbf{w}) & \mathbf{0}\\ \mathbf{0} & -\mathbf{D}_{\varepsilon w}^\top(\mathbf{w}) & \mathbf{0} & \mathbf{D}_{w\kappa}\\ \mathbf{0} & \mathbf{0} & -\mathbf{D}_{w\kappa}^\top & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\mathbf{e}_u\\ \mathbf{e}_\varepsilon\\ \mathbf{e}_w\\ \mathbf{e}_\kappa\end{pmatrix}, \tag{20}
$$

where the boundary terms have been omitted for simplicity. The mass matrices are defined as follows

$$
\begin{aligned}
M_u^{ij} &= \left\langle \xi_u^i,\, \rho A\xi_u^j\right\rangle_\Omega\,, & M_w^{ij} &= \left\langle \xi_w^i,\, \rho A\xi_w^j\right\rangle_\Omega\,,\\
M_\varepsilon^{ij} &= \left\langle \xi_\varepsilon^i,\, C_a\xi_\varepsilon^j\right\rangle_\Omega\,, & M_\kappa^{ij} &= \left\langle \xi_\kappa^i,\, C_b\xi_\kappa^j\right\rangle_\Omega\,.
\end{aligned} \tag{21}
$$

The interconnection matrices are given by

$$
\begin{aligned}
D_{\varepsilon u}^{ij} &= \left\langle \xi_\varepsilon^i,\, \partial_x\xi_u^j\right\rangle_\Omega\,,\\
D_{\varepsilon w}^{ij}(\mathbf{w}) &= \left\langle \xi_\varepsilon^i,\, \sum_{k=1}^{n_w}\partial_x\xi_w^k(x)w^k(t)\partial_x\xi_w^j\right\rangle_\Omega\,,\\
D_{w\kappa}^{ij} &= \left\langle \partial_x\xi_w^i,\, \partial_x\xi_\kappa^j\right\rangle_\Omega\,.
\end{aligned} \tag{22}
$$

5. NUMERICAL TESTS

Consider the following analytical solution for the axial and vertical displacement

$$
\begin{aligned}
u &= x(1 - x/L)\sin(t),\\
w &= \sin(\pi x/L)\sin(t),
\end{aligned} \tag{23}
$$

toghether with the boundary conditions

$$u|_0^L = 0, \quad w|_0^L = 0, \quad m_{xx}|_0^L = 0. \tag{24}$$

Then the associated axial and bending stress resultants are given by

$$\begin{aligned} n_{xx} &= EA\sin(t)\left[1 - 2x/L + \pi^2/(2L^2)\sin(t)\cos^2(\pi x/L)\right], \\ m_{xx} &= -EI\pi^2/L^2\sin(\pi x/L)\sin(t). \end{aligned} \tag{25}$$

For Eq. (23) to be the solution of (1) appropriate forcing term have to be introduced. These are given by

$$\begin{aligned} f_u &= \rho A\partial_{tt}^2 u - \partial_x n_{xx}, \\ f_w &= \rho A\partial_{tt}^2 w + \partial_{xx}^2 m_{xx} - \partial_x(n_{xx}\partial_x w), \end{aligned} \tag{26}$$

6. CONCLUSION

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