A port-Hamiltonian formulation for the full von-Kármán plate model

Andrea Brugnoli

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UNIVERSITY OF TWENTE.

Outline

Why port-Hamiltonian systems?

Von-Kármán theory of thin beams in pH form

Numerical discretization

Numerical convergence study

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A unified language for multiphysics in engineering

The port-Hamiltonian (pH) paradigm provides a language to understand multiphysics:

- **Physics** is at the core: port-Hamiltonian systems are **passive** with respect to the **energy storage function**.
- The **topological** and **metrical** structure of the equation is clearly separated (mimetic discretization).
- PH systems are **closed under interconnection**.

Finite dimensional pH systems

A theory still under developement

There is **not a unique definition** of pH systems, even in finite dimension.

Definition (Finite dimensional pH system)

The following time-invariant dynamical system is a pH system

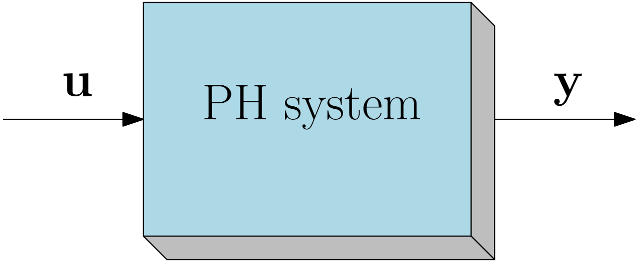
$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{B}^\top \mathbf{x}.$$

$\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$ the input and output and

- $\mathbf{J}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})^\top \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ the interconnection and control operator.
- $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{M} > 0$, the Hamiltonian.

Finite dimensional pH systems

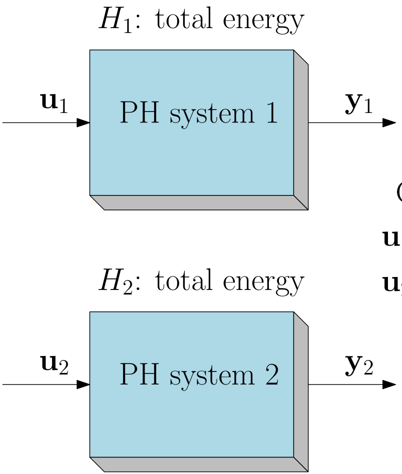
H: total energy



Lossless: $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

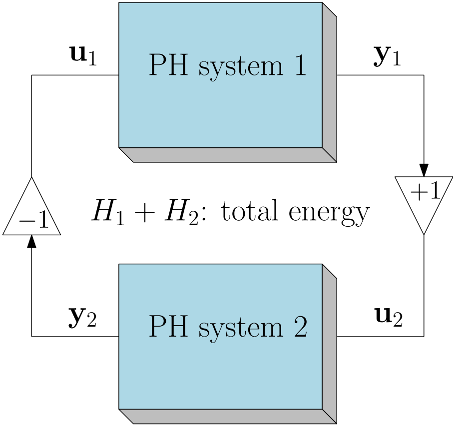
Interconnection of pH systems



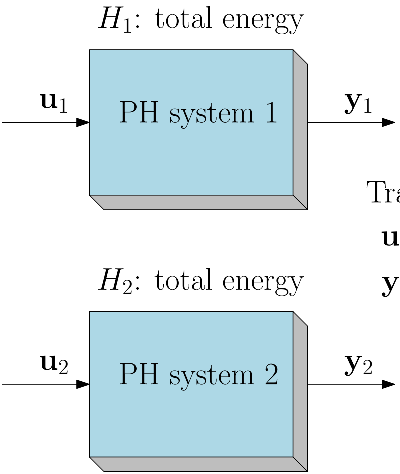
Gyrator

$\mathbf{u}_1 = -\mathbf{y}_2$

$\mathbf{u}_2 = +\mathbf{y}_1$



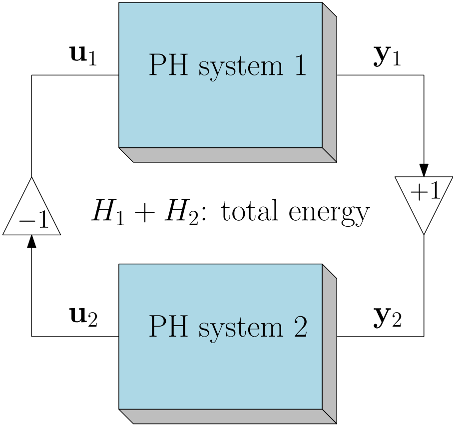
Interconnection of pH systems



Transformer

$\mathbf{u}_1 = -\mathbf{u}_2$

$\mathbf{y}_2 = +\mathbf{y}_1$



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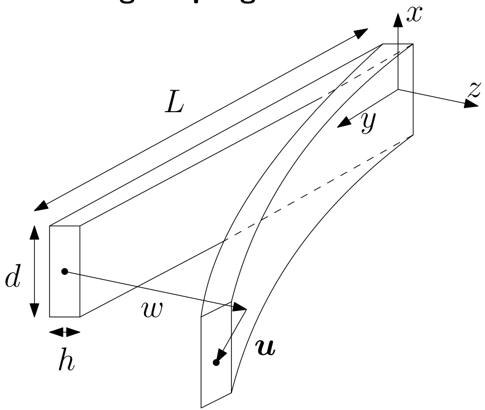
Linear vs Von-Kármán plate theory

Geometrical non-linearities allow describing bifurcations (i.e. buckling).

The von-Kármán assumption

Second-order approximation of geometrically exact beam theory **capturing the axial bending coupling.**



Basic geometric assumption

- Out of plane deflection comparable to the thickness: $w/h = \mathcal{O}(1)$.
- The squares of the in-plane stretching terms are negligible compared to the square of the rotations.

Linear isotropic plates

The axial and bending behavior are uncoupled if $w/h \ll 1$:

Axial displacement (planar elastodynamics)

$$\rho h \partial_{tt} \boldsymbol{u} = \operatorname{Div} \boldsymbol{N}, \qquad \boldsymbol{N} = D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m), \qquad \boldsymbol{\varepsilon}_m = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^\top \boldsymbol{u}) = \operatorname{Grad} \boldsymbol{u}.$$

Vertical displacement (Kirchhoff plate)

$$\rho h \partial_{tt} w = -\operatorname{div} \operatorname{Div} \boldsymbol{M}, \qquad \boldsymbol{M} = D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}), \qquad \boldsymbol{\kappa} = \operatorname{Hess} w = \operatorname{Grad} \operatorname{grad} w.$$

The linear mapping $\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\mathbf{1} + (1-\nu)\boldsymbol{A}$ is positive and preserves symmetry.

Von-Kármán plates

Decomposition strain field

$$\boldsymbol{\varepsilon} = \underbrace{\boxed{\operatorname{Grad} \boldsymbol{u}}}_{\text{Linear axial def.}} + \underbrace{\boxed{1/2 \operatorname{grad} w \otimes \operatorname{grad} w}}_{\text{Non-linear axial def.}} - z \underbrace{\boxed{\operatorname{Hess} w}}_{\text{Linear bending def.}} = \boldsymbol{\varepsilon}_m - z\boldsymbol{\kappa}.$$

Von-Kármán plate Dynamics

$$\begin{aligned}
\rho A \partial_{tt} u &= \operatorname{Div} \boxed{\boldsymbol{N}}, \\
\rho A \partial_{tt} w &= -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}(\boxed{\boldsymbol{N}} \operatorname{grad} w),
\end{aligned}$$

Total energy $H = \frac{1}{2} \int_\Omega \{ D_m \Phi(\varepsilon_m) : \boldsymbol{N} + D_b \Phi(\kappa) : \boldsymbol{M} \} \, \mathrm{d}\Omega$

Port-Hamiltonian Von-Kármán plates

Energy and coenergy variables

$$\boldsymbol{\alpha}_u = \rho h \partial_t \boldsymbol{u}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}_m,$$
$$\alpha_w = \rho h \partial_t w, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}.$$

Linear constitutive equations $\boldsymbol{e} := \delta_\alpha H = \mathcal{Q}\boldsymbol{\alpha}$ with

$$\mathcal{Q} = \mathrm{Diag}\left[(\rho h)^{-1},\ D_m \boldsymbol{\Phi},\ (\rho h)^{-1},\ D_b \boldsymbol{\Phi}\right]^{-1}.$$

The port-Hamiltonian realization

To close the system, variable w has to be accessible.

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u \\ \boldsymbol{A}_\varepsilon \\ w \\ \alpha_w \\ \boldsymbol{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div}\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{Grad}\operatorname{grad} & \mathbf{0}\end{bmatrix}}_{\mathcal{J}} \begin{pmatrix}\delta_{\boldsymbol{\alpha}_u} H \\ \delta_{\boldsymbol{A}_\varepsilon} H \\ \delta_w H \\ \delta_{\alpha_w} H \\ \delta_{\boldsymbol{A}_\kappa} H\end{pmatrix},$$

where

$$\mathcal{C}(w)(\boldsymbol{T}) = \operatorname{div}(\boldsymbol{T}\operatorname{grad} w),$$

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\operatorname{grad}(\cdot)\otimes\operatorname{grad}(w) + \operatorname{grad}(w)\otimes\operatorname{grad}(\cdot)\right].$$

Energy rate and boundary conditions

Proposition

The energy rate reads

$$\dot{H} = \langle \gamma_0 \boldsymbol{e}_u \,|\, \gamma_\perp \boldsymbol{E}_\varepsilon \rangle_{\partial\Omega} + \langle \gamma_0 \boldsymbol{e}_w \,|\, \gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0(\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w) \rangle_{\partial\Omega} + \langle \gamma_1 \boldsymbol{e}_w \,|\, \gamma_{\perp\perp} \boldsymbol{E}_\kappa \rangle_{\partial\Omega},$$

- $\gamma_0 \boldsymbol{e}_u = \boldsymbol{e}_u|_{\partial\Omega}$ *is the Dirichlet trace;*
- $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega}$ *is the normal trace;*
- $\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa = -\boldsymbol{n} \cdot \operatorname{Div} \boldsymbol{E}_\kappa - \partial_{\boldsymbol{s}}(\boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{s})|_{\partial\Omega}$ *is the effective shear force;*
- $\gamma_1 \boldsymbol{e}_w = \partial_{\boldsymbol{n}} \boldsymbol{e}_w|_{\partial\Omega}$ *is the normal derivative trace;*
- $\gamma_{\perp\perp} \boldsymbol{E}_\kappa = \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n}$ *is the normal to normal trace.*

Boundary conditions classification

BCs	Traction	Bending	
Dirichlet BCs.	$e_u\|_0^L$	$e_w\|_0^L$	$\partial_x e_w\|_0^L$
Neumann BCs.	$e_\varepsilon\|_0^L$	$e_\varepsilon \partial_x w - \partial_x e_\kappa\|_0^L$	$e_\kappa\|_0^L$

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Pure coenergy formulation

Coenergy formulation for linear constitutive equations

If the $\mathcal{Q}$ operator is inverted:

$$\begin{pmatrix} \rho A \dot{e}_u \\ C_a \dot{e}_\varepsilon \\ \rho A \dot{e}_w \\ C_b \dot{e}_\kappa \\ \dot{w} \end{pmatrix} = \begin{bmatrix} 0 & \partial_x & 0 & 0 & 0 \\ \partial_x & 0 & \partial_x w \partial_x & 0 & 0 \\ 0 & \partial_x(\cdot\, \partial_x w) & 0 & -\partial_{xx}^2 & -1 \\ 0 & 0 & \partial_{xx}^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_u \\ e_\varepsilon \\ e_w \\ e_\kappa \\ \delta_w H \end{pmatrix}.$$

In the sequel, the quantity $\delta_w H$ is removed as no displacement dependent potential (e.g. gravity) is considered

Weak formulation

Introducing test functions and integrating by parts the first, third and fourth, we get the weak formulation.

Weak formulation

Find $(e_u, e_w, e_\kappa, w) \in H^1(\Omega)$, $e_\varepsilon \in L^2(\Omega)$ such that

$$
\begin{aligned}
(\psi_u,\, \rho A\, \dot{e}_u)_\Omega &= -(\partial_x \psi_u,\, e_\varepsilon)_\Omega + (\psi_u,\, e_\varepsilon)_{\partial\Omega}. \\
(\psi_\varepsilon,\, C_a\, \dot{e}_\varepsilon)_\Omega &= (\psi_\varepsilon,\, \partial_x e_u)_\Omega + (\psi_\varepsilon,\, \partial_x w \partial_x e_w)_\Omega, \\
(\psi_w,\, \rho A \dot{e}_w)_\Omega &= -(\partial_x \psi_w \partial_x w,\, e_\varepsilon)_\Omega + (\partial_x \psi_w,\, \partial_x e_\kappa)_\Omega \\
&\quad + (\psi_w,\, e_\varepsilon \partial_x w - \partial_x e_\kappa)_{\partial\Omega}, \\
(\psi_\kappa,\, C_b\, \dot{e}_\kappa)_\Omega &= -(\partial_x \psi_\kappa,\, \partial_x e_w)_\Omega + (\psi_\kappa,\, \partial_x e_w)_{\partial\Omega}, \\
(\psi,\, \dot{w})_\Omega &= (\psi,\, e_w)_\Omega.
\end{aligned}
$$

holds $\forall (\psi_u, \psi_w, \psi_\kappa, \psi) \in H^1(\Omega)$, $\forall \psi_\varepsilon \in L^2(\Omega)$.

Weak formulation

Introducing test functions and integrating by parts the first, third and fourth, we get the weak formulation.

Weak formulation

Find $\boldsymbol{e} = (e_u, e_\varepsilon, e_w, e_\kappa) \in H^1 \times L^2 \times H^1 \times H^1$ such that

$$
\begin{aligned}
m(\psi, \partial_t \boldsymbol{e}) &= j_w(\psi, \boldsymbol{e}) + b(\psi)\mathbf{u}, \\
\partial_t w &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{e}, \\
\mathbf{y} &= b^\top(\boldsymbol{e}),
\end{aligned}
$$

$\forall \psi \in H^1 \times L^2 \times H^1 \times H^1 := X$

- m is a symmetric, coercive, bilinear form;
- j_w is a skew-symmetric bilinear form modulated by w;
- $b : X \to \mathbb{R}^6$ vector-valued functional.

Mixed finite element construction[1]

Crucial concept: Hilbert complex $H^1 \xrightarrow{\partial_x} L^2$.

Key requirements for mixed Galerkin approximation

- The subspaces $H_h^1 \subset H^1$, $L_h^2 \subset L^2$ form a subcomplex $H_h^1 \xrightarrow{\partial_x} L_h^2$ (i.e. $\partial_x H_h^1 \subset L_h^2$).
- they admit bounded linear projections $\pi_h^{H^1} : H^1 \to H_h^1$ and $\pi_h^{L^2} : L^2 \to L_h^2$ which commute with ∂_x:
$$\partial_x \pi_h^{H^1} = \pi_h^{L^2} \partial_x.$$

Satisfied for $CG_k \xrightarrow{\partial_x} DG_{k-1}$

$$\mathrm{CG}_k = \{u \in H^1(\Omega) | \, \forall \text{edge in the mesh}, \; u|_{\text{edge}} \in P_k\},$$
$$\mathrm{DG}_{k-1} = \{u \in L^2(\Omega) | \, \forall \text{edge in the mesh}, \; u|_{\text{edge}} \in P_{k-1}\},$$

where P_k space of polynomials of degree k.

[1] **arnold2006acta**.

Finite element choice and final system

For the proposed weak formulation, the FE spaces become

$$e_u^h \in \mathrm{CG}_{2k-1}, \qquad e_\varepsilon^h \in \mathrm{DG}_{2k-2}, \qquad (e_w^h,\ e_\kappa^h,\ w^h) \in CG_k, \quad k \geq 1.$$

Implications:

- Subcomplex property for the linear part: $\partial_x \mathrm{CG}_{2k-1} \subset \mathrm{DG}_{2k-2}$.
- The non linear part respects

$$\partial_x \mathrm{CG}_k \cdot \partial_x \mathrm{CG}_k \subset \mathrm{DG}_{2k-2}.$$

Finite dimensional system (Galerkin projection)

$$\begin{aligned} \mathbf{M}\dot{\mathbf{e}} &= \mathbf{J}(\mathbf{w})\mathbf{e} + \mathbf{B}\mathbf{u}, \\ \dot{\mathbf{w}} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{e}, \\ \mathbf{y} &= \mathbf{B}^\top \mathbf{e}. \end{aligned}$$

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Manufactured solution

The following manufactured solution is considered

$$u^{ex} = x^3[1-(x/L)^3]\sin(2\pi t), \qquad w^{ex} \quad = \sin(\pi x/L)\sin(2\pi t),$$

together with the boundary conditions

$$u|_0^L = 0, \quad w|_0^L = 0, \quad m_{xx}|_0^L = 0.$$

A Crank-Nicholson scheme is used for time integration.

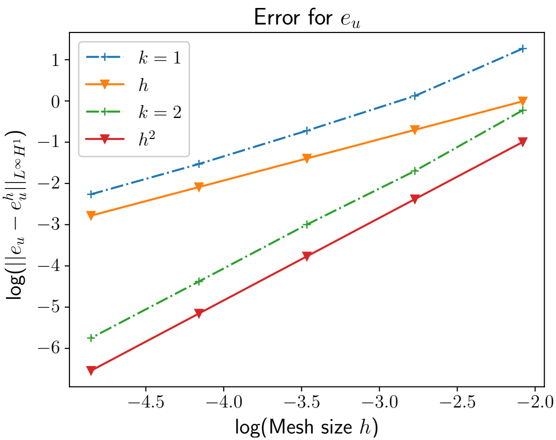
Convergence measure

The discrete time-space norm $L^\infty_{\Delta t}(\mathcal{X})(\mathcal{X} = H^1 \text{or } L^2)$ is used to measure convergence

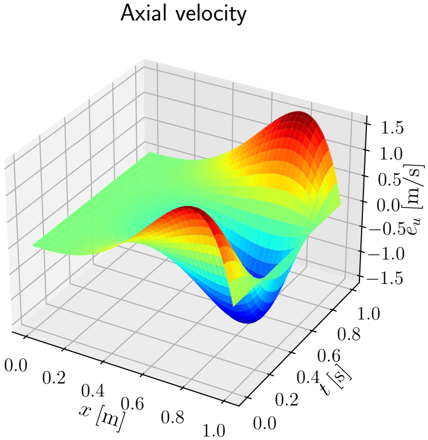
$$||\cdot||_{L^\infty(\mathcal{X})} \approx ||\cdot||_{L^\infty_{\Delta t}(\mathcal{X})} = \max_{t\in t_i} ||\cdot||_{\mathcal{X}},$$

where t_i are the discrete simulation instants.

Results

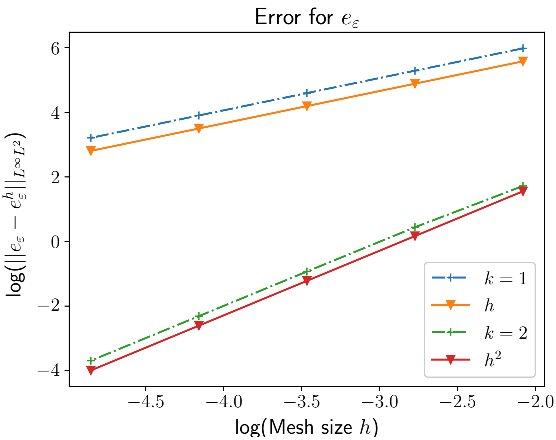


$L^\infty_{\Delta t}(H^1)$ error for e_u.

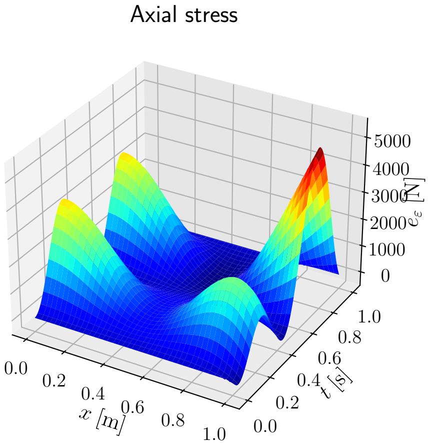


e_u^h $(h = 2^{-5}, k = 2)$.

Results

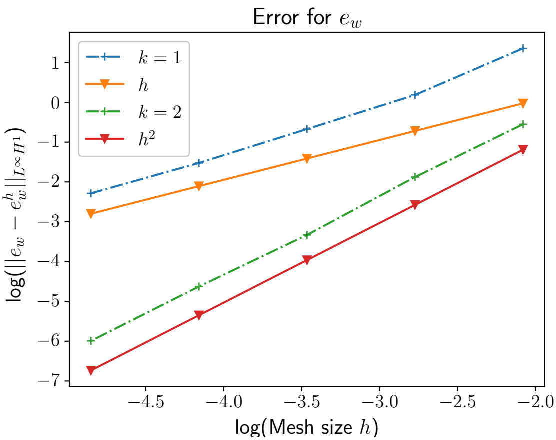


$L^\infty_{\Delta t}(L^2)$ error for e_ε.

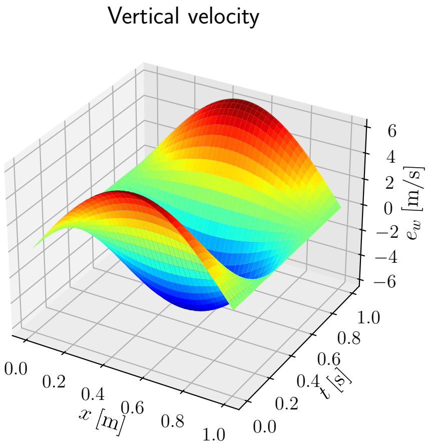


e_ε^h for $h = 2^{-5}$, $k = 2$.

Results

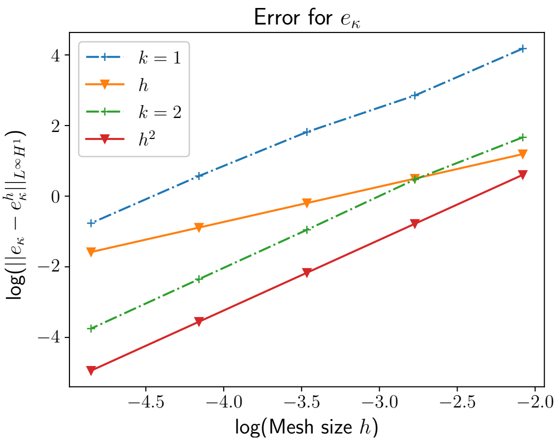


$L^\infty_{\Delta t}(H^1)$ error for e_w.

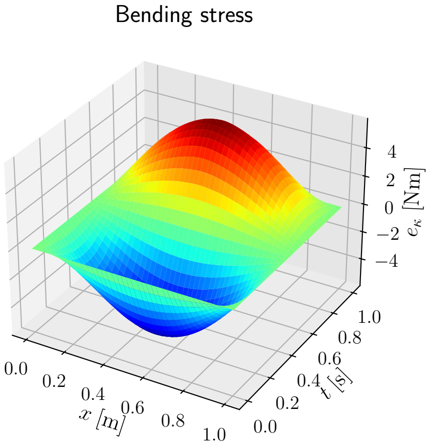


e_w^h for $h = 2^{-5}, k = 2$.

Results



$L^\infty_{\Delta t}(H^1)$ error for e_κ.



e_κ^h for $h = 2^{-5}$, $k = 2$.

Conclusion and Outlook

- First step into pH non linear mechanics. The geometrical non linearities belong to the interconnection operator.
- Natural extension for the 2D case (fancier FE).
- Can be used to study more complex phenomena. The discretization method guarantees energy conservation.

References I

Port-Hamiltonian von-Kármán plates

$$
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$$

where

$$
\mathcal{C}(w)(\boldsymbol{T}) = \operatorname{div}(\boldsymbol{T}\,\mathrm{grad}\, w),
$$

$$
\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\mathrm{grad}(\cdot)\otimes\mathrm{grad}(w) + \mathrm{grad}(w)\otimes\mathrm{grad}(\cdot)\right].
$$