International Journal of Engineering Science 47 (2009) 131–140

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

On new symplectic elasticity approach for exact free vibration solutions of rectangular Kirchhoff plates

C.W. Lim [a,*], C.F. Lü [a,b], Y. Xiang [c], W. Yao [d]

[a] Department of Building and Construction, City University of Hong Kong, Kowloon, Hong Kong, PR China
[b] Department of Civil Engineering, Zhejiang University, Hangzhou 310058, PR China
[c] School of Engineering, University of Western Sydney, Penrith South DC, New South Wales 1797, Australia
[d] State Key Laboratory of Structure Analysis of Industrial Equipment and Department of Engineering Mechanics, Dalian University of Technology, Dalian 116024, PR China

ARTICLE INFO

Article history:
Received 28 November 2007
Received in revised form 12 August 2008
Accepted 13 August 2008
Available online 27 September 2008

Communicated by K.R. Rajagopal

Keywords:
Duality
Free vibration
Kirchhoff plates
Symplectic elasticity
Hamiltonian system
Legendre's transformation
Variational principle

ABSTRACT

In the classical approach, it has been common to treat free vibration of rectangular Kirchhoff or thin plates in the Euclidian space using the Lagrange system such as the Timoshenko's method or Lévy's method and such methods are the semi-inverse methods. Because of various shortcomings of the classical approach leading to unavailability of analytical solutions in certain basic plate vibration problems, it is now proposed here a new symplectic elasticity approach based on the conservative energy principle and constructed within a new symplectic space. Employing the Hamiltonian variational principle with Legendre's transformation, exact analytical solutions within the framework of the classical Kirchhoff plate theory are established here by eigenvalue analysis and expansion of eigenfunctions in both perpendicular in-plane directions. Unlike the classical semi-inverse methods where a trial shape function required to satisfy the geometric boundary conditions is pre-determined at the outset, this symplectic approach proceeds without any shape functions and it is rigorously rational to facilitate analytical solutions which are not completely covered by the semi-inverse counterparts. Exact frequency equations for Lévy-type thin plates are presented as a special case. Numerical results are calculated and excellent agreement with the classical solutions is presented. As derivation of the formulation is independent on the assumption of displacement field, the present method is applicable not only for other types of boundary conditions, but also for thick plates based on various higher-order plate theories, as well as buckling, wave propagation, and forced vibration, etc.

1. Introduction

Thin rectangular plates are the most elementary and essential structural components in modern engineering, including civil, mechanical, electronics, aviation, marine industry, aeronautics, to name a few. Due to its practical importance, free flexural vibration of these components has been the research focus in solid mechanics for more than one century. Although analytical solutions for bending of thin plates are intensively developed during the past decades [1–5], the category of analytical solutions for thin plates are still beyond completeness. An overview of the existing literature reveals that most of the previous exact analyses are developed using the semi-inverse method and limited to Lévy-type plates since it is rather difficult to seek trial functions satisfying the non-simply supported boundary conditions, including

* Corresponding author. Tel.: +852 2788 7285; fax: +852 2788 7612.
E-mail address: bccwlim@cityu.edu.hk (C.W. Lim).

0020-7225/$ - see front matter
doi:10.1016/j.ijengsci.2008.08.003

geometric as well as natural boundary conditions. Even if it is possible in certain specific cases, the mode shape functions are excessively complicated [2].

In view of such analytical shortcomings, the symplectic approach, a mathematical method built within a symplectic space common to physicists with its development history briefly reviewed by Lim et al. [6], was introduced by Feng [7,8] and Zhong [9,10] into computational and analytical solid mechanics. It was useful for obtaining analytical solutions for some basic elasticity problems which have long been bottlenecks in the history of elasticity. In this symplectic elasticity approach, the Hamiltonian principle with Legendre's transformation is employed to derive the governing Hamiltonian canonical equation, and exact solutions could be obtained via the eigenvalue analysis and expansion of eigenfunctions. The Hamiltonian equation thus derived contains both stress and displacement components which are dual to each other and they can be interpreted physically from the energy sense. It differs, in a fundamental sense, from the governing equation derived previously in Lagrangian system involving only one kind of variables, or from that obtained using the state space method [11] in which the state variables could be arranged arbitrarily. This is mainly due to the fact that the former system with one kind of variables and the state space equation are both treated in the Euclidian space in which symplectic orthogonality is not a condition at all. Moreover, unlike the classical semi-inverse methods with pre-determined trial functions, the symplectic elasticity approach is rigorously rational without any guess functions. All geometric and natural boundary conditions are imposed on the system in a natural manner. The step-by-step derivation in the symplectic method make it possible that many basic issues, including the preliminary deformations, effects of local loading conditions, and the Saint–Venant principle, be uniformly treated and illustrated with deep insight [12].

The symplectic elasticity approach was first used by Feng [7,8] in computational solid mechanics. Subsequently, the symplectic approach was widely applied by many investigators to the static analyses of two-dimensional problems after the pioneering works of Zhong [9,10] in analytical solid mechanics using the symplectic approach. Xu et al. [13] studied the Saint–Venant problem in circular cylinders and analyzed the zero eigenvalue solutions and their Jordan normal forms of the corresponding Hamiltonian operator matrix. The similar problems in anisotropic composite laminated plates in cylindrical bending were also investigated by Zhong and Yao [14], and Yao and Yang [15]. Most recently, Lim et al. [6] proposed a new symplectic approach for the bending analysis of thin plates with two opposite edges simply supported. In their analysis, a series of bending moment functions were introduced to construct the Pro–Hellinger–Reissner variational principle, which is an analogy to plane elasticity. This new symplectic approach was subsequently extended to the bending analysis of corner-supported Kirchhoff plates due to uniformly distributed loading [16]. The exact explicit solution of the deflection was derived for the first time. Furthermore, the zero natural boundary conditions at the free edges are satisfied and the twisting moment condition at the support corners are predicted with high accuracy, which could not be exactly satisfied in many papers for more than half a century in both analytical and numerical analysis including the finite element method. As for vibration analysis of plates, Zou [17] reported an exact symplectic geometry solution for the static and dynamic analysis of Reissner plates, but it was not exactly the same as the symplectic elasticity approach described above because trial mode shape functions for the simply supported opposite edges were still adopted in his analysis.

In this paper, the Hellinger–Reissner variational principle governing free vibration of Kirchhoff rectangular plates is first transferred to the Hamiltonian principle of mixed energy with the aid of Legendre's transformation. It leads to a set of Hamiltonian dual equations. The system is then resolved using the method of variable separation and expansion of eigensolutions which are symplectic adjoint orthogonormal to each other. Examples for Lévy-type plates are presented and analyzed and the frequency equations corresponding to all combinations of the typical boundary conditions, say simple support, clamped, and free, are presented in exact forms within the framework of thin plate theory. Numerical comparisons to the classical solutions in literature are presented to validate the efficiency and accuracy of the symplectic method.

2. Theoretical formulation

Consider an isotropic rectangular Kirchhoff plate with uniform thickness h, length $2a$, width $2b$, Young's modulus E, and Poisson's ratio ν. The Cartesian coordinate system is established with the origin at the center of plate such that $-a \leqslant x \leqslant a$ and $-b \leqslant y \leqslant b$, as depicted in Fig. 1.

2.1. Basic equations

The Kirchhoff plate theory is adopted here. The positive directions of the resultant forces, resultant moments, and curvatures are defined in the same manner as that described in Hu [18]. The equations of motion for harmonic free vibration can be expressed in the frequency domain as

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\rho h \omega^2 w, \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0. \tag{1}$$

Here Q_x and Q_y, M_x and M_y, and M_{xy} are the transverse shear forces, bending moments, and twisting moment per unit length of the section, ρ the mass density, ω the circular frequency, and w the transverse displacement. The resultant moments can be expressed in terms of curvatures as

$$M_x = D(\kappa_x + \nu\kappa_y), \quad M_y = D(\kappa_y + \nu\kappa_x), \quad M_{xy} = (1-\nu)D\kappa_{xy}, \tag{2}$$

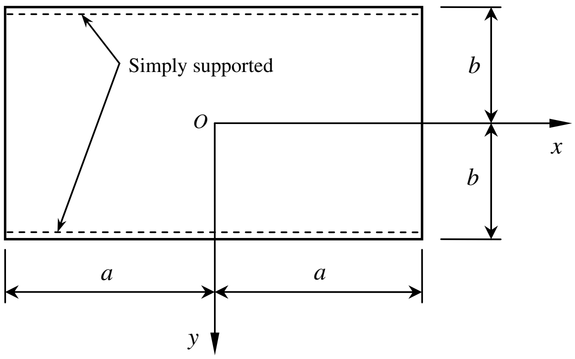

Fig. 1. Geometry and coordinate system of a rectangular Kirchhoff plate.

where $D = Eh^3/12(1-\nu^2)$ is the bending rigidity of the plate, while the curvatures κ_x, κ_y and κ_{xy} are defined as

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}. \tag{3}$$

The equivalent shear forces are determined by

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = \frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}, \tag{4a}$$

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x} = 2\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}. \tag{4b}$$

The two opposite edges at $x = \pm a$ are assumed arbitrarily constrained by simple support (S), clamped (C), or free (F), which satisfy, respectively, the following geometric and/or natural boundary conditions

$$\mathrm{S}: w = 0, \quad M_x = 0, \tag{5}$$

$$\mathrm{C}: w = 0, \quad \frac{\partial w}{\partial x} = 0, \tag{6}$$

$$\mathrm{F}: M_x = 0, \quad V_x = 0. \tag{7}$$

Obviously, the remaining two opposite edges at $y = \pm b$ can also be arbitrary constrained by the three supports in Eqs. (5)–(7).

The free vibration of a rectangular Kirchhoff plate is governed by the system of differential equations and boundary conditions above. The exact solutions to such a problem will be derived rationally using the symplectic elasticity approach in the following sections.

2.2. Hamiltonian variational principle and Hamiltonian dual equation

For static plate bending, Lim et al. [6,16] applied a set of bending moment functions and hence a Pro–Hellinger–Reissner variational principle analogous to the plane elasticity problems was derived. In contrast, the free vibration problem here will be resolved directly based on the Hellinger–Reissner variational principle [18] for thin plates. The variational principle requires

$$\delta \Pi_2 = \delta \int_{-a}^{a} \int_{-b}^{b} \left[M_x \kappa_x + 2M_{xy}\kappa_{xy} + M_y \kappa_y - U_c - \frac{1}{2}\rho h \omega^2 w \right] \mathrm{d}x\,\mathrm{d}y = 0, \tag{8}$$

where U_c is the complementary energy, defined by

$$U_\mathrm{c} = \frac{1}{2D(1-\nu^2)} \left[M_x^2 - 2\nu M_x M_y + M_y^2 + 2(1+\nu)M_{xy}^2 \right]. \tag{9}$$

To implement the symplectic analysis, a pair of dual variables should be constructed. To this end, the following variable is introduced

$$\theta = \frac{\partial w}{\partial x}, \tag{10}$$

which is defined as the slope along the x axis. Letting w, θ and V, M (V_x and M_x) be the complementary dual variables, we derived from Eq. (2) that

$$M_y = -(1-\nu^2)D\frac{\partial^2 w}{\partial y^2} + \nu M, \tag{11a}$$

$$M_{xy} = -(1-\nu)D\frac{\partial \theta}{\partial y}. \tag{11b}$$

Substituting Eq. (11) into Eq. (8), the following conservative Hamiltonian variational principle is obtained:

$$\delta \int_{-a}^{a} \int_{-b}^{b} (V\dot{w} - M\dot{\theta} - H)\mathrm{d}x\,\mathrm{d}y = 0, \tag{12}$$

where an overdot denotes differentiation with respect to x, i.e. $\dot{(\cdot)} = \partial/\partial x$, and H is the Hamiltonian function, defined by

$$H = V\theta + \frac{1}{2}\rho h\omega^2 w^2 + \nu M\frac{\partial^2 w}{\partial y^2} + \frac{1}{2D}M^2 - \frac{1-\nu^2}{2}D\left(\frac{\partial^2 w}{\partial y^2}\right)^2 - (1-\nu)D\left(\frac{\partial \theta}{\partial y}\right)^2. \tag{13}$$

It is easily obtained from Eq. (12) that

$$\dot{\mathbf{v}} = \mathbf{H}\mathbf{v}, \tag{14}$$

where $\mathbf{v} = \{w\ \theta\ V\ M\}^{\mathrm{T}}$ is the state vector in the symplectic space, and $\mathbf{H}$ is the Hamiltonian matrix operator given by

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\nu\frac{\partial^2}{\partial y^2} & 0 & 0 & -\frac{1}{D} \\ D(1-\nu^2)\frac{\partial^4}{\partial y^4} - \rho h\omega^2 & 0 & 0 & -\nu\frac{\partial^2}{\partial y^2} \\ 0 & 2D(1-\nu)\frac{\partial^2}{\partial y^2} & 1 & 0 \end{bmatrix}. \tag{15}$$

Note that the state vector in the state equation using the state space method [19,20] can be arranged arbitrarily but not restricted to the form of Eq. (14). This is due to the fact that the state equation to be directly solved using matrix algebra is constructed in the Euclidian space and thus the Hamiltonian dual properties of the equation system in the symplectic space are not required at all.

2.3. Symplectic analysis and eigenvectors

For the homogeneous Hamiltonian system in Eq. (14), it is natural to apply the method of variable separation to reduce it to an eigenvalue problem. To this end, the state vector can be expressed as

$$\mathbf{v}(x,y) = \xi(x)\boldsymbol{\psi}(y), \tag{16}$$

Substituting Eq. (16) into Eq. (14) yields

$$\xi(x) = \mathrm{e}^{\mu x}, \tag{17a}$$

and

$$\mathbf{H}\boldsymbol{\psi}(y) = \mu\boldsymbol{\psi}(y), \tag{17b}$$

where μ is the eigenvalue in the x-direction, and $\boldsymbol{\psi} = \{\bar{w}\ \bar{\theta}\ \bar{V}\ \bar{M}\}^{\mathrm{T}}$ is the eigenvector in which the elements are functions of y. Assuming $\boldsymbol{\psi} = \{A\ B\ C\ F\}^{\mathrm{T}}e^{\lambda y}$ and substituting into Eq. (17b) yield the following eigenequation for free vibration with nontrivial solution,

$$\begin{vmatrix} -\mu & 1 & 0 & 0 \\ -\nu\lambda^2 & -\mu & 0 & -\frac{1}{D} \\ D(1-\nu^2)\lambda^4 - \rho h\omega^2 & 0 & -\mu & -\nu\lambda^2 \\ 0 & 2D(1-\nu)\lambda^2 & 1 & -\mu \end{vmatrix} = 0. \tag{18}$$

There are three unknowns, i.e. λ, μ and ω, in Eq. (18) which can only yield an explicit relation between the unknowns. According to the solution procedure of symplectic elasticity [12], the eigen-problem in the y direction should be considered first. Hence, we derive from Eq. (18) for the eigenvalues of λ as

$$\lambda_{1,2} = \pm\sqrt{\omega\sqrt{\frac{\rho h}{D}} - \mu^2}, \quad \lambda_{3,4} = \pm\sqrt{-\omega\sqrt{\frac{\rho h}{D}} - \mu^2}. \tag{19}$$

Note that if all eigenvalues in Eq. (19) equal to zero, i.e. $\lambda_i = 0$, it leads to $\mu = \omega = 0$, which is obviously not the right solution for free vibration. Hence, the zero eigenvalue $\lambda = 0$ should be excluded in the present analysis, and only nonzero eigenvalues will be considered. Here, there should be two cases of nonzero eigenvalues which are discussed as follows.

(1) Particular nonzero eigenvalues

If there exists only one set of zero eigenvalues in Eq. (19), that is

$$\lambda_{1,2} = 0, \quad \mu^2 = \omega\sqrt{\rho h/D}, \quad \lambda_{3,4} = \pm\sqrt{-\beta} \tag{20a}$$

or

$$\lambda_{3,4} = 0, \quad \mu^2 = -\omega\sqrt{\rho h/D}, \quad \lambda_{1,2} = \pm\sqrt{\beta} \tag{20b}$$

where $\beta = 2\omega\sqrt{\rho h/D}$. These nonzero eigenvalues are called the particular nonzero eigenvalues. For the two sets of particular nonzero eigenvalues above, the eigenvector for Eq. (17b) can be expressed in a standard form as

$$\begin{aligned}
\bar{w} &= A_1 + A_2 y + A_3 \cosh\bar{\lambda}y + A_4 \sinh\bar{\lambda}y \\
\bar{\theta} &= B_1 + B_2 y + B_3 \cosh\bar{\lambda}y + B_4 \sinh\bar{\lambda}y \\
\overline{V} &= C_1 + C_2 y + C_3 \cosh\bar{\lambda}y + C_4 \sinh\bar{\lambda}y \\
\overline{M} &= F_1 + F_2 y + F_3 \cosh\bar{\lambda}y + F_4 \sinh\bar{\lambda}y
\end{aligned} \tag{21}$$

where $\bar{\lambda} = \sqrt{-\beta}$ when $\lambda_{1,2} = 0$, or $\bar{\lambda} = \sqrt{\beta}$ when $\lambda_{3,4} = 0$. As the eigenvector in Eq. (21) must satisfy Eq. (17b), the 16 coefficients in Eq. (21) are not at all independent. In reality, there exist only four independent coefficients. For brevity without losing generality, A_i are selected as the independent constants. Thus substitution of Eq. (21) into Eq. (17b) gives rise to

$$B_i = \mu A_i, \tag{22a}$$

$$\begin{cases} F_{1,2} = -D\mu^2 A_{1,2} \\ F_{3,4} = -D(1-2\nu)\mu^2 A_{3,4} \end{cases}, \tag{22b}$$

$$\begin{cases} C_{1,2} = -D\mu^3 A_{1,2} \\ C_{3,4} = D(3-2\nu)\mu^3 A_{3,4} \end{cases}. \tag{22c}$$

(2) General nonzero eigenvalues

Consider, in general, that all eigenvalues in Eq. (19) are nonzero, i.e. $\lambda_i \neq 0$ (called the general nonzero eigenvalues), the eigenvector can be written as

$$\begin{aligned}
\bar{w} &= A_1 \cosh\lambda_1 y + A_2 \sinh\lambda_2 y + A_3 \cosh\lambda_3 y + A_4 \sinh\lambda_4 y \\
\bar{\theta} &= B_1 \cosh\lambda_1 y + B_2 \sinh\lambda_2 y + B_3 \cosh\lambda_3 y + B_4 \sinh\lambda_4 y \\
\overline{V} &= C_1 \cosh\lambda_1 y + C_2 \sinh\lambda_2 y + C_3 \cosh\lambda_3 y + C_4 \sinh\lambda_4 y \\
\overline{M} &= F_1 \cosh\lambda_1 y + F_2 \sinh\lambda_2 y + F_3 \cosh\lambda_3 y + F_4 \sinh\lambda_4 y
\end{aligned} \tag{23}$$

Similarly, there are also only four independent constants in Eq. (23), for which the following relations are obtained:

$$B_i = \mu A_i, \tag{24a}$$

$$\begin{cases} F_{1,2} = -D(\mu^2 + \nu\lambda_1^2)A_{1,2} \\ F_{3,4} = -D(\mu^2 + \nu\lambda_3^2)A_{3,4} \end{cases}, \tag{24b}$$

$$\begin{cases} C_{1,2} = -\mu D[\mu^2 + (2-\nu)\lambda_1^2]A_{1,2} \\ C_{3,4} = -\mu D[\mu^2 + (2-\nu)\lambda_3^2]A_{3,4} \end{cases}. \tag{24c}$$

The terms containing $\cosh \lambda_i y$(A_1 and A_3) in Eqs. (21) and (23) are related to the symmetric vibration modes, and those containing $\sinh \lambda_i y$ and y (A_2 and A_4) correspond to the antisymmetric vibration modes.

3. Free vibration of plates simply supported at $y = \pm b$

There are three unknowns λ_i, μ and ω in Eq. (19) and ω should be solved from the final frequency equation. To this end, λ_i should be solved first from the boundary conditions at $y = \pm b$, and then μ can be expressed in terms of ω and the known λ_i. Next, imposing the boundary conditions at $x = \pm a$ leads to the frequency equation with only one unknown ω. This solution procedure is rather different as compared to the static bending analysis of plates [6,16].

To illustrate the application of the symplectic methodology for free vibration problems, the plate here is assumed simply supported at the opposite edges $y = \pm b$ (Lévy-type plate). The boundary conditions are

$$w = 0, \quad M_y = 0. \tag{25}$$

which should be re-expressed by virtue of the complementary variables as

$$w|_{y=\pm b} = 0, \text{ and } M_y|_{y=\pm b} = \left[-D(1-\nu^2)\frac{\partial^2 w}{\partial y^2} + \nu M\right]_{y=\pm b} = 0. \tag{26}$$

Next, the unknown λ_i should be solved analytically for the two cases of nonzero eigenvalues.

3.1. Particular nonzero eigenvalues

For this case, $\lambda_{3,4} = \pm\sqrt{-\beta}$ ($\lambda_{1,2} = 0$) or $\lambda_{1,2} = \pm\sqrt{\beta}$, ($\lambda_{3,4} = 0$). Substituting the eigensolutions in Eq. (21) into the boundary conditions in Eq. (26), and setting the determinant of the coefficient matrix to zero result in

$$\cosh\bar{\lambda}b \;\; \sinh\bar{\lambda}b = 0. \tag{27}$$

This is a transcendental equation with a unique solution $\bar{\lambda} = \pm \mathrm{i}n\pi/2b$ $(n = 1, 2, 3, \ldots)$, and hence

$$2\omega_n\sqrt{\frac{\rho h}{D}} = \frac{n^2\pi^2}{4b^2}, \quad \mu_n = \pm\frac{n\pi}{2\sqrt{2}b}, \tag{28a}$$

for $\lambda_{1,2} = 0$ and $\lambda_{3,4} = \pm\sqrt{-\beta}$; or

$$2\omega_n\sqrt{\frac{\rho h}{D}} = -\frac{n^2\pi^2}{4b^2}, \quad \mu_n = \pm\frac{n\pi}{2\sqrt{2}b}, \tag{28b}$$

for $\lambda_{3,4} = 0$ and $\lambda_{1,2} = \pm\sqrt{\beta}$. From physical viewpoint, the negative natural frequencies in Eq. (28b) are nonexistent for a plate structure and they should be dropped. Therefore, the particular nonzero eigenvalues $\lambda_{3,4} = 0$ and $\lambda_{1,2} = \pm\sqrt{\beta}$ should be eliminated, and only Eq. (28a) is preserved, which gives

$$\omega_n = \frac{n^2\pi^2}{8b^2}\sqrt{\frac{D}{\rho h}}, \quad (n = 1, 2, \ldots). \tag{29}$$

However, for this set of eigenfrequencies, only zero-eigenvector for A_i can be obtained from the boundary conditions in Eq. (26). That is, it is not feasible to seek a vibration mode which satisfies simultaneously the boundary conditions in Eq. (26) and the frequency magnitude in Eq. (29). This can also be interpreted from another point of view. The expression in Eq. (29) indicates that the natural frequencies of Lévy-type rectangular plates remain unchanged regardless of whatever the dimension in the x-direction is or whatever boundary conditions at $x = \pm a$ are. The above phenomenon is absolutely beyond the physical sense, that is, the solution in Eq. (29) is nonexistent and hence the nonzero eigenvalue $\lambda_{1,2} = \pm\sqrt{\beta}$ and $\lambda_{3,4} = 0$ should also be discarded in this problem.

Although the particular nonzero eigenvalues are proved to be nonexistent for the present simply supported edges, they are indeed the solutions to the eigenequation in Eq. (18). Hence, it is essential to consider them so as to ensure the completeness of solutions in the symplectic system. Regarding this nature of the system, the particular nonzero eigenvalues are believed to deliver rational eigenvectors provided that the plate is subjected to some proper edge supporting conditions at $y = \pm b$. This task will be fulfilled in a future endeavor.

3.2. General nonzero eigenvalues

For this case, $\lambda_i \neq 0$, substituting the eigensolutions in Eq. (23) into the boundary conditions in Eq. (26), and setting the determinant of the coefficient matrix to zero lead to

$$\cosh\lambda_1 b \;\; \sinh\lambda_1 b \;\; \cosh\lambda_3 b \;\; \sinh\lambda_3 b = 0. \tag{30}$$

This is also a transcendental equation with a unique solution $\lambda_1 = \pm \mathrm{i}n\pi/2b$ or $\lambda_3 = \pm \mathrm{i}n\pi/2b$ $(n = 1,2,3,\ldots)$. Substituting into Eq. (19) yields

$$\mu_n = \pm\sqrt{\omega_n\sqrt{\frac{\rho h}{D}} + \frac{n^2\pi^2}{4b^2}}, \quad \lambda_3 = \sqrt{-2\omega_n\sqrt{\frac{\rho h}{D}} - \frac{n^2\pi^2}{4b^2}}, \quad \left(\lambda_1 = \pm\mathrm{i}\frac{n\pi}{2b}\right), \tag{31a}$$

or

$$\mu_n' = \pm\sqrt{-\omega_n\sqrt{\frac{\rho h}{D}} + \frac{n^2\pi^2}{4b^2}}, \quad \lambda_1 = \sqrt{2\omega_n\sqrt{\frac{\rho h}{D}} - \frac{n^2\pi^2}{4b^2}}, \quad \left(\lambda_3 = \pm\mathrm{i}\frac{n\pi}{2b}\right). \tag{31b}$$

For the eigenvalues in Eq. (31a), the eigensolutions for A_i can be taken as $A_1 = \cosh\lambda_3 b$, and the eigensolution of the state variables is obtained as

$$\bar{w} = \cosh\lambda_3 b\cosh\lambda_1 y + \sinh\lambda_3 b\sinh\lambda_1 y - \cosh\lambda_1 b\cosh\lambda_3 y - \sinh\lambda_1 b\sinh\lambda_3 y, \tag{32}$$

and the remaining components of the eigenvector can be obtained accordingly. For the eigenvalues in Eq. (31b), the expression of eigensolution can be written in a similar manner. It should be emphasized that the eigenvectors corresponding to μ_n and μ_n' are symplectic orthogonal to each other, that is, they satisfy

$$\langle\boldsymbol{\psi}_n, \boldsymbol{\psi}_n'\rangle = 0 \tag{33}$$

where $\langle\boldsymbol{\psi}_n, \boldsymbol{\psi}_n'\rangle$ is the symplectic inner product for any two vectors [12].

At this stage, from the property of symplectic adjoint orthogonality of eigenvectors and expansion of eigenvectors, the state vector can be expanded as

$$\mathbf{v} = \sum_{n=1}^{\infty} [f_1(n)e^{\mu_n x}\psi_n(y) + f_2(n)e^{-\mu_n x}\psi_{-n}(y) + f_3(n)e^{\mu'_n x}\psi'_n(y) + f_4(n)e^{-\mu'_n x}\psi'_{-n}(y)], \tag{34}$$

where $f_i(n)$ are unknown functions depending on the boundary conditions at $x = \pm a$. With the expressions of μ_n and μ'_n in Eq. (31) and the eigensolutions in Eq. (32), functions $f_i(n)$ and circular frequency ω are the only unknowns in the state vector in Eq. (34). Satisfaction of the boundary conditions at $x = \pm a$ will lead to the frequency equation.

4. Frequency equations of Lévy-type plates and numerical comparison

4.1. Frequency equations of Lévy-type plates

For a fully simply supported (SSSS) plate, the two edges $x = \pm a$ are both satisfied by Eq. (5), that is

$$w(-a,y) = M(-a,y) = w(a,y) = M(a,y) = 0. \tag{35}$$

Combining Eq. (34) with Eq. (35) yields the frequency equation as

$$\sinh \mu_n a \cosh \mu_n a \sinh \mu'_n a \cosh \mu'_n a = 0. \tag{36}$$

Hence, the exact eigenfrequency is

$$\omega_n = \left[\frac{m^2\pi^2}{(2a)^2} + \frac{n^2\pi^2}{(2b)^2}\right]\sqrt{\frac{D}{\rho h}}, \quad (m,n = 1,2,\ldots). \tag{37}$$

It is seen that this analytical expression of natural frequency for SSSS plate is exactly the same as that reported in Leissa [2] using the classical semi-inverse method.

For plates with the two edges at $x = \pm a$ constrained by the combinations CC, CS, CF, SF, or FF in Eqs. (5)–(7), the corresponding frequency equations can be obtained in a similar way. The boundary conditions at $x = \pm a$ expressed in terms of complementary variables and frequency equations for these combinations are given below:

CSCS

$$w(-a,y) = \frac{\partial w(-a,y)}{\partial x} = w(a,y) = \frac{\partial w(a,y)}{\partial x} = 0, \tag{38a}$$

$$(\mu_n^2 + \mu_n'^2)\sinh(2\mu_n a)\sinh(2\mu'_n a) = 2\mu_n\mu'_n[\cosh(2\mu_n a)\cosh(2\mu'_n a) - 1], \tag{38b}$$

SSCS

$$w(-a,y) = \frac{\partial w(-a,y)}{\partial x} = w(a,y) = M(a,y) = 0, \tag{39a}$$

$$\mu_n \cosh(2\mu_n a)\sinh(2\mu'_n a) - \mu'_n \sinh(2\mu_n a)\cosh(2\mu'_n a) = 0, \tag{39b}$$

CSFS

$$w(-a,y) = \frac{\partial w(-a,y)}{\partial x} = M(a,y) = V(a,y) = 0, \tag{40a}$$

$$\begin{aligned} &2\mu_n\mu'_n[k^4 - (1-\nu)^2] + 2\mu_n\mu'_n[k^4 + (1-\nu)^2]\cosh(2\mu_n a)\cosh(2\mu'_n a) \\ &\qquad + (\mu_n^2 + \mu_n'^2)[(1-2\nu)k^4 - (1-\nu)^2]\sinh(2\mu_n a)\sinh(2\mu'_n a) = 0, \end{aligned} \tag{40b}$$

SSFS

$$w(-a,y) = M(-a,y) = M(a,y) = V(a,y) = 0, \tag{41a}$$

$$\mu_n[k^2 - (1-\nu)]^2\cosh(2\mu_n a)\sinh(2\mu'_n a) = \mu'_n[k^2 + (1-\nu)]^2\sinh(2\mu_n a)\cosh(2\mu'_n a), \tag{41b}$$

FSFS

$$M(-a,y) = V(-a,y) = M(a,y) = V(a,y) = 0, \tag{42a}$$

$$\{\mu_n^2[k^2 - (1-\nu)]^4 + \mu_n'^2[k^2 + (1-\nu)]^4\}\sinh(2\mu_n a)\sinh(2\mu'_n a) = 2\mu_n\mu'_n[k^4 - (1-\nu)^2]^2[\cosh(2\mu_n a)\cosh(2\mu'_n a) - 1], \tag{42b}$$

where $k^2 = \alpha^2/l^2$, $\alpha^2 = \omega_n\sqrt{\rho h/D}$, $l^2 = n^2\pi^2/4b^2$.

It is obviously that all formulation above, unlike the classical semi-inverse method, are derived analytically in a rational manner, and rigorously step-by-step without introducing any guess or trial functions. However, the transcendental equations for ω_n in Eqs. (38b)–(42b) are so complicated that closed form solutions to ω_n are rather difficult. Hence, these frequency expressions are normally solved numerically rather than analytically in practice.

4.2. Numerical comparison

Numerical comparison for natural frequencies of Lévy-type Kirchhoff plates are presented in this section. Table 1 exhibits the comparison of the lowest six frequency parameters $4b^2\omega\sqrt{\rho h/D}$ for square plates ($a = b$) with $\nu = 0.3$. It is seen that the symplectic results for all six combinations of boundary conditions are in excellent agreement with the classical results reported by Leissa [2]. It should be pointed out that the frequency parameters containing the factor $\sqrt{\rho h/D}$ are independent of the Poisson's ratio for plates without free edges, and hence the results for CSCS, SSCS and SSSS plates are also applicable for those with arbitrary values of Poisson's ratio.

Table 1
The lowest six natural frequency parameters $4b^2\omega\sqrt{\rho h/D}$ of Lévy-type square plates ($\nu = 0.3$)

Edges	Results	Mode number					
		1	2	3	4	5	6
CSCS	Present	28.9509	54.7431	69.3270	94.5853	102.2162	129.0955
	Leissa [2]	28.946	54.743	69.320	94.584	102.213	129.086
SSCS	Present	23.6463	51.6743	58.6464	86.1345	100.2698	113.2281
	Leissa [2]	23.646	51.674	58.641	86.126	100.259	113.217
SSSS	Present	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
	Leissa [2]	19.739	49.348	49.348	78.957	98.696	98.696
CSFS	Present	12.687	33.065	41.702	63.015	72.398	90.611
	Leissa [2]	12.69	33.06	41.70	63.01	72.40	90.61
SSFS	Present	11.685	27.756	41.197	59.066	61.861	90.294
	Leissa [2]	11.68	27.76	41.20	59.07	61.86	90.29
FSFS	Present	9.631	16.135	36.726	38.944	46.738	70.740
	Leissa [2]	9.631	16.13	36.72	38.94	46.74	70.75

Table 2
Dimensionless frequency parameters for SSCS rectangular plate

	Parameter					
	b/a	1.0	1.5	2.0	2.5	3.0
$4a^2\omega\sqrt{\rho h/D}$	Present	23.6463	18.9012	17.3318	16.2676	16.2520
	Leissa [2]	23.646	18.899	17.330	16.629	16.254
	a/b	1.0	1.5	2.0	2.5	3.0
$4b^2\omega\sqrt{\rho h/D}$	Present	23.6463	15.5783	12.9186	11.7502	11.1411
	Leissa [2]	23.646	15.573	12.918	11.754	11.142

Table 3
Dimensionless frequency parameters $4b^2\omega\sqrt{\rho h/D}$ for CSFS rectangular plate ($\nu = 0.25$)

b/a	1.0	1.1	1.2	1.3	1.4	1.5
Present	12.8618	13.5652	14.3446	15.1993	16.1286	17.1319
Leissa [2]	12.859	13.520	14.310	15.198	16.086	17.172
b/a	1.6	1.7	1.8	1.9	2.0	2.2
Present	18.2086	19.3583	20.5803	21.8745	23.2403	26.1863
Leissa [2]	18.258	19.343	20.527	21.910	23.192	26.153

Table 4
Dimensionless frequency parameters $4b^2\omega\sqrt{\rho h/D}$ for SSFS rectangular plate ($\nu = 0.25$)

b/a	0.5	0.6	0.8	1.0	1.2	1.4
Present	10.3556	10.5800	11.1367	11.8196	12.6084	13.4846
Leissa [2]	10.362	11.349	11.547	11.843	12.632	13.520
b/a	1.6	1.8	2.0	2.5	3.0	4.0
Present	14.4322	15.4378	16.4907	19.2776	22.2171	28.3603
Leissa [2]	14.409	15.396	16.481	19.244	22.205	28.324

Table 5
The lowest eight natural frequency parameters $4ab\omega\sqrt{\rho h/D}$ of Lévy-type rectangular plates ($\nu = 0.3$)

Edges	a/b	Results	Mode number							
			1	2	3	4	5	6	7	8
CSCS	0.5	Present	47.6313	57.9017	78.1785	109.486	127.0690	138.654	151.683	159.050
		Lim and Liew [21]	47.631	57.902	78.178	109.49	127.07	138.65	151.68	159.05
	2.0	Present	27.3715	47.2926	77.3879	85.1732	103.349	117.293	132.598	166.977
		Lim and Liew [21]	27.372	47.293	77.388	85.173	103.35	117.29	132.60	166.98
FSFS	0.5	Present	4.7565	13.7611	19.2628	32.2696	43.6426	52.7448	58.4160	73.4513
		Lim and Liew [21]	4.7562	13.761	19.263	32.270	43.642	52.745	58.416	73.451
	2.0	Present	19.4725	23.3691	35.3701	55.5127	78.3762	82.3933	84.7689	95.9337
		Lim and Liew [21]	19.472	23.369	35.370	55.513	78.376	82.393	84.769	95.934

Table 6
Comparisons of the present results to 3D solutions for the lowest eight natural frequency parameters $4b^2\omega/\pi^2\sqrt{\rho h/D}$ of SSSS and CSCS square plates ($h/a = 0.01$)

Mode	SSSS		CSCS	
	Present	3D [22]	Present	3D [22]
1	2.0000	2.0028	2.9333	2.9518
2	5.0000	4.9878	5.5466	5.5439
3	5.0000	5.0065	7.0243	7.046
4	8.0000	7.9873	9.5835	9.5914
5	10.000	9.9745	10.3567	10.336
6	10.000	9.9807	13.0801	13.114
7	13.000	12.966	14.2057	14.19
8	13.000	12.971	15.6821	15.689

Table 7
Comparisons of the present results to 3D solutions for selected natural frequency parameters $2b^2\omega/\pi\sqrt{\rho h/D}$ of Lévy-type rectangular thin plates ($\nu = 0.3$)

Edges	a/b	(m,n)	Present	3D solutions [23] (h/a)		CPT [23]
				0.005	0.01	
SSSS	2.0	1,1	1.963495	1.962988	1.961786	1.963495
CSCS	1.0	2,3	22.31424	22.30661	22.25712	22.31424
SSCS	0.5	1,4	32.89686	32.89453	32.87303	32.89686
SSFS	2.0	4,1	5.908996	5.898097	5.88473	5.908996
CSFS	1.0	1,2	6.637068	6.610726	6.609886	6.637068
FSFS	0.5	3,2	23.38027	23.29759	23.28252	23.38027

Tables 2–4 present the non-dimensional frequency for rectangular SCSS, SCSF and SSSF plates with different aspect ratios a/b. Tables 3 and 4 are for plates with $\nu = 0.25$. It is observed that the present results for rectangular plates are also identical to that reported in Leissa [2]. Excellent agreement is also recorded in Table 5 by comparing the present results for the lowest eight frequency parameter $4ab\omega\sqrt{\rho h/D}$ for rectangular CSCS and FSFS plates ($\nu = 0.3$) with different aspect ratios a/b to that obtained by Lim and Liew [21] using the *pb*-2 Ritz method.

For further validations, the present results are compared to that obtained based on three-dimensional (3D) elasticity solutions using the differential quadrature method [22,23]. Table 6 presents the lowest eight natural frequency parameters $4b^2\omega/\pi^2\sqrt{\rho h/D}$ for SSSS and CSCS square thin plate with an aspect ratio of $h/a = 0.01$, while Table 7 gives the frequency parameters $2b^2\omega/\pi\sqrt{\rho h/D}$ for some selected vibration modes of six types of Lévy plates with aspect ratio $h/a = 0.01$ and $h/a = 0.005$. It is seen that the present results using the proposed symplectic method agree well with 3D results for extremely thin plates. Table 7 also shows that the present symplectic results are identical to that using the differential quadrature method by Malik and Bert [23] based on the classical Kirchhoff plate theory.

All comparison studies above provide a solid demonstration that the present symplectic elasticity approach is very efficient for exact analysis of plate free vibration base on the classical Kirhhoff plate theory.

5. Conclusions

Free vibration of Kirchhoff plates was exactly solved here using a new symplectic elasticity approach based on the Hamiltonian system and constructed in the symplectic space. This new approach does not require any guess or trial shape functions that are essential in the classical semi-inverse method. The mixed energy Hamiltonian function, and hence the

Hamiltonian dual equation were derived based on the Hellinger–Reissner variational principle together with the Legendre's transformation. The Hamiltonian equation was then translated into an eigenequation in the in-plane domain of the plate, and it was then analytically solved with rigorous derivation using the method of variable separation and expansion of eigensolutions. The present vibration analysis involves not only the eigenequation in frequency domain but also the geometric domain. This is another important feature of the symplectic approach against the classical vibration analysis in Euclidian space using the Lagrangian system. It should be emphasized that the completeness of solution system of free vibration of rectangular plates were warranted using the present methodology. In a general sense, the solutions of plates with arbitrary boundary conditions other than the three typical conditions mentioned were also covered in the framework of the current formulation.

Exact frequency equations for Lévy-type plates were derived and presented as illustrative examples. Numerical results were presented and comparison shows excellent agreement with those reported in literature. As the derivation of the symplectic approach is independent on the assumptions of stress and displacement fields, exact analysis can also be conducted not only for plates of non-Lévy type for which exact solutions are hitherto unavailable, but also for thick plates based on various higher-order plate theories, which will fall into the scope of another future endeavor.

Acknowledgements

This work was supported by the Grants from the Research Scholarship Enhancement Scheme of City University of Hong Kong and the Academic Exchange Fund of City University of Hong Kong.

References

[1] A.W. Leissa, Vibration of a simply-supported elliptical plate, J. Sound Vib. 6 (1967) 145–148.
[2] A.W. Leissa, Vibration of Plates, NASA SP-160, Office of Technology Utilization, NASA, Washington DC, 1969.
[3] A.W. Leissa, Free vibration of rectangular-plates, J. Sound Vib. 31 (1973) 257–293.
[4] S. Timoshenko, S. Woinowsky-Krieger, Theory of Plates and Shells, Mcgraw-Hill, New York, 1970.
[5] J.R. Hutchinson, Vibrations of thick free circular plates, exact versus approximate solutions, ASME J. Appl. Mech. 51 (1984) 581–585.
[6] C.W. Lim, S. Cui, W.A. Yao, On new symplectic elasticity approach for exact bending solutions of rectangular thin plates with two opposite sides simply supported, Int. J. Solids Struct. 44 (2007) 5396–5411.
[7] K. Feng, On difference scheme and symplectic geometry, in: Proceedings of the Beijing Symposium Differential Geometry and Differential Equations, Beijing, 1985, pp. 42–58.
[8] K. Feng, Difference schemes for Hamiltonian formalism and symplectic geometry, J. Comput. Math. 4 (1986) 279–289.
[9] W.X. Zhong, Plane elasticity problem in strip domain and Hamiltonian system, J. Dalian Univ. Tech. 31 (1991) 373–384 (in Chinese).
[10] W.X. Zhong, On the reciprocal theorem and adjoint symplectic orthogonal relation, Acta Mech. Sin. 24 (1992) 432–437 (in Chinese).
[11] L.Y. Bahar, A sate space approach to elasticity, J. Franklin I. 299 (1975) 33–41.
[12] W.A. Yao, W.X. Zhong, C.W. Lim, Symplectic Elasticity, World Scientific Publishing, Singapore. (2008), in press.
[13] X.S. Xu, W.X. Zhong, H.W. Zhang, The Saint–Venant problem and principle in elasticity, Int. J. Solids Struct. 34 (1997) 2815–2827.
[14] W.X. Zhong, W.A. Yao, The Saint Venant solutions of multi-layered composite plates, Adv. Struct. Eng. 1 (1997) 127–133.
[15] W.A. Yao, H. Yang, Hamiltonian system based Saint Venant solutions for multi-layered composite plane anisotropic plates, Int. J. Solids Struct. 38 (2001) 5807–5817.
[16] C.W. Lim, W.A. Yao, S. Cui, Benchmarks of analytical symplectic solutions for bending of corner-supported rectangular thin plates, The IES Journal Part A. 1 (2008) 106–115.
[17] G. Zou, An exact symplectic geometry solution for the static and dynamic analysis of Reissner plates, Comput. Method Appl. Mech. Eng. 156 (1998) 171–178.
[18] H.C. Hu, Variational Principles of Theory of Elasticity with Applications, Gordon and Breach, New York, 1984.
[19] C.F. Lü, Y.Y. Lee, C. W Lim, W.Q. Chen, Free vibration of long-span continuous rectangular Kirchhoff plates with internal rigid line supports, J. Sound Vib. 297 (2006) 351–364.
[20] C.F. Lü, Z.C. Zhang, W.Q. Chen, Free vibration of generally supported rectangular Kirchhoff plates: state-space-based differential quadrature method, Int. J. Numer. Meth. Eng. 70 (2007) 1430–1450.
[21] C.W. Lim, K.M. Liew, A pb-2 Ritz formulation for flexural vibration of shallow cylindrical shells of rectangular planform, J. Sound Vib. 173 (1994) 343–375.
[22] K.M. Liew, T.M. Teo, Three-dimensional vibration analysis of rectangular plates based on differential quadrature method, J. Sound Vib. 220 (1999) 577–599.
[23] M. Malik, C.W. Bert, Three-dimensional elasticity solutions for free vibrations of rectangular plates by the differential quadrature method, Int. J. Solids Struct. 35 (1998) 299–318.