# Mixed finite elements for port-Hamiltonian von Kaŕmán beams

**Andrea Brugnoli** * **Stefano Stramigioli** *

* *University of Twente, Enschede (NL)*
*a.brugnoli@utwente.nl, s.stramigioli@utwente.nl*

**Abstract:** The port-Hamiltonian framework allows for a structured representation and interconnection of distributed parameter systems described by Partial Differential Equations (PDE) from different realms. Here, the Mindlin-Reissner model of a thick plate is presented in a tensorial formulation. Taking into account collocated boundary control and observation gives rise to an infinite-dimensional port-Hamiltonian system (pHs). The Partitioned Finite Element Method (PFEM), already presented in our previous work, allows obtaining a structure-preserving finite-dimensional port-Hamiltonian system, and accounting for boundary control in a straightforward manner. In order to illustrate the flexibility of PFEM, both types of boundary controls can be dealt with: either through forces and momenta, or through kinematic variables. The discrete model is easily implementable by using the FEniCS platform. Computation of eigenfrequencies and vibration modes, together with time-domain simulation results demonstrate the consistency of the proposed approach.

Keywords: Port-Hamiltonian systems (pHs), Geometric Discretization, Mindlin-Reissner Plate, Partitioned Finite Element Method (PFEM), Symplectic Integration

## 1. INTRODUCTION

## 2. VON KÁRMÁN BEAMS

The classical full von-Kármán dynamical model is presented in Bilbao et al. (2015). The problem, defined on an open connected set $\Omega \subset \mathbb{R}^2$, takes the dimensionless form

$$
\begin{aligned}
\ddot{\boldsymbol{u}} &= \operatorname{Div} \boldsymbol{N}, & \boldsymbol{N} &= \boldsymbol{\Phi}(\boldsymbol{\varepsilon}), & \boldsymbol{\varepsilon} &= \operatorname{Grad} \boldsymbol{u} + 1/2 \operatorname{grad} w \otimes \operatorname{grad} w, \\
\ddot{w} &= -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}\left(\boldsymbol{N} \operatorname{grad} w\right), & \boldsymbol{M} &= \boldsymbol{\Phi}(\boldsymbol{\kappa}), & \boldsymbol{\kappa} &= \operatorname{Grad} \operatorname{grad} w,
\end{aligned} \tag{1}
$$

where $\boldsymbol{u} \in \mathbb{R}^2$ is the in-plane displacement, $w$ is the vertical displacement, $\boldsymbol{\varepsilon}$ is the in-plane strain tensor, $\boldsymbol{\kappa}$ is the curvature tensor, $\boldsymbol{N}$ is the in-plane stress resultant and $\boldsymbol{M}$ is the bending stress resultant. The notation $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top$ denotes the dyadic product of two vectors. The div operator is the divergence of a vector field, and grad the gradient of a scalar field. The operator Grad $= \frac{1}{2}\left(\nabla + \nabla^\top\right)$ designates the symmetric part of the gradient (i. e. the deformation gradient in continuum mechanics). For a tensor field $\boldsymbol{U} : \Omega \rightarrow \mathbb{R}^{2\times 2}$, with components $U_{ij}$, the divergence $\operatorname{Div}(\boldsymbol{U})$ is a vector, defined column-wise as

$$
\operatorname{Div}(\boldsymbol{U}) := \sum_{i=1}^{2} \partial_{x_i} U_{ij}, \qquad \forall j = \{1, 2\}.
$$

The linear tensor mapping $\boldsymbol{\Phi}$ is positive and preserves symmetry:

$$
\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\boldsymbol{1} + (1-\nu)\boldsymbol{A}, \qquad \boldsymbol{A} = \boldsymbol{A}^\top \implies \boldsymbol{\Phi}(\boldsymbol{A}) = \boldsymbol{\Phi}(\boldsymbol{A})^\top, \qquad \text{where} \quad \boldsymbol{1} = \operatorname{Diag}(1, 1).
$$

The total energy of the model (Hamiltonian functional)

$$
H = \frac{1}{2}\int_\Omega \left\{ \|\dot{\boldsymbol{u}}\|^2 + \dot{w}^2 + \boldsymbol{N} : \boldsymbol{\varepsilon} + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \, \mathrm{d}\Omega, \qquad \text{where} \quad \boldsymbol{A} \tag{2}
$$

consists of the kinetic energy and both membrane and bending deformation energies. This model proves conservative, see Bilbao et al. (2015). Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

## 3. THE EQUIVALENT PORT-HAMILTONIAN REALIZATION

To find a suitable port-Hamiltonian system, we first select a set of new energy variables to make the Hamiltonian functional quadratic. The selection is the same as for both the linear plate problems in Brugnoli et al. (2019b,a):

$$
\boldsymbol{\alpha}_u = \dot{\boldsymbol{u}}, \qquad \alpha_w = \dot{w}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}. \tag{3}
$$

The energy is quadratic in these variables

$$
H = \frac{1}{2}\int_\Omega \left\{ \|\boldsymbol{\alpha}_u\|^2 + \alpha_w^2 + \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon + \boldsymbol{\Phi}(\boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa \right\}. \tag{4}
$$

By computing the variational derivative of the Hamiltonian, one obtains the so-called co-energy variables:

$$
\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad \boldsymbol{e}_w := \delta_{\alpha_w} H = \dot{w}, \qquad \boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon) \tag{5}
$$

Before stating the final formulation, consider the operator $\mathcal{C}(w)(\cdot) : L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \rightarrow L^2(\Omega)$ acting on symmetric tensors

$$
\mathcal{C}(w)(\boldsymbol{T}) := \operatorname{div}(\boldsymbol{T} \operatorname{grad} w). \tag{6}
$$

*Proposition 1.* The formal adjoint of the $\mathcal{C}(w)(\cdot)$ is given by

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\operatorname{grad}(\cdot)\otimes\operatorname{grad}(w) + \operatorname{grad}(w)\otimes\operatorname{grad}(\cdot)\right]. \tag{7}$$

*Proof 1.* Consider a smooth scalar field $v \in C_0^\infty(\Omega)$ and a smooth symmetric tensor field $\boldsymbol{U} \in C_0^\infty(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with compact support. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation

$$\langle v,\, \mathcal{C}(w)(\boldsymbol{U})\rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)(v)^*,\, \boldsymbol{U}\rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})}. \tag{8}$$

The proof follows from the computation

$$\begin{aligned}
\langle v,\, \mathcal{C}(w)(\boldsymbol{U})\rangle_{L^2(\Omega)} &= \langle v,\, \operatorname{div}(\boldsymbol{U}\operatorname{grad} w)\rangle_{L^2(\Omega)}, && \text{Integration by parts,}\\
&= \langle -\operatorname{grad} v,\, \boldsymbol{U}\operatorname{grad} w\rangle_{L^2(\Omega,\mathbb{R}^2)}, && \text{Dyadic product properties,}\\
&= \langle -\operatorname{grad} v \otimes \operatorname{grad} w,\, \boldsymbol{U}\rangle_{L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})}, && \text{Symmetry of } \boldsymbol{U},\\
&= \langle -1/2(\operatorname{grad} v\otimes\operatorname{grad} w + \operatorname{grad} w\otimes\operatorname{grad} v),\, \boldsymbol{U}\rangle_{L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})}
\end{aligned} \tag{9}$$

This means

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\operatorname{grad}(\cdot)\otimes\operatorname{grad}(w) + \operatorname{grad}(w)\otimes\operatorname{grad}(\cdot)\right], \tag{10}$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u\\ \boldsymbol{A}_\varepsilon\\ \alpha_w\\ \boldsymbol{A}_\kappa\end{pmatrix} = \begin{bmatrix}\mathbf{0} & \operatorname{Div} & \mathbf{0} & \mathbf{0}\\ \operatorname{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0}\\ 0 & \mathcal{C}(w) & 0 & -\operatorname{div}\operatorname{Div}\\ \mathbf{0} & \mathbf{0} & \operatorname{Grad}\operatorname{grad} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\delta_{\boldsymbol{\alpha}_u} H\\ \delta_{\boldsymbol{A}_\varepsilon} H\\ \delta_{\alpha_w} H\\ \delta_{\boldsymbol{A}_\kappa} H\end{pmatrix}, \tag{11}$$

The second line of system (11) represents the time derivative of the membrane strain tensor. To close the system, variable $w$ has to be accessible. For this reason, its dynamics has to be included. The augmented system reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u\\ \boldsymbol{A}_\varepsilon\\ w\\ \alpha_w\\ \boldsymbol{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix}\mathbf{0} & \operatorname{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \operatorname{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0}\\ 0 & 0 & 0 & 1 & 0\\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div}\operatorname{Div}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \operatorname{Grad}\operatorname{grad} & \mathbf{0}\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}\delta_{\boldsymbol{\alpha}_u} H\\ \delta_{\boldsymbol{A}_\varepsilon} H\\ \delta_w H\\ \delta_{\alpha_w} H\\ \delta_{\boldsymbol{A}_\kappa} H\end{pmatrix}. \tag{12}$$

Given the results in Brugnoli et al. (2019b,a) and Proposition 1, the operator $\mathcal{J}$ is formally skew-adjoint. If only the kinetic and deformation energies are considered, it holds $\delta_w H = 0$. In general this terms allows accommodating other potentials, for example the gravitational one. Suitable boundary variables are then obtained considering the power balance

$$\dot{H} = \langle \gamma_0 \boldsymbol{e}_u,\, \gamma_\perp \boldsymbol{E}_\varepsilon\rangle_{\partial\Omega} + \langle \gamma_0 \boldsymbol{e}_w,\, \gamma_{\perp\perp,1}\boldsymbol{E}_\kappa + \gamma_0(\boldsymbol{E}_\varepsilon \boldsymbol{n}\cdot\operatorname{grad} w)\rangle_{\partial\Omega} + \langle \gamma_1 \boldsymbol{e}_w,\, \gamma_{\perp\perp}\boldsymbol{E}_\kappa\rangle_{\partial\Omega}, \tag{13}$$

where $\gamma_0 \boldsymbol{e}_u = \boldsymbol{e}_u|_{\partial\Omega}$ is the Dirichlet trace, $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon \boldsymbol{n}|_{\partial\Omega}$ is the normal trace ($\boldsymbol{n}$ is the outward normal vector), $\gamma_{\perp\perp,1}\boldsymbol{E}_\kappa = -\boldsymbol{n}\cdot\operatorname{Div}\boldsymbol{E}_\kappa - \partial_{\boldsymbol{s}}(\boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{s})|_{\partial\Omega}$ is the effective shear force at the boundary ($\boldsymbol{s}$ is the tangent versor at the boundary), $\gamma_1 \boldsymbol{e}_w = \partial_{\boldsymbol{n}} \boldsymbol{e}_w|_{\partial\Omega}$ is the normal derivative trace and $\gamma_{\perp\perp}\boldsymbol{E}_\kappa = \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n}$ is the normal to normal trace. The boundary conditions are consistent with the ones assumed in Puel and Tucsnak (1996) for deriving a global existence result for this model.

## 4. CONCLUSION

## ACKNOWLEDGEMENTS

The authors would like to thank Denis Matignon from ISAE-SUPAERO for the fruitful and insightful discussions.

## REFERENCES

S. Bilbao, O. Thomas, C. Touzé, and M. Ducceschi. Conservative numerical methods for the full von kármán plate equations. *Numerical Methods for Partial Differential Equations*, 31(6):1948–1970, 2015. doi: 10.1002/num.21974.

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-Hamiltonian formulation and symplectic discretization of plate models Part II: Kirchhoff model for thin plates. *Applied Mathematical Modelling*, 75:961–981, 2019a.

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-Hamiltonian formulation and symplectic discretization of plate models Part I: Mindlin model for thick plates. *Applied Mathematical Modelling*, 75:940–960, 2019b.

J. P. Puel and M. Tucsnak. Global existence for the full von kármán system. *Applied Mathematics and Optimization*, 34(2):139–160, Sep 1996. ISSN 1432-0606. doi: 10.1007/BF01182621.