# A port-Hamiltonian formulation for the full von-Kármán plate model

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# Outline

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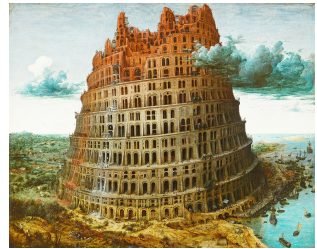
# A unified language for multiphysics in engineering

The port-Hamiltonian (pH) paradigm provides a language to understand multiphysics:

- **Physics** is at the core: pH systems are **passive** with respect to the **energy storage function**.
- The **topological** and **metrical** structure of the equations is clearly separated (mimetic discretization).
- PH systems are **closed under interconnection**.



# Finite dimensional pH systems

**A theory still under developement**

There is **not a unique definition** of pH systems, even in finite dimension.

**Definition (Finite dimensional pH system)**

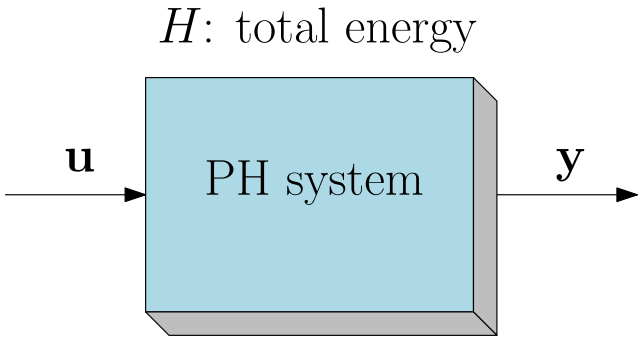
The following time-invariant dynamical system is a pH system

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{B}^\top \mathbf{x}.$$

$\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$ the input and output and

- $\mathbf{J}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})^\top \in \mathbb{R}^{n\times n}$ the interconnection operator
- $\mathbf{B} \in \mathbb{R}^{n\times m}$ the control operator.
- $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{M} > 0$, the Hamiltonian.

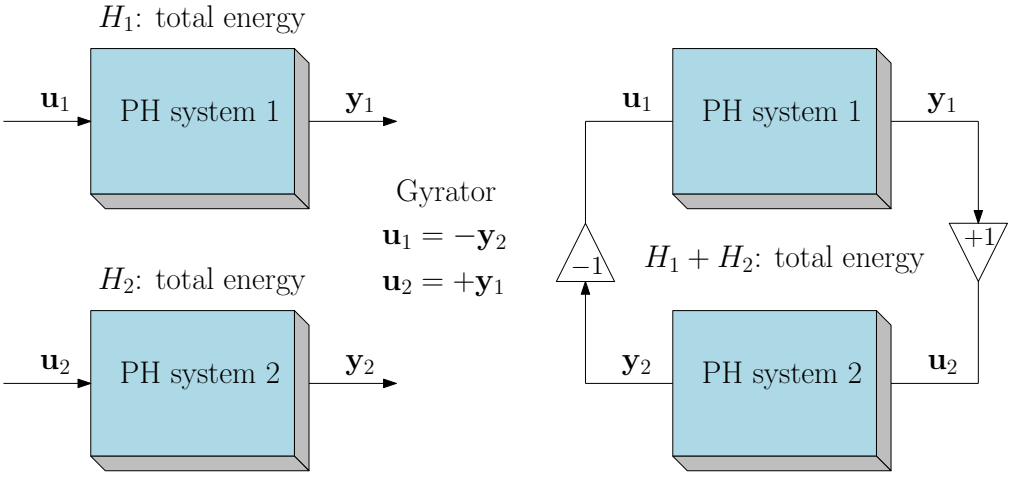
# Finite dimensional pH systems



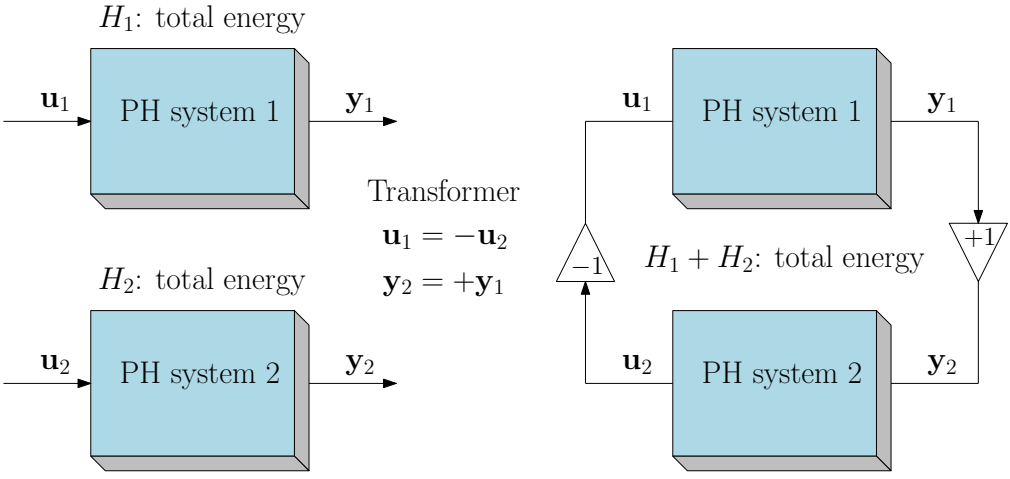

Lossless: $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

# Interconnection of pH systems



# Interconnection of pH systems



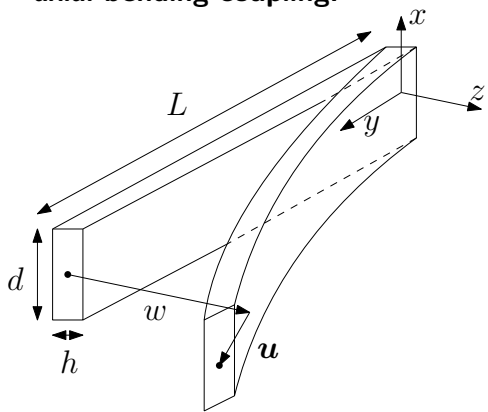
# Outline

# Linear vs Von-Kármán plate theory

Geometrical non-linearities allow describing bifurcations (i.e. buckling).

# The von-Kármán assumption

Second-order approximation of geometrically exact beam/plate theory **capturing the axial bending coupling.**




**Basic geometric assumptions**

- Out of plane deflection comparable to the thickness: $w/h = \mathcal{O}(1)$.
- The squares of the in-plane stretching terms are negligible compared to the square of the rotations.

# Linear isotropic plates ($\Omega \subset \mathbb{R}^2$)

The axial and bending behavior are uncoupled if $w/h \ll 1$:

**Membrane displacement (2D elastodynamics)**

$$\begin{aligned} \rho h \partial_{tt} \boldsymbol{u} &= \operatorname{Div} \boldsymbol{N}, \\ \boldsymbol{N} &= D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m), \\ \boldsymbol{\varepsilon}_m &= \operatorname{Sym}(\nabla \boldsymbol{u}) = \operatorname{Grad} \boldsymbol{u} \end{aligned}$$

Total membrane energy

$$H_m = \frac{1}{2} \int_\Omega \rho h ||\partial_t \boldsymbol{u}||^2 + D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m) : \boldsymbol{N} \, \mathrm{d}\Omega.$$

**Vertical displacement (Kirchhoff plate)**

$$\begin{aligned} \rho h \partial_{tt} w &= -\operatorname{div} \operatorname{Div} \boldsymbol{M}, \\ \boldsymbol{M} &= D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}), \\ \boldsymbol{\kappa} &= \operatorname{Hess} w = \operatorname{Grad} \operatorname{grad} w. \end{aligned}$$

Total bending energy

$$H_b = \frac{1}{2} \int_\Omega \rho h (\partial_t w)^2 + D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}) : \boldsymbol{M} \, \mathrm{d}\Omega.$$

The linear mapping $\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A}) \mathbf{1} + (1-\nu) \boldsymbol{A}$ is positive and preserves symmetry.

# Von-Kármán plates

Decomposition strain field

$$\boldsymbol{\varepsilon} = \underbrace{\operatorname{Grad} \boldsymbol{u}}_{\text{Linear membrane def.}} + \underbrace{1/2 \operatorname{grad} w \otimes \operatorname{grad} w}_{\text{Quadratic membrane def.}} - z \underbrace{\operatorname{Hess} w}_{\text{Linear bending def.}} = \boldsymbol{\varepsilon}_m - z\boldsymbol{\kappa}.$$

## Von-Kármán plate Dynamics

$$\begin{aligned} \rho h \partial_{tt} u &= \operatorname{Div} \ \boldsymbol{N}, \\ \rho h \partial_{tt} w &= -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}(\boldsymbol{N} \operatorname{grad} w), \end{aligned}$$

Total energy

$$H = \frac{1}{2} \int_\Omega \rho h \{ ||\partial_t \boldsymbol{u}||^2 + ||\partial_t w||^2 \} + D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m) : \boldsymbol{N} + D_b \boldsymbol{\Phi}(\kappa) : \boldsymbol{M} \, \mathrm{d}\Omega$$

# Port-Hamiltonian Von-Kármán plates

## Energy variables

The Hamiltonian functional is quadratic in the following variables

$\boldsymbol{\alpha}_u = \rho h \partial_t \boldsymbol{u}$, Axial momentum, $\alpha_w = \rho h \partial_t w$, Bending momentum,

$\boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}_m$, Membrane strain, $\boldsymbol{A}_\kappa = \boldsymbol{\kappa}$, Curvature

## Co-energy variables

The variational derivative of the Hamiltonian gives the co-energy variables

$$\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad e_w := \delta_{\alpha_w} H = \dot{w},$$
$$\boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = D_m \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon), \qquad \boldsymbol{E}_\kappa := \delta_{\boldsymbol{A}_\kappa} H = D_b \boldsymbol{\Phi}(\boldsymbol{A}_\kappa)$$

or more compactly $\boldsymbol{e} := \delta_\alpha H = \mathcal{Q}\boldsymbol{\alpha}$ with

$$\mathcal{Q} = \mathrm{Diag}\left[(\rho h)^{-1},\ D_m \boldsymbol{\Phi},\ (\rho h)^{-1},\ D_b \boldsymbol{\Phi}\right].$$

# The port-Hamiltonian realization

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u\\ \boldsymbol{A}_\varepsilon\\ \alpha_w\\ \boldsymbol{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix}\mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0}\\ \mathrm{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0}\\ \mathbf{0} & \mathcal{C}(w) & \mathbf{0} & -\operatorname{div}\mathrm{Div}\\ \mathbf{0} & \mathbf{0} & \operatorname{Grad}\operatorname{grad} & \mathbf{0}\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}\delta_{\boldsymbol{\alpha}_u}H\\ \delta_{\boldsymbol{A}_\varepsilon}H\\ \delta_{\alpha_w}H\\ \delta_{\boldsymbol{A}_\kappa}H\end{pmatrix}.$$

The operator $\mathcal{J}$ is formally skew-adjoint.
The coupling term reads

$$\begin{aligned}\mathcal{C}(w)(\cdot) : L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}}) &\to L^2(\Omega),\\ \boldsymbol{X} &\to \operatorname{div}(\boldsymbol{X}\operatorname{grad} w),\end{aligned}$$

and its formal adjoint

$$\begin{aligned}\mathcal{C}(w)^*(\cdot) : L^2(\Omega) &\to L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}}),\\ y &\to -\mathrm{Sym}\left[\operatorname{grad}(y)\otimes\operatorname{grad}(w)\right].\end{aligned}$$

# Pure coenergy formulation

## Incorporation of the constitutive equations

Once the $\mathcal{Q}$ operator (matrix) is inverted, the dynamics is expressed :

$$\begin{pmatrix} \rho h \partial_t \boldsymbol{e}_u \\ (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon \\ \rho h \partial_t e_w \\ (D_b \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & \mathcal{C}(w) & 0 & -\operatorname{div}\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathrm{Grad}\,\mathrm{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_u \\ \boldsymbol{E}_\varepsilon \\ e_w \\ \boldsymbol{E}_\kappa \end{pmatrix},$$

where the vertical position is given by

$$w(t) = w_0 + \int_0^t e_w(\tau)\, \mathrm{d}\tau.$$

# Energy rate and boundary conditions

## Proposition

*The energy rate reads*

$$\dot{H} = \langle \gamma_0 \boldsymbol{e}_u \,|\gamma_\perp \boldsymbol{E}_\varepsilon\rangle_{\partial\Omega} + \langle \gamma_0 \boldsymbol{e}_w \,|\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0(\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w)\rangle_{\partial\Omega} + \langle \gamma_1 \boldsymbol{e}_w \,|\gamma_{\perp\perp} \boldsymbol{E}_\kappa\rangle_{\partial\Omega},$$

- $\gamma_0 \boldsymbol{e}_u = \boldsymbol{e}_u|_{\partial\Omega}$ *is the Dirichlet trace;*
- $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon \boldsymbol{n}|_{\partial\Omega}$ *is the normal trace;*
- $\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa = -\boldsymbol{n} \cdot \operatorname{Div} \boldsymbol{E}_\kappa - \partial_{\boldsymbol{s}}(\boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{s})|_{\partial\Omega}$ *is the effective shear force;*
- $\gamma_1 \boldsymbol{e}_w = \partial_{\boldsymbol{n}} \boldsymbol{e}_w|_{\partial\Omega}$ *is the normal derivative trace;*
- $\gamma_{\perp\perp} \boldsymbol{E}_\kappa = \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n}$ *is the normal to normal trace.*

Boundary conditions classification

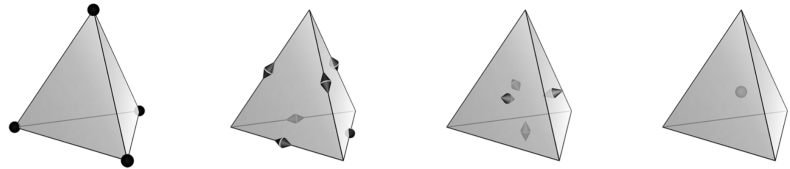
| BCs | Traction | Bending | |
|---|---|---|---|
| Kinematical/Dirichlet | $\gamma_0 \boldsymbol{e}_u$ | $\gamma_0 \boldsymbol{e}_w$ | $\gamma_1 \boldsymbol{e}_w$ |
| Dynamical/Neumann | $\gamma_\perp \boldsymbol{E}_\varepsilon$ | $\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0(\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w)$ | $\gamma_{\perp\perp} \boldsymbol{E}_\kappa$ |

# Outline

# Mixed finite elements for elasticity

Crucial concept to derive stable convergent approximations: **Hilbert complexes**.

$$
\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\text{grad}} & H^{\text{curl}}(\Omega) & \xrightarrow{\text{curl}} & H^{\text{div}}(\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\downarrow{\scriptstyle \Pi^{-,0}_{s,h}} & & \downarrow{\scriptstyle \Pi^{-,1}_{s,h}} & & \downarrow{\scriptstyle \Pi^{-,2}_{s,h}} & & \downarrow{\scriptstyle \Pi^{-,1}_{s,h}} \\
\mathrm{CG}_s(\Omega_h) & \xrightarrow{\text{grad}} & \mathrm{NED}^1_s(\Omega_h) & \xrightarrow{\text{curl}} & \mathrm{RT}_s(\Omega_h) & \xrightarrow{\text{div}} & \mathrm{DG}_{s-1}(\Omega_h)
\end{array}
$$



The Whitney forms $s = 0$ form a subcomplex of the de Rham complex

This framework is well developed for linear problems[1] and open source finite elements libraries implement them. Definitely less for non linear elasticity problems.

[1] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther (2006). "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numerica* 15, pp. 1–155.

# Weak formulation for 2D Elastodynamics

## Grad primal formulation

Suppose the traction (Neumann) boundary condition $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{g}_N$.
Find $\boldsymbol{e}_u \in H^1(\Omega;\mathbb{R}^2)$, $\boldsymbol{E}_\varepsilon \in H(\operatorname{rot}\operatorname{rot},\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}})$
such that $\forall \boldsymbol{\psi}_u \in H^1(\Omega;\mathbb{R}^2)$, $\forall \boldsymbol{\Psi}_\varepsilon \in H(\operatorname{rot}\operatorname{rot},\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}})$ the following holds

$$(\boldsymbol{\psi}_u,\, \rho h \partial_t \boldsymbol{e}_u)_\Omega = -(\operatorname{Grad}\boldsymbol{\psi}_u,\, \boldsymbol{e}_u)_\Omega + \langle \gamma_0 \boldsymbol{\psi}_u \,|\boldsymbol{g}_N\rangle_{\partial\Omega},$$
$$(\boldsymbol{\Psi}_\varepsilon,\, (D_m\boldsymbol{\Phi})^{-1}\partial_t \boldsymbol{E}_\varepsilon)_\Omega = (\boldsymbol{\Psi}_\varepsilon,\, \operatorname{Grad}\boldsymbol{e}_u)_\Omega.$$

This formulation is related to the strain elasticity complex in 2D

$$H^1(\Omega;\mathbb{R}^2) \xrightarrow{\operatorname{Grad}} H(\operatorname{rot}\operatorname{rot},\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}}) \xrightarrow{\operatorname{rot}\operatorname{rot}} L^2(\Omega).$$

# Weak formulation for 2D Elastodynamics

## Div dual formulation

Suppose the kinematic (Dirichlet) boundary condition $\gamma_0 \boldsymbol{e}_u = \boldsymbol{g}_D$.
Find $\boldsymbol{e}_u \in L^2(\Omega;\mathbb{R}^2)$, $\boldsymbol{E}_\varepsilon \in H(\mathrm{Div},\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}})$
such that $\forall \boldsymbol{\psi}_u \in L^2(\Omega;\mathbb{R}^2)$, $\forall \boldsymbol{\Psi}_\varepsilon \in H(\mathrm{Div},\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}})$ the following holds

$$(\boldsymbol{\psi}_u,\, \rho h \partial_t \boldsymbol{e}_u)_\Omega = (\boldsymbol{\psi}_u,\, \mathrm{Div}\, \boldsymbol{E}_\varepsilon)_\Omega,$$
$$(\boldsymbol{\Psi}_\varepsilon,\, (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon)_\Omega = -(\mathrm{Div}\, \boldsymbol{\Psi}_\varepsilon,\, \boldsymbol{e}_u)_\Omega + \langle \gamma_\perp \boldsymbol{E}_\varepsilon \,|\boldsymbol{g}_D\rangle_{\partial\Omega}.$$

This formulation is related to the Elasticity complex in 2D

$$H^2(\Omega) \xrightarrow{\mathrm{curl}\,\mathrm{curl}} H(\mathrm{Div},\Omega;\mathbb{S}) \xrightarrow{\mathrm{Div}} L^2(\Omega;\mathbb{R}^2).$$

# Finite element complexes

Semi discretization in space is obtained using finite elements forming a **subcomplex of original Hilbert complex**. But the theory is still under development[2] and no open source software.

So heuristic choice of the finite elements.

[2] Long Chen and Xuehai Huang (2022). "Finite Element Complexes in Two Dimensions". In: *arXiv preprint arXiv:2206.00851*.

# Finite element choice and final system

For the proposed weak formulation, the FE spaces become

$$\boldsymbol{e}_u^h \in \mathrm{CG}_{2s-1}, \qquad \boldsymbol{E}_\varepsilon^h \in \mathrm{DG}_{2s-2}, \qquad e_w^h \in \mathrm{CG}_s, \qquad \boldsymbol{E}_\kappa^h \in \mathrm{HHJ}_{s-1}, \quad s \geq 1.$$

$$\begin{aligned}
(\boldsymbol{\psi}_u^h,\, \rho h \partial_t \boldsymbol{e}_u^h)_\Omega &= -(\mathrm{Grad}\, \boldsymbol{\psi}_u^h,\, \boldsymbol{e}_u^h)_\Omega, \\
(\boldsymbol{\Psi}_\varepsilon^h,\, (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon^h)_\Omega &= (\boldsymbol{\Psi}_\varepsilon^h,\, \mathrm{Grad}\, \boldsymbol{e}_u^h)_\Omega + (\boldsymbol{\Psi}_\varepsilon^h,\, \mathrm{Sym}\{\mathrm{grad}\, e_w^h \otimes \mathrm{grad}\, w^h\})_\Omega, \\
(\psi_w^h,\, \rho b \partial_t e_w^h)_\Omega &= +d_h(\psi_w^h,\ \boldsymbol{E}_\kappa^h) - (\mathrm{Sym}\{\mathrm{grad}\, \psi_w^h \otimes \mathrm{grad}\, w^h\},\, \boldsymbol{E}_\varepsilon^h)_\Omega, \\
(\boldsymbol{\Psi}_\kappa^h,\, (D_b \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\kappa^h)_\Omega &= -d_h(e_w^h,\ \boldsymbol{\Psi}_\kappa^h).
\end{aligned}$$

## Finite dimensional system (Galerkin projection)

$$\begin{aligned}
\mathbf{M}\dot{\mathbf{e}} &= \mathbf{J}(\mathbf{w})\mathbf{e} + \mathbf{B}\mathbf{u}, \\
\dot{\mathbf{w}} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{e}, \\
\mathbf{y} &= \mathbf{B}^\top \mathbf{e}.
\end{aligned}$$

# Outline

# Manufactured solution

Consider $\Omega = [0,1] \times [0,1]$. The following manufactured solution is considered

$$\boldsymbol{u}^{\text{ex}} = \begin{pmatrix} x^4(1-x^4)\sin^2(\pi y) \\ \sin^2(\pi x) y^4 (1-y^4) \end{pmatrix} \sin(2\pi t), \qquad w^{\text{ex}} = \sin(\pi x)\sin(\pi y)\sin(2\pi t),$$

together with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad w|_{\partial\Omega} = 0, \quad \boldsymbol{n}^\top \boldsymbol{M}\boldsymbol{n}|_{\partial\Omega} = 0.$$

A Crank-Nicholson scheme is used for time integration.

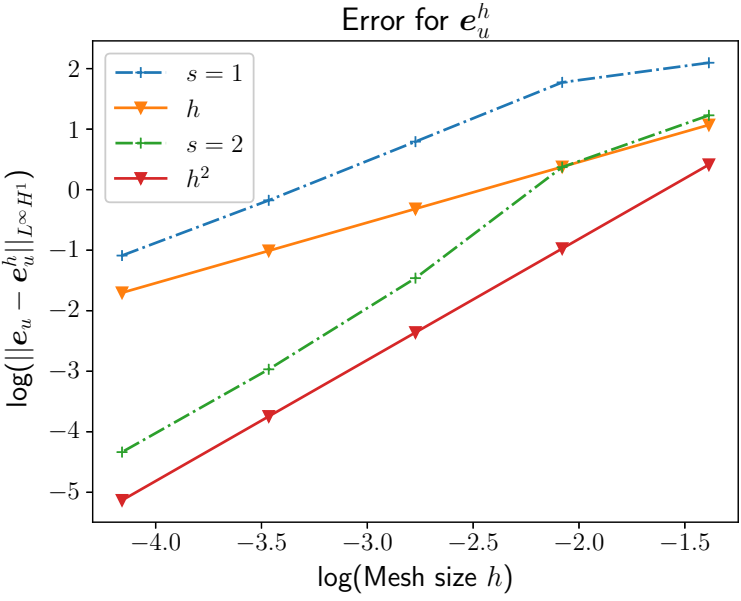
## Convergence measure

The discrete time-space norm $L^\infty_{\Delta t}(\mathcal{X})(\mathcal{X} = H^1 \text{or } L^2)$ is used to measure convergence

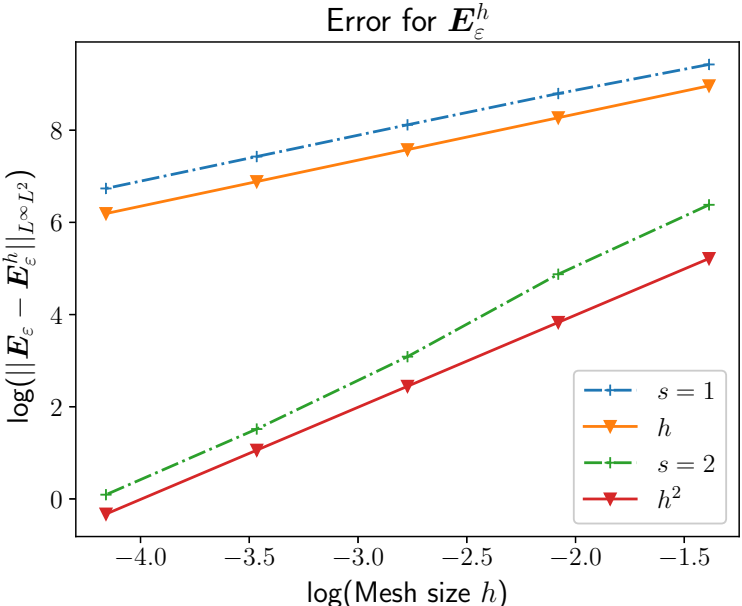
$$||\cdot||_{L^\infty(\mathcal{X})} \approx ||\cdot||_{L^\infty_{\Delta t}(\mathcal{X})} = \max_{t \in t_i} ||\cdot||_{\mathcal{X}},$$

where $t_i$ are the discrete simulation instants.

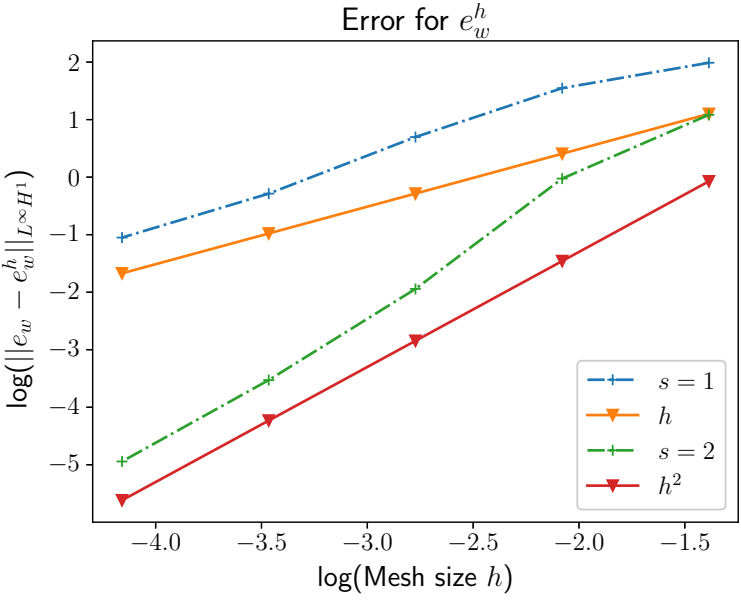
# Results


Error for $e_u^h$
$s = 1$
$h$
$s = 2$
$h^2$
$\log(||e_u - e_u^h||_{L^\infty H^1})$
2
1
0
−1
−2
−3
−4
−5
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5
log(Mesh size $h$)

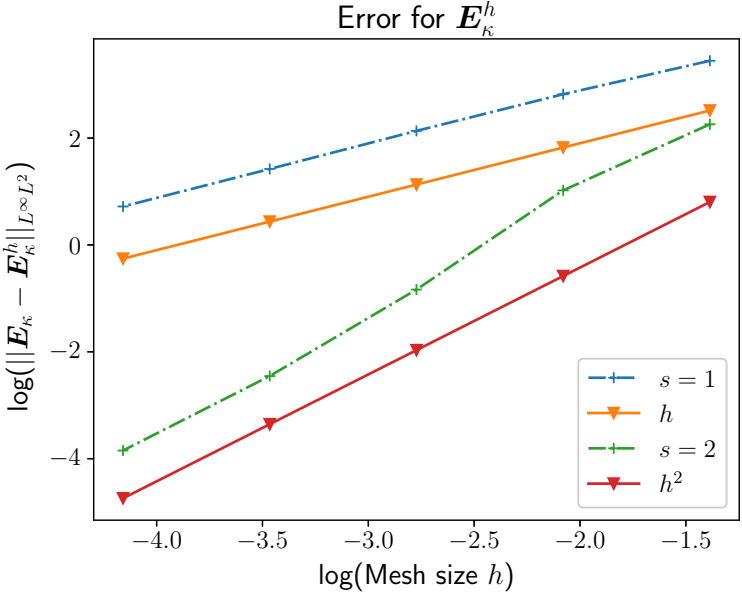
# Results


Error for $\boldsymbol{E}_\varepsilon^h$
$\log(||\boldsymbol{E}_\varepsilon - \boldsymbol{E}_\varepsilon^h||_{L^\infty L^2})$
log(Mesh size $h$)
8
6
4
2
0
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5
$s = 1$
$h$
$s = 2$
$h^2$

# Results


Error for $e_w^h$
$\log(||e_w - e_w^h||_{L^\infty H^1})$
log(Mesh size $h$)
$s = 1$
$h$
$s = 2$
$h^2$
2
1
0
−1
−2
−3
−4
−5
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5

# Results


Error for $\boldsymbol{E}_\kappa^h$
$\log(||\boldsymbol{E}_\kappa - \boldsymbol{E}_\kappa^h||_{L^\infty L^2})$
2
0
−2
−4
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5
log(Mesh size $h$)
$s = 1$
$h$
$s = 2$
$h^2$

# Conclusion and Outlook

- First step into pH non linear mechanics. The **geometrical non linearities** belong to the **interconnection operator**.
- Can be used to study more complex phenomena. The discretization method guarantees **exact discrete energy conservation** when **symplectic time integration** is used.
- **System theory perspective**: incorporates **interactions** with the environment by means of the **boundary conditions**.

# References I


Arnold, Douglas N., Richard S. Falk, and Ragnar Winther (2006). "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numerica* 15, pp. 1–155.

Chen, Long and Xuehai Huang (2022). "Finite Element Complexes in Two Dimensions". In: *arXiv preprint arXiv:2206.00851*.

# Mixed finite element construction: Kirchhoff plate

Find $(e_w, \boldsymbol{E}_\kappa) \in \mathrm{CG}_s \times \mathrm{HHJ}_{s-1}$ such that

$$(\psi_w,\ \rho b \partial_t e_w)_\Omega = +d_h(\psi_w,\ \boldsymbol{E}_\kappa), \qquad \forall\ \psi_w \in \mathrm{CG}_s,$$
$$(\boldsymbol{\Psi}_\kappa,\ (D_b\boldsymbol{\Phi})^{-1}\partial_t \boldsymbol{E}_\kappa)_\Omega = -d_h(e_w,\ \boldsymbol{\Psi}_\kappa), \qquad \forall\ \boldsymbol{\Psi}_\kappa \in \mathrm{HHJ}_{s-1}.$$

This method is non-conforming and includes inter-cells terms

$$d_h(v_w,\ \boldsymbol{E}_\kappa) := -\sum_{T\in\mathcal{T}_h} (\operatorname{Hess} v_w,\ \boldsymbol{E}_\kappa)_\Omega + \sum_{E\in\mathcal{E}_h} (\llbracket \partial_n v_w \rrbracket,\ \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n})_\Omega.$$

The $\operatorname{div}\operatorname{Div}$ distributional complex in 2D and the HHJ finite element complex:

$$\begin{array}{ccccc}
H^1(\Omega;\mathbb{R}^2) & \xrightarrow{\text{Sym curl}} & H^{-1}(\operatorname{div}\operatorname{Div},\Omega;\mathbb{R}^{2\times 2}_{\mathsf{sym}}) & \xrightarrow{\operatorname{div}\operatorname{Div}} & L^2(\Omega) \\
\downarrow{\scriptstyle \boldsymbol{I}_h} & & \downarrow{\scriptstyle \boldsymbol{\Pi}_h} & & \downarrow{\scriptstyle Q_h} \\
\mathrm{CG}_s & \xrightarrow{\text{Sym curl}} & \mathrm{HHJ}_{s-1} & \xrightarrow{\operatorname{div}\operatorname{Div}_h} & \mathrm{CG}_s,
\end{array}$$