

A port-Hamiltonian formulation for the full von-Kármán plate model

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Summary. In this contribution, a port-Hamiltonian reformulation of the full von-Kármán dynamical model for geometrically non-linear plates is detailed, including the collocated boundary control and observation. Starting from the canonical equations, a set of variables is chosen so as to make the total energy quadratic. The model, reformulated in these variables, highlights a port-Hamiltonian structure ruled by a state-modulated interconnection operator.

Classical model

The classical full von-Kármán dynamical model is presented in Bilbao et al. [2015]. The problem, defined on an open connected set $\Omega \subset \mathbb{R}^2$, takes the dimensionless form

$$\begin{aligned} \ddot{\mathbf{u}} &= \text{Div } \mathbf{N}, & \mathbf{N} &= \Phi(\varepsilon), & \varepsilon &= \text{Grad } \mathbf{u} + 1/2 \text{grad } w \otimes \text{grad } w, \\ \ddot{w} &= -\text{div Div } \mathbf{M} + \text{div}(\mathbf{N} \text{grad } w), & \mathbf{M} &= \Phi(\kappa), & \kappa &= \text{Grad grad } w, \end{aligned} \quad (1)$$

where $\mathbf{u} \in \mathbb{R}^2$ is the in-plane displacement, w is the vertical displacement, ε is the in-plane strain tensor, κ is the curvature tensor, \mathbf{N} is the in-plane stress resultant and \mathbf{M} is the bending stress resultant. The notation $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^\top$ denotes the dyadic product of two vectors. The div operator is the divergence of a vector field, and grad the gradient of a scalar field. The operator $\text{Grad} = \frac{1}{2}(\nabla + \nabla^\top)$ designates the symmetric part of the gradient (i. e. the deformation gradient in continuum mechanics). For a tensor field $\mathbf{U} : \Omega \rightarrow \mathbb{R}^{2 \times 2}$, with components U_{ij} , the divergence $\text{Div}(\mathbf{U})$ is a vector, defined column-wise as

$$\text{Div}(\mathbf{U}) := \sum_{i=1}^2 \partial_{x_i} U_{ij}, \quad \forall j = \{1, 2\}.$$

The linear tensor mapping Φ is positive and preserves symmetry:

$$\Phi(\mathbf{A}) = \nu \text{Tr}(\mathbf{A})\mathbf{1} + (1 - \nu)\mathbf{A}, \quad \mathbf{A} = \mathbf{A}^\top \implies \Phi(\mathbf{A}) = \Phi(\mathbf{A})^\top, \quad \text{where} \quad \mathbf{1} = \text{Diag}(1, 1).$$

The total energy of the model (Hamiltonian functional)

$$H = \frac{1}{2} \int_{\Omega} \left\{ \|\dot{\mathbf{u}}\|^2 + \dot{w}^2 + \mathbf{N} : \varepsilon + \mathbf{M} : \kappa \right\} d\Omega, \quad \text{where} \quad \mathbf{A} : \mathbf{B} = \text{Tr}(\mathbf{A}^\top \mathbf{B}) \quad (2)$$

consists of the kinetic energy and both membrane and bending deformation energies. This model proves conservative, see Bilbao et al. [2015]. Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

The equivalent port-Hamiltonian system (pHs)

To find a suitable port-Hamiltonian system, we first select a set of new energy variables to make the Hamiltonian functional quadratic. The selection is the same as for both the linear plate problems in Brugnoli et al. [2019a,b]:

$$\alpha_u = \dot{\mathbf{u}}, \quad \alpha_w = \dot{w}, \quad \mathbf{A}_\varepsilon = \varepsilon, \quad \mathbf{A}_\kappa = \kappa. \quad (3)$$

The energy is quadratic in these variables

$$H = \frac{1}{2} \int_{\Omega} \left\{ \|\alpha_u\|^2 + \alpha_w^2 + \Phi(\mathbf{A}_\varepsilon) : \mathbf{A}_\varepsilon + \Phi(\mathbf{A}_\kappa) : \mathbf{A}_\kappa \right\}. \quad (4)$$

By computing the variational derivative of the Hamiltonian, one obtains the so-called co-energy variables:

$$e_u := \delta_{\alpha_u} H = \dot{\mathbf{u}}, \quad e_w := \delta_{\alpha_w} H = \dot{w}, \quad \mathbf{E}_\varepsilon := \delta_{\mathbf{A}_\varepsilon} H = \Phi(\mathbf{A}_\varepsilon), \quad \mathbf{E}_\kappa := \delta_{\mathbf{A}_\kappa} H = \Phi(\mathbf{A}_\kappa). \quad (5)$$

Before stating the final formulation, consider the operator $\mathcal{C}(w)(\cdot) : L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2}) \rightarrow L^2(\Omega)$ acting on symmetric tensors

$$\mathcal{C}(w)(\mathbf{T}) = \text{div}(\mathbf{T} \text{grad } w). \quad (6)$$

Proposition 1 *The formal adjoint of the $\mathcal{C}(w)(\cdot)$ is given by*

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2} [\text{grad}(\cdot) \otimes \text{grad}(w) + \text{grad}(w) \otimes \text{grad}(\cdot)]. \quad (7)$$

Proof 1 Consider a smooth scalar field $v \in C_0^\infty(\Omega)$ and a smooth symmetric tensor field $\mathbf{U} \in C_0^\infty(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})$ with compact support. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation

$$\langle v, \mathcal{C}(w)(\mathbf{U}) \rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)(v)^*, \mathbf{U} \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})}. \quad (8)$$

The proof follows from the computation

$$\begin{aligned} \langle v, \mathcal{C}(w)(\mathbf{U}) \rangle_{L^2(\Omega)} &= \langle v, \operatorname{div}(\mathbf{U} \operatorname{grad} w) \rangle_{L^2(\Omega)}, & \text{Integration by parts,} \\ &= \langle -\operatorname{grad} v, \mathbf{U} \operatorname{grad} w \rangle_{L^2(\Omega, \mathbb{R}^2)}, & \text{Dyadic product properties,} \\ &= \langle -\operatorname{grad} v \otimes \operatorname{grad} w, \mathbf{U} \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})}, & \text{Symmetry of } \mathbf{U}, \\ &= \langle -1/2(\operatorname{grad} v \otimes \operatorname{grad} w + \operatorname{grad} w \otimes \operatorname{grad} v), \mathbf{U} \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})}. \end{aligned} \quad (9)$$

This means

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2} [\operatorname{grad}(\cdot) \otimes \operatorname{grad}(w) + \operatorname{grad}(w) \otimes \operatorname{grad}(\cdot)], \quad (10)$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_u \\ \mathbf{A}_\varepsilon \\ \alpha_w \\ \mathbf{A}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \operatorname{Div} & \mathbf{0} & \mathbf{0} \\ \operatorname{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & \mathcal{C}(w) & 0 & -\operatorname{div} \operatorname{Div} \\ \mathbf{0} & \mathbf{0} & \operatorname{Grad} \operatorname{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \delta_{\alpha_u} H \\ \delta_{\mathbf{A}_\varepsilon} H \\ \delta_{\alpha_w} H \\ \delta_{\mathbf{A}_\kappa} H \end{pmatrix}, \quad (11)$$

The second line of system (11) represents the time derivative of the membrane strain tensor. To close the system, variable w has to be accessible. For this reason, its dynamics has to be included. The augmented system reads

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_u \\ \mathbf{A}_\varepsilon \\ w \\ \alpha_w \\ \mathbf{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \operatorname{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \operatorname{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div} \operatorname{Div} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \operatorname{Grad} \operatorname{grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \delta_{\alpha_u} H \\ \delta_{\mathbf{A}_\varepsilon} H \\ \delta_w H \\ \delta_{\alpha_w} H \\ \delta_{\mathbf{A}_\kappa} H \end{pmatrix}. \quad (12)$$

Given the results in Brugnoli et al. [2019a,b] and Proposition 1, the operator \mathcal{J} is formally skew-adjoint. If only the kinetic and deformation energies are considered, it holds $\delta_w H = 0$. In general this terms allows accommodating other potentials, for example the gravitational one. Suitable boundary variables are then obtained considering the power balance

$$\dot{H} = \langle \gamma_0 \mathbf{e}_u, \gamma_\perp \mathbf{E}_\varepsilon \rangle_{\partial\Omega} + \langle \gamma_0 \mathbf{e}_w, \gamma_{\perp\perp,1} \mathbf{E}_\kappa + \gamma_0 (\mathbf{E}_\varepsilon \mathbf{n} \cdot \operatorname{grad} w) \rangle_{\partial\Omega} + \langle \gamma_1 \mathbf{e}_w, \gamma_{\perp\perp} \mathbf{E}_\kappa \rangle_{\partial\Omega}, \quad (13)$$

where $\gamma_0 \mathbf{e}_u = \mathbf{e}_u|_{\partial\Omega}$ is the Dirichlet trace, $\gamma_\perp \mathbf{E}_\varepsilon = \mathbf{E}_\varepsilon \mathbf{n}|_{\partial\Omega}$ is the normal trace (\mathbf{n} is the outward normal vector), $\gamma_{\perp\perp,1} \mathbf{E}_\kappa = -\mathbf{n} \cdot \operatorname{Div} \mathbf{E}_\kappa - \partial_s (\mathbf{n}^\top \mathbf{E}_\kappa \mathbf{s})|_{\partial\Omega}$ is the effective shear force at the boundary (\mathbf{s} is the tangent versor at the boundary), $\gamma_1 \mathbf{e}_w = \partial_n \mathbf{e}_w|_{\partial\Omega}$ is the normal derivative trace and $\gamma_{\perp\perp} \mathbf{E}_\kappa = \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n}$ is the normal to normal trace. The boundary conditions are consistent with the ones assumed in Puel and Tucsnak [1996] for deriving a global existence result for this model.

Conclusions

We have presented a pHs formulation of the full von-Kármán model. The dynamics of the system exhibits a state modulated interconnection operator, while the energy remains quadratic in the chosen variables. Of particular interest is the discretization of such a model for simulation and control purposes. The Partitioned Finite Element Method (PFEM), an extension of mixed finite elements to pHs, seems to be particularly suitable to achieve a structure-preserving discretization of this model, as in Cardoso-Ribeiro et al. [2020] for the 2D Shallow Water Equation, which exhibits the same kind of polynomial nonlinearity.

References

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