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Port-Hamiltonian Flexible Multibody Dynamics

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Abstract Insert your abstract here. Include keywords, PACS and mathematical subject classification numbers as needed.

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1 Introduction

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2 Flexible dynamics

The coupled ODE-PDE system representing the motion of a single flexible body are here recalled. The model for the classical equations derived using the principle of virtual work can be found in [1]. The small difference with respect to the derivation therein is that the equation for the translation is now written in the body frame. The dynamics is computed at a generic point P, that is not necessarily the center of mass. The body is assumed to be floating, so that the external traction τ is known. Consider an open connected set $\Omega \in \mathbb{R}^d$

$$\begin{aligned} m(\dot{v}_P + [\omega_P]_\times v_P) + [s_u]_\times^\top \dot{\omega}_P + \int_\Omega \rho \ddot{u}_f \, \mathrm{d}\Omega = \\ - [\omega_P]_\times [\omega_P]_\times s_u - \int_\Omega 2\rho [\omega_P]_\times \dot{u}_f \, \mathrm{d}\Omega + \int_\Omega \beta \, \mathrm{d}\Omega + \int_{\partial\Omega} \tau \, \mathrm{d}\Gamma \end{aligned} \tag{1}$$

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$$
\begin{aligned}
&[s_u]_\times(\dot{v}_P + [\omega_P]_\times v_P) + J_u\dot{\omega}_P + \int_\Omega \rho[x+u_f]_\times \ddot{u}_f \,\mathrm{d}\Omega - [\omega_P]_\times J_u\omega_P = \\
&-\int_\Omega 2\rho[x+u_f]_\times[\omega_P]_\times\dot{u}_f \,\mathrm{d}\Omega + \int_\Omega [x+u_f]_\times\beta \,\mathrm{d}\Omega + \int_{\partial\Omega}[x+u_f]_\times\tau \,\mathrm{d}\Gamma
\end{aligned}
\tag{2}
$$

$$
\rho(\dot{v}_P+[\omega_P]_\times v_P)+\rho([\dot{\omega}_P]_\times+[\omega_P]_\times[\omega_P]_\times)(x+u_f)+\rho(2[\omega_P]_\times\dot{u}_f+\ddot{u}_f) = \mathrm{Div}\,\Sigma+\beta,
\tag{3}
$$

where $m = \int_\Omega \rho \,\mathrm{d}\Omega$, $s_u = \int_\Omega \rho(x+u_f)\,\mathrm{d}\Omega$, $J_u = -\int_\Omega \rho[x+u_f]_\times[x+u_f]_\times \,\mathrm{d}\Omega$. Considering $v_f := \dot{u}_f$, $\dot{v}_f = \ddot{u}_f + [\omega_P]_\times\dot{u}_f$ and applying the Jacobi identity Eqs. (1), (2), (3) can be rewritten equivalenty as

Rigid translation

$$
\begin{aligned}
&m\dot{v}_P + [s_u]_\times^\top\dot{\omega}_P + \int_\Omega \rho\dot{v}_f \,\mathrm{d}\Omega = \\
&\left[mv_p + [s_u]_\times^\top\omega_P + \int_\Omega \rho v_f \,\mathrm{d}\Omega\right]_\times \omega_P + \int_\Omega \beta \,\mathrm{d}\Omega + \int_{\partial\Omega}\tau \,\mathrm{d}\Gamma,
\end{aligned}
\tag{4}
$$

Rigid rotation

$$
\begin{aligned}
&[s_u]_\times\dot{v}_P + J_u\dot{\omega}_P + \int_\Omega \rho[x+u_f]_\times\dot{v}_f \,\mathrm{d}\Omega = \\
&\left[[s_u]_\times^\top\omega_P + \int_\Omega \rho v_f \,\mathrm{d}\Omega\right]_\times v_P + \left[[s_u]_\times v_P + J_u\omega_P + \int_\Omega \rho[x+u_f]_\times\dot{v}_f \,\mathrm{d}\Omega\right]_\times \omega_P + \\
&\int_\Omega \left[\rho v_P + \rho[x+u_f]_\times^\top\omega_P + \rho v_f\right]_\times v_f \,\mathrm{d}\Omega + \int_\Omega [x+u_f]_\times\beta \,\mathrm{d}\Omega + \int_{\partial\Omega}[x+u_f]_\times\tau \,\mathrm{d}\Gamma
\end{aligned}
\tag{5}
$$

Flexibility PDE

$$
\begin{aligned}
&\rho\dot{v}_P + \rho[x+u_f]_\times^\top\dot{\omega}_P + \rho\dot{v}_f = \\
&\left[\rho v_P + [x+u_f]_\times^\top\omega_P + \rho v_f\right]_\times \omega_P + \mathrm{Div}\,\Sigma + \beta
\end{aligned}
\tag{6}
$$

3 PH flexible dynamics

In this section the flexible dynamics of a floating body is written as a coupled system of ODEs and PDEs in pH form. First of only the kinetic and deformation energy are considered for sake of simplicity. Then a generic potential is introduced, so that the generalized coordinates are included in the formulation.

3.1 Formulation without generalized coordinates

Consider the total energy (Hamiltonian), given by the sum of kinetic and deformation energy (linear elasticity):

$$H = H_{\text{kin}} + H_{\text{def}} = \frac{1}{2}\int_\Omega \rho||v_P + [\omega_P]_\times(x+u_f) + v_f||^2 + \Sigma : \varepsilon \,\mathrm{d}\Omega, \tag{7}$$

where Σ is the Cauchy stress tensor and ε is the infinitesimal stress tensor. From linear elasticity theory it is well known that $\varepsilon = \mathrm{Grad}(u)$, where $\mathrm{Grad} = \frac{1}{2}[\nabla + \nabla^\top]$ and $\Sigma = \mathbb{D}\varepsilon$, where $\mathcal{D}$ is the stiffness tensor. The inner product $A : B = \mathrm{Tr}(AB^T)$ is the tensor contraction.
The momenta (usually called energy variables in the pH framework) are then computed by derivation of the Hmailtonian. As the variables belong to finite- and infinite-dimensional spaces the derivative is either a classical gradient either a variational derivative:

$$\begin{aligned}
p_t &:= \frac{\partial H}{\partial v_P} = m v_P + [s_u]_\times^\top \omega_P + \int_\Omega \rho v_f \,\mathrm{d}\Omega,\\
p_r &:= \frac{\partial H}{\partial \omega_P} = [s_u]_\times v_P + J_u \omega_P + \int_\Omega \rho[x+u_f]_\times v_f \,\mathrm{d}\Omega,\\
p_f &:= \frac{\delta H}{\delta v_f} = \rho v_P + \rho[x+u_f]_\times^\top \omega_P + \rho v_f,\\
\varepsilon &:= \frac{\delta H}{\delta \Sigma}
\end{aligned} \tag{8}$$

The relation between energy and co-energy variable is then given by

$$\begin{bmatrix} p_t \\ p_r \\ p_f \\ \varepsilon \end{bmatrix} = \underbrace{\begin{bmatrix} m & [s_u]_\times^\top & \mathcal{I}_\rho^\Omega & 0 \\ [s_u]_\times & J_u & \mathcal{I}_{\rho x}^\Omega & 0 \\ (\mathcal{I}_\rho^\Omega)^* & (\mathcal{I}_{\rho x}^\Omega)^* & \rho & 0 \\ 0 & 0 & 0 & \mathcal{D}^{-1} \end{bmatrix}}_{\mathcal{M}\text{: Mass operator}} \begin{bmatrix} v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix}, \tag{9}$$

where the operators are defined as

$$\begin{aligned}
\mathcal{I}_\rho^\Omega &= \int_\Omega \rho(\cdot)\,\mathrm{d}\Omega, & \mathcal{I}_{\rho x}^\Omega &= \int_\Omega \rho[x+u_f]_\times(\cdot)\,\mathrm{d}\Omega,\\
(\mathcal{I}_\rho^\Omega)^* &= \rho, & (\mathcal{I}_{\rho x}^\Omega)^* &= \rho[x+u_f]_\times^\top.
\end{aligned}$$

The superscript $*$ denotes the adjoint operator. The mass operator $\mathcal{M}$ is a self-adjoint, positive operator.
If the dependency of $\mathcal{M}$ on the flexible displacement u_f is neglected then Eqs. (5), (4), (6) can be recast into port Hamiltonian form in co-energy variables

$$\mathcal{M}\frac{d}{dt}\begin{bmatrix} v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & [p_t]_\times & 0 & 0 \\ [p_t]_\times & [p_r]_\times & \mathcal{W}(p_f) & 0 \\ 0 & -\mathcal{W}^*(p_f) & 0 & \mathrm{Div} \\ 0 & 0 & \mathrm{Grad} & 0 \end{bmatrix}}_{\mathcal{J}\text{: Interconnection operator}} \begin{bmatrix} v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix} + \begin{bmatrix} \mathcal{I}^\Omega & \mathcal{I}^\Gamma \\ \mathcal{I}_x^\Omega & \mathcal{I}_x^\Gamma \\ I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \tau \end{bmatrix}, \tag{10}$$

together with the boundary conditions $\Sigma \cdot n|_{\Gamma} = \tau$, where n is the outward normal. The control operators read

$$\mathcal{I}^{\Omega} = \int_{\Omega} (\cdot)\, \mathrm{d}\Omega, \qquad \mathcal{I}_x^{\Omega} = \int_{\Omega} [x + u_f]_{\times} (\cdot)\, \mathrm{d}\Omega,$$
$$\mathcal{I}^{\Gamma} = \int_{\partial\Omega} (\cdot)\, \mathrm{d}\Gamma, \qquad \mathcal{I}_x^{\Gamma} = \int_{\partial\Omega} [x + u_f]_{\times} (\cdot)\, \mathrm{d}\Gamma.$$

The operator $\mathcal{W}(p_f) = \int_{\Omega} [p_f]_{\times} (\cdot)\, \mathrm{d}\Omega$ is given by the composition $\mathcal{W} = \mathcal{I} \circ \mathcal{A}$ where $\mathcal{A}(p_f, \cdot) = [p_f]_{\times}(\cdot)$. Its adjoint will be given by $\mathcal{W}^* = \mathcal{A}^* \circ \mathcal{I}^* = -[p_f]_{\times}(\cdot)$. It is important to highlight that Div and Grad are formally skew-adjoint operators, i.e. for homogeneous boundary conditions $\tau = 0$

$$\int_{\Omega} \Sigma : \mathrm{Grad}(v_f)\, \mathrm{d}\Omega \underbrace{=}_{\text{I.B.P.}} - \int_{\Omega} \mathrm{Div}(\Sigma) \cdot v_f\, \mathrm{d}\Omega,$$
$$\langle \Sigma,\, \mathrm{Grad}(v_f) \rangle_{\mathscr{L}^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})} \underbrace{=}_{\text{I.B.P.}} - \langle \mathrm{Div}(\Sigma),\, v_f \rangle_{\mathscr{L}^2(\Omega, \mathbb{R}^d)},$$

where $\langle , \rangle_{\mathscr{H}}$ denote an inner product over the Hilbert space $\mathscr{H}$.
$\mathscr{L}^2(\Omega, \mathbb{R}^d)$, $\mathscr{L}^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ are the spaces of square integrable vector-valued or symmetric tensors-valued functions in a geometric domain of dimension d.
For this reason operator $\mathcal{J}$ is skew-symmetric $\mathcal{J}^* = -\mathcal{J}$.

Remark 1 If case of vanishing deformation $u_f \equiv 0$ the Newton-Euler equations on $SE(3)$ [2] are retrieved

$$\frac{d}{dt} \begin{bmatrix} p_t \\ p_r \end{bmatrix} = \begin{bmatrix} 0 & [p_t]_{\times} \\ [p_t]_{\times} & [p_r]_{\times} \end{bmatrix} \begin{bmatrix} v_P \\ \omega_P \end{bmatrix} + \begin{bmatrix} f \\ m \end{bmatrix}. \tag{11}$$

This system is again in port Hamiltonian form with total energy equal to the kinetic energy.

3.2 Including the generalized coordinates

For the time being only kinetic and deformation energy were considered. If a generic potential energy contribution has to be considered one must account for the generalized coordinates

- ${}^{i}r_P$: the position of point P in the inertial frame of reference;
- R: the orientation matrix that transforms vectors from the body frame to the inertial frame (other attitude parametrization are possible, here the orientation matrix is considered for ease of presentation);
- u_f the flexible displacement;

In particular following [3] the orientation matrix is converted in a vector by concatenating its rows

$$R_{\mathrm{v}} = \mathrm{vec}(R^{\top}) = [R_x\ R_y\ R_z]^{\top},$$

where R_x, R_y, R_z are the first, second and third row of matrix R. Furthermore the corresponding cross map will be given by

$$[R_{\mathrm{v}}]_\times = \begin{bmatrix} [R_x]_\times \\ [R_y]_\times \\ [R_z]_\times \end{bmatrix} \qquad [R_{\mathrm{v}}]_\times : \mathbb{R}^9 \to \mathbb{R}^{9\times 3}$$

The total energy now includes a potential energy that depends on the generalized coordinates

$$H = H_{\mathrm{kin}} + H_{\mathrm{def}} + H_{\mathrm{pot}}$$

The overall port Hamitonian formulation is then easily constructed

$$\underbrace{\left[\begin{array}{c|c} I & 0 \\ \hline 0 & \mathcal{M} \end{array}\right]}_{\mathcal{E}} \frac{d}{dt} \underbrace{\begin{bmatrix} {}^i r_P \\ R_{\mathrm{v}} \\ u_f \\ \hline v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix}}_{e} = \underbrace{\left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [R_{\mathrm{v}}]_\times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hline -R^\top & 0 & 0 & 0 & [p_t]_\times & 0 & 0 \\ 0 & -[R_{\mathrm{v}}]_\times^\top & 0 & [p_t]_\times & [p_r]_\times & \mathcal{W}(p_f) & 0 \\ 0 & 0 & -I & 0 & -\mathcal{W}^*(p_f) & 0 & \mathrm{Div} \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Grad} & 0 \end{array}\right]}_{\mathcal{J}} \underbrace{\begin{bmatrix} \partial_{r_P} H \\ \partial_{R_{\mathrm{v}}} H \\ \delta_{u_f} H \\ \hline v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix}}_{z}, \tag{12}$$

where the contribution of external forces and torques has been omitted. For example if the gravity potential energy is considered

$$H_{\mathrm{pot}} = \int_\Omega \rho g \, {}^i r_z \, \mathrm{d}\Omega = \int_\Omega \rho g \left[{}^i r_{P,z} + R_z (x + u_f) \right] \, \mathrm{d}\Omega,$$

the co-energy variables are easily obtained

$$\begin{aligned} &\partial_{r_P} H = mg\, e_3, \quad e_i \text{ are the inertia frame versors} \\ &\partial_{R_{\mathrm{v}}} H = [0_{(3,1)},\, 0_{(3,1)},\, \int_\Omega \rho g (x + u_f)^\top \, \mathrm{d}\Omega]^\top \\ &\delta_{u_f} H = \rho g \, R_z^\top \end{aligned}$$

These correspond to the forcing terms due to gravity. System (12) fits into the framework detailed in [4] and extends it as a coupled system of ODEs and PDEs is considered. It can be rewritten compactly as follows

$$\begin{aligned} \mathcal{E}(e)\dot{e} &= \mathcal{J}(e) z + \mathcal{B}_d(e)[u_d,\, u_\partial]^\top, \\ y_d &= \mathcal{B}_d^*(e) z, \\ u_\partial &= \mathcal{B}_\partial z = \Sigma \cdot n|_\Gamma, \\ y_\partial &= \mathcal{C}_\partial z = \partial_t u_f \end{aligned} \tag{13}$$

where $u_d = \beta$, $u_\partial = \tau$. The gradient of the Hamiltonian gives $\partial_e H = \mathcal{E}^* z$. Adopting the same nomenclature as in [4], e contains the state and z contains the effort functions. In this case the operators verify $\mathcal{E} = \mathcal{E}^*$, $\mathcal{J} = -\mathcal{J}^*$. The distributed control operator $\mathcal{B}_d$ is bounded. System written in this form are called port-Hamiltonian differential algebraic systems (pHDAE).

Remark 2 For sake of simplicity the linear elastic case $H_{\mathrm{def}} = \int_\Omega \Sigma : \varepsilon \, \mathrm{d}\Omega$ has been presented. If more complex behaviors have to be accounted for (e.g. geometric stiffening) more complex expression for the deformation energy can be considered.

4 Discretization procedure

A finite-element based technique to obtain a finite dimensional pH system is illustrated. This methodology relies on the results explained in [5] and boils down to three simple steps

1. The system is put into weak form;
2. An integration by parts is applied to highlight the proper boundary control;
3. A Galerkin method is employed to obtain a finite-dimensional system.

4.1 Illustration for the Elastodynamics PDE

To explain the methodology consider the elastodynamics PDE

$$\rho \frac{\partial^2 u}{\partial t^2} - \mathrm{Div}\left(\mathcal{D}\,\mathrm{Grad}(u)\right) = u_d,$$

where a distributed control u_d (a volumetric force) is considered. The total energy is simply given by

$$H = \frac{1}{2}\int_\Omega \left\{ \rho \left(\frac{\partial u}{\partial t}\right)^2 + \Sigma : \varepsilon \right\} \mathrm{d}\Omega.$$

To get a pH representation the energy variables have to selected. The corresponding co-energies are then computed taking the variational derivative of the Hamiltonian

$$\begin{aligned} \text{Energies} \quad & x_1 = \rho\ \partial_t u, && x_2 = \varepsilon = \mathrm{Grad}(u). \\ \text{Co-energies} \quad & e_1 := \frac{\delta H}{\delta x_1} = \partial_t u, && E_2 := \frac{\delta H}{\delta x_2} = \Sigma. \end{aligned} \tag{14}$$

The port-Hamiltonian representation in co-energy variables becomes

$$\begin{bmatrix} \rho & 0 \\ 0 & \mathcal{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} e_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_d$$

The interconnection operator may be decomposed as $\mathcal{J} = \mathcal{J}_{\mathrm{Div}} + \mathcal{J}_{\mathrm{Grad}}$

$$\underbrace{\begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix}}_{\mathcal{J}} = \underbrace{\begin{bmatrix} 0 & \mathrm{Div} \\ 0 & 0 \end{bmatrix}}_{\mathcal{J}_{\mathrm{Div}}} + \underbrace{\begin{bmatrix} 0 & 0 \\ \mathrm{Grad} & 0 \end{bmatrix}}_{\mathcal{J}_{\mathrm{Grad}}} \tag{15}$$

Assuming a Neumann boundary conditions (the normal traction τ is known at the boundary), this system can be written compactly as a boundary control system

$$\begin{aligned} \mathcal{M}\frac{\partial e}{\partial t} &= \mathcal{J} e + \mathcal{B}_d u_d, \\ y_d &= \mathcal{B}_d^* e, \\ u_\partial &= E_2 \cdot n = \Sigma \cdot n|_\Gamma, \\ y_\partial &= e_1 = \partial_t u|_\Gamma. \end{aligned} \tag{16}$$

Now the system is put into weak form considering the inner product on space $\mathscr{X} = \mathscr{L}^2(\Omega, \mathbb{R}^d) \times \mathscr{L}^2(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$. Taking two elements $[a, A],\ [b, B] \in \mathscr{X}$ the inner product is computed as

$$\langle [a, A],\ [b, B] \rangle_{\mathscr{X}} = \int_\Omega a \cdot b \, \mathrm{d}\Omega + \int_\Omega A : B \, \mathrm{d}\Omega.$$

Considering a test function $w \in \mathscr{X}$ the weak form reads

$$\langle w,\ \mathcal{M}\ \partial_t e \rangle_{\mathscr{X}} = \langle w,\ \mathcal{J} e \rangle_{\mathscr{X}} + \langle w,\ \mathcal{B}_d u_d \rangle_{\mathscr{X}}$$

The bilinear form $m(w, \partial_t e) = \langle w,\ \mathcal{M}\ \partial_t e \rangle_{\mathscr{X}}$ is symmetric and coercive. The linear functional $b_\beta(w) := \langle w,\ \mathcal{B}_d \beta \rangle_{\mathscr{X}}$ takes into account distributed forces.
Now an integration by parts is applied on $\mathcal{J}_{\text{Div}}$

$$\langle w,\ \mathcal{J} e \rangle_{\mathscr{X}} = \langle w,\ \mathcal{J}_{\text{Grad}} e \rangle_{\mathscr{X}} - \langle \mathcal{J}_{\text{Grad}} w,\ e \rangle_{\mathscr{X}} + \langle w,\ u_\partial \rangle_{\mathscr{L}^2(\Gamma)} \tag{17}$$

The expression $j_{\text{Grad}}(w, e) := \langle w,\ \mathcal{J}_{\text{Grad}} e \rangle_{\mathscr{X}} - \langle \mathcal{J}_{\text{Grad}} w,\ e \rangle_{\mathscr{X}}$ is a skew symmetric bilinear form as it holds $j_{\text{Grad}}(w, e) = -j_{\text{Grad}}(e, w)$. The bilinear form $b_\partial(w, u_\partial) := \langle w,\ u_\partial \rangle_{\mathscr{L}^2(\Gamma)}$ imposes weakly the Neumann condition. System (16) is now rewritten in weak form

$$m(w, \partial_t e) = j_{\text{Grad}}(w, e) + b_\beta(w) + b_\partial(w, u_\partial). \tag{18}$$

The output equation is discretized considering test function w_∂ defined over the boundary

$$\langle w_\partial,\ y_\partial \rangle_{\mathscr{X}} = \langle w_\partial,\ e_1 \rangle_{\mathscr{X}} \tag{19}$$

If a Galerkin method is applied then corresponding test and trial functions are discretized using the same basis

$$\begin{aligned} w_1 &= \boldsymbol{\phi}_1^\top \boldsymbol{v}_1, \qquad & W_2 &= \boldsymbol{\Phi}_2^\top \boldsymbol{W}_2, \\ e_1 &= \boldsymbol{\phi}_1^\top \boldsymbol{e}_1, \qquad & E_2 &= \boldsymbol{\Phi}_2^\top \boldsymbol{E}_2. \end{aligned}$$

A finite dimensional pH system is readily obtained

$$\begin{aligned} \boldsymbol{M} \frac{d\boldsymbol{e}}{dt} &= \boldsymbol{J}\boldsymbol{e} + \boldsymbol{B}_d \boldsymbol{u}_d + \boldsymbol{B}_\partial \boldsymbol{u}_\partial, \\ \boldsymbol{M}_d \boldsymbol{y}_d &= \boldsymbol{B}_d^\top \boldsymbol{e}, \\ \boldsymbol{M}_\partial \boldsymbol{y}_\partial &= \boldsymbol{B}_\partial^\top \boldsymbol{e}. \end{aligned} \tag{20}$$

Remark 3 Stable mixed finite elements for the elastodynamics problem are detailed in [6]. However the formulation therein is based on a weak form obtained by integration by parts of the $\mathcal{J}_{\text{Div}}$ operator. The mixed finite element method for such a problem are then stable in the sense of Brezzi thanks to the properties of L^2/H^{Div} finite element spaces.

4.2 Discretized rigid-flexible Hamiltonian dynamics

The same methodology is applied to system (13). If corresponding test functions w, state e and the effort functions z are discretized using the same bases

$$w = \boldsymbol{\phi}^\top \boldsymbol{w}, \quad e = \boldsymbol{\phi}^\top \boldsymbol{e}, \quad z = \boldsymbol{\phi}^\top \boldsymbol{z},$$

then a finite-dimensional pHDAE system is obtained (after integration by parts of the $\mathcal{J}_{\mathrm{Div}}$ operator as)

$$\begin{aligned} \boldsymbol{E}\frac{d\boldsymbol{e}}{dt} &= \boldsymbol{J}\boldsymbol{z} + \boldsymbol{B}_d\boldsymbol{u}_d + \boldsymbol{B}_\partial\boldsymbol{u}_\partial, \\ \boldsymbol{M}_d\boldsymbol{y}_d &= \boldsymbol{B}_d^\top\boldsymbol{e}, \\ \boldsymbol{M}_\partial\boldsymbol{y}_\partial &= \boldsymbol{B}_\partial^\top\boldsymbol{e}. \end{aligned} \tag{21}$$

The actual computation of vector $\boldsymbol{z}$ relies on the Hamiltonian:

$$\frac{\partial H_d}{\partial \boldsymbol{e}} = \boldsymbol{E}^\top \boldsymbol{z}, \qquad H = H_{d,\mathrm{kin}} + H_{d,\mathrm{def}} + H_{d,\mathrm{pot}},$$

For the deformation and kinetic energy it is straightforward to find the between the state and effort functions as those energy are quadratic in the state variable:

$$H_{d,\mathrm{kin}} + H_{d,\mathrm{def}} = \frac{1}{2}\boldsymbol{e}_{\mathrm{kd}}^\top \boldsymbol{M}_{\mathrm{kd}}\, \boldsymbol{e}_{\mathrm{kd}}, \tag{22}$$

where $\boldsymbol{e}_{\mathrm{kd}} = [\boldsymbol{v}_P; \boldsymbol{\omega}_P; \boldsymbol{v}_f; \boldsymbol{\Sigma}]$ and $\boldsymbol{M}_{\mathrm{kd}}$ is the discretization of the mass operator $\mathcal{M}$ given in Eq (9). Therefore it holds that $\boldsymbol{z}_{\mathrm{kd}} = \boldsymbol{e}_{\mathrm{kd}}$ The only term that requires additional care is the potential energy and particularly the variational derivative of the Hamiltonian with respect to the deformation displacement $z_u = \delta_{u_f} H$. Consider the continuous power balance associate to

$$\dot{H} = \int_\Omega \frac{\partial u_f}{\partial t} \cdot z_u \,\mathrm{d}\Omega = \int_\Omega \frac{\partial u_f}{\partial t} \cdot \frac{\delta H}{\delta u_f} \,\mathrm{d}\Omega$$

The deformation velocity and its corresponding effort variable are discretized using the same basis, i.e. $u_f = \boldsymbol{\phi}_u^\top \boldsymbol{u}_f$, $z_u = \boldsymbol{\phi}_u^\top \boldsymbol{z}_u$. The discrete Hamiltonian rate assumes two equivalent expressions

$$\dot{H}_d(\boldsymbol{u}_f) = \begin{cases} \dot{\boldsymbol{u}}_f^\top \boldsymbol{M}_u\, \boldsymbol{z}_u, \\ \dot{\boldsymbol{u}}_f^\top \dfrac{\partial H_d}{\partial \boldsymbol{u}_f}, \end{cases}$$

where $\boldsymbol{M}_u = \int_\Omega \boldsymbol{\phi}_u\, \boldsymbol{\phi}_u^\top \,\mathrm{d}\Omega$. To preserve the power balance at a discrete level it must hold $\boldsymbol{z}_u = \boldsymbol{M}_u^{-1} \frac{\partial H_d}{\partial \boldsymbol{u}_f}$.

Another crucial point to be considered when handling the discretized model is that since no constraint has been considered for the deformation velocity the matrix $\boldsymbol{M}_{\mathrm{kd}}$ is singular since the deformation field contains rigid body motions. In order to get a suitable representation of the dynamics of a floating flexible system the flexible displacement and velocity are assumed to be zero at point P. This is equivalent to consider a cantilever condition when using shape functions to described the deformation field. Practically, the rows and columns associated to the degrees of freedom for u_f and v_f at point P are removed from matrices $\boldsymbol{E}$, $\boldsymbol{J}$, as well as the rows of matrices $\boldsymbol{B}_d$, $\boldsymbol{B}_\partial$.

Table 1 Please write your table caption here

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number	number	number
number	number	number

5 Multibody system in pH form

In §3, §4 the pH formulation of a single flexible floating body and the discretized system were presented. By

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