Dissipative Dynamical Systems

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Outline

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Why dissipative dynamical systems?

All engineering systems exhibit dissipation.

- Electrical networks with resistors;
- Mechanical systems (viscoelastic or Coulomb friction);
- Thermodynamic systems: dissipation leads to an increase in entropy.

The notion of dissipativity establishes a natural link between the properties of input-output and state-space models. Many modern computational tools for the analysis and synthesis of control systems are based on it.

Jan C. Willems. "Dissipative dynamical systems Part I: General theory". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 321–351

Jan C. Willems. "Dissipative dynamical systems Part II: Linear systems with quadratic supply rates". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 352–393

Arjan van der Schaft. *L2-gain and passivity in nonlinear control*. Springer-Verlag, 1999

Some mathematical notation

$\mathbb{R}_+ = [0, \infty)$ denotes the set of positive reals.
$\mathbb{R}^2_+ := \{(t_1, t_2) \in \mathbb{R}^2 |\ t_2 \geq t_1\}$ (causal triangular sector of $\mathbb{R}^2$).
Let V be a finite dimensional normed liner space with norm $||\cdot||_V$.
(If $V = \mathbb{R}^n$ then the Euclidean norm is denoted by$||x||_2 = \sqrt{x^\top x}$)

Definition (Locally integrable functions L^p_{loc})

For each positive integer $p \in 1, 2, \ldots,$ the set $L^p_{\text{loc}}(\mathbb{R}, V)$ consists of all functions $f : \mathbb{R} \to V$, which are measurable and satisfy

$$\int_a^b ||f(t)||_V^p \,\mathrm{d}t < \infty, \qquad \forall\, a, b \in \mathbb{R}.$$

The case $p = \infty$ consists of all bounded measurable functions on compact intervals, i.e. $\sup_{t \in [a,b]} f(t) < \infty$.

General setting

Consider the state-space system with inputs and outputs

$$\Sigma : \quad \begin{aligned} \dot{x} &= f(x,u), \qquad & u(t) &\in U, \\ y &= h(x,u), \qquad & y(t) &\in Y, \end{aligned}$$

where $x(t) \in \mathcal{X}$. In general $\mathcal{X}$ is a manifold and U, Y vector spaces.
For sake simplicity, assume $\mathcal{X} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$.

Theorem

Suppose f, h to be Lipschitz continuous in x and u jointly.
Then system Σ has a unique solution $\forall x(t_0) \in \mathcal{X},\ u(\cdot) \in L^2_{loc}(\mathbb{R}, U)$ with $x(\cdot) \in L^2_{loc}(\mathbb{R}, \mathcal{X}),\ y(\cdot) \in L^2_{loc}(\mathbb{R}, Y)$.

Reachability and controllability

Definition (State transition function)

Given the system Σ, the state transition function ϕ is the map

$$\phi(t_1, t_0, x(t_0), u(\cdot)) : \mathbb{R}_+^2 \times \mathcal{X} \times L^2_{\mathsf{loc}}(\mathbb{R}, U) \to \mathcal{X}$$

such that $x(t_1) = \phi(t_1, t_0, x(t_0), u(\cdot))$.

Definition (Reachability)

The state space $\mathcal{X}$ of system Σ is said to be **reachable** from x_{-1} if

$$\forall\, x \in \mathcal{X},\ \exists\, t_{-1} \leq 0,\ \exists\, u(\cdot) \in L^2_{\mathsf{loc}}(\mathbb{R}, U) \text{ such that } x = \phi(0, t_{-1}, x_{-1}, u(\cdot)).$$

Outline

The mathematical definition of dissipativity

On the combined space $U \times Y$ consider the supply rate function $s : U \times Y \to \mathbb{R}$.

Definition (Dissipative state space system)

A state space system Σ is said to be dissipative w.r.t. the supply rate s if there exists a function $S : \mathcal{X} \to \mathbb{R}_+$ (the storage function), such that $\forall\, x(t_0) \in \mathcal{X}$ at any time t_0, $\forall\, u(\cdot)$ and $\forall\, t_1 \geq t_0$, the following inequality holds

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t, \qquad \text{Dissipation Inequality.} \tag{1}$$

If equality holds then the system is called conservative (w.r.t. the supply rate s).

Corollary (Convexity of the storage functions set)

Given two storage functions S_1 and S_2 then any convex combination $\alpha S_1 + (1-\alpha) S_2,\ \alpha = [0, 1]$ is also a storage function.

Passive systems and L^2 finite gain

Two important class of supply rate functions:

- passive systems $s(u, y) = u^\top y$;
- finite L^2 gain $s(u, y) = \frac{1}{2}\gamma ||u||_2^2 - \frac{1}{2}||y||_2^2, \quad \gamma \geq 0.$

Definition (Passive system)

Σ with $U = Y = \mathbb{R}^m$ is **passive** if it is dissipative w.r.t.

$$s(u, y) = u^\top y.$$

Σ is **input strictly passive** if $\exists\, \delta > 0$ such that Σ is dissipative w.r.t.

$$s(u, y) = u^\top y - \delta ||u||_2^2.$$

Σ is **output strictly passive** if $\exists\, \varepsilon > 0$ such that Σ is dissipative w.r.t.

$$s(u, y) = u^\top y - \varepsilon ||y||_2^2$$

Σ is **lossless** if it is conservative with respect to $s(u, y) = u^\top y$.

Passive systems and L^2 finite gain

Two important class of supply rate functions:

- passive systems $s(u,y) = u^\top y$;
- finite L^2 gain $s(u,y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2, \quad \gamma \geq 0.$

Definition (L^2 finite gain)

A system Σ with $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$ has L^2-gain $\leq \gamma$ ($\gamma \geq 0$) if it is dissipative w.r.t.

$$s(u,y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2.$$

The L^2-gain of Σ is defined as

$$\gamma(\Sigma) := \inf\{\gamma |\ \Sigma \text{ has } L^2\text{-gain} \leq \gamma\}.$$

How to establish dissipativity? The available storage

Theorem (Necessary and sufficient conditions for dissipativity)

Consider system Σ and supply rate $s(u,y)$. Σ is dissipative with respect to s iff

$$S_a(x) := \sup_{\substack{u(\cdot)\\ T\geq 0}} - \int_0^T s(u(t), y(t))\,\mathrm{d}t, \qquad x(0) = x, \tag{2}$$

is finite $\forall\, x \in \mathcal{X}$.

Furthermore, if S_a is finite $\forall\, x \in \mathcal{X}$ then S_a is a storage function, called the ***available storage****, and all other possible storage functions S satisfy*

$$S_a(x) \leq S(x) - \inf_x S(x), \qquad \forall\, x \in \mathcal{X}$$

Moreover $\inf_x S_a(x) = 0$.

The available storage is the minimal storage function.

Proof

- (If) Suppose S_a is finite. Then $S_a \geq 0$ (sup of a set that contains 0). Compare $S_a(x(t_0))$ and $S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))\,\mathrm{d}t$ with $s(u,y)$ evaluated on a trajectory generated by $u : [t_0, t_1] \to \mathbb{R}^m$ that drives $x(t_0)$ at t_0 to $x(t_1)$ at t_1. Since S_a is the supremum over all $u(\cdot)$ it follows

$$S_a(x(t_0)) \geq S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))\,\mathrm{d}t \implies S_a \text{ is a storage function.}$$

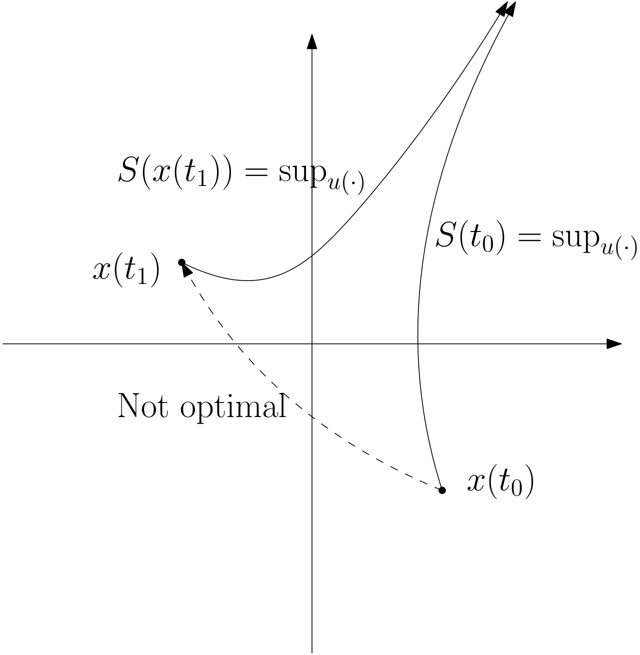
- (Only if) Suppose Σ dissipative. Then $\exists\, S \geq 0$ such that $\forall\, u(\cdot)$

$$S(x(0)) + \int_0^T s(u(t), y(t))\,\mathrm{d}t \geq S(x(T)) \geq 0.$$

  This implies that

$$S(x(0)) \geq \sup_{\substack{u(\cdot)\\ T\geq 0}} - \int_0^T s(u(t), y(t))\,\mathrm{d}t = S_a(x(0)) \implies S_a(x(0)) < \infty$$

Then $S' = S - \inf_x S(x)$ satisfy the dissipation inequality so $S'(x) \geq S_a(x), \forall x$ and $\inf_x S'(x) = 0$ (and hence $\inf S_a(x) = 0$).


$S(x(t_1)) = \sup_{u(\cdot)}$
$S(t_0) = \sup_{u(\cdot)}$
$x(t_1)$
Not optimal
$x(t_0)$

Reachability and Storage functions

If the system is reachable from x^*, the finiteness of S_a needs to be checked only in x^*

Proposition

Assume that Σ is reachable from $x^* \in \mathcal{X}$. Then Σ is dissipative iff $S_a(x^*) < \infty$.

Proof

(If) Suppose there exists $x \in \mathcal{X}$ such that $S_a(x) = \infty$. Since by reachability x can be reached from x^* in finite time, this would imply (by time invariance) that also $S_a(x^*) = \infty$.

The maximal storage: the required supply

If Σ is reachable from x^*, there exists another canonically defined storage function.

Theorem

Assume that Σ is reachable from $x^* \in \mathcal{X}$.
Define the required supply (from x^*) $S_r : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ as

$$S_r(x) := \inf_{\substack{u(\cdot) \\ T \geq 0}} \int_{-T}^{0} s(u(t), y(t))\, \mathrm{d}t, \qquad x(-T) = x^*, \quad x(0) = x. \tag{3}$$

Then the following holds:

1. S_r *satisfies the dissipation inequality.*
2. Σ *is dissipative iff* $\exists\, K > -\infty$ *such that* $S_r(x) \geq K$, $\forall\, x \in \mathcal{X}$.
3. *If S is a storage function for Σ, then*
$$S(x) - S(x^*) \leq S_r(x), \qquad x \in \mathcal{X},$$
and $S_r(x) + S(x^*)$ is itself a storage function (and in particular $S_r(x) + S_a(x^*)$).

Proof

1. To steer the system from x^* at $-T$ to $x(t_1)$ consider $u(\cdot) : [-T, t_1] \to U$ which first take x^* to $x(t_0)$ at time $t_0 \leq t_1$, and then equal to a given input $u(\cdot) : [t_0, t_1] \to U$ transferring $x(t_0)$ to $x(t_1)$. This is a suboptimal policy, so
$$S_r(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t \geq S_r(x(t_1)).$$
2. For the second claim, by definition of S_a and S_r
$$S_a(x^*) = \sup_x -S_r(x),$$
then Σ is dissipative iff $\exists K > -\infty$ such that $S_r(x) \geq -K$, $\forall x$.
3. Let S satisfy the dissipation inequality.
Then for any $u(\cdot) : [-T, 0] \to U$ such that $x(-T) = x^*$ to $x(0) = x$ it holds
$$S(x) - S(x^*) \leq \int_{-T}^{0} s(u(t), y(t))\, \mathrm{d}t.$$
Taking the infimum on the right-hand side over all $u(\cdot)$ proves the claim.
If $S \geq 0$, then $S_r + S(x^*) \geq 0$ is a storage function.

The a priori bounds

The available storage

It is the amount of internal storage which may be recovered from the system.

The required supply

It is the amount of supply which has to be delivered to the system in order to transfer it from a state of minimum storage to a given state.

Alternative definition of dissipativity

If Σ is dissipative with a storage function S for which $x^* = \arg\min_x S(x)$, then also $S - S(x^*)$ is a storage function, which is zero at x^*. Motions starting from x^* verify

$$\int_0^T s(u(t), y(t)) \geq 0, \qquad x(0) = x^*, \quad \forall T \geq 0. \tag{4}$$

Definition (Dissipativity from x^*)

A system Σ with supply rate s is called dissipative from x^* if (4) holds.

Proposition

A dissipative system Σ is dissipative from x^* iff its storage function satisfies $S(x^*) = 0$. If additionally the system is reachable from x^* then the system is dissipative and its required supply satisfies $S_r(x^*) = 0$.

Proof (Only if) Assume Σ is dissipative from x^*. By definition of S_a if holds $S_a(x^*) = 0$. If is reachable from x^* then by the previous proposition the system is dissipative, and $S_r(x^*) = 0$.

Theorem

Let Σ be dissipative and dissipative from x^*. Suppose that s is such that

$$\exists u(x) \;\text{ such that }\; s(u(x), h(x, u(x))) \leq 0, \qquad x \in \mathcal{X}. \tag{5}$$

for which x^* is a globally asymptotically equilibrium for the closed-loop system $\dot{x} = f(x, u(x))$. Then any storage function S attains its minimum at x^* and

$$S_a(x) \leq S(x) - S(x^*), \qquad \forall x \in \mathcal{X}.$$

Proof Consider the dissipation inequality for any S, rewritten as

$$-\int_0^T s(u(t), y(t))\, \mathrm{d}t \leq S(x) - S(x(T)), \qquad x(0) = x.$$

Extend $u(\cdot) : [0, T] \to U$ to the infinite time interval $[0, \infty)$ by considering on (T, ∞) a feedback $u(x)$ verifying (5) such that x^* is a globally asymptotical equilibrium. Since $s(u(x), h(x, u(x))) \leq 0$ and convergence of $x(t)$ to x^* for $t \to \infty$ that

$$-\int_0^T s(u(t), y(t))\, \mathrm{d}t \leq S(x) - S(x^*)$$

Taking the supremum at the left-hand side for $u(\cdot) : [0, T] \to U$ concludes the proof.

Corollary

Consider a system Σ *that is dissipative and reachable from* x^**, and for which* s *verifies* (5)*, such that* x^* *is a global asymptotical equilibrium for* $\dot{x} = f(x, u(x))$*.*
Then any storage function S *attains its minimum at* x^* *and the storage function* $S'(x) := S(x) - S(x^*)$ *satisfies*

$$S_a(x) \leq S'(x) \leq S_r(x), \qquad \forall x \in \mathcal{X},$$

where $S_a(x^*) = S_r(x^*) = 0$*.*

Outline

Reminder on Lyapunov stability

Consider $\dot{x} = f(x)$, $x \in \mathcal{X}$ with f locally Lipschitz continuous.
Denote $x(t; x_0)$ the solution for $x(0) = x_0$ with $t \in [0, T(x_0))$ and $T(x_0) > 0$ maximal.

Definition (Stability)

Let x^* be an equilibrium $f(x^*) = 0$, and thus $x(t; x^*) = x^*$, $\forall t$. The equilibrium x^* is

1. **stable**, if for each $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that
$$||x_0 - x^*|| \leq \delta(\varepsilon) \implies ||x(t; x_0) - x^*|| < \varepsilon, \qquad \forall t \geq 0.$$
2. **asymptotically stable**, if it is stable and additionally there exists $\widehat{\delta}$ such that
$$||x_0 - x^*|| \leq \widehat{\delta} \implies \lim_{t \to \infty} x(t; x_0) = x^*$$
3. **globally asymptotically stable**, if it is stable and
$$\lim_{t \to \infty} x(t; x_0) = x^*, \qquad \forall x_0 \in \mathcal{X}.$$
4. **unstable**, if it is not stable.

Reminder on Lyapunov stability

Definition (Lyapunov Functions)

Let x^* be an equilibrium of $\dot{x} = f(x)$. A C^1 function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying

$$V(x^*) = 0, \quad V(x) > 0, \quad x \neq x^*,$$

that is V is positive definite at x^*, and

$$\dot{V}(x) := \nabla V(x) \cdot f(x) \leq 0, \qquad x \in \mathcal{X},$$

is called a Lyapunov function for the equilibrium x^*

Theorem

Let x^* be an equilibrium. If there exists a Lyapunov function V for the equilibrium x^*, then x^* is a stable equilibrium. If moreover

$$\dot{V}(x) < 0, \qquad \forall x \in \mathcal{X}, \quad x \neq x^*,$$

then x^* is an asymptotically stable equilibrium, which is globally asymptotically stable if V is proper (that is, the sets $\{x \in \mathcal{X} | \, 0 \leq V(x) \leq c\}$ are compact for every $c \in \mathbb{R}_+$, equivalent to V is radially unbounded if $\mathcal{X} = \mathbb{R}^n$).

First stability result

Assume $S(x) \in C^1(\mathcal{X}, \mathbb{R}_+)$. Then it holds

$$\nabla S(x) \cdot f(x,u) \le s(u, h(x,u)), \qquad \forall x, u.$$

Proposition

Let $s(u,y)$ be a supply rate, and $S : \mathcal{X} \to R_+$ be a C^1 storage function for Σ. Assume that s satisfies

$$s(0,y) \le 0, \qquad \forall y \in Y,$$

Assume that $x^* \in \mathcal{X}$ is an equilibrium for the unforced system $x = f(x,0)$. Then x^* is a stable equilibrium of the unforced system with Lyapunov function $V(x) := S(x) - S(x^*)$ for x around x^*, while $s(0, h(x^*,0)) = 0$. If additionally, $\dot{S}(x) < 0$, $\forall x \neq x^*$, then x^* is an asymptotically stable equilibrium

Proof Since $\nabla S(x) \cdot f(x,0) \le s(0, h(x,0)) \le 0$, S is nonincreasing along solutions of $\dot{x} = f(x,0)$. Since $f(x^*,0) = 0$, it holds $s(0, h(x^*,0)) = 0$. The rest follows from Lyapunov stability theorem.

Refinement via LaSalle

The condition $\dot{S} < 0$ can be relaxed by using the LaSalle invariance principle.

Definition (Invariant set)

A set $\mathcal{N} \subset \mathcal{X}$ is invariant for $\dot{x} = f(x)$ if $x(t; x_0) \in \mathcal{N}$, $\forall x_0 \in \mathcal{N}$, $\forall t \in \mathbb{R}$, and is positively invariant if this holds $\forall t \geq 0$

Theorem (LaSalle's invariance principle)

Let $V : X \to \mathbb{R}$ be a C^1 function for which $\dot{V}(x) := \nabla V(x) \cdot f(x) \leq 0$, $\forall\, x \in \mathcal{X}$. Suppose there exists a compact set $\mathcal{C}$ which is positively invariant for $\dot{x} = f(x)$. Then for any $x_0 \in C$ the solution $x(t; x_0)$ converges for $t \to \infty$ to the largest subset $\mathcal{I}$ of $\mathcal{A} = \{x \in \mathcal{X} |\, \dot{V}(x) = 0\} \cap \mathcal{C}$ that is invariant for $\dot{x} = f(x)$.

The pendulum example

Dynamics:

$\dot{x}_1 = x_2,$

$\dot{x}_2 = -\dfrac{g}{l}\sin(x_1) - \dfrac{r}{ml^2}x_2$

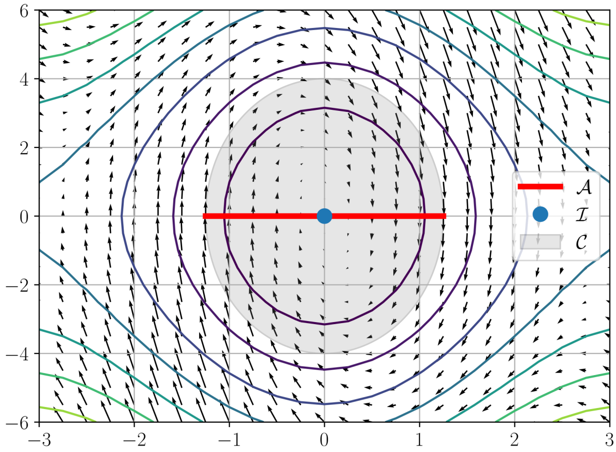
Sets:

$\mathcal{C} = \{x \in \mathcal{X} |\, V(x_1, x_2) \le k\},$

$\mathcal{A} = \{x \in \mathcal{X} |\, \dot{V} = 0\} \cap \mathcal{C},$

$\quad = \{x_2 = 0\} \cap \mathcal{C},$

$\mathcal{I} = \{(0, 0)\}$



$$V(x_1, x_2) = mgl(1 - \cos x_1) + \frac{1}{2}\, ml^2 x_2^2$$

Proposition

Let $S : \mathcal{X} \to R_+$ *be a* C^1 *storage function for* Σ*. Assume that* s *satisfies*

$$s(0, y) \leq 0, \qquad \forall y \in Y$$

Assume that $x^* \in \mathcal{X}$ *is a strict local minimum for* S*. Assume also that no solution of* $\dot{x} = f(x, 0)$ *other than* $x(t) \equiv x^*$ *remains in* $\{x \in \mathcal{X} |\ s(0, h(x, 0)) = 0\}$, $\forall t$*. Then* x^* *is an asymptotically stable equilibrium of* $\dot{x} = f(x, 0)$*, which is globally asymptotically stable if* $V(x) := S(x) - S(x^*) \geq 0$ *is proper.*

Proof $\dot{S}(x) = 0 \implies s(0, h(x, 0)) = 0$. The statement now directly follows from LaSalle's Invariance principle.

Outline

The open character of dissipativity theory

Consider k systems Σ_i with input, state, and output spaces U_i, $\mathcal{X}_i$, Y_i, $i = 1, \ldots, k$.
Suppose Σ_i are dissipative with respect to the supply rates

$$s_i(u_i, y_i), \qquad u_i \in U_i,\ y_i \in Y_i,\ i = 1, \ldots, k,$$

and storage functions $S_i(x_i), i = 1, \ldots, k$.
Now consider an interconnection of $\Sigma_i, i = 1, \ldots, k$, defined through

$$I \subset U_1 \times Y_1 \times \cdots \times U_k \times Y_k \times U_e \times Y_e,$$

where U_e, Y_e are spaces of external input and output.

Proposition

Suppose the supply rates $s_1, \ldots, s_k$ *and the interconnection subset* I *are such that* $\exists s_e : U_e \times Y_e \to \mathbb{R}$ *for which*

$$s_1(u_1, y_1) + \cdots + s_k(u_k, y_k) \leq s_e(u_e, y_e),$$
$$\forall ((u_1, y_1), \ldots, (u_k, y_k), (u_e, y_e)) \in I.$$

Then the interconnected system Σ_I *is dissipative with respect to the supply rate* s_e*, with storage function* $S(x_1, \ldots, x_k) := S_1(x_1) + \cdots + S_k(x_k)$

The Lyapunov function of interconnectd systems

For simplicity the spaces of external inputs and outputs are removed.

Proposition

Suppose the supply rates $s_1, \ldots, s_k$ *and the interconnection subset* I *are such that there exist positive constants* $\alpha_1, \ldots, \alpha_k$ *for which*

$$\alpha_1 s_1(u_1, y_1) + \cdots + \alpha_k s_k(u_k, y_k) \leq 0, \qquad \forall((u_1, y_1), \ldots, (u_k, y_k)) \in I. \tag{6}$$

Then the function

$$S_\alpha(x_1, \ldots, x_k) := \alpha_1 S_1(x_1) + \cdots + \alpha^k S_k(x_k)$$

satisfies $\dot{S}_\alpha \leq 0$ *along all solutions of the interconnected system* Σ_I.

Proof It suffices to multiply each dissipation inequality by α_1, add them and use the inequality (6).

Outline

Conclusions

Some important considerations:

- The definition of a dissipative dynamical system postulates the existence of a storage function. The dynamical equations are insufficient to specify the storage function uniquely.
- The storage function satisfies an a priori bound. It is bounded from below by the available storage and from above by the required supply. These bounds possess a variational characterization.
- In dissipative systems states for which the storage function attains a local minimum are locally stable and the storage function is a suitable Lyapunov function.
- Immeadiate extension to interconnected systems: the sum of the storage functions of the individual subsystems is a storage function for the interconnected system.

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