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Portwings Internal Meeting
Challenges and outlook for the numerics

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What numerics for the portwings project?

Methods should preserve the continuous structure at the discrete level. Which structure?

1. Cohomology: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$;
2. Variational structure $\delta \int l = 0$, (l Lagrangian density);
3. Hamiltonian structure $\dot{\mathcal{F}} = \{F, H\}$, $\{\cdot,\cdot\}$ Poisson brackets.
4. ...

Recent developments:

- splitting of topological and metric operators (Bauer and Behrens 2018);
- Lie group structure and underlying variational formulation (Gawlik and Gay-Balmaz 2020);
- connection with algebraic topology, i.e. de Rham complex and more general Hilbert complexes, e.g. elasticity (Bochev and Hyman 2006; Arnold, Falk, and Winther 2006; Palha et al. 2014);

Principle behind split discretization

- Fluid equation written in covariant form (exterior calculus);
- Split Hamiltonian form $\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}$.
  - Topological braket depending on d (exterior derivative) or ι_v (interior product).
  - Metric dependent $\mathcal{H}$, since it depends on $*$

Linear shallow water waves in Hamiltonian form

- $\mathcal{H} = \frac{1}{2}\int_\Omega \left\{\bar{h}\,\|\boldsymbol{u}\|^2 + gh^2\right\}\,\mathrm{d}\Omega$, with $\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} = \bar{h}\boldsymbol{u}$, $\frac{\delta\mathcal{H}}{\delta h} = gh$, where g gravity acc. and $\bar{h}$ equilibrium fluid height.
- $\{\mathcal{F}, \mathcal{G}\} = -(\frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}, \nabla\frac{\delta\mathcal{G}}{\delta h}) - (\frac{\delta\mathcal{F}}{\delta h}, \nabla\cdot\frac{\delta\mathcal{G}}{\delta\boldsymbol{u}})$.

$$\mathcal{F} = \int \boldsymbol{u}\,\mathrm{d}\Omega : \ \dot{\mathcal{F}} = -\left(\frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}, \nabla\frac{\delta\mathcal{H}}{\delta h}\right) \rightarrow \partial_t\boldsymbol{u} = -g\nabla h.$$

$$\mathcal{F} = \int h\,\mathrm{d}\Omega : \ \dot{\mathcal{F}} = -\left(\frac{\delta\mathcal{F}}{\delta h}, \nabla\cdot\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\right) \rightarrow \partial_t h = -\bar{h}\nabla\cdot\boldsymbol{u}.$$

Split and weak (or mixed) form

De Rham complex: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$.

Split form

$$\begin{tikzcd} h \in V^0 \arrow[r, "\nabla"] \arrow[d, "\tilde{h}=\tilde{*}h"'] & V^1 \ni \boldsymbol{u} \arrow[d, "\boldsymbol{u}=\tilde{*}\boldsymbol{u}"] \\ \tilde{h} \in V^3 & V^2 \ni \tilde{\boldsymbol{u}} \arrow[l, "\nabla\cdot"'] \end{tikzcd}$$

- Assume full $\tilde{*}$ in metric eqs.
- Both $\nabla, \nabla\cdot$ are imposed strongly.
- Both diff. eqs exact

Weak form (Mixed FE)

$$\begin{tikzcd} h \in V^3 \arrow[r, "\widehat{\nabla}"] \arrow[d, "\tilde{h}=h"'] & V^2 \ni \boldsymbol{u} \arrow[d, "\tilde{\boldsymbol{u}}=\boldsymbol{u}"] \\ \tilde{h} \in V^3 & V^2 \ni \tilde{\boldsymbol{u}} \arrow[l, "\nabla\cdot"'] \end{tikzcd}$$

- Assume $\tilde{*} = \mathrm{Id}$, i.e. $\tilde{\boldsymbol{u}} = \boldsymbol{u}$, $\tilde{h} = h$.
- Weak gradient $\widehat{\nabla}$.
- Moment Eq. weak

Split decomposition in practice

1. First projection of the strong form. For the 1D case:

$$\partial_t \mathbf{u}_e^{(1)} + g\mathbf{D}^{en}\mathbf{h}_n^{(0)} = \mathbf{0}, \qquad \mathbf{u}_e^{(1)} : 1 \text{ form}, \quad \mathbf{h}_n^{(0)} : 0 \text{ form},$$

$$\partial_t \widetilde{\mathbf{h}}_e^{(1)} + \bar{h}\mathbf{D}^{en}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{0}, \qquad \widetilde{\mathbf{h}}_e^{(1)} : 1 \text{ form}, \quad \widetilde{\mathbf{u}}_n^{(0)} : 0 \text{ form}.$$

   $\mathbf{D}^{en}$ metric free approximation of exterior derivative.

2. Project the metric closure relations:
   - High accuracy $CG_1^u - CG_1^h$ spaces:

$$\mathbf{M}^{nn}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{P}^{ne}\mathbf{u}_e^{(1)}, \qquad \mathbf{M}^{nn}\mathbf{h}_n^{(0)} = \mathbf{P}^{ne}\widetilde{\mathbf{h}}_e^{(1)};$$

   - Low accuracy $CG_0^u - CG_0^h$ spaces:

$$\mathbf{M}^{en}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{I}^{ee}\mathbf{u}_e^{(1)}, \qquad \mathbf{M}^{en}\mathbf{h}_n^{(0)} = \mathbf{I}^{ee}\widetilde{\mathbf{h}}_e^{(1)};$$

   - Medium accuracy $CG_1^u - CG_0^h$ spaces:

$$\mathbf{M}^{nn}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{P}^{ne}\mathbf{u}_e^{(1)}, \qquad \mathbf{M}^{en}\mathbf{h}_n^{(0)} = \mathbf{I}^{ee}\widetilde{\mathbf{h}}_e^{(1)};$$

   $\mathbf{M}^{nn}$, $\mathbf{M}^{en}$ metric dependent, $\mathbf{P}^{ne}$ metric free averaging op.

Weak (mixed) form

Weak formulation: find $\boldsymbol{u} \in V^2(\mathbb{R}^3),\ h \in V^3(\mathbb{R})$

$$
\begin{aligned}
(\boldsymbol{v}_u, \partial_t \boldsymbol{u})_{L^2} &= -(\boldsymbol{v}_u, g\widehat{\nabla} h)_{L^2} = (\nabla \cdot \boldsymbol{v}_u, gh)_{L^2}, \qquad && \forall \boldsymbol{v}_u \in V^2, \\
(v_h, \partial_t h)_{L^2} &= -(v_h, \bar{h}\nabla \cdot \boldsymbol{u})_{L^2}, && \forall v_h \in V^3.
\end{aligned}
$$

After projection

$$
\begin{bmatrix} \mathbf{M}^{nn} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{ee} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{h}_e \end{pmatrix} = \begin{bmatrix} \mathbf{0} & g\mathbf{D}^{ne} \\ -\bar{h}\mathbf{D}^{en} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{h}_e \end{pmatrix}, \qquad \mathbf{D}^{ne} = \mathbf{D}^{en\top}.
$$

Another formulation with weak divergence: find
$\boldsymbol{u} \in V^1(\mathbb{R}^3),\ h \in V^0(\mathbb{R})$

$$
\begin{aligned}
(\boldsymbol{v}_u, \partial_t \boldsymbol{u})_{L^2} &= -(\boldsymbol{v}_u, g\nabla h)_{L^2}, && \forall \boldsymbol{v}_u \in V^1, \\
(v_h, \partial_t h)_{L^2} &= -(v_h, \bar{h}\widehat{\nabla \cdot} \boldsymbol{u})_{L^2} = (\nabla v_h, \bar{h}\boldsymbol{u})_{L^2}, \qquad && \forall v_h \in V^0.
\end{aligned}
$$

Connection with algebraic topology

$$
\begin{tikzcd}
& f^*E_V \arrow[rr] \arrow[dl] \arrow[dd] & & E_V \arrow[dl] \arrow[dd] \\
f^*E \arrow[rr] \arrow[dd] & & E \arrow[dd] & \\
& U \arrow[rr] \arrow[dl] & & V \arrow[dl] \\
M \arrow[rr] & & N &
\end{tikzcd}
$$

Tools available

Firedrake: https://www.firedrakeproject.org/.

FEniCS: https://fenicsproject.org/.

Fluid Structure Interaction in Fenics: Bergersen et al. 2020.

Mesh morphing in FEniCS: https://bitbucket.org/Epoxid/femorph/src/c7317791c8f00d70fe16d593344cb164a53cad9b/?at=dokken%2Frestructuring

PyDec: https://github.com/hirani/pydec

Learning Python for scientific computing https://faculty.math.illinois.edu/~hirani/cbmg/index.html