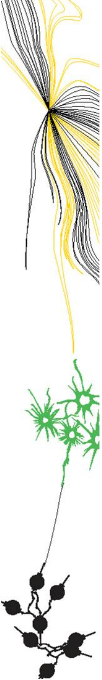

Improving multiphysics simulation through port-Hamiltonian system theory

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Outline

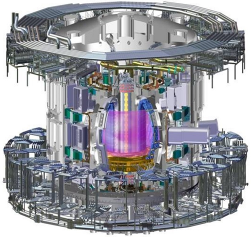
Outline

Challenges in muliphysics problems

Multiphysics problems are commonly found in industrial applications.

Aeroelasticity

Thermoelasticity



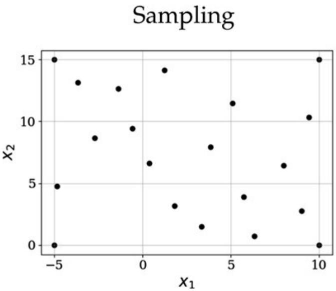
Magnetohydrodynamics

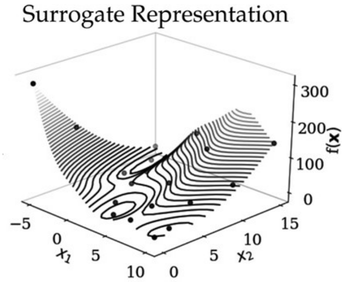
Challenges:

- Coupling between different models.
- Huge computational cost due to the large size of the models.
- Multidisciplinary optimization for dynamical systems.

Typical workflow in industry

- **Specific modelling** and numerical methods for each physical domain.
  - The **open character** of systems is **not considered**.
  - Numerical methods do not preserve the structure required to interconnect systems.
- Model reduction via statistical methods.
  - The **physical structure is lost** and first principles are violated.
  - This methodology **does not generalize** to different problems.

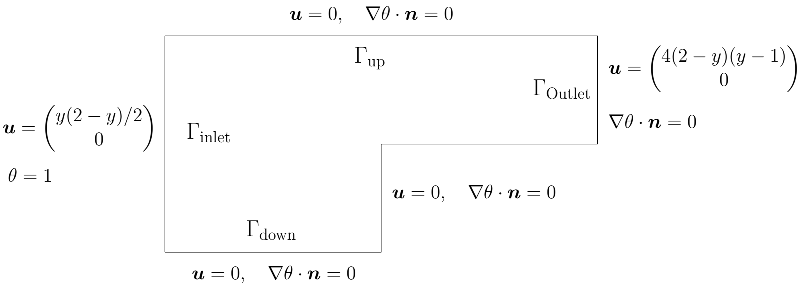




Example: convection dominated transport

Convection dominated transport of a passive scalar field in a Stokes flow[1]

$$\begin{aligned} \nu\Delta\boldsymbol{u} + \nabla p &= 0, & \boldsymbol{u} &: \text{Velocity},\\ \nabla\cdot\boldsymbol{u} &= 0, & p &: \text{Pressure},\\ -\varepsilon\Delta\theta + \boldsymbol{u}\cdot\nabla\theta &= 0. & \theta &: \text{Temperature.} \end{aligned}$$



Geometry and boundary conditions

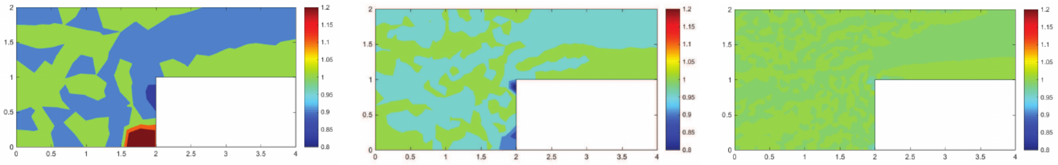
[1] Volker John et al. "On the Divergence Constraint in Mixed Finite Element Methods for Incompressible Flows". In: *SIAM Review* 59.3 (2017), pp. 492–544. DOI: 10.1137/15M1047696.

When multiphysics goes wrong

Exact solution for the temperature $\theta_{\text{ex}} = 1$.

- $(\boldsymbol{u}_h, p_h)$ represented using the Taylor-Hood element $\mathbb{P}_2/\mathbb{P}_1$;
- θ_h obtained via Voronoi finite volume method.

The Taylor-Hood element does not lead to divergence free velocity $||\nabla \cdot \boldsymbol{u}_h||_{L^2(\Omega)} \neq 0$.



Refinement 1 Refinement 2 Refinement 3

Figure: Discrete temperature field θ_h obtained

Outline

A unified language for multiphysics in engineering

The port-Hamiltonian (pH) paradigm provides a language to understand multiphysics:

- **Physics** is at the core: port-Hamiltonian systems are **passive** with respect to the **energy storage function**.
- The **topological** and **metrical** structure of the equation is clearly separated (mimetic discretization).
- PH systems are **closed under interconnection**.

A quest for duality

The concept of **interconnection** and the port behavior of pH systems is mathematically formalized as **duality pairing**. How exactly is that defined ?

Finite dimensional pH systems

A theory still under developement

There is **not a unique definition** of pH systems, even in finite dimension.

Definition (Finite dimensional pH system)

The time-invariant dynamical system

$$\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\, \nabla_{\mathbf{x}} H + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{B}^\top\, \nabla_{\mathbf{x}} H,$$

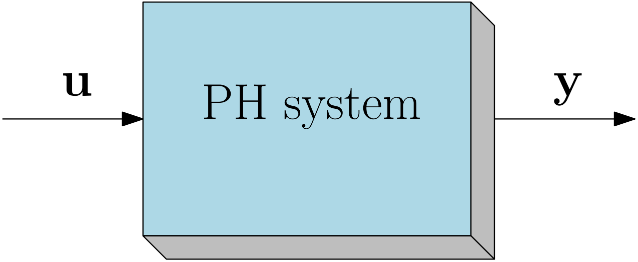
where $\mathbf{x}$ is the state, $\mathbf{u}$ the control input, $\mathbf{y}$ the collocated output and

- $H(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, the Hamiltonian, is bounded from below.
- $\mathbf{J} = -\mathbf{J}^\top$ the interconnection operator.
- $\mathbf{R} = \mathbf{R}^\top \in \mathbb{R}^{n\times n}$, $\mathbf{R} \geq 0$ the resistive operator.
- $\mathbf{B} \in \mathbb{R}^{n\times m}$ the control operator.

is a pH system.

Finite dimensional pH systems

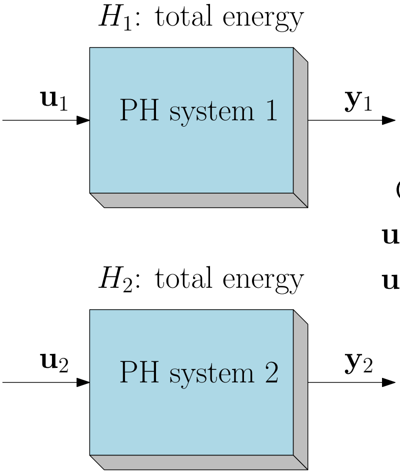
H: total energy



Lossless ($\mathbf{R} = \mathbf{0}$): $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

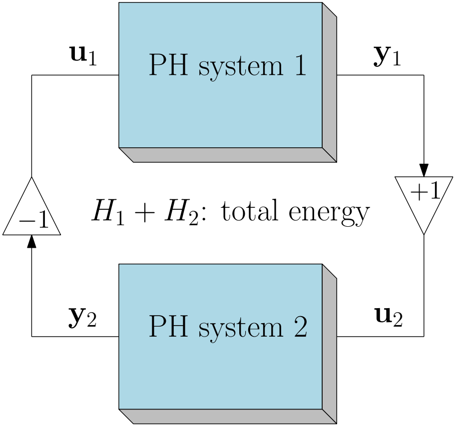
Energy preserving interconnection of pHs



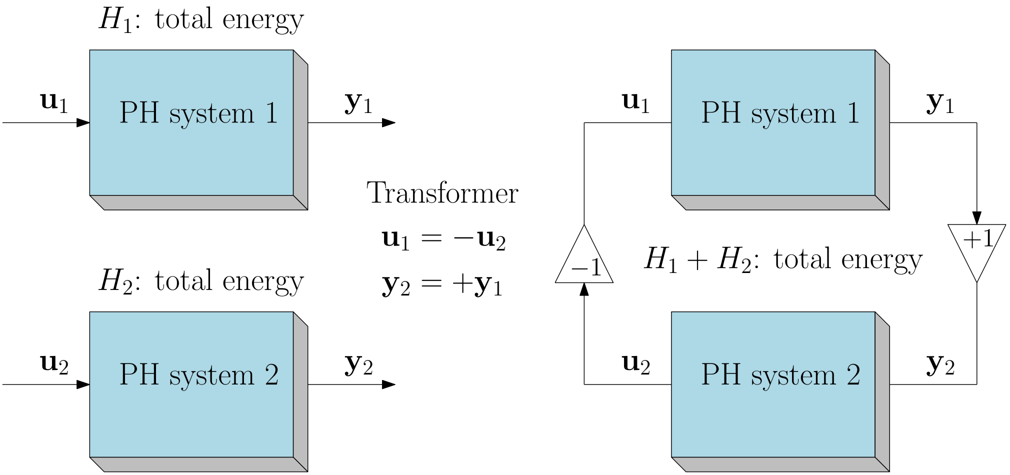
Gyrator

$\mathbf{u}_1 = -\mathbf{y}_2$

$\mathbf{u}_2 = +\mathbf{y}_1$



Energy preserving interconnection of pHs



The geometric structure of pH systems[2]

Definition (Dirac structure)

Given a vector space F and its dual $E = F'$ with respect to the duality product $\langle \cdot \,|\, \cdot \rangle : E \times F \to \mathbb{R}$, consider the symmetric bilinear form:

$$\langle\langle\, (f_1, e_1), (f_2, e_2)\,\rangle\rangle := \langle e_1 \,|\, f_2 \rangle + \langle e_2 \,|\, f_1 \rangle, \qquad (f_i, e_i) \in B = F \times E, \; i = 1, 2.$$

A Dirac structure is a linear subspace $\mathcal{D} \subset B$ which equals its orthogonal companion with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, *i.e.* $\mathcal{D} = \mathcal{D}^{[\perp]}$, where:

$$\mathcal{D}^{[\perp]} := \left\{ (f, e) \in B \;\mid\; \langle\langle\, (f, e), (\widehat{f}, \widehat{e})\,\rangle\rangle = 0, \; \forall\, (\widehat{f}, \widehat{e}) \in \mathcal{D} \right\}.$$

Theorem (Finite dimensional Dirac structure)

A subspace $\mathcal{D} \subset F \times E$ where F is a finite-dimensional vector space F and $E = F'$ its dual is a Dirac structure if and only if $\langle \mathbf{e} \,|\, \mathbf{f} \rangle = 0$ and $\dim D = \dim F$.

[2] T. J. Courant. "Dirac manifolds". In: *Transactions of the American Mathematical Society* 319.2 (1990), pp. 631–661. ISSN: 0002-9947. DOI: 10.2307/2001258.

Dirac structure and pH systems

From classical matrix factorization $\exists\, \mathbf{G} \in \mathbb{R}^{k\times n}$ and $\mathbf{K} = \mathbf{K}^\top \in \mathbb{R}^{k\times k}$, $\mathbf{K} \geq 0$ such that $\mathbf{R} = \mathbf{G}^\top\, \mathbf{K}\mathbf{G}$.

Dirac structure representation

Considering the following **port behavior**:

- the **storage ports** $(\mathbf{f}_x, \mathbf{e}_x) := (-\dot{\mathbf{x}}, \nabla_{\mathbf{x}} H) \in \mathbb{R}^n \times \mathbb{R}^n$;
- the **resistive ports** $(\mathbf{f}_r, \mathbf{e}_r) \in \mathbb{R}^k \times \mathbb{R}^k$;
- the **interconnection ports** $(\mathbf{f}_u, \mathbf{e}_u) := (\mathbf{y}, \mathbf{u}) \in \mathbb{R}^m \times \mathbb{R}^m$.

Given this port behavior, the pH system rewrites

$$\begin{pmatrix} -\dot{\mathbf{x}} \\ \mathbf{f}_r \\ \mathbf{f}_u \end{pmatrix} = \underbrace{\begin{bmatrix} -\mathbf{J} & \mathbf{G}^\top & -\mathbf{B} \\ \mathbf{G} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{J}_e} \begin{pmatrix} \nabla_{\mathbf{x}} H \\ \mathbf{e}_r \\ \mathbf{e}_u \end{pmatrix}, \qquad \mathbf{e}_r = \mathbf{K}\mathbf{f}_r.$$

Since $\mathbf{J}_e$ is skewsymmetric its graph $\{(\mathbf{f}, \mathbf{e}) \in F \,|\, \mathbf{f} = \mathbf{J}_e \mathbf{e}\}$ is a Dirac structure.

A simple definition[3]

Definition (Port-Hamiltonian system)

Let $X_\mathcal{S},\ X_\mathcal{R},\ X_\mathcal{P}$ be Banach spaces. A port-Hamiltonian system is a triple $(\mathcal{D}, H, \mathcal{R})$:

- $\mathcal{D} \subset (X_\mathcal{S},\ X_\mathcal{R},\ X_\mathcal{P}) \times (X'_\mathcal{S},\ X'_\mathcal{R},\ X'_\mathcal{P})$ is a Dirac structure.
- $H : U \to \mathbb{R}$ (with $U \subset X_\mathcal{S}$ open) is a Hamiltonian.
- $\mathcal{R} \subset X_\mathcal{R} \times X'_\mathcal{R}$ is a resistive relation.

The behavior of the pH system on an interval $\mathbb{I} \subset \mathbb{R}$ consists of all $(x, f_\mathcal{R}, f_\mathcal{P}, e_\mathcal{R}, e_\mathcal{P})$

- $x \in W^{1,2}_{\mathrm{loc}}(\mathbb{I}, X_\mathcal{S})$, and $x(t) \in U,\ \forall t \in \mathbb{I}$,
- $(f_\mathcal{R}, e_\mathcal{R}) \in L^2_{\mathrm{loc}}(\mathbb{I}; X_\mathcal{R} \times X'_\mathcal{R})$ and $(f_\mathcal{P}, e_\mathcal{P}) \in L^2_{\mathrm{loc}}(\mathbb{I}; X_\mathcal{P} \times X'_\mathcal{P})$

that fulfill the differential inclusion

$$(-\partial_t x,\ f_\mathcal{R},\ f_\mathcal{P},\ D_x H,\ e_\mathcal{R},\ e_\mathcal{P}) \in \mathcal{D}, \qquad (f_\mathcal{R}, e_\mathcal{R}) \in \mathcal{R}, \qquad \text{for almost all } t \in \mathbb{I}.$$

[3] Timo Reis. "Some notes on port-Hamiltonian systems on Banach spaces". In: *IFAC-PapersOnLine* 54.19 (2021). 7th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2021, pp. 223–229. DOI: 10.1016/j.ifacol.2021.11.082.

Dirac structure

Let X be a Banach space. A subspace $\mathcal{D} \subset X \times X'$ is called a Dirac structure, if $\forall\, f \in X, e \in X'$, it holds

$$(f, e) \in \mathcal{D} \iff \Big(\langle \widehat{e} \,|\, f \rangle + \langle e \,|\, \widehat{f} \rangle = 0, \quad \forall\, (\widehat{f}, \widehat{e}) \in \mathcal{D} \Big).$$

Hamiltonian

Let X be a Banach space and $U \subset X$ be open. A mapping $H : U \to \mathbb{R}$ is a Hamiltonian if it is locally Lipschitz continuous and Gâteaux differentiable

Resistive relation

Let X be a Banach space. A relation $\mathcal{R} \subset X \times X'$ is called resistive, if

$$\langle e \,|\, f \rangle \leq 0, \qquad \forall\, (f, e) \in \mathcal{R}.$$

Operators

If $J \in \mathcal{L}(X^{'}, X)$ is a skew-dual operator $\langle w \,|Jv\rangle = \langle v \,| - Jw\rangle \;\forall\, v, w \in X^{'}$ then $\mathcal{D} = \{(Je, e) : e \in X^{'}\}$ is a Dirac structure[4].

If $K : X \to X^{'}$ is dissipative $\langle K(x) \,|x\rangle \leq 0, \;\forall\, x \in X$, then $\mathcal{R} = \{(K(f), f) : e \in X^{'}\}$ is a resistive relation.

$$\begin{pmatrix} -\partial_t x \\ f_{\mathcal{R}} \\ f_{\mathcal{P}} \end{pmatrix} = J \begin{pmatrix} D_x \mathcal{H} \\ e_{\mathcal{R}} \\ e_{\mathcal{P}} \end{pmatrix}, \qquad e_{\mathcal{R}} = K(f_{\mathcal{R}}).$$

[4] T. Reis and T. Stykel. "Passivity, Port-Hamiltonian Formulation and Solution Estimates for a Coupled Magneto-Quasistatic System". In: *arXiv preprint arXiv:2205.15259* (2022).

Example: the wave equation

Consider the Hamiltonian

$$H = (p,\, \kappa p)_{L^2(\Omega)} + (\boldsymbol{v},\, \rho^{-1}\boldsymbol{v})_{L^2(\Omega,\mathbb{R}^3)}.$$

where κ is the Bulk modulus and ρ is the density.

The wave equation on $\mathbb{R}^3$ with distributed input

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ \boldsymbol{v} \end{pmatrix} = -\begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad}_w & 0 \end{bmatrix}\begin{pmatrix} D_p H \\ D_{\boldsymbol{v}} H \end{pmatrix} + \begin{bmatrix} \mathrm{Id} \\ 0 \end{bmatrix} u, \qquad \mathrm{grad}_w \text{ is the weak gradient,}$$

$$y = \begin{bmatrix} \mathrm{Id} & 0 \end{bmatrix}\begin{pmatrix} D_p H \\ D_{\boldsymbol{v}} H \end{pmatrix},$$

Spaces: $X_{\mathcal{S}} = L^2(\mathbb{R}^3) \times H^{\mathrm{div}}(\mathbb{R}^3)',\ X_{\mathcal{R}} = \emptyset,\ X_{\mathcal{P}} = L^2(\mathbb{R}^3).$

$$J = \begin{bmatrix} 0 & \mathrm{div} & -\mathrm{Id} \\ \mathrm{grad}_w & 0 & 0 \\ \mathrm{Id} & 0 & 0 \end{bmatrix}.$$

Example: the Maxwell equations

Consider the Hamiltonian:

$$H = \frac{1}{2}(\boldsymbol{D},\, \varepsilon^{-1}\boldsymbol{D})_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}(\boldsymbol{B},\, \mu^{-1}\boldsymbol{B})_{L^2(\Omega;\mathbb{R}^3)}.$$

where ε is the electric permittivity and μ is the magnetic permeability.

The Maxwell equation on $\Omega \subset \mathbb{R}^3$ with conducting boundary condition

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{D}\\ \boldsymbol{B}\end{pmatrix} = \begin{bmatrix}0 & \mathrm{curl}_w\\ -\mathrm{curl} & 0\end{bmatrix}\begin{pmatrix}D_{\boldsymbol{D}}H\\ D_{\boldsymbol{B}}H\end{pmatrix}, \qquad \begin{aligned}\nabla\cdot\boldsymbol{D} = 0, \qquad \nabla\cdot\boldsymbol{B} = 0,\\ D_{\boldsymbol{D}}H \times \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{E}\times\boldsymbol{n}|_{\partial\Omega} = 0,\end{aligned}$$

where curl_w corresponds to a weak curl operator
Spaces: $H_0^{\mathrm{curl}}(\Omega|\,\mathrm{div}=0)' \times X_{\mathcal{S}} = L^2(\Omega;\mathbb{R}^3|\,\mathrm{div}=0),\ X_{\mathcal{R}} = \emptyset,\ X_{\mathcal{P}} = \emptyset.$

$$J = \begin{bmatrix}0 & -\mathrm{curl}_w\\ \mathrm{curl} & 0\end{bmatrix}.$$

And many more

The same framework applies to

- Linear and non-linear solid mechanics (beams, plates, shells, etc.).
- Fluid dynamics.
- Chemical reactions.

However some aspects need further clarification:

- how to describe generic boundary conditions?
- how to represent the duality in a discrete setting?

Outline

Geometric Dirac structure[5]

Dirac structure for differential forms

On a Riemannian oriented manifold Ω consider

- the flows $(f_1^p, f_2^q, f_\partial^{n-p}) \in F = \Lambda^p(\Omega) \times \Lambda^q(\Omega) \times \Lambda^{n-p}(\partial\Omega)$,
- the efforts $(e_1^{n-p}, e_2^{n-q}, e_\partial^{n-q}) \in E = \Lambda^{n-p}(\Omega) \times \Lambda^{n-q}(\Omega) \times \Lambda^{n-q}(\partial\Omega)$,

with $p + q = n + 1$ and $\Lambda^k(\Omega)$ is the space of smooth k-forms.
The following subset $\mathcal{D} \subset F \times E$ defines a Dirac structure

$$\begin{pmatrix} f_1^p \\ f_2^q \end{pmatrix} = \begin{bmatrix} 0 & (-1)^{pq+1}\,\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{pmatrix} e_1^{n-p} \\ e_2^{n-q} \end{pmatrix}, \qquad \begin{pmatrix} f_\partial^{n-p} \\ e_\partial^{n-q} \end{pmatrix} = \begin{bmatrix} \mathrm{tr} & 0 \\ 0 & (-1)^p\,\mathrm{tr} \end{bmatrix} \begin{pmatrix} e_1^{n-p} \\ e_2^{n-q} \end{pmatrix}.$$

The key is that this subset verify the following power balance (Stokes formula)

$$\langle e_1^{n-p} \,|\, f_1^p \rangle_\Omega + \langle e_2^{n-q} \,|\, f_2^q \rangle_\Omega + \langle e_\partial^{n-q} \,|\, f_\partial^{n-p} \rangle_{\partial\Omega} = 0.$$

[5] A.J. van der Schaft and B.M. Maschke. “Hamiltonian formulation of distributed-parameter systems with boundary energy flow”. In: *Journal of Geometry and Physics* 42.1 (2002), pp. 166–194. DOI: 10.1016/S0393-0440(01)00083-3.

Hyperbolic dynamical systems

Consider the following dynamical system with boundary input and observation

$$\begin{pmatrix} \partial_t \alpha^p \\ \partial_t \beta^q \end{pmatrix} = - \begin{bmatrix} 0 & (-1)^{pq+1}\,\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{pmatrix} \delta_\alpha H^{n-p} \\ \delta_\beta H^{n-q} \end{pmatrix}, \qquad (-1)^p \,\mathrm{tr}\, \delta_\beta H^{n-q} = u^{n-q},$$

$$y^{n-p} = \mathrm{tr}\, \delta_\alpha H^{n-p},$$

where the variational derivative is defined by

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} H(\mu^k + \varepsilon \delta\mu^k) = \langle \delta_\mu H^{n-p} \,|\delta\mu^k\rangle_\Omega,$$

Considering the following **port behavior**:

- the **storage ports** $(f_1^p,\, f_2^q,\, e_1^{n-p},\, e_2^{n-q}) := (-\partial_t\alpha^p,\, -\partial_t\beta^q,\, \delta_\alpha H^{n-p},\, \delta_\beta H^{n-q})$;
- the **interconnection ports** $(f_\partial^{n-p}, e_\partial^{n-q}) := (y^{n-p}, u^{n-q})$.

Then the dynamical system defines a Dirac structure.

The geometric wave and Maxwell equations

Assume $\Omega \subset \mathbb{R}^3$.

Case $p = 3,\ q = 1$ Wave equation.

$$\begin{pmatrix} \partial_t p^3 \\ \partial_t \boldsymbol{u}^1 \end{pmatrix} = -\begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{pmatrix} \delta_p H^0 \\ \delta_{\boldsymbol{u}} H^2 \end{pmatrix}, \qquad -\delta_{\boldsymbol{u}} H^2 \cdot \boldsymbol{n}|_{\partial\Omega} = u^2,$$
$$y^0 = \delta_p H^0|_{\partial\Omega}.$$

Case $p = 2,\ q = 2$ Maxwell equation

$$\begin{pmatrix} \partial_t \boldsymbol{D}^2 \\ \partial_t \boldsymbol{B}^2 \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{bmatrix} \begin{pmatrix} \delta_{\boldsymbol{D}} H^1 \\ \delta_{\boldsymbol{B}} H^1 \end{pmatrix}, \qquad \boldsymbol{n} \times (\delta_{\boldsymbol{B}} H^1 \times \boldsymbol{n})|_{\partial\Omega} = u^1,$$
$$y^1 = \delta_{\boldsymbol{D}} H^1 \times \boldsymbol{n}|_{\partial\Omega}.$$

What about the functional analytic structure?

Consider the Sobolev space

$$H\Lambda^k(\Omega) := \{\mu^k \in L^2\Lambda^k(\Omega) |\ \mathrm{d}\mu^k \in L^2\Omega^{k+1}(\Omega)\}, \qquad k = 0, \dots, n-1.$$

One could replace spaces of smooth forms with forms living in this Sobolev space. However, the **Stokes formula** has only been proven when **more regularity** is present.

Theorem (D. Arnold *Finite Element Exterior calculus*)

In a manifold Ω with Lipschitz boundary it holds

$$\int_\Omega \mathrm{d}\mu \wedge \lambda + (-1)^k \int_\Omega \mu \wedge \mathrm{d}\lambda = \int_{\partial\Omega} \mathrm{tr}\, \mu \wedge \mathrm{tr}\, \lambda, \qquad \mu \in H^1\Lambda^k(\Omega), \quad \lambda \in H\Lambda^{n-k-1}(\Omega),$$

where $H^1\Lambda^k(\Omega)$ is the space of k-forms with coefficient in $H^1(\Omega)$.

Nevertheless, using conforming finite elements for $H\Lambda^k$, one can obtained a discrete version of the Dirac structure.

Outline

Multiphysics problems

Port-Hamiltonian systems as a unified language for multiphysics

Mimetic discretization of port-Hamiltonian systems

The adjoint pH structure

Using the **Hodge isomorphism**, an adjoint pH structure (associated to an adjoint Dirac structure) is computed

Adjoint pH system

$$\begin{pmatrix} \partial_t \alpha^{n-p} \\ \partial_t \beta^{n-q} \end{pmatrix} = - \begin{bmatrix} 0 & (-1)^{a_0} \,\mathrm{d}^* \\ (-1)^{a_1} \,\mathrm{d}^* & 0 \end{bmatrix} \begin{pmatrix} \delta_\alpha \widehat{H}^p \\ \delta_\beta \widehat{H}^q \end{pmatrix}, \qquad (-1)^{a_3} \operatorname{tr} \star \delta_\beta \widehat{H}^q = u^{n-q},$$

$$y^{n-p} = \operatorname{tr} \star \delta_\alpha \widehat{H}^p,$$

where d^* is the codifferential and a_i are coefficients due to the Hodge star.

Mimetic dual-field discretization[6]

- Combining the port-Hamiltonian system and its adjoint, **two dynamical systems**, whose dynamics is governed **by skew-adjoint operators** are constructed.
- The two systems are put into **weak form** considering variables that live in $H\Lambda^k(\Omega)$ using the L^2 inner product. The codifferential is interpreted weakly using the integration by parts formula.
- Conforming finite element $\mathcal{V}^k \subset H\Lambda^k(\Omega)$ are used for the variables.
- Time integration performed with symplectic Runge-Kutta method based on Gauss-Legendre collocation points.

[6] A. Brugnoli, R. Rashad, and S. Stramigioli. "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus". In: *arXiv preprint arXiv:2202.04390* (2022). Under Review.

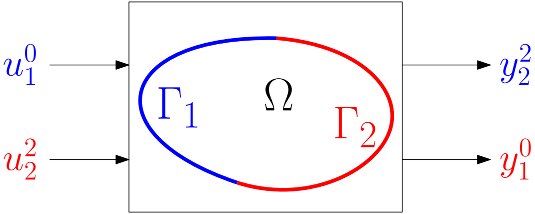
The key ingredient for discretization: the De Rham complex

$$
\begin{tikzcd}
H\Omega^0(M) \arrow[r, "\mathrm{d}"] & H\Omega^1(M) \arrow[r, "\mathrm{d}"] \arrow[d, shift left, "\sharp"] & H\Omega^2(M) \arrow[r, "\mathrm{d}"] \arrow[d, shift left, "\beta^{-1}"] & H\Omega^3(M) \arrow[d, shift left, "\star"] \\
H^1(M) \arrow[u, leftrightarrow, "Id"'] \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi^{-,0}_{s,h}"] & H^{\mathrm{curl}}(M) \arrow[u, shift left, "\flat"] \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^{-,1}_{s,h}"] & H^{\mathrm{div}}(M) \arrow[u, shift left, "\beta"] \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^{-,2}_{s,h}"] & L^2(M) \arrow[u, shift left, "\star^{-1}"] \arrow[d, "\Pi^{-,1}_{s,h}"] \\
\mathrm{CG}_s(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] & \mathrm{NED}^1_s(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] & \mathrm{RT}_s(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] & \mathrm{DG}_{s-1}(\mathcal{T}_h)
\end{tikzcd}
$$

Illustration: the wave equation in 3D

$$\begin{pmatrix} \partial_t p^3 \\ \partial_t \boldsymbol{u}^1 \end{pmatrix} = -\begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{pmatrix} \delta_p H^0 \\ \delta_{\boldsymbol{u}} H^2 \end{pmatrix}, \qquad \begin{aligned} \delta_p H^0|_{\Gamma_1} &= u_1^0, \\ -\delta_{\boldsymbol{u}} H^2 \cdot \boldsymbol{n}|_{\Gamma_2} &= u_2^2, \end{aligned}$$

$$\begin{aligned} y_2^2 &= -\delta_{\boldsymbol{u}} H^2 \cdot \boldsymbol{n}|_{\Gamma_1}, \\ y_1^0 &= \delta_p H^0|_{\Gamma_2}. \end{aligned}$$



Wave equation with mixed Dirichlet and Neumann boundary conditions

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