

Port-Hamiltonian flexible multibody dynamics

Andrea Brugnoli¹

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- 1 Previous work on multibody systems and the pH formalism
- 2 PH formulation of a floating body
 - Canonical and modified momenta
 - Towards a pH formulation
 - The pHDAE system
 - Discretization
- 3 Construction of multibody chain
 - General procedure for planar beams
 - The linear case
 - Example: boundary interconnection of the Kirchhoff plate

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Using Lie Algebra and differential forms a pH model of a flexible link has already been proposed¹. This model can be embedded in a complex multibody system².

Advantages:

- Modular construction of flexible systems;
- Large deformations naturally considered.

Drawbacks:

- Implementation really does not look trivial;
- Limited to one-dimensional systems;
- Numerical analysis not feasible;
- Model reduction techniques not easily applicable.

¹A. Macchelli, C. Melchiorri, and S. Stramigioli. "Port-Based Modeling of a Flexible Link". In: *IEEE Transactions on Robotics* 23 (2007), pp. 650–660. DOI: [10.1109/TR0.2007.898990](https://doi.org/10.1109/TR0.2007.898990).

²A. Macchelli, C. Melchiorri, and S. Stramigioli. "Port-Based Modeling and Simulation of Mechanical Systems With Rigid and Flexible Links". In: *IEEE Transactions on Robotics* 25.5 (2009), pp. 1016–1029. DOI: [10.1109/TR0.2009.2026504](https://doi.org/10.1109/TR0.2009.2026504).

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Floating frame of reference approach

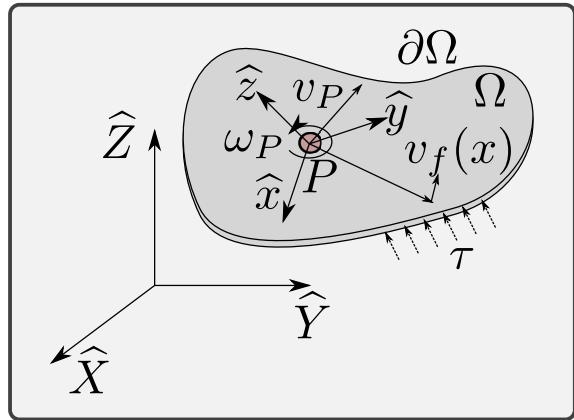
The elastic motion is described in a reference frame that follows the large rigid motion.

Advantages

- Most used paradigm in multibody dynamics;
- For control other approaches are too complex;
- Linear model reduction techniques are applicable.

Drawbacks:

- Not suitable for large deformations.



The velocity of a generic point is expressed by considering a small flexible displacement superimposed to the rigid motion

$$\mathbf{v} = \mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times}(\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f.$$

where the cross map $[\mathbf{a}]_{\times}$ denotes the skew-symmetric matrix associated to vector \mathbf{a} .
This equation is expressed in the body reference frame $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

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Kinematic variables:

- \mathbf{x} is the position vector of the current point;
- $\mathbf{v}_P, \boldsymbol{\omega}_P$ are the linear and angular velocities of point P ;
- $\mathbf{v}_f := \dot{\mathbf{u}}_f$ the deformation velocity.

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Inertia terms:

- $m := \int_{\Omega} \rho \, d\Omega$ the total mass;
- $\mathbf{s}_u := \int_{\Omega} \rho(\mathbf{x} + \mathbf{u}_f) \, d\Omega$ the static moment;
- $\mathbf{J}_u := \int_{\Omega} \rho[\mathbf{x} + \mathbf{u}_f]_{\times}^{\top}[\mathbf{x} + \mathbf{u}_f]_{\times} \, d\Omega$ the inertia matrix.

Canonical momenta

Consider the total energy (Hamiltonian), given by the sum of kinetic and deformation energy:

$$H = H_{\text{kin}} + H_{\text{def}} = \frac{1}{2} \int_{\Omega} \left\{ \rho \| \mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times} (\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f \|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} d\Omega.$$

Canonical momenta

$$\mathbf{p}_t := \frac{\partial H}{\partial \mathbf{v}_P} = m \mathbf{v}_P + [\mathbf{s}_u]_{\times}^{\top} \boldsymbol{\omega}_P + \int_{\Omega} \rho \mathbf{v}_f d\Omega,$$

$$\mathbf{p}_r := \frac{\partial H}{\partial \boldsymbol{\omega}_P} = [\mathbf{s}_u]_{\times} \mathbf{v}_P + \mathbf{J}_u \boldsymbol{\omega}_P + \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \mathbf{v}_f d\Omega,$$

$$\mathbf{p}_f := \frac{\delta H}{\delta \mathbf{v}_f} = \rho \mathbf{v}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \boldsymbol{\omega}_P + \rho \mathbf{v}_f,$$

$$\boldsymbol{\varepsilon} := \frac{\delta H}{\delta \boldsymbol{\Sigma}} = \boldsymbol{\mathcal{D}}^{-1} \boldsymbol{\Sigma},$$

Canonical momenta

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$$H = H_{\text{kin}} + H_{\text{def}} = \frac{1}{2} \int_{\Omega} \left\{ \rho \| \mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times} (\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f \|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} d\Omega.$$

Canonical momenta

$$\begin{bmatrix} \mathbf{p}_t \\ \mathbf{p}_r \\ \mathbf{p}_f \\ \boldsymbol{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} m\mathbf{I}_{3 \times 3} & [\mathbf{s}_u]_{\times}^{\top} & \mathcal{I}_{\rho}^{\Omega} & 0 \\ [\mathbf{s}_u]_{\times} & \mathbf{J}_u & \mathcal{I}_{\rho x}^{\Omega} & 0 \\ (\mathcal{I}_{\rho}^{\Omega})^* & (\mathcal{I}_{\rho x}^{\Omega})^* & \rho & 0 \\ 0 & 0 & 0 & \mathcal{D}^{-1} \end{bmatrix}}_{\mathcal{M}: \text{Mass operator}} \begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega}_P \\ \mathbf{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}, \quad \begin{aligned} \mathcal{I}_{\rho}^{\Omega} &:= \int_{\Omega} \rho(\cdot) d\Omega, \\ \mathcal{I}_{\rho x}^{\Omega} &:= \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times}(\cdot). \end{aligned}$$

The mass operator \mathcal{M} is a self-adjoint, positive operator. It holds

$$H_{\text{kin}} + H_{\text{def}} = \frac{1}{2} \langle \mathbf{e}_{\text{kd}}, \mathcal{M} \mathbf{e}_{\text{kd}} \rangle, \quad \mathbf{e}_{\text{kd}} = [\mathbf{v}_P; \boldsymbol{\omega}_P; \mathbf{v}_f; \boldsymbol{\Sigma}]$$

Canonical momenta

Consider the total energy (Hamiltonian), given by the sum of kinetic and deformation energy:

$$H = H_{\text{kin}} + H_{\text{def}} = \frac{1}{2} \int_{\Omega} \left\{ \rho \| \mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times} (\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f \|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} d\Omega.$$

Modified canonical momenta

$$\begin{aligned} \hat{\mathbf{p}}_t &:= m \mathbf{v}_P + [\mathbf{s}_u]_{\times}^{\top} \boldsymbol{\omega}_P + 2 \int_{\Omega} \rho \mathbf{v}_f d\Omega, & \hat{\mathbf{p}}_f &:= \rho \mathbf{v}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \boldsymbol{\omega}_P + 2 \rho \mathbf{v}_f, \\ \hat{\mathbf{p}}_r &:= [\mathbf{s}_u]_{\times} \mathbf{v}_P + \mathbf{J}_u \boldsymbol{\omega}_P + 2 \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \mathbf{v}_f d\Omega, & \mathcal{I}_{p_f}^{\Omega} &:= \int_{\Omega} \{ [\mathbf{p}_f]_{\times} + [\hat{\mathbf{p}}_f]_{\times} \} (\cdot) d\Omega, \end{aligned}$$

Notice that the kinetic energy also depends on the flexible displacement

$$\frac{\delta H_{\text{kin}}}{\delta \mathbf{u}_f} = [\mathbf{p}_f]_{\times} \boldsymbol{\omega}_P.$$

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The equations are obtained by application of the virtual work principle³.

Linear momentum balance

$$\begin{aligned} m(\dot{\mathbf{v}}_P + [\boldsymbol{\omega}_P]_{\times} \mathbf{v}_P) + [\mathbf{s}_u]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho \ddot{\mathbf{u}}_f \, d\Omega = \\ - [\boldsymbol{\omega}_P]_{\times} [\boldsymbol{\omega}_P]_{\times} \mathbf{s}_u - \int_{\Omega} 2\rho [\boldsymbol{\omega}_P]_{\times} \dot{\mathbf{u}}_f \, d\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, d\Gamma, \end{aligned}$$

³Bernd Simeon. *Computational flexible multibody dynamics*. Springer, 2013, Chapter 4.

Equations of motion

The equations are obtained by application of the virtual work principle³.

Linear momentum balance

$$m\dot{\mathbf{v}}_P + [\mathbf{s}_u]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho \dot{\mathbf{v}}_f \, d\Omega =$$
$$\left[m\mathbf{v}_P + [\mathbf{s}_u]_{\times}^{\top} \boldsymbol{\omega}_P + 2 \int_{\Omega} \rho \mathbf{v}_f \, d\Omega \right]_{\times} \boldsymbol{\omega}_P + \int_{\partial\Omega} \boldsymbol{\tau} \, d\Gamma.$$

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Linear momentum balance

$$m\dot{\mathbf{v}}_P + [\mathbf{s}_u]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho \dot{\mathbf{v}}_f \, d\Omega = [\hat{\mathbf{p}}_t]_{\times} \boldsymbol{\omega}_P + \int_{\partial\Omega} \boldsymbol{\tau} \, d\Gamma.$$

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Angular momentum balance

$$[\mathbf{s}_u]_{\times}(\dot{\mathbf{v}}_P + [\boldsymbol{\omega}_P]_{\times}\mathbf{v}_P) + \mathbf{J}_u\dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho[\mathbf{x} + \mathbf{u}_f]_{\times}\ddot{\mathbf{u}}_f \, d\Omega + [\boldsymbol{\omega}_P]_{\times}\mathbf{J}_u\boldsymbol{\omega}_P = \\ - \int_{\Omega} 2\rho[\mathbf{x} + \mathbf{u}_f]_{\times}[\boldsymbol{\omega}_P]_{\times}\dot{\mathbf{u}}_f \, d\Omega + \int_{\partial\Omega} [\mathbf{x} + \mathbf{u}_f]_{\times}\boldsymbol{\tau} \, d\Gamma,$$

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Angular momentum balance

$$\begin{aligned} & [\mathbf{s}_u]_{\times} \dot{\mathbf{v}}_P + \mathbf{J}_u \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \dot{\mathbf{v}}_f \, d\Omega = \\ & \left[[\mathbf{s}_u]_{\times}^{\top} \boldsymbol{\omega}_P + 2 \int_{\Omega} \rho \mathbf{v}_f \, d\Omega \right]_{\times} \mathbf{v}_P + \left[[\mathbf{s}_u]_{\times} \mathbf{v}_P + \mathbf{J}_u \boldsymbol{\omega}_P + 2 \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \mathbf{v}_f \, d\Omega \right]_{\times} \boldsymbol{\omega}_P + \\ & \int_{\Omega} 2 \left[\rho \mathbf{v}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \boldsymbol{\omega}_P \right]_{\times} \mathbf{v}_f \, d\Omega + \int_{\partial\Omega} [\mathbf{x} + \mathbf{u}_f]_{\times} \boldsymbol{\tau} \, d\Gamma. \end{aligned}$$

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Angular momentum balance

$$[\mathbf{s}_u]_{\times} \dot{\mathbf{v}}_P + \mathbf{J}_u \dot{\boldsymbol{\omega}}_P + \int_{\Omega} \rho [\mathbf{x} + \mathbf{u}_f]_{\times} \dot{\mathbf{v}}_f \, d\Omega =$$
$$[\hat{\mathbf{p}}_t]_{\times} \mathbf{v}_P + [\hat{\mathbf{p}}_r]_{\times} \boldsymbol{\omega}_P + \mathcal{I}_{p_f}^{\Omega} \mathbf{v}_f + \int_{\partial\Omega} [\mathbf{x} + \mathbf{u}_f]_{\times} \boldsymbol{\tau} \, d\Gamma.$$

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The equations are obtained by application of the virtual work principle³.

Flexibility PDE

$$\rho(\dot{\mathbf{v}}_P + [\boldsymbol{\omega}_P]_{\times} \mathbf{v}_P) + \rho([\dot{\boldsymbol{\omega}}_P]_{\times} + [\boldsymbol{\omega}_P]_{\times} [\boldsymbol{\omega}_P]_{\times})(\mathbf{x} + \mathbf{u}_f) + \rho(2[\boldsymbol{\omega}_P]_{\times} \dot{\mathbf{u}}_f + \ddot{\mathbf{u}}_f) = \text{Div } \boldsymbol{\Sigma},$$

together with boundary conditions

Neumann condition $\boldsymbol{\Sigma} \cdot \mathbf{n}|_{\Gamma_N} = \boldsymbol{\tau}|_{\Gamma_N}$, \mathbf{n} is the outward normal,

Dirichlet condition $\mathbf{u}_f|_{\Gamma_D} = \bar{\mathbf{u}}_f|_{\Gamma_D}$,

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Equations of motion

The equations are obtained by application of the virtual work principle³.

Flexibility PDE

$$\rho \dot{\mathbf{v}}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \rho \dot{\mathbf{v}}_f = \left[\rho \mathbf{v}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \boldsymbol{\omega}_P + 2\rho \mathbf{v}_f \right]_{\times} \boldsymbol{\omega}_P + \text{Div } \boldsymbol{\Sigma}.$$

together with boundary conditions

Neumann condition $\boldsymbol{\Sigma} \cdot \mathbf{n}|_{\Gamma_N} = \boldsymbol{\tau}|_{\Gamma_N}$, \mathbf{n} is the outward normal,

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Equations of motion

The equations are obtained by application of the virtual work principle³.

Flexibility PDE

$$\rho \dot{\mathbf{v}}_P + \rho [\mathbf{x} + \mathbf{u}_f]_{\times}^{\top} \dot{\boldsymbol{\omega}}_P + \rho \dot{\mathbf{v}}_f = -\delta_{\mathbf{u}_f} H - \mathcal{I}_{pf}^* \boldsymbol{\omega}_P + \text{Div } \boldsymbol{\Sigma}.$$

together with boundary conditions

Neumann condition $\boldsymbol{\Sigma} \cdot \mathbf{n}|_{\Gamma_N} = \boldsymbol{\tau}|_{\Gamma_N}$, \mathbf{n} is the outward normal,

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Generalized coordinates are required for a complete formulation:

- ${}^i\mathbf{r}_P$ the position of point P in the inertial frame of reference;
- \mathbf{R} the direction cosine matrix (other attitude parametrizations are possible);
- \mathbf{u}_f the flexible displacement;

The direction cosine matrix is converted into a vector by concatenating its rows

$$\mathbf{R}_v = \text{vec}(\mathbf{R}^\top) = [\mathbf{R}_1 \ \mathbf{R}_2 \ \mathbf{R}_3]^\top,$$

where $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ are the rows of matrix \mathbf{R} . Furthermore the corresponding cross map will be given by

$$[\mathbf{R}_v]_\times = \begin{bmatrix} [\mathbf{R}_1]_\times \\ [\mathbf{R}_2]_\times \\ [\mathbf{R}_3]_\times \end{bmatrix}, \quad [\mathbf{R}_v]_\times : \mathbb{R}^9 \rightarrow \mathbb{R}^{9 \times 3}.$$

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The overall port-Hamiltonian formulation (without including the boundary traction τ)

$$\underbrace{\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathcal{M} \end{bmatrix}}_{\mathcal{E}} \frac{\partial}{\partial t} \underbrace{\begin{bmatrix} {}^i r_P \\ \mathbf{R}_v \\ u_f \\ v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix}}_e = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \mathbf{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\mathbf{R}_v]_{\times} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_{3 \times 3} & 0 \\ -\mathbf{R}^{\top} & 0 & 0 & 0 & [\hat{\mathbf{p}}_t]_{\times} & 0 & 0 \\ 0 & -[\mathbf{R}_v]_{\times}^{\top} & 0 & [\hat{\mathbf{p}}_t]_{\times} & [\hat{\mathbf{p}}_r]_{\times} & \mathcal{I}_{p_f}^{\Omega} & 0 \\ 0 & 0 & -\mathbf{I}_{3 \times 3} & 0 & -(\mathcal{I}_{p_f}^{\Omega})^* & 0 & \text{Div} \\ 0 & 0 & 0 & 0 & 0 & \text{Grad} & 0 \end{bmatrix}}_{\mathcal{J}} \underbrace{\begin{bmatrix} \partial_{r_P} H \\ \partial_{\mathbf{R}_v} H \\ \delta_{u_f} H \\ v_P \\ \omega_P \\ v_f \\ \Sigma \end{bmatrix}}_z.$$

Final pHDAE system

This system fits into the framework detailed in⁴ and extends it.

$$\mathcal{E}(e)\partial_t e = \mathcal{J}(e)z(e) + \mathcal{B}_r(e)u_\partial,$$

$$y_r = \mathcal{B}_r^*(e)z(e),$$

$$u_\partial = \mathcal{B}_\partial z(e) = \Sigma \cdot n|_{\partial\Omega} = \tau|_{\partial\Omega},$$

$$y_\partial = \mathcal{C}_\partial z(e) = v_f|_{\partial\Omega},$$

with $y_r = (v_P + [x + u_f]^\top_\times \omega_P)|_{\partial\Omega}$.

Operator \mathcal{E} is positive self-adjoint, \mathcal{J} is formally skew-symmetric. The Hamiltonian satisfies

$$\partial_e H = \mathcal{E}^* z.$$

⁴Volker Mehrmann and Riccardo Morandin. "Structure-preserving discretization for port-Hamiltonian descriptor systems". In: *Proceedings of the 59th IEEE Conference on Decision and Control*. 2019, pp. 6663–6868.

Power balance

The power balance equals the power due to the surface traction

$$\begin{aligned}\dot{H}(\mathbf{e}) &= \langle \partial_{\mathbf{e}} H, \partial_t \mathbf{e} \rangle_X, \\ &= \langle \mathbf{z}, \mathcal{E} \partial_t \mathbf{e} \rangle_X, \quad \text{Adjoint definition,} \\ &= \langle \mathbf{y}_{\partial}, \mathbf{u}_{\partial} \rangle_{\partial\Omega} + \langle \mathcal{B}_r^* \mathbf{z}, \mathbf{u}_{\partial} \rangle_{\partial\Omega}, \quad \text{I.B.P. on } \mathcal{J}, \\ &= \int_{\partial\Omega} \mathbf{u}_{\partial} \cdot (\mathbf{y}_{\partial} + \mathbf{y}_r) \, d\Omega, \\ &= \int_{\partial\Omega} \boldsymbol{\tau} \cdot \mathbf{v} \, d\Gamma,\end{aligned}$$

where $\mathbf{y}_{\partial} + \mathbf{y}_r := (\mathbf{v}_P + [\boldsymbol{\omega}_P]_{\times}(\mathbf{x} + \mathbf{u}_f) + \mathbf{v}_f)|_{\partial\Omega} = \mathbf{v}|_{\partial\Omega}$ is the velocity field at the boundary.

Some remarks

- Generic linear elastic model can be included.
- Conservative forces are easily accounted for by introducing an appropriate potential energy. The gravitational potential

$$H_{\text{pot}} = \int_{\Omega} \rho g {}^i r_z \, d\Omega = \int_{\Omega} \rho g \left[{}^i r_{P,z} + \mathbf{R}_z(\mathbf{x} + \mathbf{u}_f) \right] d\Omega.$$

- Geometric stiffening could be considered by adding a potential energy associated to centrifugal forces or using a substructuring technique.
- If case of vanishing deformations $\mathbf{u}_f \equiv 0$, the Newton-Euler equations on the Euclidean group $SE(3)$ are retrieved

$$\frac{d}{dt} \begin{pmatrix} {}^i \mathbf{r}_P \\ \mathbf{R}_v \\ \mathbf{p}_t \\ \mathbf{p}_r \end{pmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & [\mathbf{R}_v]_{\times} \\ -\mathbf{R}^{\top} & 0 & 0 & [\mathbf{p}_t]_{\times} \\ 0 & -[\mathbf{R}_v]_{\times}^{\top} & [\mathbf{p}_t]_{\times} & [\mathbf{p}_r]_{\times} \end{bmatrix} \begin{bmatrix} \partial_{\mathbf{r}_P} H \\ \partial_{\mathbf{R}_v} H \\ \partial_{\mathbf{p}_t} H \\ \partial_{\mathbf{p}_r} H \end{bmatrix}.$$

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Same procedure as always but the integration by parts is applied to the Div operator to highlight the Neumann condition.

Finite-dimensional pHDAE system

After integration by parts of the Div operator

$$\mathbf{E}(\mathbf{e})\dot{\mathbf{e}} = \mathbf{J}(\mathbf{e})\mathbf{z}(\mathbf{e}) + \mathbf{B}_d(\mathbf{e})\mathbf{u}_d + \mathbf{B}_\partial(\mathbf{e})\mathbf{u}_\partial,$$

$$\mathbf{y}_d := \mathbf{M}_d\tilde{\mathbf{y}}_d = \mathbf{B}_d^\top \mathbf{z}(\mathbf{e}),$$

$$\mathbf{y}_\partial := \mathbf{M}_\partial\tilde{\mathbf{y}}_\partial = \mathbf{B}_\partial^\top \mathbf{z}(\mathbf{e}).$$

Dirichlet conditions

The set Γ_D for the Dirichlet condition has to be non empty, otherwise the deformation field is allowed for rigid movement, leading to a singular mass matrix. Test and state shape functions must verify an homogeneous Dirichlet condition⁵.

⁵O.P. Agrawal and A.A. Shabana. "Application of deformable-body mean axis to flexible multibody system dynamics". In: *Computer Methods in Applied Mechanics and Engineering* 56.2 (1986), pp. 217–245.

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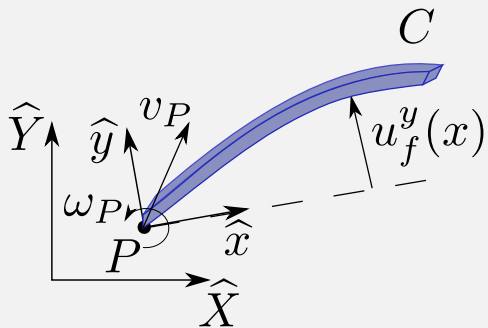


Figure: Floating beam.

Beam discretized system

Neglecting the dependence on the deformation field in the mass matrix ($\mathbf{M} = \text{const}$)

$$\mathbf{E}\dot{\mathbf{e}} = \mathbf{J}(\mathbf{e})\mathbf{z}(\mathbf{e}) + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{B}^\top \mathbf{z},$$

with boundary variables

$$\mathbf{u} = [F_P^x, F_P^y, T_P^z, F_C^x, F_C^y, T_C^z]^\top,$$

$$\mathbf{y} = [v_P^x, v_P^y, \omega_P^z, v_C^x, v_C^y, \omega_C^z]^\top.$$

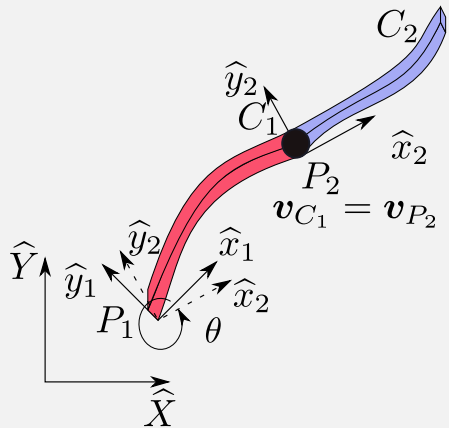


Figure: Two hinged beams.

The interconnection variables are

$$\mathbf{u}_1^{\text{int}} = [F_{C_1}^x, F_{C_1}^y]^\top := \mathbf{F}_{C_1},$$

$$\mathbf{u}_2^{\text{int}} = [F_{P_2}^x, F_{P_2}^y]^\top := \mathbf{F}_{P_2},$$

$$\mathbf{y}_1^{\text{int}} = [v_{C_1}^x, v_{C_1}^y]^\top := \mathbf{v}_{C_1},$$

$$\mathbf{y}_2^{\text{int}} = [v_{P_2}^x, v_{P_2}^y]^\top := \mathbf{v}_{P_2}.$$

Hinged interconnected beams

The transformer interconnection

$$\mathbf{u}_1^{\text{int}} = -\mathbf{R}(\theta)\mathbf{u}_2^{\text{int}}, \quad \mathbf{y}_2^{\text{int}} = \mathbf{R}(\theta)^\top \mathbf{y}_1^{\text{int}},$$

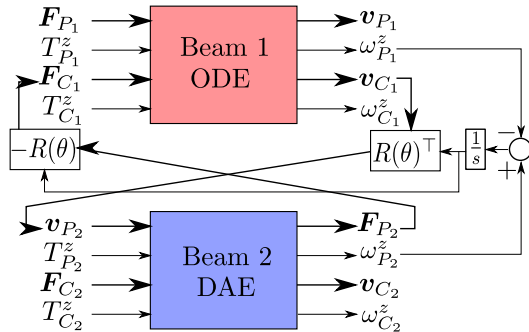
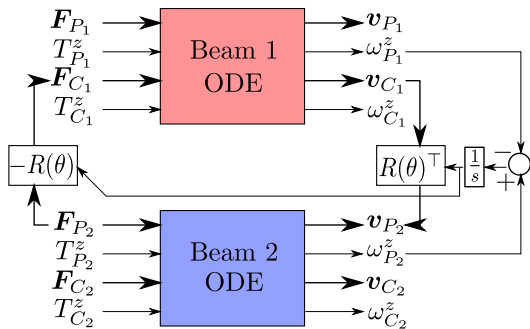
where $\mathbf{R}(\theta)$ is the relative rotation matrix, imposes the constraints on the velocity level and gives rise to a quasi-linear index 2 pHDAE.

$$\begin{bmatrix} \mathbf{E}_1 & 0 & 0 \\ 0 & \mathbf{E}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1(\mathbf{e}_1) & 0 & -\mathbf{B}_1^{\text{int}}\mathbf{R} \\ 0 & \mathbf{J}_2(\mathbf{e}_2) & \mathbf{B}_2^{\text{int}} \\ \mathbf{R}^\top \mathbf{B}_1^{\text{int}\top} & -\mathbf{B}_2^{\text{int}\top} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{\partial 1}^{\text{ext}} & 0 \\ 0 & \mathbf{B}_{\partial 2}^{\text{ext}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\text{ext}} \\ \mathbf{u}_2^{\text{ext}} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{y}_1^{\text{ext}} \\ \mathbf{y}_2^{\text{ext}} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\partial 1}^{\text{ext}\top} & 0 & 0 \\ 0 & \mathbf{B}_{\partial 2}^{\text{ext}\top} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \lambda \end{bmatrix}.$$

Equivalence of gyrator and transformer interconnection

The same result can be obtained by using a pHDAE system and a gyrator interconnection. It is sufficient to interchange the role of output and input of the second system $\mathbf{u}_2^{\text{int}} \leftrightarrow \mathbf{y}_2^{\text{int}}$.



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Hypothesis:

- 1 small angular velocities;
- 2 small relative configuration.

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf} & 0 \\ \mathbf{M}_{fr} & \mathbf{M}_{ff} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_r \\ \dot{\mathbf{p}}_f \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{G}_r^\top \\ 0 & \mathbf{J}_{ff} & \mathbf{G}_f^\top \\ -\mathbf{G}_r & -\mathbf{G}_f & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_r \\ \mathbf{p}_f \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_r \\ \mathbf{B}_f \\ 0 \end{bmatrix} \mathbf{u}.$$

with Hamiltonian $H = \frac{1}{2} \mathbf{p}^\top \mathbf{M} \mathbf{p}$. The modular construction of complex multi-body systems is then analogous to a sub-structuring technique⁶.

⁶D. De Klerk, D. J. Rixen, and S. N. Voormeeren. "General Framework for Dynamic Substructuring: History, Review and Classification of Techniques". In: *AIAA Journal* 46.5 (2008), pp. 1169–1181. DOI: 10.2514/1.33274.

Model reduction

Such system can be reduced using linear model reduction methods directly in the DAE⁷.

Vector \mathbf{p}_f is projected on a meaningful subspace $\mathbf{p}_f \approx \mathbf{V}_f^{\text{red}} \mathbf{p}_f^{\text{red}}$

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{rf}^{\text{red}} & 0 \\ \mathbf{M}_{fr}^{\text{red}} & \mathbf{M}_{ff}^{\text{red}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_r \\ \dot{\mathbf{p}}_f^{\text{red}} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathbf{G}_r^{\top} \\ 0 & \mathbf{J}_{ff}^{\text{red}} & \mathbf{G}_f^{\text{red}\top} \\ -\mathbf{G}_r & -\mathbf{G}_f^{\text{red}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_r \\ \mathbf{p}_f^{\text{red}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{B}_r \\ \mathbf{B}_f^{\text{red}} \\ 0 \end{bmatrix} \mathbf{u},$$

⁷H. Egger et al. “On Structure-Preserving Model Reduction for Damped Wave Propagation in Transport Networks”. In: *SIAM Journal on Scientific Computing* 40.1 (2018), A331–A365. DOI: 10.1137/17M1125303.

Index reduction

$$\begin{aligned} \mathbf{M}\dot{\mathbf{p}} &= \mathbf{J}\mathbf{p} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{B}\mathbf{u}, \\ \mathbf{0} &= \mathbf{G}\mathbf{p}, \end{aligned}$$

A null space matrix can be employed to eliminate the Lagrange multiplier and preserve the port-Hamiltonian structure.

$$\text{range}\{\mathbf{P}\} = \text{null}\{\mathbf{G}\}.$$

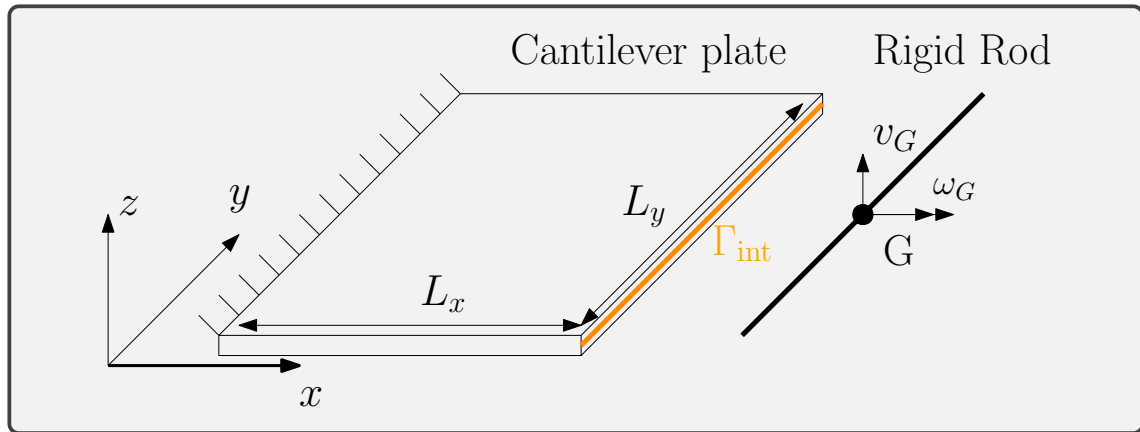
Then, the range of \mathbf{P} automatically satisfies the constraints. Considering the transformation $\hat{\mathbf{p}} = \mathbf{P}\mathbf{p}$ and pre-multiplying the system by \mathbf{P}^\top an equivalent ODE is obtained

$$\widehat{\mathbf{M}} \dot{\hat{\mathbf{p}}} = \widehat{\mathbf{J}} \hat{\mathbf{p}} + \widehat{\mathbf{B}} \mathbf{u},$$

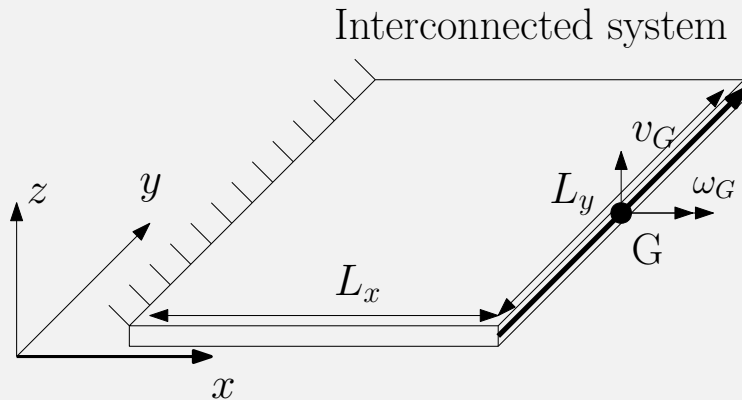
with $\widehat{\mathbf{M}} = \mathbf{P}^\top \mathbf{M} \mathbf{P}$, $\widehat{\mathbf{J}} = \mathbf{P}^\top \mathbf{J} \mathbf{P}$, $\widehat{\mathbf{B}} = \mathbf{P}^\top \mathbf{B}$.

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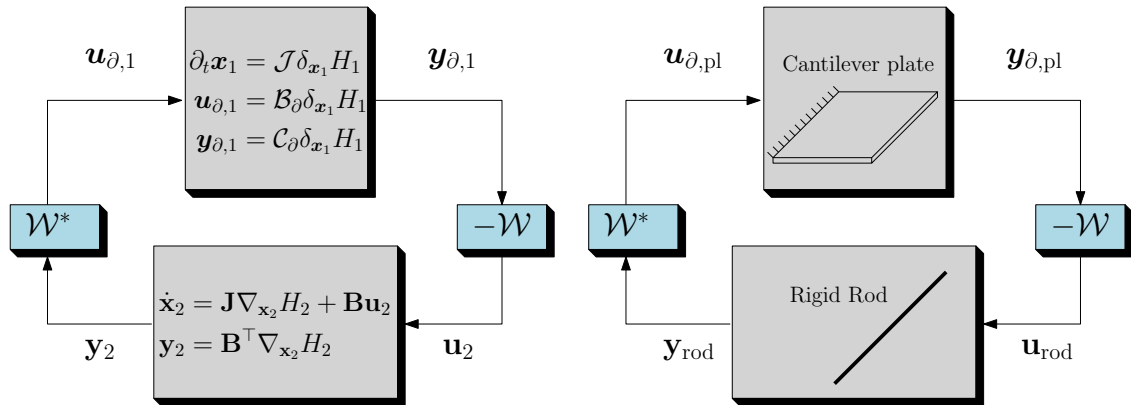
Rigid rod welded to a cantilever plate



Rigid rod welded to a cantilever plate



Boundary interconnection



$$\begin{array}{ll}
 \text{Plate } (\Omega = [0, L_x] \times [0, L_y]) & \text{Rigid rod} \\
 \begin{bmatrix} \rho h & 0 \\ 0 & \mathcal{C}_b \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} e_w \\ \mathbf{E}_\kappa \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} e_w \\ \mathbf{E}_\kappa \end{bmatrix} & \begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \mathbf{u}_{\text{rod}}, \\
 u_{\partial, \text{pl}} = e_w(x = L_x, y), & \mathbf{y}_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix}, \\
 y_{\partial, \text{pl}} = \tilde{q}_n(x = L_x, y). &
 \end{array}$$

The interconnection operator provides the total force and torque acting on the rigid rod

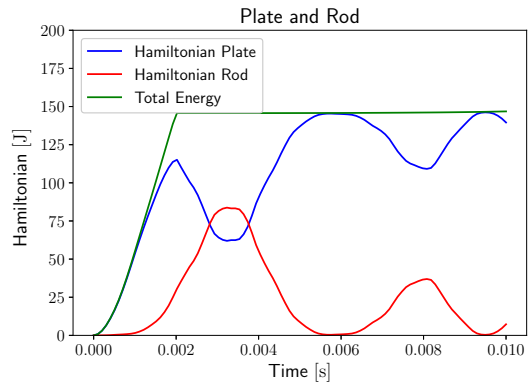
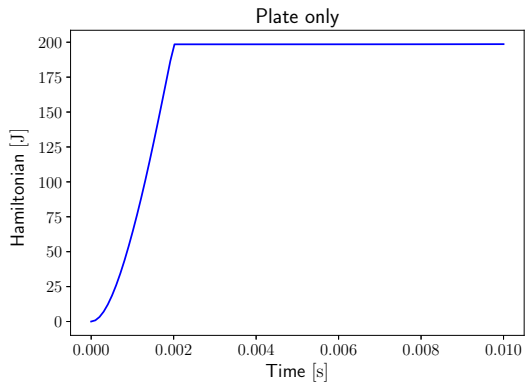
$$\mathcal{W} y_{\partial, \text{pl}} = - \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial, \text{pl}} \, ds \\ \int_{\Gamma_{\text{int}}} (y - L_y/2) y_{\partial, \text{pl}} \, ds \end{pmatrix}.$$

The adjoint operator provides a rigid movement as the plate input at Γ_{int}

$$\begin{aligned}
 \langle \mathcal{W} y_{\partial, \text{pl}}, \mathbf{y}_{\text{rod}} \rangle_{\mathbb{R}^m} &= \langle y_{\partial, \text{pl}}, \mathcal{W}^* \mathbf{y}_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})}, \\
 \mathcal{W}^* \mathbf{y}_{\text{rod}} &= v_G + \omega_G (y - L_y/2).
 \end{aligned}$$

Load

$$f = \begin{cases} 10^5 \left[y + 10 (y - L_y/2)^2 \right] \text{ [Pa]}, & \forall t < 2 \text{ [ms]}, \\ 0, & \forall t \geq 2 \text{ [ms]}, \end{cases} \quad t_{\text{end}} = 10 \text{ [ms]}.$$



Summarizing:

- PH formulation of floating bodies;
- Finite element discretization;
- Interconnection of subcomponents;
- Linearized case.

Some open questions:

- Stability and convergence of FE⁷;
- Time discretization;
- Non-linear model reduction of pHDAE;
- Control strategies.

A. Brugnoli, V. Alazard D.and Pommier-Budinger, and D. Matignon. “Port-Hamiltonian flexible multibody dynamics”. In: *Multibody System Dynamics* (2020). DOI: [10.1007/s11044-020-09758-6](https://doi.org/10.1007/s11044-020-09758-6)

⁷G. Cohen and S. Fauqueux. “Mixed Spectral Finite Elements for the Linear Elasticity System in Unbounded Domains”. In: *SIAM Journal on Scientific Computing* 26.3 (2005), pp. 864–884. DOI: [10.1137/S1064827502407457](https://doi.org/10.1137/S1064827502407457).

Thanks for your attention
Questions?

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