December, the 13th, 2019

Interconnection of the Kirchhoff plate within the port-Hamiltonian framework

Andrea Brugnoli[1]

[1]ISAE-SUPAERO, Toulouse

Outline

Outline

1. The Kirchhoff plate as a port-Hamiltonian system
   - Classical formulation
   - Port-Hamiltonian formulation
2. Structure preserving discretization
3. Interconnection with rigid elements
4. Stabilization by boundary injection

The classical Kirchhoff model

Classical bilaplacian formulation

For an homogeneous isotropic material

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\Delta^2 w = p, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2.$$

$\Delta^2 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$ is the bilaplacian operator

- ρ $[\mathrm{kg/m^3}]$ is the mass density;
- h $[\mathrm{m}]$ is the plate thickness;
- p $[\mathrm{N/m^2}]$ is an external distributed force;
- D $[\mathrm{Pa\,m}]$ is the bending stiffness;

The classical Kirchhoff model

Bending moment formulation

$$\rho h \frac{\partial^2 w}{\partial t^2} + \operatorname{div} \operatorname{Div}(\boldsymbol{M}) = p, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2.$$

Where $\boldsymbol{M} = \mathbb{D} \nabla^2 w \in \mathbb{R}^{2\times 2}_{\text{sym}}$ is the bending moment tensor and $\nabla^2 = \operatorname{Grad} \circ \operatorname{grad}$ the Hessian.

$$\operatorname{div} \operatorname{Div}(\boldsymbol{M}) = \partial_{xx} M_{11} + 2\partial_{xy} M_{12} + \partial_{yy} M_{22}$$

- ρ $[\mathrm{kg}/\mathrm{m}^3]$ is the mass density;
- h $[\mathrm{m}]$ is the plate thickness;
- p $[\mathrm{N}/\mathrm{m}^2]$ is an external distributed force;
- $\mathbb{D}$ is the bending rigidity tensor (symmetric, positive). For an homogeneous isotropic material

$$\mathbb{D}\boldsymbol{A} = D \left\{ (1-\nu)\boldsymbol{A} + \nu \operatorname{Tr}(\boldsymbol{A}) \boldsymbol{I} \right\};$$

Boundary conditions

For the boundary variables consider the definitions

$$
\begin{aligned}
\text{Flexural moment} \quad & M_{nn} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}), \\
\text{Torsional moment} \quad & M_{ns} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{s}), \\
\text{Effective shear force} \quad & \widetilde{q}_n = -(\mathrm{Div}\, \boldsymbol{M}) \cdot \boldsymbol{n} - \partial_{\boldsymbol{s}} M_{ns},
\end{aligned}
$$

where $\boldsymbol{n}$, $\boldsymbol{s}$ are the normal and tangential versors along the boundary $\partial\Omega$.
$\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j} A_{ij} B_{ij}$ is the tensor contraction and $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top \in \mathbb{R}^{2\times 2}$ is the dyadic product between vectors.
Consider a partition of the boundary: $\partial\Omega = \Gamma_c \cup \Gamma_s \cup \Gamma_f$.

- Γ_c is the clamped part, i.e. w, $\partial_{\boldsymbol{n}} w$ known;
- Γ_s is the simply supported part, i.e. w, M_{nn} known;
- Γ_f is the free part, i.e. M_{nn}, q_n known;

Outline

Hamiltonian and energy variables

The total energy of the system is given by the sum of kinetic and deformation energy

$$H = \frac{1}{2}\int_{\Omega} \left\{ \rho \left(\partial_t w\right)^2 + \mathbb{D}\nabla^2 w : \nabla^2 w \right\} \, \mathrm{d}\Omega$$

Consider the following choice for the energy variables

$$\begin{aligned} \alpha_1 &:= \rho \partial_t w, \quad \text{Linear momentum} \\ \boldsymbol{A}_2 &:= \nabla^2 w, \quad \text{Curvature} \end{aligned}$$

This leads to the following co-energy variables

$$\begin{aligned} e_1 &:= \frac{\delta H}{\delta \alpha_1} = \partial_t w = (\rho h)^{-1} \alpha_1, \quad \text{Velocity} \\ \boldsymbol{E}_2 &:= \frac{\delta H}{\delta \boldsymbol{A}_2} = \boldsymbol{M} = \mathbb{D} \boldsymbol{A}_2, \quad \text{Bending moment} \end{aligned}$$

Port Hamiltonian formulation

The system

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_1 \\ \boldsymbol{A}_2\end{pmatrix} = \underbrace{\begin{bmatrix}0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}e_1 \\ \boldsymbol{E}_2\end{pmatrix}, \qquad \begin{pmatrix}e_1 \\ \boldsymbol{E}_2\end{pmatrix} = \underbrace{\begin{bmatrix}(\rho h)^{-1} & 0 \\ 0 & \mathbb{D}\end{bmatrix}}_{\mathcal{Q}}\begin{pmatrix}\alpha_1 \\ \boldsymbol{A}_2\end{pmatrix}$$

with homogeneous boundary conditions

$$w|_{\Gamma_c} = \partial_{\boldsymbol{n}} w|_{\Gamma_c} = 0, \quad w|_{\Gamma_s} = M_{nn}|_{\Gamma_s} = 0, \quad q_n|_{\Gamma_f} = M_{nn}|_{\Gamma_f} = 0$$

defines a Stokes-Dirac structure.

Notice that $D(\mathcal{J}) = H^2_{\Gamma_c \cup \Gamma_s}(\Omega) \times H^{\operatorname{div}\operatorname{Div}}_{\Gamma_f \cup \Gamma_s}(\Omega)$:

$$H^2_{\Gamma_c \cup \Gamma_s}(\Omega) := \left\{ w \in L^2(\Omega) |\ \nabla^2 w \in L^2(\Omega,\ \mathbb{R}^{2\times 2}_{\text{sym}}),\ w|_{\Gamma_c \cup \Gamma_s} = \partial_{\boldsymbol{n}} w|_{\Gamma_c} = 0 \right\},$$

$$H^{\operatorname{div}\operatorname{Div}}_{\Gamma_f \cup \Gamma_s}(\Omega) := \left\{ \boldsymbol{M} \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}})) |\ \operatorname{div}\operatorname{Div} \boldsymbol{M} \in L^2(\Omega),\ M_{nn}|_{\Gamma_f \cup \Gamma_s} = \widetilde{q}_n|_{\Gamma_f} = 0 \right\}$$

Port-Hamiltonian formulation

Consider the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$,

$$\mathcal{F} = \mathcal{E} := L^2(\Omega) \times L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathsf{sym}}), \qquad (\partial_t \alpha_1, \ \partial_t \boldsymbol{A}_2) = \boldsymbol{f} \in \mathcal{F}, \quad (e_1, \ \boldsymbol{E}_2) = \boldsymbol{e} \in \mathcal{E}$$

It is necessary to show that the set

$$\mathcal{D}_{\mathcal{J}} := \{(\boldsymbol{f}, \boldsymbol{e}) \in \mathsf{Graph}(\mathcal{J}) | \ \boldsymbol{e} \in D(\mathcal{J})\} \subset \mathcal{B}$$

equals its orthogonal complement

$$\mathcal{D}_{\mathcal{J}}^{\perp} = \{b \in \mathcal{B} | \ \langle \boldsymbol{b}, \boldsymbol{b}' \rangle_+ = 0, \ \forall \ \boldsymbol{b}' \in \mathcal{D}_{\mathcal{J}}\}$$

with respect to the canonical symmetrical pairing

$$\langle \boldsymbol{b}^1, \boldsymbol{b}^2 \rangle_+ = \langle \boldsymbol{f}^1, \boldsymbol{e}^2 \rangle_{L^2} + \langle \boldsymbol{e}^1, \boldsymbol{f}^2 \rangle_{L^2}, \quad \boldsymbol{b}^i = (\boldsymbol{f}^i, \boldsymbol{e}^i) \in \mathcal{B}, \quad i = 1, 2$$

The proof is readily obtained considering that the following holds

$$(\operatorname{div} \operatorname{Div})^* = \nabla^2$$

This means that the operator $\mathcal{J}$ is formally skew-adjoint. By application of the Stokes theorem it is obtained $\mathcal{D}_{\mathcal{J}} = \mathcal{D}_{\mathcal{J}}^{\perp}$.

Inhomogeneous boundary conditions can be considered as well, but boundary variables have to be included in $\mathcal{D}_{\mathcal{J}}$.

It is worth noticing that the boundary variables are defined by the power balance

$$\dot{H} = \int_{\partial\Omega} \left\{ \partial_t w \, \widetilde{q}_n + \partial_{\boldsymbol{n}} (\partial_t w) \, M_{nn} \right\} \, \mathrm{d}s.$$

Outline

How to discretize pH systems?

Infinite dimensional pHs

PDE:

$$\partial_t x(z,t) = \mathcal{J}\delta_x H + B u(z,t), \qquad y(z,t) = B^* \delta_x H.$$

Boundary conditions:

$$u_\partial = \mathcal{B}\ \delta_x H, \quad y_\partial = \mathcal{C}\ \delta_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t) y(z,t)\ \mathrm{d}\Omega$$

Finite dimensional pHs

ODE:

$$\dot{x} = J \partial_x H + B_d u_d + B_\partial u_\partial, \qquad y_d = B_d^T \partial_x H, \qquad y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

Available methods

- Spectral methods (Moulla 2012);
- Finite differences (Trenchant 2018);
- Finite elements based (Golo 2004, Kotyczka 2018, Cardoso-Riberio 2019);

The partitioned finite element method

General form of a linear pH system in co-energy variables

$$\mathcal{M}\frac{\partial e}{\partial t} = \mathcal{J}e, \qquad \mathcal{M} = \mathcal{Q}^{-1}$$

General procedure for PFEM

1. Put the system into weak form:
$$\left(v, \mathcal{M}\frac{\partial e}{\partial t}\right)_\Omega = (v, \mathcal{J}e)_\Omega .$$
2. Apply integration by part on a partition of $\mathcal{J}$:
$$(v, \mathcal{J}e)_\Omega \overbrace{=}^{i.b.p.} j(v, e)_\Omega + b(v, u_\partial)_{\partial\Omega},$$
so that $j(v, e)_\Omega$ is a skew-symmetric bilinear form.
3. Discretization by Galerkin method (same basis function for test and co-energy variables)

Application to the Kirchhoff plate

Consider the Kirchhoff plate operators

$$\mathcal{M} = \mathsf{Diag}(\rho,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \mathsf{Diag}(\rho, \ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the first line of the operator $\mathcal{J}$ is integrated by parts

$$\begin{aligned}(v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{-v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1\right\} \, \mathrm{d}\Omega \\ &= \underbrace{\int_\Omega \left\{-\nabla^2 v_1 : \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1\right\} \, \mathrm{d}\Omega}_{j_{\mathsf{Hess}}(\boldsymbol{v}, \boldsymbol{e})} + \underbrace{\int_{\partial\Omega} \left\{v_1 q_n + \partial_{\boldsymbol{n}} v_1 M_{nn}\right\} \, \mathrm{d}s}_{b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}}\end{aligned}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \mathsf{Diag}(\rho,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the second line of the operator $\mathcal{J}$ is integrated by parts

$$\begin{aligned}(v, \mathcal{J} e)_\Omega &= \int_\Omega \left\{-v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1\right\} \, \mathrm{d}\Omega \\ &= \underbrace{\int_\Omega \{-v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \operatorname{div}\operatorname{Div} \boldsymbol{V}_2 \ e_1\} \ \mathrm{d}\Omega}_{j_{\operatorname{div}\operatorname{Div}}(\boldsymbol{v},\boldsymbol{e})} + \underbrace{\int_{\partial\Omega} \{v_{q_n} \partial_t w + v_{m_n} \partial_{\boldsymbol{n}} \partial_t w\} \ \mathrm{d}s}_{b_D(\boldsymbol{v},\boldsymbol{u}_\partial)_{\partial\Omega}},\end{aligned}$$

where $v_{q_n} = -(\operatorname{Div} \boldsymbol{V}_2) \cdot \boldsymbol{n} - \partial_{\boldsymbol{s}} v_{m_s}, \ v_{m_s} = \boldsymbol{V}_2 : (\boldsymbol{n} \otimes \boldsymbol{s}), \ v_{m_n} = \boldsymbol{V}_2 : (\boldsymbol{n} \otimes \boldsymbol{n})$

Application to the Kirchhoff plate

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

The selection depends on the control variables. For Neumann control the first line is integrated by parts. For Dirichlet control the second.

Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the followng weak form is obtained:

$$m(\boldsymbol{v}, \partial_t \boldsymbol{e}) = j_{\mathsf{Hess}}(\boldsymbol{v}, \boldsymbol{e}) + b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}$$

For both $\boldsymbol{e}_1, \boldsymbol{E}_2$ the H^2 conforming Bell elements are selected. For the boundary variables Lagrange polynomials of order two are selected. Dirichlet boundary conditions are enforced by Lagrange multipliers

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

Outline

Boundary interconnection of the Kirchhoff plate

The system is composed by a cantilever plate connected to a rigid rod. The interconnection is given by a compact operator.

$$
\text{dpH} \begin{cases} \dfrac{\partial x_1}{\partial t} = \mathcal{J} \dfrac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \dfrac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \dfrac{\delta H_1}{\delta x_1} \end{cases} \qquad \text{pH} \begin{cases} \dfrac{dx_2}{dt} = J \dfrac{\partial H_2}{\partial x_2} + B u_2 \\ y_2 = B^T \dfrac{\partial H_2}{\partial x_2} + D u_2 \end{cases},
$$

where $x_1 \in \mathscr{X}$, $u_{\partial,1} \in \mathscr{U}$, $y_{\partial,1} \in \mathscr{Y} = \mathscr{U}'$ belong to some Hilbert spaces (the prime denotes the topological dual of a space) and $x_2 \in \mathbb{R}^n, u, y \in \mathbb{R}^m$. The duality pairings for the boundary ports are denoted by

$$
\langle u_{\partial,1},\ y_{\partial,1} \rangle_{\mathscr{U} \times \mathscr{Y}}, \qquad \langle u_2,\ y_2 \rangle_{\mathbb{R}^m}.
$$

For the interconnection, consider the compact operator $\mathcal{W} : \mathscr{Y} \to \mathbb{R}^m$ and the following power preserving interconnection

$$
u_2 = -\mathcal{W}\, y_{\partial,1}, \qquad u_{\partial,1} = \mathcal{W}^*\, y_2,
$$

Boundary interconnection of the Kirchhoff plate

Plate $(\Omega = [0, L_x] \times [0, L_y])$

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix}$$

$$u_{\partial,\mathsf{pl}} = \partial_t w(x = L_x, y),$$

$$y_{\partial,\mathsf{pl}} = \widetilde{q}_n(x = L_x, y).$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \boldsymbol{u}_{\mathsf{rod}},$$

$$\boldsymbol{y}_{\mathsf{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space $\mathscr{Y}$ is the space of square-integrable functions with support on $\Gamma_{\mathsf{int}} = \{(x, y) |\, x = L_x, 0 \leq y \leq L_y\}$. The compact interconnection operator then reads

$$\mathcal{W} y_{\partial,\mathsf{pl}} = \begin{pmatrix} \int_{\Gamma_{\mathsf{int}}} y_{\partial,\mathsf{pl}} \, \mathrm{d}s \\ \int_{\Gamma_{\mathsf{int}}} (y - L_y/2) \, y_{\partial,\mathsf{pl}} \, \mathrm{d}s \end{pmatrix}.$$

The adjoint operator is then obtained considering that $\boldsymbol{u}_{\mathsf{rod}} = \mathcal{W} y_{\partial,\mathsf{pl}}$ and that the inner product of $\mathbb{R}^m$ is easily converted to an inner product on the space $L^2(\Gamma_{\mathsf{int}})$

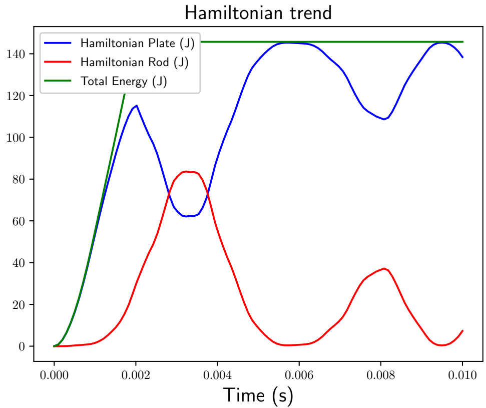
$$\langle \mathcal{W} y_{\partial,\mathsf{pl}}, \, \boldsymbol{y}_{\mathsf{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial,\mathsf{pl}}, \, \mathcal{W}^* \boldsymbol{y}_{\mathsf{rod}} \rangle_{L^2(\Gamma_{\mathsf{int}})},$$

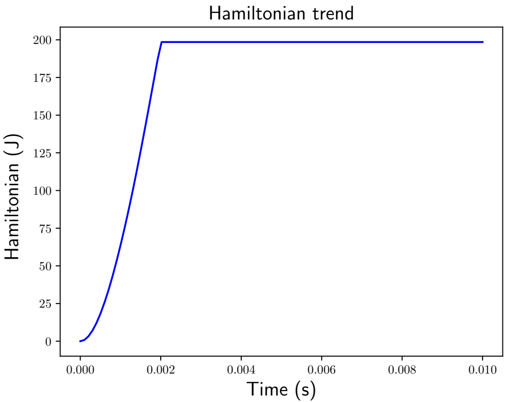
$$\mathcal{W}^* y_{\mathsf{rod}} = v_G + \omega_G \, (y - L_y/2) \, .$$

Distributed load ($t_{\text{end}} = 10\,[\text{ms}]$)

$$p = \begin{cases} 10^5 \left[y + 10\,(y - L_y/2)^2\right] [Pa], & \forall\, t < 2\,[\text{ms}], \\ 0, & \forall\, t \geq 2\,[\text{ms}]. \end{cases}$$

Play/Pause


Hamiltonian trend
Hamiltonian Plate (J)
Hamiltonian Rod (J)
Total Energy (J)
0
20
40
60
80
100
120
140
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)


Hamiltonian trend
Hamiltonian (J)
0
25
50
75
100
125
150
175
200
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)

Outline

Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} \qquad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

subjected to the following boundary conditions

$$\begin{aligned} \partial_t w|\Gamma_D &= 0, \\ \partial_x \partial_t w|\Gamma_D &= 0, \end{aligned} \qquad \Gamma_D = \{x = 0\}$$

$$\begin{aligned} \boldsymbol{M} : (n \otimes n)|\Gamma_N &= u_M, \\ \mathcal{D}\boldsymbol{M}|\Gamma_N := \widetilde{q}|\Gamma_N &= u_F, \end{aligned} \qquad \Gamma_N = \{x = 0,\ x = 1,\ y = 1\}$$

with initial conditions (compatible with the constraints):

$$w_t(x, y, 0) = x^2; \qquad \Sigma(x, y, 0) = 0.$$

Obtain a finite-dimensional uncontrolled system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \boldsymbol{u},$$

$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

Apply the control law $u = -Ky, \ K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

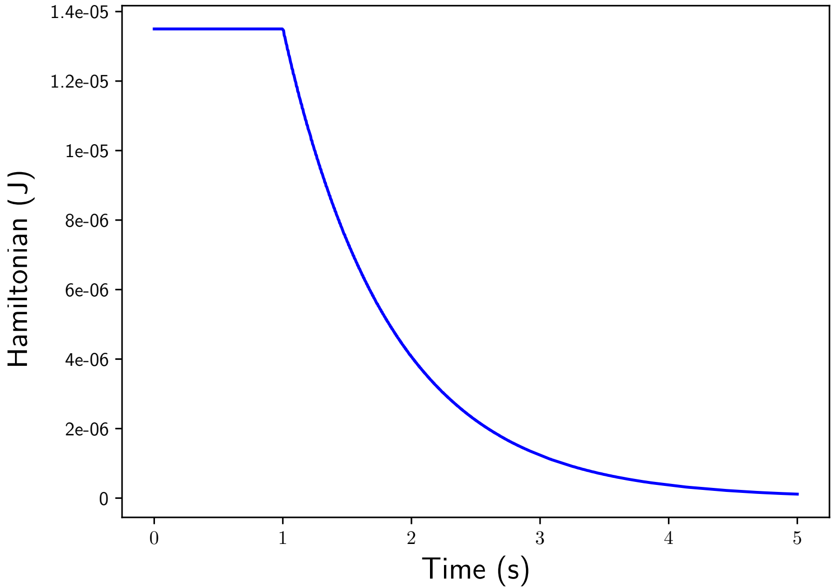
with $R = BKB^T \succeq 0$.
The Hamiltonian $\dot{H} = -e^T Re \leq 0$ is a non increasing function and by La Salle principle the equilibrium point $e = 0$ is asymptotically stable.

Control parameter ($t_{\text{end}} = 5[\mathrm{s}]$)

$$K = \begin{cases} 0, & \forall t < 1\,[\mathrm{s}], \\ 100, & \forall t \geq 1\,[\mathrm{s}]. \end{cases}$$

Play/Pause

Stabilization by boundary injection



Conclusion

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic interconnections;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

- The Hellan-Herrmann-Johnson method[1], but difficulties when dealing with inhomogeneous bcs;
- New discretization method capable that handles inhomogeneous bcs[2]

[1] H. Blum and R. Rannacher. "On mixed finite element methods in plate bending analysis". In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: 10.1007/BF00350239.

[2] Katharina. Rafetseder and Walter. Zulehner. "A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach". In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: 10.1137/17M1118427.

Thanks for your attention
Questions?

References I

- C. Beattie et al. “Linear port-Hamiltonian descriptor systems”. In: *Mathematics of Control, Signals, and Systems* 30.4 (2018), p. 17. URL: https://link.springer.com/article/10.1007%2Fs00498-018-0223-3.
- K. Bell. “A refined triangular plate bending finite element”. In: *International Journal for Numerical Methods in Engineering* 1.1 (1969), pp. 101–122. DOI: 10.1002/nme.1620010108.
- H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: 10.1007/BF00350239.
- A. Brugnoli et al. “Port-Hamiltonian formulation and symplectic discretization of plate models. Part II: Kirchhoff model for thin plates”. In: *Applied Mathematical Modelling* 75 (2019), pp. 961 –981. ISSN: 0307-904X. DOI: https://doi.org/10.1016/j.apm.2019.04.036.

References II

F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. "A structure-preserving Partitioned Finite Element Method for the 2D wave equation". In: *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*. Valparaíso, CL, 2018, pp. 1–6. URL: https://doi.org/10.1016/j.ifacol.2018.06.033.

V. Duindam et al. *Modeling and Control of Complex Physical Systems*. Springer Verlag, 2009. URL: https://www.springer.com/us/book/9783642031953.

G. Golo et al. "Hamiltonian Discretization of Boundary Control Systems". In: *Automatica* 40.5 (May 2004), pp. 757–771. DOI: http://dx.doi.org/10.1016/j.automatica.2003.12.017.

B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Operator Theory: Advances and Applications 223. Germany: Springer Verlag, 2012. DOI: 10.1007/978-3-0348-0399-1. URL: https://link.springer.com/book/10.1007/978-3-0348-0399-1.

- P. Kotyczka, B. Maschke, and L. Lefèvre. "Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems". In: *Journal of Computational Physics* 361 (2018), pp. 442 –476. ISSN: 0021-9991. DOI: https://doi.org/10.1016/j.jcp.2018.02.006.
- A. Macchelli, C. Melchiorri, and L. Bassi. "Port-based Modelling and Control of the Mindlin Plate". In: *Proceedings of the 44th IEEE Conference on Decision and Control.* 2005, pp. 5989–5994. DOI: 10.1109/CDC.2005.1583120.
- R. Moulla, L. Lefevre, and B. Maschke. "Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws". In: *Journal of Computational Physics* 231.4 (2012), pp. 1272–1292. URL: https://doi.org/10.1016/j.jcp.2011.10.008.
- Dirk Pauly and Walter Zulehner. "On Closed and Exact Grad-grad-and div-Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D". In: *arXiv preprint arXiv:1609.05873* (2016).

References IV

Katharina. Rafetseder and Walter. Zulehner. "A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach". In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: 10.1137/17M1118427.

F. Rathgeber et al. "Firedrake: automating the finite element method by composing abstractions". In: *ACM Transactions on Mathematical Software (TOMS)* 43.3 (2017), p. 24.

Institut Supérieur de l'Aéronautique et de l'Espace

10 avenue Édouard Belin – BP 54032

31055 Toulouse Cedex 4 – France

Phone: +33 5 61 33 80 80

www.isae-supaero.fr