

# Tensorial Formulations for thin and thick plates

## Weak Formulation and Discretization Procedure

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**INFIDHEM**

# Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
  - Strong form
  - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
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# The corresponding 1D models

## Timoshenko beam

- Valid for thick beams
- Dimension of the PH model: 4
- Differential operator  $J$  of order 1

$$\begin{aligned}\alpha &= [\rho v, I_\rho \omega_x, \frac{\partial \phi_x}{\partial x}, \frac{\partial w}{\partial x} - \phi_x]^T \\ e &= [v, \omega_x, M_{xx}, T_x]^T \\ J &= \begin{pmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 \end{pmatrix}\end{aligned}$$

## Euler-Bernoulli beam

- Valid for thin beams
- Dimension of the PH model: 2
- Differential operator  $J$  of order 2

$$\begin{aligned}\alpha &= [\rho v, \frac{\partial^2 w}{\partial x^2}]^T \\ e &= [v, M_{xx}]^T \\ J &= \begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}\end{aligned}$$

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# Energy and co-energy variables

Linear momenta and curvature are taken as energy variables. Additional the shear strain are considered, leading to

$$\boldsymbol{\alpha} = \left( \rho h v, \rho \frac{h^3}{12} \omega_x, \rho \frac{h^3}{12} \omega_y, \kappa_{xx}, \kappa_{yy}, \kappa_{xy}, \gamma_{xz}, \gamma_{yz} \right)^T$$

where  $v = \frac{\partial w}{\partial t}$ ,  $\omega_x = \frac{\partial \psi_x}{\partial t}$ ,  $\omega_y = \frac{\partial \psi_y}{\partial t}$ . The Hamiltonian density is quadratic in the energy variables

$$\mathcal{H} = \frac{1}{2} \boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{D}_b & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_s \end{bmatrix} \boldsymbol{\alpha}$$

The variational derivative provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v, \omega_x, \omega_y, M_{xx}, M_{yy}, M_{xy}, Q_x, Q_y)^T$$

## Definition of $J$ and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 & 0 & 0 & 0 & \\ \frac{\partial}{\partial y} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \alpha}{\partial t} = J \mathbf{e}.$$

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# Main References

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon.

Port-hamiltonian formulation and symplectic discretization of plate models. Part I : Mindlin model for thick plates.

*arXiv preprint arXiv:1809.11131*, 2018.

Under Review

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon.

Port-hamiltonian formulation and symplectic discretization of plate models. Part II : Kirchhoff model for thin plates.

*arXiv preprint arXiv:1809.11136*, 2018.

Under Review

# A scalar-vector-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now  $\kappa_{xy}$  now is half the value of the one in the vectorial case. The curvatures tensor is the linear deformation tensor applied to the rotation vector  $\boldsymbol{\theta} = (\psi_x, \psi_y)^T$

$$\mathbb{K} = \text{Grad}(\boldsymbol{\theta}) = \frac{1}{2} \left( \nabla \otimes \boldsymbol{\theta} + \left( \nabla \otimes \boldsymbol{\theta} \right)^T \right),$$

where Grad is the symmetric gradient operator applied to a vector, which gives rise to a symmetric tensor.

The momenta are found by introducing a fourth order tensor  $\mathbb{D}$ , such that  $M_{ij} = \sum_{k,l} \mathbb{D}_{ijkl} K_{kl}$ , a linear relation between  $\mathbb{M}$  and  $\mathbb{K}$ .

# Energy Variables

The energy variables are now distinguished with respect to their different nature

$$\begin{aligned}\alpha_w &= \rho h \frac{\partial w}{\partial t}, & \alpha_\theta &= \frac{\rho h^3}{12} \frac{\partial \theta}{\partial t}, \\ \mathbb{A}_\kappa &= \mathbb{K}, & \alpha_{\epsilon_s} &= \epsilon_s.\end{aligned}$$

The Hamiltonian is expressed as follows

$$H = \int_{\Omega} \frac{1}{2} \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + \frac{\rho h^3}{12} \frac{\partial \theta}{\partial t} \cdot \frac{\partial \theta}{\partial t} + \mathbb{M} : \mathbb{K} + \mathbf{Q} \cdot \epsilon_s \right\} d\Omega,$$

where  $\mathbb{M} : \mathbb{K}$  denotes the tensor contraction operation.

The momenta  $\mathbb{M}$  depends linearly on  $\mathbb{K}$ , hence  $\frac{1}{2}\mathbb{M} : \mathbb{K}$  is a quadratic form in  $\mathbb{K}$ .

## Co-energy variables

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$\begin{aligned}e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, & e_\theta &:= \frac{\delta H}{\delta \alpha_\theta} = \frac{\partial \theta}{\partial t}, \\ \mathbb{E}_\kappa &:= \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}, & e_{\epsilon_s} &:= \frac{\delta H}{\delta \epsilon_s} = \mathbf{Q}.\end{aligned}$$

where now the  $\epsilon_s$  and  $\mathbf{Q}$  are the shear strain and stress respectively.

**Proposition** (see [1] for the proof)

*The variational derivative of the Hamiltonian with respect to the curvatures tensor is the momenta tensor  $\frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}$ .*

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# Strong form and Interconnection structure

From  $\text{div}$ , the scalar divergence of a vector, we construct  $\text{Div}$ , the vector-valued divergence of a symmetric tensor, defined by

$$\boldsymbol{\varepsilon} = \text{Div}(\mathbb{E}) \quad \text{with } \varepsilon_i = \text{div}(\mathbb{E}_{ji}) = \sum_{j=1}^n \frac{\partial \mathbb{E}_{ji}}{\partial x_j}.$$

The port-Hamiltonian system is expressed as follows

$$\begin{cases} \frac{\partial \alpha_w}{\partial t} &= \text{div}(\mathbf{e}_{\epsilon_s}), \\ \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} &= \text{Div}(\mathbb{E}_\kappa) + \mathbf{e}_{\epsilon_s}, \\ \frac{\partial \mathbb{A}_\kappa}{\partial t} &= \text{Grad}(\mathbf{e}_\theta), \\ \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} &= \text{grad}(e_w) - \mathbf{e}_\theta \end{cases}.$$

If the variables are concatenated together, the formally skew-symmetric operator  $J$  can be highlighted

### Strong form for the Mindlin plate

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \mathbb{A}_\kappa \\ \boldsymbol{\alpha}_{\epsilon_s} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \text{div} \\ 0 & 0 & \text{Div} & \mathbf{I}_{2 \times 2} \\ 0 & \text{Grad} & 0 & 0 \\ \text{grad} & -\mathbf{I}_{2 \times 2} & 0 & 0 \end{bmatrix}}_J \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbb{E}_\kappa \\ e_{\epsilon_s} \end{pmatrix},$$

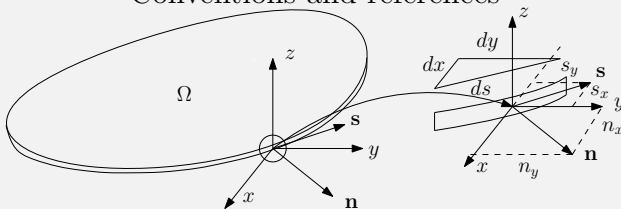
where all zeros are intended as nullifying operators from the space of input variables to the space of output variables.

Theorem (See [1] for additional details)

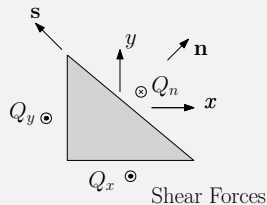
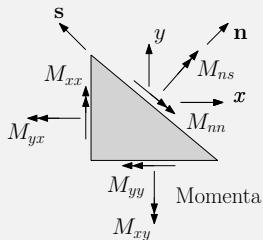
*The adjoint of the tensor divergence Div is  $-\text{Grad}$ , the opposite of the symmetric gradient.*

# Boundary Variables

## Conventions and references



## Boundary variables





# Energy flow

Again the boundary values can be found by evaluating the time derivative of the Hamiltonian

$$\begin{aligned}
 \dot{H} &= \int_{\Omega} \left\{ \frac{\partial \alpha_w}{\partial t} e_w + \frac{\partial \alpha_{\theta}}{\partial t} \cdot \mathbf{e}_{\theta} + \frac{\partial \mathbb{A}_{\kappa}}{\partial t} : \mathbb{E}_{\kappa} + \frac{\partial \alpha_{\epsilon_s}}{\partial t} \cdot \mathbf{e}_{\epsilon_s} \right\} d\Omega \\
 &= \int_{\Omega} \{ \operatorname{div}(\mathbf{e}_{\epsilon_s}) e_w + [\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}] \cdot \mathbf{e}_{\theta} + \operatorname{Grad}(\mathbf{e}_{\theta}) : \mathbb{E}_{\kappa} + (\operatorname{grad}(e_w) - \mathbf{e}_{\theta}) \cdot \mathbf{e}_{\epsilon_s} \} d\Omega \\
 &= \int_{\partial\Omega} \left\{ \underbrace{(\mathbf{n} \cdot \mathbf{e}_{\epsilon_s})}_{Q_n} e_w + (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \cdot \mathbf{e}_{\theta} \right\} d\Omega, \\
 &= \int_{\partial\Omega} \{ Q_n e_w + (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \cdot (\omega_n \mathbf{n} + \omega_s \mathbf{s}) \} d\Omega, \\
 &= \int_{\partial\Omega} \left\{ Q_n e_w + \omega_n \underbrace{\mathbf{n}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{nn}} + \omega_s \underbrace{\mathbf{s}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{ns}} \right\} d\Omega, \\
 &= \int_{\partial\Omega} \{ \textcolor{blue}{Q}_n \textcolor{red}{e}_w + \textcolor{blue}{\omega}_n \textcolor{red}{M}_{nn} + \textcolor{blue}{\omega}_s \textcolor{red}{M}_{ns} \} d\Omega.
 \end{aligned}$$

Same result as in the vectorial case but with intrinsic operators.

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# Weak Formulation

The first line is multiplied by  $v_w$  (multiplication by a scalar)

$$\int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega = \int_{\Omega} v_w \operatorname{div}(\mathbf{e}_{\epsilon_s}) \, d\Omega,$$

the second and the fourth lines by  $\mathbf{v}_{\theta}$ ,  $\mathbf{v}_{\epsilon_s}$  (scalar product of  $\mathbb{R}^2$ )

$$\begin{aligned} \int_{\Omega} \mathbf{v}_{\theta} \cdot \frac{\partial \boldsymbol{\alpha}_{\theta}}{\partial t} \, d\Omega &= \int_{\Omega} \mathbf{v}_{\theta} \cdot (\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}) \, d\Omega, \\ \int_{\Omega} \mathbf{v}_{\epsilon_s} \cdot \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} \, d\Omega &= \int_{\Omega} \mathbf{v}_{\epsilon_s} \cdot (\operatorname{grad}(e_w) - \mathbf{e}_{\theta}) \, d\Omega, \end{aligned}$$

the third one by  $\mathbb{V}_{\kappa}$  (tensor contraction)

$$\int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} \, d\Omega = \int_{\Omega} \mathbb{V}_{\kappa} : \operatorname{Grad}(\mathbf{e}_{\theta}) \, d\Omega.$$

# Choice I: Boundary control through forces and momenta

The first two line of the system in weak form are integrated by parts

$$\int_{\Omega} v_w \operatorname{div}(\mathbf{e}_{\epsilon_s}) \, d\Omega = \int_{\partial\Omega} v_w \underbrace{\mathbf{n} \cdot \mathbf{e}_{\epsilon_s}}_{Q_n} \, ds - \int_{\Omega} \operatorname{grad}(v_w) \cdot \mathbf{e}_{\epsilon_s} \, d\Omega,$$

$$\int_{\Omega} \mathbf{v}_{\theta} \cdot (\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}) \, d\Omega = \int_{\partial\Omega} \mathbf{v}_{\theta} \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds - \int_{\Omega} \{ \operatorname{Grad}(\mathbf{v}_{\theta}) : \mathbb{E}_{\kappa} - \mathbf{v}_{\theta} \cdot \mathbf{e}_{\epsilon_s} \} \, d\Omega.$$

The usual additional manipulation is performed on the boundary term containing the momenta, so that the proper boundary values arise

$$\begin{aligned} \int_{\partial\Omega} \mathbf{v}_{\theta} \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds &= \int_{\partial\Omega} \left\{ \underbrace{(\mathbf{v}_{\theta} \cdot \mathbf{n})}_{v_{\omega_n}} \mathbf{n} + \underbrace{(\mathbf{v}_{\theta} \cdot \mathbf{s})}_{v_{\omega_s}} \mathbf{s} \right\} \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds \\ &= \int_{\partial\Omega} \{ v_{\omega_n} M_{nn} + v_{\omega_s} M_{ns} \} \, ds. \end{aligned}$$

The final system exhibits as **control inputs** the boundary forces and momenta

## Weak Form with forces and momenta as inputs

$$\begin{cases} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} d\Omega &= - \int_{\Omega} \text{grad}(v_w) \cdot e_{\epsilon_s} d\Omega + \int_{\partial\Omega} v_w Q_n ds \\ \int_{\Omega} v_{\theta} \cdot \frac{\partial \alpha_{\theta}}{\partial t} d\Omega &= - \int_{\Omega} \{ \text{Grad}(v_{\theta}) : \mathbb{E}_{\kappa} - v_{\theta} \cdot e_{\epsilon_s} \} d\Omega + \int_{\partial\Omega} \{ v_{\omega_n} M_{nn} + v_{\omega_s} M_{ns} \} ds \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} d\Omega &= \int_{\Omega} \mathbb{V}_{\kappa} : \text{Grad}(e_{\theta}) d\Omega \\ \int_{\Omega} v_{\epsilon_s} \cdot \frac{\partial \alpha_{\epsilon_s}}{\partial t} d\Omega &= \int_{\Omega} v_{\epsilon_s} \cdot (\text{grad}(e_w) - e_{\theta}) d\Omega \end{cases} .$$

In this first case, the boundary controls  $\mathbf{u}_{\partial}$  and the corresponding output  $\mathbf{y}_{\partial}$  are

$$\mathbf{u}_{\partial} = \begin{pmatrix} Q_n \\ M_{nn} \\ M_{ns} \end{pmatrix}_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \begin{pmatrix} e_w \\ \omega_n \\ \omega_s \end{pmatrix}_{\partial\Omega} .$$

## Choice II: Boundary control through kinematic variables

If instead the last two lines are integrated by parts

Weak Form with linear and rotational velocities as inputs

$$\left\{ \begin{array}{lcl} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} d\Omega & = & \int_{\Omega} v_w \operatorname{div}(\mathbf{e}_{\epsilon_s}) d\Omega \\ \int_{\Omega} \mathbf{v}_{\theta} \cdot \frac{\partial \boldsymbol{\alpha}_{\theta}}{\partial t} d\Omega & = & \int_{\Omega} \mathbf{v}_{\theta} \cdot (\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}) d\Omega \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} d\Omega & = & - \int_{\Omega} \operatorname{Div}(\mathbb{V}_{\kappa}) \cdot \mathbf{e}_{\theta} d\Omega + \int_{\partial\Omega} \{v_{M_{nn}} \boldsymbol{\omega}_n + v_{M_{ns}} \boldsymbol{\omega}_s\} ds \\ \int_{\Omega} \mathbf{v}_{\epsilon_s} \cdot \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} d\Omega & = & - \int_{\Omega} \{\operatorname{div}(\mathbf{v}_{\epsilon_s}) \mathbf{e}_w + \mathbf{v}_{\epsilon_s} \cdot \mathbf{e}_{\theta}\} d\Omega + \int_{\partial\Omega} v_{Q_n} \mathbf{e}_w ds \end{array} \right.$$

where  $v_{M_{nn}} = \mathbf{n}^T \mathbb{V}_{\kappa} \mathbf{n}$ ,  $v_{M_{ns}} = \mathbf{s}^T \mathbb{V}_{\kappa} \mathbf{n}$  and  $v_{Q_n} = \mathbf{v}_{\epsilon_s} \cdot \mathbf{n}$ . In this second case, the boundary controls  $\mathbf{u}_{\partial}$  and corresponding output  $\mathbf{y}_{\partial}$  are

$$\mathbf{u}_{\partial} = \begin{pmatrix} \mathbf{e}_w \\ \boldsymbol{\omega}_n \\ \boldsymbol{\omega}_s \end{pmatrix}_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \begin{pmatrix} Q_n \\ M_{nn} \\ M_{ns} \end{pmatrix}_{\partial\Omega}.$$

# Summary for the Mindlin Plate

Main points:

- variational derivative w.r.t. a tensor quantity;
- strong tensorial form;
- structure preserving weak formulation with different choices of the boundary inputs;
- only two choices for the boundary inputs were shown but others are possible.

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# Energy and co-energy variables

This model is the 2D extension of the Bernoulli beam. It is logical to select as energy variable the linear momentum, together with the curvatures

$$\boldsymbol{\alpha} = (\mu v, \kappa_{xx}, \kappa_{yy}, \kappa_{xy})^T$$

where  $v = \frac{\partial w}{\partial t}$ . The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2} \boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & \mathbf{D} \end{bmatrix} \boldsymbol{\alpha}$$

So the variational derivative of the total Hamiltonian  $H = \int_{\Omega} \mathcal{H} d\Omega$  provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v, M_{xx}, M_{yy}, M_{xy})^T$$

## Definition of J and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -\left(\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial x \partial y}\right) \\ \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial x \partial y} & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \alpha}{\partial t} = J \mathbf{e}.$$

From the Schwarz theorem for  $C^2$  functions the mixed derivative could be expressed as  $2\frac{\partial^2}{\partial x \partial y}$ , instead of  $\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial x \partial y}$ . However, in this way the symmetry intrinsically present in  $\gamma_{xy} = -z \left( \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 w}{\partial x \partial y} \right)$  would be lost. The mixed derivative is here split to reestablish the symmetric nature of curvatures and momenta (that are of tensorial nature).

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# A scalar-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now  $\kappa_{xy}$  now is half the value of the one in the vectorial case. The curvatures tensor is the linear deformation tensor applied to the rotation vector  $\boldsymbol{\theta} = \text{grad}(w)$

$$\mathbb{K} = \text{Grad}(\boldsymbol{\theta}) = \text{Grad}(\text{grad}(w)).$$

The momenta are found by introducing a fourth order tensor  $\mathbb{D}$ , such that  $\mathbb{M}_{ij} = \mathbb{D}_{ijkl} \mathbb{K}_{kl}$

For what concerns the choice of the energy variables a scalar and a tensor variable are grouped together

$$\alpha_w = \mu \frac{\partial w}{\partial t} \quad \mathbb{A}_\kappa = \mathbb{K}$$

The Hamiltonian energy is written as

$$H = \int_{\Omega} \left\{ \frac{1}{2} \mu \left( \frac{\partial w}{\partial t} \right)^2 + \frac{1}{2} \mathbb{M} : \mathbb{K} \right\} d\Omega,$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \quad \mathbb{E}_\kappa := \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}.$$

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# Interconnection structure

The formally skew-symmetric operator  $J$  can be highlighted

## Strong form for the Kirchhoff plate

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \mathbb{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\text{div} \circ \text{Div} \\ \text{Grad} \circ \text{grad} & 0 \end{bmatrix}}_J \begin{pmatrix} e_w \\ \mathbb{E}_\kappa \end{pmatrix}.$$

where all zeros are intended as nullifying operator from the space of input variables to the space of output variables.

Theorem (See [2] for additional details)

*The adjoint of  $\text{div} \circ \text{Div}$  is  $\text{Grad} \circ \text{grad}$  (i.e. the Hessian operator)*

## Remark

*The interconnection structure  $J$  now resembles that of the Bernoulli beam. The double divergence and the double gradient coincide, in dimension one, with the second derivative.*

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# Weak Formulation

The first line is multiplied by  $v_w$  (scalar multiplication)

$$\int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega = \int_{\Omega} -v_w \operatorname{div}(\operatorname{Div}(\mathbb{E}_{\kappa})) \, d\Omega,$$

the second line by  $\mathbb{V}_{\kappa}$  (tensor contraction)

$$\int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} \, d\Omega = \int_{\Omega} \mathbb{V}_{\kappa} : \operatorname{Grad}(\operatorname{grad}(e_w)) \, d\Omega.$$

Now depending on which line is integrated by parts different boundary control term can be selected.

# Choice I: Boundary control through forces and momenta

The first line has to be integrated by parts **twice**

$$\int_{\Omega} -v_w \operatorname{div}(\operatorname{Div}(\mathbb{E}_{\kappa})) \, d\Omega = \int_{\partial\Omega} \underbrace{-\mathbf{n} \cdot \operatorname{Div}(\mathbb{E}_{\kappa})}_{Q_n} v_w \, ds + \int_{\Omega} \operatorname{grad}(v_w) \cdot \operatorname{Div}(\mathbb{E}_{\kappa}) \, d\Omega$$

Applying again the integration by parts

$$\int_{\Omega} \operatorname{grad}(v_w) \cdot \operatorname{Div}(\mathbb{E}_{\kappa}) \, d\Omega = \int_{\partial\Omega} \operatorname{grad}(v_w) \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds - \int_{\Omega} \operatorname{Grad}(\operatorname{grad}(v_w)) : \mathbb{E}_{\kappa} \, d\Omega$$

The usual additional manipulation is performed on the boundary terms, so that the proper boundary values arise

$$\begin{aligned} \int_{\partial\Omega} \operatorname{grad}(v_w) \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds &= \int_{\partial\Omega} \left( \frac{\partial v_w}{\partial n} \mathbf{n} + \frac{\partial v_w}{\partial s} \mathbf{s} \right) \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds \\ &= \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} \underbrace{\mathbf{n}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{nn}} + \frac{\partial v_w}{\partial s} \underbrace{\mathbf{s}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{ns}} \right\} \, ds \\ &= \sum_{\Gamma_i \subset \partial\Omega} [M_{ns} v_w]_{\partial\Gamma_i} + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} - v_w \frac{\partial M_{ns}}{\partial s} \right\} \, ds \end{aligned}$$

Defining the effective shear stress as  $\tilde{Q}_n = Q_n - \frac{\partial M_{ns}}{\partial s}$  If the boundary is regular the final expression becomes

$$\int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega = - \int_{\Omega} \text{Grad}(\text{grad}(v_w)) : \mathbb{E}_{\kappa} \, d\Omega + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} + v_w \tilde{Q}_n \right\} \, ds.$$

Weak form with forces and momenta as inputs

$$\begin{cases} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega &= - \int_{\Omega} \text{Grad}(\text{grad}(v_w)) : \mathbb{E}_{\kappa} \, d\Omega + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} + v_w \tilde{Q}_n \right\} \, ds, \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} \, d\Omega &= \int_{\Omega} \mathbb{V}_{\kappa} : \text{Grad}(\text{grad}(e_w)) \, d\Omega. \end{cases}$$

The control input  $\mathbf{u}_{\partial}$  and the corresponding conjugate outputs  $\mathbf{y}_{\partial}$  are

$$\mathbf{u}_{\partial} = \begin{pmatrix} \tilde{Q}_n \\ M_{nn} \end{pmatrix}_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \begin{pmatrix} e_w \\ \frac{\partial e_w}{\partial n} \end{pmatrix}_{\partial\Omega}.$$

## Choice II: Boundary control through kinematic variables

The same procedure can be performed on the second line of the system

Weak form with linear and angular velocities as inputs

$$\begin{cases} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} d\Omega &= \int_{\Omega} -v_w \operatorname{div}(\operatorname{Div}(\mathbb{E}_{\kappa})) d\Omega, \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} d\Omega &= \int_{\Omega} \operatorname{div}(\operatorname{Div}(\mathbb{V}_{\kappa})) e_w d\Omega + \int_{\partial\Omega} \left\{ v_{M_{nn}} \frac{\partial e_w}{\partial n} + v_{\tilde{Q}_n} e_w \right\} ds. \end{cases}$$

where  $v_{M_{nn}} = \mathbf{n}^T \mathbb{V}_{\kappa} \mathbf{n}$  and  $v_{\tilde{Q}_n} = -\operatorname{Div}(\mathbb{V}_{\kappa}) \cdot \mathbf{n} - \frac{\partial(\mathbf{s}^T \mathbb{V}_{\kappa} \mathbf{n})}{\partial s}$ .

The control input  $\mathbf{u}_{\partial}$  are now the kinematic boundary conditions, the corresponding conjugate outputs  $\mathbf{y}_{\partial}$  are the dynamic boundary condition

$$\mathbf{u}_{\partial} = \left( \frac{e_w}{\frac{\partial e_w}{\partial n}} \right)_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \left( \frac{\tilde{Q}_n}{M_{nn}} \right)_{\partial\Omega}.$$

# Summary for the Kirchhoff Plate

Main points:

- new distributed port-Hamiltonian system involving second order differential operator in space dimension 2 for  $J$ ;
- strong tensorial form where  $\text{div} \circ \text{Div} = (\text{Grad} \circ \text{grad})^*$ ;
- structure preserving weak formulation with different choices of the boundary inputs;
- only two choices for the boundary variables were shown but others are possible;

Other contribution (not presented here):

- definition of the underlying Stokes-Dirac structure for the vectorial formulation of the Kirchhoff plate [2].

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F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. [A structure-preserving partitioned finite element method for the 2d wave equation.](#)

In *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 1–6, Valparaíso, CL, 2018.

<http://oatao.univ-toulouse.fr/19965>

# General principles of the structure preserving discretization

The energy, co-energy and test functions of the *same* index are discretized by using the *same* bases:

$$\begin{aligned}\alpha_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x, y) \alpha_i^k(t), & e_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x, y) e_i^k(t), & v_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x, y) v_i^k, \\ \alpha_i^{ap} &:= \phi_i(x, y)^T \boldsymbol{\alpha}_i(t), & e_i^{ap} &:= \phi_i(x, y)^T \mathbf{e}_i(t), & v_i^{ap} &:= \phi_i(x, y)^T \mathbf{v}_i.\end{aligned}$$

The same procedure is applied for the boundary terms with a specific basis  $\boldsymbol{\psi}$

$$u_{\partial,i} \approx u_{\partial,i}^{ap} := \sum_{k=1}^{n_{\partial,i}} \psi_i^k(s) u_{\partial,i}^k(t) = \boldsymbol{\psi}_i(s)^T \mathbf{u}_{\partial,i}(t).$$

## Remark

*The functions  $\boldsymbol{\psi}_i(s)$  can be selected as the restriction of functions  $\boldsymbol{\phi}$  over the boundary  $\boldsymbol{\psi}(s) = \boldsymbol{\phi}(x(s), y(s))$  or in other ways.*



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## Mindlin Plate: discretized operators

The formally skew-symmetric operator  $J$  is replaced with the following skew-symmetric matrix

$$J_d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -D_{x,71}^T & -D_{y,81}^T \\ 0 & 0 & 0 & -D_{x,42}^T & 0 & -D_{y,62}^T & D_{0,27} & 0 \\ 0 & 0 & 0 & 0 & -D_{y,53}^T & -D_{x,63}^T & 0 & D_{0,38} \\ 0 & D_{x,42} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{y,53} & 0 & 0 & 0 & 0 & 0 \\ 0 & D_{y,62} & D_{x,63} & 0 & 0 & 0 & 0 & 0 \\ D_{x,71} & -D_{0,27}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ D_{y,81} & 0 & -D_{0,38}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As a consequence of the integration by parts, a control input is included in the finite-dimensional system. Matrix  $B$  is defined by

$$B = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \\ 0_{\nu_4 \times n_{\partial,1}} & 0_{\nu_4 \times n_{\partial,2}} & 0_{\nu_4 \times n_{\partial,3}} \end{bmatrix}.$$

# Final discretized system

The final system is written as

$$\begin{aligned}M\dot{\boldsymbol{\alpha}} &= J_d \boldsymbol{e} + B \boldsymbol{u}_{\partial}, \\ \boldsymbol{y}_{\partial} &= B^T \boldsymbol{e}\end{aligned}$$

where

- $M = \text{Diag}[M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8]$  is a block diagonal matrix,
- $\boldsymbol{\alpha}$  is simply the concatenation of the  $\boldsymbol{\alpha}_i$  (just like  $\boldsymbol{e}$ )
- $\boldsymbol{u}_{\partial}$  is the concatenation of the  $\boldsymbol{u}_{\partial,i}$  ( $\boldsymbol{u}_{\partial,1} = \boldsymbol{Q}_n$ ,  $\boldsymbol{u}_{\partial,2} = \boldsymbol{M}_{nn}$  and  $\boldsymbol{u}_{\partial,3} = \boldsymbol{M}_{ns}$ ).

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# Kirchhoff Plate: discretized operators

The discretized system is written as

$$\begin{pmatrix} M_1 \dot{\alpha}_1 \\ M_2 \dot{\alpha}_2 \\ M_3 \dot{\alpha}_3 \\ M_4 \dot{\alpha}_4 \end{pmatrix} = \begin{bmatrix} 0 & -D_{xx}^T & -D_{yy}^T & -2D_{xy}^T \\ D_{xx} & 0 & 0 & 0 \\ D_{yy} & 0 & 0 & 0 \\ 2D_{xy} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_{\partial,1} \\ u_{\partial,2} \end{pmatrix},$$

where  $M_i$  are square matrices (of size  $N_i \times N_i$ ),  $D_{xx}$  is an  $N_2 \times N_1$  matrix,  $D_{yy}$  is an  $N_3 \times N_1$  matrix,  $D_{xy}$  is an  $N_4 \times N_1$  matrix,  $B_1$  is an  $N_1 \times N_{\partial,1}$  matrix and finally  $B_2$  is an  $N_2 \times N_{\partial,2}$  matrix. The collocated output are defined as

$$y_{\partial} = \begin{bmatrix} B_1^T & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}.$$

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# Variational formulation using PFEM

The general variational form can be now stated:

- Energy variables belong to the mixed space  $\alpha \in V_\alpha = V_{\alpha_1} \times \cdots \times V_{\alpha_n}$ , where  $\alpha_i \in V_{\alpha_i}$ ;
- Coenergy variables  $e = \mathcal{Q}\alpha$  where  $\mathcal{Q}$  is a coercive symmetrical operator  $\mathcal{Q} : V_\alpha \rightarrow V_\alpha$ ;
- Test functions belong to the same space as  $\alpha$   $v \in V_\alpha$

## General structure preserving weak form with known inputs

If the boundary input  $u_\partial$  is known (i.e for the Mindlin plate free or clamped conditions) then

$$m \left( v, \frac{\partial \alpha}{\partial t} \right) = j(v, e) + b_{u_\partial}(v_{y_\partial}) \quad \forall v \in V$$

where  $m$  is the mass bilinear, symmetric and coercive form,  $j$  is the interconnection bilinear and antisymmetric form and  $b$  is a linear functional.

# Dealing with algebraic constraints

If the boundary input vector is partly unknown then Lagrange multipliers and corresponding test function have to be introduced. Variables  $\lambda_{u_\partial}, v_{u_\partial} \in V_\lambda$  defined over  $\Gamma_\lambda$ , the boundary subset where the input are unknown and the conjugated output are set to zero (e.g. clamped plate with a free side).

## General structure preserving weak form with unknown inputs

$$\begin{cases} m\left(v, \frac{\partial \alpha}{\partial t}\right) = j(v, e) + b(v_{y_\partial}, \lambda_{u_\partial}) \\ 0 = c(v_{u_\partial}, e_{y_\partial}) \end{cases} \quad \forall v \in V, \forall v_{u_\partial} \in V_\lambda$$

where  $v_{y_\partial} = \mathcal{B}_{y_\partial} v$  and  $e_{y_\partial} = \mathcal{B}_{y_\partial} e$  ( $\mathcal{B}$  is the boundary variables operator). Now  $b$  is a bilinear form on spaces  $V_\alpha \times V_\lambda$  and  $c$  is defined over  $V_\lambda \times V_\alpha$  and is obtained from  $b$  by swapping variables and applying  $\mathcal{B}_{y_\partial}$  on  $e$  instead of  $v$ .



# Numerical Study of the Mindlin Plate

This PH system can be discretized using a FE software. These numerical results were obtained using Fenics. To compare the eigenvalues publications [4, 5, 6] were used :

R. Durán, L. Hervella-Nieto, E. Liberman, and J. Solomin. [Approximation of the vibration modes of a plate by Reissner-Mindlin equations.](#)

*Mathematics of Computation of the American Mathematical Society*, 68(228):1447–1463, 1999

D.J. Dawe and O.L. Roufaeil. [Rayleigh-Ritz vibration analysis of Mindlin plates.](#)

*Journal of Sound and Vibration*, 69(3):345–359, 1980

H.C. Huang and E. Hinton. [A nine node Lagrangian Mindlin plate element with enhanced shear interpolation.](#)

*Engineering Computations*, 1(4):369–379, 1984

In the next tables these references will be denoted respectively by D-H, D-R, H-H. Our results are denoted by B-A.

# Study case

## Properties of the plate

- Square plate with  $l_x = l_y = 1$ ;
- Young modulus  $E = 70 \text{ GPa}$ ;
- Density  $\rho = 2000 \text{ kg/m}^3$ ;
- Poisson modulus  $\nu = 0.3$ .

## Boundary conditions considered

- All clamped CCCC ( $w = 0, \omega_s = 0, \omega_n = 0$ );
- Simply supported hard SSSS ( $w = 0, \omega_s = 0$ );
- Half clamped half simply supported SCSC;
- All clamped but one side free CCCF (F stands for free, i.e.  $M_{nn} = 0, M_{ns} = 0, Q_n = 0$ ).

A scalar viable discretized using  $n$   $P^1$  elements for each side gives rise to  $(n + 1) \times (n + 1)$  dofs and one discretized using  $n$   $P^2$  elements contains  $(2n + 1) \times (2n + 1)$  dofs. Hence since the system is of dimension 8:

- if 10  $P^1$  elements are used for each side the final system contains 968 states;
- if 10  $P^2$  elements are used for each side the final system contains 3528 states.






# Benchmark variables

The variables are discretized by using Lagrange polynomials of order 1 (same order for each variable) or 2. Normalized eigenfrequency found in [5] are used as benchmark

$$\hat{\omega}_{mn} = \omega_{mn}^h L \left( \frac{2(1+\nu)\rho}{E} \right)^{\frac{1}{2}} \quad \text{Final result independent on } E, \rho,$$

$m$  and  $n$  being the numbers of half-waves occurring in the modes shapes in the  $x$  and  $y$  directions, respectively.

Colors used for comparing of the first 4 eigenvalues

-  benchmark results;
-  1% error or less;
-  up to 5% error;
-  up to 15% error;
-  spurious eigenvalue;

# Eigenvalues for $t/L = 0.1$ using $P^1$

| BCs  | Mode                          | $N : 10(\text{B-A})$ | $N : 10(\text{D-H})$ | $N : 20(\text{B-A})$ | $N : 20(\text{D-H})$ | H-H   | D-R   |
|------|-------------------------------|----------------------|----------------------|----------------------|----------------------|-------|-------|
| CCCC | $\hat{\omega}_{11}$           | 1.5999               | 1.5947               | 1.5917               | 1.5921               | 1.591 | 1.594 |
|      | $\hat{\omega}_{21}$           | 3.0615               | 3.1181               | 3.0410               | 3.0595               | 3.039 | 3.046 |
|      | $\hat{\omega}_{12}$           | 3.0615               | 3.1181               | 3.0410               | 3.0595               | 3.039 | 3.046 |
|      | $\hat{\omega}_{22}$           | 4.3161               | 4.4477               | 4.2682               | 4.3106               | 4.263 | 4.285 |
| SSSS | $\hat{\omega}_{11}$           | 0.9324               | 0.9384               | 0.9324               | 0.9323               | 0.930 | 0.930 |
|      | $\hat{\omega}_{21}$           | 2.2227               | 2.2893               | 2.2226               | 2.2366               | 2.219 | 2.219 |
|      | $\hat{\omega}_{12}$           | 2.2227               | 2.2893               | 2.2226               | 2.2366               | 2.219 | 2.219 |
|      | $\hat{\omega}_{22}$           | 3.4142               | 3.5657               | 3.3608               | 3.4450               | 3.405 | 3.406 |
| SCSC | $\hat{\omega}_{11}$           | 1.3111               | 1.3060               | 1.3013               | 1.3016               | 1.300 | 1.302 |
|      | $\hat{\omega}_{21}$           | 2.4155               | 2.4664               | 2.3966               | 2.4120               | 2.394 | 2.398 |
|      | $\hat{\omega}_{12}$           | 2.9082               | 2.9617               | 2.8871               | 2.9043               | 2.885 | 2.888 |
|      | $\hat{\omega}_{22}$           | 3.8906               | 4.0126               | 3.8458               | 3.8830               | 3.839 | 3.852 |
| CCCF | $\hat{\omega}_{\frac{1}{2}1}$ | 1.0855               | 1.0812               | 1.0982               | 1.0848               | 1.081 | 1.089 |
|      | $\hat{\omega}_{\frac{3}{2}1}$ | 1.7636               | 1.7759               | 1.7461               | 1.7525               | 1.744 | 1.758 |
|      | $\hat{\omega}_{\frac{1}{2}2}$ | 2.6696               | 2.7413               | 2.6575               | 2.6787               | 2.657 | 2.673 |
|      | $\hat{\omega}_{\frac{5}{2}1}$ | 3.2248               | 3.3186               | 3.1997               | 3.2282               | 3.197 | 3.216 |

# Eigenvalues for $t/L = 0.01$ using $P^1$

| BCs  | Mode                          | $N : 10(\text{B-A})$ | $N : 10(\text{D-H})$ | $N : 20(\text{B-A})$ | $N : 20(\text{D-H})$ | H-H   | D-R   |
|------|-------------------------------|----------------------|----------------------|----------------------|----------------------|-------|-------|
| CCCC | $\hat{\omega}_{11}$           | 0.1967               | .1754                | .1765                | .1754                | .1754 | .1754 |
|      | $\hat{\omega}_{21}$           | 0.4030               | .3668                | .3604                | .3599                | .3574 | .3576 |
|      | $\hat{\omega}_{12}$           | 0.4030               | .3668                | .3604                | .3599                | .3574 | .3576 |
|      | $\hat{\omega}_{22}$           | 0.6431               | .5487                | .5358                | .5323                | .5264 | .5274 |
| SSSS | $\hat{\omega}_{11}$           | 0.1706               | .0972                | .1128                | .0965                | .0963 | .0963 |
|      | $\hat{\omega}_{21}$           | 0.3576               | .2486                | .2660                | .2426                | .2406 | .2406 |
|      | $\hat{\omega}_{12}$           | 0.3576               | .2486                | .2660                | .2426                | .2406 | .2406 |
|      | $\hat{\omega}_{22}$           | 0.5803               | .4035                | .3865                | .3893                | .3847 | .3848 |
| SCSC | $\hat{\omega}_{11}$           | 0.1864               | .1417                | .1487                | .1413                | .1411 | .1411 |
|      | $\hat{\omega}_{21}$           | 0.3649               | .2748                | .2829                | .2688                | .2668 | .2668 |
|      | $\hat{\omega}_{12}$           | 0.3987               | .3474                | .3485                | .3402                | .3377 | .3377 |
|      | $\hat{\omega}_{22}$           | 0.6075               | .4814                | .4933                | .4657                | .4604 | .4608 |
| CCCF | $\hat{\omega}_{\frac{1}{2}1}$ | 0.1238               | .1182                | .1166                | .1170                | .1166 | .1171 |
|      | $\hat{\omega}_{\frac{3}{2}1}$ | 0.2207               | .1977                | .1954                | .1956                | .1949 | .1951 |
|      | $\hat{\omega}_{\frac{1}{2}2}$ | 0.3204               | .3193                | .3078                | .3109                | .3080 | .3093 |
|      | $\hat{\omega}_{\frac{5}{2}1}$ | 0.4144               | .3874                | .3751                | .3771                | .3736 | .3740 |

# Eigenvalues for $t/L = 0.1$ using $P^2$

| BCs  | Mode                          | $N = 5$ (B-A) | $N = 10$ (B-A) | H-H   | D-R   |
|------|-------------------------------|---------------|----------------|-------|-------|
| CCCC | $\hat{\omega}_{11}$           | 1.5976        | 1.5914         | 1.591 | 1.594 |
|      | $\hat{\omega}_{21}$           | 3.0584        | 3.0405         | 3.039 | 3.046 |
|      | $\hat{\omega}_{12}$           | 3.0677        | 3.0405         | 3.039 | 3.046 |
|      | $\hat{\omega}_{22}$           | 4.3109        | 4.2662         | 4.263 | 4.285 |
| SSSS | $\hat{\omega}_{11}$           | 0.9304        | 0.9302         | 0.930 | 0.930 |
|      | $\hat{\omega}_{21}$           | 2.2223        | 2.2194         | 2.219 | 2.219 |
|      | $\hat{\omega}_{12}$           | 2.2224        | 2.2194         | 2.219 | 2.219 |
|      | $\hat{\omega}_{22}$           | 3.4128        | 3.4061         | 3.405 | 3.406 |
| SCSC | $\hat{\omega}_{11}$           | 1.3053        | 1.3004         | 1.300 | 1.302 |
|      | $\hat{\omega}_{21}$           | 2.4040        | 2.3946         | 2.394 | 2.398 |
|      | $\hat{\omega}_{12}$           | 2.9060        | 2.8858         | 2.885 | 2.888 |
|      | $\hat{\omega}_{22}$           | 3.8721        | 3.8415         | 3.839 | 3.852 |
| CCCF | $\hat{\omega}_{\frac{1}{2}1}$ | 1.0845        | 1.0797         | 1.081 | 1.089 |
|      | $\hat{\omega}_{\frac{3}{2}1}$ | 1.7559        | 1.7425         | 1.744 | 1.758 |
|      | $\hat{\omega}_{\frac{1}{2}2}$ | 2.6762        | 2.6547         | 2.657 | 2.673 |
|      | $\hat{\omega}_{\frac{5}{2}1}$ | 3.2186        | 3.1954         | 3.197 | 3.216 |

# Eigenvalues for $t/L = 0.01$ using $P^2$

| BCs  | Mode                          | $N = 5$ (B-A) | $N = 10$ (B-A) | H-H    | D-R    |
|------|-------------------------------|---------------|----------------|--------|--------|
| CCCC | $\hat{\omega}_{11}$           | 0.1872        | 0.1762         | 0.1754 | 0.1754 |
|      | $\hat{\omega}_{21}$           | 0.3725        | 0.3598         | 0.3574 | 0.3576 |
|      | $\hat{\omega}_{12}$           | 0.4055        | 0.3598         | 0.3574 | 0.3576 |
|      | $\hat{\omega}_{22}$           | 0.6043        | 0.5335         | 0.5264 | 0.5274 |
| SSSS | $\hat{\omega}_{11}$           | 0.0963        | 0.0963         | 0.0963 | 0.0963 |
|      | $\hat{\omega}_{21}$           | 0.2422        | 0.2406         | 0.2406 | 0.2406 |
|      | $\hat{\omega}_{12}$           | 0.2430        | 0.2406         | 0.2406 | 0.2406 |
|      | $\hat{\omega}_{22}$           | 0.3874        | 0.3848         | 0.3847 | 0.3848 |
| SCSC | $\hat{\omega}_{11}$           | 0.1492        | 0.1418         | 0.1411 | 0.1411 |
|      | $\hat{\omega}_{21}$           | 0.2827        | 0.2683         | 0.2668 | 0.2668 |
|      | $\hat{\omega}_{12}$           | 0.3608        | 0.3394         | 0.3377 | 0.3377 |
|      | $\hat{\omega}_{22}$           | 0.4940        | 0.4654         | 0.4604 | 0.4608 |
| CCCF | $\hat{\omega}_{\frac{1}{2}1}$ | 0.1197        | 0.1169         | 0.1166 | 0.1171 |
|      | $\hat{\omega}_{\frac{3}{2}1}$ | 0.2092        | 0.1960         | 0.1949 | 0.1951 |
|      | $\hat{\omega}_{\frac{1}{2}2}$ | 0.3188        | 0.3089         | 0.3080 | 0.3093 |
|      | $\hat{\omega}_{\frac{5}{2}1}$ | 0.3938        | 0.3757         | 0.3736 | 0.3740 |



# Temporal Simulation

Consider a uniform specific load acting on a clamped plate at rest

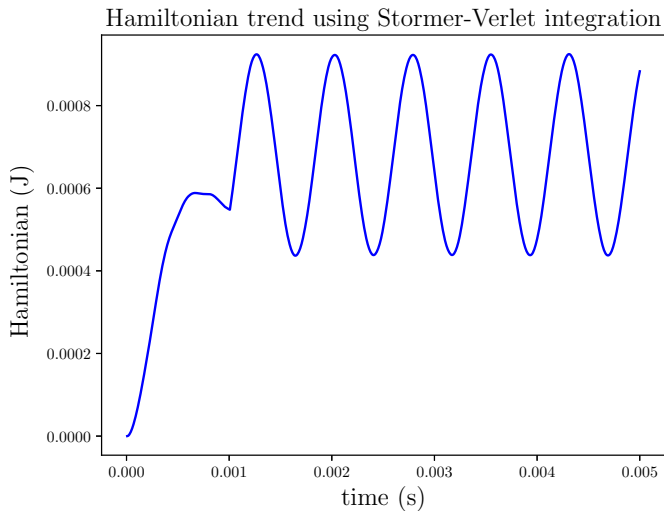
$$\mathbf{f}(t) = \begin{cases} -1 \text{ m/s}^2 \hat{\mathbf{z}} & t < 0.01 \text{ s} \\ 0 & t > 0.01 \text{ s} \end{cases}$$

Corresponding linear form:  $l(v) = \int \rho h v_w f \, d\Omega$ .

Parameters:

- Total simulation time:  $T = 0.05$
- Time step:  $dt = 6.85 \cdot 10^{-6} \text{ s}$
- Integration in time: Stormer-Verlet

**Simulation output  $e_w = \frac{\partial w}{\partial t}$**   
**(scale factor  $\times 500$ , 100 frame per second)**



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Thank you for your attention. Questions?