Numerical discretization of port-Hamiltonian plate models ⋆

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Abstract:

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1. INTRODUCTION

2. PROBLEM STATEMENT

In this section the models under consideration are recalled. The details can be found in Brugnoli et al. (2019a,b).

2.1 Notations

For a scalar field $u : \mathbb{R}^d \to \mathbb{R}$ the gradient is defined as

$$\mathrm{grad}(u) = \nabla u := (\partial_{x_1} u \dots \partial_{x_d} u)^\top .$$

For a vector field $\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d$, with components u_j, the gradient is defined as

$$\mathrm{grad}(\boldsymbol{u})_{ij} := (\nabla \boldsymbol{u})_{ij} = \partial_{x_i} u_j .$$

The symmetric part of the gradient operator Grad (i. e. the deformation gradient in continuum mechanics) is given by

$$\mathrm{Grad}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla^\top \boldsymbol{u}\right).$$

The Hessian operator of u is then computed as follows

$$\mathrm{Hess}(u) = \nabla^2 u = \mathrm{Grad}(\mathrm{grad}(u)),$$

For a tensor field $\boldsymbol{U} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, with components u_{ij}, the divergence is a vector, defined column-wise as

$$\mathrm{Div}(\boldsymbol{U}) = \nabla \cdot \boldsymbol{U} := \left(\sum_{i=1}^d \partial_{x_i} u_{ij}\right)_{j=1,\dots,d}.$$

The double divergence of a tensor field $\boldsymbol{U}$ is then a scalar field defined as

$$\mathrm{div}(\mathrm{Div}(\boldsymbol{U})) := \sum_{i,j=1}^d \partial_{x_i}\partial_{x_j} u_{ij} .$$

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2.2 Plate models in port-Hamiltonian form

2.2.0.1. Mindlin-Reissner plate The Mindlin model is a generalization to the 2D case of the Timoshenko beam model and is expressed by a system of three coupled PDEs (Timoshenko and Woinowsky-Krieger (1959))

$$\begin{cases} \rho h \dfrac{\partial^2 w}{\partial t^2} &= \mathrm{div}(\boldsymbol{q}), \\ \dfrac{\rho h^3}{12}\dfrac{\partial^2 \boldsymbol{\theta}}{\partial t^2} &= \boldsymbol{q} + \mathrm{Div}(\boldsymbol{M}), \end{cases} \tag{1}$$

where ρ is the mass density, h the plate thickness, w the vertical displacement, $\boldsymbol{\theta} = (\theta_x, \theta_y)^\top$ collects the deflection of the cross section along axes x and y respectively. Variables $\boldsymbol{M}, \boldsymbol{q}$ represent the momenta tensor and the shear stress. The Hooke law relates those to the curvature tensor and shear deformation vector

$$\begin{aligned} \boldsymbol{M} &:= \mathcal{D}\boldsymbol{K}, & \boldsymbol{K} &:= \mathrm{Grad}(\boldsymbol{\theta}), \\ \boldsymbol{q} &:= \mathcal{C}\boldsymbol{\gamma}, & \boldsymbol{\gamma} &:= \mathrm{grad}(w) - \boldsymbol{\theta}, \end{aligned}$$

where $\mathcal{D}, \mathcal{C}$ are symmetric positive tensors. The kinetic and potential energy density $\mathcal{K}$ and $\mathcal{U}$ read

$$\begin{aligned} \mathcal{K} &= \frac{1}{2}\left\{\rho h \left(\frac{\partial w}{\partial t}\right)^2 + \frac{\rho h^3}{12}\frac{\partial \boldsymbol{\theta}}{\partial t}\cdot\frac{\partial \boldsymbol{\theta}}{\partial t}\right\}, \\ \mathcal{U} &= \frac{1}{2}\left\{\boldsymbol{M} : \boldsymbol{K} + \boldsymbol{q}\cdot\boldsymbol{\gamma}\right\}, \end{aligned} \tag{2}$$

where $\boldsymbol{M} : \boldsymbol{K} := \sum_{i,j} m_{ij}\kappa_{ij}$ is the tensor contraction. The Hamiltonian is easily written as

$$H = \int_\Omega (\mathcal{K} + \mathcal{U}) \ \mathrm{d}\Omega. \tag{3}$$

To get a port-Hamiltonian formulation suitable energy variables must be selected. The appropriate set is the following

$$\begin{aligned} \alpha_w &= \rho h \frac{\partial w}{\partial t}, & \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12}\frac{\partial \boldsymbol{\theta}}{\partial t}, \\ \boldsymbol{A}_\kappa &= \boldsymbol{K}, & \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}. \end{aligned} \tag{4}$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$
\begin{aligned}
e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \qquad \boldsymbol{e}_\theta := \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t}, \\
\boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, \qquad \boldsymbol{e}_\gamma := \frac{\delta H}{\delta \boldsymbol{\alpha}_\gamma} = \boldsymbol{q}.
\end{aligned} \tag{5}
$$

Energy and co-energy are relative by a positive symmetric operator $\boldsymbol{\alpha} = \mathcal{Q}\boldsymbol{e}$

$$
\mathcal{Q} = \mathrm{diag}(\frac{1}{\rho h}, \frac{12}{\rho h^3}, \mathcal{D}, \mathcal{C})
$$

The port-Hamiltonian system is expressed as follows

$$
\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ 0 & \mathrm{Grad} & 0 & 0 \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}}_{J} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}, \tag{6}
$$

This system defines a Stokes-Dirac structure, therefore, the boundary values can be found by evaluating the time derivative of the Hamiltonian.

2.2.0.2. Kirchhoff plate The Kirchhoff Model is a generalization to the 2D case of the Euler-Bernoulli beam model. The classical equations for this model Timoshenko and Woinowsky-Krieger (1959) are

$$
\rho h \frac{\partial^2 w}{\partial t^2} = -\mathrm{div}(\mathrm{Div}(\boldsymbol{M})) \tag{7}
$$

The bending moment tensor and the curvature are related as in the Mindlin model $\boldsymbol{M} = \boldsymbol{D}\boldsymbol{K}$. Following the Kirchhoff assumption the curvature tensor is the Hessian of the vertical displacement

$$
\boldsymbol{K} := \mathrm{Grad}(\mathrm{grad}(w)).
$$

The kinetic and potential energy densities $\mathcal{K}$ and $\mathcal{U}$ read

$$
\mathcal{K} = \frac{1}{2}\rho h \left(\frac{\partial w}{\partial t}\right)^2, \quad \mathcal{U} = \frac{1}{2}\boldsymbol{M} : \boldsymbol{K}, \tag{8}
$$

The Hamiltonian is easily written as

$$
H = \int_\Omega (\mathcal{K} + \mathcal{U}) \, \mathrm{d}\Omega. \tag{9}
$$

Selecting as energy variables

$$
\begin{aligned}
\alpha_w &= \rho h \frac{\partial w}{\partial t}, && \text{Linear momentum}, \\
\boldsymbol{A}_\kappa &= \boldsymbol{K}, && \text{Curvature tensor}.
\end{aligned} \tag{10}
$$

co-energy variables are found by computing the variational derivative of the Hamiltonian

$$
\begin{aligned}
e_w &:= \frac{\delta H}{\delta \alpha_w} = w_t, && \text{Vertical Velocity}, \\
\boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, && \text{Momenta tensor},
\end{aligned} \tag{11}
$$

where $w_t := \frac{\partial w}{\partial t}$ for compactness. The coercive operator linking energy and co-energies reads

$$
\mathcal{Q} = \mathrm{diag}(\frac{1}{\rho h}, \mathcal{D})
$$

The port-Hamiltonian system is expressed as follows

$$
\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\mathrm{div} \circ \mathrm{Div} \\ \mathrm{Grad} \circ \mathrm{grad} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}, \tag{12}
$$

3. CONCLUSION

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