Dissipative Dynamical Systems

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Outline

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Why dissipative dynamical systems?

All engineering systems exhibit dissipation.

- Electrical networks with resistors;
- Mechanical systems (viscoelastic or Coulomb friction);
- Thermodynamic systems: dissipation leads to an increase in entropy.

The notion of dissipativity establishes a natural link between the properties of input-output and state-space models. Many modern computational tools for the analysis and synthesis of control systems are based on it.

Jan C. Willems. "Dissipative dynamical systems Part I: General theory". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 321–351

Jan C. Willems. "Dissipative dynamical systems Part II: Linear systems with quadratic supply rates". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 352–393

Arjan van der Schaft. *L2-gain and passivity in nonlinear control*. Springer-Verlag, 1999

Some mathematical notation

$\mathbb{R}_+ = [0, \infty)$ denotes the set of positive reals.

Let V be a finite dimensional normed liner space with norm $||\cdot||_V$.

(If $V = \mathbb{R}^n$ then the Euclidean norm is denoted by$||x||_2 = \sqrt{x^\top x}$)

Definition (L^p Banach spaces)

For each positive integer $p \in 1, 2, \ldots$, the set $L^p(\mathbb{R}_+, V)$ consists of all functions $f : \mathbb{R}_+ \to V$, which are measurable and satisfy

$$\int_0^\infty ||f(t)||_V^p \, \mathrm{d}t < \infty,$$

The case $p = \infty$ consists of all bounded measurable functions, i.e. $\sup_{t \in \mathbb{R}_+} f(t) < \infty$.

The L^p spaces are Banach spaces (complete normed linear spaces) w.r.t. the norm

$$||f||_{L^p} = \left(\int_0^\infty ||f(t)||_V^p \, \mathrm{d}t \right)^{\frac{1}{p}}, \quad q = 1, 2, \ldots \qquad ||f||_{L^\infty} = \sup_{t \in \mathbb{R}^+} |f(t)|, \quad q = \infty.$$

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Definition (Extended L^p Banach spaces)

For each $T \in \mathbb{R}_+$ the function $f_T : \mathbb{R}_+ \to V$ defined by

$$f_T = \begin{cases} f(t), & 0 \leq t < T, \\ 0, & t \geq T \end{cases}$$

is called the truncation of f.

For $q = 1, 2, \ldots, \infty$ the set $L^{pe}(\mathbb{R}_+, V)$ consists of all measurable functions $f : \mathbb{R}_+ \to V$ such that $f_T \in L^p(\mathbb{R}_+, V), \quad \forall T,\ 0 \leq T < \infty$.

The spaces L^{pe} are called the extended L^p spaces. It holds $L^p(\mathbb{R}_+, V) \subset L^{pe}(\mathbb{R}_+, V)$.

General setting

Consider the state-space system with inputs and outputs

$$\Sigma : \quad \begin{array}{ll} \dot{x} = f(x,u), & u(t) \in U, \\ y = h(x,u), & y(t) \in Y. \end{array}$$

- $x(t)$ belong to the state manifold $\mathcal{X}$ ($\dim \mathcal{X} = n$ and $(x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$ are local coordinates).
- U and Y are linear spaces with $\dim U = m$, $\dim Y = p$.
  For simplicity it is assumed that $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$.

Assumption

There exists a unique solution trajectory $x(\cdot)$ on the infinite time interval $t \in \mathbb{R}^+$ of the differential equation $\dot{x} = f(x,u), \quad \forall\, x(0) \in \mathcal{X},\ \forall\, u(\cdot) \in L^{2e}(U)$.
Furthermore, it will be assumed that the thus generated output functions $y(\cdot) = h(x(\cdot), u(\cdot))$ are in $L^{2e}(Y)$.

Reachability and controllability

Notation: $\mathbb{R}^2_+ := \{(t_1, t_2) \in \mathbb{R}^2 | \, t_2 \geq t_1\}$ (causal triangular sector of $\mathbb{R}^2$).
Given two sets A, B the notation B^A indicates the set of functions $f : A \to B$.

Definition (State transition function)

Given a state space system Σ, the state transition function ϕ is the map

$$\phi(t_1, t_0, x(t_0), u) : \mathbb{R}^2_+ \times \mathcal{X} \times U^{\mathbb{R}} \to \mathcal{X}$$

such that $x(t_1) = \phi(t_1, t_0, x(t_0), u)$.

Definition (Reachability and controllability)

The state space $\mathcal{X}$ of system Σ is said to be reachable from x_{-1} if
$\forall \, x \in \mathcal{X}, \; \exists \, t_{-1} \leq 0, \; \exists u(\cdot) \in U^{\mathbb{R}}$ such that $x = \phi(0, t_{-1}, x_{-1}, u(\cdot))$.
It is said to be controllable to x_1 if for any $x \in \mathcal{X}, \; \exists t_1 > 0$ and $u(\cdot) \in \mathcal{U}$ such that $x_1 = \phi(0, t_{-1}, x_{-1}, u(\cdot))$.

Outline

The mathematical definition of dissipativity

On the combined space $U \times Y$ consider the supply rate function $s : U \times Y \to \mathbb{R}$.

Definition (Dissipative state space system)

A state space system Σ is said to be dissipative w.r.t. the supply rate s if there exists a function $S : \mathcal{X} \to \mathbb{R}_+$ (the storage function), such that $\forall\, x(t_0) \in \mathcal{X}$ at any time t_0, and $\forall\, u(\cdot)$ and $\forall\, t_1 \geq t_0$ and the following inequality holds

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t, \qquad \text{Dissipation Inequality.}$$

It equality holds then the system is called conservative (w.r.t. the supply rate s).

Corollary (Convexity of the storage functions set)

Given two storage functions S_1 and S_2 then any convex combination $\alpha S_1 + (1-\alpha) S_2,\ \alpha = [0,1]$ is also a storage function.

Passive systems and L^2 finite gain

Two important class of supply rate functions:

- passive systems $s(u,y) = u^\top y$;
- finite L^2 gain $s(u,y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2, \quad \gamma \geq 0.$

Definition (Passive system)

A system Σ with $U = Y = \mathbb{R}^m$ is passive if it is dissipative w.r.t. $s(u,y) = u^\top y$.

Σ is input strictly passive if $\exists \delta > 0$ such that Σ is dissipative w.r.t. $s(u,y) = u^\top y - \delta||u||_2^2$.

Σ is output strictly passive if there exists $\varepsilon > 0$ such that Σ is dissipative with respect to $s(u,y) = u^\top y - \varepsilon||y||_2^2$.

Σ is lossless if it is conservative with respect to $s(u,y) = u^\top y$.

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Definition (L^2 finite gain)

A system Σ with $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$ has L^2-gain $\leq \gamma$ ($\gamma \geq 0$) if it is dissipative w.r.t.

$$s(u, y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2.$$

The L^2-gain of Σ is defined as

$$\gamma(\Sigma) := \inf\{\gamma | \ \Sigma \text{ has } L^2\text{-gain} \leq \gamma\}.$$

Σ is said to have L^2-gain $< \gamma$ if $\exists\, \tilde{\gamma} \leq \gamma$ such that Σ has L^2-gain $\leq \tilde{\gamma}$.

Σ is called inner if it is conservative with respect to $s(u, y) = \frac{1}{2}||u||_2^2 - \frac{1}{2}||y||_2^2$.

How to establish dissipativity?

Theorem (Necessary and sufficient conditions for dissipativity)

Consider system Σ and supply rate $s(u,y)$. Σ is dissipative with respect to s iff

$$S_a(x) := \sup_{\substack{u(\cdot)\\ T\geq 0}} -\int_0^T s(u(t),y(t))\,\mathrm{d}t, \qquad x(0)=x$$

is finite $\forall\, x\in\mathcal{X}$. Furthermore, if S_a is finite $\forall\, x\in\mathcal{X}$ then S_a is a storage function, called the available storage, and all other possible storage functions S satisfy

$$S_a(x) \leq S(x) - \inf_x S(x), \qquad \forall\, x\in\mathcal{X}$$

Moreover $\inf_x S_a(x) = 0$.

Proof

- ($\Longrightarrow$) Suppose S_a is finite. Then $S_a \geq 0$ (supremum of a set that contains 0). Given $u : [t_0, t_1] \to \mathbb{R}^m$ compare $S(x(t_0))$ and $S(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t$. Since S_a is the supremum over all $u(\cdot)$ it follows

$$S_a(x(t_0)) \geq S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t \implies S_a \text{ is a storage function.}$$

- ($\Longleftarrow$) Suppose Σ dissipative. Then $\exists\, S \geq 0$ such that $\forall\, u(\cdot)$

$$S(x(t)) + \int_0^T s(u(t), y(t))\, \mathrm{d}t \geq S(x(T)) \geq 0.$$

  This implies that

$$S(x(0)) \geq \sup_{\substack{u(\cdot)\\ T \geq 0}} - \int_0^T s(u(t), y(t))\, \mathrm{d}t = S_a(x(0)) \implies S_a(x(0)) < \infty$$

Reachability and Storage functions

If the system is reachable from some state, then the finiteness of S_a needs only to be checked for this initial condition.

Theorem

Assume that Σ is reachable from $x^* \in \mathcal{X}$. Then Σ is dissipative iff $S_a(x^*) < \infty$.

Bibliography

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- Willems, Jan C. “Dissipative dynamical systems Part I: General theory”. In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 321–351.
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