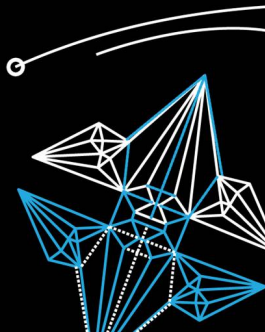Portwings Internal Meeting
Challenges and outlook for the numerics

Andrea Brugnoli

Department of Robotics and Mechatronics,
University of Twente

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# Overview

1. What numerics for portwings?
   - Split discretization
   - Algebraic topology: from continuous to discrete

2. What about mechanics?

# What numerics for the portwings project?

Methods should preserve the continuous structure at the discrete level. Which structure?

1. Cohomology: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla \times} V^2(\mathbb{R}^3) \xrightarrow{\nabla \cdot} V^3(\mathbb{R})$;
2. Variational structure $\delta \int l = 0$, ($l$ Lagrangian density);
3. Hamiltonian structure $\dot{\mathcal{F}} = \{F, H\}$, $\{\cdot, \cdot\}$ Poisson brackets.
4. $\ldots$

Recent developments:

- ▶ splitting of topological and metric operators (Bauer and Behrens 2018);
- ▶ Lie group structure and underlying variational formulation (Gawlik and Gay-Balmaz 2020);
- ▶ connection with algebraic topology, i.e. de Rham complex and more general Hilbert complexes, e.g. elasticity (Bochev and Hyman 2006; Arnold, Falk, and Winther 2006; Palha et al. 2014);

## Principle behind split discretization

▶ Fluid equation written in covariant form (exterior calculus);

▶ Split Hamiltonian form $\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}$.

  ▶ Topological braket depending on $d$ (exterior derivative) or $\iota_v$ (interior product).

  ▶ Metric dependent $\mathcal{H}$, since it depends on $*$

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Linear shallow water waves in Hamiltonian form

▶ $\mathcal{H} = \frac{1}{2} \int_M \{\bar{h} \|\boldsymbol{u}\|^2 + gh^2\} \, \mathrm{d}x$, with $\frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} = \bar{h}\boldsymbol{u}$, $\frac{\delta \mathcal{H}}{\delta h} = gh$, where $g$ gravity acc. and $\bar{h}$ equilibrium fluid height.

▶ $\{\mathcal{F}, \mathcal{G}\} = -(\frac{\delta \mathcal{F}}{\delta \boldsymbol{u}}, \nabla \frac{\delta \mathcal{G}}{\delta h})_{L^2} - (\frac{\delta \mathcal{F}}{\delta h}, \nabla \cdot \frac{\delta \mathcal{G}}{\delta \boldsymbol{u}})_{L^2}$.

$$\mathcal{F} = \int \boldsymbol{u} \, \mathrm{d}\Omega : \ \dot{\mathcal{F}} = -\left(\frac{\delta \mathcal{F}}{\delta \boldsymbol{u}}, \nabla \frac{\delta \mathcal{H}}{\delta h}\right) \to \partial_t \boldsymbol{u} = -g\nabla h.$$

$$\mathcal{F} = \int h \, \mathrm{d}\Omega : \ \dot{\mathcal{F}} = -\left(\frac{\delta \mathcal{F}}{\delta h}, \nabla \cdot \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}}\right) \to \partial_t h = -\bar{h}\nabla \cdot \boldsymbol{u}.$$

## Split and weak (or mixed) form

De Rham complex: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$.

### Split form

$$
\begin{array}{ccc}
h \in V^0 & \xrightarrow{\nabla} & V^1 \ni \boldsymbol{u} \\
{\scriptstyle \tilde{h}=\tilde{*}h}\downarrow & & \downarrow{\scriptstyle \boldsymbol{u}=\tilde{*}\boldsymbol{u}} \\
\tilde{h} \in V^3 & \xleftarrow{\nabla\cdot} & V^2 \ni \tilde{\boldsymbol{u}}
\end{array}
$$

- ▶ Assume full $\tilde{*}$ in metric eqs.
- ▶ Both $\nabla, \nabla\cdot$ are imposed strongly.
- ▶ Both diff. eqs exact

### Weak form (Mixed FE)

$$
\begin{array}{ccc}
h \in V^3 & \xrightarrow{\widehat{\nabla}} & V^2 \ni \boldsymbol{u} \\
{\scriptstyle \tilde{h}=h}\downarrow & & \downarrow{\scriptstyle \tilde{\boldsymbol{u}}=\boldsymbol{u}} \\
\tilde{h} \in V^3 & \xleftarrow{\nabla\cdot} & V^2 \ni \tilde{\boldsymbol{u}}
\end{array}
$$

- ▶ Assume $\tilde{*} = \mathrm{Id}$, i.e. $\tilde{\boldsymbol{u}} = \boldsymbol{u}$, $\tilde{h} = h$.
- ▶ Weak gradient $\widehat{\nabla}$.
- ▶ Moment Eq. weak

## Split decomposition in practice

1. First projection of the strong form. For the 1D case:

   $$\partial_t u_e^{(1)} + g D^{en} h_n^{(0)} = 0, \qquad u_e^{(1)} : 1 \text{ form}, \quad h_n^{(0)} : 0 \text{ form},$$

   $$\partial_t \widetilde{h}_e^{(1)} + \bar{h} D^{en} \widetilde{u}_n^{(0)} = 0, \qquad \widetilde{h}_e^{(1)} : 1 \text{ form}, \quad \widetilde{u}_n^{(0)} : 0 \text{ form}.$$

   $D^{en}$ metric free approximation of exterior derivative.

2. Project the metric closure relations:
   - High accuracy $CG_1^u - CG_1^h$ spaces:

     $$M^{nn} \widetilde{u}_n^{(0)} = P^{ne} u_e^{(1)}, \qquad M^{nn} h_n^{(0)} = P^{ne} \widetilde{h}_e^{(1)};$$

   - Low accuracy $DG_0^u - DG_0^h$ spaces:

     $$M^{en} \widetilde{u}_n^{(0)} = I^{ee} u_e^{(1)}, \qquad M^{en} h_n^{(0)} = I^{ee} \widetilde{h}_e^{(1)};$$

   - Medium accuracy $CG_1^u - DG_0^h$ spaces:

     $$M^{nn} \widetilde{u}_n^{(0)} = P^{ne} u_e^{(1)}, \qquad M^{en} h_n^{(0)} = I^{ee} \widetilde{h}_e^{(1)};$$

   $M^{nn}$, $M^{en}$ metric dependent, $P^{ne}$ metric free averaging op.

## Weak (mixed) form

Weak formulation: find $\boldsymbol{u} \in V^2(\mathbb{R}^3)$, $h \in V^3(\mathbb{R})$

$$(\boldsymbol{v}_u, \partial_t \boldsymbol{u})_{L^2} = -(\boldsymbol{v}_u, g\widehat{\nabla}h)_{L^2} = (\nabla \cdot \boldsymbol{v}_u, gh)_{L^2}, \qquad \forall \boldsymbol{v}_u \in V^2,$$
$$(v_h, \partial_t h)_{L^2} = -(v_h, \bar{h}\nabla \cdot \boldsymbol{u})_{L^2}, \qquad\qquad\qquad \forall v_h \in V^3.$$

After projection

$$\begin{bmatrix} \mathsf{M}^{nn} & 0 \\ 0 & \mathsf{M}^{ee} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathsf{u}_n \\ \mathsf{h}_e \end{pmatrix} = \begin{bmatrix} 0 & g\mathsf{D}^{ne} \\ -\bar{h}\mathsf{D}^{en} & 0 \end{bmatrix} \begin{pmatrix} \mathsf{u}_n \\ \mathsf{h}_e \end{pmatrix}, \qquad \mathsf{D}^{ne} = \mathsf{D}^{en\top}.$$

Another formulation with weak divergence: find
$\boldsymbol{u} \in V^1(\mathbb{R}^3)$, $h \in V^0(\mathbb{R})$

$$(\boldsymbol{v}_u, \partial_t \boldsymbol{u})_{L^2} = -(\boldsymbol{v}_u, g\nabla h)_{L^2}, \qquad\qquad \forall \boldsymbol{v}_u \in V^1,$$
$$(v_h, \partial_t h)_{L^2} = -(v_h, \bar{h}\widehat{\nabla \cdot}\boldsymbol{u})_{L^2} = (\nabla v_h, \bar{h}\boldsymbol{u})_{L^2}, \qquad \forall v_h \in V^0.$$

# Connection with algebraic topology (n=2)

$$
\begin{array}{ccccc}
& C^0(D) & \xrightarrow{\ \delta\ } & C^1(D) & \xrightarrow{\ \delta\ } & C^2(D) \\
\mathcal{I}^0 \;\; \mathcal{R}^0 & & \mathcal{I}^1 \;\; \mathcal{R}^1 & & \mathcal{I}^2 \;\; \mathcal{R}^2 \\
\Lambda^0(M) & \xrightarrow{\ d\ } & \Lambda^1(M) & \xrightarrow{\ d\ } & \Lambda^2(M)
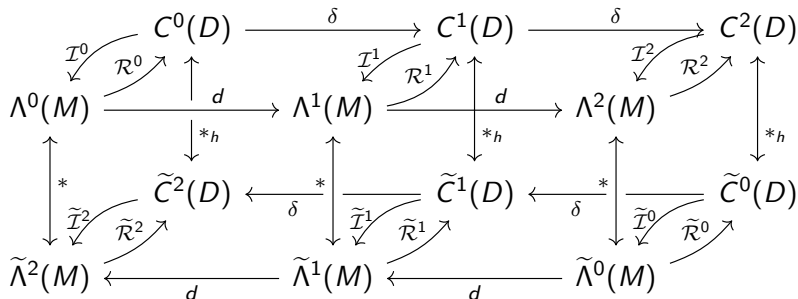\end{array}
$$

Diagram taken from Palha et al. 2014.

- $C^k(D)$ space of cochains on the primal grid $D$;
- $\widetilde{C}^k$ space of cochains on the dual grid $\widetilde{D}$;
- $\delta$ coboundary operator (dual of $\partial$ on chains);
- $\mathcal{I}^k, \widetilde{\mathcal{I}}^k$ interpolation operators;
- $\mathcal{R}^k, \widetilde{\mathcal{R}}^k$ reduction operators;

## Connection with Finite Elements and Finite Volumes

How to discretize the Hodge Laplacian in in physics-compatible fashion? The exterior derivative can be discretize exactly on one grid, but not the codifferential $d^* = (-1)^{n(k+1)+1} * d*$.

Two different solutions:

- ▶ Dual grid discretization: the Hodge star is treated numerically by considering a dual grid and leads to the explicit construction of the Hodge matrix (analogously to finite volumes);

- ▶ Single grid discretization: the Hodge operator is incorporated implicitly by using an inner product. The codifferential is converted into an exterior derivative by using the integration by parts (equivalent to mixed finite element method).

## Continuum mechanics and differential geometry

In Arnold, Falk, and Winther 2006 the authors present linear elasticity using vector-valued forms and its associated complex.

The wing can be modeled as a thin shell structure. For thin structures (Euler-Bernoulli beam and Kirchhoff plate) the metric enters the interconnection operator.

Open question for Kirchhoff plate: Hess $\cong \nabla \circ d$, where $\nabla$ covariant derivative?

## Tools available

Firedrake: `https://www.firedrakeproject.org/`.

FEniCS: `https://fenicsproject.org/`.

Fluid Structure Interaction in Fenics: Bergersen et al. 2020.

Mesh morphing in FEniCS: `https://bitbucket.org/Epoxid/femorph/src/c7317791c8f00d70fe16d593344cb164a53cad9b/?at=dokken%2Frestructuring`

PyDec: `https://github.com/hirani/pydec`

Learning Python for scientific computing `https://faculty.math.illinois.edu/~hirani/cbmg/index.html`