### Université Fédérale
### Toulouse Midi-Pyrénées

# THÈSE

**En vue de l'obtention du**

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

**Délivré par :** *l'Institut Supérieur de l'Aéronautique et de l'Espace (ISAE)*

**Présentée et soutenue le** *30 Octobre 2020* **par :**
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## A port-Hamiltonian formulation of flexible structures
## Modelling and symplectic finite element discretization

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# Abstract

This thesis aims at extending the port-Hamiltonian (pH) approach to continuum mechanics in higher geometrical dimensions (particularly in 2D). The pH formalism has a strong multiphysics character and represents a unified framework to model, analyze and control both finite- and infinite-dimensional systems. Despite the large literature on this topic, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an intrinsic, i.e. coordinate free, and equivalent pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems in port-Hamiltonian form requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

# Résumé

Cette thèse vise à étendre l'approche port-Hamiltonienne (pH) à la mécanique des milieux continus dans des dimensions géométriques plus élevées (en particulier on se focalise sur la dimension deux). Le formalisme pH, avec son fort caractère multiphysique, représente un cadre unifié pour modéliser, analyser et contrôler les systèmes de dimension finie et infinie. Malgré l'abondante littérature sur ce sujet, les problèmes d'élasticité en deux ou trois dimensions géométriques n'ont presque jamais été considérés. Dans ce travail de thèse la connexion entre problèmes d'élasticité et systèmes distribués port-Hamiltoniens est établie. L'originalité apportée réside dans trois contributions majeures. Tout d'abord, une nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d'élasticité écrits en forme port-Hamiltonienne nécessite l'utilisation d'éléments finis non standards. Néanmoins, l'implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

# Acknowledgments

# Remerciements

# Ringraziamenti

*Alla mia famiglia*

# Contents

# List of Figures

# List of Tables

# List of Acronyms

**DAE**      *Differential-Algebraic Equation*

**dpHs**      *distributed port-Hamiltonian systems*

**FEM**      *Finite Element Method*

**IDA-PBC**   *Interconnection and Damping Assignment Passivity Based Control*

**PDE**      *Partial Differential Equation*

**PFEM**      *Partitioned Finite Element Method*

**pH**      *port-Hamiltonian*

**pHs**      *port-Hamiltonian systems*

**pHDAE**      *port-Hamiltonian Descriptor System*

# Part I

# Introduction and state of the art

# Introduction

*I was born not knowing and have had only a little time to change that here and there.*

*Richard Feynman*
*Letter to Armando Garcia J.*

## Contents

## 1.1   Motivation and context

## 1.2   Overview of chapters

## 1.3   Contributions

# Literature review

> Books serve to show a man that those original thoughts of his aren't very new after all.
>
> *Abraham Lincoln*

## 2.1 Port-Hamiltonian distributed systems

For 1D linear PH systems with a generalized skew-adjoint system operator, [LGZM05] gives conditions on the assignment of boundary inputs and outputs for the system operator to generate a contraction semigroup. The latter is instrumental to show well-posedness of a linear PH system, see [JZ12]. Essentially, at most half the number of boundary port variables can be imposed as control inputs for a well-posed PH system in 1D. The complete characterization of pH in arbitrary dimension is still an open research field. Two notable exceptions [KZ15, Skr19] provide partial answers to this problem. The first demonstrate the well-posedness of the linear wave equation in arbitrary geometrical dimensions. The second generalizes this result to treat the case of generic first order linear pHs in arbitrary geometrical dimensions.

## 2.2 Structure-preserving discretization

## 2.3 Mixed finite element for elasticity

## 2.4 Multibody dynamics

# Reminder on port-Hamiltonian systems

## Contents

The main mathematical aspects behind the pH formalism are recalled in this chapter. First, the finite dimensional case is considered. The geometric concept of Dirac structure [Cou90] is first presented. Finite dimensional port-Hamiltonian system are then introduced by making clear their intimate connection with the concept of Dirac structure. Second, the infinite dimensional case is recalled. The equivalent of Dirac structures for the infinite-dimensional case is the concept of Stokes-Dirac structure. Analogously to what happens in the finite-dimensional case, infinite-dimensional (or distributed) port-Hamiltonian systems are intimately related to the concept of Stokes-Dirac structure.

This notion of Stokes-Dirac structure was first introduced in the literature by making use of a differential geometry approach [vdSM02]. Despite being really insightful in terms of geometrical structure, this approach does not encompass the case of higher-order differential operators. An extension in this sense is still an open question. Since bending problems in elasticity introduce higher-order differential operators, the language of PDE will be privileged over the one of differential forms.

7

## 3.1    Finite dimensional setting

Finite dimensional port-Hamiltonian are characterized by geometrical structures called Dirac structures. It is important to define this geometric concept and see how pHs relate to it.

### 3.1.1    Dirac structure

Consider a finite dimensional space $F$ over the field $\mathbb{R}$ and $E \equiv F'$ its dual, i.e. the space of linear operator $\mathbf{e} : F \to \mathbb{R}$. The elements of $F$ are called flows, while the elements of $E$ are called efforts. Those are port variables and their combination gives the power flowing inside the system. The space $B = F \times E$ is called the bond space of power variables. Therefore the power is defined as $\langle \mathbf{e}, \mathbf{f} \rangle = e(\mathbf{f})$, where $\langle \mathbf{e}, \mathbf{f} \rangle$ is the dual product between $\mathbf{f}$ and $\mathbf{e}$.

**Definition 1** (Dirac Structure [Cou90], Def. 1.1.1)

*Given the finite-dimensional space $F$ and its dual $E$ with respect to the inner product $\langle \cdot, \cdot \rangle_{E \times F} : F \times E \to \mathbb{R}$, consider the symmetric bilinear form:*

$$\langle \langle (\mathbf{f}_1, \mathbf{e}_1), (\mathbf{f}_2, \mathbf{e}_2) \rangle \rangle := \langle \mathbf{e}_1, \mathbf{f}_2 \rangle_{E \times F} + \langle \mathbf{e}_2, \mathbf{f}_1 \rangle_{E \times F}, \qquad where \qquad \mathbf{f}_i, \mathbf{e}_i \in B, \ i = 1, 2 \quad (3.1)$$

*A Dirac structure on $B := F \times E$ is a subspace $D \subset B$, which is maximally isotropic under $\langle \langle \cdot, \cdot \rangle \rangle$. Equivalently, a Dirac structure on $B := F \times E$ is a subspace $D \subset B$ which equals its orthogonal complement with respect to $\langle \langle \cdot, \cdot \rangle \rangle : D = D^\perp$.*

This definition can be extended to consider distributed forces and dissipation [Vil07].

**Proposition 1** (Characterization of Dirac structures)

*Consider the space of power variables $F \times E$ and let $X$ denote an $n$-dimensional space, the space of energy variables. Suppose that $F := F_s \times F_e$ and that $E := E_s \times E_e$, with $dim F_s = dim E_s = n$ and $dim F_e = dim E_e = m$. Moreover, let $\mathbf{J}(\mathbf{x})$ denote a skew-symmetric matrix of dimension $n$ and $\mathbf{B}(\mathbf{x})$ a matrix of dimension $n \times m$. Then, the set*

$$D := \left\{ (\mathbf{f}_s, \mathbf{f}_e, \mathbf{e}_s, \mathbf{e}_e) \in F \times E \mid \quad \mathbf{f}_s = \mathbf{J}(\mathbf{x})\mathbf{e}_s + \mathbf{B}(\mathbf{x})\mathbf{f}_e, \ \mathbf{e}_e = -\mathbf{B}(\mathbf{x})^\top \mathbf{e}_s \right\} \qquad (3.2)$$

*is a Dirac structure.*

It is now possible to make the connection between Dirac structures and pH system explicit.

### 3.1.2 Finite dimensional port-Hamiltonian systems

Consider the time-invariant dynamical system:

$$
\begin{cases}
\dot{\mathbf{x}} & = \mathbf{J}(\mathbf{x})\nabla H(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}, \\
\mathbf{y} & = \mathbf{B}(\mathbf{x})^\top \nabla H(\mathbf{x}),
\end{cases}
\tag{3.3}
$$

where $H(\mathbf{x}) : X \subset \mathbb{R}^n \to \mathbb{R}$, the Hamiltonian, is a real-valued function bounded from below. Such a system is called port-Hamiltonian, as it arises from the Hamiltonian modelling of a physical system and it interacts with the environment through the input $\mathbf{u}$, included in the formulation. The connection with the concept of Dirac structure is achieved by considering the following port behavior:

$$
\begin{aligned}
\mathbf{f}_s = \dot{\mathbf{x}}, & \qquad \mathbf{e}_s = \nabla H(\mathbf{x}), \\
\mathbf{f}_e = \mathbf{u}, & \qquad \mathbf{e}_e = -\mathbf{y}.
\end{aligned}
\tag{3.4}
$$

With this choice of the port variables, system (3.3) defines, by Proposition 1, a Dirac structure. Dissipation and distributed forces can be included and the corresponding system defines an extended Dirac structure, once the proper port variables have been introduced.

System 3.3 is a pH system in canonical form. Recently, finite dimensional differential algebraic port-Hamiltonian systems (pHDAE) have been introduced both for linear [BMXZ18] and non linear systems [MM19]. This enriched description share all the crucial features of ordinary pHs, but easily account for algebraic constraints, time-dependent transformations and explicit dependence on time in the Hamiltonian. The application of the proposed discretization method lead naturally to pHDAE systems.

## 3.2 Infinite dimensional setting

Infinite dimensional spaces appears whenever differential operators have to be considered. In this section we first explain what defines a differential operator. Then Stokes-Dirac structures, characterized by a skew-symmetric differential operator, are introduced. Finally distributed port-Hamiltonian systems and their connection to the concept of Stokes-Dirac structure are illustrated.

Before starting we recall how inner products of square integrable function are computed. Let $\Omega$ denote a compact subset of $\mathbb{R}^d$ and let $L^2(\Omega, \mathbb{A})$ be the space of square integrable functions over the set $\mathbb{A}$ in $\Omega$, with inner product denoted by $\langle \cdot, \cdot \rangle_{L^2(\Omega, \mathbb{A})}$. The set $\mathbb{A}$ can either denote scalars $\mathbb{R}$, vectors $\mathbb{R}^d$, tensors $\mathbb{R}^{d \times d}$ or a Cartesian product of those. For scalars $(a, b) \in L^2(\Omega)$, vectors $(\boldsymbol{a}, \boldsymbol{b}) \in L^2(\Omega, \mathbb{R}^d)$ and tensors $(\boldsymbol{A}, \boldsymbol{B}) \in L^2(\Omega, \mathbb{R}^{d \times d})$ the $L^2$ inner

product is given by

$$\langle a, b\rangle_{L^2(\Omega)} = \int_\Omega ab \, \mathrm{d}\Omega, \qquad \langle \boldsymbol{a}, \boldsymbol{b}\rangle_{L^2(\Omega,\mathbb{R}^d)} = \int_\Omega \boldsymbol{a}\cdot\boldsymbol{b} \, \mathrm{d}\Omega, \qquad \langle \boldsymbol{A}, \boldsymbol{B}\rangle_{L^2(\Omega,\mathbb{R}^{d\times d})} = \int_\Omega \boldsymbol{A}:\boldsymbol{B} \, \mathrm{d}\Omega.$$
(3.5)

The notation $\boldsymbol{A}:\boldsymbol{B} = \sum_{i,j} A_{ij}B_{ij}$ denotes the tensor contraction. Furthermore, the space of square integrable vector-valued functions over the boundary of $\Omega$ is indicated by $L^2(\partial\Omega,\mathbb{R}^m)$. This space is endowed with the inner product

$$\langle \boldsymbol{a}_\partial, \boldsymbol{b}_\partial\rangle_{L^2(\partial\Omega,\mathbb{R}^m)} = \int_{\partial\Omega} \boldsymbol{a}_\partial\cdot\boldsymbol{b}_\partial \, \mathrm{d}S, \qquad \boldsymbol{a}_\partial, \boldsymbol{b}_\partial \in \mathbb{R}^m.$$
(3.6)

### 3.2.1   Linear differential operators

Let $\Omega$ denote a compact subset of $\mathbb{R}^d$ representing the spatial domain of the distributed parameter system. Consider two function space $F_1, F_2$ over the sets $\mathbb{A}$, $\mathbb{B}$ defined on $\Omega \subset \mathbb{R}^d$ and a map $\mathcal{L}$ relating the two

$$\mathcal{L} : F_1(\Omega, \mathbb{A}) \longrightarrow F_2(\Omega, \mathbb{B}),$$
$$\boldsymbol{u} \longrightarrow \boldsymbol{v}.$$
(3.7)

Sets $\mathbb{A}, \mathbb{B}$ can either denote scalars $\mathbb{R}$, vectors $\mathbb{R}^d$, tensors $\mathbb{R}^{d\times d}$ or a Cartesian product of those. Given $\boldsymbol{u} \in F_1$, $\boldsymbol{v} \in F_2$ The map $\mathcal{L}$ is a linear differential operator if it can be represented by a linear combination of derivatives of $\boldsymbol{u}$

$$\boldsymbol{v} = \mathcal{L}\boldsymbol{u} \iff \boldsymbol{v} := \sum_{|\alpha|=0}^n \mathcal{P}_\alpha \partial^\alpha \boldsymbol{u},$$
(3.8)

where $\alpha := (\alpha_1,\ldots,\alpha_d)$ is a multi-index of order $|\alpha| := \sum_{i=1}^d \alpha_i$ and $\partial^\alpha := \partial_{x_1}^{\alpha_1}\ldots\partial_{x_d}^{\alpha_d}$ is a differential operator of order $|\alpha|$ resulting from a combination of spatial derivatives. $\mathcal{P}_\alpha : \mathbb{A} \to \mathbb{B}$ is a constant algebraic operator from set $\mathbb{A}$ to $\mathbb{B}$.

**Example 1** (Divergence operator in $\mathbb{R}^d$)
*Given $\boldsymbol{u} \in C^\infty(\Omega,\mathbb{R}^d)$, $v \in C^\infty(\Omega)$, where $C^\infty(\Omega,\mathbb{R}^d)$, $C^\infty(\Omega)$ denotes the set of smooth vector- and scalar-valued function defined on $\Omega$, the divergence operator in Cartesian coordinate is expressed as*

$$v = \operatorname{div}\boldsymbol{u} = \sum_{i=1}^d \boldsymbol{e}_i \cdot \partial_{x_i}\boldsymbol{u},$$
(3.9)

*where $\boldsymbol{e}_i$ is the $i-$th element of the canonical basis in $\mathbb{R}^d$.*

The differential operators employed in this thesis are reported in Appendix A. A very important notion related to a differential operator is the one of formal adjoint.

**Definition 2** (Formal Adjoint)

314  *Let $\mathcal{L} = L^2(\Omega, \mathbb{A}) \to L^2(\Omega, \mathbb{B})$ be a differential operator and $\boldsymbol{u} \in C_0^\infty(\Omega, \mathbb{A})$, $\boldsymbol{v}(\Omega, \mathbb{B})$ be*
315  *smooth variables with compact support on $\Omega$. The formal adjoint of the differential operator*
316  *$\mathcal{L}$, denoted by $\mathcal{L}^* = L^2(\Omega, \mathbb{B}) \to L^2(\Omega, \mathbb{A})$, is defined by the relation*

$$\langle \mathcal{L}\boldsymbol{u}, \, \boldsymbol{v} \rangle_{L^2(\Omega, \mathbb{B})} = \langle \boldsymbol{u}, \, \mathcal{L}^*\boldsymbol{v} \rangle_{L^2(\Omega, \mathbb{A})} . \tag{3.10}$$

317  This definition represent an extension to generic sets $\mathbb{A}$, $\mathbb{B}$ of Def. 5.80 in [RR04] (reported
318  in Appendix A).

**Remark 1** (Differences between adjoint and formal adjoint)
*The definition of formal adjoint is such that the integration by parts formula is respected.*
*Contrarily to the adjoint of an operator, the formal adjoint definition does not regard the*
*actual domain of the operator nor the boundary conditions. For example, the differential*
*operators* div, grad *are unbounded in the $L^2$ topology. Whenever unbounded operators are*
*considered, it is important to define their domain. To avoid the need of specifying domains,*
*the notion of formal adjoint is used. The formal adjoint respects the integration by parts*
*formula and is defined only for sufficiently smooth functions with compact support. In this*
*sense the formal adjoint of* div *is* $-$ grad, *since for smooth functions with compact support, it*
*holds*

$$\langle \text{div} \, \boldsymbol{y}, \, x \rangle_{L^2(\Omega, \mathbb{R})} \underbrace{=}_{I.B.P.} - \langle \boldsymbol{y}, \, \text{grad} \, x \rangle_{L^2(\Omega, \mathbb{R}^d)} ,$$

319  *for $\boldsymbol{y} \in C_0^\infty(\Omega, \mathbb{R}^d)$, $x \in C_0^\infty(\Omega)$ (I.B.P. stands for integration by parts). The definition of*
320  *the domain of the operators, that requires the knowledge of the boundary conditions, has not*
321  *been specified.*

322  For pHs formally skew-adjoint operators (or simply skew-symmetric) plays a fundamental
323  role.

**Definition 3** (Formally skew-adjoint operator)
325  *Let $\mathcal{J} : L^2(\Omega, \mathbb{F}) \to L^2(\Omega, \mathbb{F})$ be a linear differential operator. Notice that the set $\mathbb{F}$ in the*
326  *domain and co-domain is the same. Then, $\mathcal{J}$ is formally skew-adjoint (or skew-symmetric)*
327  *if and only if $\mathcal{J} = -\mathcal{J}^*$.*
328

329  *If functions with compact support are considered, i.e. $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C_0^\infty(\Omega, \mathbb{F})$ a formally skew-*
330  *adjoint operators is characterized by the relation*

$$\langle \mathcal{J}\boldsymbol{u}_1, \, \boldsymbol{u}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \boldsymbol{u}_1, \, \mathcal{J}\boldsymbol{u}_2 \rangle_{L^2(\Omega, \mathbb{A})} = 0. \tag{3.11}$$

### 3.2.2 Constant Stokes-Dirac structures

332  Constant Stokes-Dirac structures are the infinite-dimensional generalization of constant Dirac
333  structures (i.e. Dirac structures for which the matrices $\mathbf{J}$, $\mathbf{B}$ in (3.3) are constant). Stokes-

Dirac structure are characterized by the fact that they they are equal their orthogonal comple-
ment with respect to a bilinear product. So we recall the definition of orthogonal companion
for the case of smooth functions.

**Definition 4** (Orthogonal complement)
*Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be an open connected set and $C^\infty(\partial\Omega, \mathbb{R}^m)$ the space of smooth*
*functions over its boundary. Consider the space*

$$B = C^\infty(\Omega, \mathbb{F}) \times C^\infty(\partial\Omega, \mathbb{R}^m) \times C^\infty(\Omega, \mathbb{F}) \times C^\infty(\partial\Omega, \mathbb{R}^m) \qquad (3.12)$$

*and the bilinear pairing defined by*

$$\langle\langle \cdot, \cdot \rangle\rangle : B \times B \longrightarrow \mathbb{R},$$

$$(\boldsymbol{a},\ \boldsymbol{a}_\partial,\ \boldsymbol{b},\ \boldsymbol{b}_\partial) \times (\boldsymbol{c},\ \boldsymbol{c}_\partial,\ \boldsymbol{d},\ \boldsymbol{d}_\partial) \longrightarrow \begin{matrix} \langle \boldsymbol{a},\ \boldsymbol{d} \rangle_{L^2(\Omega, \mathbb{F})} + \langle \boldsymbol{b},\ \boldsymbol{c} \rangle_{L^2(\Omega, \mathbb{F})} + \\ \langle \boldsymbol{a}_\partial,\ \boldsymbol{d}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \boldsymbol{b}_\partial,\ \boldsymbol{c}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{matrix} \qquad (3.13)$$

*Given a linear subspace $W \subset B$, its orthogonal complement is the set*

$$W^\perp = \{\boldsymbol{v} \in B | \langle\langle \boldsymbol{v}, \boldsymbol{w} \rangle\rangle = 0,\ \forall \boldsymbol{w} \in W\} \qquad (3.14)$$

We can now define what a Stokes-Dirac structure is.

**Definition 5** (Stokes-Dirac structure)
*A subset $D \subset B$, with $B$ defined in (3.12), is a Stokes-Dirac structure iff*

$$D = D^\perp, \qquad (3.15)$$

*where the orthogonal complement has been defined in Eq. (3.14)*

For a subset to be a Stokes-Dirac structures a link between flow and effort variables must
hold. Consider $\boldsymbol{f} \in C^\infty(\Omega, \mathbb{F})$ and $\boldsymbol{e} \in C^\infty(\Omega, \mathbb{F})$ and te following relation between the two

$$\boldsymbol{f} = \mathcal{J}\boldsymbol{e}, \qquad \mathcal{J} = -\mathcal{J}^*, \qquad (3.16)$$

where $\mathcal{J}$ is a formally skew-adjoint operator. A Stokes-Dirac structure requires the specifi-
cation of boundary variables in order to express a general power conservation property for
*open* physical systems. We make therefore the following assumption, over the existence of
appropriate boundary operators.

**Assumption 1** (Existence of boundary operators)
*Assume that exist two linear boundary operators $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ such that for $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in C^\infty(\Omega, \mathbb{F})$ the*
*following integration by parts formula holds*

$$\langle \mathcal{J}\boldsymbol{u}_1,\ \boldsymbol{u}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \boldsymbol{u}_1,\ \mathcal{J}\boldsymbol{u}_2 \rangle_{L^2(\Omega, \mathbb{A})} = \langle \mathcal{B}_\partial \boldsymbol{u}_1,\ \mathcal{C}_\partial \boldsymbol{u}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \mathcal{B}_\partial \boldsymbol{u}_2,\ \mathcal{C}_\partial \boldsymbol{u}_1 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}.$$
$$(3.17)$$

This assumption is necessary to appropriately define a Stokes-Dirac structure. Only few particular cases, like the transport equation, do not verify it. We can now characterize generic Stokes-Dirac structure for smooth functions spaces.

**Proposition 2** (Characterization of Stokes-Dirac structures)
*Let $B$ be defined as in Eq. (3.12) and $\mathcal{J}$ be a formally skew adjoint operator verifying Assumption 1. The set*

$$D_{\mathcal{J}} = \{(\boldsymbol{f},\ \boldsymbol{f}_{\partial},\ \boldsymbol{e},\ \boldsymbol{e}_{\partial}) \in B | \boldsymbol{f} = \mathcal{J}\boldsymbol{e},\ \boldsymbol{f}_{\partial} = \mathcal{B}_{\partial}\boldsymbol{e},\ \boldsymbol{e}_{\partial} = -\mathcal{C}_{\partial}\boldsymbol{e}\} \tag{3.18}$$

*is a Stokes-Dirac structure with respect to the bilinear pairing (3.13).*

*Proof.* A Stokes-Dirac is characterized by the fact that $D_{\mathcal{J}} = D_{\mathcal{J}}^{\perp}$. Then one has to show that $D_{\mathcal{J}} \subset D_{\mathcal{J}}^{\perp}$ and $D_{\mathcal{J}}^{\perp} \subset D_{\mathcal{J}}$. Following [LGZM05], the proof is obtained following three steps.

*Step 1.* To show that $D_{\mathcal{J}} \subset D_{\mathcal{J}}^{\perp}$, take $(\boldsymbol{f},\ \boldsymbol{f}_{\partial},\ \boldsymbol{e},\ \boldsymbol{e}_{\partial}) \in D_{\mathcal{J}}$. Then

$$
\begin{aligned}
\langle\langle\,(\boldsymbol{f},\boldsymbol{f}_{\partial},\boldsymbol{e},\boldsymbol{e}_{\partial}),(\boldsymbol{f},\boldsymbol{f}_{\partial},\boldsymbol{e},\boldsymbol{e}_{\partial})\,\rangle\rangle =&\, 2\,\langle\boldsymbol{e},\,\boldsymbol{f}\rangle_{L^2(\Omega,\mathbb{F})} + 2\,\langle\boldsymbol{e}_{\partial},\,\boldsymbol{f}_{\partial}\rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \\
=&\, 2\,\langle\boldsymbol{e},\,\mathcal{J}\boldsymbol{e}\rangle_{L^2(\Omega,\mathbb{F})} + 2\,\langle\boldsymbol{e}_{\partial},\,\boldsymbol{f}_{\partial}\rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \\
\overset{\text{Eq. (3.17)}}{=}&\, 2\,\langle\mathcal{B}_{\partial}\boldsymbol{e},\,\mathcal{C}_{\partial}\boldsymbol{e}\rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + 2\,\langle\boldsymbol{e}_{\partial},\,\boldsymbol{f}_{\partial}\rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \\
\overset{\text{Eq. (3.18)}}{=}&\, 2\,\langle\mathcal{B}_{\partial}\boldsymbol{e},\,\mathcal{C}_{\partial}\boldsymbol{e}\rangle_{L^2(\partial\Omega,\mathbb{R}^m)} - 2\,\langle\mathcal{B}_{\partial}\boldsymbol{e},\,\mathcal{C}_{\partial}\boldsymbol{e}\rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \\
=&\, 0.
\end{aligned}
$$

This implies $D_{\mathcal{J}} \subset D_{\mathcal{J}}^{\perp}$.

*Step 2.* Take $(\boldsymbol{\phi},\ \boldsymbol{\phi}_{\partial},\ \boldsymbol{\epsilon},\ \boldsymbol{\epsilon}_{\partial}) \in D_{\mathcal{J}}^{\perp}$ and $\boldsymbol{e}_0 \in C_0^{\infty}(\Omega,\mathbb{F})$. This implies $\mathcal{B}_{\partial}\boldsymbol{e}_0 = (\boldsymbol{0},\boldsymbol{0})$ and $\mathcal{C}_{\partial}\boldsymbol{e}_0 = (\boldsymbol{0},\boldsymbol{0})$. Taking $(\mathcal{J}\boldsymbol{e}_0, \boldsymbol{0}, \boldsymbol{e}_0, \boldsymbol{0}) \in D_{\mathcal{J}}$ then

$$\langle\langle\,(\boldsymbol{\phi},\boldsymbol{\phi}_{\partial},\boldsymbol{\epsilon},\boldsymbol{\epsilon}_{\partial}),(\mathcal{J}\boldsymbol{e}_0,\boldsymbol{0},\boldsymbol{e}_0,\boldsymbol{0})\,\rangle\rangle = \langle\boldsymbol{\epsilon},\,\mathcal{J}\boldsymbol{e}_0\rangle_{L^2(\Omega,\mathbb{F})} + \langle\boldsymbol{e}_0,\,\boldsymbol{\phi}\rangle_{L^2(\Omega,\mathbb{F})} = 0, \quad \forall\boldsymbol{e}_0 \in C_0^{\infty}(\Omega,\mathbb{F}).$$

It follows that $\boldsymbol{\epsilon} \in C_0^{\infty}(\Omega,\mathbb{F})$ and $\boldsymbol{\phi} = \mathcal{J}\boldsymbol{\epsilon}$.

*Step 3.* Take $(\boldsymbol{\phi},\ \boldsymbol{\phi}_{\partial},\ \boldsymbol{\epsilon},\ \boldsymbol{\epsilon}_{\partial}) \in D_{\mathcal{J}}^{\perp}$ and $(\boldsymbol{f},\ \boldsymbol{f}_{\partial},\ \boldsymbol{e},\ \boldsymbol{e}_{\partial}) \in D_{\mathcal{J}}$. From step 2 and (3.17)

$$0 = \langle \mathcal{J}\boldsymbol{e}, \boldsymbol{\epsilon} \rangle_{L^2(\Omega,\mathbb{F})} + \langle \boldsymbol{e}, \mathcal{J}\boldsymbol{\epsilon} \rangle_{L^2(\Omega,\mathbb{F})} + \langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)},$$

$$\overset{\text{Eq. } (3.17)}{=} \langle \mathcal{B}_\partial \boldsymbol{e}, \mathcal{C}_\partial \boldsymbol{\epsilon} \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \mathcal{B}_\partial \boldsymbol{\epsilon}, \mathcal{C}_\partial \boldsymbol{e} \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)},$$

$$= \langle \mathcal{B}_\partial \boldsymbol{e}, \mathcal{C}_\partial \boldsymbol{\epsilon} \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \mathcal{B}_\partial \boldsymbol{\epsilon}, \mathcal{C}_\partial \boldsymbol{e} \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle -\mathcal{C}_\partial \boldsymbol{e}, \boldsymbol{\phi}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \boldsymbol{\epsilon}_\partial, \mathcal{B}_\partial \boldsymbol{e} \rangle_{L^2(\partial\Omega,\mathbb{R}^m)},$$

$$= \langle \mathcal{B}_\partial \boldsymbol{e}, \mathcal{C}_\partial \boldsymbol{\epsilon} + \boldsymbol{\epsilon}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} + \langle \mathcal{B}_\partial \boldsymbol{\epsilon} - \boldsymbol{\phi}_\partial, \mathcal{C}_\partial \boldsymbol{e} \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \qquad \text{By linearity,}$$

$$= \langle \boldsymbol{e}_\partial, \mathcal{C}_\partial \boldsymbol{\epsilon} + \boldsymbol{\epsilon}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} - \langle \mathcal{B}_\partial \boldsymbol{\epsilon} - \boldsymbol{\phi}_\partial, \boldsymbol{f}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}.$$

Given the fact that $\boldsymbol{e}_\partial$, $\boldsymbol{f}_\partial$ are arbitrary then

$$\boldsymbol{\phi}_\partial = \mathcal{B}_\partial \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon}_\partial = -\mathcal{C}_\partial \boldsymbol{\epsilon},$$

meaning that $D_{\mathcal{J}}^\perp \subset D_{\mathcal{J}}$. □

### 3.2.3   Distributed port-Hamiltonian system

A distributed lossless port-Hamiltonian system is defined by a set of variables that describes the unknowns, by a formally skew-adjoint differential operator, an energy functional and a set of boundary inputs and corresponding conjugated outputs. Such a system is described by the following set of equations, defined on an open connected set $\Omega \subset \mathbb{R}^d$

$$\begin{aligned} \partial_t \boldsymbol{\alpha} &= \mathcal{J}\,\delta_{\boldsymbol{\alpha}} H, & \boldsymbol{\alpha} &\in C^\infty(\Omega,\mathbb{F}), \\ \boldsymbol{u}_\partial &= \mathcal{B}_\partial\,\delta_{\boldsymbol{\alpha}} H, & \boldsymbol{u}_\partial &\in \mathbb{R}^m, \\ \boldsymbol{y}_\partial &= \mathcal{C}_\partial\,\delta_{\boldsymbol{\alpha}} H, & \boldsymbol{y}_\partial &\in \mathbb{R}^m. \end{aligned} \tag{3.19}$$

The unknowns $\boldsymbol{\alpha}$ are called energy variables in the port-Hamiltonian framework, the formally skew-adjoint operator $\mathcal{J}$ is named interconnection operator (see Def. 3 for a precise definition of formal skew adjointness). $\mathcal{B}_\partial, \mathcal{C}_\partial$ are boundary operators, that provide the boundary input $\boldsymbol{u}_\partial$ and output $\boldsymbol{y}_\partial$ [TW09, Chapter 4]. The functional $H(\boldsymbol{\alpha}) : C^\infty(\Omega,\mathbb{F}) \to \mathbb{R}$ corresponds to the Hamiltonian functional and in all the examples considered in this thesis coincide with the total energy of the system. Notation $\delta_{\boldsymbol{\alpha}} H$ indicates the variational derivative of $H$.

**Definition 6** (Variational derivative, Def. 4.1 in [Olv93])
*Consider a functional $H(\boldsymbol{\alpha}) : C^\infty(\Omega,\mathbb{F}) \to \mathbb{R}$*

$$H(\boldsymbol{\alpha}) = \int_\Omega \mathcal{H}(\boldsymbol{\alpha})\,\mathrm{d}\Omega.$$

*Given a variation $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}} + \eta\boldsymbol{\delta\alpha}$ the variational derivative $\frac{\delta H}{\delta\boldsymbol{\alpha}}$ is defined as*

$$H(\bar{\boldsymbol{\alpha}} + \eta\boldsymbol{\delta\alpha}) = H(\bar{\boldsymbol{\alpha}}) + \eta\langle \delta_{\boldsymbol{\alpha}} H, \boldsymbol{\delta\alpha} \rangle_{L^2(\Omega,\mathbb{F})} + O(\eta^2).$$

**Remark 2**

*If the integrand does not contain derivative of the argument $\boldsymbol{\alpha}$ then the variational derivative is equal to the partial derivative of the Hamiltonian density $\mathcal{H}$*

$$\frac{\delta H}{\delta \boldsymbol{\alpha}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\alpha}}.$$

**Remark 3** (Co-energy variables)
*The variational derivative of the Hamiltonian defines the co-energy variables $\boldsymbol{e} := \delta_{\boldsymbol{\alpha}} H$. These are equivalent to the effort variables of the Stokes-Dirac structure as we will immediately show.*

Suppose that operators $\mathcal{J}$, $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ in Eq. 3.19 verify Ass. 1. Then, System (3.19) is lossless since the energy rate is given by

$$\begin{aligned}
\dot{H} &= \langle \delta_{\boldsymbol{\alpha}} H, \partial_t \boldsymbol{\alpha} \rangle_{L^2(\Omega,\mathbb{F})}, \\
&\overset{Eq.(3.17)}{=} \langle \mathcal{B}_\partial \delta_{\boldsymbol{\alpha}} H, \mathcal{C}_\partial \delta_{\boldsymbol{\alpha}} H \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \\
&= \langle \boldsymbol{u}_\partial, \boldsymbol{y}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}.
\end{aligned} \tag{3.20}$$

The connection between the concept of Stokes-Dirac structure and dpHs becomes clear if the following port behavior is considered

$$\begin{aligned}
\boldsymbol{f} &= \partial_t \boldsymbol{\alpha}, & \boldsymbol{e} &= \delta_{\boldsymbol{\alpha}} H, \\
\boldsymbol{f}_\partial &= \boldsymbol{u}_\partial, & \boldsymbol{e}_\partial &= -\boldsymbol{y}_\partial.
\end{aligned} \tag{3.21}$$

By proposition (2) System (3.19) under the port behavior (3.21) defines a Stokes-Dirac structure. No rigorous characterization has been given so far for operators $\mathcal{J}$, $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ in system (3.19). A formal characterization of these operators has been given in [LGZM05] for pH of generic order only in one geometrical dimensional. In Chapter 7 the operator $\mathcal{J}$ will be better characterize using an appropriate partition. By applying a general integration by parts formula, the operators $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ associated to $\mathcal{J}$ can be defined as well. The following examples clarifies this assertion for some known pHs.

### 3.2.3.1 Wave equation

Given an open bounded connected set $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, the propagation of sound in air can be described by the following model [TRLGK18]

$$\begin{aligned}
\chi_s \partial_t p(\boldsymbol{x}, t) &= -\operatorname{div} \boldsymbol{v}, \\
\mu_0 \partial_t \boldsymbol{v}(\boldsymbol{x}, t) &= -\operatorname{grad} p,
\end{aligned} \tag{3.22}$$

where the scalars $\chi_s$, $\mu_0$ are the constant adiabatic compressibility factor and the steady state mass density respectively. The scalar field $p \in \mathbb{R}$ and vector field $\boldsymbol{v} \in \mathbb{R}^2$ represents the variation of pressure and velocity from the steady state. The Hamiltonian (total energy) reads

$$H = \frac{1}{2} \int_\Omega \left\{ \chi_s p^2 + \mu_0 \|\boldsymbol{v}\|^2 \right\} \, \mathrm{d}\Omega.$$

To recast (3.22) in pH form the energy variables has to be introduced $\boldsymbol{\alpha} = [\alpha_p,\ \boldsymbol{\alpha}_v]^{\top}$

$$\alpha_p := \chi_s p, \qquad \boldsymbol{\alpha}_v := \mu_0 \boldsymbol{v}.$$

The Hamiltonian is rewritten as

$$H = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\chi_s} \alpha_p^2 + \frac{1}{\mu_0} \|\boldsymbol{\alpha}_v\|^2 \right\}\ \mathrm{d}\Omega.$$

By definition, the co-energy are

$$e_p = \frac{\delta H}{\delta \alpha_p} = \frac{1}{\chi_s} \alpha_p = p, \qquad \boldsymbol{e}_v = \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \frac{1}{\mu_0} \boldsymbol{\alpha}_v = \boldsymbol{v}.$$

Equation (3.22) can be recast in port-Hamiltonian form

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_p \\ \boldsymbol{\alpha}_v \end{pmatrix} = - \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_p \\ \boldsymbol{e}_v \end{pmatrix}.$$

From the energy rate it is possible to identify the boundary variables.

$$
\begin{aligned}
\dot{H} &= + \int_{\Omega} \left\{ e_p\, \partial_t \alpha_p + \boldsymbol{e}_v \cdot \partial_t \boldsymbol{\alpha}_v \right\}\ \mathrm{d}\Omega, \\
&= - \int_{\Omega} \left\{ e_p\, \mathrm{div}\, \boldsymbol{e}_v + \boldsymbol{e}_v \cdot \mathrm{grad}\, e_p \right\}\ \mathrm{d}\Omega, & \text{Chain rule,} \\
&= - \int_{\Omega} \mathrm{div}(e_p\, \boldsymbol{e}_v)\ \mathrm{d}\Omega, & \text{Stokes theorem,} \\
&= - \int_{\partial\Omega} e_p\, \boldsymbol{e}_v \cdot \boldsymbol{n}\ \mathrm{d}S = - \left\langle e_p,\ \boldsymbol{e}_v \cdot \boldsymbol{n} \right\rangle_{L^2(\partial\Omega, \mathbb{R}^2)}.
\end{aligned}
$$

The boundary term $\left\langle e_p,\ \boldsymbol{e}_v \cdot \boldsymbol{n} \right\rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$ pairs two power variables. One is taken as control input, the other plays the role of power-conjugated output. The assignment of these roles to the boundary power variables is referred to as causality of the boundary port [KML18],[Kot19, Chapter 2]. Under uniform causality assumption, either $e_p$ or $\boldsymbol{e}_v$ can assume the role of (distributed) boundary input, but not both. This leads to two possible selections:

- First case $u_\partial = e_p, \quad y_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}$.
  This imposes the variable $e_p := p$ as boundary input and corresponds to a classical Dirichlet condition. The boundary operator for this case are given by

$$\mathcal{B}_\partial \begin{pmatrix} e_p \\ \boldsymbol{e}_v \end{pmatrix} = e_p|_{\partial\Omega}, \qquad \mathcal{C}_\partial \begin{pmatrix} e_p \\ \boldsymbol{e}_v \end{pmatrix} = \boldsymbol{e}_v \cdot \boldsymbol{n}|_{\partial\Omega},$$

  corresponding to the standard trace and normal trace operators.

- Second case $u_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}, \quad y_\partial = e_p$.
  This imposes the variable $\boldsymbol{e}_v \cdot \boldsymbol{n} := \boldsymbol{v} \cdot \boldsymbol{n}$ as boundary input and corresponds to a Neumann condition. The boundary operators are therefore switched with respect to

411     the previous case

$$\mathcal{B}_\partial \begin{pmatrix} e_p \\ \boldsymbol{e}_v \end{pmatrix} = \boldsymbol{e}_v \cdot \boldsymbol{n}|_{\partial\Omega}, \qquad \mathcal{C}_\partial \begin{pmatrix} e_p \\ \boldsymbol{e}_v \end{pmatrix} = e_p|_{\partial\Omega}.$$

### 412     3.2.3.2    Euler Bernoulli beam

413     The Euler-Bernoulli beam model consists of one PDE, describing the vertical displacement
414     along the beam length:

$$\rho A(x) \frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) = 0, \quad x \in \Omega = \{0, L\}, \tag{3.23}$$

415     where $w(x,t)$ is the transverse displacement of the beam. The coefficients $\rho(x), A(x)E(x)$
416     and $I(x)$ are the mass density, cross section, Young's modulus of elasticity and the moment
417     of inertia of a cross section. The energy variables are then chosen as follows:

$$\alpha_w = \rho A(x)\frac{\partial w}{\partial t}(x,t), \quad \text{Linear Momentum}, \quad \alpha_\kappa = \frac{\partial^2 w}{\partial x^2}(x,t), \quad \text{Curvature.} \tag{3.24}$$

418     Those variables are collected in the vector $\boldsymbol{\alpha} = (\alpha_w,\ \alpha_\kappa)^T$, so that the Hamiltonian can
419     be written as a quadratic functional in the energy variables:

$$H = \frac{1}{2}\int_\Omega \left\{ \frac{1}{\rho A}\alpha_w^2 + EI\alpha_\kappa^2 \right\}\ \mathrm{d}\Omega \tag{3.25}$$

420     The co-energy variables are found by computing the variational derivative of the Hamil-
421     tonian:

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}(x,t), & \text{Vertical velocity,} \\ e_\kappa &:= \frac{\delta H}{\delta \alpha_\kappa} = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t), & \text{Flexural momentum.} \end{aligned} \tag{3.26}$$

422     The underlying interconnection structure is then found to be:

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \alpha_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix}\begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}. \tag{3.27}$$

423     The power flow gives access to the boundary variables:

$$\begin{aligned} \dot{H} &= \int_\Omega \left\{ e_w\partial_t\alpha_w + e_\kappa\partial_t\alpha_\kappa \right\}\ \mathrm{d}\Omega, \\ &= \int_\Omega \left\{ -e_w\partial_{xx}e_\kappa + e_\kappa\partial_{xx}e_w \right\}\ \mathrm{d}\Omega, \qquad \text{Integration by parts,} \\ &= \int_{\partial\Omega}\left\{ -e_w\partial_x e_\kappa + e_\kappa\partial_x e_w \right\}\ \mathrm{d}s = \langle -e_w|_{\partial\Omega},\ \partial_x e_\kappa|_{\partial\Omega}\rangle_{\mathbb{R}^4} + \langle e_\kappa|_{\partial\Omega},\ \partial_x e_w|_{\partial\Omega}\rangle_{\mathbb{R}^4} \end{aligned} \tag{3.28}$$

Since the system is of differential order two, two pairing appears, giving rise to four combination of uniform boundary causality

- First case $u_{\partial,1} = e_w$, $\quad u_{\partial,2} = \partial_x e_w$, $\quad y_{\partial,1} = -\partial_x e_\kappa$, $\quad y_{\partial,2} = e_\kappa$.

  This imposes the vertical $e_w := \partial_t w$ and angular velocity $\partial_x e_w := \partial_{xt} w$ as boundary inputs

$$
\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \qquad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \qquad (3.29)
$$

  If the inputs are null a clamped boundary condition is obtained.

- Second case $u_{\partial,1} = e_w$, $\quad u_{\partial,2} = e_\kappa$, $\quad y_{\partial,1} = -\partial_x e_\kappa$, $\quad y_{\partial,2} = \partial_x e_w$.

  This imposes the vertical velocity and flexural momentum $e_\kappa := EI\partial_{xx} w$ as boundary inputs

$$
\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \qquad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \qquad (3.30)
$$

  Zero inputs lead to a simply supported condition.

- Third case $u_{\partial,1} = -\partial_x e_\kappa$, $\quad u_{\partial,2} = e_\kappa$, $\quad y_{\partial,1} = e_w$, $\quad y_{\partial,2} = \partial_x e_w$.

  This imposes the shear force $\partial_x e_\kappa := \partial_x(EI\partial_{xx} w)$ and flexural momentum as boundary inputs

$$
\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \qquad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \qquad (3.31)
$$

  Null inputs correspond to a free condition.

- Fourth case $u_{\partial,1} = -\partial_x e_\kappa$, $\quad u_{\partial,2} = \partial_x e_w$, $\quad y_{\partial,1} = e_w$, $\quad y_{\partial,2} = e_\kappa$.

  This imposes the shear force and angular velocity as boundary inputs

$$
\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \qquad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \qquad (3.32)
$$

### 3.2.3.3   2D shallow water equations

This formulation may be found in [CR16, Section 6.2]. This model describes a thin fluid layer of constant density in hydrostatic balance, like the propagation of a tsunami wave far from

shore. Consider an open bounded connected set $\Omega \subset \mathbb{R}^2$ and a constant bed profile. The mass conservation implies

$$\frac{\partial h}{\partial t} + \mathrm{div}(h\boldsymbol{v}) = 0,$$

where $h(x, y, t) \in \mathbb{R}$ is a scalar field representing the fluid height, $\boldsymbol{v}(x, y, t) \in \mathbb{R}^2$ is the fluid velocity field. The conservation of linear momentum reads

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla(\rho g h) = 0,$$

where $\rho$ is the mass density and $g$ the gravitational acceleration constant. Using the identity

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla(\|\boldsymbol{v}\|^2) + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v},$$

where $\nabla\times$ is the rotational of $\boldsymbol{v}$ (also denoted $\mathrm{curl}\,\boldsymbol{v}$), the momentum is rearranged as follows

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = -\nabla\left(\frac{1}{2}\rho\|\boldsymbol{v}\|^2 + \rho g h\right) - \rho(\nabla \times \boldsymbol{v}) \times \boldsymbol{v}.$$

The last term on the right-hand side can be rewritten

$$\rho(\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = \begin{bmatrix} 0 & -\rho\omega \\ \rho\omega & 0 \end{bmatrix} \boldsymbol{v},$$

with $\omega = \partial_x v_y - \partial_y v_x$ the local vorticity term. To derive a suitable pH formulation, the total energy, made up of kinetic and potential contribution, has to be invoked

$$H = \frac{1}{2}\int_\Omega \left\{\rho h\|\boldsymbol{v}\|^2 + \rho g h^2\right\}\,\mathrm{d}\Omega.$$

As energy variable the fluid height and the linear momentum are chosen

$$\alpha_h = h, \qquad \boldsymbol{\alpha}_v = \rho\boldsymbol{v}. \tag{3.33}$$

The Hamiltonian is a non separable functional of the energy variables

$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2}\int_\Omega \left\{\frac{1}{\rho}\alpha_h\|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h^2\right\}\,\mathrm{d}\Omega. \tag{3.34}$$

The co-energy variables are given by

$$e_h := \frac{\delta H}{\delta \alpha_h} = \frac{1}{2\rho}\|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h, \qquad \boldsymbol{e}_v := \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \frac{1}{\rho}\alpha_h\boldsymbol{\alpha}_v. \tag{3.35}$$

The mass and momentum conservation are then rewritten as follows

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div} \\ -\mathrm{grad} & \boldsymbol{\mathcal{G}} \end{bmatrix}\begin{pmatrix} e_h \\ \boldsymbol{e}_v \end{pmatrix}, \tag{3.36}$$

The gyroscopic skew-symmetric term $\boldsymbol{\mathcal{G}}$ introduces a non-linearity as it depends on the energy variables

$$\boldsymbol{\mathcal{G}}(\alpha_h, \boldsymbol{\alpha}_v) = \frac{\omega}{\alpha_h} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \omega = \partial_x \alpha_{v,y} - \partial_y \alpha_{v,x}.$$

Despite the non-standard formulation, the energy rate provides anyway the boundary variables

$$
\begin{aligned}
\dot{H} &= + \int_\Omega \left\{ e_h \, \partial_t \alpha_h + \boldsymbol{e}_v \cdot \partial_t \boldsymbol{\alpha}_v \right\} \, \mathrm{d}\Omega, \\
&= - \int_\Omega \left\{ e_h \, \operatorname{div} \boldsymbol{e}_v + \boldsymbol{e}_v \cdot (\operatorname{grad} e_h - \mathcal{G} \boldsymbol{e}_v) \right\} \, \mathrm{d}\Omega, && \text{skew-symmetry of } \mathcal{G}, \\
&= - \int_\Omega \left\{ e_h \, \operatorname{div} \boldsymbol{e}_v + \boldsymbol{e}_v \cdot \operatorname{grad} e_h \right\} \, \mathrm{d}\Omega, && \text{Chain rule,} && (3.37) \\
&= - \int_\Omega \operatorname{div}(e_h \, \boldsymbol{e}_v) \, \mathrm{d}\Omega, && \text{Stokes theorem,} \\
&= - \int_{\partial\Omega} e_h \, \boldsymbol{e}_v \cdot \boldsymbol{n} \, \mathrm{d}S = - \left\langle e_h, \, \boldsymbol{e}_v \cdot \boldsymbol{n} \right\rangle_{\partial\Omega}.
\end{aligned}
$$

Again two possible cases of uniform boundary causality arise:

- First case $u_\partial = e_h, \quad y_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}$.
  This imposes the variable $e_h := h$ as boundary input and corresponds to a given water level for a fluid boundary.

- Second case $u_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}, \quad y_\partial = e_p$.
  This imposes the variable $\boldsymbol{e}_v \cdot \boldsymbol{n} := h\boldsymbol{v} \cdot \boldsymbol{n}$ as boundary input and corresponds to a given volumetric flow rate.

## 3.3   Conclusion

In this chapter, the main mathematical tools needed to understand infinite-dimensional pHs were recalled. A general characterization of the underlying operators behind a boundary control pH system is still an open topic. In Chapter 7, these operators are characterized, in connection to the discretization method developed.

# Part II

# Port-Hamiltonian elasticity and thermoelasticity

# Elasticity in port-Hamiltonian form

> I try not to break the rules but merely to test their elasticity.
>
> *Bill Veeck*

## Contents

ontinuum mechanics is the mathematical description of how materials behave kine-matically under external excitations. In this framework, the microscopic structure of a material body is neglected and a macroscopic viewpoint, that describes the body as a continuum, is adopted. This leads to a PDE based model. In this chapter, the general linear elastodynamics problem is recalled. A suitable port-Hamiltonian formulation is then derived.

## 4.1 Continuum mechanics

In this section, the main concepts behind a deformable continuum are briefly recalled following [Lee12]. For a detailed discussion on this topic, the reader may consult [Abe12, LPKL12].

### 4.1.1 Non linear formulation of elasticity

The bounded region of $\mathbb{R}^d$, $d \in \{2, 3\}$ occupied by a solid is called configuration. The reference configuration $\Omega$ is the domain that a bodies occupies at the initial state. To describe how the

body deforms in time the deformation map $\mathbf{\Phi} : \Omega \times [0, T_f] \to \Omega' \subset \mathbb{R}^d$ is introduced. This map is differentiable and orientation preserving, and the image of $\Omega$ under $\mathbf{\Phi}(\cdot, t)$ $\forall t \in [0, T_f]$ is called the deformed configuration $\Omega_t$. Given a specific point in the reference frame its image is denoted by $\mathbf{y} = \mathbf{\Phi}(\mathbf{x}, t)$. The gradient of the deformation map is called the deformation gradient $\mathbf{F} := \nabla_x \mathbf{\Phi} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$. A rigid deformation maps a point $\mathbf{x} \in \Omega \to \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$, where $\mathbf{A}(t)$ is an orthogonal matrix and $\mathbf{b}(t) \in \mathbb{R}^d$ a vector. A differentiable deformation map $\mathbf{\Phi}$ is a rigid deformation iff $\mathbf{F}^\top \mathbf{F} - \mathbf{I} = 0$, where $\mathbf{I}$ is the identity in $\mathbb{R}^{d \times d}$ (for the proof see [Cia88], page 44). For this reason, a suitable measure of the deformation is the Green-St.Venant strain tensor $\frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I})$.

A quantity of interest is the displacement $\mathbf{u} : \Omega \times [0, T_f] \to \mathbb{R}^d$ with respect to the reference configuration. It is defined as $\mathbf{u}(\mathbf{x}, t) = \mathbf{\Phi}(\mathbf{x}, t) - \mathbf{x}$. The gradient of the displacement verifies $\nabla_x \mathbf{u} = \mathbf{F} - \mathbf{I}$. The strain tensor can now be written in terms of the displacement

$$\frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}) = \frac{1}{2}\left[ (\nabla_x \mathbf{u} + \mathbf{I})^\top (\nabla_x \mathbf{u} + \mathbf{I}) - \mathbf{I} \right]$$
$$= \frac{1}{2}\left[ \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^\top + (\nabla_x \mathbf{u})^\top (\nabla_x \mathbf{u}) \right],$$

or in components

$$\frac{1}{2}(F_{ik}^\top F_{kj} - I_{ij}) = \frac{1}{2}\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \right).$$

To state the balance laws the actual deformed configuration is considered. The linear and angular momenta in a subdomain $\omega_t \subset \Omega_t$ are computed as

$$\int_{\omega_t} \rho\, \mathbf{v}\, \mathrm{d}\omega_t, \quad \text{and} \quad \int_{\omega_t} \rho\, \mathbf{y} \times \mathbf{v}\, \mathrm{d}\omega_t,$$

where $\rho$ is the mass density and the velocity $\mathbf{v} = \frac{D\mathbf{u}}{Dt}(\mathbf{y}, t) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t)$ is the material time derivative of the displacement (see [Abe12, Chapter 1]). Let $\omega_{t,1}$, $\omega_{t,2}$ be two subregions in a deformed continuum $\Omega_t$ with contacting surface $S_{12}$. There is a force acting on this surface for a continuum that is called stress vector or traction. If $\mathbf{n}$ is the outward normal at $\mathbf{y}$ on $S_{12}$ with respect to $\omega_{t,1}$, then the surface force that $\omega_{t,1}$ exerts on $\omega_{t,2}$ is denoted by $\mathbf{t}(\mathbf{y}, \mathbf{n}) \in \mathbb{R}^d$. By the Newton third law, the surface force that $\omega_{t,2}$ applies on $\omega_{t,1}$ is given by $\mathbf{t}(\mathbf{y}, -\mathbf{n}) = -\mathbf{t}(\mathbf{y}, \mathbf{n})$. It is assumed that the linear and angular momentum balance hold for any subregion $\omega_t \in \Omega_t$

$$\frac{d}{dt}\int_{\omega_t} \rho\mathbf{v}\, \mathrm{d}\omega_t = \int_{\partial\omega_t} \mathbf{t}(\mathbf{y}, \mathbf{n})\, \mathrm{d}S + \int_{\omega_t} \mathbf{f}\, \mathrm{d}\omega_t,$$
$$\frac{d}{dt}\int_{\omega_t} \rho\mathbf{y} \times \mathbf{v}\, \mathrm{d}\omega_t = \int_{\partial\omega_t} \mathbf{y} \times \mathbf{t}(\mathbf{y}, \mathbf{n})\, \mathrm{d}S + \int_{\omega_t} \mathbf{y} \times \mathbf{f}\, \mathrm{d}\omega_t,$$

where $\partial\omega_t$ stands for the boundary surface of the subdomain $\omega_t$, $\mathbf{n}$ is the outward normal to the surface $\partial\omega_t$ and $\mathbf{f}$ represents an exterior body force. The following theorem characterizes the stress vector (see [Cia88, Chapter 2]):

491 **Theorem 1** (Cauchy's theorem)
492 *If the linear and angular momenta balance hold, then there exists a matrix-valued function $\mathbf{\Sigma}$*
493 *from $\Omega_t$ to $\mathbb{S}$ such that $\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) = \mathbf{\Sigma}(\boldsymbol{y})\boldsymbol{n}$, $\forall \boldsymbol{y} \in \Omega_t$ where the right-hand side is the matrix-*
494 *vector multiplication.*

The set $\mathbb{S} = \mathbb{R}^{d \times d}_{\text{sym}}$ denotes the field of symmetric matrices in $\mathbb{R}^{d \times d}$. The symmetry of the stress tensor $\mathbf{\Sigma}$ is due to the balance of angular momentum. The divergence theorem can then be applied

$$\int_{\partial \omega_t} \mathbf{\Sigma}\, \boldsymbol{n}\, \mathrm{d}S = \int_{\omega_t} \nabla_y \cdot \mathbf{\Sigma}\, \mathrm{d}\omega_t,$$

where $\nabla_y \cdot$ is the tensor divergence with respect to the deformed configuration, $\nabla_y \cdot \mathbf{\Sigma} = \sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial y_i}$. Because the considered subregion $\omega_t$ is arbitrary, using the linear balance momentum and the conservation of mass, the following PDE is found

$$\rho \frac{D\boldsymbol{v}}{Dt} - \nabla_y \cdot \mathbf{\Sigma} = \boldsymbol{f}, \qquad \boldsymbol{y} \in \Omega_t.$$

495 This equation is written with respect to the deformed configuration $\Omega_t$. For a detailed deriva-
496 tion of this equation the reader may consult [Abe12, Chapter 4]. To obtain a closed formu-
497 lation, the constitutive law, namely the link between $\mathbf{\Sigma}$ and the strain tensor $\frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I})$,
498 has to be introduced. In the next section such relation will be discussed for the case of linear
499 elasticity.

500 ### 4.1.2 The linear elastodynamics problem

Whenever deformations are small, $\|\nabla_x \boldsymbol{u}\| \ll 1$, then the reference and deformed configurations are almost indistinguishable $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u} = \boldsymbol{x} + O(\nabla_x \boldsymbol{u}) \approx \boldsymbol{x}$. This allows writing the linear momentum balance in the reference configuration

$$\rho \frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x}, \boldsymbol{t}) - \mathrm{Div}\, \mathbf{\Sigma}(\boldsymbol{x}, t) = \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega.$$

The material derivative simplifies to a partial one. The operator Div is the divergence of a tensor field with respect to the reference configuration (see Appendix A for a description of the differential operators)

$$\mathrm{Div}\, \mathbf{\Sigma}(\boldsymbol{x}, t) = \nabla_x \cdot \mathbf{\Sigma}(\boldsymbol{x}, t) = \left( \sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial x_i} \right)_{1 \leq j \leq d}.$$

Furthermore, the non-linear terms in the Green-St. Venant strain tensor can be dropped

$$\frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\left[ \nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top + (\nabla_x \boldsymbol{u})^\top (\nabla_x \boldsymbol{u}) \right] \approx \frac{1}{2}\left[ \nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top \right].$$

The linearized strain tensor (also called infinitesimal strain tensor) is the symmetric gradient of the displacement

$$\boldsymbol{\varepsilon} := \operatorname{Grad} \boldsymbol{u}, \qquad \text{where} \qquad \operatorname{Grad} \boldsymbol{u} = \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top\right]. \tag{4.1}$$

To obtain a closed system of equations, it is now necessary to characterize the relation between stress and strain. This relation is normally called *constitutive law*. In the following, the particular case of elastic materials is considered. These are able to resist distorting excitations and return to its original size and shape when these excitations are removed. For this class of materials, the stress tensor is solely determined by the deformed configuration at a given time (Hooke's law)

$$\boldsymbol{\Sigma}(\boldsymbol{x}) = \boldsymbol{\mathcal{D}}(\boldsymbol{x})\,\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})).$$

The *stiffness tensor* or *elasticity tensor* $\boldsymbol{\mathcal{D}} : \mathbb{S} \to \mathbb{S}$ is a rank 4 tensor that is symmetric positive definite and uniformly bounded above and below. Because of symmetry, its components satisfy

$$\mathcal{D}_{ijkl} = \mathcal{D}_{jikl} = \mathcal{D}_{klij}.$$

From the uniform boundedness of $\boldsymbol{\mathcal{D}}$, the map $\boldsymbol{\mathcal{D}} : L^2(\Omega, \mathbb{S}) \to L^2(\Omega, \mathbb{S})$ is a symmetric positive definite bounded linear operator ($L^2(\Omega, \mathbb{S})$ is the space of square integrable symmetric tensor-valued functions). The compliance tensor $\boldsymbol{\mathcal{C}}$ is defined by $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{D}}^{-1}$ . Thus $\boldsymbol{\mathcal{C}} : \mathbb{S} \to \mathbb{S}$ is as well symmetric positive definite and uniformly bounded above and below. An isotropic elastic medium has the same kinematic properties in any direction and at each point. If an elastic medium is isotropic, then the stiffness and compliance tensors assume the form

$$\boldsymbol{\mathcal{D}}(\cdot) = 2\mu(\cdot) + \lambda \operatorname{Tr}(\cdot)\boldsymbol{I}, \qquad \boldsymbol{\mathcal{C}}(\cdot) = \frac{1}{2\mu}\left[(\cdot) - \frac{\lambda}{2\mu + d\lambda} \operatorname{Tr}(\cdot)\boldsymbol{I}\right], \qquad d = \{2, 3\}, \tag{4.2}$$

where Tr is the trace operator and the positive scalar functions $\mu, \lambda$, defined on $\Omega$, are called the Lamé coefficients. In engineering applications it is easier to compute experimentally two other parameters: the Young modulus $E$ and Poisson's ratio $\nu$. Those are expressed in terms of the Lamé coefficients as

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \tag{4.3}$$

and conversely

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \qquad \mu = \frac{E}{2(1 + \nu)}. \tag{4.4}$$

The stiffness and compliant tensor are expressed as

$$\boldsymbol{\mathcal{D}}(\cdot) = \frac{E}{1 + \nu}\left[(\cdot) + \frac{\nu}{1 - 2\nu} \operatorname{Tr}(\cdot)\boldsymbol{I}\right], \tag{4.5}$$

$$\boldsymbol{\mathcal{C}}(\cdot) = \frac{1 + \nu}{E}\left[(\cdot) - \frac{\nu}{1 + \nu(d - 2)} \operatorname{Tr}(\cdot)\boldsymbol{I}\right]. \tag{4.6}$$

514 The linear elastodynamics problem is formulated through a vector-valued PDE

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{Div}(\boldsymbol{\mathcal{D}}\,\mathrm{Grad}\,\boldsymbol{u}) = \boldsymbol{f}. \tag{4.7}$$

515 The classical elastodynamics problem is expressed considering the displacement $\boldsymbol{u}$ as the
516 unknown. This PDE goes together with appropriate boundary conditions that will be specified
517 in 4.2.

## 4.2 Port-Hamiltonian formulation of linear elasticity

519 In this section a port-Hamiltonian formulation for elasticity is deduced from the classical elas-
520 todynamics problem. It must be highlighted that already in the seventies a purely hyperbolic
521 formulation for elasticity was detailed [HM78]. The missing point is the clear connection
522 with the theory of Hamiltonian PDEs. An Hamiltonian formulation can be found in [Gri15,
523 Chapter 16], but without any connection to the concept of Stokes-Dirac structure induced by
524 the underlying geometry.
525

### 4.2.1 Energy and co-energy variables

527 Consider an open connected set $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$. The displacement within a deformable
528 continuum is given by Eq. (4.7).

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{Div}(\boldsymbol{\mathcal{D}}\,\mathrm{Grad}\,\boldsymbol{u}) = 0, \qquad \boldsymbol{x} \in \Omega. \tag{4.8}$$

529 The contribution of the body force $\boldsymbol{f}$ has been removed for ease of presentation. To derive a
530 pH formulation, the total energy, that includes the kinetic and deformation energy, is needed

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho \, \|\partial_t \boldsymbol{u}\|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} \, \mathrm{d}\Omega. \tag{4.9}$$

The notation $\boldsymbol{A} : \boldsymbol{B} = \mathrm{Tr}(\boldsymbol{A}^\top \boldsymbol{B}) = \sum_{i,j} A_{ij} B_{ij}$ denotes the tensor contraction. Recall that
$\boldsymbol{\varepsilon} = \mathrm{Grad}\,\boldsymbol{u}$ and $\boldsymbol{\Sigma} = \boldsymbol{\mathcal{D}}\boldsymbol{\varepsilon}$. The energy variables are then the linear momentum and the
deformation field

$$\boldsymbol{\alpha}_v = \rho \boldsymbol{v}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon},$$

531 where $\boldsymbol{v} := \partial_t \boldsymbol{u}$. The Hamiltonian can be rewritten as a quadratic functional in the energy
532 variables

$$H = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho} \boldsymbol{\alpha}_v^2 + (\boldsymbol{\mathcal{D}} \boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon \right\} \, \mathrm{d}\Omega. \tag{4.10}$$

533 The co-energy variables are given by

$$\boldsymbol{e}_v := \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \boldsymbol{v}, \qquad \boldsymbol{E}_\varepsilon := \frac{\delta H}{\delta \boldsymbol{A}_\varepsilon} = \boldsymbol{\Sigma}. \tag{4.11}$$

534     The tensor-valued co-energy $\boldsymbol{E}_\varepsilon$ is obtained by taking the variational derivative with
535 respect to a tensor.

536 **Proposition 3**
537 *The variational derivative of the Hamiltonian with respect to the strain tensor is the stress*
538 *tensor $\delta_{\boldsymbol{A}_\varepsilon} H = \boldsymbol{\Sigma}$.*

539 *Proof.* Let $\mathbb{S} : \mathbb{R}^{d \times d}_{\mathrm{sym}}$ be the space of symmetric tensor and $L^2(\Omega, \mathbb{S})$ the space of the square
540 integrable symmetric tensors endowed with the tensor contraction as inner product

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_{L^2(\Omega, \mathbb{S})} = \int_\Omega \boldsymbol{A} : \boldsymbol{B} \ \mathrm{d}\Omega. \tag{4.12}$$

The contribution due to the deformation part in Hamiltonian is given by:

$$H_{\mathrm{def}}(\boldsymbol{A}_\varepsilon) = \frac{1}{2} \int_\Omega (\boldsymbol{\mathcal{D}} \boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon \ \mathrm{d}\Omega.$$

A variation $\boldsymbol{\Delta A}_\varepsilon$ of the strain tensor with respect to a given value $\bar{\boldsymbol{A}}_\varepsilon$ leads to:

$$\begin{aligned} H_{\mathrm{def}}(\bar{\boldsymbol{A}}_\varepsilon + \eta \boldsymbol{\Delta A}_\varepsilon) = &+ \frac{1}{2} \int_\Omega (\boldsymbol{\mathcal{D}} \bar{\boldsymbol{A}}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon \ \mathrm{d}\Omega \\ &+ \eta \frac{1}{2} \int_\Omega \left\{ (\boldsymbol{\mathcal{D}} \bar{\boldsymbol{A}}_\varepsilon) : \boldsymbol{\Delta A}_\varepsilon + (\boldsymbol{\mathcal{D}} \boldsymbol{\Delta A}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon \right\} \ \mathrm{d}\Omega + O(\eta^2). \end{aligned}$$

The term $(\boldsymbol{\mathcal{D}} \boldsymbol{\Delta A}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon$ can be further rearranged using the symmetry of $\boldsymbol{\mathcal{D}}$ and the commutativity of the tensor contraction

$$(\boldsymbol{\mathcal{D}} \boldsymbol{\Delta A}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon = (\boldsymbol{\mathcal{D}} \bar{\boldsymbol{A}}_\varepsilon) : \boldsymbol{\Delta A}_\varepsilon,$$

so that

$$H_{\mathrm{def}}(\bar{\boldsymbol{A}}_\varepsilon + \eta \boldsymbol{\Delta A}_\varepsilon) = \frac{1}{2} \int_\Omega (\boldsymbol{\mathcal{D}} \bar{\boldsymbol{A}}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon \ \mathrm{d}\Omega + \eta \int_\Omega (\boldsymbol{\mathcal{D}} \bar{\boldsymbol{A}}_\varepsilon) : \boldsymbol{\Delta A}_\varepsilon \ \mathrm{d}\Omega + O(\eta^2).$$

By definition of variational derivative it can be written:

$$H_{\mathrm{def}}(\bar{\boldsymbol{A}}_\varepsilon + \eta \boldsymbol{\Delta A}_\varepsilon) = H_{\mathrm{def}}(\bar{\boldsymbol{A}}_\varepsilon) + \eta \left\langle \frac{\delta H}{\delta \boldsymbol{A}_\varepsilon}, \boldsymbol{\Delta A}_\varepsilon \right\rangle_{L^2(\Omega, \mathbb{S})} + O(\eta^2),$$

Then, by identification

$$\frac{\delta H_{\mathrm{def}}}{\delta \boldsymbol{A}_\varepsilon} = \boldsymbol{\mathcal{D}} \bar{\boldsymbol{A}}_\varepsilon = \boldsymbol{\Sigma}.$$

541 Since the Hamiltonian is separable then $\delta_{\boldsymbol{A}_\varepsilon} H_{\mathrm{def}} = \delta_{\boldsymbol{A}_\varepsilon} H$, leading to the final result.     $\square$

## 4.2.2  Final system and associated Stokes-Dirac structure

543 It is now possible to state the final pH form

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathrm{Div} \\ \mathrm{Grad} & \mathbf{0} \end{bmatrix}\begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \end{pmatrix}. \tag{4.13}$$

544 The first equation of the system is the conservation of linear momentum. The second repre-
545 sents a compatibility condition

$$\begin{aligned} \partial_t \boldsymbol{A}_\varepsilon &= \mathrm{Grad}(\boldsymbol{e}_v), \\ \partial_t \boldsymbol{\varepsilon} &= \mathrm{Grad}(\boldsymbol{v}), \\ \partial_t \, \mathrm{Grad}\, \boldsymbol{u} &= \mathrm{Grad}(\partial_t \boldsymbol{u}). \end{aligned} \tag{4.14}$$

546    Assuming that $\boldsymbol{u} \in C^2$, higher order derivatives commute (Schwarz theorem). Hence,
547 the equation is verified. The following theorem ensures the differential operator is formally
548 skew-adjoint (one can also find this result in the recent article [PZ20, Lemma 3.3], available
549 as arXiv preprint).

550 **Theorem 2**
551 *The formal adjoint of the tensor divergence* Div *is* $-$Grad, *the opposite of the symmetric*
552 *gradient.*

*Proof.* We denote by $\mathbb{V} = \mathbb{R}^d$ the space of vector field in $\mathbb{R}^d$ and by $\mathbb{S} = \mathbb{R}^{d \times d}$ hte space of symmetric tensor field in $\mathbb{R}^{d \times d}$. Let us consider the Hilbert space of the square integrable symmetric tensors $L^2(\Omega, \mathbb{S})$ with scalar product is defined in (4.12). Moreover consider the Hilbert space of the square integrable vector function $L^2(\Omega, \mathbb{V})$, endowed with the usual scalar product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{L^2(\Omega, \mathbb{V})} = \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b} \; \mathrm{d}\Omega.$$

Let us consider the tensor divergence operator defined as:

$$\mathrm{Div}: \; L^2(\Omega, \mathbb{S}) \to L^2(\Omega, \mathbb{V}), \qquad \text{with } \psi_j = \mathrm{div}(\Psi_{ij}) = \sum_{i=1}^d \frac{\partial \Psi_{ij}}{\partial x_i}.$$
$$\boldsymbol{\Psi} \to \mathrm{Div}\, \boldsymbol{\Psi} = \boldsymbol{\psi},$$

We try to identify $\mathrm{Div}^*$

$$\mathrm{Div}^*: \; L^2(\Omega, \mathbb{V}) \to L^2(\Omega, \mathbb{S}),$$
$$\boldsymbol{\phi} \to \mathrm{Div}^*\boldsymbol{\phi} = \boldsymbol{\Phi},$$

such that

$$\langle \mathrm{Div}\, \boldsymbol{\Psi},\, \boldsymbol{\phi} \rangle_{L^2(\Omega, \mathbb{V})} = \langle \boldsymbol{\Psi},\, \mathrm{Div}^*\boldsymbol{\phi} \rangle_{L^2(\Omega, \mathbb{S})}, \qquad \begin{aligned} &\forall \boldsymbol{\Psi} \in \mathrm{Dom}(\mathrm{Div}) \subset L^2(\Omega, \mathbb{S}), \\ &\forall \boldsymbol{\phi} \in \mathrm{Dom}(\mathrm{Div}^*) \subset L^2(\Omega, \mathbb{V}). \end{aligned}$$

Now let us take $\boldsymbol{\Psi} \in C_0^1(\Omega, \mathbb{S}) \subset \mathrm{Domain}(\mathrm{Div})$ the space of differentiable symmetric tensors

with compact support in $\Omega$. Additionally $\phi$ will belong to $C_0^1(\Omega, \mathbb{V}) \subset \text{Dom}(\text{Div}^*)$, the space of differentiable vector functions with compact support in $\Omega$. Then

$$\langle \text{Div}\, \boldsymbol{\Psi}, \phi \rangle_{L^2(\Omega, \mathbb{V})} = \int_\Omega \boldsymbol{\psi} \cdot \phi \ \text{d}\Omega,$$

$$= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \frac{\partial \Psi_{ij}}{\partial x_i} \phi_j \ \text{d}\Omega,$$

$$= -\int_\Omega \sum_{i=1}^d \sum_{j=1}^d \Psi_{ij} \frac{\partial \phi_j}{\partial x_i} \ \text{d}\Omega, \qquad \text{since the functions vanish at the boundary,}$$

$$= -\int_\Omega \sum_{i=1}^d \sum_{j=1}^d \Psi_{ij} F_{ij} \ \text{d}\Omega, \qquad \text{where} \qquad F_{ij} = \frac{\partial \phi_j}{\partial x_i},$$

$$= -\langle \boldsymbol{\Psi}, \boldsymbol{F} \rangle_{L^2(\Omega, \mathbb{S})}, \qquad\qquad\qquad\qquad \boldsymbol{F} = \text{grad}\, \phi.$$

But in this latter case, it could not be stated that $\boldsymbol{F} \in L^2(\Omega, \mathbb{S})$. Now, since $\boldsymbol{\Psi} \in L^2(\Omega, \mathbb{S})$, $\Psi_{ji} = \Psi_{ij}$, thus the last equality can be further decomposed as

$$\sum_{i,j} \Psi_{ij} \frac{\partial \phi_j}{\partial x_i} = \sum_{i,j} \Psi_{ij} \frac{1}{2}\left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right) = \sum_{i,j} \Psi_{ij} \Phi_{ij}, \qquad \text{with } \Phi_{ij} := \frac{1}{2}\left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right).$$

Thus $\boldsymbol{\Phi} = \text{Grad}\, \phi \in L^2(\Omega, \mathbb{S})$ and it can be stated that:

$$\langle \text{Div}\, \boldsymbol{\Psi},\, \phi \rangle_{L^2(\Omega, \mathbb{V})} = -\int_\Omega \sum_{i,j} \Psi_{ij} \frac{1}{2}\left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right) \ \text{d}\Omega$$

$$= -\int_\Omega \sum_{i,j} \Psi_{ij} \Phi_{ij} \ \text{d}\Omega = \langle \boldsymbol{\Psi},\, -\text{Grad}\, \phi \rangle_{L^2(\Omega, \mathbb{S})}.$$

It can be concluded that the formal adjoint of Div is $\text{Div}^* = -\text{Grad}$. $\qquad\qquad\qquad\square$

The boundary values are then found by evaluating the energy rate

$$\dot{H} = \int_\Omega \{ \boldsymbol{e}_v \cdot \partial_t \boldsymbol{\alpha}_v + \boldsymbol{E}_\varepsilon : \partial_t \boldsymbol{A}_\varepsilon \} \ \text{d}\Omega,$$

$$= \int_\Omega \{ \boldsymbol{e}_v \cdot \text{Div}\, \boldsymbol{E}_\varepsilon + \boldsymbol{E}_\varepsilon : \text{Grad}\, \boldsymbol{e}_v \} \ \text{d}\Omega,$$

$$= \int_\Omega \text{div}(\boldsymbol{E}_\varepsilon\, \boldsymbol{e}_v) \ \text{d}\Omega, \qquad\qquad \text{Stokes theorem (see Appendix A Eq. (A.6)),}$$

$$= \int_{\partial\Omega} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \boldsymbol{n}) \ \text{d}S = \langle \boldsymbol{e}_v,\, \boldsymbol{E}_\varepsilon\, \boldsymbol{n} \rangle_{L^2(\partial\Omega, \mathbb{R}^d)}.$$

$$\tag{4.15}$$

The imposition of the velocity field along the boundary $\boldsymbol{e}_v = \partial_t \boldsymbol{u}$ corresponds to a Dirichlet condition. Setting $\boldsymbol{E}_\varepsilon\, \boldsymbol{n} = \boldsymbol{\Sigma}\, \boldsymbol{n}$ (the traction) corresponds to a Neumann condition. Consider
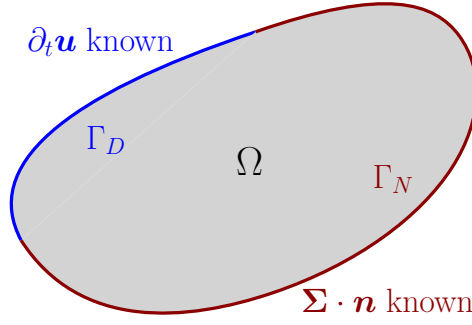
Figure 4.1: A 2D continuum with Neumann and Dirichlet boundary conditions

a partition of the boundary $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$ and $\Gamma_N \cap \Gamma_D = \{\emptyset\}$, where a Dirichlet and a Neumann condition applies on the open subset $\Gamma_D$ and $\Gamma_N$ respectively (see Fig. 4.1). Then the final pH formulation reads

$$
\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon\end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathrm{Div} \\ \mathrm{Grad} & \mathbf{0}\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}\boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon\end{pmatrix},
$$

$$
\boldsymbol{u}_\partial = \underbrace{\begin{bmatrix} \boldsymbol{\gamma}_0^{\Gamma_D} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}_n^{\Gamma_N}\end{bmatrix}}_{\mathcal{B}_\partial}\begin{pmatrix}\boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon\end{pmatrix}, \tag{4.16}
$$

$$
\boldsymbol{y}_\partial = \underbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{\gamma}_n^{\Gamma_D} \\ \boldsymbol{\gamma}_0^{\Gamma_N} & \mathbf{0}\end{bmatrix}}_{\mathcal{C}_\partial}\begin{pmatrix}\boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon\end{pmatrix},
$$

where $\boldsymbol{\gamma}_0^{\Gamma_*}$ denotes the trace over the set $\Gamma_*$, namely $\boldsymbol{\gamma}_0^{\Gamma_*}\boldsymbol{e}_v = \boldsymbol{e}_v|_{\Gamma_*}$. Furthermore, $\boldsymbol{\gamma}_n^{\Gamma_*}$ denotes the normal trace over the set $\Gamma_*$, namely $\boldsymbol{\gamma}_n^{\Gamma_*}\boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon\boldsymbol{n}|_{\Gamma_*}$.

**Conjecture 1** (Stokes-Dirac structure for elastodynamics)
*Let $H^{\mathrm{Grad}}(\Omega, \mathbb{V})$ the space of vectors with symmetric gradient in $L^2(\Omega, \mathbb{S})$ and $H^{\mathrm{Div}}(\Omega, \mathbb{S})$ denote the space of symmetric tensors with divergence in $L^2(\Omega, \mathbb{V})$. Consider the following definitions*

$$
\begin{aligned}
H &:= H^{\mathrm{Grad}}(\Omega, \mathbb{V}) \times H^{\mathrm{Div}}(\Omega, \mathbb{S}), \\
F &:= L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{S}), \\
F_\partial &:= L^2(\Gamma_D, \mathbb{V}) \times L^2(\Gamma_N, \mathbb{V}).
\end{aligned}
$$

*The set*

$$
D_{\mathcal{J}} = \left\{ \begin{pmatrix}\boldsymbol{f} \\ \boldsymbol{f}_\partial \\ \boldsymbol{e} \\ \boldsymbol{e}_\partial\end{pmatrix} \;\middle|\; \boldsymbol{e} \in H,\; \boldsymbol{f} = \mathcal{J}\boldsymbol{e},\; \boldsymbol{f}_\partial = \mathcal{B}_\partial\boldsymbol{e},\; \boldsymbol{e}_\partial = -\mathcal{C}_\partial\boldsymbol{e} \right\}, \tag{4.17}
$$

where $\boldsymbol{e} = (\boldsymbol{e}_v, \boldsymbol{E}_\varepsilon)$ and $\mathcal{J}, \mathcal{B}_\partial, \mathcal{C}_\partial$ are defined in (4.16), is a Stokes–Dirac structure with respect to the pairing

$$\left\langle\!\left\langle (\boldsymbol{f}^1, \boldsymbol{f}_\partial^1, \boldsymbol{e}^1, \boldsymbol{e}_\partial^1), (\boldsymbol{f}^2, \boldsymbol{f}_\partial^2, \boldsymbol{e}^2, \boldsymbol{e}_\partial^2) \right\rangle\!\right\rangle := \left\langle \boldsymbol{e}^1, \boldsymbol{f}^2 \right\rangle_F + \left\langle \boldsymbol{e}^2, \boldsymbol{f}^1 \right\rangle_F + \left\langle \boldsymbol{e}_\partial^1, \boldsymbol{f}_\partial^2 \right\rangle_{F_\partial} + \left\langle \boldsymbol{e}_\partial^2, \boldsymbol{f}_\partial^1 \right\rangle_{F_\partial},$$

(4.18)

where

$$\left\langle (\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{d}) \right\rangle_{F_\partial} = \int_{\Gamma_D} \boldsymbol{a} \cdot \boldsymbol{c} \, \mathrm{d}S + \int_{\Gamma_N} \boldsymbol{b} \cdot \boldsymbol{d} \, \mathrm{d}S, \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{V}.$$

**Crucial points to obtain a rigorous proof**    *The crucial point that needs to be elucidated is where the boundary variables live. These variables belong to the fractional Sobolev spaces $H^{\frac{1}{2}}(\partial\Omega, \mathbb{V})$, $H^{-\frac{1}{2}}(\partial\Omega, \mathbb{V})$ linked by duality with respect to the pivot space $L^2(\partial\Omega, \mathbb{V})$. This is why a $L^2$ inner product has been assumed as boundary inner product.*

*Furthermore, the partition of the boundary due to the non uniform boundary control complicates the proof, since one has to properly connect the two partitions at their interconnection.*

**Elements to support the conjecture**    *A Stokes-Dirac is characterized by the fact that $D_\mathcal{J} = D_\mathcal{J}^\perp$. Then one has to show that $D_\mathcal{J} \subset D_\mathcal{J}^\perp$ and $D_\mathcal{J}^\perp \subset D_\mathcal{J}$. The main steps of Theorem 3.6 in [LGZM05] are followed here to support the substantiation of the conjecture. The integration by parts formula is applied as in (4.15).*

*Step 1. To show that $D_\mathcal{J} \subset D_\mathcal{J}^\perp$, take $(\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial) \in D_\mathcal{J}$. Then*

$$\begin{aligned}
\left\langle\!\left\langle (\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial), (\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial) \right\rangle\!\right\rangle &= 2 \left\langle \boldsymbol{e}, \boldsymbol{f} \right\rangle_F + 2 \left\langle \boldsymbol{e}_\partial, \boldsymbol{f}_\partial \right\rangle_{F_\partial}, \\
&= 2 \left\langle \boldsymbol{e}, \mathcal{J}\boldsymbol{e} \right\rangle_F + 2 \left\langle \boldsymbol{e}_\partial, \boldsymbol{f}_\partial \right\rangle_{F_\partial}, \\
&= + 2 \int_\Omega \left\{ \boldsymbol{e}_v \cdot \mathrm{Div}\, \boldsymbol{E}_\varepsilon + \boldsymbol{E}_\varepsilon : \mathrm{Grad}\, \boldsymbol{e}_v \right\} \, \mathrm{d}\Omega \\
&\quad - 2 \int_{\Gamma_D} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \, \boldsymbol{n}) \, \mathrm{d}S - 2 \int_{\Gamma_N} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \, \boldsymbol{n}) \, \mathrm{d}S, \\
&= + 2 \int_\Omega \left\{ \boldsymbol{e}_v \cdot \mathrm{Div}\, \boldsymbol{E}_\varepsilon + \boldsymbol{E}_\varepsilon : \mathrm{Grad}\, \boldsymbol{e}_v \right\} \, \mathrm{d}\Omega \\
&\quad - 2 \int_{\partial\Omega} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \, \boldsymbol{n}) \, \mathrm{d}S, = 0, \quad \textit{from (4.15)}.
\end{aligned}$$

*This implies $D_\mathcal{J} \subset D_\mathcal{J}^\perp$.*

*Step 2. Take $(\boldsymbol{\phi}, \boldsymbol{\phi}_\partial, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_\partial) \in D_\mathcal{J}^\perp$ and $\boldsymbol{e}_0 \in H$ with compact support on $\Omega$. This implies $\mathcal{B}_\partial \boldsymbol{e}_0 = (\boldsymbol{0}, \boldsymbol{0})$ and $\mathcal{C}_\partial \boldsymbol{e}_0 = (\boldsymbol{0}, \boldsymbol{0})$. Taking $(\mathcal{J}\boldsymbol{e}_0, \boldsymbol{0}, \boldsymbol{e}_0, \boldsymbol{0}) \in D_\mathcal{J}$ then*

$$\left\langle\!\left\langle (\boldsymbol{\phi}, \boldsymbol{\phi}_\partial, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_\partial), (\mathcal{J}\boldsymbol{e}_0, \boldsymbol{0}, \boldsymbol{e}_0, \boldsymbol{0}) \right\rangle\!\right\rangle = \left\langle \boldsymbol{\epsilon}, \mathcal{J}\boldsymbol{e}_0 \right\rangle_F + \left\langle \boldsymbol{e}_0, \boldsymbol{\phi} \right\rangle_F = 0, \quad \forall \boldsymbol{e}_0 \in H.$$

*It follows that $\boldsymbol{\epsilon} \in H$ and $\boldsymbol{\phi} = \mathcal{J}\boldsymbol{\epsilon}$.*

*Step 3. Take $(\boldsymbol{\phi}, \boldsymbol{\phi}_\partial, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_\partial) \in D_{\mathcal{J}}^\perp$ and $(\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial) \in D_{\mathcal{J}}$. Variables $\boldsymbol{e}, \boldsymbol{\epsilon}$ are indeed tuples containing a vector and a tensor, namely $\boldsymbol{e} = (\boldsymbol{e}_v, \boldsymbol{E}_\varepsilon)$, $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_v, \boldsymbol{\mathcal{E}}_\varepsilon)$. From step 2 and* (4.18)

$$0 = \langle \boldsymbol{e}, \mathcal{J}\boldsymbol{\epsilon} \rangle_F + \langle \mathcal{J}\boldsymbol{e}, \boldsymbol{\epsilon} \rangle_F + \langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial \rangle_{F_\partial} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial \rangle_{F_\partial},$$

$$= \int_{\partial\Omega} \{ \boldsymbol{e}_v \cdot (\boldsymbol{\mathcal{E}}_\varepsilon\, \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot (\boldsymbol{E}_\varepsilon\, \boldsymbol{n}) \}\; \mathrm{d}S + \langle -\mathcal{C}_\partial \boldsymbol{e}, \boldsymbol{\phi}_\partial \rangle_{F_\partial} + \langle \boldsymbol{\epsilon}_\partial, \mathcal{B}_\partial \boldsymbol{e} \rangle_{F_\partial}$$

*Consider the splitting of the boundary $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$*

$$\int_{\partial\Omega} \{ \boldsymbol{e}_v \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n}) \}\; \mathrm{d}S = + \int_{\Gamma_N} \{ \boldsymbol{e}_{\partial,2} \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot \boldsymbol{f}_{\partial,2} \}\; \mathrm{d}S,$$

$$+ \int_{\Gamma_D} \{ \boldsymbol{f}_{\partial,1} \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot \boldsymbol{e}_{\partial,1} \}\; \mathrm{d}S,$$

*where the elements of the vectors $\boldsymbol{f}_\partial = (\boldsymbol{f}_{\partial,1}, \boldsymbol{f}_{\partial,2})$, $\boldsymbol{e}_\partial = (\boldsymbol{e}_{\partial,1}, \boldsymbol{e}_{\partial,2})$ have been considered. By expanding of the terms $\langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial \rangle_{F_\partial} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial \rangle_{F_\partial}$ and given the fact that $\boldsymbol{e}$ is arbitrary then*

$$\boldsymbol{\phi}_\partial = \begin{bmatrix} \gamma_0^{\Gamma_D} & \mathbf{0} \\ \mathbf{0} & \gamma_n^{\Gamma_N} \end{bmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_v \\ \boldsymbol{\mathcal{E}}_\varepsilon \end{pmatrix}, \qquad \boldsymbol{\epsilon}_\partial = - \begin{bmatrix} \mathbf{0} & \gamma_n^{\Gamma_D} \\ \gamma_0^{\Gamma_N} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_v \\ \boldsymbol{\mathcal{E}}_\varepsilon \end{pmatrix},$$

*meaning that $D_{\mathcal{J}}^\perp \subset D_{\mathcal{J}}$.*

Linear elasticity falls within the assumption of [Skr19]. Therefore, it is a well posed boundary control pH system. A question that naturally arises is how to reformulate this system using the language of differential geometry. This is possible through the usage of vector-valued differential forms. The interested reader may consult [Bre08].

## 4.3 Conclusion

In this chapter, the pH formulation of elasticity have been obtained. This model represents a generalization of the wave equation to higher dimensional variables. This leads to the introduction of symmetric tensorial quantities describing the state of stress and deformation within the body.

For a plane continuum with moderate thickness, it is possible to reduce the general three-dimensional mode to two uncoupled systems: one representing the in plane behavior ruled by 2D elasticity and one representing the out-of-plane deflection. This will be the object of the next chapter dedicated to the study of a pH formulation of plate bending. It is important to remember that plate models are just particular cases of three-dimensional elasticity.

# Port-Hamiltonian plate theory

You get tragedy where the tree, instead of bending, breaks.

*Culture and Value*
*Ludwig Wittgenstein*

## Contents

Lates are plane structural elements with a small thickness compared to the planar dimension. Thanks to this feature, it is not necessary to model plate structures using three-dimensional elasticity. Dimensional reduction strategies are employed to describe plate structures as two-dimensional problems. These strategies rely on an educated guess of the displacement field. For beams and plates this field is expressed in terms of unknown functions $\phi_i^j(x, y, t)$ that solely depends on the midplane coordinates $(x, y)$

$$u_i(x, y, z, t) = \sum_{j=0}^{m} (z)^j \phi_i^j(x, y, t).$$

where $u_i$, $i = \{x, y, z\}$ are the components of the displacement field. A first-order approximation is commonly used, meaning that a linear dependence on $z$ is considered. Two main models arise from such a framework:

- the Mindlin-Reissner model for thick plates;

35

622          • the Kirchhoff-Love model for thin plates.

623      In this chapter it its shown how to formulate first-order plate models as pHs.

## 624  5.1   First order plate theory

As previously stated, first order theories assume a linear dependence on the vertical coordinate (cf. [Red06])

$$u_i(x, y, z, t) = \phi_i^0(x, y, t) + z\phi_i^1(x, y, t).$$

625  This hypothesis implies that the fibers, i.e. segments perpendicular to the mid-plane before
626  deformation, remain straight after deformation. Additionally, for plate with moderate thick-
627  ness the fibers are considered inextensible, meaning that $\phi_z^1 = 0$. These assumptions lead to
628  the following displacement field

$$
\begin{aligned}
u_x(x, y, z, t) &= u_x^0(x, y, t) - z\theta_x(x, y, t), \\
u_y(x, y, z, t) &= u_y^0(x, y, t) - z\theta_y(x, y, t), \\
u_z(x, y, z, t) &= u_z^0(x, y, t),
\end{aligned}
\tag{5.1}
$$

where $u_i(x, y, t) = \phi_i^0(x, y, t)$, $\theta_i(x, y, t) = -\phi_i^1(x, y, t)$. Assuming a linear elastic behavior, the 3D strain tensor for such a displacement field takes the form

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right) - z\frac{1}{2}\left(\partial_\beta \theta_\alpha + \partial_\alpha \theta_\beta\right) = \varepsilon_{\alpha\beta}^0 - z\kappa_{\alpha\beta}, \tag{5.2}$$

$$\varepsilon_{\alpha z} = \frac{1}{2}\left(\partial_a u_z - \theta_\alpha\right) = \frac{1}{2}\gamma_\alpha, \tag{5.3}$$

where $\alpha = \{x, y\}$, $\beta = \{x, y\}$. The tensors $\boldsymbol{\varepsilon}^0$, $\boldsymbol{\kappa}$, $\boldsymbol{\gamma}$ are called membrane, bending (or curvature) and shear strain tensor

$$\boldsymbol{\varepsilon}^0 = \operatorname{Grad} \boldsymbol{u}^0, \tag{5.4}$$

$$\boldsymbol{\kappa} = \operatorname{Grad} \boldsymbol{\theta}, \tag{5.5}$$

$$\boldsymbol{\gamma} = \operatorname{grad} u_z - \boldsymbol{\theta}. \tag{5.6}$$

where $\boldsymbol{u}^0 = (u_x, u_y)^\top$, $\boldsymbol{\theta} = (\theta_x, \theta_y)^\top$. For now, it is assumed that the material is isotropic, linear elastic (in Section §5.3 this hypothesis is removed). Recall the Hooke's law for 3D continua (see Eq. (4.5))

$$\boldsymbol{\Sigma} = \frac{E}{1 + \nu}\left[\boldsymbol{\varepsilon} + \frac{\nu}{1 - 2\nu}\operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I}_{3\times 3}\right].$$

where $E$, $\nu$ are the Young modulus and Poisson ratio. The hypothesis of inextensible fibers implies $\varepsilon_{zz} = 0$. However, imposing a plane strain condition provides a model that is too stiff. Rather than a plain strain assumption, a plain stress hypothesis is used to derive the constitutive law for plates. The displacement field (5.1) is left unchanged, but, instead of $\varepsilon_{zz}$,

$\Sigma_{zz}$ is set to zero. If $\Sigma_{zz} = 0$, one gets

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).$$

Consequently, it is computed

$$\text{Tr}(\varepsilon) = \frac{1-2\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).$$

The constitutive law for the in-plane stress takes the form

$$\mathbf{\Sigma}_{2D} = \mathbf{\mathcal{D}}_{2D}\,\boldsymbol{\varepsilon}_{2D},$$

where $\mathbf{\Sigma}_{2D} = \Sigma_{\alpha\beta}$, $\boldsymbol{\varepsilon}_{2D} = \varepsilon_{\alpha\beta}$ and

$$\mathbf{\mathcal{D}}_{2D} = \frac{E}{1-\nu^2}\left[(1-\nu)(\cdot) + \nu\,\text{Tr}(\cdot)\mathbf{I}_{2\times2}\right]. \tag{5.7}$$

Concerning the shear deformation, the constitutive law reduces to

$$\boldsymbol{\sigma}_s = G\boldsymbol{\gamma}, \tag{5.8}$$

where $\boldsymbol{\sigma}_s := \mathbf{\Sigma}_{\alpha,3}$ and $G = \frac{E}{2(1+\nu)}$ is the shear modulus. In the following sections, the most common plate models will be presented.

### 5.1.1  Mindlin-Reissner model

The Mindlin-Reissner model [Rei47, Min51] represents a first-order shear deformation theory for describing the bending of plate. The in-plane midplane displacement are zero $\boldsymbol{u}^0(x,y) = \mathbf{0}$ for an isotropic plate that experiences only bending. Hence, the displacement field reduces to

$$
\begin{aligned}
u_x(x,y,z) &= -z\partial_x\theta_x, \\
u_y(x,y,z) &= -z\partial_y\theta_y, \\
u_z(x,y,z) &= u_z^0(x,y).
\end{aligned}
\tag{5.9}
$$

In pure bending, the strain tensor is given by

$$\boldsymbol{\varepsilon}_b := \boldsymbol{\varepsilon}_{2D}(\boldsymbol{u}^0 = \mathbf{0}) = -z\boldsymbol{\kappa},$$

with $\boldsymbol{\kappa}$ given by (5.5). Consequently, the stress tensor reads

$$\mathbf{\Sigma}_b := \mathbf{\Sigma}_{2D}(\boldsymbol{u}^0 = \mathbf{0}) = -z\mathbf{\mathcal{D}}_{2D}\boldsymbol{\kappa},$$

where $\mathbf{\mathcal{D}}_{2D}$ is defined in Eq. (5.7).

The undeformed middle plane of the plate is denoted by $\Omega$. The total domain of the

plate is the product $\Omega \times (-h/2, h/2)$, where $h$ is the constant thickness. To effectively reduce the problem from three- to two-dimensional, the stresses have to be integrated along the fibers. Since the stress varies linearly across the thickness, the stress has to be multiplied by $z$ before the integration to get a non null contribution. The resulting quantity is called bending momenta tensor and is given by

$$\boldsymbol{M} := -\int_{-h/2}^{h/2} z\boldsymbol{\Sigma}_b \, \mathrm{d}z = \boldsymbol{\mathcal{D}}_b \, \boldsymbol{\kappa}, \tag{5.10}$$

where

$$\boldsymbol{\mathcal{D}}_b = D_b \left[(1-\nu)(\cdot) + \nu \operatorname{Tr}(\cdot)\boldsymbol{I}_{2\times 2}\right], \qquad \text{where} \qquad D_b = \frac{Eh^3}{12(1-\nu^2)}. \tag{5.11}$$

The shear stress has to be integrated along the fibers as well. Given the excessive rigidity of the shear contribution, a correction factor $k = 5/6$ [Red06, Chapter 10] is introduced

$$\boldsymbol{q} = \int_{-h/2}^{h/2} k\boldsymbol{\sigma_s} = kGh\boldsymbol{\gamma}, \tag{5.12}$$

where $\boldsymbol{\gamma}$ is defined in Eq. (5.6). The equations of motion can be obtained using Hamilton's principle. It consists in minimizing the total Lagrangian, given by $L = E_{\mathrm{def}} - E_{\mathrm{kin}}$, where $E_{\mathrm{def}}, E_{\mathrm{kin}}$ are the deformation and kinetic energy

$$E_{\mathrm{def}} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}\Omega \mathrm{d}z = \frac{1}{2}\int_\Omega \left\{\boldsymbol{M} : \boldsymbol{\kappa} + \boldsymbol{q}\cdot\boldsymbol{\gamma}\right\} \, \mathrm{d}\Omega, \tag{5.13}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \rho \, \|\partial_t \boldsymbol{u}\|^2 \, \mathrm{d}\Omega \mathrm{d}z = \frac{1}{2}\int_\Omega \left\{\frac{\rho h^3}{12}\|\partial_t \boldsymbol{\theta}\|^2 + \rho h(\partial_t u_z)^2\right\} \, \mathrm{d}\Omega, \tag{5.14}$$

where $\rho$ is the mass density. The Hamilton principle states that

$$\int_0^T \delta L \, \mathrm{d}t = \int_0^T \left\{\delta E_{\mathrm{def}} - \delta E_{\mathrm{kin}}\right\} \, \mathrm{d}t = 0.$$

The final result is the following system of PDEs (for the detailed computations see [Red06, Chapter 10])

$$\begin{aligned}
\rho h \frac{\partial^2 u_z}{\partial t^2} &= \operatorname{div}\boldsymbol{q}, \qquad (x,y) \in \Omega, \\
\frac{\rho h^3}{12}\frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} &= \operatorname{Div}\boldsymbol{M} + \boldsymbol{q},
\end{aligned} \tag{5.15}$$

with $\boldsymbol{M} = \boldsymbol{\mathcal{D}}_b \operatorname{Grad}\boldsymbol{\theta}$ and $\boldsymbol{q} = kGh\left(\operatorname{grad} u_z - \boldsymbol{\theta}\right)$. This PDE goes together with specified boundary conditions. Those will be detailed in 5.2.1.

## 5.1.2   Kirchhoff-Love model

The Kirchhoff model was formulated around 1850 and it is referred to as classical plate theory. The hypotheses on the displacement field consist of the following three points (see Fig. 5.1):
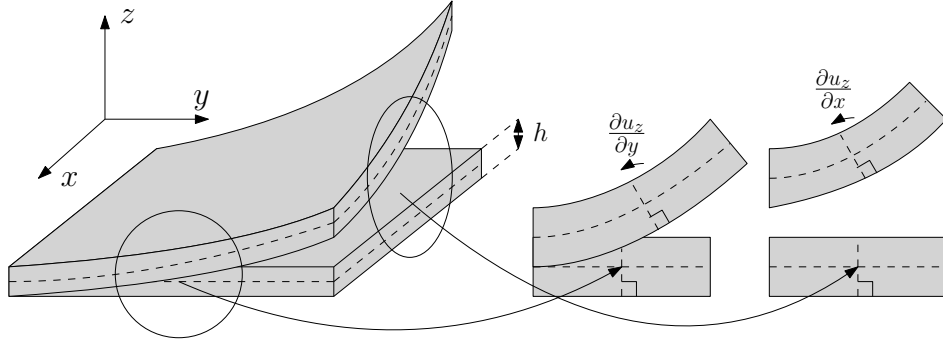
Figure 5.1: Kinematic assumption for the Kirchhoff plate

1. The fibers, segments perpendicular to the mid-plane before deformation, remain straight after deformation.

2. The fibers are inextensible.

3. While rotating, fibers remain perpendicular to the middle surface after deformation.

While the first two points are valid also for the Mindlin plate, the third assumption is specific to the Kirchhoff-Love model. Such an approximation is valid for plates having span-to-thickness ratio of the order of $L/h \approx 100-1000$ and implies zero transverse shear deformation

$$\boldsymbol{\gamma} = 0 \implies \varepsilon_{xz} = -\theta_x + \frac{\partial u_z}{\partial x} = 0, \qquad \varepsilon_{yz} = -\theta_y + \frac{\partial u_z}{\partial y} = 0.$$

The rotation vector is then related to the vertical displacement $\boldsymbol{\theta} = \operatorname{grad} u_z$. Plugging this into (5.5), it is found

$$\boldsymbol{\kappa} = \operatorname{Grad} \operatorname{grad} u_z = \operatorname{Hess} u_z. \tag{5.16}$$

Since the focus is on bending behavior, the in-plane displacement of the mid-plane are assumed to be zero $\boldsymbol{u}^0(x,y) = \boldsymbol{0}$. Hence, the displacement field assumes the form

$$
\begin{aligned}
u_x(x,y,z) &= -z\partial_x u_z, \\
u_y(x,y,z) &= -z\partial_y u_z, \\
u_z(x,y,z) &= u_z^0(x,y).
\end{aligned}
\tag{5.17}
$$

For the Kirchhoff plate, the same link between the momenta and bending tensor holds

$$\boldsymbol{M} = \boldsymbol{\mathcal{D}}_b \boldsymbol{\kappa},$$

where $\boldsymbol{\mathcal{D}}_b$ and $\boldsymbol{\kappa}$ are given in (5.11), (5.16) respectively. The equations of motion can be obtained using Hamilton's principle [Red06, Chapter 2]. The deformation energy, kinetic

energy and external work read

$$E_{\mathrm{def}} = \frac{1}{2} \int_\Omega \int_{-h/2}^{h/2} \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \,\mathrm{d}\Omega\mathrm{d}z = \frac{1}{2} \int_\Omega \{\boldsymbol{M} : \boldsymbol{\kappa}\} \,\mathrm{d}\Omega, \tag{5.18}$$

$$E_{\mathrm{kin}} = \frac{1}{2} \int_\Omega \int_{-h/2}^{h/2} \rho \,\|\partial_t \boldsymbol{u}\|^2 \,\mathrm{d}\Omega\mathrm{d}z \approx \frac{1}{2} \int_\Omega \rho h (\partial_t u_z)^2 \,\mathrm{d}\Omega. \tag{5.19}$$

**Remark 4** (Rotational energy)
*For the kinetic energy the rotational contribution*

$$E_{rot} = \frac{1}{2} \int_\Omega \int_{-h/2}^{h/2} \left\{ \rho \,(\partial_t u_x)^2 + (\partial_t u_y)^2 \right\} \,\mathrm{d}\Omega\mathrm{d}z = \frac{h^3}{24} \int_\Omega \rho \left\{ (\partial_{tx} u_z)^2 + (\partial_{ty} u_z)^2 \right\} \,\mathrm{d}\Omega = O(h^3),$$

*is neglected given the small thickness assumption.*

The final result from the Hamilton's principle is the following PDE (for the detailed computations the reader may consult [Red06, Chapter 3])

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = -\operatorname{div}\operatorname{Div}(\boldsymbol{\mathcal{D}}_b \operatorname{Grad}\operatorname{grad} u_z), \qquad (x,y) \in \Omega. \tag{5.20}$$

Developing the calculations, one obtains

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = -D_b \Delta^2 u_z, \qquad (x,y) \in \Omega,$$

where $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4}$ is the bi-Laplacian. Appropriate boundary conditions for this problem will be detailed in 5.2.2.

## 5.2   Port-Hamiltonian formulation of isotropic plates

In this section the pH formulation of the isotropic Mindlin and Kirchhoff plate models is detailed. In [MMB05], the Mindlin plate model was put in pH form by appropriate selection of the energy variables. However, the final system does not consider the nature of the different variables that come into play, leading to a non intrinsic final formulation. Additionally, this model was presented using the jet bundle formalism in [SS17]. The Kirchhoff model was never explored in the pH framework and represents an original contribution of this thesis. The interested reader can find in [RZ18] a rigorous mathematical treatment of the biharmonic problem and its decomposition in 2D geometries, but only for the static case (the 3D case, that does not relate to plate bending, is treated in [DZ18]).

### 5.2.1   Port-Hamiltonian Mindlin plate

Let $w := u_z$ denote the vertical displacement of the plate. Consider a bounded, connected domain $\Omega \subset \mathbb{R}^2$ and the Hamiltonian (total energy)

$$H = \frac{1}{2} \int_\Omega \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + \frac{\rho h^3}{12} \left\| \frac{\partial \boldsymbol{\theta}}{\partial t} \right\|^2 + \boldsymbol{M} : \boldsymbol{\kappa} + \boldsymbol{q} \cdot \boldsymbol{\gamma} \right\} \, \mathrm{d}\Omega, \tag{5.21}$$

where $\boldsymbol{M}$, $\boldsymbol{\kappa}$, $\boldsymbol{q}$, $\boldsymbol{\gamma}$ are defined in Eqs. (5.10), (5.5), (5.12), (5.6) respectively. The choice of the energy variables is the same as in [MMB05] but here scalar-, vector- and tensor-valued variables are gathered together:

$$\begin{aligned}
\alpha_w &= \rho h \frac{\partial w}{\partial t}, & \text{Linear momentum,} && \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t}, & \text{Angular momentum,} \\
\boldsymbol{A}_\kappa &= \boldsymbol{\kappa}, & \text{Curvature tensor,} && \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}. & \text{Shear deformation.}
\end{aligned} \tag{5.22}$$

The energy is now a quadratic function of the energy variables

$$H = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho h} \alpha_w^2 + \frac{12}{\rho h^3} \left\| \boldsymbol{\alpha}_\theta \right\|^2 + (\boldsymbol{\mathcal{D}}_b \boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa + (\boldsymbol{\mathcal{D}}_s \boldsymbol{\alpha}_\gamma) \cdot \boldsymbol{\alpha}_\gamma \right\} \, \mathrm{d}\Omega, \tag{5.23}$$

where $\boldsymbol{\mathcal{D}}_s := G h k \boldsymbol{I}_{2\times 2}$ and $G$ is the shear modulus $k$ the correction factor. The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned}
e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, & \text{Linear velocity,} && \boldsymbol{e}_\theta &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t}, & \text{Angular velocity,} \\
\boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, & \text{Momenta tensor,} && \boldsymbol{e}_\gamma &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\gamma} = \boldsymbol{q} & \text{Shear stress.}
\end{aligned} \tag{5.24}$$

**Proposition 4**

*The variational derivative of the Hamiltonian with respect to the curvature tensor is the momenta tensor $\frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}$.*

*Proof.* The proof is analogous to the one already detailed in Prop. 3                     □

Once the variables are concatenated together, the pH system is expressed as follows

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ \boldsymbol{0} & \boldsymbol{0} & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ \boldsymbol{0} & \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}. \tag{5.25}$$

The first two equations are equivalent to (5.15). The last two equations, like (4.14) for 3D elasticity, represent the fact the higher order derivatives commute. We shall now establish the total energy balance in terms of boundary variables as they will be part of the underlying
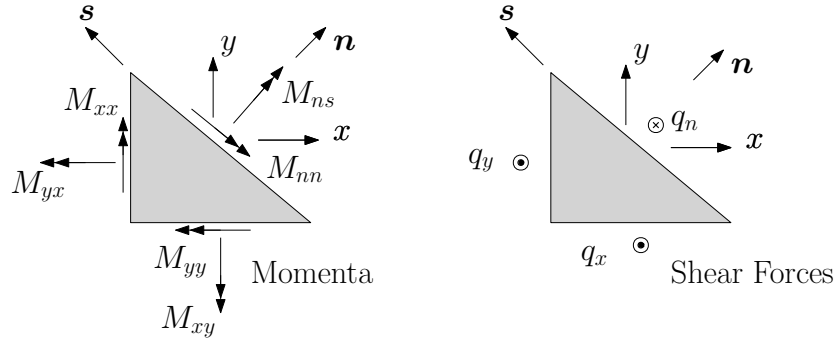
Figure 5.2: Cauchy law for momenta and forces at the boundary.
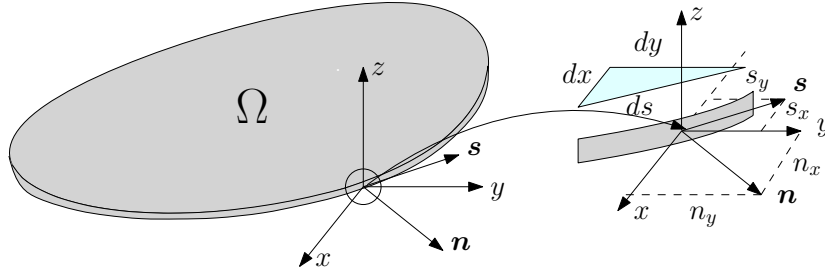


Figure 5.3: Reference frames and notations.

Stokes-Dirac structure of this model. The energy rate reads

$$
\begin{aligned}
\dot{H} &= \int_{\Omega} \left\{ \frac{\partial \alpha_w}{\partial t} e_w + \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \cdot \boldsymbol{e}_\theta + \frac{\partial \boldsymbol{A}_\kappa}{\partial t} : \boldsymbol{E}_\kappa + \frac{\partial \boldsymbol{\alpha}_\gamma}{\partial t} \cdot \boldsymbol{e}_\gamma \right\} \, \mathrm{d}\Omega \\
&= \int_{\Omega} \left\{ \operatorname{div}(\boldsymbol{e}_\gamma) e_w + \operatorname{Div}(\boldsymbol{E}_\kappa) \cdot \boldsymbol{e}_\theta + \operatorname{Grad}(\boldsymbol{e}_\theta) : \boldsymbol{E}_\kappa + \operatorname{grad}(e_w) \cdot \boldsymbol{e}_\gamma \right\} \, \mathrm{d}\Omega \qquad \text{Stokes theorem,} \\
&= \int_{\partial\Omega} \left\{ w_t\, q_n + \omega_n\, M_{nn} + \omega_s\, M_{ns} \right\} \, \mathrm{d}s,
\end{aligned}
$$

$$(5.26)$$

where $s$ is the curvilinear abscissa. The last integral is obtained by applying the Stokes theorem. The boundary variables appearing in the last line of (5.26) and illustrated in Fig. 5.2 are defined as follows:

$$
\begin{aligned}
\text{Shear force} \qquad & q_n := \boldsymbol{q} \cdot \boldsymbol{n} = \boldsymbol{e}_\gamma \cdot \boldsymbol{n}, \\
\text{Flexural momentum} \qquad & M_{nn} := \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}) = \boldsymbol{E}_\kappa : (\boldsymbol{n} \otimes \boldsymbol{n}), \\
\text{Torsional momentum} \qquad & M_{ns} := \boldsymbol{M} : (\boldsymbol{s} \otimes \boldsymbol{n}) = \boldsymbol{E}_\kappa : (\boldsymbol{s} \otimes \boldsymbol{n}),
\end{aligned}
$$

$$(5.27)$$

Vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ designate the normal and tangential unit vectors to the boundary, as shown in Fig. 5.3. Given two vectors $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^m$, the notation $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top \in \mathbb{R}^{n \times m}$ denotes the
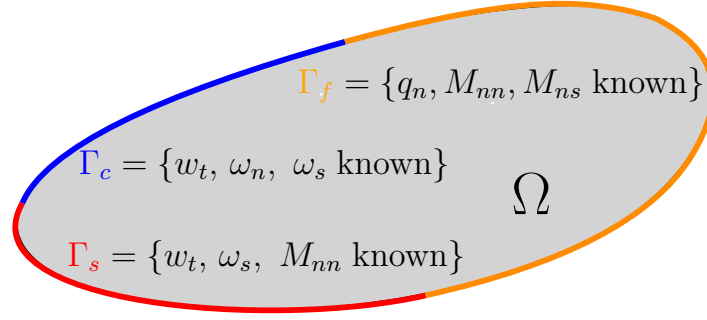
Figure 5.4: Boundary conditions for the Mindlin plate.

outer (or dyadic) product of two vectors. The corresponding power conjugated variables are

$$
\begin{aligned}
\text{Vertical velocity} \quad w_t &:= \frac{\partial w}{\partial t} = e_w, \\
\text{Flexural rotation} \quad \omega_n &:= \frac{\partial \boldsymbol{\theta}}{\partial t} \cdot \boldsymbol{n} = \boldsymbol{e}_\theta \cdot \boldsymbol{n}, \\
\text{Torsional rotation} \quad \omega_s &:= \frac{\partial \boldsymbol{\theta}}{\partial t} \cdot \boldsymbol{s} = \boldsymbol{e}_\theta \cdot \boldsymbol{s}.
\end{aligned}
\tag{5.28}
$$

Consider a partition of the boundary $\partial\Omega = \overline{\Gamma}_C \cup \overline{\Gamma}_S \cup \overline{\Gamma}_F$, $\Gamma_C \cap \Gamma_S \cap \Gamma_F = \{\emptyset\}$. The open subset $\Gamma_C$, $\Gamma_S$, $\Gamma_F$ could be empty. Given definitions (5.27), (5.28), the boundary conditions for the Mindlin plate [DHNLS99] (see Fig. 5.4) that are considered are:

- Clamped (C) on $\Gamma_C \subseteq \partial\Omega$ : $w_t$, $\omega_n$, $\omega_s$ known;

- Simply supported hard (S) on $\Gamma_S \subseteq \partial\Omega$: $w_t$, $\omega_s$, $M_{nn}$ known;

- Free (F) on $\Gamma_F \subseteq \partial\Omega$: $M_{nn}$, $M_{ns}$, $q_n$ known.

Then the final pH formulation reads

$$
\frac{\partial}{\partial t}
\begin{pmatrix}
\alpha_w \\
\boldsymbol{\alpha}_\theta \\
\boldsymbol{A}_\kappa \\
\boldsymbol{\alpha}_\gamma
\end{pmatrix}
=
\underbrace{
\begin{bmatrix}
0 & 0 & 0 & \mathrm{div} \\
\boldsymbol{0} & \boldsymbol{0} & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\
\boldsymbol{0} & \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} \\
\mathrm{grad} & -\boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0}
\end{bmatrix}
}_{\mathcal{J}}
\begin{pmatrix}
e_w \\
\boldsymbol{e}_\theta \\
\boldsymbol{E}_\kappa \\
\boldsymbol{e}_\gamma
\end{pmatrix},
$$

$$
\boldsymbol{u}_\partial =
\underbrace{
\begin{bmatrix}
\gamma_0^{\Gamma_C} & 0 & 0 & 0 \\
0 & \gamma_n^{\Gamma_C} & 0 & 0 \\
0 & \gamma_s^{\Gamma_C} & 0 & 0 \\
\gamma_0^{\Gamma_S} & 0 & 0 & 0 \\
0 & \gamma_s^{\Gamma_S} & 0 & 0 \\
0 & 0 & \gamma_{nn}^{\Gamma_S} & 0 \\
0 & 0 & \gamma_{nn}^{\Gamma_F} & 0 \\
0 & 0 & \gamma_{ns}^{\Gamma_F} & 0 \\
0 & 0 & 0 & \gamma_n^{\Gamma_F}
\end{bmatrix}
}_{\mathcal{B}_\partial}
\begin{pmatrix}
e_w \\
\boldsymbol{e}_\theta \\
\boldsymbol{E}_\kappa \\
\boldsymbol{e}_\gamma
\end{pmatrix},
\qquad (5.29)
$$

$$
\boldsymbol{y}_\partial =
\underbrace{
\begin{bmatrix}
0 & 0 & 0 & \gamma_n^{\Gamma_C} \\
0 & 0 & \gamma_{nn}^{\Gamma_C} & 0 \\
0 & 0 & \gamma_{ns}^{\Gamma_C} & 0 \\
0 & 0 & 0 & \gamma_n^{\Gamma_S} \\
0 & 0 & \gamma_{ns}^{\Gamma_S} & 0 \\
0 & \gamma_n^{\Gamma_S} & 0 & 0 \\
0 & \gamma_n^{\Gamma_F} & 0 & 0 \\
0 & \gamma_s^{\Gamma_F} & 0 & 0 \\
\gamma_0^{\Gamma_F} & 0 & 0 & 0
\end{bmatrix}
}_{\mathcal{C}_\partial}
\begin{pmatrix}
e_w \\
\boldsymbol{e}_\theta \\
\boldsymbol{E}_\kappa \\
\boldsymbol{e}_\gamma
\end{pmatrix},
$$

where $\gamma_0^{\Gamma_*} a = a|_{\Gamma_*}$ denotes the trace over the set $\Gamma_*$. Furthermore, notations $\gamma_n^{\Gamma_*}\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{n}|_{\Gamma_*}$, $\gamma_s^{\Gamma_*}\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{s}|_{\Gamma_*}$ indicate the normal and tangential trace over the set $\Gamma_*$ respectively. Symbols $\gamma_{nn}^{\Gamma_*}, \gamma_{ns}^{\Gamma_*}$ denote the normal-normal trace and the normal-tangential trace of tensor-valued functions, $\gamma_{nn}^{\Gamma_*}\boldsymbol{A} = \boldsymbol{A} : (\boldsymbol{n} \otimes \boldsymbol{n})|_{\Gamma_*}$, $\gamma_{ns}^{\Gamma_*}\boldsymbol{A} = \boldsymbol{A} : (\boldsymbol{n} \otimes \boldsymbol{s})|_{\Gamma_*}$.

**Remark 5**

*It can be observed that the interconnection structure given by $\mathcal{J}$ in (5.29) mimics that of the Timoshenko beam [JZ12, Chapter 7].*

**Conjecture 2** (Stokes-Dirac structure for the Mindlin plate)

*Consider $\mathbb{V} = \mathbb{R}^2$, $\mathbb{S} = \mathbb{R}_{sym}^{2\times 2}$ and let $H^1(\Omega)$ be the space of functions with gradient in $L^2(\Omega, \mathbb{V})$ and $H^{\mathrm{div}}(\Omega, \mathbb{V})$ the space of vector-valued functions with divergence in $L^2(\Omega)$. Furthermore, $H^1(\Omega, \mathbb{V})$ is the space of vectors with symmetric gradient in $L^2(\Omega, \mathbb{S})$ and $H^{\mathrm{Div}}(\Omega, \mathbb{S})$ denote*

*the space of symmetric tensors with divergence in $L^2(\Omega, \mathbb{V})$. Consider the definitions*

$$H := H^1(\Omega) \times H^{\mathrm{Grad}}(\Omega, \mathbb{V}) \times H^{\mathrm{Div}}(\Omega, \mathbb{S}) \times H^{\mathrm{div}}(\Omega, \mathbb{V}),$$
$$F := L^2(\Omega) \times L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{V}),$$
$$F_\partial := L^2(\Gamma_C, \mathbb{R}^3) \times L^2(\Gamma_S, \mathbb{R}^3) \times L^2(\Gamma_F, \mathbb{R}^3).$$

716     *The set*

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{f}_\partial \\ \boldsymbol{e} \\ \boldsymbol{e}_\partial \end{pmatrix} \mid \boldsymbol{e} \in H, \ \boldsymbol{f} = \mathcal{J}\boldsymbol{e}, \ \boldsymbol{f}_\partial = \mathcal{B}_\partial\boldsymbol{e}, \ \boldsymbol{e}_\partial = -\mathcal{C}_\partial\boldsymbol{e} \right\}, \tag{5.30}$$

717     *where $\boldsymbol{e} = (e_w, \boldsymbol{e}_\theta, \boldsymbol{E}_\kappa, \boldsymbol{e}_\gamma)$ and $\mathcal{J}, \mathcal{B}_\partial, \mathcal{C}_\partial$ are defined in (5.29), is a Stokes–Dirac structure*
718     *with respect to the pairing*

$$\left\langle\!\left\langle (\boldsymbol{f}^1, \boldsymbol{f}_\partial^1, \boldsymbol{e}^1, \boldsymbol{e}_\partial^1), (\boldsymbol{f}^2, \boldsymbol{f}_\partial^2, \boldsymbol{e}^2, \boldsymbol{e}_\partial^2) \right\rangle\!\right\rangle := \left\langle \boldsymbol{e}^1, \boldsymbol{f}^2 \right\rangle_F + \left\langle \boldsymbol{e}^2, \boldsymbol{f}^1 \right\rangle_F + \left\langle \boldsymbol{e}_\partial^1, \boldsymbol{f}_\partial^2 \right\rangle_{F_\partial} + \left\langle \boldsymbol{e}_\partial^2, \boldsymbol{f}_\partial^1 \right\rangle_{F_\partial},$$
$$\tag{5.31}$$

*where $\boldsymbol{e}_\partial^i = (e_{\partial,1}^i, \ e_{\partial,2}^i, \ e_{\partial,3}^i)$, $\boldsymbol{f}_\partial^i = (f_{\partial,1}^i, \ f_{\partial,2}^i, \ f_{\partial,3}^i)$ and*

$$\langle (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}), (\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}) \rangle_{F_\partial} = \int_{\Gamma_C} \boldsymbol{a} \cdot \boldsymbol{d} \, \mathrm{d}S + \int_{\Gamma_S} \boldsymbol{b} \cdot \boldsymbol{e} \, \mathrm{d}S + \int_{\Gamma_F} \boldsymbol{c} \cdot \boldsymbol{f} \, \mathrm{d}S, \quad \boldsymbol{a}, \ \boldsymbol{b}, \ \boldsymbol{c}, \ \boldsymbol{d}, \ \boldsymbol{e}, \ \boldsymbol{f} \in \mathbb{R}^3.$$

719     **Crucial points and elements in favor of the conjecture**   *Analogously to what was*
720     *stated in Conjecture 1, the boundary spaces have to properly defined. If the integration by*
721     *parts is carried out as in Eq. (5.26), one can follow the same lines of Conjecture 1 to support*
722     *the present Conjecture.*

723     The Mindlin plate falls within the assumption of [Skr19], hence it is a well posed boundary
724     control pH systems.

725     ## 5.2.2   Port-Hamiltonian Kirchhoff plate

726     Again the starting point is the Hamiltonian (total energy)

$$H = \frac{1}{2} \int_\Omega \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \, \mathrm{d}\Omega, \tag{5.32}$$

727     where $\boldsymbol{M}$, $\boldsymbol{\kappa}$ are defined in Eqs. (5.10), (5.16). For what concerns the choice of the energy
728     variables, a scalar and a tensor variable are considered:

$$\alpha_w = \rho h \frac{\partial w}{\partial t}, \quad \text{Linear momentum}, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}, \quad \text{Curvature tensor}. \tag{5.33}$$

729    The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \quad \text{Linear velocity}, \qquad \boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, \quad \text{Curvature tensor.} \qquad (5.34)$$

730        The port-Hamiltonian system is then written as

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \circ \operatorname{Div} \\ \operatorname{Grad} \circ \operatorname{grad} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}. \qquad (5.35)$$

The first equation is equivalent to (5.20). The last equation represent the fact the higher order derivatives commute

$$\partial_t \boldsymbol{A}_\kappa = \operatorname{Grad} \operatorname{grad} e_w,$$
$$\partial_t \boldsymbol{\kappa} = \operatorname{Grad} \operatorname{grad} \partial_t w,$$
$$\partial_t \operatorname{Grad} \operatorname{grad} w = \operatorname{Grad} \operatorname{grad} \partial_t w,$$

731    The last equation holds for $w \in C^3(\Omega)$.

732    **Theorem 3**
733    *The operator* $\operatorname{Grad} \circ \operatorname{grad}$*, corresponding to the Hessian operator, is the adjoint of the double*
734    *divergence* $\operatorname{div} \circ \operatorname{Div}$*.*

*Proof.* Let $\mathbb{S} = \mathbb{R}^{d \times d}_{\text{sym}}$ and consider the Hilbert space of the square integrable symmetric square tensors $L^2(\Omega, \mathbb{S})$ over an open connected set $\Omega$ (its inner product is defined in (4.12)). Consider the Hilbert space $L^2(\Omega)$ of scalar square integrable functions, endowed with the standard inner product. Consider the double divergence operator defined as:

$$\operatorname{div} \operatorname{Div} : \ L^2(\Omega, \mathbb{S}) \to L^2(\Omega), \qquad \text{with } \psi = \operatorname{div} \operatorname{Div} \boldsymbol{\Psi} = \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 \boldsymbol{\Psi}_{ij}}{\partial x_i \, \partial x_j}.$$
$$\boldsymbol{\Psi} \to \operatorname{div} \operatorname{Div} \boldsymbol{\Psi} = \psi,$$

We shall identify $\operatorname{div} \operatorname{Div}^*$

$$\operatorname{div} \operatorname{Div}^* : \ L^2(\Omega) \to L^2(\Omega, \mathbb{S}),$$
$$f \to \operatorname{div} \operatorname{Div}^* f = \boldsymbol{F},$$

such that

$$\langle \operatorname{div} \operatorname{Div} \boldsymbol{\Psi}, f \rangle_{L^2(\Omega)} = \langle \boldsymbol{\Psi}, \operatorname{div} \operatorname{Div}^* f \rangle_{L^2(\Omega, \mathbb{S})}, \qquad \begin{aligned} &\forall \, \boldsymbol{\Psi} \in \operatorname{Dom}(\operatorname{div} \operatorname{Div}) \subset L^2(\Omega, \mathbb{S}) \\ &\forall \, f \in \operatorname{Dom}(\operatorname{div} \operatorname{Div}^*) \subset L^2(\Omega) \end{aligned}$$

The function have to belong to the operator domain, so for instance $f \in C_0^2(\Omega) \in \operatorname{Dom}(\operatorname{div} \operatorname{Div}^*)$ the space of twice differentiable scalar functions with compact support and $\boldsymbol{\Psi}$ can be chosen in the set $C_0^2(\Omega, \mathbb{S}) \in \operatorname{Dom}(\operatorname{div} \operatorname{Div})$, the space of twice differentiable symmetric

tensors with compact support on $\Omega$. A classical result is the fact that the adjoint of the vector divergence is $\mathrm{div}^* = -\mathrm{grad}$ as stated in [KZ15]. By theorem 2, it holds $\mathrm{Div}^* = -\mathrm{Grad}$. Considering that $\mathrm{div}\,\mathrm{Div} = \mathrm{div} \circ \mathrm{Div}$ is the composition of two different operators and that the adjoint of a composed operator is the adjoint of each operator in reverse order, i.e. $(B \circ C)^* = C^* \circ B^*$, then it can be stated

$$(\mathrm{div} \circ \mathrm{Div})^* = \mathrm{Div}^* \circ \mathrm{div}^* = \mathrm{Grad} \circ \mathrm{grad}\,.$$

735 Since only formal adjoints are being looked for, this concludes the proof. $\qquad\square$

736     The energy rate provides the boundary port variables

$$
\begin{aligned}
\dot{H} &= \int_\Omega \{\partial_t \alpha_w\, e_w + \partial_t \boldsymbol{A}_\kappa : \boldsymbol{E}_\kappa\}\ \mathrm{d}\Omega \\
&= \int_\Omega \{-\,\mathrm{div}\,\mathrm{Div}\,\boldsymbol{E}_\kappa\, e_w + \mathrm{Grad}\,\mathrm{grad}\, e_w : \boldsymbol{E}_\kappa\}\ \mathrm{d}\Omega, &&\text{Stokes theorem} \\
&= \int_{\partial\Omega} \{-\boldsymbol{n} \cdot \mathrm{Div}\,\boldsymbol{E}_\kappa\, e_w + (\boldsymbol{n} \otimes \mathrm{grad}\, e_w) : \boldsymbol{E}_\kappa\}\ \mathrm{d}s, \\
&= \int_{\partial\Omega} \{-\boldsymbol{n} \cdot \mathrm{Div}\,\boldsymbol{E}_\kappa\, e_w + \partial_{\boldsymbol{n}} e_w\, (\boldsymbol{n} \otimes \boldsymbol{n}) : \boldsymbol{E}_\kappa + \partial_{\boldsymbol{s}} e_w\, (\boldsymbol{n} \otimes \boldsymbol{s}) : \boldsymbol{E}_\kappa\}\ \mathrm{d}s, &&\text{Dyadic properties} \\
&= \int_{\partial\Omega} \{\widehat{q}_n\, w_t + \partial_{\boldsymbol{n}} w_t\, M_{nn} + \partial_{\boldsymbol{s}} w_t\, M_{ns}\}\ \mathrm{d}s.
\end{aligned}
$$
(5.36)

737 where $s$ is the curvilinear abscissa, $w_t := \partial_t w$ and $\partial_{\boldsymbol{s}} w_t$ denotes the directional derivative
738 along the tangential versor at the boundary. Additionally, the following definitions have been
739 introduced

$$\widehat{q}_n := -\boldsymbol{n} \cdot \mathrm{Div}(\boldsymbol{E}_\kappa), \quad M_{nn} := (\boldsymbol{n} \otimes \boldsymbol{n}) : \boldsymbol{E}_\kappa, \quad M_{ns} := (\boldsymbol{n} \otimes \boldsymbol{s}) : \boldsymbol{E}_\kappa.$$
(5.37)

740 Variables $w_t$ and $\partial_{\boldsymbol{s}} w_t$ are not independent as they are differentially related with respect to
741 derivation along $\boldsymbol{s}$ (see for instance [TWK59, Chapter 4]). The tangential derivative has to be
742 moved on the torsional momentum $M_{ns}$. For sake of simplicity, $\partial\Omega$ is supposed to be regular.
743 Then the integration by parts provides

$$\int_{\partial\Omega} \partial_{\boldsymbol{s}} w_t\, M_{ns}\ \mathrm{d}s = -\int_{\partial\Omega} \partial_{\boldsymbol{s}} M_{ns}\, w_t\ \mathrm{d}s.$$
(5.38)

744 The final energy balance reads

$$\dot{H} = \int_{\partial\Omega} \{w_t\, \widetilde{q}_n + \partial_{\boldsymbol{n}} w_t\, M_{nn}\}\ \mathrm{d}s,$$
(5.39)

745 where the boundary variables are

$$
\begin{aligned}
&\text{Effective shear force} && \widetilde{q}_n := \widehat{q}_n - \partial_{\boldsymbol{s}} M_{ns}, \\
&\text{Flexural momentum} && M_{nn} := \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}) = \boldsymbol{E}_\kappa : (\boldsymbol{n} \otimes \boldsymbol{n}),
\end{aligned}
$$
(5.40)
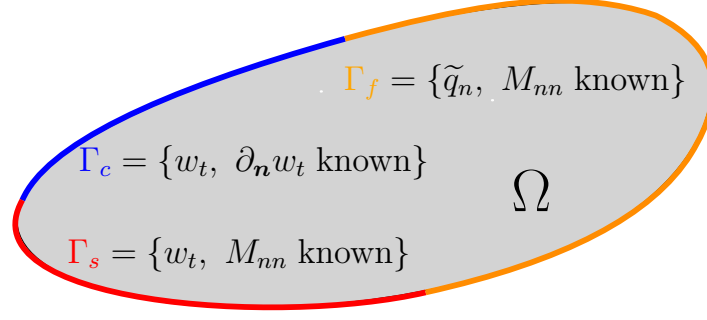
Figure 5.5: Boundary conditions for the Kirchhoff plate.

and $\widehat{q}_n$ is defined in (5.37). The corresponding power conjugated variables are:

$$\text{Vertical velocity} \quad w_t := \frac{\partial w}{\partial t} = e_w,$$
$$\text{Flexural rotation} \quad \partial_{\boldsymbol{n}} w_t := \nabla e_w \cdot \boldsymbol{n}. \tag{5.41}$$

Consider a partition of the boundary $\partial\Omega = \overline{\Gamma}_C \cup \overline{\Gamma}_S \cup \overline{\Gamma}_F$, $\Gamma_C \cap \Gamma_S \cap \Gamma_F = \{\emptyset\}$, where $\Gamma_C, \Gamma_S, \Gamma_F$ are open subset of $\partial\Omega$. Given definitions (5.40), (5.41), the boundary conditions for the Kirchhoff plate [GSV18] are the following (see Fig. 5.5):

- Clamped (C) on $\Gamma_C \subseteq \partial\Omega$ : $w_t$, $\partial_{\boldsymbol{n}} w_t$ known;

- Simply supported (S) on $\Gamma_S \subseteq \partial\Omega$: $w_t$, $M_{nn}$ known;

- Free (F) on $\Gamma_F \subseteq \partial\Omega$: $\widetilde{q}_n$, $M_{nn}$ known.

Then the final pH formulation reads

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\operatorname{div}\circ\operatorname{Div} \\ \operatorname{Grad}\circ\operatorname{grad} & \boldsymbol{0} \end{bmatrix}}_{\mathcal{J}}\begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix},$$

$$\boldsymbol{u}_\partial = \underbrace{\begin{bmatrix} \gamma_0^{\Gamma_C} & 0 \\ \gamma_1^{\Gamma_C} & 0 \\ \gamma_0^{\Gamma_S} & 0 \\ 0 & \gamma_{nn}^{\Gamma_S} \\ 0 & \gamma_{nn,1}^{\Gamma_F} \\ 0 & \gamma_{nn}^{\Gamma_F} \end{bmatrix}}_{\mathcal{B}_\partial}\begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix},$$

$$\boldsymbol{y}_\partial = \underbrace{\begin{bmatrix} 0 & \gamma_{nn,1}^{\Gamma_C} \\ 0 & \gamma_{nn}^{\Gamma_C} \\ 0 & \gamma_{nn,1}^{\Gamma_S} \\ \gamma_1^{\Gamma_S} & 0 \\ \gamma_0^{\Gamma_F} & 0 \\ \gamma_1^{\Gamma_F} & 0 \end{bmatrix}}_{\mathcal{C}_\partial}\begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix},$$
(5.42)

where $\gamma_0^{\Gamma_*}a = a|_{\Gamma_*}$ and $\gamma_1^{\Gamma_*}a = \partial_n a|_{\Gamma_*}$ denote the standard and the normal derivative trace over the set $\Gamma_*$ respectively. The symbol $\gamma_{nn,1}^{\Gamma_*}$ denotes the map $\gamma_{nn,1}^{\Gamma_*}\boldsymbol{A} = -\boldsymbol{n}\cdot\operatorname{Div}\boldsymbol{A} - \partial_s(\boldsymbol{A}\colon (\boldsymbol{n}\otimes\boldsymbol{s}))|_{\Gamma_*}$, while $\gamma_{nn}^{\Gamma_*}\boldsymbol{A} = \boldsymbol{A}\colon(\boldsymbol{n}\otimes\boldsymbol{n})|_{\Gamma_*}$ indicates the normal-normal trace of a tensor-valued function.

**Remark 6**

*The interconnection structure $\mathcal{J}$ in (5.42) mimics that of the Bernoulli beam [CRMPB17]. The double divergence and the Hessian coincide, in dimension one, with the second derivative.*

**Conjecture 3** (Stokes-Dirac structure for the Kirchhoff plate)
*Consider $\mathbb{S} = \mathbb{R}^{2\times 2}_{sym}$ and let $H^2(\Omega)$ be the space of functions with Hessian in $L^2(\Omega,\mathbb{S})$ and $H^{\operatorname{div}\operatorname{Div}}(\Omega,\mathbb{S})$ the space of vector-valued functions with double divergence in $L^2(\Omega)$. Consider the definitions*

$$
\begin{aligned}
H &:= H^2(\Omega) \times H^{\operatorname{div}\operatorname{Div}}(\Omega,\mathbb{S}), \\
F &:= L^2(\Omega) \times L^2(\Omega,\mathbb{S}), \\
F_\partial &:= L^2(\Gamma_C,\mathbb{R}^2) \times L^2(\Gamma_S,\mathbb{R}^2) \times L^2(\Gamma_F,\mathbb{R}^2).
\end{aligned}
$$

*The set*

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{f}_\partial \\ \boldsymbol{e} \\ \boldsymbol{e}_\partial \end{pmatrix} \mid \boldsymbol{e}\in H,\ \boldsymbol{f}=\mathcal{J}\boldsymbol{e},\ \boldsymbol{f}_\partial=\mathcal{B}_\partial\boldsymbol{e},\ \boldsymbol{e}_\partial=-\mathcal{C}_\partial\boldsymbol{e} \right\},$$
(5.43)

*where $\boldsymbol{e} = (e_w, \boldsymbol{E}_\kappa)$ and $\mathcal{J},\mathcal{B}_\partial,\mathcal{C}_\partial$ are defined in (5.42), is a Stokes–Dirac structure with*
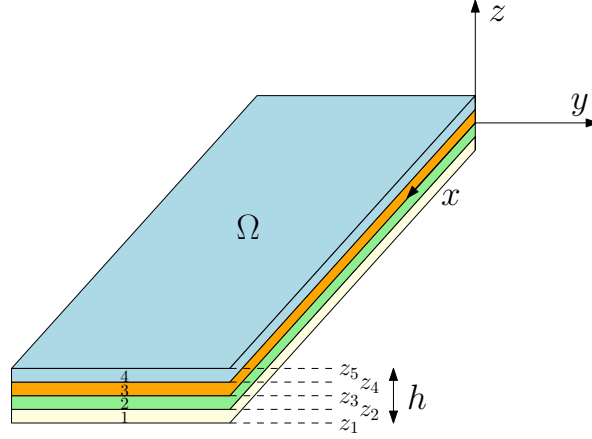
Figure 5.6: Laminated plate with 4 layers.

*respect to the pairing*

$$\left\langle\!\left\langle (\boldsymbol{f}^1, \boldsymbol{f}_\partial^1, \boldsymbol{e}^1, \boldsymbol{e}_\partial^1), (\boldsymbol{f}^2, \boldsymbol{f}_\partial^2, \boldsymbol{e}^2, \boldsymbol{e}_\partial^2) \right\rangle\!\right\rangle := \left\langle \boldsymbol{e}^1,\, \boldsymbol{f}^2 \right\rangle_F + \left\langle \boldsymbol{e}^2,\, \boldsymbol{f}^1 \right\rangle_F + \left\langle \boldsymbol{e}_\partial^1,\, \boldsymbol{f}_\partial^2 \right\rangle_{F_\partial} + \left\langle \boldsymbol{e}_\partial^2,\, \boldsymbol{f}_\partial^1 \right\rangle_{F_\partial},$$
$$(5.44)$$

*where $\boldsymbol{e}_\partial^i = (\boldsymbol{e}_{\partial,1}^i,\ \boldsymbol{e}_{\partial,2}^i)$, $\boldsymbol{f}_\partial^i = (\boldsymbol{f}_{\partial,1}^i,\ \boldsymbol{f}_{\partial,2}^i)$ and*

$$\langle (\boldsymbol{a},\, \boldsymbol{b},\, \boldsymbol{c}),\, (\boldsymbol{d},\, \boldsymbol{e},\, \boldsymbol{f}) \rangle_{F_\partial} = \int_{\Gamma_C} \boldsymbol{a} \cdot \boldsymbol{d} \,\mathrm{d}S + \int_{\Gamma_S} \boldsymbol{b} \cdot \boldsymbol{e} \,\mathrm{d}S + \int_{\Gamma_F} \boldsymbol{c} \cdot \boldsymbol{f} \,\mathrm{d}S, \quad \boldsymbol{a},\, \boldsymbol{b},\, \boldsymbol{c},\, \boldsymbol{d},\, \boldsymbol{e},\, \boldsymbol{f} \in \mathbb{R}^2.$$

**Validity of the conjecture**   *The integration by parts has to be carried as in Eq.* (5.36) *to retrieve a similar discussion to the one in Conjecture 1.*

## 5.3   Laminated anisotropic plates

Until now homogeneous isotropic materials have been considered. For this class of materials, the membrane and bending problems are decoupled. In aeronautical applications, structure are made up of laminae of different materials to enhance the mechanical properties of the resulting structure. In some cases, a certain coupling is desired, to increase the aerodynamical performance of the wing as it deforms.

Consider again the deformation field given by (5.1)

$$\boldsymbol{u}(x, y, z, t) = \boldsymbol{u}^0(x, y, t) - z\boldsymbol{\theta}(x, y, t),$$
$$u_z(x, y, z, t) = u_z^0(x, y, t),$$

where $\boldsymbol{u} = (u_x, u_y)$. The link between in-plane deformation (5.2) and the membrane and

774  bending contribution (5.4), (5.5).

$$\boldsymbol{\varepsilon}_{2D} = \boldsymbol{\varepsilon}^0 - z\boldsymbol{\kappa} \qquad \text{where} \qquad \boldsymbol{\varepsilon}^0 = \text{Grad}\,\boldsymbol{u}^0, \quad \boldsymbol{\kappa} = \text{Grad}\,\boldsymbol{\theta}. \qquad (5.45)$$

775  Assume that each layer is an anisotropic material under plane stress condition. Then, it
776  holds (see [Red03, Chapter 1] for details)

$$\boldsymbol{\Sigma}_{2D}^i = \boldsymbol{\mathcal{D}}_{2D}^i \boldsymbol{\varepsilon}_{2D}^i,$$

777  where $i$ indicates the layer under consideration. The matrix $\boldsymbol{\mathcal{D}}_{2D}^i$ depends on the properties
778  of each material. To reduce the problem to bi-dimensional, the stresses have to be integrated
779  along the thickness. Differently from isotropic plate, for laminated anisotropic plates the
780  membrane and bending behavior are coupled. To see this consider the membrane and bending
781  resultant of the stress

$$\boldsymbol{N} := \int_{-h/2}^{h/2} \boldsymbol{\Sigma}_{2D}\,\mathrm{d}z, \qquad \boldsymbol{M} := \int_{-h/2}^{h/2} -z\boldsymbol{\Sigma}_{2D}\,\mathrm{d}z. \qquad (5.46)$$

782  Since the stress are discontinuous due to the change of constitutive law along the thickness,
783  the integration has to be performed lamina-wise. Once the computations are carried out, it
784  is found

$$\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{D}}_m & \boldsymbol{\mathcal{D}}_c \\ \boldsymbol{\mathcal{D}}_c & \boldsymbol{\mathcal{D}}_b \end{bmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{pmatrix}, \qquad (5.47)$$

785  where

$$\boldsymbol{\mathcal{D}}_m = \sum_{i=1}^{n_{\text{layer}}} \boldsymbol{\mathcal{D}}_{2D}^i (z_{i+1} - z_i), \quad \boldsymbol{\mathcal{D}}_c = -\frac{1}{2}\sum_{i=1}^{n_{\text{layer}}} \boldsymbol{\mathcal{D}}_{2D}^i (z_{i+1}^2 - z_i^2), \quad \boldsymbol{\mathcal{D}}_b = \frac{1}{3}\sum_{i=1}^{n_{\text{layer}}} \boldsymbol{\mathcal{D}}_{2D}^i (z_{i+1}^3 - z_i^3),$$
$$(5.48)$$

786  and $n_{\text{layer}}$ is the number of layers and $z_i$ represents the height of the $i^{\text{th}}$ layer (see Fig. 5.6).
787  The coupling term $\boldsymbol{\mathcal{D}}_c$ disappears if a symmetric configuration is considered. For the shear
788  contribution it is obtained

$$\boldsymbol{q} := \int_{-h/2}^{h/2} \boldsymbol{\sigma}_s\,\mathrm{d}z = \boldsymbol{\mathcal{D}}_s\boldsymbol{\gamma}, \qquad \text{where} \qquad \boldsymbol{\gamma} = \text{grad}\,u_z - \boldsymbol{\theta}. \qquad (5.49)$$

789  The tensor $\boldsymbol{\mathcal{D}}_s$ is not diagonal as in the isotropic case, cf. §5.2.1.
790

791  In the following section it is shown how anisotropic laminated plates can be formulated
792  as pHs.

### 5.3.1    Port-Hamiltonian laminated Mindlin plate

For a shear deformable laminated plate the kinetic and deformation energy read

$$E_{\text{kin}} = \frac{1}{2} \int_\Omega \left\{ \rho h \left\| \frac{\partial \boldsymbol{u}^0}{\partial t} \right\|^2 + \rho h \left( \frac{\partial u_z}{\partial t} \right)^2 + \frac{\rho h^3}{12} \left\| \frac{\partial \boldsymbol{\theta}}{\partial t} \right\|^2 \right\} \, \mathrm{d}\Omega,$$

$$E_{\text{def}} = \frac{1}{2} \int_\Omega \left\{ \boldsymbol{N} : \boldsymbol{\varepsilon}^0 + \boldsymbol{M} : \boldsymbol{\kappa} + \boldsymbol{q} \cdot \boldsymbol{\gamma} \right\} \, \mathrm{d}\Omega.$$

By using Hamilton's principle the equations of motion are retrieved (see [Red03, Chapter 3] for an exhaustive explanation)

$$\rho h \frac{\partial^2 \boldsymbol{u}^0}{\partial t^2} = \text{Div } \boldsymbol{N},$$

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = \text{div } \boldsymbol{q}, \tag{5.50}$$

$$\frac{\rho h^3}{12} \frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} = \text{Div } \boldsymbol{M} + \boldsymbol{q},$$

where $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{q}$ are defined in Eqs. (5.47), (5.49). To get a port-Hamiltonian formulation, the following energy variable are chosen

$$\boldsymbol{\alpha}_u = \rho h \frac{\partial \boldsymbol{u}^0}{\partial t}, \qquad \alpha_w = \rho h \frac{\partial u_z}{\partial t}, \qquad \boldsymbol{\alpha}_\theta = \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t},$$
$$\boldsymbol{A}_{\varepsilon^0} = \boldsymbol{\varepsilon}^0, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}, \qquad \boldsymbol{\alpha}_\gamma = \boldsymbol{\gamma}. \tag{5.51}$$

This choice highlights the nature of the problem in which the membrane part (equivalent to a 2D elasticity problem) and the bending part interact. The total energy $H = E_{\text{kin}} + E_{\text{def}}$ is now a quadratic function of the energy variables

$$E_{\text{kin}} = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho h} \left\| \frac{\partial \boldsymbol{\alpha}_u}{\partial t} \right\|^2 + \frac{1}{\rho h} \left( \frac{\partial \alpha_w}{\partial t} \right)^2 + \frac{12}{\rho h^3} \left\| \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \right\|^2 \right\} \, \mathrm{d}\Omega,$$

$$E_{\text{def}} = \frac{1}{2} \int_\Omega \left\{ (\boldsymbol{\mathcal{D}}_m \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{\mathcal{D}}_c \boldsymbol{A}_\kappa) : \boldsymbol{A}_{\varepsilon^0} + (\boldsymbol{\mathcal{D}}_c \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{\mathcal{D}}_b \boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa + (\boldsymbol{\mathcal{D}}_s \boldsymbol{\alpha}_\gamma) \cdot \boldsymbol{\alpha}_\gamma \right\} \, \mathrm{d}\Omega,$$

The co-energies are equal to

$$e_w := \frac{\delta H}{\delta \boldsymbol{\alpha}_u} = \frac{\partial \boldsymbol{u}^0}{\partial t}, \qquad e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial u_z}{\partial t}, \qquad \boldsymbol{e}_\theta := \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t},$$
$$\boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_{\varepsilon^0}} = \boldsymbol{N}, \qquad \boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, \qquad e_\gamma := \frac{\delta H}{\delta \boldsymbol{\alpha}_\gamma} = \boldsymbol{q} \tag{5.52}$$

799 The final pH formulation is found as usual considering the dynamics (5.50) and fact that
800 higher derivatives commute

$$
\frac{\partial}{\partial t}
\begin{pmatrix}
\boldsymbol{\alpha}_u \\
\alpha_w \\
\boldsymbol{\alpha}_\theta \\
\boldsymbol{A}_{\varepsilon^0} \\
\boldsymbol{A}_\kappa \\
\boldsymbol{\alpha}_\gamma
\end{pmatrix}
=
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} \\
0 & 0 & 0 & 0 & 0 & \mathrm{div} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\
\mathrm{Grad} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathrm{Grad} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\begin{pmatrix}
\boldsymbol{e}_u \\
e_w \\
\boldsymbol{e}_\theta \\
\boldsymbol{E}_{\varepsilon^0} \\
\boldsymbol{E}_\kappa \\
\boldsymbol{e}_\gamma
\end{pmatrix}.
\tag{5.53}
$$

801 The coupling between the membrane and bending part is clear when considering the link
802 between energy and co-energy variables

$$
\begin{pmatrix}
\boldsymbol{e}_u \\
e_w \\
\boldsymbol{e}_\theta \\
\boldsymbol{E}_{\varepsilon^0} \\
\boldsymbol{E}_\kappa \\
\boldsymbol{e}_\gamma
\end{pmatrix}
=
\begin{bmatrix}
\frac{1}{\rho h}\boldsymbol{I}_{2\times 2} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
0 & \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\
\mathbf{0} & 0 & \frac{12}{\rho h^3}\boldsymbol{I}_{2\times 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & 0 & \mathbf{0} & \boldsymbol{\mathcal{D}}_m & \boldsymbol{\mathcal{D}}_c & \mathbf{0} \\
\mathbf{0} & 0 & \mathbf{0} & \boldsymbol{\mathcal{D}}_c & \boldsymbol{\mathcal{D}}_b & \mathbf{0} \\
\mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\mathcal{D}}_s
\end{bmatrix}
\begin{pmatrix}
\boldsymbol{\alpha}_u \\
\alpha_w \\
\boldsymbol{\alpha}_\theta \\
\boldsymbol{A}_{\varepsilon^0} \\
\boldsymbol{A}_\kappa \\
\boldsymbol{\alpha}_\gamma
\end{pmatrix}.
\tag{5.54}
$$

803 Again appropriate boundary variables and a suitable Stokes-Dirac structure can be found for
804 this model. The final formulation is just a superposition of systems (4.16) and (5.29).

805 ## 5.3.2 Port-Hamiltonian laminated Kirchhoff plate

According to the Kirchhoff hypotheses the kinetic and deformation energies reduce to

$$
E_{\mathrm{kin}} = \frac{1}{2}\int_\Omega \left\{ \rho h \left\| \frac{\partial \boldsymbol{u}^0}{\partial t} \right\|^2 + \rho h \left( \frac{\partial u_z}{\partial t} \right)^2 \right\} \, \mathrm{d}\Omega,
$$

$$
E_{\mathrm{def}} = \frac{1}{2}\int_\Omega \left\{ \boldsymbol{N} : \boldsymbol{\varepsilon}^0 + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \, \mathrm{d}\Omega,
$$

806 where $\boldsymbol{\kappa}$ is defined in Eq. (5.5). Furthermore, as stated in Remark 4, the rotational contri-
807 bution in the kinetic energy has been neglected. The equations of motion are (see [Red03,
808 Chapter 3] for an exhaustive explanation)

$$
\begin{aligned}
\rho h \frac{\partial^2 \boldsymbol{u}^0}{\partial t^2} &= \mathrm{Div}\,\boldsymbol{N}, \\
\rho h \frac{\partial^2 u_z}{\partial t^2} &= -\mathrm{div}\,\mathrm{Div}\,\boldsymbol{M},
\end{aligned}
\tag{5.55}
$$

809   where $\boldsymbol{N}$, $\boldsymbol{M}$ are defined in Eqs. (5.47). To get a port-Hamiltonian formulation, the following
810   energy variable are chosen

$$\boldsymbol{\alpha}_u = \rho h \frac{\partial \boldsymbol{u}^0}{\partial t}, \qquad \alpha_w = \rho h \frac{\partial u_z}{\partial t},$$
$$\boldsymbol{A}_{\varepsilon^0} = \boldsymbol{\varepsilon}^0, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}. \tag{5.56}$$

The total energy $H = E_{\text{kin}} + E_{\text{def}}$ is now a quadratic function of the energy variables

$$E_{\text{kin}} = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho h} \left\| \frac{\partial \boldsymbol{\alpha}_u}{\partial t} \right\|^2 + \frac{1}{\rho h} \left( \frac{\partial \alpha_w}{\partial t} \right)^2 \right\} \, \mathrm{d}\Omega,$$
$$E_{\text{def}} = \frac{1}{2} \int_\Omega \left\{ (\boldsymbol{D}_m \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{D}_c \boldsymbol{A}_\kappa) : \boldsymbol{A}_{\varepsilon^0} + (\boldsymbol{D}_c \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{D}_b \boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa \right\} \, \mathrm{d}\Omega,$$

811   The co-energies are equal to

$$e_w := \frac{\delta H}{\delta \boldsymbol{\alpha}_u} = \frac{\partial \boldsymbol{u}^0}{\partial t}, \qquad e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial u_z}{\partial t},$$
$$\boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_{\varepsilon^0}} = \boldsymbol{N}, \qquad \boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, \tag{5.57}$$

812   The final pH formulation is found as usual considering the dynamics (5.55) and fact that
813   higher derivatives commute

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_u \\ \alpha_w \\ \boldsymbol{A}_{\varepsilon^0} \\ \boldsymbol{A}_\kappa \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \mathrm{Div} & \boldsymbol{0} \\ 0 & 0 & 0 & -\mathrm{div} \circ \mathrm{Div} \\ \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{Grad} \circ \mathrm{grad} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_u \\ e_w \\ \boldsymbol{E}_{\varepsilon^0} \\ \boldsymbol{E}_\kappa \end{pmatrix}. \tag{5.58}$$

814   Again, the coupling appears when considering the link between energy and co-energy variables

$$\begin{pmatrix} \boldsymbol{e}_u \\ e_w \\ \boldsymbol{E}_{\varepsilon^0} \\ \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} \frac{1}{\rho h} \boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ 0 & \frac{1}{\rho h} & 0 & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_m & \boldsymbol{D}_c \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_c & \boldsymbol{D}_b \end{bmatrix} \begin{pmatrix} \boldsymbol{\alpha}_u \\ \alpha_w \\ \boldsymbol{A}_{\varepsilon^0} \\ \boldsymbol{A}_\kappa \end{pmatrix}. \tag{5.59}$$

815   The energy rate provides the appropriate boundary conditions from which one can construct
816   the Stokes-Dirac structure. The necessary computations are not performed here as the final
817   result is just a juxtaposition of systems (4.16), (5.42).

## 818   **5.4   Conclusion**

819   In this chapter, a pH formulation for the most commonly used plate models has been detailed.
820   Many open questions remain. In particular, how to generalize the results to shell problems,
821   for which the domain is a surface embedded in the three dimensional space (a manifold).
822   Computations get more involved in this case since the usage of differential geometry concepts

is unavoidable. These models are important since they are widely used in the aerospace industry and ubiquitous in nature.

The reformulation of plate models using the language of differential geometry is another open research topic. Indeed, while for the Mindlin plate it should be possible to use vector-valued forms to obtain an equivalent system, for the Kirchhoff plate the task appears more involved. An interesting reference that can provide some ideas in this direction is [Yao11].

# Thermoelasticity in port-Hamiltonian form

*Eh bien, mon ami, la terre sera un jour ce cadavre refroidi. Elle deviendra inhabitable et sera inhabitée comme la lune, qui depuis longtemps a perdu sa chaleur vitale.*

*Vingt mille lieues sous les mers*
*Jules Verne*

## Contents

Hermoelasticity is the study of deformable bodies undergoing thermal excitations. It is a clear example of a multiphysics phenomenon since the heat transfer and elastic vibrations within the body mutually interact. In this chapter, a linear model of thermoelasticity is obtained under the pH formalism. Each physics is described separately and the final system is obtained considering a power-preserving interconnection of two pHs.

## 6.1 Port-Hamiltonian linear coupled thermoelasticity

In this section, a pH formulation of heat transfer is first introduced. The classical model of thermoelasticity is then recalled. The same model is found by interconnecting the heat equation and the linear elastodynamics problem seen as pHs. It is shown that the interconnection

preserves a quadratic functional that plays the role of a fictitious energy. The resulting system is dissipative with respect to this functional. The construction makes use of the intrinsic modularity of pHs [KZvdSB10].

### 6.1.1   The heat equation as a pH descriptor system

Consider the heat equation in a bounded connected set $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, describing the evolution of the temperature field $T(\boldsymbol{x}, t)$

$$\rho c_\epsilon \frac{\partial T}{\partial t} = k\Delta T + r_Q, \qquad \boldsymbol{x} \in \Omega, \tag{6.1}$$

where $\rho$, $c_\epsilon$, $k$, $r_Q$ are the mass density, the specific heat density at constant strain, the thermal diffusivity and an heat source. Symbol $\Delta$ denotes the Laplacian in $\mathbb{R}^d$. The Dirichlet and Neumann condition of this problem are

$$T \text{ known on } \Gamma_D^T, \qquad \text{Dirichlet condition,}$$
$$-k \text{ grad } T \cdot \boldsymbol{n} \text{ known on } \Gamma_N^T, \qquad \text{Neumann condition,}$$

where a partition of the boundary $\partial\Omega = \Gamma_D^T \cup \Gamma_N^T$ has been considered. This model can be put in pH form by means of a canonical interconnection structure. An algebraic relationship that describes the Fourier law has to be incorporated in the model (cf. [Kot19, Chapter 2]). Here, a differential-algebraic formulation is exploited to obtain the same system.

Let $T_0$ be a constant reference temperature (the introduction of this variables is instrumental for coupled thermoelasticity). The functional

$$H_T = \frac{1}{2} \int_\Omega \rho c_\epsilon T_0 \left(\frac{T - T_0}{T_0}\right)^2 \mathrm{d}\Omega$$

has the physical dimension of an energy and represents a Lyapunov functional of this system. Even though it does not represent the internal energy, it has some important properties. Select as energy variable

$$\alpha_T := \rho c_\epsilon (T - T_0),$$

whose corresponding co-energy is

$$e_T := \frac{\delta H_T}{\delta \alpha_T} = \frac{\alpha_T}{\rho c_\epsilon T_0} = \frac{T - T_0}{T_0} =: \theta.$$

Introducing the heat flux $\boldsymbol{j}_Q := -k \text{ grad } T$ as additional variable, the heat equation (6.1) is

equivalently reformulated as

$$
\begin{bmatrix} 1 & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_T \\ \boldsymbol{j}_Q \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & -(T_0 k)^{-1} \end{bmatrix} \begin{pmatrix} e_T \\ \boldsymbol{j}_Q \end{pmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} u_T,
$$

$$
y_T = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_T \\ \boldsymbol{j}_Q \end{pmatrix}.
\tag{6.2}
$$

with $u_T := r_Q$ and $y_T$ represents the corresponding power-conjugated variable. In matrix notation, it is obtained

$$
\mathcal{E}_T \partial_t \boldsymbol{\alpha}_T = (\mathcal{J}_T - \mathcal{R}_T) \, e_T + \mathcal{B}_T \, u_T,
$$

$$
y_d = \mathcal{B}_T^* \, e_T
\tag{6.3}
$$

where $\boldsymbol{\alpha}_T = (\alpha_T, \ \boldsymbol{j}_Q)$, $e_T = (e_T, \ \boldsymbol{j}_Q)$ and

$$
\mathcal{E}_T = \begin{bmatrix} 1 & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathcal{J}_T = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \mathbf{0} \end{bmatrix}, \quad \mathcal{R}_T = \begin{bmatrix} 0 & 0 \\ \mathbf{0} & (T_0 k)^{-1} \end{bmatrix}, \quad \mathcal{B}_T = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}.
$$

The system is an example of pH descriptor system (cf. [BMXZ18] for the finite dimensional case). The Hamiltonian reads

$$
H_T = \frac{1}{2} \int_\Omega e_T \cdot \mathcal{E}_T \boldsymbol{\alpha}_T \, \mathrm{d}\Omega.
\tag{6.4}
$$

The power rate is then deduced

$$
\begin{aligned}
\dot{H}_T &= \int_\Omega e_T \cdot \mathcal{E}_T \, \partial_t \boldsymbol{\alpha}_T \, \mathrm{d}\Omega, \\
&= \int_\Omega e_T \cdot \{ (\mathcal{J}_T - \mathcal{R}_T) e + \mathcal{B}_T u_T \} \, \mathrm{d}\Omega, \\
&= \int_\Omega u_T \, y_T \, \mathrm{d}\Omega - \int_\Omega \left( e_T \operatorname{div} \boldsymbol{j}_Q + \boldsymbol{j}_Q \operatorname{grad} e_T + \frac{\|\boldsymbol{j}_Q\|^2}{k T_0} \right) \, \mathrm{d}\Omega, \\
&\leq \int_\Omega u_T \, y_T \, \mathrm{d}\Omega - \int_{\partial\Omega} e_T \, \boldsymbol{j}_Q \cdot \boldsymbol{n} \, \mathrm{d}S.
\end{aligned}
\tag{6.5}
$$

This choice of Hamiltonian allows retrieving the classical boundary conditions and leads to a dissipative system. Other formulations, based on an entropy or internal energy functionals, are possible for the heat equation [DMSB09, SHM19a]. These provide an accrescent or a lossless system. Unfortunately these formulations are non linear and their discretization is a difficult task [SHM19b].

### 6.1.2   Classical thermoelasticity

The derivation of the classical theory of thermoelasticity is not carried out here. The reader may consult in [HE09, Chapter 1] or [Abe12, Chapter 8] for a detailed discussion on this topic.
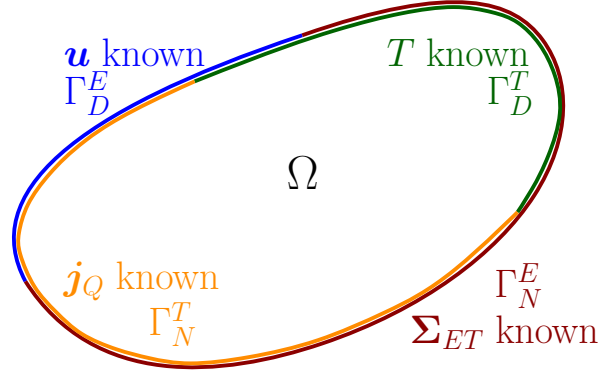
Figure 6.1: Boundary conditions for the thermoelastic problem.

Consider a bounded connected set $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. The classical equations for linear fully-coupled thermoelasticity for an isotropic thermoelastic material are [Bio56, Car73]

$$
\begin{aligned}
\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} &= \mathrm{Div}(\boldsymbol{\Sigma}_{ET}), \\
\rho c_\epsilon \frac{\partial T}{\partial t} &= -\mathrm{div}(\boldsymbol{j}_Q) - \mathcal{C}_\beta : \frac{\partial \boldsymbol{\varepsilon}}{\partial t}, \\
\boldsymbol{\Sigma}_{ET} &= \boldsymbol{\Sigma}_E + \boldsymbol{\Sigma}_T, \\
\boldsymbol{\Sigma}_E &= 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I}_{d\times d}, \\
\boldsymbol{\Sigma}_T &= -\mathcal{C}_\beta \theta, \\
\boldsymbol{\varepsilon} &= \mathrm{Grad}(\boldsymbol{u}), \\
\boldsymbol{j}_Q &= -k\,\mathrm{grad}\,T.
\end{aligned}
\tag{6.6}
$$

For simplicity the coupling term

$$
\mathcal{C}_\beta := T_0\beta(2\mu + d\lambda)\boldsymbol{I}_{d\times d}
$$

has been introduced. Field $\boldsymbol{u}$ is the displacement, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\boldsymbol{\Sigma}_E, \boldsymbol{\Sigma}_T$ are the stress tensor contribution due to mechanical deformation and a thermal field. Coefficients $\lambda$, $\mu$ are the Lamé parameters, and $\beta$ the thermal expansion coefficient. Given a partition of the boundary $\partial\Omega = \Gamma_D^E \cup \Gamma_N^E = \Gamma_D^T \cup \Gamma_N^T$ for the elastic and thermal domain. The general boundary conditions read (see Fig. 6.1)

$$
\begin{aligned}
\boldsymbol{u} \text{ known on } \Gamma_D^E \times (0, +\infty), && T \text{ known on } \Gamma_D^T \times (0, +\infty), \\
\boldsymbol{\Sigma}_{ET} \cdot \boldsymbol{n} \text{ known on } \Gamma_N^E \times (0, +\infty), && \boldsymbol{j}_Q \cdot \boldsymbol{n} \text{ known on } \Gamma_N^T \times (0, +\infty).
\end{aligned}
\tag{6.7}
$$

In the following section an equivalent system is constructed by interconnecting the heat equation and the elastodynamics system in a structured manner.

### 6.1.3 Thermoelasticity as two coupled pHs

Consider again the equation of elasticity on $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ (cf. Eq. (4.16)), together with a distributed input $\boldsymbol{u}_E$ that plays the role of a distributed force

$$
\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \mathrm{Div} \\ \mathrm{Grad} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \end{pmatrix} + \begin{bmatrix} \boldsymbol{I}_{d \times d} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u}_E,
$$
$$
\boldsymbol{y}_E = \begin{bmatrix} \boldsymbol{I}_{d \times d} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \end{pmatrix},
\tag{6.8}
$$

with Hamiltonian

$$
H_E = \frac{1}{2} \int_\Omega \left\{ \boldsymbol{\alpha}_v \cdot \boldsymbol{e}_v + \boldsymbol{A}_\varepsilon : \boldsymbol{E}_\varepsilon \right\} \, \mathrm{d}\Omega.
$$

Recall the pH formulation of the heat equation (6.2)

$$
\begin{bmatrix} 1 & 0 \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_T \\ \boldsymbol{j}_Q \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div} \\ -\mathrm{grad} & -(T_0 k)^{-1} \end{bmatrix} \begin{pmatrix} e_T \\ \boldsymbol{j}_Q \end{pmatrix} + \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix} u_T,
$$
$$
y_T = \begin{bmatrix} 1 & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} e_T \\ \boldsymbol{j}_Q \end{pmatrix},
\tag{6.9}
$$

with Hamiltonian $H_T$ defined in (6.4). The linear thermoelastic problem can be expressed as a coupled port-Hamiltonian system. Consider the following interconnection

$$
\boldsymbol{u}_E = -\mathrm{Div}(\mathcal{C}_\beta \, y_T), \qquad u_T = -\mathcal{C}_\beta : \mathrm{Grad}(\boldsymbol{y}_E).
\tag{6.10}
$$

The interconnection is power preserving as it can be compactly written as

$$
\boldsymbol{u}_E = \mathcal{A}_\beta(y_T), \qquad u_T = -\mathcal{A}_\beta^*(\boldsymbol{y}_E).
$$

where $\mathcal{A}_\beta^*$ denotes the formal adjoint. The assertion is justified by the following proposition.

**Proposition 5**
*Let $C_0^\infty(\Omega)$, $C_0^\infty(\Omega, \mathbb{R}^d)$ be the space of smooth functions and vector-valued functions respectively. Given $y_T \in C_0^\infty(\Omega)$, $\boldsymbol{y}_E \in C_0^\infty(\Omega, \mathbb{R}^d)$, the coupling operator*

$$
\mathcal{A}_\beta : C_0^\infty(\Omega) \to C_0^\infty(\Omega, \mathbb{R}^d),
$$
$$
y_T \to -\mathrm{Div}(\mathcal{C}_\beta y_T)
\tag{6.11}
$$

*has formal adjoint*

$$
\mathcal{A}_\beta^* : C_0^\infty(\Omega, \mathbb{R}^d) \to C_0^\infty(\Omega)
$$
$$
\boldsymbol{y}_E \to -\mathcal{C}_\beta : \mathrm{Grad}(\boldsymbol{y}_E)
\tag{6.12}
$$

*Proof.* It is necessary to show

$$
\langle \boldsymbol{y}_E, \mathcal{A}_\beta y_T \rangle_{L^2(\Omega, \mathbb{R}^d)} = \left\langle \mathcal{A}_\beta^* \boldsymbol{y}_E, y_T \right\rangle_{L^2(\Omega)},
\tag{6.13}
$$

908  where for $\boldsymbol{u}_E, \boldsymbol{y}_E \in C_0^\infty(\Omega)$, $u_T, y_T \in C_0^\infty(\Omega)$

$$\langle \boldsymbol{u}_E, \boldsymbol{y}_E \rangle_{L^2(\Omega, \mathbb{R}^d)} = \int_{\Omega_E} \boldsymbol{u}_E \cdot \boldsymbol{y}_E \, \mathrm{d}\Omega, \qquad \langle u_T, y_T \rangle_{L^2(\Omega)} = \int_{\Omega_T} u_T y_T \, \mathrm{d}\Omega. \tag{6.14}$$

909  The proof is a simple application of Theorem 5

$$\begin{aligned}
\langle \boldsymbol{y}_E, \mathcal{A}_\beta y_T \rangle_{L^2(\Omega, \mathbb{R}^d)} &= - \int_\Omega \boldsymbol{y}_E \cdot \mathrm{Div}(\mathcal{C}_\beta y_T) \, \mathrm{d}\Omega, \\
&= - \int_\Omega \mathrm{Grad}(\boldsymbol{y}_E) : \mathcal{C}_\beta y_T \, \mathrm{d}\Omega, \\
&= \int_\Omega \mathcal{A}_\beta^*(\boldsymbol{y}_E) y_T \, \mathrm{d}\Omega, \\
&= \left\langle \mathcal{A}_\beta^* \boldsymbol{y}_E, y_T \right\rangle_{L^2(\Omega)}.
\end{aligned} \tag{6.15}$$

910  This concludes the proof.                                                                                  □

911     If the compact support assumption is removed, it is obtained

$$\begin{aligned}
\langle u_T, y_T \rangle_{L^2(\Omega)} + \langle \boldsymbol{u}_E, \boldsymbol{y}_E \rangle_{L^2(\Omega, \mathbb{R}^3)} &= - \int_\Omega \{ (\mathcal{C}_\beta : \mathrm{Grad}\, \boldsymbol{e}_v) e_T + \mathrm{Div}(\mathcal{C}_\beta e_T) \cdot \boldsymbol{e}_v \} \, \mathrm{d}\Omega, \\
&= - \int_\Omega \mathrm{div}(e_T \mathcal{C}_\beta \cdot \boldsymbol{e}_v) \, \mathrm{d}\Omega, \\
&= - \int_{\partial\Omega} (e_T \mathcal{C}_\beta \cdot \boldsymbol{n}) \cdot \boldsymbol{e}_v \, \mathrm{d}S.
\end{aligned} \tag{6.16}$$

Using the expression of $y_T, \boldsymbol{y}_E$, considering that $T_0$ is constant and applying Schwarz theorem for smooth function, the inputs are equal to

$$\boldsymbol{u}_E = \mathrm{Div}(\boldsymbol{\Sigma}_T), \qquad u_T = -\mathcal{C}_\beta : \mathrm{Grad}(\boldsymbol{v}) = -\mathcal{C}_\beta : \frac{\partial \varepsilon}{\partial t}.$$

912     The coupled thermoelastic problem can now be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon \\ \alpha_T \\ \boldsymbol{j}_Q \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \mathrm{Div} & \mathcal{A}_\beta & \boldsymbol{0} \\ \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\mathcal{A}_\beta^* & 0 & 0 & -\mathrm{div} \\ \boldsymbol{0} & \boldsymbol{0} & -\mathrm{grad} & -(T_0 k)^{-1} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \\ e_T \\ \boldsymbol{j}_Q \end{pmatrix}, \tag{6.17}$$

with total energy given by $H = H_E + H_T$. The power balance for each subsystem is given by

$$\dot{H}_E = \int_\Omega \boldsymbol{u}_E \cdot \boldsymbol{y}_E \, \mathrm{d}\Omega + \int_{\partial\Omega} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n}) \, \mathrm{d}S, \tag{6.18}$$

$$\dot{H}_T \leq \int_\Omega u_T \, y_T \, \mathrm{d}\Omega - \int_{\partial\Omega} \theta \, \boldsymbol{j}_Q \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{6.19}$$

The overall power balance is easily computed considering Eqs. (6.18) (6.19) and (6.16)

$$\dot{H} = \dot{H}_E + \dot{H}_T \leq \int_{\partial\Omega} \{[\boldsymbol{E}_\varepsilon - e_T \mathcal{C}_\beta] \cdot \boldsymbol{n}\} \cdot \boldsymbol{e}_v \, \mathrm{d}S - \int_{\partial\Omega} \theta \, \boldsymbol{j}_Q \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{6.20}$$

This result is the same stated in [Car73], page 332. From the power balance the classical boundary conditions are retrieved. This allows defining appropriate boundary operators for the thermoelastic problem

$$\boldsymbol{u}_\partial = \underbrace{\begin{bmatrix} \boldsymbol{\gamma}_0^{\Gamma_D^E} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}_n^{\Gamma_N^E} & -\boldsymbol{\gamma}_n^{\Gamma_N^E}(\mathcal{C}_\beta \cdot) & \mathbf{0} \\ 0 & 0 & \gamma_0^{\Gamma_D^T} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\gamma}_n^{\Gamma_N^T} \end{bmatrix}}_{\mathcal{B}_\partial} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \\ e_T \\ \boldsymbol{j}_Q \end{pmatrix}, \quad \boldsymbol{y}_\partial = \underbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{\gamma}_n^{\Gamma_D^E} & -\boldsymbol{\gamma}_n^{\Gamma_D^E}(\mathcal{C}_\beta \cdot) & \mathbf{0} \\ \boldsymbol{\gamma}_0^{\Gamma_N^E} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \gamma_n^{\Gamma_D^T} \\ 0 & 0 & \gamma_0^{\Gamma_N^T} & 0 \end{bmatrix}}_{\mathcal{C}_\partial} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \\ e_T \\ \boldsymbol{j}_Q \end{pmatrix}.$$

$$\tag{6.21}$$

System (6.17) together with (6.21) is a pH system with boundary control and observation. Indeed, the classical thermoelastic problem can be modeled as two coupled systems, demonstrating the modularity of the pH paradigm.

## 6.2 Thermoelastic port-Hamiltonian bending

In this section, the thermoelastic bending of thin beam and plate structures is described as coupled interconnection pf pHs. Starting from classical thermoelastic models a suitable pH formulation can be obtained. This couples a mechanical system defined on a reduced domain (uni-dimensional for beams, bi-dimensional for plates), to a thermal domain defined in the three-dimensional space.

### 6.2.1 Thermoelastic Euler-Bernoulli beam

The model for the linear thermoelastic vibrations of an isotropic thin rod is detailed in [Cha62, LR00]. The domain of the beam is uni-dimensional $\Omega_E = \{0, L\}$, while the thermal domain is three-dimensional $\Omega_T = \{0, L\} \times S$, where $S$ is the set representing the beam cross section. The set $S$ is assumed to constant along the axis for simplicity. The ruling equations are

$$\begin{aligned} \rho A \frac{\partial^2 w}{\partial t^2} &= -EI \frac{\partial^4 w}{\partial x^4} - \beta E T_0 \frac{\partial^2}{\partial x^2} \int_S z\theta \, \mathrm{d}x \, \mathrm{d}y, & x \in \{0, L\} = \Omega_E, \\ \rho c_{\epsilon,B} T_0 \frac{\partial\theta}{\partial t} &= k T_0 \Delta\theta + \beta T_0 E z \frac{\partial^3 w}{\partial x^2 \partial t}, & (x, y, z) \in \Omega_E \times S = \Omega_T, \end{aligned} \tag{6.22}$$

where $w(x,t)$ is the vertical displacement of the beam $I = \int_S z^2 \, \mathrm{d}x \, \mathrm{d}y$ the second moment of area, $E$ the Young modulus and $A$ the cross section. The constant $c_{\epsilon,B}$ is due to the thermoelastic coupling (cf. [Cha62, LR00] for a detailed explanation). The other terms have

meaning than in Section §6.1. Since the normalized temperature $\theta(x, y, z, t)$ depends on all spatial coordinates, the symbol $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$ is the Laplacian in three dimensions. The physical constants are assumed to be constant for simplicity.

The coupling operator is defined as

$$\mathcal{A}_{\beta,B}(y_T) := -\beta E T_0 \partial_{xx} \left( \int_S z y_T \, \mathrm{d}x \, \mathrm{d}y \right). \tag{6.23}$$

To unveil an interconnection that is power with respect to a certain function, the formal adjoint of the coupling operator is needed.

**Proposition 6**

*Let $C_0^\infty(\Omega_T)$, $C_0^\infty(\Omega_E)$ be the space of smooth functions with compact support defined on $\Omega_T$ and $\Omega_E$ respectively. Given $y_T \in C_0^\infty(\Omega_T)$, $y_E \in C_0^\infty(\Omega_E)$ the formal adjoint of the coupling operator is*

$$\mathcal{A}_{\beta,B}^*(y_E) = -\beta E T_0 z \, \partial_{xx} y_E. \tag{6.24}$$

*Proof.* The formal adjoint is defined by the relation

$$\langle y_E, \mathcal{A}_{\beta,B} y_T \rangle_{L^2(\Omega_E)} = \left\langle \mathcal{A}_{\beta,B}^* y_E, \, y_T \right\rangle_{L^2(\Omega_T)}, \tag{6.25}$$

where for $u_E, y_E \in C_0^\infty(\Omega_E)$, $u_T, y_T \in C_0^\infty(\Omega_T)$

$$\langle u_E, y_E \rangle_{L^2(\Omega_E)} = \int_{\Omega_E} u_E y_E \, \mathrm{d}x, \qquad \langle u_T, y_T \rangle_{L^2(\Omega_T)} = \int_{\Omega_T} y_T y_T \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{6.26}$$

Using Def. (6.23) and the integration by parts, one finds

$$
\begin{aligned}
\langle y_E, \mathcal{A}_{\beta,B} y_T \rangle_{L^2(\Omega_E)} &= \int_{\Omega_E} y_E \mathcal{A}_{\beta,B} y_T \, \mathrm{d}x, \\
&= -\int_{\Omega_E} y_E \beta E T_0 \partial_{xx} \left( \int_S z y_T \, \mathrm{d}x \, \mathrm{d}y \right) \, \mathrm{d}x, \\
&= -\int_{\Omega_E} (\partial_{xx} y_E) \beta E T_0 \left( \int_S z y_T \, \mathrm{d}x \, \mathrm{d}y \right) \, \mathrm{d}x,
\end{aligned}
\tag{6.27}
$$

Since $\Omega_T = \Omega_E \times S$ and from the properties of multiple integrals, it is found

$$
\begin{aligned}
-\int_{\Omega_E} \partial_{xx}(y_E) \beta E T_0 \left( \int_S z \, y_T \, \mathrm{d}x \, \mathrm{d}y \right) \, \mathrm{d}x &= -\int_{\Omega_E} \int_S (\partial_{xx} y_E) \beta E T_0 z \, y_T \, \mathrm{d}x \, \mathrm{d}x \, \mathrm{d}y, \\
&= -\int_{\Omega_T} (\partial_{xx} y_E) \beta E T_0 z \, y_T \, \mathrm{d}x \, \mathrm{d}x \, \mathrm{d}y, \tag{6.28} \\
&= \left\langle \mathcal{A}_{\beta,B}^* y_E, \, y_T \right\rangle_{L^2(\Omega_T)}.
\end{aligned}
$$

This concludes the proof.                                                                 $\square$

Using Eqs. (6.23) and (6.24), System (6.22), is rewritten as

$$
\begin{aligned}
\rho A \frac{\partial^2 w}{\partial t^2} &= -EI \frac{\partial^4 w}{\partial x^4} + \mathcal{A}_{\beta,B}\theta, \\
\rho c_{\epsilon,B} T_0 \frac{\partial \theta}{\partial t} &= k T_0 \Delta\theta - \mathcal{A}_{\beta,B}^* \frac{\partial w}{\partial t}.
\end{aligned}
\tag{6.29}
$$

Consider the Hamiltonian functional

$$
H = H_E + H_T = \frac{1}{2}\int_{\Omega_E}\left\{\rho A\left(\frac{\partial w}{\partial t}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right\}\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_T}\rho c_{\epsilon,B} T_0 \theta^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{6.30}
$$

The energy variables are chosen to make the Hamiltonian functional quadratic

$$
\alpha_w = \rho A \partial_t w, \qquad \alpha_\kappa = \partial_{xx} w, \qquad \alpha_T = \rho c_{\epsilon,B} T_0 \theta. \tag{6.31}
$$

The corresponding co-energy variables evaluate to

$$
e_w := \frac{\delta H}{\delta \alpha_w} = \partial_t w, \qquad e_\kappa := \frac{\delta H}{\delta \alpha_\kappa} = EI\partial_{xx}w, \qquad e_T := \frac{\delta H}{\delta \alpha_T} = \theta. \tag{6.32}
$$

System (6.29) can now be rewritten as

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_\kappa \\ \alpha_T \\ \boldsymbol{j}_Q \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} & \mathcal{A}_{\beta,B} & 0 \\ \partial_{xx} & 0 & 0 & 0 \\ -\mathcal{A}_{\beta,B}^* & 0 & 0 & -\operatorname{div} \\ \mathbf{0} & \mathbf{0} & -\operatorname{grad} & -(kT_0)^{-1} \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \\ e_T \\ \boldsymbol{j}_Q \end{pmatrix}, \tag{6.33}
$$

This system is the equivalent of (6.17) for bending of beams. Hence, following the same reasoning, it can be obtained starting from each subsystem in pH form by means of an appropriate interconnection.

### 6.2.2 Thermoelastic Kirchhoff plate

For the bending of thin plate, several different models have been proposed [Cha62, Lag89, Sim99, Nor06]. Here, the Chadwick model [Cha62] is considered. The thin plate occupies the open connected set $\Omega_E \times \left\{-\frac{h}{2}, \frac{h}{2}\right\}$, where $h$ is the plate thickness. The system of equations describe the midplane vertical displacement and the evolution of the temperature in the 3D domain

$$
\begin{aligned}
\rho h \frac{\partial^2 w}{\partial t^2} &= -D_b \Delta_{2D}^2 w - \frac{\beta T_0 E}{1-\nu}\Delta_{2D}\left(\int_{-h/2}^{h/2} z\theta\,\mathrm{d}z\right), & (x,y) \in \Omega_E, \\
\rho c_{\epsilon,P} T_0 \frac{\partial \theta}{\partial t} &= -kT_0 \Delta_{3D} + \frac{\beta T_0 E z}{1-\nu}\Delta_{2D}\left(\frac{\partial w}{\partial t}\right), & (x,y,z) \in \Omega_E \times \left\{-\frac{h}{2}, \frac{h}{2}\right\} = \Omega_T,
\end{aligned}
\tag{6.34}
$$

where $w(x,y,t)$ is the vertical deflection, $D_b = \frac{Eh^3}{12(1-\nu^2)}$ the bending rigidity (cf. Eq. (5.11)), $\nu$ the Poisson modulus and $c_{\epsilon,P}$ a constant (depending on the heat capacity at constant strain

and other coupling parameters, cf. [Cha62]). Symbols $\Delta_{2D} = \partial_{xx} + \partial_{yy}$, $\Delta_{3D} = \partial_{xx} + \partial_{yy} + \partial_{zz}$ are the two- and three-dimensional Laplacian.

The coupling operator is here defined as

$$\mathcal{A}_{\beta,P}(y_T) := -\frac{\beta T_0 E}{1-\nu} \Delta_{2D} \left( \int_{-h/2}^{h/2} z y_T \, \mathrm{d}z \right). \tag{6.35}$$

Analouglsy with respect to the Euler-Bernoulli beam its formal adjoint is sought for.

**Proposition 7**

*Let $C_0^\infty(\Omega_T)$, $C_0^\infty(\Omega_E)$ be the space of smooth functions with compact support defined on $\Omega_T$ and $\Omega_E$ respectively. Given $y_T \in C_0^\infty(\Omega_T)$, $y_E \in C_0^\infty(\Omega_E)$ the formal adjoint of the coupling operator is*

$$\mathcal{A}_{\beta,B}^*(y_E) = -\frac{\beta T_0 E z}{1-\nu} \Delta_{2D} y_E. \tag{6.36}$$

*Proof.* The proof is completely identical to Prop. 6.                                            $\square$

System 6.34 is rewritten as

$$\begin{aligned}
\rho h \frac{\partial^2 w}{\partial t^2} &= -D_b \Delta_{2D}^2 w + \mathcal{A}_{\beta,P}\theta, \\
\rho c_{\epsilon,P} T_0 \frac{\partial \theta}{\partial t} &= -k T_0 \Delta_{3D}\theta - \mathcal{A}_{\beta,P}^*(\frac{\partial w}{\partial t}),
\end{aligned} \tag{6.37}$$

The Hamiltonian functional equals

$$\begin{aligned}
H = H_E + H_T =& \frac{1}{2} \int_{\Omega_E} \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + (\boldsymbol{\mathcal{D}}_b \mathrm{Hess}_{2D} w) : \mathrm{Hess}_{2D} w \right\} \, \mathrm{d}x \, \mathrm{d}y \\
&+ \frac{1}{2} \int_{\Omega_T} \rho c_{\epsilon,P} T_0 \theta^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,
\end{aligned} \tag{6.38}$$

where $\mathrm{Hess}_{2D}$ is the Hessian in two dimensions and $\boldsymbol{\mathcal{D}}_b$ was defined in (5.11) (cf. Sec. §5.1.1). The energy and co-energy variables are

$$\begin{aligned}
\alpha_w &= \rho h \partial_t w, & \boldsymbol{A}_\kappa &= \mathrm{Hess}_{2D} w, & \alpha_T &= \rho c_{\epsilon,P} T_0 \theta, \\
e_w &= \partial_t w, & \boldsymbol{E}_\kappa &= \boldsymbol{\mathcal{D}}_b \mathrm{Hess}_{2D} w, & e_T &= \theta.
\end{aligned} \tag{6.39}$$

System (6.37) is rewritten as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \boldsymbol{A}_\kappa \\ \alpha_T \\ \boldsymbol{j}_Q \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div}\,\mathrm{Div}_{2D} & \mathcal{A}_{\beta,P} & 0 \\ \mathrm{Hess}_{2D} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_{\beta,P}^* & 0 & 0 & -\mathrm{div}_{3D} \\ \mathbf{0} & \mathbf{0} & -\mathrm{grad}_{3D} & -(kT_0)^{-1} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \\ e_T \\ \boldsymbol{j}_Q \end{pmatrix}, \tag{6.40}$$

The subscript $2D$, $3D$ refers to two- and three-dimensional operators respectively. The final system reproduces the same structured coupling already observed for (6.17), (6.33).

**Remark 7**

*The thermoelastic bending can be reduced to two problems defined on the same domain (cf. [HZ97] for beams and [AL00] for plates) by introducing the following approximation of the temperature field*

$$\theta(x, y, z) = \theta_0 + z\theta_1, \tag{6.41}$$

*where $\theta_0 = \theta_0(x)$, $\theta_1 = \theta_1(x)$ for beams and $\theta_0 = \theta_0(x, y)$, $\theta_1 = \theta_1(x, y)$ for plates. However, this introducing a strong simplication as the thermal phenomena typically occur in the whole three-dimensional space.*

## 6.3   Conclusion

In this chapter, it was shown classical linear thermoelastic problem are equivalent to two coupled port-Hamiltonian systems. This is especially interesting for the simulation of thermoelastic phenomena: each subsystem can be discretized separately and then coupled to the other using the discretized coupling operator. This allows to track easily how the energy flows within the two physics.

# Part III

# Finite element structure preserving discretization

# Partitioned finite element method

> Every truth is simple... is that not doubly a lie?
>
> *Twilight of the Idols*
> *Friedrich Nietzsche*

## Contents

Iscretization is the process of transferring continuous models into discrete counterparts. The discrete model should be faithful to the continuous one. To this aim, it is usually essential that the main properties of the continuous system are preserved at the discrete level. An algorithm that is capable of conserving properties at the discrete level is called structure-preserving [CMKO11]. In this chapter, a method to spatially discretize infinite-dimensional pHs into finite-dimensional ones in a structure preserving manner is illustrated.

## 7.1 Discretization under uniform boundary condition

A discrete version of a infinite-dimensional pH system is meant to preserve the underlying properties related to power continuity. To achieve this purpose, the discretization procedure consists of two steps [KML18]:

- Finite-dimensional approximation of the Stokes-Dirac structure, i.e. the formally skew symmetric differential operator that defines the structure. The duality of the power

variables has to be mapped onto the finite approximation. The subspace of the discrete variables will be represented by a Dirac structure.

- The Hamiltonian requires as well a suitable discretization, which gives rise to a discrete Hamiltonian.

A structure-preserving discretization is able to construct an equivalent pH system that possess the structural properties of the original model:

| Infinite dimensional pH system | Structure-preserving discretization |
|---|---|
| PDE with distributed inputs: $$\frac{\partial \boldsymbol{\alpha}}{\partial t}(\boldsymbol{x},t) = \mathcal{J}\frac{\delta H}{\delta \boldsymbol{\alpha}} + \mathcal{B}\boldsymbol{u}_\Omega(\boldsymbol{x},t),$$ $$\boldsymbol{y}_\Omega(\boldsymbol{x},t) = \mathcal{B}^*\frac{\delta H}{\delta \boldsymbol{\alpha}}.$$ | Resulting ODE: $$\dot{\boldsymbol{\alpha}}_d = \mathbf{J}\,\nabla H_d + \mathbf{B}_\Omega \mathbf{u}_\Omega + \mathbf{B}_\partial \mathbf{u}_\partial,$$ $$\mathbf{y}_\Omega = \mathbf{B}_\Omega^\top \nabla H_d,$$ $$\mathbf{y}_\partial = \mathbf{B}_\partial^\top \nabla H_d.$$ |
| Boundary conditions: $$\boldsymbol{u}_\partial = \mathcal{B}_\partial\frac{\delta H}{\delta \boldsymbol{\alpha}}, \quad \boldsymbol{y}_\partial = \mathcal{C}_\partial\frac{\delta H}{\delta \boldsymbol{\alpha}}.$$ | Discretized Hamiltonian: $$H_d := H(\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_d).$$ |
| Power balance (Stokes Theorem): $$\dot{H} = \int_{\partial\Omega} \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial \,\mathrm{d}S + \int_\Omega \boldsymbol{u}_\Omega \cdot \boldsymbol{y}_\Omega \,\mathrm{d}\Omega.$$ | Power balance: $$\dot{H} = \mathbf{u}_\partial^\top \mathbf{y}_\partial + \mathbf{u}_\Omega^\top \mathbf{y}_\Omega.$$ |

In this thesis the Partitioned Finite Element Method (PFEM), originally presented in [CRML18, CRML19], is chosen to obtain discretized models of dpHs. This procedure boils down to three simple steps

1. The system is written in weak form;

2. An integration by parts is applied to highlight the appropriate boundary control;

3. A Galerkin method is employed to obtain a finite-dimensional system. For the approximation basis the finite element method is here employed but spectral methods can be used as well.

Once the system has been put into weak form, a subset of the equations is integrated by parts, so that boundary variables are naturally included into the formulation and appear as control inputs, the collocated outputs being defined accordingly. The discretization of energy and co-energy variables (and the associated test functions) leads directly to a full rank representation for the finite-dimensional pH system. This approach makes possible the usage of FEM software, like FEniCS [LMW+12], or Firedrake [RHM+17]. The procedure is universal, as it relies on a general integration by parts formula that characterizes multi-dimensional pHs. This is why the methodology is illustrated in all its generality and then detailed for

some particular examples.

This methodology is easily applicable under a uniform causality assumption. The case of mixed boundary conditions requires additional care and will be treated in the subsequent Section §7.2.

### 7.1.1 General procedure

Given an open connected set $\Omega \in \mathbb{R}^d$, $d \in \{1, 2, 3\}$, consider a generic pH system defined on $\Omega$

$$\partial_t \boldsymbol{\alpha} = \mathcal{J} \boldsymbol{e}, \qquad \boldsymbol{\alpha} \in L^2(\Omega, \mathbb{F}), \quad \mathcal{J} : L^2(\Omega, \mathbb{F}) \to L^2(\Omega, \mathbb{F}) | \, \mathcal{J} = -\mathcal{J}^*, \tag{7.1a}$$

$$\boldsymbol{e} := \delta_{\boldsymbol{\alpha}} H, \qquad \boldsymbol{e} \in H^{\mathcal{J}} := \left\{ \boldsymbol{e} \in L^2(\Omega, \mathbb{F}) | \, \mathcal{J} \boldsymbol{e} \in L^2(\Omega, \mathbb{F}) \right\}, \tag{7.1b}$$

$$\boldsymbol{u}_\partial = \mathcal{B}_\partial \boldsymbol{e}, \qquad \boldsymbol{u}_\partial \in \mathbb{R}^m, \tag{7.1c}$$

$$\boldsymbol{y}_\partial = \mathcal{C}_\partial \boldsymbol{e}, \qquad \boldsymbol{y}_\partial \in \mathbb{R}^m. \tag{7.1d}$$

The operator $\mathcal{J} : L^2(\Omega, \mathbb{F}) \to L^2(\Omega, \mathbb{F})$ is a differential, formally skew adjoint operator $\mathcal{J} = -\mathcal{J}^*$ over the space $L^2(\Omega, \mathbb{F})$. The set $\mathbb{F}$ is an appropriate Cartesian product of either scalar, vectorial or tensorial quantities. Its precise definition depends on the example upon consideration. For scalars $(a, b) \in L^2(\Omega)$, vectors $(\boldsymbol{a}, \boldsymbol{b}) \in L^2(\Omega, \mathbb{R}^d)$ and tensors $(\boldsymbol{A}, \boldsymbol{B}) \in L^2(\Omega, \mathbb{R}^{d \times d})$ the $L^2$ inner product is given by

$$\langle a, b \rangle_{L^2(\Omega)} = \int_\Omega ab \, \mathrm{d}\Omega, \qquad \langle \boldsymbol{a}, \boldsymbol{b} \rangle_{L^2(\Omega, \mathbb{R}^d)} = \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}\Omega, \qquad \langle \boldsymbol{A}, \boldsymbol{B} \rangle_{L^2(\Omega, \mathbb{R}^{d \times d})} = \int_\Omega \boldsymbol{A} : \boldsymbol{B} \, \mathrm{d}\Omega. \tag{7.2}$$

For scalars $a_\partial, b_\partial \in L^2(\partial\Omega)$ and vectors $\boldsymbol{a}_\partial, \boldsymbol{b}_\partial \in L^2(\partial\Omega, \mathbb{R}^m)$ defined on the boundary the inner product is defined as

$$\langle a_\partial, b_\partial \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} a_\partial b_\partial \, \mathrm{d}S, \qquad \langle \boldsymbol{a}_\partial, \boldsymbol{b}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} = \int_{\partial\Omega} \boldsymbol{a}_\partial \cdot \boldsymbol{b}_\partial \, \mathrm{d}S. \tag{7.3}$$

The Hamiltonian functional of Eq. (7.1b) is allowed to be non linear in the energy variables

$$H = \int_\Omega \mathcal{H}(\boldsymbol{\alpha}) \, \mathrm{d}\Omega,$$

where $\mathcal{H}(\boldsymbol{\alpha}) : L^2(\Omega, \mathbb{F}) \to \mathbb{R}$ is a non linear function.

To applied this methodology the non linearities are restricted to the Hamiltonian and a uniform causality condition is supposed to characterize the system. It is required as well that the system admits a partition of the variables. This requirement is always encounter in the following examples. These hypotheses are resumed in the following assumptions.

**Assumption 2** (Partitioning of the system)

1072 *Consider system* (7.1a). *It is assumed that the Hilbert space* $L^2(\Omega, \mathbb{F}) := L^2(\Omega, \mathbb{F})$ *admits the*
1073 *splitting* $L^2(\Omega, \mathbb{F}) = L^2(\Omega, \mathbb{A}) \times L^2(\Omega, \mathbb{B})$ *This means that* $\mathbb{F} = \mathbb{A} \times \mathbb{B}$.

1074

1075 *The operator $\mathcal{J}$ is assumed to be skew-symmetric (or formally skew-adjoint) on $L^2(\Omega, \mathbb{F})$*
1076 *and linear:*

$$\mathcal{J} = \mathcal{J}_a + \mathcal{J}_d, \tag{7.4}$$

1077 *where $\mathcal{J}_a$ is the algebraic contribution (a skew-symmetric matrix) and $\mathcal{J}_d$ the differential*
1078 *contribution. The algebraic part is assumed to take the form*

$$\mathcal{J}_a = \begin{bmatrix} 0 & -\boldsymbol{L}^\top \\ \boldsymbol{L} & 0 \end{bmatrix}, \qquad \begin{array}{l} \boldsymbol{L}^\top : L^2(\Omega, \mathbb{B}) \to L^2(\Omega, \mathbb{A}), \\ \boldsymbol{L} \ : L^2(\Omega, \mathbb{A}) \to L^2(\Omega, \mathbb{B}), \end{array} \tag{7.5}$$

1079 *where $\boldsymbol{L}$ is a bounded operator. Analogously, the linear differential operator $\mathcal{J}_d$ is assumed to*
1080 *be of the form*

$$\mathcal{J}_d = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix}, \qquad \begin{array}{l} \mathcal{L}^* : L^2(\Omega, \mathbb{B}) \to L^2(\Omega, \mathbb{A}), \\ \mathcal{L} \ : L^2(\Omega, \mathbb{A}) \to L^2(\Omega, \mathbb{B}), \end{array} \tag{7.6}$$

1081 *where $\mathcal{L}^*$ denotes the formal adjoint of the linear differential operator $\mathcal{L}$. The operator $\mathcal{L}$ is*
1082 *unbounded and can be either a first or a second order differential operator (in the latter case*
1083 *it can be expressed as $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2$). Given the splitting $L^2(\Omega, \mathbb{A}) \times L^2(\Omega, \mathbb{B}) = L^2(\Omega, \mathbb{F})$ the*
1084 *Hilbert space $H^{\mathcal{J}}$ can be split as well as*

$$H^{\mathcal{J}} = H^{\mathcal{L}} \times H^{-\mathcal{L}^*}, \qquad \begin{array}{l} H^{\mathcal{L}} := \left\{ \boldsymbol{u}_1 \in L^2(\Omega, \mathbb{A}) |\ \mathcal{L}\boldsymbol{u}_1 \in L^2(\Omega, \mathbb{B}) \right\}, \\ H^{-\mathcal{L}^*} := \left\{ \boldsymbol{u}_2 \in L^2(\Omega, \mathbb{B}) |\ -\mathcal{L}^*\boldsymbol{u}_2 \in L^2(\Omega, \mathbb{A}) \right\} \end{array} \tag{7.7}$$

1085 The boundary operators are then supposed to fulfill the following assumption, that guar-
1086 antees a uniform causality condition.

1087 **Assumption 3** (General integration by parts formula)
1088 *Assume that there exist two boundary operators $\mathcal{N}_{\partial,1}$, $\mathcal{N}_{\partial,2}$ such that for $(\boldsymbol{u}_1, \boldsymbol{u}_2) \in H^{\mathcal{L}} \times H^{-\mathcal{L}^*}$*
1089 *a general integration by parts formula holds*

$$\langle \boldsymbol{u}_2, \mathcal{L}\,\boldsymbol{u}_1 \rangle_{L^2(\Omega, \mathbb{B})} - \langle \mathcal{L}^*\,\boldsymbol{u}_2, \boldsymbol{u}_1 \rangle_{L^2(\Omega, \mathbb{A})} = \langle \mathcal{N}_{\partial,1}\boldsymbol{u}_1, \mathcal{N}_{\partial,2}\boldsymbol{u}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \tag{7.8}$$

1090 *The boundary operators $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ of Eqs. (7.1c), (7.1d), are then assumed to verify, in an*
1091 *exclusive manner, either*

$$\mathcal{B}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}, \quad \mathcal{C}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix}, \tag{7.9}$$

1092 *or*

$$\mathcal{B}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix}, \quad \mathcal{C}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}. \tag{7.10}$$

1093 **Remark 8** (Duality pairing for rigged Hilbert spaces)
1094 *The integration by part formula establishes a duality pairing between Sobolev spaces. This*

1095  *duality pairing is then compatible with an $L^2$ inner product in presence of a rigged Hilbert space*
1096  *(Gelfand triple). Without entering into technical details, we shall always use this equivalence*
1097  *of representation. Therefore, the boundary integrals are expressed as $L^2$ inner product over*
1098  *the boundary.*

Thanks to Assumption 2, System (7.1) is rewritten as

$$\partial_t \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\boldsymbol{L}^\top - \mathcal{L}^* \\ \boldsymbol{L} + \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{matrix} \boldsymbol{\alpha}_1 \in L^2(\Omega, \mathbb{A}), \\ \boldsymbol{\alpha}_2 \in L^2(\Omega, \mathbb{B}), \end{matrix} \tag{7.11a}$$

$$\begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} := \begin{pmatrix} \delta_{\boldsymbol{\alpha}_1} H \\ \delta_{\boldsymbol{\alpha}_2} H \end{pmatrix}, \qquad \begin{matrix} \boldsymbol{e}_1 \in H^\mathcal{L}, \\ \boldsymbol{e}_2 \in H^{-\mathcal{L}^*}. \end{matrix} \tag{7.11b}$$

1099  In light of Assumption 3, if Eq. (7.9) holds the boundary variables are given by

$$\boldsymbol{u}_\partial = \mathcal{N}_2 \boldsymbol{e}_2, \qquad \boldsymbol{y}_\partial = \mathcal{N}_1 \boldsymbol{e}_1, \qquad \boldsymbol{u}_\partial, \, \boldsymbol{y}_\partial \in \mathbb{R}^m. \tag{7.12}$$

1100  Otherwise, if Eq. (7.10) applies, then

$$\boldsymbol{u}_\partial = \mathcal{N}_1 \boldsymbol{e}_1, \qquad \boldsymbol{y}_\partial = \mathcal{N}_2 \boldsymbol{e}_2, \qquad \boldsymbol{u}_\partial, \, \boldsymbol{y}_\partial \in \mathbb{R}^m. \tag{7.13}$$

1101  In both cases, the power balance reads

$$\begin{aligned} \dot{H} &= \langle \boldsymbol{e}_1, \, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \boldsymbol{e}_2, \, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= \langle \boldsymbol{e}_1, \, -\mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \boldsymbol{e}_2, \, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= \langle \mathcal{N}_{\partial,1} \boldsymbol{e}_1, \, \mathcal{N}_{\partial,2} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &= \langle \boldsymbol{y}_\partial, \, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \tag{7.14}$$

1102  We are now in a position to illustrate the methodology.

1103  **Step 1**  First consider the weak form of system (7.11a), obtained by taking the $L^2$ inner
1104  product introducing an appropriate test function $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \mathbb{A} \times \mathbb{B} = \mathbb{F}$ and integrating
1105  over the domain $\Omega$

$$\begin{aligned} \langle \boldsymbol{v}_1, \, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \boldsymbol{v}_1, \, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \boldsymbol{v}_1, \, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}, \\ \langle \boldsymbol{v}_2, \, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \, \boldsymbol{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \boldsymbol{v}_2, \, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}. \end{aligned} \tag{7.15}$$

1106  To obtain a closed system, the constitutive law (7.11b) and the output variables (7.1d)
1107  are put in weak form

$$\begin{aligned} \langle \boldsymbol{v}_1, \, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= \langle \boldsymbol{v}_1, \, \delta_{\boldsymbol{\alpha}_1} H \rangle_{L^2(\Omega, \mathbb{A})}, \\ \langle \boldsymbol{v}_2, \, \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \, \delta_{\boldsymbol{\alpha}_2} H \rangle_{L^2(\Omega, \mathbb{B})}, \\ \langle \boldsymbol{v}_\partial, \, \boldsymbol{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \boldsymbol{v}_\partial, \, \mathcal{C}_\partial \boldsymbol{e} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \end{aligned} \tag{7.16}$$

1108  where the test function $\boldsymbol{v}_\partial \in L^2(\partial\Omega, \mathbb{R}^m)$ is defined on the boundary $\partial\Omega$ and $\mathcal{C}_\partial$ is defined
1109  either by Eq. (7.9) or (7.10).

1110  **Step 2**   Next the integration by part has to be carried out. The choice is dictated by the
1111  boundary control to be imposed on the system. Consider again Eq. (7.15). The integration
1112  by parts can be carried out either on term $-\langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})}$, or on term $\langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})}$.
1113  Depending on which line undergoes the integration by parts (this is why the name Partitioned
1114  Finite Element method), two structure preserving weak forms are obtained. These differ by
1115  the boundary causality imposed to the system.

1116  **Integration by parts of the term** $-\langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})}$   In this case case, using Eq.
1117  (7.8), it is obtained

$$-\langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})} = -\langle \mathcal{L}\boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \mathcal{N}_{\partial,1}\boldsymbol{v}_1, \mathcal{N}_{\partial,2}\boldsymbol{e}_2 \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}. \tag{7.17}$$

1118  Then the weak form of the system dynamics reads

$$\begin{aligned}
\langle \boldsymbol{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega,\mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})} - \langle \mathcal{L}\boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \mathcal{N}_{\partial,1}\boldsymbol{v}_1, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}, \\
\langle \boldsymbol{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega,\mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})},
\end{aligned} \tag{7.18}$$

1119  The following proposition is crucial as the lossless character of the infinite-dimensional system
1120  (due to the formally skew-adjoint operator) translates into an equivalent property for the
1121  corresponding bilinear form in the weak form.

**Proposition 8**
*Given the Hilbert space $H_2^{\mathcal{L}} := H^{\mathcal{L}} \times L^2(\Omega, \mathbb{B})$ and variables $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in H_2^{\mathcal{L}}$, $\boldsymbol{e} = (\boldsymbol{e}_1, \boldsymbol{e}_2) \in H_2^{\mathcal{L}}$, the bilinear form*

$$\begin{aligned}
j_{\mathcal{L}} : H_2^{\mathcal{L}} \times H_2^{\mathcal{L}} &\longrightarrow \mathbb{R}, \\
(\boldsymbol{v}, \boldsymbol{e}) &\longrightarrow -\langle \mathcal{L}\boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})}
\end{aligned}$$

1122  *is skew-symmetric.*

*Proof.* The proof is obtained by the following computation

$$\begin{aligned}
j_{\mathcal{L}}(\boldsymbol{v}, \boldsymbol{e}) &= -\langle \mathcal{L}\boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})}, \\
&= -\left( -\langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \mathcal{L}\boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} \right), \\
&= -\left( -\langle \mathcal{L}\boldsymbol{e}_1, \boldsymbol{v}_2 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \boldsymbol{e}_2, \mathcal{L}\boldsymbol{v}_1 \rangle_{L^2(\Omega,\mathbb{B})} \right) = -j_{\mathcal{L}}(\boldsymbol{e}, \boldsymbol{v}).
\end{aligned}$$

1123                                                                                                          $\square$

1124  Now assume that the system satisfies the boundary causality condition 7.12. Then, this

choice of the integration by parts lead to the following weak formulation

$$
\begin{aligned}
\langle \boldsymbol{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathcal{L} \boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \boldsymbol{v}_1, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
\langle \boldsymbol{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\
\langle \boldsymbol{v}_1, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= \langle \boldsymbol{v}_1, \delta_{\boldsymbol{\alpha}_1} H \rangle_{L^2(\Omega, \mathbb{A})}, \\
\langle \boldsymbol{v}_2, \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \delta_{\boldsymbol{\alpha}_2} H \rangle_{L^2(\Omega, \mathbb{B})}, \\
\langle \boldsymbol{v}_\partial, \boldsymbol{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \boldsymbol{v}_\partial, \mathcal{N}_{\partial,1} \boldsymbol{e}_1 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}.
\end{aligned}
\tag{7.19}
$$

**Integration by parts of the term** $\langle \boldsymbol{v}_2, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}$  Using Eq. (7.8), it is obtained

$$
\langle \boldsymbol{v}_2, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} = \langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2, \mathcal{N}_{\partial,1} \boldsymbol{e}_1 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}.
\tag{7.20}
$$

Then the weak form of the system dynamics reads

$$
\begin{aligned}
\langle \boldsymbol{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}, \\
\langle \boldsymbol{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)},
\end{aligned}
\tag{7.21}
$$

Again the bilinear form arising from the formally skew-adjoint operator is skew-symmetric.

**Proposition 9**

*Given the Hilbert space $H_1^{-\mathcal{L}^*} = L^2(\Omega, \mathbb{A}) \times H^{-\mathcal{L}^*}$ and variables $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in H_1^{-\mathcal{L}^*}$, $\boldsymbol{e} = (\boldsymbol{e}_1, \boldsymbol{e}_2) \in H_1^{-\mathcal{L}^*}$, the bilinear form*

$$
\begin{aligned}
j_{-\mathcal{L}^*} : H_1^{-\mathcal{L}^*} \times H_1^{-\mathcal{L}^*} &\longrightarrow \mathbb{R}, \\
(\boldsymbol{v}, \boldsymbol{e}) &\longrightarrow -\langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})}
\end{aligned}
$$

*is skew-symmetric.*

*Proof.* The proof follows from the computation

$$
\begin{aligned}
j_{-\mathcal{L}^*}(\boldsymbol{v}, \boldsymbol{e}) &= -\langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})}, \\
&= -\left( -\langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} \right), \\
&= -\left( -\langle \boldsymbol{e}_1, \mathcal{L}^* \boldsymbol{v}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{L}^* \boldsymbol{e}_2, \boldsymbol{v}_1 \rangle_{L^2(\Omega, \mathbb{A})} \right) = -j_{-\mathcal{L}^*}(\boldsymbol{e}, \boldsymbol{v}).
\end{aligned}
$$

$\square$

Now assume that the system satisfies the boundary causality condition (7.13). Then, the

1132  final weak formulation reads

$$\langle \boldsymbol{v}_1,\, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega,\mathbb{A})} = -\langle \boldsymbol{v}_1,\, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})} - \langle \boldsymbol{v}_1,\, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})} ,$$

$$\langle \boldsymbol{v}_2,\, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega,\mathbb{B})} = \langle \boldsymbol{v}_2,\, \boldsymbol{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \mathcal{L}^* \boldsymbol{v}_2,\, \boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{A})} + \langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2,\, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} ,$$

$$\langle \boldsymbol{v}_1,\, \boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{A})} = \langle \boldsymbol{v}_1,\, \delta_{\boldsymbol{\alpha}_1} H \rangle_{L^2(\Omega,\mathbb{A})} , \tag{7.22}$$

$$\langle \boldsymbol{v}_2,\, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} = \langle \boldsymbol{v}_2,\, \delta_{\boldsymbol{\alpha}_2} H \rangle_{L^2(\Omega,\mathbb{B})} ,$$

$$\langle \boldsymbol{v}_\partial,\, \boldsymbol{y}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} = \langle \boldsymbol{v}_\partial,\, \mathcal{N}_{\partial,2} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} .$$

1133  **Galerkin discretization**    To conclude the illustration of this methodology, a Galerkin dis-
1134  cretization is introduced. This means that test, energy and co-energy functions are discretized
1135  using the same basis. Furthermore the boundary variables are discretized as well using bases
1136  defined over the boundary

$$\boldsymbol{v}_1 \approx \sum_{i=1}^{n_1} \boldsymbol{\phi}_1^i(\boldsymbol{x}) v_1^i, \qquad \boldsymbol{\alpha}_1 \approx \sum_{i=1}^{n_1} \boldsymbol{\phi}_1^i(\boldsymbol{x}) \alpha_1^i(t), \qquad \boldsymbol{e}_1 \approx \sum_{i=1}^{n_1} \boldsymbol{\phi}_1^i(\boldsymbol{x}) e_1^i(t), \quad \boldsymbol{x} \in \Omega,$$

$$\boldsymbol{v}_2 \approx \sum_{i=1}^{n_2} \boldsymbol{\phi}_2^i(\boldsymbol{x}) v_2^i, \qquad \boldsymbol{\alpha}_2 \approx \sum_{i=1}^{n_2} \boldsymbol{\phi}_2^i(\boldsymbol{x}) \alpha_2^i(t), \qquad \boldsymbol{e}_2 \approx \sum_{i=1}^{n_2} \boldsymbol{\phi}_2^i(\boldsymbol{x}) e_2^i(t), \quad \boldsymbol{x} \in \Omega, \tag{7.23}$$

$$\boldsymbol{v}_\partial \approx \sum_{i=1}^{n_\partial} \boldsymbol{\phi}_\partial^i(\boldsymbol{s}) v_\partial^i, \qquad \boldsymbol{u}_\partial \approx \sum_{i=1}^{n_\partial} \boldsymbol{\phi}_\partial^i(\boldsymbol{s}) u_\partial^i(t), \qquad \boldsymbol{y}_\partial \approx \sum_{i=1}^{n_\partial} \boldsymbol{\phi}_\partial^i(\boldsymbol{s}) y_\partial^i(t), \quad \boldsymbol{s} \in \partial\Omega,$$

1137  where $\boldsymbol{\phi}_1^i \in \mathbb{A}$, $\boldsymbol{\phi}_2^i \in \mathbb{B}$, $\boldsymbol{\phi}_\partial^i \in \mathbb{R}^m$.

1138  **Discretization of the weak form** (7.19)    Plugging the approximation into the weak
1139  form (7.19) and consider that the resulting equation holds $\forall v_1^i,\, v_2^j,\, v_\partial^k$ ($i \in \{1, n_1\}$, $j \in$
1140  $\{1, n_2\}$, $k \in \{1, n_\partial\}$), the finite dimensional system is obtained

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,$$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{bmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{bmatrix}, \tag{7.24}$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

1141  Vectors $\boldsymbol{\alpha}_{d,1}$, $\boldsymbol{\alpha}_{d,2}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{u}_\partial$, $\mathbf{y}_\partial$ are given by the column-wise concatenation of their respec-
1142  tive degrees of freedom. The matrices are defined as follows

$$M_1^{ij} = \left\langle \boldsymbol{\phi}_1^i,\, \boldsymbol{\phi}_1^j \right\rangle_{L^2(\Omega,\mathbb{A})} , \qquad D_0^{mi} = \left\langle \boldsymbol{\phi}_2^m,\, \boldsymbol{L}\boldsymbol{\phi}_1^i \right\rangle_{L^2(\Omega,\mathbb{B})} , \qquad B_1^{ik} = \left\langle \mathcal{N}_{\partial,1} \boldsymbol{\phi}_1^i,\, \boldsymbol{\phi}_\partial^k \right\rangle_{L^2(\partial\Omega,\mathbb{R}^m)} ,$$

$$M_2^{mn} = \left\langle \boldsymbol{\phi}_2^m,\, \boldsymbol{\phi}_2^n \right\rangle_{L^2(\Omega,\mathbb{B})} , \qquad D_{\mathcal{L}}^{mi} = \left\langle \boldsymbol{\phi}_2^m,\, \mathcal{L}\boldsymbol{\phi}_1^i \right\rangle_{L^2(\Omega,\mathbb{B})} , \qquad M_\partial^{lk} = \left\langle \boldsymbol{\phi}_\partial^l,\, \boldsymbol{\phi}_\delta^k \right\rangle_{L^2(\partial\Omega,\mathbb{R}^m)} , \tag{7.25}$$

where $i, j \in \{1, n_1\}$, $m, n \in \{1, n_2\}$, $l, k \in \{1, n_\partial\}$. Introducing the definitions

$$\delta_{\boldsymbol{\alpha}_{d,1}} H_d := \delta_{\boldsymbol{\alpha}_1} H \left( \boldsymbol{\alpha}_1 = \sum_{i=1}^{n_1} \phi_1^i \alpha_1^i, \ \boldsymbol{\alpha}_2 = \sum_{i=1}^{n_1} \phi_2^i \alpha_2^i \right),$$

$$\delta_{\boldsymbol{\alpha}_{d,2}} H_d := \delta_{\boldsymbol{\alpha}_2} H \left( \boldsymbol{\alpha}_1 = \sum_{i=1}^{n_1} \phi_1^i \alpha_1^i, \ \boldsymbol{\alpha}_2 = \sum_{i=1}^{n_1} \phi_2^i \alpha_2^i \right),$$

the discretized gradient of the Hamiltonian read

$$\begin{aligned}
\partial_{\alpha_{d,1}^i} H_d(\boldsymbol{\alpha}_d) &= \left\langle \boldsymbol{\phi}_1^i, \ \delta_{\boldsymbol{\alpha}_{d,1}} H_d \right\rangle_{L^2(\Omega, \mathbb{A})}, & i &\in \{1, n_1\}, \\
\partial_{\alpha_{d,2}^j} H_d(\boldsymbol{\alpha}_d) &= \left\langle \boldsymbol{\phi}_2^j, \ \delta_{\boldsymbol{\alpha}_{d,2}} H_d \right\rangle_{L^2(\Omega, \mathbb{B})}, & j &\in \{1, n_2\}.
\end{aligned} \tag{7.26}$$

A pH system in canonical form is found observing that Sys. (7.24) is compactly rewritten as

$$\mathbf{M}\dot{\boldsymbol{\alpha}}_d = \mathbf{J}_{\mathcal{L}}\mathbf{e} + \mathbf{B}\mathbf{u}_\partial, \tag{7.27}$$

$$\mathbf{M}\mathbf{e} = \nabla H_d(\boldsymbol{\alpha}_d), \tag{7.28}$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \mathbf{B}^\top \mathbf{e}, \tag{7.29}$$

where $\boldsymbol{\alpha}_d = (\boldsymbol{\alpha}_{d,1}^\top \ \boldsymbol{\alpha}_{d,2}^\top)^\top$, $\mathbf{e} = (\mathbf{e}_1^\top \ \mathbf{e}_2^\top)^\top$, $\nabla H_d(\boldsymbol{\alpha}_d) = (\partial_{\boldsymbol{\alpha}_{d,1}}^\top H_d(\boldsymbol{\alpha}_d) \ \partial_{\boldsymbol{\alpha}_{d,2}}^\top H_d(\boldsymbol{\alpha}_d))^\top$ and

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix}, \qquad \mathbf{J}_{\mathcal{L}} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}. \tag{7.30}$$

Plugging (7.28) into (7.27), a pH system in canonical form is obtained

$$\begin{aligned}
\dot{\boldsymbol{\alpha}}_d &= \mathbf{J}\,\nabla H_d(\boldsymbol{\alpha}_d) + \mathbf{B}\,\mathbf{u}_\partial, & \text{where} & & \mathbf{J} &= \mathbf{M}^{-1}\mathbf{J}_{\mathcal{L}}\mathbf{M}^{-1}, \\
\widehat{\mathbf{y}}_\partial &= \mathbf{B}^\top \nabla H_d(\boldsymbol{\alpha}_d), & \text{where} & & \widehat{\mathbf{y}}_\partial &= \mathbf{M}_\partial \mathbf{y}_\partial.
\end{aligned} \tag{7.31}$$

The structure preserving character of the method is evident from the preservation at the discrete level of the power balance. The finite dimensional counterpart of the energy rate is given by

$$\begin{aligned}
\dot{H}_d &= \nabla^\top H_d(\boldsymbol{\alpha}_d)\dot{\boldsymbol{\alpha}}_d, \\
&= \nabla^\top H_d(\boldsymbol{\alpha}_d)\mathbf{J}\,\nabla H_d(\boldsymbol{\alpha}_d) + \nabla^\top H_d(\boldsymbol{\alpha}_d)\mathbf{B}\,\mathbf{u}_\partial, & \text{Skew-symmetry of } \mathbf{J} \\
&= \widehat{\mathbf{y}}_\partial^\top \mathbf{u}_\partial.
\end{aligned} \tag{7.32}$$

This result mimics its infinite dimensional equivalent (7.14).

**Discretization of the weak form** (7.22)  Plugging the approximation into the weak form (7.22) a finite dimensional system with a different causality is obtained

$$
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top + \mathbf{D}_{-\mathcal{L}^*} \\ \mathbf{D}_0 - \mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial,
$$
$$
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{pmatrix}, \tag{7.33}
$$
$$
\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.
$$

The differences with respect to formulation (7.24) reside in matrices $\mathbf{D}_{-\mathcal{L}^*}$, $\mathbf{B}_2$, whose definitions are

$$
D_{-\mathcal{L}^*}^{im} = \left\langle \boldsymbol{\phi}_1^i, -\mathcal{L}^* \boldsymbol{\phi}_2^m \right\rangle_{L^2(\Omega, \mathbb{A})}, \quad B_2^{mk} = \left\langle \mathcal{N}_{\partial,2} \boldsymbol{\phi}_2^m, \boldsymbol{\phi}_\partial^k \right\rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \tag{7.34}
$$

where $i \in \{1, n_1\}$, $m \in \{1, n_2\}$, $k \in \{1, n_\partial\}$. System (7.33) can be put in canonical form by replacing the co-energy variables by the discretized gradient.

**Example: the irrotational shallow water equations** Consider as an example the shallow water equations detailed in Sec. §3.2.3.3. The flow is assumed to be irrotational $(\nabla \times \boldsymbol{v} = 0)$. As a consequence the term $\boldsymbol{\mathcal{G}}$ in Eq. (3.36) vanishes. To fulfill Assumption 3, the incoming volumetric flow is known at the boundary, so that a uniform Neumann condition is imposed. This lead to the following boundary control system, defined on an open connected set $\Omega \subset \mathbb{R}^2$

$$
\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} &= -\begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_h \\ \boldsymbol{e}_v \end{pmatrix}, & \alpha_h &\in L^2(\Omega), \\
\begin{pmatrix} e_h \\ \boldsymbol{e}_v \end{pmatrix} &:= \begin{pmatrix} \delta_{\alpha_h} H \\ \delta_{\boldsymbol{\alpha}_v} H \end{pmatrix} = \begin{pmatrix} \frac{1}{2\rho} \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h, \\ \frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v. \end{pmatrix}, & \begin{aligned} \boldsymbol{\alpha}_v &\in L^2(\Omega, \mathbb{R}^2), \\ e_h &\in H^1(\Omega), \\ \boldsymbol{e}_v &\in H^{\mathrm{div}}(\Omega, \mathbb{R}^2), \end{aligned} \\
u_\partial &= -\boldsymbol{e}_v \cdot \boldsymbol{n}, & u_\partial &\in \mathbb{R}, \\
y_\partial &= e_h, & y_\partial &\in \mathbb{R},
\end{aligned} \tag{7.35}
$$

where the Hamiltonian is a non linear functional in the energy variables

$$
H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho} \alpha_h \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h^2 \right\} \, \mathrm{d}\Omega.
$$

The energy and co-energy variables are related to the physical variables (fluid height and velocity) through Eqs. (3.33), (3.35). In this case $\mathbb{A} = \mathbb{R}$, $\mathbb{B} = \mathbb{R}^2$ and $\mathcal{L} = \mathrm{grad}$, $-\mathcal{L}^* = \mathrm{div}$. This implies $H^{\mathcal{L}} = H^1(\Omega)$, $H^{-\mathcal{L}^*} = H^{\mathrm{div}}(\Omega, \mathbb{R}^2)$. As shown in (3.37), the energy rate equals

$$
\dot{H} = -\langle \boldsymbol{e}_v, \mathrm{grad}\, e_h \rangle_{L^2(\Omega, \mathbb{R}^2)} - \langle \mathrm{div}\, \boldsymbol{e}_v, e_h \rangle_{L^2(\Omega)} = \langle -\boldsymbol{e}_v \cdot \boldsymbol{n}, e_h \rangle_{L^2(\partial\Omega)}. \tag{7.36}
$$

The boundary operators are therefore given by

$$
\begin{aligned}
u_\partial &= \mathcal{N}_{\partial,2} \boldsymbol{e}_v = -\gamma_n \boldsymbol{e}_v = -\boldsymbol{e}_v \cdot \boldsymbol{n}|_{\partial\Omega}, \\
y_\partial &= \mathcal{N}_{\partial,1} e_h = \gamma_0 e_h = e_h|_{\partial\Omega}.
\end{aligned} \tag{7.37}
$$

1166   This system represents a particular example of the general formulation of the general frame-
1167   work (7.11), together with boundary conditions (7.12). To obtain a finite dimensional system,
1168   the test variables $v_h$, $\boldsymbol{v}_v$ are introduced and the integration by parts is performed on the div
1169   operator, leading to the weak form

$$
\begin{aligned}
\langle v_h,\, \partial_t \alpha_h \rangle_{L^2(\Omega)} &= \langle \operatorname{grad} v_h,\, \boldsymbol{e}_v \rangle_{L^2(\Omega,\mathbb{R}^2)} + \langle \gamma_0 v_h,\, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega)}, \\
\langle \boldsymbol{v}_v,\, \partial_t \boldsymbol{\alpha}_v \rangle_{L^2(\Omega,\mathbb{R}^2)} &= -\langle \boldsymbol{v}_v,\, \operatorname{grad} e_h \rangle_{L^2(\Omega,\mathbb{R}^2)}, \\
\langle v_h,\, e_h \rangle_{L^2(\Omega)} &= \left\langle v_h,\, \frac{1}{2\rho}\|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h \right\rangle_{L^2(\Omega)}, \\
\langle \boldsymbol{v}_v,\, \boldsymbol{e}_v \rangle_{L^2(\Omega,\mathbb{R}^2)} &= \left\langle \boldsymbol{v}_v,\, \frac{1}{\rho}\alpha_h \boldsymbol{\alpha}_v \right\rangle_{L^2(\Omega,\mathbb{R}^2)}, \\
\langle v_\partial,\, y_\partial \rangle_{L^2(\partial\Omega)} &= \langle v_\partial,\, \gamma_0 e_h \rangle_{L^2(\partial\Omega)}.
\end{aligned}
\tag{7.38}
$$

1170   Introducing a Galerkin approximation as in (7.23)

$$
\begin{aligned}
v_h &\approx \sum_{i=1}^{n_h} \phi_h^i(\boldsymbol{x}) v_h^i, & \alpha_h &\approx \sum_{i=1}^{n_h} \phi_h^i(\boldsymbol{x}) \alpha_h^i(t), & e_h &\approx \sum_{i=1}^{n_h} \phi_h^i(\boldsymbol{x}) e_h^i(t), & \boldsymbol{x} &\in \Omega, \\
\boldsymbol{v}_v &\approx \sum_{i=1}^{n_v} \boldsymbol{\phi}_v^i(\boldsymbol{x}) v_v^i, & \boldsymbol{\alpha}_v &\approx \sum_{i=1}^{n_{2v}} \boldsymbol{\phi}_v^i(\boldsymbol{x}) \alpha_v^i(t), & \boldsymbol{e}_v &\approx \sum_{i=1}^{n_v} \boldsymbol{\phi}_v^i(\boldsymbol{x}) e_v^i(t), & \boldsymbol{x} &\in \Omega, \\
v_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(\boldsymbol{s}) v_\partial^i, & u_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(\boldsymbol{s}) u_\partial^i(t), & y_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(\boldsymbol{s}) y_\partial^i(t), & \boldsymbol{s} &\in \partial\Omega,
\end{aligned}
\tag{7.39}
$$

1171   the finite dimensional system is obtained

$$
\begin{aligned}
\begin{bmatrix} \mathbf{M}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,h} \\ \dot{\boldsymbol{\alpha}}_{d,v} \end{pmatrix} &= -\begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\operatorname{grad}}^\top \\ \mathbf{D}_{\operatorname{grad}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_h \\ \mathbf{e}_v \end{pmatrix} + \begin{bmatrix} \mathbf{B}_h \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\
\begin{bmatrix} \mathbf{M}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v \end{bmatrix} \begin{pmatrix} \mathbf{e}_h \\ \mathbf{e}_v \end{pmatrix} &= \begin{bmatrix} \partial_{\boldsymbol{\alpha}_{d,h}} H_d(\boldsymbol{\alpha}_{d,h},\, \boldsymbol{\alpha}_{d,v}) \\ \partial_{\boldsymbol{\alpha}_{d,v}} H_d(\boldsymbol{\alpha}_{d,h},\, \boldsymbol{\alpha}_{d,v}) \end{bmatrix}, \\
\mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_h^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_h \\ \mathbf{e}_v \end{pmatrix}.
\end{aligned}
\tag{7.40}
$$

1172   The matrices are defined as follows

$$
\begin{aligned}
M_h^{ij} &= \left\langle \phi_h^i,\, \phi_h^j \right\rangle_{L^2(\Omega)}, & D_{\operatorname{grad}}^{mi} &= \left\langle \boldsymbol{\phi}_v^m,\, \operatorname{grad} \phi_h^i \right\rangle_{L^2(\Omega,\mathbb{R}^2)}, \\
M_v^{mn} &= \langle \boldsymbol{\phi}_v^m,\, \boldsymbol{\phi}_v^n \rangle_{L^2(\Omega,\mathbb{R}^2)}, & B_h^{ik} &= \left\langle \gamma_0 \phi_h^i,\, \phi_\partial^k \right\rangle_{L^2(\partial\Omega)}, \\
M_\partial^{lk} &= \left\langle \phi_\partial^l,\, \phi_\partial^k \right\rangle_{L^2(\partial\Omega)},
\end{aligned}
\tag{7.41}
$$

where $i, j \in \{1, n_h\}$, $m, n \in \{1, n_v\}$, $l, k \in \{1, n_\partial\}$. The discretized gradient of the Hamiltonian read

$$
\begin{aligned}
\partial_{\alpha^i_{d,h}} H_d(\boldsymbol{\alpha}_{d,h},\, \boldsymbol{\alpha}_{d,v}) &= \left\langle \boldsymbol{\phi}^i_h,\, \frac{1}{2\rho} \left\| \sum_{r=1}^{n_2} \boldsymbol{\phi}^r_v \alpha^r_v \right\|^2 + \rho g \sum_{r=1}^{n_1} \phi^r_h \alpha^r_h \right\rangle_{L^2(\Omega)}, &\qquad i \in \{1, n_h\}, \\
\partial_{\alpha^m_{d,v}} H_d(\boldsymbol{\alpha}_{d,h},\, \boldsymbol{\alpha}_{d,v}) &= \left\langle \boldsymbol{\phi}^m_v,\, \frac{1}{\rho} \left( \sum_{r=1}^{n_1} \phi^r_h \alpha^r_h \right) \left( \sum_{r=1}^{n_2} \boldsymbol{\phi}^r_v \alpha^r_v \right) \right\rangle_{L^2(\Omega,\mathbb{R}^2)}, &\qquad m \in \{1, n_v\}.
\end{aligned}
\tag{7.42}
$$

One possible finite element discretization for this problem can be found in [Pir89]. The non linear nature of the problem strongly complicates the analysis. The presence of shocks has to accounted for in the numerical discretization. The proposed methodology has to copy with finite time shocks to become a valid alternative to already well established strategies.

### 7.1.2   Linear case

The general framework detailed in Sec. 7.1.1 is valid for both linear and non linear system. However, in the linear case a major simplification occurs since the constitutive law connecting energy and co-energy variables is easily invertible. This allows a description based on co-energy variables only.

To make the system linear, The additional assumption is introduced.

**Assumption 4**
*The Hamiltonian is assumed to be a positive quadratic functional in the energy variables $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$. Furthermore, the Hamiltonian is considered to be separable with respect to $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ (this hypothesis is always met for the systems under consideration). Therefore, it can be expressed as*

$$
H = \frac{1}{2} \langle \boldsymbol{\alpha}_1,\, \mathcal{Q}_1 \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega,\mathbb{A})} + \frac{1}{2} \langle \boldsymbol{\alpha}_2,\, \mathcal{Q}_2 \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega,\mathbb{B})},
\tag{7.43}
$$

*where $\mathcal{Q}_1$, $\mathcal{Q}_2$ are positive symmetric operators, bounded from below and above*

$$
m_1 \boldsymbol{I}_\mathbb{A} \le \mathcal{Q}_1 \le M_1 \boldsymbol{I}_\mathbb{A}, \qquad m_2 \boldsymbol{I}_\mathbb{B} \le \mathcal{Q}_2 \le M_2 \boldsymbol{I}_\mathbb{B}, \qquad m_1 > 0,\ m_2 > 0,\ M_1 > 0,\ M_2 > 0,
$$

where $\boldsymbol{I}_\mathbb{A}$, $\boldsymbol{I}_2$ are the identity operator in $\mathbb{A}$, $\mathbb{B}$ respectively. Because of Assumption 4, the co-energy variables are given by

$$
\boldsymbol{e}_1 := \delta_{\alpha_1} H = \mathcal{Q}_1 \boldsymbol{\alpha}_1, \qquad \boldsymbol{e}_2 := \delta_{\alpha_2} H = \mathcal{Q}_2 \boldsymbol{\alpha}_2
\tag{7.44}
$$

Since $\mathcal{Q}_1$, $\mathcal{Q}_2$ are positive bounded from below and above, it is possible to invert them to obtain

$$
\boldsymbol{\alpha}_1 = \mathcal{Q}_1^{-1} \boldsymbol{e}_1 = \mathcal{M}_1 \boldsymbol{e}_1, \qquad \boldsymbol{\alpha}_2 = \mathcal{Q}_2^{-1} \boldsymbol{e}_2 = \mathcal{M}_2 \boldsymbol{e}_2, \qquad \mathcal{M}_1 := \mathcal{Q}_1^{-1},\ \mathcal{M}_2 := \mathcal{Q}_2^{-1}.
\tag{7.45}
$$

1195 The Hamiltonian is then written in terms of co-energy variables as

$$H = \frac{1}{2} \langle \boldsymbol{e}_1, \mathcal{M}_1 \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \frac{1}{2} \langle \boldsymbol{e}_2, \mathcal{M}_2 \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} . \tag{7.46}$$

1196 Under assumptions 2, 3, 4, a pH linear system is expressed as

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\boldsymbol{L}^\top - \mathcal{L}^* \\ \boldsymbol{L} + \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{e}_1 &\in H^{\mathcal{L}}, \\ \boldsymbol{e}_2 &\in H^{-\mathcal{L}^*}. \end{aligned} \tag{7.47}$$

1197 If Eq. (7.9) holds the boundary variables equal

$$\boldsymbol{u}_\partial = \mathcal{N}_2 \boldsymbol{e}_2, \qquad \boldsymbol{y}_\partial = \mathcal{N}_1 \boldsymbol{e}_1, \qquad \boldsymbol{u}_\partial, \boldsymbol{y}_\partial \in \mathbb{R}^m. \tag{7.48}$$

1198 Whereas if Eq. (7.10) holds, then

$$\boldsymbol{u}_\partial = \mathcal{N}_1 \boldsymbol{e}_1, \qquad \boldsymbol{y}_\partial = \mathcal{N}_2 \boldsymbol{e}_2, \qquad \boldsymbol{u}_\partial, \boldsymbol{y}_\partial \in \mathbb{R}^m. \tag{7.49}$$

1199 From equation (7.46), the power balance reads

$$\begin{aligned} \dot{H} &= \langle \boldsymbol{e}_1, \mathcal{M}_1 \partial_t \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \boldsymbol{e}_2, \mathcal{M}_2 \partial_t \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} , \\ &= \langle \boldsymbol{e}_1, -\mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \boldsymbol{e}_2, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} , \\ &= \langle \mathcal{N}_{\partial, 1} \boldsymbol{e}_1, \mathcal{N}_{\partial, 2} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} , \\ &= \langle \boldsymbol{y}_\partial, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} . \end{aligned} \tag{7.50}$$

1200 To get a finite dimensional approximation the same procedure detailed in Sec. §7.1.1 is
1201 followed. The only difference is that there is no need to discretize the constitutive relations
1202 as those are already incorporated in the dynamics.

1203 Once the system is put into weak form, if the operator $-\mathcal{L}^*$ is integrated by parts, one
1204 obtains the weak form

$$\begin{aligned} \langle \boldsymbol{v}_1, \mathcal{M}_1 \partial_t \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathcal{L} \boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial, 1} \boldsymbol{v}_1, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} , \\ \langle \boldsymbol{v}_2, \mathcal{M}_2 \partial_t \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} , \\ \langle \boldsymbol{v}_\partial, \boldsymbol{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \boldsymbol{v}_\partial, \mathcal{N}_{\partial, 1} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} . \end{aligned} \tag{7.51}$$

1205 Otherwise, if operator $\mathcal{L}$ is integrated by parts, it is computed

$$\begin{aligned} \langle \boldsymbol{v}_1, \mathcal{M}_1 \partial_t \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} , \\ \langle \boldsymbol{v}_2, \mathcal{M}_2 \partial_t \boldsymbol{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L} \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial, 2} \boldsymbol{v}_2, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} , \quad (7.52) \\ \langle \boldsymbol{v}_\partial, \boldsymbol{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \boldsymbol{v}_\partial, \mathcal{N}_{\partial, 2} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} . \end{aligned}$$

1206 After introducing a Galerkin approximation as in (7.23), the discretized version of the weak
1207 form (7.51) reads

$$\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \tag{7.53}$$

1208   The only difference with respect to Eq. (7.24) concerns the mass matrices

$$M_{\mathcal{M}_1}^{ij} = \left\langle \boldsymbol{\phi}_1^i, \, \mathcal{M}_1 \boldsymbol{\phi}_1^j \right\rangle_{L^2(\Omega,\mathbb{A})}, \qquad M_{\mathcal{M}_2}^{mn} = \left\langle \boldsymbol{\phi}_2^m, \, \mathcal{M}_2 \boldsymbol{\phi}_2^n \right\rangle_{L^2(\Omega,\mathbb{B})} \qquad i,j \in \{1, n_1\}, \; m,n \in \{1, n_2\}. \tag{7.54}$$

1209   If the Galerkin approximation is applied to the weak form (7.52), it is obtained

$$\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top + \mathbf{D}_{-\mathcal{L}^*} \\ \mathbf{D}_0 - \mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial,$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \tag{7.55}$$

1210   In both cases, it is easy to verify that the Hamiltonian

$$H_d = \frac{1}{2} \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \mathbf{e}_2, \tag{7.56}$$

1211   once differentiated in time, provides the energy rate

$$\dot{H}_d = \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial = \widehat{\mathbf{y}}_\partial^\top \mathbf{u}_\partial, \qquad \text{where} \qquad \widehat{\mathbf{y}}_\partial := \mathbf{M}_\partial \mathbf{y}_\partial. \tag{7.57}$$

1212   This result mimics its finite dimensional counterpart (7.50).

### 1213   7.1.3   Linear flexible structures

1214   In this section, some linear example from the elasticity realms are considered. We restrict
1215   the discussion to linear problems. This case is anyway significant, as these examples are
1216   frequently encountered in engineering applications.

### 1217   7.1.3.1   Euler-Bernoulli beam

1218   We reconsider the example discussed in Sec. §3.2.3.2. The relation between energy and
1219   co-energy variables is given by Eqs. (3.24), (3.26)

$$\alpha_w = \rho A \, e_w, \qquad \alpha_\kappa = \frac{1}{EI} \, e_\kappa \tag{7.58}$$

1220   The coefficients $\rho, A, E$ and $I$ are the mass density, the cross section area, Young's modulus
1221   of elasticity and the moment of inertia of the cross section.

1222 **Control through forces and torques** Given an interval $\Omega = (0, L)$, a thin beam under
1223 free boundary condition (forces and torques imposed at the boundary) can be modeled in
1224 terms of co-energy variables by the following system

$$\begin{bmatrix} \rho A & 0 \\ 0 & (EI)^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \qquad \begin{aligned} e_w \in H^2(\Omega), \\ e_\kappa \in H^2(\Omega), \end{aligned} \tag{7.59a}$$

$$\boldsymbol{u}_\partial = \begin{bmatrix} 0 & \boldsymbol{\gamma}_0 \\ 0 & -\boldsymbol{\gamma}_1 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \qquad \boldsymbol{u}_\partial \in \mathbb{R}^4, \tag{7.59b}$$

$$\boldsymbol{y}_\partial = \begin{bmatrix} \boldsymbol{\gamma}_1 & 0 \\ \boldsymbol{\gamma}_0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \qquad \boldsymbol{y}_\partial \in \mathbb{R}^4. \tag{7.59c}$$

1225 The boundary operator $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1$ denote the trace and the first derivative trace along the bound-
1226 ary. In a one-dimensional domain the boundary degenerates to two single points

$$\boldsymbol{\gamma}_0 a = a|_{\partial\Omega} = \begin{pmatrix} -a(0) \\ +a(L) \end{pmatrix}, \qquad \boldsymbol{\gamma}_1 a = \partial_x a|_{\partial\Omega} = \begin{pmatrix} -\partial_x a(0) \\ +\partial_x a(L) \end{pmatrix}. \tag{7.60}$$

1227 In this case $\mathbb{A} = \mathbb{B} = \mathbb{R}$. The operators $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{L}$, $N_{\partial,1}$, $N_{\partial,2}$ read

$$\mathcal{M}_1 = \rho A, \qquad \mathcal{M}_2 = (EI)^{-1}, \qquad \mathcal{L} = \partial_{xx}, \qquad N_{\partial,1} = \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_0 \end{bmatrix}, \qquad N_{\partial,2} = \begin{bmatrix} \boldsymbol{\gamma}_0 \\ -\boldsymbol{\gamma}_1 \end{bmatrix}. \tag{7.61}$$

1228    The Hamiltonian is given by

$$H = \frac{1}{2} \int_\Omega \left\{ \rho A \, e_w^2 + (EI)^{-1} \, e_\kappa^2 \right\} \, \mathrm{d}\Omega. \tag{7.62}$$

1229 Applying twice the integration by parts formula, one obtains the power balance

$$\begin{aligned} \dot{H} &= \langle e_w, \rho A \, \partial_t e_w \rangle_{L^2(\Omega)} + \left\langle e_\kappa, (EI)^{-1} \, \partial_t e_\kappa \right\rangle_{L^2(\Omega)}, \\ &= \langle e_w, -\partial_{xx} e_\kappa \rangle_{L^2(\Omega)} + \langle e_\kappa, \partial_{xx} e_w \rangle_{L^2(\Omega)}, \\ &= \langle \boldsymbol{\gamma}_1 e_w, \boldsymbol{\gamma}_0 e_\kappa \rangle_{\mathbb{R}^2} + \langle \boldsymbol{\gamma}_0 e_w, -\boldsymbol{\gamma}_1 e_\kappa \rangle_{\mathbb{R}^2}, \\ &= \langle \boldsymbol{y}_\partial, \boldsymbol{u}_\partial \rangle_{\mathbb{R}^4}. \end{aligned} \tag{7.63}$$

1230 Given the test functions $v_w$, $v_\kappa$, the weak form is readily obtained as

$$\begin{aligned} \langle v_w, \rho A \partial_t e_w \rangle_{L^2(\Omega)} &= \langle v_w, -\partial_{xx} e_\kappa \rangle_{L^2(\Omega)}, \\ \left\langle v_\kappa, (EI)^{-1} \partial_t e_\kappa \right\rangle_{L^2(\Omega)} &= \langle v_\kappa, \partial_{xx} e_w \rangle_{L^2(\Omega)}. \end{aligned} \tag{7.64}$$

1231  If the integration by parts is applied twice to the first line of Eq. (7.59a), it is obtained

$$\langle v_w, \, \rho A \partial_t e_w \rangle_{L^2(\Omega)} = - \langle \partial_{xx} v_w, \, e_\kappa \rangle_{L^2(\Omega)} + \langle \boldsymbol{\gamma}_1 v_w, \, (u_{\partial,1}, u_{\partial,2}) \rangle_{\mathbb{R}^2} + \langle \boldsymbol{\gamma}_0 v_w, \, (u_{\partial,3}, u_{\partial,4}) \rangle_{\mathbb{R}^2},$$
$$\left\langle v_\kappa, \, (EI)^{-1} \partial_t e_\kappa \right\rangle_{L^2(\Omega)} = \langle v_\kappa, \, \partial_{xx} e_w \rangle_{L^2(\Omega)}.$$
(7.65)

1232  Introducing a Galerkin discretization for test and efforts functions

$$v_w = \sum_{i=1}^{n_w} \phi_w^i v_w^i, \qquad e_w = \sum_{i=1}^{n_w} \phi_w^i e_w^i(t), \qquad v_\kappa = \sum_{i=1}^{n_\kappa} \phi_\kappa^i v_\kappa^i, \qquad e_\kappa = \sum_{i=1}^{n_\kappa} \phi_\kappa^i e_\kappa^i(t), \qquad (7.66)$$

1233  and considering that $\boldsymbol{u}_\partial \in \mathbb{R}^4, \boldsymbol{y}_\partial \in \mathbb{R}^4$, the following is obtained

$$\begin{bmatrix} \mathbf{M}_{\rho A} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{EI^{-1}} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\partial_{xx}}^\top \\ \mathbf{D}_{\partial_{xx}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix} + \begin{bmatrix} \mathbf{B}_w \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,$$
$$\mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_w^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix}.$$
(7.67)

1234  The matrices $\mathbf{M}_{\rho A}$, $\mathbf{M}_{EI^{-1}}$, $\mathbf{D}_{\partial_{xx}}$ are defined as ($i, j \in \{1, n_w\}$, $m, n \in \{1, n_\kappa\}$)

$$M_{\rho A}^{ij} = \left\langle \phi_w^i, \, \rho A \phi_w^j \right\rangle_{L^2(\Omega)}, \qquad M_{EI^{-1}}^{mn} = \left\langle \phi_\kappa^m, \, (EI)^{-1} \phi_\kappa^n \right\rangle_{L^2(\Omega)}, \qquad D_{\partial_{xx}}^{mi} = \left\langle \phi_\kappa^m, \, \partial_{xx} \phi_w^i \right\rangle_{L^2(\Omega)}.$$
(7.68)

1235  The $\mathbf{B}_w$ is composed of four column vectors $\mathbf{B}_w = [\mathbf{b}_w^1 \ \mathbf{b}_w^2 \ \mathbf{b}_w^3 \ \mathbf{b}_w^4]$

$$b_w^{1,i} = -\partial_x \phi_w^i(0), \qquad b_w^{2,i} = \partial_x \phi_w^i(L), \qquad b_w^{3,i} = -\phi_w^i(0), \qquad b_w^{4,i} = \phi_w^i(L), \qquad i \in \{1, n_w\}.$$
(7.69)

**Control through linear and angular velocities**  Equivalently, the second line of Eq. (7.59a) could have been integrated by parts to control trough the linear and angular velocities at the extremities. Consider the system with known forces and torques at the extremities

$$\begin{bmatrix} \rho A & 0 \\ 0 & (EI)^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \qquad \begin{array}{c} e_w \in H^2(\Omega), \\ e_\kappa \in H^2(\Omega), \end{array} \qquad (7.70a)$$

$$\boldsymbol{u}_\partial = \begin{bmatrix} \gamma_1 & 0 \\ \gamma_0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \qquad \boldsymbol{u}_\partial \in \mathbb{R}^4, \qquad (7.70b)$$

$$\boldsymbol{y}_\partial = \begin{bmatrix} 0 & \gamma_0 \\ 0 & -\gamma_1 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \qquad \boldsymbol{y}_\partial \in \mathbb{R}^4. \qquad (7.70c)$$

1236  Once the system is put into weak form and the second line of Eq. (7.70a) is integrated twice,
1237  it is computed

$$\langle v_w, \, \rho A \partial_t e_w \rangle_{L^2(\Omega)} = \langle v_w, \, -\partial_{xx} e_\kappa \rangle_{L^2(\Omega)},$$
$$\left\langle v_\kappa, \, (EI)^{-1} \partial_t e_\kappa \right\rangle_{L^2(\Omega)} = \langle \partial_{xx} v_\kappa, \, e_w \rangle_{L^2(\Omega)} + \langle \boldsymbol{\gamma}_0 v_\kappa, \, (u_{\partial,1}, u_{\partial,2}) \rangle_{\mathbb{R}^2} + \langle -\boldsymbol{\gamma}_1 v_\kappa, \, (u_{\partial,3}, u_{\partial,4}) \rangle_{\mathbb{R}^2}.$$
(7.71)

Replacing a Galerkin approximation, it is obtained

$$\begin{bmatrix} \mathbf{M}_{\rho A} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{EI^{-1}} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\partial_{xx}} \\ -\mathbf{D}_{-\partial_{xx}}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_\kappa \end{bmatrix} \mathbf{u}_\partial,$$
$$\mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_\kappa^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix}.$$
(7.72)

The matrix $\mathbf{D}_{-\partial_{xx}}$ is defined as

$$D_{-\partial_{xx}}^{im} = \left\langle \phi_w^i, -\partial_{xx}\phi_\kappa^m \right\rangle_{L^2(\Omega)}, \qquad i, \in \{1, n_w\}, \ m \in \{1, n_\kappa\}.$$
(7.73)

The $\mathbf{B}_\kappa$ is composed of four column vectors $\mathbf{B}_\kappa = [\mathbf{b}_\kappa^1 \ \mathbf{b}_\kappa^2 \ \mathbf{b}_\kappa^3 \ \mathbf{b}_\kappa^4]$

$$b_\kappa^{1,m} = -\phi_\kappa^m(0), \quad b_\kappa^{2,m} = \phi_\kappa^m(L), \quad b_\kappa^{3,m} = \partial_x\phi_\kappa^m(0), \quad b_\kappa^{4,m} = -\partial_x\phi_\kappa^m(L), \quad m \in \{1, n_\kappa\}.$$
(7.74)

Both discretization require the use of Hermite polynomials to meet the regularity requirement. Indeed, to lower the regularity requirement for the finite elements employed in the discretization, both lines can be integrated by parts. This will be discussed in Chap. 8.

### 7.1.3.2 Kirchhoff plate

The link beetween the energy and co-energy variables for the isotropic Kirchhoff model is the following (5.33)

$$\alpha_w = \rho h e_w, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\mathcal{C}}_b \boldsymbol{E}_\kappa, \qquad \text{where} \qquad \boldsymbol{\mathcal{C}}_b := \boldsymbol{\mathcal{D}}_b^{-1}$$
(7.75)

where $\rho$ is the mass density, $h$ the plate thickness and $\boldsymbol{\mathcal{D}}_b$, the bending rigidity tensor, cf. Eq. (5.11). The bending compliance is given by

$$\boldsymbol{\mathcal{C}}_b = \frac{12}{Eh^3}[(1+\nu)(\cdot) - \nu\operatorname{Tr}(\cdot)\boldsymbol{I}_{2\times 2}].$$
(7.76)

Given an open connected set $\Omega \subset \mathbb{R}^2$, the Kirchhoff plate model (5.42) in co-energy form controlled by forces and momenta is then expressed as

$$\begin{bmatrix} \rho h & 0 \\ 0 & \boldsymbol{\mathcal{C}}_b \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \operatorname{Hess} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}, \qquad \begin{array}{l} e_w \in H^2(\Omega), \\ \boldsymbol{E}_\kappa \in H^{\operatorname{div}\operatorname{Div}}(\Omega, \mathbb{R}_{\operatorname{sym}}^{2\times 2}), \end{array}$$
(7.77a)

$$\boldsymbol{u}_\partial = \begin{bmatrix} 0 & \gamma_{nn,1} \\ 0 & \gamma_{nn} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}, \qquad \boldsymbol{u}_\partial \in \mathbb{R}^2,$$
(7.77b)

$$\boldsymbol{y}_\partial = \begin{bmatrix} \gamma_0 & 0 \\ \gamma_1 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}, \qquad \boldsymbol{y}_\partial \in \mathbb{R}^2,$$
(7.77c)

We recall the expressions of the trace maps

$$
\begin{aligned}
\gamma_0 a &= a|_{\partial\Omega}, & \gamma_{nn,1}\boldsymbol{A} &= -\boldsymbol{n}\cdot\mathrm{Div}\,\boldsymbol{A} - \partial_s(\boldsymbol{A}:(\boldsymbol{n}\otimes\boldsymbol{s}))|_{\partial\Omega}, \\
\gamma_1 a &= \partial_{\boldsymbol{n}}a|_{\partial\Omega}, & \gamma_{nn}\boldsymbol{A} &= \boldsymbol{A}:(\boldsymbol{n}\otimes\boldsymbol{n})|_{\partial\Omega}.
\end{aligned}
\tag{7.78}
$$

In this case, the sets are $\mathbb{A} = \mathbb{R}$, $\mathbb{B} = \mathbb{R}^{2\times2}_{\mathrm{sym}}$. The operators $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{L}$, $N_{\partial,1}$, $N_{\partial,2}$ are

$$
\mathcal{M}_1 = \rho h, \qquad \mathcal{M}_2 = \boldsymbol{\mathcal{C}}_b, \qquad \mathcal{L} = \mathrm{Hess}, \qquad N_{\partial,1} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}, \qquad N_{\partial,2} = \begin{bmatrix} \gamma_{nn,1} \\ \gamma_{nn} \end{bmatrix}.
\tag{7.79}
$$

The energy rate from Eq. (5.39) equals $\dot{H} = \langle \boldsymbol{y}_\partial, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega,\mathbb{R}^2)}$. Introducing the test functions $(v_w, \boldsymbol{V}_\kappa)$ and integrating by parts twice the first line of (7.77a) one gets

$$
\begin{aligned}
\langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} &= -\langle \mathrm{Hess}\,v_w, \boldsymbol{E}_\kappa \rangle_{L^2(\Omega,\mathbb{R}^{2\times2}_{\mathrm{sym}})} + \langle \gamma_0 v_w, u_{\partial,1} \rangle_{L^2(\partial\Omega)} + \langle \gamma_1 v_w, u_{\partial,2} \rangle_{L^2(\partial\Omega)}, \\
\langle \boldsymbol{V}_\kappa, \boldsymbol{\mathcal{C}}_{\boldsymbol{b}}\partial_t \boldsymbol{V}_\kappa \rangle_{L^2(\Omega,\mathbb{R}^{2\times2}_{\mathrm{sym}})} &= \langle \boldsymbol{V}_\kappa, \mathrm{Hess}\,e_w \rangle_{L^2(\Omega,\mathbb{R}^{2\times2}_{\mathrm{sym}})}.
\end{aligned}
\tag{7.80}
$$

Introducing a Galerkin discretization for test and efforts functions

$$
\begin{aligned}
v_w &= \sum_{i=1}^{n_w} \phi_w^i v_w^i, & \boldsymbol{V}_\kappa &= \sum_{i=1}^{n_\kappa} \boldsymbol{\Phi}_\kappa^i v_\kappa^i, & \boldsymbol{v}_\partial &= \sum_{i=1}^{n_\partial} \phi_\partial^i v_\partial^i, \\
e_w &= \sum_{i=1}^{n_w} \phi_w^i e_w^i, & \boldsymbol{E}_\kappa &= \sum_{i=1}^{n_\kappa} \boldsymbol{\Phi}_\kappa^i e_\kappa^i & \boldsymbol{u}_\partial &= \sum_{i=1}^{n_\partial} \phi_\partial^i u_\partial^i,
\end{aligned} \qquad \boldsymbol{y}_\partial = \sum_{i=1}^{n_\partial} \phi_\partial^i y_\partial^i.
\tag{7.81}
$$

the following finite dimensional system is obtained

$$
\begin{aligned}
\begin{bmatrix} \mathbf{M}_{\rho h} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\boldsymbol{\mathcal{C}}_b} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\kappa \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\mathrm{Hess}}^\top \\ \mathbf{D}_{\mathrm{Hess}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix} + \begin{bmatrix} \mathbf{B}_w & \mathbf{B}_{\partial_{\boldsymbol{n}}w} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\
\mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_w^\top & \mathbf{0} \\ \mathbf{B}_{\partial_{\boldsymbol{n}}w}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix}.
\end{aligned}
\tag{7.82}
$$

The matrices $\mathbf{M}_{\rho h}$, $\mathbf{M}_{\boldsymbol{\mathcal{C}}_b}$, $\mathbf{D}_{\mathrm{Hess}}$ are defined as ($i,j \in \{1, n_w\}$, $m,n \in \{1, n_\kappa\}$)

$$
M_{\rho h}^{ij} = \left\langle \phi_w^i, \rho h \phi_w^j \right\rangle_{L^2(\Omega)}, \quad M_{\boldsymbol{\mathcal{C}}_b}^{mn} = \langle \boldsymbol{\Phi}_\kappa^m, \mathcal{C}_b \boldsymbol{\Phi}_\kappa^n \rangle_{L^2(\Omega,\mathbb{R}^{2\times2}_{\mathrm{sym}})}, \quad D_{\mathrm{Hess}}^{mi} = \left\langle \boldsymbol{\Phi}_\kappa^m, \mathrm{Hess}\,\phi_w^i \right\rangle_{L^2(\Omega)}.
\tag{7.83}
$$

Matrices $\mathbf{B}_w$, $\mathbf{B}_{\partial_{\boldsymbol{n}}w}$ are given by

$$
B_w^{il} = \left\langle \gamma_0 \phi_w^i, \phi_{\partial,1}^l \right\rangle_{L^2(\partial\Omega)}, \qquad B_{\partial_{\boldsymbol{n}}w}^{il} = \left\langle \gamma_1 \phi_w^i, \phi_{\partial,2}^l \right\rangle_{L^2(\partial\Omega)}, \qquad l \in \{1, n_\partial\}.
\tag{7.84}
$$

This kind of discretization requires $H^2$ conforming element. The construction of those is rather involved [AFS68, Bel69] and they are computationally expensive. Nevertheless, this kind of discretization is able to handle generic boundary conditions [GSV18]. For this reason, it is the most adapted for the pH framework.

1261

1262     To lower the regularity requirement for the finite elements many non conforming dis-
1263 cretization have been proposed. The most employed is the Hellan-Herrmann-Johnson element
1264 [AB85, BR90]. However, this method does not handle generic non homogeneous boundary
1265 conditions. Given the unavailability of the boundary for interconnections, the modularity
1266 feature of pHs cannot be fully exploited.

1267

1268 **Remark 9** (On the $H^{\mathrm{div\,Div}}$ space)
1269 *Equivalently, the second line of Eq. (7.77a) can be integrated by parts twice to obtain a*
1270 *discretized system whose input are the linear velocity and the angular velocity at the boundary.*
1271 *However, while for the $H^2$ space conforming finite elements are available, for the $H^{\mathrm{div\,Div}}$ no*
1272 *conforming finite elements have been proposed. This makes the discretization unfeasible.*

1273 ### 7.1.3.3   Mindlin plate

1274 Using Eqs. (5.22) and (5.24), the relation between co-energy and energy variables for the
1275 isotropic Mindlin plate is found to be

$$\begin{aligned}
\alpha_w &= \rho h e_w, & \boldsymbol{\alpha}_\theta &= I_\theta \boldsymbol{e}_\theta, & I_\theta &:= \rho h^3/12, \\
\boldsymbol{A}_\kappa &= \boldsymbol{\mathcal{C}}_b \boldsymbol{E}_\kappa, & \boldsymbol{\alpha}_\gamma &= C_s \boldsymbol{e}_\gamma, & C_s &:= 1/(kGh),
\end{aligned} \tag{7.85}$$

1276 where $k$ is the shear correction factor, $G$ the shear modulus. The other variables have the
1277 same meaning as in Sec. §7.1.3.2.

1278 **Control through forces and torques**   A pH representation in co-energy variables with
1279 known forces and momenta at the boundary is given by the system

$$\begin{bmatrix} \rho h & 0 & 0 & 0 \\ 0 & I_\theta & 0 & 0 \\ 0 & 0 & \boldsymbol{\mathcal{C}}_b & 0 \\ 0 & 0 & 0 & C_s \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ 0 & \mathrm{Grad} & 0 & 0 \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}, \qquad \begin{aligned} & e_w \in H^1(\Omega), \\ & \boldsymbol{e}_\theta \in H^{\mathrm{Grad}}(\Omega, \mathbb{R}^2), \\ & \boldsymbol{E}_\kappa \in H^{\mathrm{Div}}(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \\ & \boldsymbol{e}_\gamma \in H^{\mathrm{div}}(\Omega, \mathbb{R}^2), \end{aligned}$$

(7.86a)

$$\boldsymbol{u}_\partial = \begin{bmatrix} 0 & 0 & 0 & \gamma_n \\ 0 & 0 & \gamma_{nn} & 0 \\ 0 & 0 & \gamma_{ns} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}, \qquad \boldsymbol{u}_\partial \in \mathbb{R}^3, \tag{7.86b}$$

$$\boldsymbol{y}_\partial = \begin{bmatrix} \gamma_0 & 0 & 0 & 0 \\ 0 & \gamma_n & 0 & 0 \\ 0 & \gamma_s & 0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}, \qquad \boldsymbol{y}_\partial \in \mathbb{R}^3. \tag{7.86c}$$

The trace operators are defined as

$$\gamma_0 a = a|_{\partial\Omega}, \qquad \begin{aligned} \gamma_n \boldsymbol{a} &= \boldsymbol{a} \cdot \boldsymbol{n}|_{\partial\Omega}, & \gamma_{nn}\boldsymbol{A} &= \boldsymbol{A} : (\boldsymbol{n} \otimes \boldsymbol{n})|_{\partial\Omega}, \\ \gamma_s \boldsymbol{a} &= \boldsymbol{a} \cdot \boldsymbol{s}|_{\partial\Omega}, & \gamma_{ns}\boldsymbol{A} &= \boldsymbol{A} : (\boldsymbol{n} \otimes \boldsymbol{s})|_{\partial\Omega}. \end{aligned} \tag{7.87}$$

The variables assume value in the sets $\mathbb{A} = \mathbb{R} \times \mathbb{R}^2$, $\mathbb{B} = \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^2$. The mass operators are given by

$$\mathcal{M}_1 = \begin{bmatrix} \rho h & 0 \\ 0 & I_\theta \end{bmatrix}, \qquad \mathcal{M}_2 = \begin{bmatrix} \boldsymbol{\mathcal{C}}_b & \boldsymbol{0} \\ \boldsymbol{0} & C_s \end{bmatrix}. \tag{7.88}$$

The $\boldsymbol{L}$, $\mathcal{L}$, $\mathcal{N}_{\partial,1}$, $\mathcal{N}_{\partial,1}$ operators are

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I}_{2\times 2} \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} \boldsymbol{0} & \mathrm{Grad} \\ \mathrm{grad} & \boldsymbol{0} \end{bmatrix}, \qquad \mathcal{N}_{\partial,1} = \begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_n \\ 0 & \gamma_s \end{bmatrix}, \qquad \mathcal{N}_{\partial,2} = \begin{bmatrix} 0 & \gamma_n \\ \gamma_{nn} & 0 \\ \gamma_{ns} & 0 \end{bmatrix}. $$

(7.89)

The energy rate is retrieved from Eq. (5.26) $\dot{H} = \langle \boldsymbol{y}_\partial, \boldsymbol{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$. Introducing the test functions $(v_w, \boldsymbol{v}_\theta, \boldsymbol{V}_\kappa, \boldsymbol{v}_\gamma)$ and integrating by parts the first two lines of (7.86a) one gets

$$\langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} = - \langle \mathrm{grad}\, v_w, \boldsymbol{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \gamma_0 v_w, u_{\partial,1} \rangle_{L^2(\partial\Omega)},$$

$$\langle \boldsymbol{v}_\theta, I_\theta \partial_t \boldsymbol{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} = - \langle \mathrm{Grad}\, \boldsymbol{v}_\theta, \boldsymbol{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \langle \boldsymbol{v}_\theta, \boldsymbol{e}_\gamma \rangle_{L^2(\Omega)} + \langle \boldsymbol{\gamma}_0 \boldsymbol{v}_\theta, \boldsymbol{\gamma}_n \boldsymbol{E}_\kappa \rangle_{L^2(\partial\Omega, \mathbb{R}^2)},$$

$$\langle \boldsymbol{V}_\kappa, \boldsymbol{\mathcal{C}_b} \partial_t \boldsymbol{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})} = \langle \boldsymbol{V}_\kappa, \mathrm{Grad}\, \boldsymbol{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})},$$

$$\langle \boldsymbol{v}_\gamma, C_s \partial_t \boldsymbol{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} = \langle \boldsymbol{v}_\gamma, \mathrm{grad}\, e_w \rangle_{L^2(\Omega, \mathbb{R}^2)} - \langle \boldsymbol{v}_\gamma, \boldsymbol{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)}.$$

(7.90)

1286    The term $\langle \boldsymbol{\gamma}_0 \boldsymbol{v}_\theta, \, \boldsymbol{u}_{\partial,2} \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$ can be decomposed in its tangential and normal components

$$\langle \boldsymbol{\gamma}_0 \boldsymbol{v}_\theta, \, \boldsymbol{\gamma}_n \boldsymbol{E}_\kappa \rangle_{L^2(\partial\Omega, \mathbb{R}^2)} = \langle \gamma_n \boldsymbol{v}_\theta, \, u_{\partial,2} \rangle_{L^2(\partial\Omega)} + \langle \gamma_s \boldsymbol{v}_\theta, \, u_{\partial,3} \rangle_{L^2(\partial\Omega)} \tag{7.91}$$

1287    Introducing a Galerkin discretization for test and efforts functions

$$
\begin{aligned}
v_w &= \sum_{i=1}^{n_w} \phi_w^i v_w^i, & \boldsymbol{v}_\theta &= \sum_{i=1}^{n_\theta} \boldsymbol{\phi}_\theta^i v_\theta^i, & \boldsymbol{V}_\kappa &= \sum_{i=1}^{n_\kappa} \boldsymbol{\Phi}_\kappa^i v_\kappa^i, & \boldsymbol{v}_\gamma &= \sum_{i=1}^{n_\gamma} \boldsymbol{\phi}_\gamma^i v_\gamma^i, & \boldsymbol{v}_\partial &= \sum_{i=1}^{n_\partial} \boldsymbol{\phi}_\partial^i v_\partial^i, \\
e_w &= \sum_{i=1}^{n_w} \phi_w^i e_w^i, & \boldsymbol{e}_\theta &= \sum_{i=1}^{n_\theta} \boldsymbol{\phi}_\theta^i e_\theta^i, & \boldsymbol{E}_\kappa &= \sum_{i=1}^{n_\kappa} \boldsymbol{\Phi}_\kappa^i e_\kappa^i, & \boldsymbol{e}_\gamma &= \sum_{i=1}^{n_\gamma} \boldsymbol{\phi}_\gamma^i e_\gamma^i, & \boldsymbol{u}_\partial &= \sum_{i=1}^{n_\partial} \boldsymbol{\phi}_\partial^i u_\partial^i, \\
& & & & & & & & \boldsymbol{y}_\partial &= \sum_{i=1}^{n_\partial} \boldsymbol{\phi}_\partial^i y_\partial^i.
\end{aligned}
\tag{7.92}
$$

1288    the following finite dimensional system is obtained

$$
\mathrm{Diag}
\begin{bmatrix} \mathbf{M}_{\rho h} \\ \mathbf{M}_{I_\theta} \\ \mathbf{M}_{\mathcal{C}_b} \\ \mathbf{M}_{C_s} \end{bmatrix}
\begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\theta \\ \dot{\mathbf{e}}_\kappa \\ \dot{\mathbf{e}}_\gamma \end{pmatrix}
=
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{\mathrm{grad}}^\top \\
\mathbf{0} & \mathbf{0} & -\mathbf{D}_{\mathrm{Grad}}^\top & -\mathbf{D}_0^\top \\
\mathbf{0} & \mathbf{D}_{\mathrm{Grad}} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}_{\mathrm{grad}} & \mathbf{D}_0 & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}
+
\begin{bmatrix}
\mathbf{B}_w & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{\theta_n} & \mathbf{B}_{\theta_s} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\mathbf{u}_\partial,
$$

$$
\mathbf{M}_\partial \mathbf{y}_\partial =
\begin{bmatrix}
\mathbf{B}_w^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{\theta_n}^\top & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{\theta_s}^\top & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}.
\tag{7.93}
$$

1289    The notation Diag denotes a block diagonal matrix. The mass matrices $\mathbf{M}_{\rho h}$, $\mathbf{M}_{I_\theta}$, $\mathbf{M}_{\mathcal{C}_b}$, $\mathbf{M}_{C_s}$
1290 are computed as

$$
\begin{aligned}
M_{\rho h}^{ij} &= \left\langle \phi_w^i, \, \rho h \phi_w^j \right\rangle_{L^2(\Omega)}, & M_{\mathcal{C}_b}^{pq} &= \left\langle \boldsymbol{\Phi}_\kappa^p, \, \mathcal{C}_b \boldsymbol{\Phi}_\kappa^q \right\rangle_{L^2(\Omega, \mathbb{R}^{2\times2}_{\mathrm{sym}})}, \\
M_{I_\theta}^{mn} &= \left\langle \phi_\kappa^m, \, I_\theta \phi_\kappa^n \right\rangle_{L^2(\Omega, \mathbb{R}^2)}, & M_{C_s}^{rs} &= \left\langle \phi_\gamma^r, \, C_s \phi_\gamma^s \right\rangle_{L^2(\Omega, \mathbb{R}^2)},
\end{aligned}
\tag{7.94}
$$

1291 where $i, j \in \{1, n_w\}$, $m, n \in \{1, n_\theta\}$, $p, q \in \{1, n_\kappa\}$, $r, s \in \{1, n_\gamma\}$. Matrices $\mathbf{D}_{\mathrm{grad}}$, $\mathbf{D}_{\mathrm{Grad}}$, $\mathbf{D}_0$
1292 assume the form

$$
\begin{aligned}
D_{\mathrm{grad}}^{rj} &= \left\langle \boldsymbol{\phi}_\gamma^r, \, \mathrm{grad}\, \phi_w^j \right\rangle_{L^2(\Omega, \mathbb{R}^2)}, & D_0^{rn} &= -\left\langle \boldsymbol{\phi}_\gamma^r, \, \boldsymbol{\phi}_\theta^n \right\rangle_{L^2(\Omega, \mathbb{R}^2)}. \\
D_{\mathrm{Grad}}^{pn} &= \left\langle \boldsymbol{\Phi}_\kappa^p, \, \mathrm{Grad}\, \phi_\theta^n \right\rangle_{L^2(\Omega, \mathbb{R}^{2\times2}_{\mathrm{sym}})}, &
\end{aligned}
\tag{7.95}
$$

1293    Matrices $\mathbf{B}_w$, $\mathbf{B}_{\theta_n}$, $\mathbf{B}_{\theta_s}$ are computed as ($l \in \{1, n_\partial\}$)

$$
B_w^{il} = \left\langle \gamma_0 \phi_w^i, \, \phi_{\partial,1}^l \right\rangle_{L^2(\partial\Omega)}, \qquad B_{\theta_n}^{ml} = \left\langle \gamma_n \boldsymbol{\phi}_\theta^m, \, \phi_{\partial,2}^l \right\rangle_{L^2(\partial\Omega)}, \qquad B_{\theta_s}^{ml} = \left\langle \gamma_s \boldsymbol{\phi}_\theta^m, \, \phi_{\partial,3}^l \right\rangle_{L^2(\partial\Omega)}.
\tag{7.96}
$$

**Control through linear and angular velocities**   If instead the opposite causality is considered, the continuous system read

$$
\begin{bmatrix}
\rho h & 0 & 0 & 0 \\
\mathbf{0} & I_\theta & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boldsymbol{\mathcal{C}}_b & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & C_s
\end{bmatrix}
\frac{\partial}{\partial t}
\begin{pmatrix}
e \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma
\end{pmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & \mathrm{div} \\
\mathbf{0} & \mathbf{0} & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\
\mathbf{0} & \mathrm{Grad} & \mathbf{0} & \mathbf{0} \\
\mathrm{grad} & -\boldsymbol{I}_{2\times 2} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\begin{pmatrix}
e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma
\end{pmatrix},
\tag{7.97a}
$$

$$
\boldsymbol{u}_\partial =
\begin{bmatrix}
\gamma_0 & 0 & 0 & 0 \\
0 & \gamma_n & 0 & 0 \\
0 & \gamma_s & 0 & 0
\end{bmatrix}
\begin{pmatrix}
e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma
\end{pmatrix},
\qquad
\boldsymbol{u}_\partial \in \mathbb{R}^3,
\tag{7.97b}
$$

$$
\boldsymbol{y}_\partial =
\begin{bmatrix}
0 & 0 & 0 & \gamma_n \\
0 & 0 & \gamma_{nn} & 0 \\
0 & 0 & \gamma_{ns} & 0
\end{bmatrix}
\begin{pmatrix}
e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma
\end{pmatrix},
\qquad
\boldsymbol{y}_\partial \in \mathbb{R}^3.
\tag{7.97c}
$$

Integrating by parts the last two lines of (7.97a) one gets

$$
\begin{aligned}
\langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} &= \langle v_w, \mathrm{div}\, \boldsymbol{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)}, \\
\langle \boldsymbol{v}_\theta, I_\theta \partial_t \boldsymbol{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} &= \langle \boldsymbol{v}_\theta, \mathrm{Div}\, \boldsymbol{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \boldsymbol{v}_\theta, \boldsymbol{e}_\gamma \rangle_{L^2(\Omega)}, \\
\langle \boldsymbol{V}_\kappa, \boldsymbol{\mathcal{C}}_b \partial_t \boldsymbol{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})} &= -\langle \mathrm{Div}\, \boldsymbol{V}_\kappa, \boldsymbol{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \boldsymbol{\gamma}_n \boldsymbol{V}_\kappa, \boldsymbol{\gamma}_0 \boldsymbol{e}_\theta \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}, \\
\langle \boldsymbol{v}_\gamma, C_s \partial_t \boldsymbol{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} &= -\langle \mathrm{div}\, \boldsymbol{v}_\gamma, e_w \rangle_{L^2(\Omega)} - \langle \boldsymbol{v}_\gamma, \boldsymbol{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \gamma_0 v_w, u_{\partial,1} \rangle_{L^2(\partial\Omega)}.
\end{aligned}
\tag{7.98}
$$

The term $\langle \boldsymbol{\gamma}_n \boldsymbol{V}_\kappa, \boldsymbol{\gamma}_0 \boldsymbol{e}_\theta \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$ can be decomposed in its tangential and normal components

$$
\langle \boldsymbol{\gamma}_n \boldsymbol{V}_\kappa, \boldsymbol{\gamma}_0 \boldsymbol{e}_\theta \rangle_{L^2(\partial\Omega, \mathbb{R}^2)} = \langle \gamma_{nn} \boldsymbol{V}_\kappa, u_{\partial,2} \rangle_{L^2(\partial\Omega)} + \langle \gamma_{ns} \boldsymbol{V}_\kappa, u_{\partial,3} \rangle_{L^2(\partial\Omega)}.
\tag{7.99}
$$

Plugging approximation (7.92) into this system, one computes

$$
\mathrm{Diag}
\begin{bmatrix}
\mathbf{M}_{\rho h} \\ \mathbf{M}_{I_\theta} \\ \mathbf{M}_{\mathcal{C}_b} \\ \mathbf{M}_{C_s}
\end{bmatrix}
\begin{pmatrix}
\dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\theta \\ \dot{\mathbf{e}}_\kappa \\ \dot{\mathbf{e}}_\gamma
\end{pmatrix}
=
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathrm{div}} \\
\mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathrm{Div}} & -\mathbf{D}_0^\top \\
\mathbf{0} & -\mathbf{D}_{\mathrm{Div}}^\top & \mathbf{0} & \mathbf{0} \\
-\mathbf{D}_{\mathrm{div}}^\top & \mathbf{D}_0 & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\begin{pmatrix}
\mathbf{e}_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma
\end{pmatrix}
+
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{M_{nn}} & \mathbf{B}_{M_{ns}} \\
\mathbf{B}_{q_n} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\mathbf{u}_\partial,
$$

$$
\mathbf{M}_\partial \mathbf{y}_\partial =
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{q_n}^\top \\
\mathbf{0} & \mathbf{0} & \mathbf{B}_{M_{nn}}^\top & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{B}_{M_{ns}}^\top & \mathbf{0}
\end{bmatrix}
\begin{pmatrix}
\mathbf{e}_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma
\end{pmatrix}.
\tag{7.100}
$$

Matrices $\mathbf{D}_{\mathrm{div}}$, $\mathbf{D}_{\mathrm{Div}}$ assume the form ($i, j \in \{1, n_w\}$, $m, n \in \{1, n_\theta\}$, $p, q \in \{1, n_\kappa\}$, $r, s \in \{1, n_\gamma\}$)

$$
D_{\mathrm{div}}^{is} = \left\langle \phi_w^i, \mathrm{div}\, \boldsymbol{\phi}_\gamma^s \right\rangle_{L^2(\Omega)}, \qquad D_{\mathrm{Div}}^{mq} = \langle \boldsymbol{\phi}_\theta^m, \mathrm{Div}\, \boldsymbol{\Phi}_\kappa^q \rangle_{L^2(\Omega, \mathbb{R}^2)}.
\tag{7.101}
$$

Matrix $\mathbf{B}_{q_n}$, $\mathbf{B}_{M_{nn}}$, $\mathbf{B}_{M_{ns}}$ are computed as ($l \in \{1, n_\partial\}$)

$$B_{q_n}^{rl} = \left\langle \gamma_n \boldsymbol{\phi}_\gamma^r, \, \phi_{\partial,1}^l \right\rangle_{L^2(\partial\Omega)}, \quad B_{M_{nn}}^{pl} = \left\langle \gamma_{nn} \boldsymbol{\Phi}_\kappa^p, \, \phi_{\partial,2}^l \right\rangle_{L^2(\partial\Omega)}, \quad B_{M_{ns}}^{pl} = \left\langle \gamma_{ns} \boldsymbol{\Phi}_\kappa^p, \, \phi_{\partial,3}^l \right\rangle_{L^2(\partial\Omega)}.$$
(7.102)

This finite dimensional system represents a purely mixed discretization of the problem and is really close to the plane elasticity system. Conforming finite elements for the plane elasticity system on simplicial meshes have been constructed in [AW02]. The resulting element is rather cumbersome and computationally expensive as the stress tensor has at least 24 degrees of freedom on a triangle For this reason, many finite element discretization imposes the symmetry of the stress tensor weakly [AFW07]. To actually implement the discretization, in Chap. 8 the Mindlin plate problem is going to be reformulated so that the momenta tensor is only weakly symmetric.

## 7.2 Mixed boundary conditions

In this section Assumption 3 on uniform boundary condition is modified to account for general non homogeneous boundary conditions. The discretization of Stokes-Dirac structure under mixed causality has been already treated in [KML18]. However, to satisfy the power balance at a discrete level, some additional parameters are introduced. This makes the employment of this methodology not simple and dependent on the considered application. Furthermore, elasticity models do not fall within the required assumptions.

We propose here two methodologies to tackle mixed boundary conditions within the Partitioned Finite Element Method. The first introduces Lagrange multipliers, and therefore algebraic constraints, to enforce the mixed causality. Finite dimensional differential algebraic port-Hamiltonian systems (pHDAE) have been introduced in [BMXZ18] for linear systems and in [MM19] for non linear systems. This enriched description share all the crucial features of ordinary pHs, but easily account for algebraic constraints, time-dependent transformations and explicit dependence on time in the Hamiltonian. The second method employs a domain decomposition technique to interconnect systems with different causalities. For the sake of simplicity The illustration is restrained to the linear case.

The open connected set $\Omega \subset \mathbb{R}^d$, $d = \{1, 2, 3\}$, with Lipschitz boundary $\partial\Omega$ represent the spatial domain. The boundary is split into two partition $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \{\emptyset\}$. The set $\Gamma_1$, $\Gamma_2$ are considered to be connected, cf. Fig. 7.1.

**Remark 10** (Connectedness of $\Gamma_1, \Gamma_2$)
*Disconnected set can be handled as well. This requires the introduction of an heavy notation and complicates the illustration. For sake of simplicity, the connectedness hypothesis applies.*
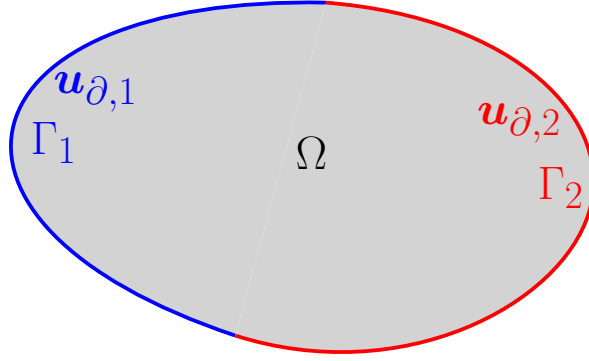
Figure 7.1: Partition of boundary into two connected sets.

For scalars $a_{\partial,*}, b_{\partial,*} \in L^2(\Gamma_*)$ and vectors $\boldsymbol{a}_{\partial,*}, \boldsymbol{b}_{\partial,*} \in L^2(\Gamma_*, \mathbb{R}^m)$ defined on the boundary partition $\Gamma_*$ the inner product is defined as

$$\langle a_{\partial,*},\, b_{\partial,*}\rangle_{L^2(\Gamma_*)} = \int_{\Gamma_*} a_{\partial,*} b_{\partial,*} \,\mathrm{d}\Gamma_*, \qquad \langle \boldsymbol{a}_{\partial,*},\, \boldsymbol{b}_{\partial,*}\rangle_{L^2(\Gamma_*,\mathbb{R}^m)} = \int_{\Gamma_*} \boldsymbol{a}_{\partial,*} \cdot \boldsymbol{b}_{\partial,*} \,\mathrm{d}\Gamma_*. \quad (7.103)$$

Consider now the following boundary control linear pH system in co-energy form

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\boldsymbol{L}^\top - \mathcal{L}^* \\ \boldsymbol{L} + \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{e}_1 \in H^{\mathcal{L}}, \\ \boldsymbol{e}_2 \in H^{-\mathcal{L}^*}, \end{aligned} \qquad (7.104a)$$

$$\begin{pmatrix} \boldsymbol{u}_{\partial,1} \\ \boldsymbol{u}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_2} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{u}_{\partial,1} \in \mathbb{R}^m, \\ \boldsymbol{u}_{\partial,2} \in \mathbb{R}^m, \end{aligned} \qquad (7.104b)$$

$$\begin{pmatrix} \boldsymbol{y}_{\partial,1} \\ \boldsymbol{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_1} \\ \mathcal{N}_{\partial,1}^{\Gamma_2} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{y}_{\partial,1} \in \mathbb{R}^m, \\ \boldsymbol{y}_{\partial,2} \in \mathbb{R}^m. \end{aligned} \qquad (7.104c)$$

The operator $\mathcal{N}_{\partial,*}^{\Gamma_\circ}$ with $*, \circ \in \{1,2\}$ represent now the restriction of operator $\mathcal{N}_{\partial,*}$, defined in Eq. (7.8), over the subset $\Gamma_\circ$. The boundary inputs and output are now vectors $\mathbb{R}^{2m}$. This does not mean that the boundary conditions have been doubled, but only that the components of $\boldsymbol{u}_\partial, \boldsymbol{y}_\partial$ are only defined on the subsets $\Gamma_1$, $\Gamma_2$ of the overall boundary. This corresponds to a slight modification of Assumption 3.

Given the additive property of the integral, it is possible to write

$$\langle \mathcal{N}_{\partial,1}\boldsymbol{e}_1,\, \mathcal{N}_{\partial,2}\boldsymbol{e}_2 \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_1}\boldsymbol{e}_1,\, \mathcal{N}_{\partial,2}^{\Gamma_1}\boldsymbol{e}_2 \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2}\boldsymbol{e}_1,\, \mathcal{N}_{\partial,2}^{\Gamma_2}\boldsymbol{e}_2 \right\rangle_{L^2(\Gamma_2,\mathbb{R}^m)},$$
$$= \langle \boldsymbol{u}_{\partial,1},\, \boldsymbol{y}_{\partial,1}\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \langle \boldsymbol{y}_{\partial,2},\, \boldsymbol{u}_{\partial,2}\rangle_{L^2(\Gamma_2,\mathbb{R}^m)}.$$
$$(7.105)$$

The continuous power balance is obtained using Eqs. (7.50) and (7.105)

$$\dot{H} = \langle \boldsymbol{u}_{\partial,1},\, \boldsymbol{y}_{\partial,1}\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \langle \boldsymbol{y}_{\partial,2},\, \boldsymbol{u}_{\partial,2}\rangle_{L^2(\Gamma_2,\mathbb{R}^m)}. \qquad (7.106)$$

### 7.2.1 Solution using Lagrange multipliers

This solution introduces a Lagrange multiplier for the boundary control that does not arise explicitly in the weak form. To illustrate the idea, consider again the weak form 7.51 (obtained by integration by parts of the $-\mathcal{L}^*$ partition) of Sys. 7.104

$$
\langle \boldsymbol{v}_1, \mathcal{M}_1 \partial_t \boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{A})} = -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{A})} - \langle \mathcal{L}\boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \mathcal{N}_{\partial,1}\boldsymbol{v}_1, \mathcal{N}_{\partial,2}\boldsymbol{e}_2 \rangle_{L^2(\partial\Omega,\mathbb{R}^m)},
$$
$$
\langle \boldsymbol{v}_2, \mathcal{M}_2 \partial_t \boldsymbol{e}_2 \rangle_{L^2(\Omega,\mathbb{B})} = \langle \boldsymbol{v}_2, \boldsymbol{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})}.
$$
$$(7.107)$$

The term $\langle \mathcal{N}_{\partial,1}\boldsymbol{v}_1, \mathcal{N}_{\partial,2}\boldsymbol{e}_2 \rangle_{L^2(\partial\Omega,\mathbb{R}^m)}$ can be split into the two boundary contributions, as in Eq. (7.105). The variable $\boldsymbol{y}_{\partial,1}$ plays here the role of a Lagrange multiplier $\boldsymbol{y}_{\partial,1} = \boldsymbol{\lambda}_{\partial,1}$

$$
\langle \mathcal{N}_{\partial,1}\boldsymbol{v}_1, \mathcal{N}_{\partial,2}\boldsymbol{e}_2 \rangle_{L^2(\partial\Omega,\mathbb{R}^m)} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_1}\boldsymbol{v}_1, \mathcal{N}_{\partial,2}^{\Gamma_1}\boldsymbol{e}_2 \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2}\boldsymbol{v}_1, \mathcal{N}_{\partial,2}^{\Gamma_2}\boldsymbol{e}_2 \right\rangle_{L^2(\Gamma_2,\mathbb{R}^m)},
$$
$$
= \left\langle \mathcal{N}_{\partial,1}^{\Gamma_1}\boldsymbol{v}_1, \boldsymbol{y}_{\partial,1} \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2}\boldsymbol{v}_1, \boldsymbol{u}_{\partial,2} \right\rangle_{L^2(\Gamma_2,\mathbb{R}^m)},
$$
$$
= \left\langle \mathcal{N}_{\partial,1}^{\Gamma_1}\boldsymbol{v}_1, \boldsymbol{\lambda}_{\partial,1} \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2}\boldsymbol{v}_1, \boldsymbol{u}_{\partial,2} \right\rangle_{L^2(\Gamma_2,\mathbb{R}^m)},
$$
$$(7.108)$$

If test function $\boldsymbol{v}_{\partial,1}$, $\boldsymbol{v}_{\partial,2} \in \mathbb{R}^m$ are introduced, the input and outputs definitions

$$
\boldsymbol{u}_{\partial,1} = \mathcal{N}_{\partial,1}^{\Gamma_1}\boldsymbol{e}_1, \qquad \boldsymbol{y}_{\partial,1} = \boldsymbol{\lambda}_{\partial,1}, \qquad \boldsymbol{y}_{\partial,2} = \mathcal{N}_{\partial,1}^{\Gamma_2}\boldsymbol{e}_1,
$$
$$(7.109)$$

can be put into weak form to obtain

$$
\langle \boldsymbol{v}_{\partial,1}, \boldsymbol{u}_{\partial,1} \rangle_{L^2(\Gamma_1,\mathbb{R}^m)} = \left\langle \boldsymbol{v}_{\partial,1}, \mathcal{N}_{\partial,1}^{\Gamma_1}\boldsymbol{e}_1 \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)},
$$
$$
\langle \boldsymbol{v}_{\partial,1}, \boldsymbol{y}_{\partial,1} \rangle_{L^2(\Gamma_1,\mathbb{R}^m)} = \langle \boldsymbol{v}_{\partial,1}, \boldsymbol{\lambda}_{\partial,1} \rangle_{L^2(\Gamma_1,\mathbb{R}^m)},
$$
$$
\langle \boldsymbol{v}_{\partial,2}, \boldsymbol{y}_{\partial,2} \rangle_{L^2(\Gamma_1,\mathbb{R}^m)} = \left\langle \boldsymbol{v}_{\partial,2}, \mathcal{N}_{\partial,1}^{\Gamma_2}\boldsymbol{e}_1 \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)}.
$$
$$(7.110)$$

As usual, a Galerkin approximation is introduced

$$
\boldsymbol{v}_1 \approx \sum_{i=1}^{n_1} \boldsymbol{\phi}_1^i(\boldsymbol{x}) v_1^i, \qquad \boldsymbol{e}_1 \approx \sum_{i=1}^{n_1} \boldsymbol{\phi}_1^i(\boldsymbol{x}) e_1^i(t), \qquad \triangle_{\partial,1} \approx \sum_{i=1}^{n_{\partial,1}} \boldsymbol{\phi}_{\partial,1}^i(\boldsymbol{s}_1) \triangle_{\partial,1}^i, \quad \boldsymbol{s}_1 \in \Gamma_1,
$$
$$
\boldsymbol{v}_2 \approx \sum_{i=1}^{n_2} \boldsymbol{\phi}_2^i(\boldsymbol{x}) v_2^i, \qquad \boldsymbol{e}_2 \approx \sum_{i=1}^{n_2} \boldsymbol{\phi}_2^i(\boldsymbol{x}) e_2^i(t), \qquad \square_{\partial,2} \approx \sum_{i=1}^{n_{\partial,2}} \boldsymbol{\phi}_{\partial,2}^i(\boldsymbol{s}_2) \square_{\partial,2}^i(t), \quad \boldsymbol{s}_2 \in \Gamma_2.
$$
$$(7.111)$$

where $\triangle$ stays for $v, u, y, \lambda$ and $\square$ for $v, u, y$. Replacing the approximation 7.111 into Eqs. 7.107, 7.108, 7.110, the following differential-algebraic system is constructed

$$
\text{Diag} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,1} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top & \mathbf{B}_{1,\Gamma_1} \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_{1,\Gamma_1}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_{1,\Gamma_2} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{\partial,1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,1} \\ \mathbf{B}_{1,\Gamma_2}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix}.
$$

$$(7.112)$$

Apart for matrices $\mathbf{M}_{\partial,1}, \mathbf{M}_{\partial,2}, \mathbf{B}_{1,\Gamma_1}, \mathbf{B}_{1,\Gamma_2}$,

$$
\begin{aligned}
M_{\partial,1}^{lk} &= \left\langle \boldsymbol{\phi}_{\partial,1}^l, \, \boldsymbol{\phi}_{\partial,1}^k \right\rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \quad (l,k) \in \{1, n_{\partial,1}\}, \quad B_{1,\Gamma_1}^{ik} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_1} \boldsymbol{\phi}_1^i, \, \boldsymbol{\phi}_{\partial,1}^k \right\rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \\
M_{\partial,2}^{fg} &= \left\langle \boldsymbol{\phi}_{\partial,2}^f, \, \boldsymbol{\phi}_{\partial,2}^g \right\rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \quad (f,g) \in \{1, n_{\partial,2}\}, \quad B_{1,\Gamma_2}^{ig} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2} \boldsymbol{\phi}_1^i, \, \boldsymbol{\phi}_{\partial,2}^g \right\rangle_{L^2(\Gamma_2, \mathbb{R}^m)},
\end{aligned} \quad i \in \{1, n_1\},
$$

$$(7.113)$$

the other matrices keep the same definition as in (7.53). The discrete Hamiltonian, whose expression is [BMXZ18]

$$
H_d = \frac{1}{2} \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \mathbf{e}_2. \tag{7.114}
$$

gives rise to the discrete power balance

$$
\begin{aligned}
\dot{H}_d &= \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \dot{\mathbf{e}}_1 + \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \dot{\mathbf{e}}_2, \\
&= -\mathbf{e}_1^\top (\mathbf{D}_0 + \mathbf{D}_{\mathcal{L}})^\top \mathbf{e}_2 + \mathbf{e}_2^\top (\mathbf{D}_0 + \mathbf{D}_{\mathcal{L}}) \mathbf{e}_1 + \mathbf{e}_1^\top (\mathbf{B}_{1,\Gamma_1} \boldsymbol{\lambda}_{\partial,1} + \mathbf{B}_{1,\Gamma_2} \mathbf{u}_{\partial,2}), \\
&= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}, \\
&= \widehat{\mathbf{y}}_{\partial,1}^\top \mathbf{u}_{\partial,1} + \widehat{\mathbf{y}}_{\partial,2}^\top \mathbf{u}_{\partial,2}, \qquad \text{where} \qquad \widehat{\mathbf{y}}_{\partial,1} := \mathbf{M}_{\partial,1} \mathbf{y}_{\partial,1}, \quad \widehat{\mathbf{y}}_{\partial,2} := \mathbf{M}_{\partial,2} \mathbf{y}_{\partial,2}.
\end{aligned} \tag{7.115}
$$

This result is the finite dimensional equivalent of (7.106).

Equivalently, the weak form Eq.7.52 may be used as a starting point. The computation follows in a completely analogous manner. The only difference is that $\boldsymbol{y}_{\partial,2} = \boldsymbol{\lambda}_{\partial,2}$ plays the role of the Lagrange multiplier. The final finite dimensional system then is given by

$$
\text{Diag} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top + \mathbf{D}_{-\mathcal{L}^*} & \mathbf{0} \\ \mathbf{D}_0 - \mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} & \mathbf{B}_{2,\Gamma_2} \\ \mathbf{0} & -\mathbf{B}_{2,\Gamma_2}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{2,\Gamma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,\Gamma_1}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix}.
$$

$$(7.116)$$

where $\mathbf{B}_{2,\Gamma_1}$, $\mathbf{B}_{2,\Gamma_2}$ are given by

$$
B_{2,\Gamma_1}^{mk} = \left\langle \mathcal{N}_{\partial,2}^{\Gamma_1} \boldsymbol{\phi}_2^m, \, \boldsymbol{\phi}_{\partial,1}^k \right\rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \quad B_{2,\Gamma_2}^{mg} = \left\langle \mathcal{N}_{\partial,2}^{\Gamma_2} \boldsymbol{\phi}_2^m, \, \boldsymbol{\phi}_{\partial,2}^g \right\rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \tag{7.117}
$$

where $m \in \{1, n_2\}$, $k \in \{1, n_{\partial,1}\}$, $g \in \{1, n_{\partial,2}\}$. This solution can be applied to incorporate
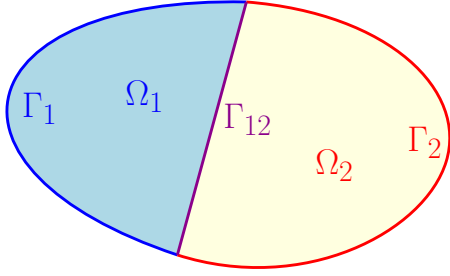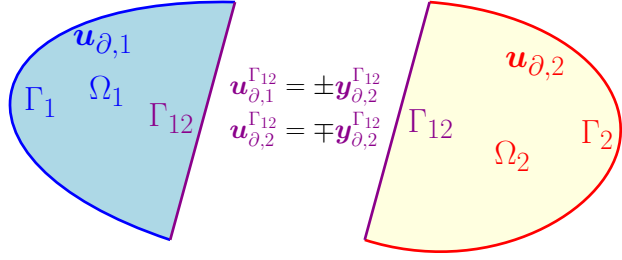
Figure 7.2: Splitting of the domain.



Figure 7.3: Interconnection at the interface $\Gamma_{12}$.

all possible mixed boundary conditions in a systematic manner. However the finite element discretization is required to satisfy the inf-sup condition. Simulating the resulting system is harder, since the algebraic constrains pose additional difficulties for the time integration.

### 7.2.2 Virtual domain decomposition

Since the boundary subsets $\Gamma_1$, $\Gamma_2$ are supposed to be connected set, a single interface is sufficient to decompose the system appropriately. In Fig. 7.2 the splitting of the domain is accomplished by introducing the interface $\Gamma_{12}$. This separation line that separates the domain is an additional degree of freedom, as it can be freely drawn. If the finite element method is used for the basis functions, the interface should be drawn so that the meshing of the subdomains does not generate excessively skewed triangles.

The idea is based on the fact that System 7.104 can be split into two systems with uniform causality. The following set of boundary variables is used for $\Omega_1$ subdomain

$$\begin{pmatrix} \boldsymbol{u}_{\partial,1} \\ \boldsymbol{u}_{\partial,1}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ \mathcal{N}_{\partial,1}^{\Gamma_{12}} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{y}_{\partial,1} \\ \boldsymbol{y}_{\partial,1}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_1} \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_{12}} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}. \tag{7.118}$$

Whereas for the $\Omega_2$ subdomain, the boundary variables are

$$\begin{pmatrix} \boldsymbol{u}_{\partial,2} \\ \boldsymbol{u}_{\partial,2}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_2} \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_{12}} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{y}_{\partial,2} \\ \boldsymbol{y}_{\partial,2}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ \mathcal{N}_{\partial,1}^{\Gamma_{12}} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}. \tag{7.119}$$

The following relations then hold (cf. Fig. 7.3)

$$\boldsymbol{u}_{\partial,1}^{\Gamma_{12}} = \pm \boldsymbol{y}_{\partial,2}^{\Gamma_{12}}, \qquad \boldsymbol{u}_{\partial,2}^{\Gamma_{12}} = \mp \boldsymbol{y}_{\partial,1}^{\Gamma_{12}}. \tag{7.120}$$

The plus or minus sign is due to the fact that either $\mathcal{N}_{\partial,1}^{\Gamma_{12}}$ or $\mathcal{N}_{\partial,2}^{\Gamma_{12}}$ contains a scalar product with the outgoing normal (or the tangent unit vector) at $\Gamma_{12}$ (that has opposite direction depending on which subdomain is considered). These relations are at the core of the methodology, since they state the equivalence between a problem with mixed causalities and the interconnection of two problems with uniform causality.

To obtain a final system with the desired causality, the weak form has to be carried out separately on each subdomain. In particular, on subdomain $\Omega_1$ the $\mathcal{L}$ operator is integrated by parts, whereas on subdomain $\Omega_2$ the $-\mathcal{L}^*$ operator undergoes the integration by parts. Consequently, on subdomains $\Omega_1$ ($\Omega_2$) the boundary input $\boldsymbol{u}_{\partial,1}$ ($\boldsymbol{u}_{\partial,2}$) explicitly appears. Let $L^2(\Omega_*, \mathbb{A})$ be the $L^2(\Omega, \mathbb{A})$ space restricted to the subdomain $\Omega_*$, and let $L^2(\Omega_*, \mathbb{B})(\Omega_*)$ be the restriction of $L^2(\Omega, \mathbb{B})$ to $\Omega_*$ for $* \in \{1, 2\}$. The weak form of the dynamics (7.104a) for the $\Omega_1$ contribution reads

$$
\begin{aligned}
\langle \boldsymbol{v}_1, \mathcal{M}_1 \partial_t \boldsymbol{e}_1 \rangle_{L^2(\Omega_1,\mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega_1,\mathbb{A})} - \langle \boldsymbol{v}_1, \mathcal{L}^* \boldsymbol{e}_2 \rangle_{L^2(\Omega_1,\mathbb{A})} \\
\langle \boldsymbol{v}_2, \mathcal{M}_2 \partial_t \boldsymbol{e}_2 \rangle_{L^2(\Omega_1,\mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega_1,\mathbb{B})} + \langle \mathcal{L}^* \boldsymbol{v}_2, \boldsymbol{e}_1 \rangle_{L^2(\Omega_1,\mathbb{A})} + \langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2, \mathcal{N}_{\partial,1} \boldsymbol{e}_1 \rangle_{L^2(\partial\Omega_1,\mathbb{R}^m)} .
\end{aligned}
\tag{7.121}
$$

For $\Omega_2$, we get

$$
\begin{aligned}
\langle \boldsymbol{v}_1, \mathcal{M}_1 \partial_t \boldsymbol{e}_1 \rangle_{L^2(\Omega_2,\mathbb{A})} &= -\langle \boldsymbol{v}_1, \boldsymbol{L}^\top \boldsymbol{e}_2 \rangle_{L^2(\Omega_2,\mathbb{A})} - \langle \mathcal{L} \boldsymbol{v}_1, \boldsymbol{e}_2 \rangle_{L^2(\Omega_2,\mathbb{B})} + \langle \mathcal{N}_{\partial,1} \boldsymbol{v}_1, \mathcal{N}_{\partial,2} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega_2,\mathbb{R}^m)} , \\
\langle \boldsymbol{v}_2, \mathcal{M}_2 \partial_t \boldsymbol{e}_2 \rangle_{L^2(\Omega_2,\mathbb{B})} &= \langle \boldsymbol{v}_2, \boldsymbol{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega_2,\mathbb{B})} + \langle \boldsymbol{v}_2, \mathcal{L}\boldsymbol{e}_1 \rangle_{L^2(\Omega_2,\mathbb{B})} .
\end{aligned}
\tag{7.122}
$$

Since $\partial\Omega_1 = \overline{\Gamma}_1 \cup \overline{\Gamma}_{12}$ and $\partial\Omega_2 = \overline{\Gamma}_2 \cup \overline{\Gamma}_{12}$, the boundary terms can be decomposed

$$
\begin{aligned}
\langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2, \mathcal{N}_{\partial,1} \boldsymbol{e}_1 \rangle_{L^2(\partial\Omega_1,\mathbb{R}^m)} &= \langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2, \mathcal{N}_{\partial,1} \boldsymbol{e}_1 \rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \langle \mathcal{N}_{\partial,2} \boldsymbol{v}_2, \mathcal{N}_{\partial,1} \boldsymbol{e}_1 \rangle_{L^2(\Gamma_{12},\mathbb{R}^m)} , \\
&= \left\langle \mathcal{N}_{\partial,2}^{\Gamma_1} \boldsymbol{v}_2, \mathcal{N}_{\partial,1}^{\Gamma_1} \boldsymbol{e}_1 \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,2}^{\Gamma_{12}} \boldsymbol{v}_2, \mathcal{N}_{\partial,1}^{\Gamma_{12}} \boldsymbol{e}_1 \right\rangle_{L^2(\Gamma_{12},\mathbb{R}^m)} , \\
&= \left\langle \mathcal{N}_{\partial,2}^{\Gamma_1} \boldsymbol{v}_2, \boldsymbol{u}_{\partial,1} \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,2}^{\Gamma_{12}} \boldsymbol{v}_2, \boldsymbol{u}_{\partial,1}^{\Gamma_{12}} \right\rangle_{L^2(\Gamma_{12},\mathbb{R}^m)} .
\end{aligned}
\tag{7.123}
$$

Analogously, for the remaining boundary term we find

$$
\langle \mathcal{N}_{\partial,1} \boldsymbol{v}_1, \mathcal{N}_{\partial,2} \boldsymbol{e}_2 \rangle_{L^2(\partial\Omega_2,\mathbb{R}^m)} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2} \boldsymbol{v}_1, \boldsymbol{u}_{\partial,2} \right\rangle_{L^2(\Gamma_2,\mathbb{R}^m)} + \left\langle \mathcal{N}_{\partial,1}^{\Gamma_{12}} \boldsymbol{v}_1, \boldsymbol{u}_{\partial,2}^{\Gamma_{12}} \right\rangle_{L^2(\Gamma_{12},\mathbb{R}^m)} .
\tag{7.124}
$$

A Galerkin approximation, analogous to (7.111), is used for each subdomain

$$
\begin{aligned}
\boldsymbol{v}_{1,1} &\approx \sum_{i=1}^{n_{1,1}} \boldsymbol{\phi}_{1,1}^i(\boldsymbol{x}_1) v_{1,1}^i, \quad \boldsymbol{x}_1 \in \Omega_1, & \boldsymbol{v}_{1,2} &\approx \sum_{i=1}^{n_{1,2}} \boldsymbol{\phi}_{1,2}^i(\boldsymbol{x}_2) v_{1,2}^i, \quad \boldsymbol{x}_2 \in \Omega_2, \\
\boldsymbol{v}_{2,1} &\approx \sum_{i=1}^{n_{2,1}} \boldsymbol{\phi}_{2,1}^i(\boldsymbol{x}_1) v_{2,1}^i, & \boldsymbol{v}_{2,2} &\approx \sum_{i=1}^{n_{2,2}} \boldsymbol{\phi}_{2,2}^i(\boldsymbol{x}_2) v_{2,2}^i, \\
\boldsymbol{e}_{1,1} &\approx \sum_{i=1}^{n_{1,1}} \boldsymbol{\phi}_{1,1}^i(\boldsymbol{x}_1) e_{1,1}^i(t), & \boldsymbol{e}_{1,2} &\approx \sum_{i=1}^{n_{1,2}} \boldsymbol{\phi}_{1,2}^i(\boldsymbol{x}_2) e_{1,2}^i(t), \\
\boldsymbol{e}_{2,1} &\approx \sum_{i=1}^{n_{2,1}} \boldsymbol{\phi}_{2,1}^i(\boldsymbol{x}_1) e_{2,1}^i(t), & \boldsymbol{e}_{2,2} &\approx \sum_{i=1}^{n_{2,2}} \boldsymbol{\phi}_{2,2}^i(\boldsymbol{x}_2) e_{2,2}^i(t).
\end{aligned}
\tag{7.125}
$$

For the boundary variables, additional terms for the common interface are needed

$$
\begin{aligned}
\Box_{\partial,1} &\approx \sum_{i=1}^{n_{\partial,1}} \phi_{\partial,1}^i(\boldsymbol{s}_1) \Box_{\partial,1}^i(t), \quad \boldsymbol{s}_1 \in \Gamma_1, \qquad \Box_{\partial,1}^{\Gamma_{12}} \approx \sum_{i=1}^{n_{\partial,12}} \phi_{\partial,12}^i(\boldsymbol{s}_{12}) \Box_{\partial,1}^{i,\Gamma_{12}}(t), \\
\Box_{\partial,2} &\approx \sum_{i=1}^{n_{\partial,2}} \phi_{\partial,2}^i(\boldsymbol{s}_2) \Box_{\partial,2}^i(t), \quad \boldsymbol{s}_2 \in \Gamma_2. \qquad \Box_{\partial,2}^{\Gamma_{12}} \approx \sum_{i=1}^{n_{\partial,12}} \phi_{\partial,12}^i(\boldsymbol{s}_{12}) \Box_{\partial,2}^{i,\Gamma_{12}}(t),
\end{aligned}
\qquad \boldsymbol{s}_{12} \in \Gamma_{12}.
$$

$$(7.126)$$

where $\Box$ stays for $v$, $u$, $y$.

**Remark 11** (Choice of the interface basis functions)
*Notice that the same basis functions $\phi_{\partial,12}$ are used for both interface variables. This is necessary in order to dispose of the same degrees of freedom for the interconnection.*

Replacing approximations 7.111, 7.126 into Eqs. 7.121, 7.123, 7.118, a finite dimensional system for the $\Omega_1$ subdomain is obtained

$$
\begin{aligned}
\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1}^{\Omega_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2}^{\Omega_1} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_{1,1} \\ \dot{\mathbf{e}}_{2,1} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^{\Omega_1\top} + \mathbf{D}_{-\mathcal{L}^*}^{\Omega_1} \\ \mathbf{D}_0^{\Omega_1} - \mathbf{D}_{-\mathcal{L}^*}^{\Omega_1\top} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,1} \\ \mathbf{e}_{2,1} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{2,\Gamma_1}^{\Omega_1} & \mathbf{B}_{2,\Gamma_{12}}^{\Omega_1} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,1}^{\Gamma_{12}} \end{pmatrix}, \\
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,12} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,1}^{\Gamma_{12}} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,\Gamma_1}^{\Omega_1\top} \\ \mathbf{0} & \mathbf{B}_{2,\Gamma_{12}}^{\Omega_1\top} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,1} \\ \mathbf{e}_{2,1} \end{pmatrix}.
\end{aligned}
$$

$$(7.127)$$

The mass and interconnection operator matrices are the restriction to the subdomain of the matrices given in (7.116)

$$
\begin{aligned}
M_{\mathcal{M}_1}^{\Omega_1,ij} &= \left\langle \phi_{1,1}^i, \mathcal{M}_1 \phi_{1,1}^j \right\rangle_{L^2(\Omega_1,\mathbb{A})}, \quad D_0^{\Omega_1,mj} = \left\langle \phi_{2,1}^i, \boldsymbol{L}\phi_{1,1}^j \right\rangle_{L^2(\Omega_1,\mathbb{B})}, \qquad i,j \in \{1,n_{1,1}\}, \\
M_{\mathcal{M}_2}^{\Omega_1,mn} &= \left\langle \phi_{2,1}^m, \mathcal{M}_2 \phi_{2,1}^n \right\rangle_{L^2(\Omega_1,\mathbb{B})}, \quad D_{-\mathcal{L}^*}^{\Omega_1,in} = \left\langle \phi_{1,1}^m, -\mathcal{L}^* \phi_{2,1}^n \right\rangle_{L^2(\Omega_1,\mathbb{A})}, \quad m,n \in \{1,n_{2,1}\}.
\end{aligned}
$$

$$(7.128)$$

Matrices $\mathbf{M}_{\partial,1}$ is constructed as in Eq. (7.116). Matrix $\mathbf{M}_{\partial,12}$ is similarly built

$$
M_{\partial,12}^{lk} = \left\langle \phi_{\partial,12}^l, \phi_{\partial,12}^k \right\rangle_{L^2(\Gamma_{12},\mathbb{R}^m)}, \qquad l,k \in \{1,n_{\partial,12}\}.
$$

$$(7.129)$$

The novel matrices $\mathbf{B}_{2,\Gamma_1}^{\Omega_1}$, $\mathbf{B}_{1,\Gamma_{12}}^{\Omega_1}$ have elements

$$
\begin{aligned}
B_{2,\Gamma_1}^{\Omega_1,mh} &= \left\langle \mathcal{N}_{\partial,2}^{\Gamma_1} \phi_{2,1}^m, \phi_{\partial,1}^h \right\rangle_{L^2(\Gamma_1,\mathbb{R}^m)}, \qquad m \in \{1,n_{2,1}\}, \quad h \in \{1,n_{\partial,1}\}, \\
B_{2,\Gamma_{12}}^{\Omega_1,mk} &= \left\langle \mathcal{N}_{\partial,2}^{\Gamma_{12}} \phi_{2,1}^m, \phi_{\partial,12}^k \right\rangle_{L^2(\Gamma_{12},\mathbb{R}^m)}, \qquad\qquad\qquad k \in \{1,n_{\partial,12}\}.
\end{aligned}
$$

$$(7.130)$$

If instead the approximations are plugged into Eqs. 7.122, 7.124, 7.119, a finite dimensional system for the $\Omega_2$ subdomain is computed

$$\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1}^{\Omega_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2}^{\Omega_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_{1,2} \\ \dot{\mathbf{e}}_{2,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^{\Omega_2\top} - \mathbf{D}_{\mathcal{L}}^{\Omega_2\top} \\ \mathbf{D}_0^{\Omega_2} + \mathbf{D}_{\mathcal{L}}^{\Omega_2} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,2} \\ \mathbf{e}_{2,2} \end{pmatrix} + \begin{bmatrix} \mathbf{B}_{1,\Gamma_2}^{\Omega_2} & \mathbf{B}_{1,\Gamma_{12}}^{\Omega_2} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{\partial,2} \\ \mathbf{u}_{\partial,2}^{\Gamma_{12}} \end{pmatrix},$$

$$\begin{bmatrix} \mathbf{M}_{\partial,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,12} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,2} \\ \mathbf{y}_{\partial,2}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} \mathbf{B}_{1,\Gamma_2}^{\Omega_2\top} & \mathbf{0} \\ \mathbf{B}_{1,\Gamma_{12}}^{\Omega_2\top} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,2} \\ \mathbf{e}_{2,2} \end{pmatrix}.$$

$$(7.131)$$

The mass and interconnection operator matrices are the restriction to the subdomain of the matrices given in (7.112)

$$M_{\mathcal{M}_1}^{\Omega_2,ij} = \left\langle \phi_{1,2}^i, \, \mathcal{M}_1\phi_{1,2}^j \right\rangle_{L^2(\Omega_2,\mathbb{A})}, \quad D_0^{\Omega_2,mj} = \left\langle \phi_{2,2}^i, \, \boldsymbol{L}\phi_{1,2}^j \right\rangle_{L^2(\Omega_2,\mathbb{B})}, \quad i,j \in \{1, n_{1,2}\},$$

$$M_{\mathcal{M}_2}^{\Omega_2,mn} = \left\langle \phi_{2,2}^m, \, \mathcal{M}_2\phi_{2,2}^n \right\rangle_{L^2(\Omega_2,\mathbb{B})}, \quad D_{\mathcal{L}}^{\Omega_2,mj} = \left\langle \phi_{2,2}^m, \, \mathcal{L}\phi_{1,2}^n \right\rangle_{L^2(\Omega_2,\mathbb{B})}, \quad m,n \in \{1, n_{2,2}\}.$$

$$(7.132)$$

Matrix $\mathbf{M}_{\partial,2}$ is constructed as in (7.112). The elements of matrices $\mathbf{B}_{1,\Gamma_2}$, $\mathbf{B}_{1,\Gamma_{12}}$ are computed as

$$B_{1,\Gamma_2}^{ig} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2}\phi_{1,2}^i, \, \phi_{\partial,2}^g \right\rangle_{L^2(\Gamma_2)}, \quad i \in \{1, n_{1,2}\}, \quad g \in \{1, n_{\partial,2}\},$$

$$B_{1,\Gamma_{12}}^{ik} = \left\langle \mathcal{N}_{\partial,1}^{\Gamma_{12}}\phi_{1,2}^i, \, \phi_{\partial,12}^k \right\rangle_{L^2(\Gamma_{12})}, \quad\quad k \in \{1, n_{\partial,12}\}.$$

$$(7.133)$$

Systems (7.127), (7.131) are compactly rewritten as

| System (7.127) | System (7.131) |
|---|---|
| $\mathbf{M}_{\Omega_1}\dot{\mathbf{e}}_{\Omega_1} = \mathbf{J}_{\Omega_1}\mathbf{e}_{\Omega_1} + \mathbf{B}_{\Gamma_1}^{\Omega_1}\mathbf{u}_{\partial,1} + \mathbf{B}_{\Gamma_{12}}^{\Omega_1}\mathbf{u}_{\partial,1}^{\Gamma_{12}},$ | $\mathbf{M}_{\Omega_2}\dot{\mathbf{e}}_{\Omega_2} = \mathbf{J}_{\Omega_2}\mathbf{e}_{\Omega_2} + \mathbf{B}_{\Gamma_2}^{\Omega_2}\mathbf{u}_{\partial,2} + \mathbf{B}_{\Gamma_{12}}^{\Omega_2}\mathbf{u}_{\partial,2}^{\Gamma_{12}},$ |
| $\mathbf{M}_{\partial,1}\mathbf{y}_{\partial,1} = \mathbf{B}_{\Gamma_1}^{\Omega_1\top}\mathbf{e}_{\Omega_1},$ | $\mathbf{M}_{\partial,2}\mathbf{y}_{\partial,2} = \mathbf{B}_{\Gamma_2}^{\Omega_2\top}\mathbf{e}_{\Omega_2},$ |
| $\mathbf{M}_{\partial,12}\mathbf{y}_{\partial,1}^{\Gamma_{12}} = \mathbf{B}_{\Gamma_{12}}^{\Omega_1\top}\mathbf{e}_{\Omega_1}.$ | $\mathbf{M}_{\partial,12}\mathbf{y}_{\partial,2}^{\Gamma_{12}} = \mathbf{B}_{\Gamma_{12}}^{\Omega_2\top}\mathbf{e}_{\Omega_2}.$ |
| (7.134) | (7.135) |
| with Hamiltonian $H_{d,1} = \frac{1}{2}\mathbf{e}_{\Omega_1}^\top\mathbf{M}_{\Omega_1}\mathbf{e}_{\Omega_1}$ | with Hamiltonian $H_{d,2} = \frac{1}{2}\mathbf{e}_{\Omega_2}^\top\mathbf{M}_{\Omega_2}\mathbf{e}_{\Omega_2}$ |

To obtain a system with the desired causality, an interconnection is employed to connect the two Systems (7.134), (7.135) along the shared boundary $\Gamma_{12}$. Given (7.120), the gyrator interconnection is computed as

$$\mathbf{u}_{\partial,1}^{\Gamma_{12}} = \pm\mathbf{y}_{\partial,2}^{\Gamma_{12}} = \pm\mathbf{M}_{\partial,12}^{-1}\mathbf{B}_{\Gamma_{12}}^{\Omega_2\top}\mathbf{e}_{\Omega_2},$$

$$\mathbf{u}_{\partial,2}^{\Gamma_{12}} = \mp\mathbf{y}_{\partial,2}^{\Gamma_{12}} = \mp\mathbf{M}_{\partial,12}^{-1}\mathbf{B}_{\Gamma_{12}}^{\Omega_1\top}\mathbf{e}_{\Omega_1},$$

$$(7.136)$$

The coupling matrix is then defined by

$$\mathbf{C} := \mathbf{B}_{\Gamma_{12}}^{\Omega_1}\mathbf{M}_{\partial,12}^{-1}\mathbf{B}_{\Gamma_{12}}^{\Omega_2\top}.$$

$$(7.137)$$

Plugging Eq. (7.136) into 7.134, 7.135, the final system with mixed causality is obtained

$$\begin{bmatrix} \mathbf{M}_{\Omega_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\Omega_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_{\Omega_1} \\ \dot{\mathbf{e}}_{\Omega_2} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_{\Omega_1} & \pm\mathbf{C} \\ \mp\mathbf{C}^\top & \mathbf{J}_{\Omega_2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{\Omega_1} \\ \mathbf{e}_{\Omega_2} \end{pmatrix} + \begin{bmatrix} \mathbf{B}_{\Gamma_1}^{\Omega_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\Gamma_2}^{\Omega_2} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{pmatrix},$$

$$\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{B}_{\Gamma_1}^{\Omega_1\top} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\Gamma_2}^{\Omega_2\top} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{\Omega_1} \\ \mathbf{e}_{\Omega_2} \end{pmatrix}. \tag{7.138}$$

The total Hamiltonian is the sum

$$H_d = H_{d,1} + H_{d,2} = \frac{1}{2}\mathbf{e}_{\Omega_1}^\top \mathbf{M}_{\Omega_1} \mathbf{e}_{\Omega_1} + \frac{1}{2}\mathbf{e}_{\Omega_2}^\top \mathbf{M}_{\Omega_2} \mathbf{e}_{\Omega_2}. \tag{7.139}$$

So, the power rate is

$$\begin{aligned} \dot{H}_d &= \mathbf{e}_{\Omega_1}^\top \mathbf{M}_{\Omega_1} \dot{\mathbf{e}}_{\Omega_1} + \mathbf{e}_{\Omega_2}^\top \mathbf{M}_{\Omega_2} \dot{\mathbf{e}}_{\Omega_2}, \\ &= \mathbf{e}_{\Omega_1}^\top \mathbf{J}_{\Omega_1} \mathbf{e}_{\Omega_1} + \mathbf{e}_{\Omega_2}^\top \mathbf{J}_{\Omega_2} \mathbf{e}_{\Omega_2} \pm \mathbf{e}_{\Omega_1}^\top \mathbf{C}\, \mathbf{e}_{\Omega_2} \mp \mathbf{e}_{\Omega_2}^\top \mathbf{C}^\top \mathbf{e}_{\Omega_1} + \mathbf{e}_{\Omega_1}^\top \mathbf{B}_{\Gamma_1}^{\Omega_1} \mathbf{u}_{\partial,1} + \mathbf{e}_{\Omega_2}^\top \mathbf{B}_{\Gamma_2}^{\Omega_2} \mathbf{u}_{\partial,2}, \\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}, \\ &= \widehat{\mathbf{y}}_{\partial,1}^\top \mathbf{u}_{\partial,1} + \widehat{\mathbf{y}}_{\partial,2}^\top \mathbf{u}_{\partial,2}, \qquad \text{where} \qquad \widehat{\mathbf{y}}_{\partial,1} := \mathbf{M}_{\partial,1}\mathbf{y}_{\partial,1}, \quad \widehat{\mathbf{y}}_{\partial,2} := \mathbf{M}_{\partial,2}\mathbf{y}_{\partial,2}. \end{aligned} \tag{7.140}$$

Again this results mimics its corresponding infinite-dimensional (7.106).

This technique allows obtaining a system with the correct causality, but has some draw-backs. Suitable finite elements are required for both kind of discretization detailed in Sec. 7.1.1, but the two are not always available (see Remark 9). A rigorous numerical convergence analysis of this technique appears rather involved. Some cases of mixed conditions, in particular conditions on single components of vectors, cannot be handled by this technique. For example, the simply supported condition in beams and plates imposes zero normal component of the traction at the boundary. Furthermore two different meshes are required and the interconnection has to manipulate carefully the degrees of freedom. This makes the implementation heavier than the Lagrange multiplier solution §7.2.1.

## 7.3   Conclusion

In this chapter a universal discretization method for multi-dimensional pHs has been detailed. The underlying Assumptions 2, 3 are indeed those that characterize the well-posedness of multi-dimensional pHs [Skr19]. For the time being, it has being shown that this technique is capable of constructing a finite-dimensional pHs from an infinite-dimensional one. For this reason, it is a structure-preserving method. The questions of numerical convergence and choice of approximation basis (in this thesis the focus is on the finite element method but spectral methods can be employed as well) are addressed in the next chapter, for the linear case only.

# Convergence numerical study

## 8.1 Plate problems using known mixed finite elements

## 8.2 Non-standard discretization of flexible structures

First of all we construct a family of finite elements capable of discretizing problem (**??**). Consider a regular triangulation $\mathcal{T}_h$ with elements $T$. The space of polynomials of order $k$ on a mesh cell is denoted by $P_k$. The following family of finite elements is conforming to the weak formulation (**??**)

$$
\begin{aligned}
H_h^1(\Omega) &= \{w_h \in H^1(\Omega)|\ \forall T,\ w_h|_T \in P_k\}, \\
H_h^{\mathrm{Grad}}(\Omega, \mathbb{R}^2) &= \{\boldsymbol{\theta}_h \in H^{\mathrm{Grad}}(\Omega, \mathbb{R}^2)|\ \forall T,\ \boldsymbol{\theta}_h|_T \in (P_k)^2\}, \\
L_h^2(\Omega, \mathbb{S}) &= \{\boldsymbol{M}_h \in L^2(\Omega, \mathbb{S})|\ \forall T, \boldsymbol{M}_h|_T \in (P_{k-1})_{\mathrm{sym}}^{2\times 2}\}, \\
L_h^2(\Omega, \mathbb{R}^2) &= \{\boldsymbol{q}_h \in L^2(\Omega, \mathbb{R}^2)|\ \forall T,\ \boldsymbol{q}_h|_T \in (P_{k-1})^2\},
\end{aligned}
\tag{8.1}
$$

To approximate spaces $H_h^1(\Omega)$, $H_h^{\mathrm{Grad}}(\Omega, \mathbb{R}^2)$ Lagrange polynomials of order $k$ are selected. For spaces $L_h^2(\Omega, \mathbb{S})$, $L_h^2(\Omega, \mathbb{R}^2)$ Discontinous Galerkin polynomials of order $k-1$ are employed. We use the acronym CGDG to denote this combination of elements

$$
\mathrm{CGDG} = \mathrm{CG} \times \mathrm{CG}(\mathbb{R}^2) \times \mathrm{DG}(\mathbb{S}) \times \mathrm{DG}(\mathbb{R}^2)
$$

This selection of finite elements can be seen as a standard discretization of the problem combined with a reduced integration of the stress tensor. For this reason, the following conjecture on the error estimates is proposed.

**Conjecture 4**
*Assuming a smooth solution to problem (**??**), the following error estimates hold*

$$
\begin{aligned}
||e_w - e_w^h||_{L^\infty(H^1(\Omega))} &\lesssim h^k, \quad ||\boldsymbol{E}_\kappa - \boldsymbol{E}_\kappa^h||_{L^\infty(L^2)} \lesssim h^k, \\
||\boldsymbol{e}_\theta - \boldsymbol{e}_\theta^h||_{L^\infty(H^{\mathrm{Grad}}(\Omega, \mathbb{R}^2)} &\lesssim h^k, \quad\ \ ||\boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma^h||_{L^\infty(L^2)} \lesssim h^k,
\end{aligned}
\tag{8.2}
$$

*where the notation $a \lesssim b$ means $a \leq Cb$. The constant $C$ depends only on the true solution and on the final time.*

To validate the method first we test a finite element combinations on an analytic solution. Constructing an analytical solution for a vibrating Mindlin plate is far from trivial. Therefore, the solution for the static case [**?**] is exploited.

**Step 1**  Consider a distributed static force given by

$$f_s(x, y) = \frac{E_Y}{12(1 - \nu^2)}\{12y(y - 1)(5x^2 - 5x + 1)$$
$$\times[2y^2(y - 1)2 + x(x - 1)(5y^2 - 5y + 1)] + 12x(x - 1)$$
$$\times(5y^2 - 5y + 1)[2x^2(x - 1)2 + y(y - 1)(5x^2 - 5x + 1)]\}.$$

The static displacement and rotation are given by

$$w_s(x, y) = \frac{1}{3}x^3(x - 1)^3 y^3(y - 1)^3 - \frac{2b^2}{5(1 - \nu)}[y^3(y - 1)^3 x(x - 1)(5x^2 - 5x + 1).$$

$$\boldsymbol{\theta}_s(x, y) = \begin{pmatrix} y^3(y - 1)^3 \ x^2(x - 1)^2(2x - 1) \\ x^3(x - 1)^3 \ y^2(y - 1)^2(2y - 1) \end{pmatrix}$$

The static solution solves the following problem defined on the square domain $\Omega = (0, 1) \times (0, 1)$:

$$0 = \text{div } \boldsymbol{q}_s + f_s, \qquad \boldsymbol{\mathcal{D}}_b^{-1}\boldsymbol{M}_s = \text{Grad } \boldsymbol{\theta}_s, \tag{8.3}$$
$$0 = \text{Div}\boldsymbol{M}_s + \boldsymbol{q}_s, \qquad D_s^{-1}\boldsymbol{q}_s = \text{grad } w_s - \boldsymbol{\theta}_s.$$

**Step 2**  Given the linear nature of the system a solution for the dynamic problem is found by multiplying the static solution by a time dependent term. For simplicity a sinus function is chosen

$$w_d(x, y, t) = w_s(x, y)\sin(t), \quad \boldsymbol{\theta}_d(x, y, t) = \boldsymbol{\theta}_s(x, y)\sin(t).$$

For the port-Hamiltonian system velocities are needed

$$e_w^{\text{ex}}(x, y, t) = w_s(x, y)\cos(t), \quad \boldsymbol{e}_\theta^{\text{ex}}(x, y, t) = \boldsymbol{\theta}_s(x, y)\cos(t).$$

The momenta and shear force are then defined by

$$\boldsymbol{M}_d = \boldsymbol{E}_\kappa^{\text{ex}} = \boldsymbol{\mathcal{D}}_b \text{ Grad } \boldsymbol{\theta}_d, \quad \boldsymbol{q}_d = \boldsymbol{e}_\gamma^{\text{ex}} = D_s(\text{grad } w_d - \boldsymbol{\theta}_d)$$

**Step 3**  Appropriate forcing terms have to be introduced (i.e. $f, \boldsymbol{\tau}$ in (**??**)). The force and torque in the dynamical case become

$$f_d = f_s \sin(t) + \rho b \partial_{tt} w_d, \qquad \boldsymbol{\tau}_d = \frac{\rho b^3}{12}\partial_{tt}\boldsymbol{\theta}_d.$$

Variables $(e_w^{\text{ex}}, \boldsymbol{e}_\theta^{\text{ex}}, \boldsymbol{E}_\kappa^{\text{ex}}, \boldsymbol{e}_\gamma^{\text{ex}})$ under excitations $(f_d, \boldsymbol{\tau}_d)$ solve problem (7.86a). The solution being smooth, the conjectured error estimates 4 should hold. The numerical values of the physical parameters are reported in Table 8.1. To integrate the equations in time a Crank-Nicholson scheme has been used. The time step is set to $\Delta t = h/10$ to have a lower impact of the time discretization error with respect to the spatial error. The final time is set to one $t_f = 1[\text{s}]$.

| Plate parameters | | | | |
|---|---|---|---|---|
| $E$ | $\rho$ | $\nu$ | $k$ | $h$ |
| 1 [Pa] | 1 [kg/m$^3$] | 0.3 | 5/6 | 0.1 [m] |

Table 8.1: Physical parameters for the Mindlin plate.

| $\frac{1}{h}$ | $||e_w - e_w^h||_{L^\infty(H^1)}$ | | $||\boldsymbol{e}_\theta - \boldsymbol{e}_\theta^h||_{L^\infty(H^{\mathrm{Grad}})}$ | | $||\boldsymbol{E}_\kappa - \boldsymbol{E}_\kappa^h||_{L^\infty(L^2)}$ | | $||\boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma^h||_{L^\infty(L^2)}$ | |
|---|---|---|---|---|---|---|---|---|
| | Error | Order | Error | Order | Error | Order | Error | Order |
| 8 | 7.30e-05 | — | 5.52e-04 | — | 3.99e-08 | — | 9.02e-07 | — |
| 16 | 3.13e-05 | 1.22 | 2.26e-04 | 1.28 | 1.88e-08 | 1.08 | 5.47e-07 | 0.72 |
| 32 | 1.57e-05 | 0.99 | 1.11e-04 | 1.02 | 8.84e-09 | 1.09 | 2.94e-07 | 0.89 |
| 64 | 7.87e-06 | 0.99 | 5.57e-05 | 0.99 | 4.31e-09 | 1.03 | 1.50e-07 | 0.97 |
| 128 | 3.94e-06 | 0.99 | 2.78e-05 | 0.99 | 2.14e-09 | 1.01 | 7.55e-08 | 0.99 |

Table 8.2: Mindlin plate convergence result $k = 1$.

| $\frac{1}{h}$ | $||e_w - e_w^h||_{L^\infty(H^1)}$ | | $||\boldsymbol{e}_\theta - \boldsymbol{e}_\theta^h||_{L^\infty(H^{\mathrm{Grad}})}$ | | $||\boldsymbol{E}_\kappa - \boldsymbol{E}_\kappa^h||_{L^\infty(L^2)}$ | | $||\boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma^h||_{L^\infty(L^2)}$ | |
|---|---|---|---|---|---|---|---|---|
| | Error | Order | Error | Order | Error | Order | Error | Order |
| 8 | 9.78e-06 | — | 1.04e-04 | — | 7.30e-09 | — | 1.77e-07 | — |
| 16 | 2.53e-06 | 1.95 | 2.49e-05 | 2.07 | 1.85e-09 | 1.97 | 4.93e-08 | 1.84 |
| 32 | 6.35e-07 | 1.99 | 6.06e-06 | 2.04 | 4.63e-10 | 1.99 | 1.27e-08 | 1.95 |
| 64 | 1.58e-07 | 1.99 | 1.50e-06 | 2.01 | 1.15e-10 | 2.00 | 3.21e-09 | 1.98 |
| 128 | 3.97e-08 | 2.00 | 3.74e-07 | 2.00 | 2.89e-11 | 2.00 | 8.06e-10 | 1.99 |

Table 8.3: Mindlin plate convergence result $k = 2$.

| $\frac{1}{h}$ | $||e_w - e_w^h||_{L^\infty(H^1)}$ | | $||\boldsymbol{e}_\theta - \boldsymbol{e}_\theta^h||_{L^\infty(H^{\mathrm{Grad}})}$ | | $||\boldsymbol{E}_\kappa - \boldsymbol{E}_\kappa^h||_{L^\infty(L^2)}$ | | $||\boldsymbol{e}_\gamma - \boldsymbol{e}_\gamma^h||_{L^\infty(L^2)}$ | |
|---|---|---|---|---|---|---|---|---|
| | Error | Order | Error | Order | Error | Order | Error | Order |
| 8 | 1.38e-06 | — | 1.24e-05 | — | 8.24e-10 | — | 2.24e-08 | — |
| 16 | 1.79e-07 | 2.94 | 1.51e-06 | 3.03 | 1.03e-10 | 2.99 | 2.90e-09 | 2.94 |
| 32 | 2.26e-08 | 2.98 | 1.88e-07 | 3.00 | 1.28e-11 | 3.00 | 3.64e-10 | 2.99 |
| 64 | 2.83e-09 | 2.99 | 2.36e-08 | 2.99 | 1.60e-12 | 3.00 | 4.54e-11 | 3.00 |
| 128 | 3.54e-10 | 2.99 | 2.95e-09 | 2.99 | 2.00e-13 | 3.00 | 5.67e-12 | 3.00 |

Table 8.4: Mindlin plate convergence result $k = 3$.

(a) $L^{\infty}_{\Delta t}(H^1(\Omega))$ error for $e_w$

(b) $L^{\infty}_{\Delta t}(H^{\mathrm{Grad}}(\Omega, \mathbb{R}^2))$ error for $\boldsymbol{e}_{\theta}$

(c) $L^{\infty}_{\Delta t}(L^2(\Omega, \mathbb{S}))$ error for $\boldsymbol{E}_{\kappa}$

(d) $L^{\infty}_{\Delta t}(L^2)$ error for $\boldsymbol{e}_{\gamma}$
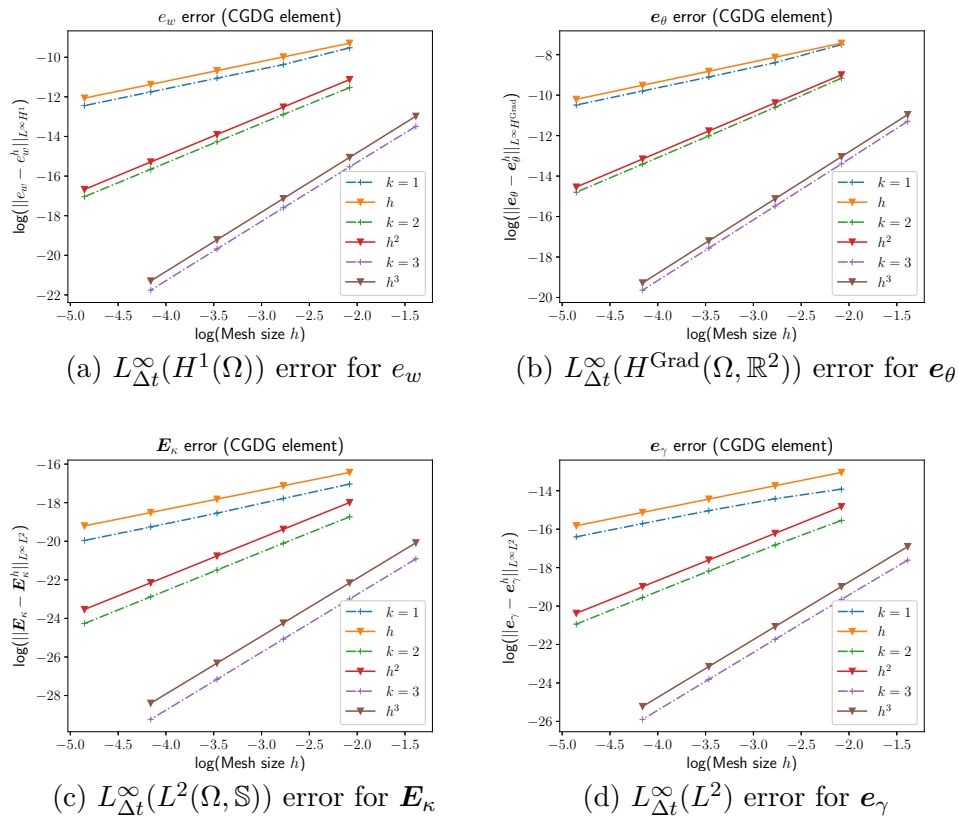
Figure 8.1: Error for the Mindlin plate using the CGDG elements

Chapter 9

# Numerical applications

## 9.1 Boundary stabilization

## 9.2 Thermoelastic wave propagation

## 9.3 Mixed boundary conditions

### 9.3.1 Trajectory tracking of a thin beam

### 9.3.2 Vibroacoustic under mixed boundary conditions

## 9.4 Modal analysis of plates

# Part IV

# Port-Hamiltonian flexible multibody dynamics

# Modular multibody systems in port-Hamiltonian form

## 10.1 Reminder of the rigid case

## 10.2 Flexible floating body

## 10.3 Modular construction of multibody systems

# Validation

## 11.1 Beam systems

### 11.1.1 Modal analysis of a flexible mechanism

### 11.1.2 Non-linear crank slider

### 11.1.3 Hinged beam

## 11.2 Plate systems

### 11.2.1 Boundary interconnection with a rigid element

### 11.2.2 Actuated plate

# Conclusion

# Conclusions and future directions

Je n'ai cherché de rien prouver, mais de bien peindre et d'éclairer bien ma
peinture.

*André Gide*
*Préface de L'Immoraliste*

Appendix A

# Mathematical tools

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## A.1  Differential operators

The space of all, symmetric and skew-symmetric $d \times d$ matrices are denoted by $\mathbb{M}$, $\mathbb{S}$, $\mathbb{K}$ respectively. The space of $\mathbb{R}^d$ vectors is denoted by $\mathbb{V}$. $\Omega \subset \mathbb{R}^d$ is an open connected set. For a scalar field $u : \Omega \to \mathbb{R}$ the gradient is defined as

$$\mathrm{grad}(u) = \nabla u := \left( \partial_{x_1} u \ldots \partial_{x_d} u \right)^\top.$$

For a vector field $\boldsymbol{u} : \Omega \to \mathbb{V}$, with components $u_i$, the gradient (Jacobian) is defined as

$$\mathrm{grad}(\boldsymbol{u})_{ij} := (\nabla \boldsymbol{u})_{ij} = \partial_{x_i} u_j.$$

The symmetric part of the gradient operator Grad (i. e. the deformation gradient in continuum mechanics) is thus given by

$$\mathrm{Grad}(\boldsymbol{u}) := \frac{1}{2} \left( \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top \right) \in \mathbb{S}.$$

The Hessian operator of $u$ is then computed as follows

$$\mathrm{Hess}(u) = \nabla^2 u = \mathrm{Grad}(\mathrm{grad}(u)),$$

For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{M}$, with components $u_{ij}$, the divergence is a vector, defined column-wise as

$$\mathrm{Div}(\boldsymbol{U}) = \nabla \cdot \boldsymbol{U} := \left( \sum_{i=1}^{d} \partial_{x_i} u_{ij} \right)_{j=1,\ldots,d}.$$

The double divergence of a tensor field $\boldsymbol{U}$ is then a scalar field defined as

$$\mathrm{div}(\mathrm{Div}(\boldsymbol{U})) := \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_{x_i} \partial_{x_j} u_{ij}.$$

**Definition 7** (Formal adjoint, Def. 5.80 [RR04])
*Consider the differential operator defined on $\Omega$*

$$\mathcal{L}(\boldsymbol{x}, \partial) = \sum_{|\alpha| \leq k} a_\alpha(\boldsymbol{x}) \partial^\alpha, \tag{A.1}$$

1509  *where $\alpha := (\alpha_1, \ldots, \alpha_d)$ is a multi-index of order $|\alpha| := \sum_{i=1}^{d} \alpha_i$, $a_\alpha$ are a set of real scalars*
1510  *and $\partial^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$ is a differential operator of order $|\alpha|$ resulting from a combination of*
1511  *spatial derivatives. The formal adjoint of $\mathcal{L}$ is the operator defined by*

$$\mathcal{L}^*(\boldsymbol{x}, \partial)u = \sum_{|\alpha| \leq k} (-1)^\alpha \partial^\alpha (a_\alpha(\boldsymbol{x})u(\boldsymbol{x})). \tag{A.2}$$

1512      The importance of this definition lies in the fact that

$$\langle \phi, \mathcal{L}(\boldsymbol{x}, \partial)\psi \rangle_\Omega = \langle \mathcal{L}^*(\boldsymbol{x}, \partial)\phi, \psi \rangle_\Omega \tag{A.3}$$

1513  for every $\phi, \psi \in C_0^\infty(\Omega)$. If the assumption of compact support is removed, then (A.3) no
1514  longer holds; instead the integration by parts yields additional terms involving integrals over
1515  the boundary $\partial\Omega$. However, these boundary terms vanish if $\phi$ and $\psi$ satisfy certain restrictions
1516  on the boundary.

## A.2   Integration by parts

1518  **Theorem 4** (Integration by parts for tensors)
1519  *Consider a smooth tensor-valued function $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ and vector-valued function $\boldsymbol{b} \in \mathbb{V} = \mathbb{R}^d$.*
1520  *The following integration by parts formula holds*

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{A}) \cdot \boldsymbol{b} + \boldsymbol{A} : \mathrm{grad}(\boldsymbol{b})\} \ \mathrm{d}\Omega = \int_\Omega \mathrm{div}(\boldsymbol{A}\boldsymbol{b}) \ \mathrm{d}\Omega = \int_{\partial\Omega} (\boldsymbol{A}^\top \boldsymbol{n}) \cdot \boldsymbol{b} \ \mathrm{d}S, \tag{A.4}$$

1521  *where $\boldsymbol{n}$ is the outward normal at the boundary and $\mathrm{d}S$ the infinitesimal surface.*

1522  *Proof.* Consider the components expression of Eq. (A.4)

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{A}) \cdot \boldsymbol{b} + \boldsymbol{A} : \mathrm{grad}(\boldsymbol{b})\} \ \mathrm{d}\Omega = \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} \{(\partial_{x_i} A_{ij})b_j + A_{ij}(\partial_{x_i} b_j)\} \ \mathrm{d}\Omega,$$

$$= \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_{x_i}(A_{ij}b_j) \ \mathrm{d}\Omega = \int_\Omega \mathrm{div}(\boldsymbol{A}\boldsymbol{b}) \ \mathrm{d}\Omega, \tag{A.5}$$

$$= \int_{\partial\Omega} \sum_{i=1}^{d} \sum_{j=1}^{d} (n_i A_{ij})b_j \ \mathrm{d}S = \int_{\partial\Omega} (\boldsymbol{A}^\top \boldsymbol{n}) \cdot \boldsymbol{b} \ \mathrm{d}S.$$

1523                                                                                                                    $\square$

1524      The previous result can be specialized for symmetric tensor field [BBF$^+$13, Chapter 1].

1525  **Theorem 5** (Integration by parts for symmetric tensors)
1526  *Consider a smooth tensor-valued function $\boldsymbol{M} \in \mathbb{S} = \mathbb{R}_{sym}^{d \times d}$ and vector-valued function $\boldsymbol{b} \in \mathbb{V} =$*

1527 $\mathbb{R}^d$. *Then, it holds*

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{S}) \cdot \boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\} \; \mathrm{d}\Omega = \int_\Omega \mathrm{div}(\boldsymbol{M}\boldsymbol{b}) \; \mathrm{d}\Omega = \int_{\partial\Omega} (\boldsymbol{M}\,\boldsymbol{n}) \cdot \boldsymbol{b} \; \mathrm{d}S. \tag{A.6}$$

1528 *Proof.* Consider the components expression of Eq. (A.6)

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{M}) \cdot \boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\} \; \mathrm{d}\Omega = \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} \left\{ (\partial_{x_i} M_{ij}) b_j + M_{ij} \frac{1}{2} (\partial_{x_i} b_j + \partial_{x_j} b_i) \right\} \; \mathrm{d}\Omega, \tag{A.7}$$

1529 The term $M_{ij}\frac{1}{2}(\partial_{x_i} b_j + \partial_{x_j} b_i)$ can be manipulated exploiting the symmetry of the tensor $\boldsymbol{M}$

$$\begin{aligned}
\sum_{i=1}^{d} \sum_{j=1}^{d} \frac{1}{2}(M_{ij}\partial_{x_i} b_j + M_{ij}\partial_{x_j} b_i) &= \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{1}{2}(M_{ij}\partial_{x_i} b_j + M_{ji}\partial_{x_i} b_j), \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{1}{2}(M_{ij} + M_{ji})\partial_{x_i} b_j \qquad \text{Since } \boldsymbol{M} \text{ is symmetric,} \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} M_{ij}\partial_{x_i} b_j = \boldsymbol{M} : \mathrm{grad}(\boldsymbol{b})
\end{aligned} \tag{A.8}$$

1530 Then it holds

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{M}) \cdot \boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\} \; \mathrm{d}\Omega = \int_\Omega \{\mathrm{Div}(\boldsymbol{M}) \cdot \boldsymbol{b} + \boldsymbol{M} : \mathrm{grad}(\boldsymbol{b})\} \; \mathrm{d}\Omega \tag{A.9}$$

1531 Using Eq (A.4) then

$$\begin{aligned}
\int_\Omega \{\mathrm{Div}(\boldsymbol{M}) \cdot \boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\} \; \mathrm{d}\Omega &= \int_\Omega \{\mathrm{Div}(\boldsymbol{M}) \cdot \boldsymbol{b} + \boldsymbol{M} : \mathrm{grad}(\boldsymbol{b})\} \; \mathrm{d}\Omega, \\
&= \int_{\partial\Omega} (\boldsymbol{M}^\top \boldsymbol{n}) \cdot \boldsymbol{b} \; \mathrm{d}S, \qquad \text{Since } \boldsymbol{M} \text{ is symmetric,} \\
&= \int_{\partial\Omega} (\boldsymbol{M}\,\boldsymbol{n}) \cdot \boldsymbol{b} \; \mathrm{d}S.
\end{aligned} \tag{A.10}$$

1532 This concludes the proof. $\qquad\square$

1533 ## A.3  Bilinear forms

**Definition 8** (Skew-symmetric bilinear form)
*A bilinear form on the Hilbert space $H$*

$$\begin{aligned}
b : H \times H &\longrightarrow \mathbb{R}, \\
(\boldsymbol{v}, \boldsymbol{u}) &\longrightarrow b(\boldsymbol{v}, \boldsymbol{u}),
\end{aligned}$$

*is skew-symmetric iff*

$$b(\boldsymbol{v}, \boldsymbol{u}) = -b(\boldsymbol{u}, \boldsymbol{v}).$$

# Finite elements gallery

# Implementation using FEniCS and Firedrake

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**Résumé** — Malgré l'abondante littérature sur le formalisme pH, les problèmes d'élasticité en deux ou trois dimensions géométriques n'ont presque jamais été considérés. Cette thèse vise à étendre l'approche port-Hamiltonienne (pH) à la mécanique des milieux continus. L'originalité apportée réside dans trois contributions majeures. Tout d'abord, la nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d'élasticité nécessite l'utilisation d'éléments finis non standard. Néanmoins, l'implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

**Mots clés :** Systèmes port-Hamiltonien, méchanique des solides, discretisation symplectique, méthode des éléments finis, dynamique multicorps

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**Abstract** — Despite the large literature on pH formalism, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an equivalent and intrinsic, i.e. coordinate free, pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

**Keywords:** Port-Hamiltonian systems, continuum mechanics, structure preserving discretization, finite element method, multibody dynamics.

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