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Numerics for Physics-Based PDEs with Boundary Control
The Partitioned Finite Element Method for PHs

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Outline

1 Introduction

2 Structure preserving discretization through mixed finite elements
- Uniform boundary conditions
- The linear case
- Mixed boundary conditions

3 Applications
- Boundary control of the irrotational shallow water equations
- Boundary control of the cantilever Kirchhoff plate

4 Conclusion

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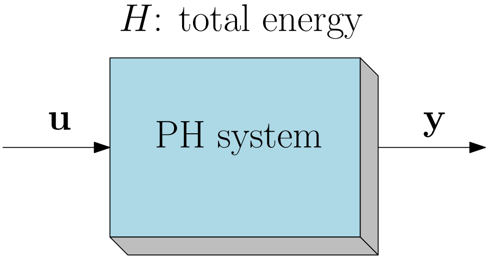
Why port-Hamiltonian systems?



Lossless: $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

PH systems are:

- Physically motivated;
- Lumped (ODEs) or distributed (PDEs);
- Passive (passivity based control);
- Closed under interconnection (modular multiphysics modelling);

Necessity of numerical methods

To tackle complex models and for control implementation, numerical methods are needed.

State of the art and this contribution

Discretization of port-Hamiltonian systems:

- Mixed finite elements for differential forms[1,2];
- Spectral methods[3];
- Finite differences[4].

This contribution

Mixed finite element for hyperbolic PDEs in port-Hamiltonian form under uniform or mixed boundary conditions.

[1] **golo2004hamiltonian**.
[2] **kotyczka2018weak**.
[3] **moulla2012pseudo**.
[4] **trenchant2018**.

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Structure preserving discretization

Infinite-dimensional pH system

PDE with boundary control:

$$\frac{\partial \boldsymbol{\alpha}}{\partial t}(\boldsymbol{x}, t) = \mathcal{J}\delta_{\boldsymbol{\alpha}} H.$$

Boundary conditions:

$$\boldsymbol{u}_\partial = \mathcal{B}_\partial \delta_{\boldsymbol{\alpha}} H, \qquad \boldsymbol{y}_\partial = \mathcal{C}_\partial \delta_{\boldsymbol{\alpha}} H.$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial \, \mathrm{d}S.$$

Structure-preserving discretization

Resulting ODE:

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_d &= \mathbf{J}\,\nabla H_d + \mathbf{B}_\partial \mathbf{u}_\partial, \\ \mathbf{y}_\partial &= \mathbf{B}_\partial^\top \nabla H_d. \end{aligned}$$

Discretized Hamiltonian:

$$H_d := H(\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_d).$$

Power balance:

$$\dot{H} = \mathbf{u}_\partial^\top \mathbf{y}_\partial.$$

Underlying hypotheses of the method

Assumption (Partitioned structure of the pH system)

The pH system has the partitioned form

$$\partial_t \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} := \begin{pmatrix} \delta_{\boldsymbol{\alpha}_1} H \\ \delta_{\boldsymbol{\alpha}_2} H \end{pmatrix},$$

$$\begin{aligned} \boldsymbol{\alpha}_1 &\in L^2(\Omega, \mathbb{A}), \\ \boldsymbol{\alpha}_2 &\in L^2(\Omega, \mathbb{B}), \\ \boldsymbol{e}_1 &\in H^{\mathcal{L}} := \left\{ \boldsymbol{u}_1 \in L^2(\Omega, \mathbb{A}) \middle| \, \mathcal{L}\boldsymbol{u}_1 \in L^2(\Omega, \mathbb{B}) \right\}, \\ \boldsymbol{e}_2 &\in H^{\mathcal{L}^*} := \left\{ \boldsymbol{u}_2 \in L^2(\Omega, \mathbb{B}) \middle| \, \mathcal{L}^*\boldsymbol{u}_2 \in L^2(\Omega, \mathbb{A}) \right\}. \end{aligned}$$

The sets $\mathbb{A}, \mathbb{B}$ are Cartesian product of either scalar, vectorial or tensorial quantities.

Wave-like equations (e.g. linear elastic models) possess this structure[5].

[5] **joly2003variational**.

Underlying hypotheses of the method

Assumption (Abstract integration by parts formula)

There exists two boundary operators $\mathcal{N}_{\partial,1}$, $\mathcal{N}_{\partial,2}$ such that a general integration by parts formula holds $\forall \boldsymbol{e}_1 \in H^{\mathcal{L}}$ and $\forall \boldsymbol{e}_2 \in H^{\mathcal{L}^*}$

$$\langle \boldsymbol{e}_2, \, \mathcal{L}\, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} - \langle \mathcal{L}^*\, \boldsymbol{e}_2, \, \boldsymbol{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} = \langle \mathcal{N}_{\partial,1} \boldsymbol{e}_1, \, \mathcal{N}_{\partial,2} \boldsymbol{e}_2 \rangle_{\partial\Omega} \, .$$

where $\langle \cdot, \, \cdot \rangle_{\partial\Omega}$ denotes an appropriate duality pairing.

Assumption (Uniform boundary condition)

The boundary operators $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ are then assumed to verify, in an exclusive manner, either

$$\mathcal{B}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}, \qquad \mathcal{C}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix},$$

or

$$\mathcal{B}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix}, \qquad \mathcal{C}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}.$$

The partitioned finite element method

This discretization procedure represents the application of mixed finite elements to port-Hamiltonian systems:

1.
2.
3.

The partitioned finite element method

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1. The system is written in weak form;
2. An integration by parts is applied to highlight the appropriate boundary control;
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The partitioned finite element method

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1. The system is written in weak form;
2. An integration by parts is applied to highlight the appropriate boundary control;
3. A Galerkin method is employed to obtain a finite-dimensional system. For the approximation basis the Finite Element Method FEM (large sparse matrices) is here employed but Spectral Methods SM (small full matrices) can be used as well.

The partitioned finite element method

This discretization procedure represents the application of mixed finite elements to port-Hamiltonian systems:

1. The system is written in weak form;
2. An integration by parts is applied to highlight the appropriate boundary control;
3. A Galerkin method is employed to obtain a finite-dimensional system. For the approximation basis the Finite Element Method FEM (large sparse matrices) is here employed but Spectral Methods SM (small full matrices) can be used as well.

The discretized system

Consider the causality

$$\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1, \qquad \boldsymbol{y}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2.$$

By integrating by parts $\mathcal{L}$ the appropriate causality is obtained for the discretized system.

Finite dimensional system for $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\mathcal{L}^*} \\ -\mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial,$$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{pmatrix},$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

The discretized system

Consider the causality

$$\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2, \qquad \boldsymbol{y}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1.$$

By integrating by parts $-\mathcal{L}^*$ the appropriate causality is obtained for the discretized system.

Finite dimensional system for $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,$$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{bmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{bmatrix},$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Discrete power balance

The power balance

$$\dot{H}_d = \partial_{\boldsymbol{\alpha}_{d,1}}^\top H_d(\boldsymbol{\alpha}_d)\dot{\boldsymbol{\alpha}}_{d,1} + \partial_{\boldsymbol{\alpha}_{d,2}}^\top H_d(\boldsymbol{\alpha}_d)\dot{\boldsymbol{\alpha}}_{d,2}$$

mimics the continuous one.

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1$

$$\begin{aligned}\dot{H}_d &= \mathbf{e}_1^\top \mathbf{D}_{-\mathcal{L}^*}\mathbf{e}_2 - \mathbf{e}_2^\top \mathbf{D}_{-\mathcal{L}^*}^\top \mathbf{e}_1 + \mathbf{e}_2^\top \mathbf{B}_2 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial\end{aligned}$$

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2$

$$\begin{aligned}\dot{H}_d &= -\mathbf{e}_1^\top \mathbf{D}_{\mathcal{L}}^\top \mathbf{e}_2 + \mathbf{e}_2^\top \mathbf{D}_{\mathcal{L}}\mathbf{e}_1 + \mathbf{e}_1^\top \mathbf{B}_1 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial.\end{aligned}$$

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The linear case

Assumption (Quadratic separable Hamiltonian)

The Hamiltonian is assumed to be a positive quadratic separable functional in $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$

$$H = \frac{1}{2} \langle \boldsymbol{\alpha}_1, \, \mathcal{Q}_1 \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \frac{1}{2} \langle \boldsymbol{\alpha}_2, \, \mathcal{Q}_2 \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})},$$

where $\mathcal{Q}_1, \mathcal{Q}_2$ *are positive symmetric bounded operators*

$$m_1 \boldsymbol{I}_{\mathbb{A}} \leq \mathcal{Q}_1 \leq M_1 \boldsymbol{I}_{\mathbb{A}}, \qquad m_2 \boldsymbol{I}_{\mathbb{B}} \leq \mathcal{Q}_2 \leq M_2 \boldsymbol{I}_{\mathbb{B}}, \qquad m_1 > 0, \; m_2 > 0, \; M_1 > 0, \; M_2 > 0.$$

PH linear system

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{e}_1 &\in H^{\mathcal{L}}, \\ \boldsymbol{e}_2 &\in H^{\mathcal{L}^*}, \end{aligned}$$

where $\mathcal{M}_1 := \mathcal{Q}_1^{-1}$, $\mathcal{M}_2 := \mathcal{Q}_2^{-1}$. **Constitutive laws** have been included in the dynamics.

The linear discretized system

Finite dimensional system for $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1,\ \boldsymbol{y}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2$

$$\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\mathcal{L}^*} \\ -\mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial,$$
$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Finite dimensional system for $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2,\ \boldsymbol{y}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1$

$$\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,$$
$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Power balance

The power balance

$$\dot{H}_d = \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \dot{\mathbf{e}}_1 + \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \dot{\mathbf{e}}_2$$

mimics the continuous one.

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1} \boldsymbol{e}_1$

$$\begin{aligned}\dot{H}_d &= \mathbf{e}_1^\top \mathbf{D}_{-\mathcal{L}^*} \mathbf{e}_2 - \mathbf{e}_2^\top \mathbf{D}_{-\mathcal{L}^*}^\top \mathbf{e}_1 + \mathbf{e}_2^\top \mathbf{B}_2 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial\end{aligned}$$

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2} \boldsymbol{e}_2$

$$\begin{aligned}\dot{H}_d &= -\mathbf{e}_1^\top \mathbf{D}_{\mathcal{L}}^\top \mathbf{e}_2 + \mathbf{e}_2^\top \mathbf{D}_{\mathcal{L}} \mathbf{e}_1 + \mathbf{e}_1^\top \mathbf{B}_1 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial.\end{aligned}$$

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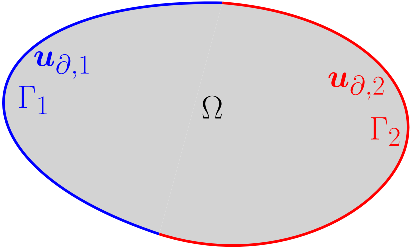
Mixed boundary conditions (linear system)

Consider now the following boundary-controlled linear pH system in co-energy form

$$
\begin{aligned}
\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} &= \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \\
\begin{pmatrix} \boldsymbol{u}_{\partial,1} \\ \boldsymbol{u}_{\partial,2} \end{pmatrix} &= \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_2} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \\
\begin{pmatrix} \boldsymbol{y}_{\partial,1} \\ \boldsymbol{y}_{\partial,2} \end{pmatrix} &= \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_1} \\ \mathcal{N}_{\partial,1}^{\Gamma_2} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}.
\end{aligned}
$$



The operator $\mathcal{N}_{\partial,*}^{\Gamma_\circ}$ with $*, \circ \in \{1, 2\}$ represents the restriction of operator $\mathcal{N}_{\partial,*}$ over the subset $\Gamma_\circ \subset \partial\Omega$.

Lagrange multiplier method

A Lagrange multiplier can be introduced to include the input that does not explicitly appear in the weak formulation, i.e. to enforce the essential boundary condition.

Integration by parts of $-\mathcal{L}^*$ ($\boldsymbol{\lambda}_{\partial,1} = \boldsymbol{y}_{\partial,1}$)

$$
\mathrm{Diag}\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,1} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\mathcal{L}}^\top & \mathbf{B}_{1,\Gamma_1} \\ \mathbf{D}_{\mathcal{L}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_{1,\Gamma_1}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_{1,\Gamma_2} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{\partial,1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,1} \\ \mathbf{B}_{1,\Gamma_2}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix}.
$$

A pH differential-algebraic system is obtained in this case (pHDAE).

Lagrange multiplier method

A Lagrange multiplier can be introduced to include the input that does not explicitly appear in the weak formulation, i.e. to enforce the essential boundary condition.

Integration by parts of $\mathcal{L}$ ($\boldsymbol{\lambda}_{\partial,2} = \boldsymbol{y}_{\partial,2}$)

$$
\mathrm{Diag}\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\mathcal{L}^*} & \mathbf{0} \\ -\mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} & \mathbf{B}_{2,\Gamma_2} \\ \mathbf{0} & -\mathbf{B}_{2,\Gamma_2}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{2,\Gamma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,\Gamma_1}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix}.
$$

A pH differential-algebraic system is obtained in this case (pHDAE).

Power balance

The energy balance

$$\dot{H}_d = \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \dot{\mathbf{e}}_1 + \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \dot{\mathbf{e}}_2$$

mimics the continuous counterpart.

Integration by parts of $-\mathcal{L}^*$ ($\boldsymbol{\lambda}_{\partial,1} = \boldsymbol{u}_{\partial,1}$)

$$\begin{aligned}\dot{H}_d &= -\mathbf{e}_1^\top \mathbf{D}_{\mathcal{L}}^\top \mathbf{e}_2 + \mathbf{e}_2^\top \mathbf{D}_{\mathcal{L}} \mathbf{e}_1 + \mathbf{e}_1^\top (\mathbf{B}_{1,\Gamma_1} \boldsymbol{\lambda}_{\partial,1} + \mathbf{B}_{1,\Gamma_2} \mathbf{u}_{\partial,2}), \\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}.\end{aligned}$$

Integration by parts of $\mathcal{L}$ ($\boldsymbol{\lambda}_{\partial,2} = \boldsymbol{u}_{\partial,2}$)

$$\begin{aligned}\dot{H}_d &= \mathbf{e}_1^\top \mathbf{D}_{-\mathcal{L}^*}^\top \mathbf{e}_2 - \mathbf{e}_2^\top \mathbf{D}_{-\mathcal{L}^*} \mathbf{e}_1 + \mathbf{e}_2^\top (\mathbf{B}_{2,\Gamma_2} \boldsymbol{\lambda}_{\partial,2} + \mathbf{B}_{2,\Gamma_1} \mathbf{u}_{\partial,1}), \\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}.\end{aligned}$$

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   - Boundary control of the irrotational shallow water equations
   - Boundary control of the cantilever Kirchhoff plate

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Irrotational shallow water equations

The Hamiltonian is a non-quadratic and non-separable functional

$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2}\int_\Omega \left\{\frac{1}{\rho}\alpha_h \left\|\boldsymbol{\alpha}_v\right\|^2 + \rho g \alpha_h^2\right\} \,\mathrm{d}\Omega.$$

Variables:

- α_h the fluid height;
- $\boldsymbol{\alpha}_v$ the linear momentum;

Parameters:

- ρ density;
- g gravity acceleration

Dynamics:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_h \\ \boldsymbol{\alpha}_v\end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \mathbf{0}\end{bmatrix}\begin{pmatrix} e_h \\ \boldsymbol{e}_v\end{pmatrix}, \qquad (x, y) \in \Omega = \{x^2 + y^2 \leq R\},$$

$$\begin{pmatrix} e_h \\ \boldsymbol{e}_v\end{pmatrix} = \begin{pmatrix}\delta_{\alpha_h} H \\ \delta_{\boldsymbol{\alpha}_v} H\end{pmatrix} = \begin{pmatrix}\frac{1}{2\rho}\left\|\boldsymbol{\alpha}_v\right\|^2 + \rho g \alpha_h \\ \frac{1}{\rho}\alpha_h \boldsymbol{\alpha}_v\end{pmatrix},$$

Proportional control law

Consider a uniform Neumann bc

$$u_\partial = -\boldsymbol{e}_v \cdot \boldsymbol{n}|_{\partial\Omega}.$$

Conjugated output

$$y_\partial = e_h|_{\partial\Omega}.$$

Proportional control: La Salle argument

Proportional control for stabilization around a given fluid height h^{des}

$$u_\partial = -k(y_\partial - y_\partial^{\text{des}}), \qquad y_\partial^{\text{des}} = \rho g h^{\text{des}}, \quad k > 0.$$

The control law ensures that the Lyapunov functional

$$V = \frac{1}{2}\int_\Omega \left\{ \frac{1}{2}\rho g(\alpha_h - h^{\text{des}})^2 + \frac{1}{2\rho}\alpha_h \left\|\boldsymbol{\alpha}_v\right\|^2 \right\} d\Omega \geq 0,$$

has negative semi definite time derivative

$$\dot{V} = -k\int_{\partial\Omega} \left(y_\partial - y_\partial^{\text{des}}\right)^2 \,\mathrm{d}\Gamma \leq 0.$$

Discretization strategy

- The div operator is integrated by parts to highlight the appropriate the Neumann boundary control.
- FEniCS is used to generate the matrices.

Parameters	
ρ	$1000\ [\mathrm{kg}\cdot\mathrm{m}^3]$
g	$10\ [\mathrm{m/s}^2]$
R	$1\ [\mathrm{m}]$
h^{des}	$1\ [\mathrm{m}]$

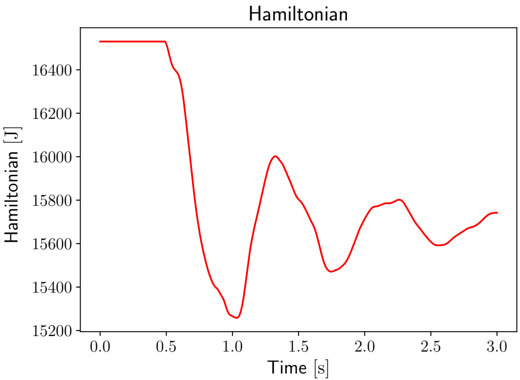
Simulation Settings	
Integrator	Runge-Kutta 45
$\mathrm{N}^{\circ}_{\text{dof}}$	3973
FE spaces	$(\alpha_h \approx \mathrm{CG}_1) \times (\boldsymbol{\alpha}_v \approx \mathrm{DG}_0) \times (u_\partial \approx \mathrm{DG}_0)$
t_{end}	$3\ [\mathrm{s}]$

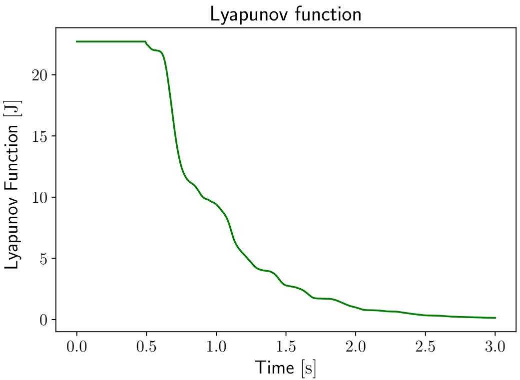
Control parameter $$k = \begin{cases} 0, & \forall t < 0.5\ [\mathrm{s}], \\ 10^{-3}, & \forall t \geq 0.5\ [\mathrm{s}]. \end{cases}$$

Results irrotational SWE

Control parameter $$k = \begin{cases} 0, & \forall t < 0.5\,[\mathrm{s}], \\ 10^{-3}, & \forall t \geq 0.5\,[\mathrm{s}]. \end{cases}$$

Results irrotational SWE





Outline

1 Introduction

2 Structure preserving discretization through mixed finite elements

3 Applications
- Boundary control of the irrotational shallow water equations
- Boundary control of the cantilever Kirchhoff plate

4 Conclusion

Cantilever Kirchhoff plate

The Hamiltonian is a quadratic functional (linear case), hence a co-energy formulation is used

$$H(e_w, \boldsymbol{E}_\kappa) = \frac{1}{2}\int_\Omega \left\{\rho h e_w^2 + \boldsymbol{\mathcal{D}}_b^{-1}(\boldsymbol{E}_\kappa) : \boldsymbol{E}_\kappa\right\} \, \mathrm{d}\Omega, \qquad \text{where} \qquad \boldsymbol{A} : \boldsymbol{B} = \sum_{ij} A_{ij} B_{ij}.$$

Variables:

- e_w the vertical velocity;
- $\boldsymbol{E}_\kappa$ the bending stress tensor;

Parameters:

- ρ density, h plate thickness;
- $\boldsymbol{\mathcal{D}}_b^{-1}$ the bending compliance tensor

$$\begin{bmatrix} \rho h & 0 \\ \boldsymbol{0} & \boldsymbol{\mathcal{D}}_b^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \operatorname{Hess} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix} \qquad (x, y) \in \Omega = [0, 1] \times [0, 1],$$

Damping injection control strategy

Consider mixed Dirichlet homogeneous conditions and Neumann boundary control

$$
\begin{aligned}
e_w|_{\Gamma_D} &= 0, \\
\partial_x e_w|_{\Gamma_D} &= 0,
\end{aligned}
\qquad \Gamma_D = \{x = 0\}, \qquad
\begin{aligned}
u_{\partial,q} &= \widetilde{q}_n|_{\Gamma_N}, \\
u_{\partial,m} &= M_{nn}|_{\Gamma_N}.
\end{aligned}
\qquad \Gamma_N = \{y = 0 \cup x = 1 \cup y = 1\}.
$$

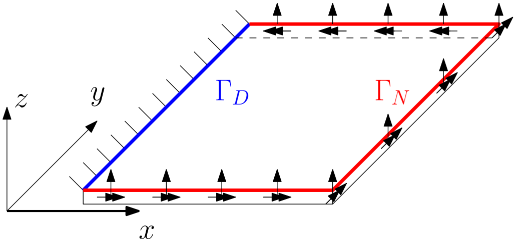
where M_{nn} is the flexural moment and $\widetilde{q}_n$ is the effective shear force.

The corresponding boundary outputs read

$$
\begin{aligned}
y_{\partial,q} &= e_w|_{\Gamma_N}, \\
y_{\partial,m} &= \partial_{\boldsymbol{n}} e_w|_{\Gamma_N}.
\end{aligned}
$$

The following control law stabilizes the system[5]

$$
\begin{aligned}
u_{\partial,q} &= -k y_{\partial,q}, \\
u_{\partial,m} &= -k y_{\partial,m},
\end{aligned}
\qquad k > 0.
$$



[5] **Iagnese1989**.

Discretization strategy

- The $\operatorname{div}\operatorname{Div}$ operator is integrated by parts twice to enforce weekly the Neumann bc.
- The FIREDRAKE library is used to generate the matrices.
- The Dirichlet condition is imposed weakly through a Lagrange multiplier (strong imposition of boundary conditions for H^2 conforming elements is not trivial[6]).

Plate Parameters	
E	70 [GPa]
ρ	2700 [kg $\cdot$ m^3]
ν	0.35
h/L	0.05
$L_x = L_y$	1 [m]

Simulation Settings	
Integrator	Störmer-Verlet
Δt	1 [μs]
N$^\circ_{\text{dof}}$	2574
FE spaces	$(e_w \approx \text{Argyris}) \times (\boldsymbol{E}_\kappa \approx \text{DG}_3) \times (\boldsymbol{\lambda} \approx \text{CG}_2)$
t_{end}	5 [s]

Control parameter
$$k = \begin{cases} 0, & \forall t < 1 \text{ [s]}, \\ 10, & \forall t \geq 1 \text{ [s]}. \end{cases}$$

[6] **kirby2019**.

Results cantilever Kirchoff plate

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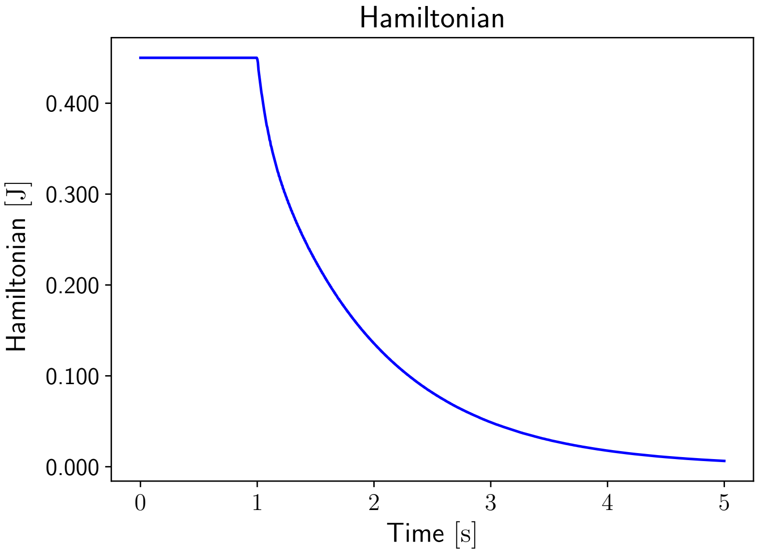
Results cantilever Kirchoff plate



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Open problems and Outlook

Open problem:

Developments:

[7] **chen2020divDiv**.
[8] **egger2018**.
[9] **toledo2020**.
[10] **wu2020reduced**.

Open problems and Outlook

Open problem:

- Finite element space for the Lagrange multiplier, satisfying the inf-sup condition;

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- Discretization: efficient finite element for the Kirchhoff plate based on div-div conforming elements[7];

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Developments:

- Discretization: efficient finite element for the Kirchhoff plate based on div-div conforming elements[7];
- Model reduction: POD methods, H_2-optimal strategies or Krilov subspace methods[8];
- Observer based boundary control[9] and reduced LQG design for distributed control[10].

[7] **chen2020divDiv**.
[8] **egger2018**.
[9] **toledo2020**.
[10] **wu2020reduced**.

Additional information

Jupiter Notebooks for the wave and heat equation and the Mindlin plate model are available:

brugnoli2020zenodo.

Flexible multibody dynamics for pHs based on the proposed discretization:

brugnoli2020msd.

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