Dissipative Dynamical Systems

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Outline

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Why dissipative dynamical systems?

All engineering systems exhibit dissipation.

- Electrical networks with resistors;
- Mechanical systems (viscoelastic or Coulomb friction);
- Thermodynamic systems: dissipation leads to an increase in entropy.

The notion of dissipativity establishes a natural link between the properties of input-output and state-space models. Many modern computational tools for the analysis and synthesis of control systems are based on it.

Jan C. Willems. "Dissipative dynamical systems Part I: General theory". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 321–351

Jan C. Willems. "Dissipative dynamical systems Part II: Linear systems with quadratic supply rates". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 352–393

Arjan van der Schaft. *L2-gain and passivity in nonlinear control*. Springer-Verlag, 1999

Some mathematical notation

$\mathbb{R}_+ = [0, \infty)$ denotes the set of positive reals.
Let V be a finite dimensional normed liner space with norm $||\cdot||_V$.
(If $V = \mathbb{R}^n$ then the Euclidean norm is denoted by$||x||_2 = \sqrt{x^\top x}$)

Definition (Local L^p_{loc} Banach spaces)

For each positive integer $p \in 1, 2, \ldots$, the set $L^p_{\text{loc}}(\mathbb{R}, V)$ consists of all functions $f : \mathbb{R} \to V$, which are measurable and satisfy

$$\int_a^b ||f(t)||_V^p \, \mathrm{d}t < \infty, \qquad \forall\, a, b \in \mathbb{R}.$$

The case $p = \infty$ consists of all bounded measurable functions on compact intervals, i.e. $\sup_{t\in[a,b]} f(t) < \infty$.

General setting

Consider the state-space system with inputs and outputs

$$\Sigma : \quad \begin{aligned} \dot{x} &= f(x,u), \qquad & u(t) &\in U, \\ y &= h(x,u), \qquad & y(t) &\in Y, \end{aligned}$$

where $x(t) \in \mathcal{X}$. In general $\mathcal{X}$ is a manifold and U, Y vector spaces.
For sake simplicity, assume $\mathcal{X} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$.

Theorem

Suppose f, h to be Lipschitz continuous in x and u jointly. Then system Σ has a unique solution $\forall x(t_0) \in \mathbb{R}^n$, $u(\cdot) \in L^2_{loc}(\mathbb{R}, U)$ with $x(\cdot) \in L^2_{loc}(\mathbb{R}, \mathcal{X})$, $y(\cdot) \in L^2_{loc}(\mathbb{R}, Y)$.

Reachability and controllability

Notation: $\mathbb{R}^2_+ := \{(t_1, t_2) \in \mathbb{R}^2 |\ t_2 \geq t_1\}$ (causal triangular sector of $\mathbb{R}^2$).

Definition (State transition function)

Given the system Σ, the state transition function ϕ is the map

$$\phi(t_1, t_0, x(t_0), u(\cdot)) : \mathbb{R}^2_+ \times \mathcal{X} \times L^2_{\text{loc}}(\mathbb{R}, U) \to \mathbb{R}^n$$

such that $x(t_1) = \phi(t_1, t_0, x(t_0), u(\cdot))$.

Definition (Reachability and controllability)

The state space $\mathcal{X}$ of system Σ is said to be **reachable** from x_{-1} if

$$\forall\, x \in \mathcal{X},\ \exists\, t_{-1} \leq 0,\ \exists\, u(\cdot) \in L^2_{\text{loc}}(\mathbb{R}, U) \text{ such that } x = \phi(0, t_{-1}, x_{-1}, u(\cdot)).$$

It is said to be **controllable** to x_1 if

$$\forall\, x \in \mathcal{X},\ \exists\, t_1 > 0,\ \exists\, u(\cdot) \in L^2_{\text{loc}}(\mathbb{R}, U) \text{ such that } x_1 = \phi(t_1, 0, x, u(\cdot)).$$

Outline

The mathematical definition of dissipativity

On the combined space $U \times Y$ consider the supply rate function $s : U \times Y \to \mathbb{R}$.

Definition (Dissipative state space system)

A state space system Σ is said to be dissipative w.r.t. the supply rate s if there exists a function $S : \mathcal{X} \to \mathbb{R}_+$ (the storage function), such that $\forall\, x(t_0) \in \mathcal{X}$ at any time t_0, and $\forall\, u(\cdot)$ and $\forall\, t_1 \geq t_0$ and the following inequality holds

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t, \qquad \text{Dissipation Inequality.}$$

It equality holds then the system is called conservative (w.r.t. the supply rate s).

Corollary (Convexity of the storage functions set)

Given two storage functions S_1 *and* S_2 *then any convex combination* $\alpha S_1 + (1-\alpha) S_2,\ \alpha = [0, 1]$ *is also a storage function.*

Passive systems and L^2 finite gain

Two important class of supply rate functions:

- passive systems $s(u,y) = u^\top y$;
- finite L^2 gain $s(u,y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2, \quad \gamma \geq 0.$

Definition (Passive system)

Σ with $U = Y = \mathbb{R}^m$ is **passive** if it is dissipative w.r.t.

$$s(u,y) = u^\top y.$$

Σ is **input strictly passive** if $\exists\, \delta > 0$ such that Σ is dissipative w.r.t.

$$s(u,y) = u^\top y - \delta||u||_2^2.$$

Σ is **output strictly passive** if $\exists\, \varepsilon > 0$ such that Σ is dissipative w.r.t.

$$s(u,y) = u^\top y - \varepsilon||y||_2^2$$

Σ is **lossless** if it is conservative with respect to $s(u,y) = u^\top y$.

Passive systems and L^2 finite gain

Two important class of supply rate functions:

- passive systems $s(u, y) = u^\top y$;
- finite L^2 gain $s(u, y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2, \quad \gamma \geq 0.$

Definition (L^2 finite gain)

A system Σ with $U = \mathbb{R}^m,\ Y = \mathbb{R}^p$ has L^2-gain $\leq \gamma$ ($\gamma \geq 0$) if it is dissipative w.r.t.

$$s(u, y) = \frac{1}{2}\gamma||u||_2^2 - \frac{1}{2}||y||_2^2.$$

The L^2-gain of Σ is defined as

$$\gamma(\Sigma) := \inf\{\gamma |\ \Sigma \text{ has } L^2\text{-gain} \leq \gamma\}.$$

Σ is said to have L^2-gain $< \gamma$ if $\exists\, \tilde{\gamma} \leq \gamma$ such that Σ has L^2-gain $\leq \tilde{\gamma}$.

Σ is called inner if it is conservative with respect to $s(u, y) = \frac{1}{2}||u||_2^2 - \frac{1}{2}||y||_2^2$.

How to establish dissipativity? The available storage

Theorem (Necessary and sufficient conditions for dissipativity)

Consider system Σ *and supply rate* $s(u, y)$. Σ *is dissipative with respect to* s *iff*

$$S_a(x) := \sup_{\substack{u(\cdot)\\ T \geq 0}} -\int_0^T s(u(t), y(t))\,\mathrm{d}t, \qquad x(0) = x,$$

is finite $\forall\, x \in \mathcal{X}$. *Furthermore, if* S_a *is finite* $\forall\, x \in \mathcal{X}$ *then* S_a *is a storage function, called the* ***available storage****, and all other possible storage functions* S *satisfy*

$$S_a(x) \leq S(x) - \inf_x S(x), \qquad \forall\, x \in \mathcal{X}$$

Moreover $\inf_x S_a(x) = 0$.

The available storage is the minimal storage function.

Proof

- ($\Longrightarrow$) Suppose S_a is finite. Then $S_a \geq 0$ (supremum of a set that contains 0). Compare $S(x(t_0))$ and $S(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))\,\mathrm{d}t$ with $s(u, y)$ evaluated on a trajectory generated by $u : [t_0, t_1] \to \mathbb{R}^m$ that drives $x(t_0)$ at t_0 to $x(t_1)$ at t_1. Since S_a is the supremum over all $u(\cdot)$ it follows

$$S_a(x(t_0)) \geq S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t))\,\mathrm{d}t \implies S_a \text{ is a storage function.}$$

- ($\Longleftarrow$) Suppose Σ dissipative. Then $\exists\, S \geq 0$ such that $\forall\, u(\cdot)$

$$S(x(t)) + \int_0^T s(u(t), y(t))\,\mathrm{d}t \geq S(x(T)) \geq 0.$$

This implies that

$$S(x(0)) \geq \sup_{\substack{u(\cdot)\\ T\geq 0}} -\int_0^T s(u(t), y(t))\,\mathrm{d}t = S_a(x(0)) \implies S_a(x(0)) < \infty$$

Reachability and Storage functions

If the system is reachable from x^*, the finiteness of S_a needs to be checked only in x^*

Theorem

Assume that Σ is reachable from $x^* \in \mathcal{X}$. Then Σ is dissipative iff $S_a(x^*) < \infty$.

Proof

$(\Longleftarrow)$ By contradiction. Suppose there exists $x \in \mathcal{X}$ such that $S_a(x) = \infty$. Since by reachability x can be reached from x^* in finite time, this would imply (by time invariance) that also $S_a(x^*) = \infty$.

The maximal storage: the required supply

If Σ is reachable from x^*, there exists another canonically defined storage function.

Theorem

Assume that Σ is reachable from $x^* \in \mathcal{X}$. Define the required supply (from x^*) $S_r : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ as

$$S_r(x) := \inf_{\substack{u(\cdot) \\ T \geq 0}} \int_{-T}^{0} s(u(t), y(t))\,\mathrm{d}t, \qquad x(-T) = x^*, \quad x(0) = x.$$

Then S_r satisfies the dissipation inequality. Furthermore, Σ is dissipative iff $\exists\, K > -\infty$ such that $S_r(x) \geq K$, $\forall\, x \in \mathcal{X}$. Moreover, if S is a storage function for Σ, then

$$S(x) \leq S_r(x) + S(x^*), \qquad x \in \mathcal{X},$$

and $S_r(x) + S(x^*)$ is itself a storage function (and in particular $S_r(x) + S_a(x^*)$).

Proof

To steer the system from x^* at $-T$ to $x(t_1)$ consider $u(\cdot) : [-T, t_1] \to U$ which first take x^* to $x(t_0)$ at time $t_0 \leq t_1$, and then equal to a given input $u(\cdot) : [t_0, t_1] \to U$ transferring $x(t_0)$ to $x(t_1)$. This is a suboptimal control policy, whence

$$S_r(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))\, \mathrm{d}t \geq S_r(x(t_1)).$$

For the second claim, note that by definition of S_a and S_r

$$S_a(x^*) = \sup_x -S_r(x),$$

By the previous theorem, Σ is dissipative iff $\exists K > -\infty$ such that $S_r(x) \geq -K$, $\forall x$.
Finally, let S satisfy the dissipation inequality. Then for any $u(\cdot) : [-T, 0] \to U$ transferring $x(-T) = x^*$ to $x(0) = x$ we have by the dissipation inequality

$$S(x) - S(x^*) \leq \int_{-T}^{0} s(u(t), y(t))\, \mathrm{d}t.$$

Taking the infimum on the right-hand side over all $u(\cdot)$ proves the claim.
If $S \geq 0$, then $S_r + S(x^*) \geq 0$, and also $S_r + S(x^*)$ satisfies the dissipation inequality.

Bibliography

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