Discretization of the wave equation for vibroacoustics

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Abstract

In this note we present an exterior calculus representation of the acoustic wave equation, which is particularly suitable for (partitioned) finite element discretization. We present the finite element approximation based on integration by parts of only one equation, which yields two finite-dimensional models with different causality, i. e. with different co-state variables imposed as boundary conditions.

1 Governing equations

In classical notation, the propagation of sound in air is modeled by

$$-\frac{\partial}{\partial t}\begin{bmatrix}\chi_s p\\ \mu_0 \mathbf{v}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathrm{div}\\ \mathrm{grad} & \mathbf{0}^T\end{bmatrix}\begin{bmatrix}p\\ \mathbf{v}\end{bmatrix}. \tag{1}$$

on $\Omega \subset \mathbb{R}^n$. $p \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ denote the variations of pressure and velocity from a steady state, $\mu 0$ is the steady state mass density, and χ_s represents a constant adiabatic compressibility factor (see e. g. [7] − find other references from acoustics). Typically, a 3D domain, $n = 3$, but sometimes, exploiting symmetry, we might consider the reduced problem on a 2D domain, $n = 2$.

1.1 Equations in terms of differential forms

We write the equations in terms of differential forms applying the identities (see e. g. [5] or [6] for the 3D case)

$$\mathrm{div}\,\mathbf{v} = *\mathrm{d}(*\mathbf{v}^\flat) \quad \text{and} \quad \mathrm{grad}\,p = (\mathrm{d}p)^\sharp \tag{2}$$

and the rule for the two-fold application of the Hodge star

$$**\alpha = (-1)^{k(n-k)}\alpha, \qquad \alpha \in \Lambda^k(\Omega). \tag{3}$$

Remark 1. We verify the validity of (2) for the 2D case. Given the vector field $\mathbf{v}$, the associated one-form (index lowering) is

$$\mathbf{v}^\flat = v_1 dx + v_2 dy. \tag{4}$$

Applying the Hodge star gives

$$*\mathbf{v}^\flat = v_1 dy - v_2 dx. \tag{5}$$

For the exterior derivative of this expression, we apply the following rule[1]. Given the k-form

$$\omega^k = \sum a_{i_1,\ldots,i_k} dx_{i_i} \wedge \ldots \wedge dx_{i_k}, \tag{6}$$

its exterior derivative is

$$\mathrm{d}\omega^k = \sum \mathrm{d}a_{i_1,\ldots,i_k} \wedge dx_{i_i} \wedge \ldots \wedge dx_{i_k}, \tag{7}$$

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[1] See [2], Theorem 36.C.

i. e. we formally apply exterior differentiation to the coefficient functions. Applied to (5), we get

$$\begin{aligned} \mathrm{d}{*}\mathbf{v}^\flat &= \frac{\partial v_1}{\partial x} dx \wedge dy - \frac{\partial v_2}{\partial y} dy \wedge dx \\ &= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) dx \wedge dy. \end{aligned} \tag{8}$$

Finally, the application of the Hodge star ($*d\mathrm{vol} = 1$) gives for the 2D case

$$\begin{aligned} {*}\mathrm{d}{*}\mathbf{v}^\flat &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \\ &= \operatorname{div} \mathbf{v}. \end{aligned} \tag{9}$$

We can now write the first equation of (1), after application of the Hodge star (note that $\mathrm{d}{*}\mathbf{v}^\flat \in \Lambda^n$, as

$$\begin{aligned} -\chi_s \frac{\partial}{\partial t} {*}p &= {*}{*}\mathrm{d}{*}\mathbf{v}^\flat \\ &= \mathrm{d}{*}\mathbf{v}^\flat. \end{aligned} \tag{10}$$

Second equation, applying the musical isomorphism and the Hodge star

$$\begin{aligned} -\mu_0 \frac{\partial}{\partial t} {*}\mathbf{v}^\flat &= {*}(\operatorname{grad} p)^\flat \\ &= {*}\mathrm{d}p. \end{aligned} \tag{11}$$

Combining these two equations, we get

$$-\frac{\partial}{\partial t} \begin{bmatrix} \chi_s {*}p \\ \mu_0 {*}\mathbf{v}^\flat \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{n-1}\mathrm{d}{*} \\ {*}\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} p \\ (-1)^{n-1}\mathbf{v}^\flat \end{bmatrix}. \tag{12}$$

The inclusion of the factor $(-1)^{n-1}(-1)^{n-1} = 1$ will become clear in a moment.

In the sequel, we write (12) in terms of the differential forms

$$u_1 := {*}p \in L^2\Lambda^n(\Omega) =: V_n, \qquad u_2 := {*}\mathbf{v}^\flat \in H^1\Lambda^{n-1}(\Omega) =: V_{n-1}. \tag{13}$$

V_{n-1} and V_n represent a subsequence of the de Rham complex, in that $\mathrm{d}V_{n-1} \subset V_n$. The choice of the functional spaces follows from inspection of the first equation in (12). Applying the Hodge star to u_1 and u_2 gives

$${*}u_1 := {*}{*}p = (-1)^{0\cdot n} p = p, \qquad {*}u_2 := {*}{*}\mathbf{v}^\flat = (-1)^{1\cdot(n-1)}\mathbf{v}^\flat = (-1)^{n-1}\mathbf{v}^\flat. \tag{14}$$

This allows to rewrite (12) in terms of u_1 and u_2:

$$-\frac{\partial}{\partial t} \begin{bmatrix} \chi_s u_1 \\ \mu_0 u_2 \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{n-1}\mathrm{d}{*} \\ {*}\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} {*}u_1 \\ {*}u_2 \end{bmatrix}. \tag{15}$$

1.2 Relation with the canonical form of two conservation laws

Choosing the canonical states

$$x_1 = \chi_s {*}p = \chi_s u_1 \in \Lambda^n(\Omega), \qquad x_2 = \mu_0 \mathbf{v}^\flat = (-1)^{n-1}\mu_0 {*}u_2 \in \Lambda^1(\Omega), \tag{16}$$

and accordingly

$${*}x_1 = \chi_s {*}u_1 \in \Lambda^0(\Omega), \qquad {*}x_2 = \mu_0 (-1)^{n-1} {*}{*}u_2 = \mu_0 u_2 \in \Lambda^{n-1}(\Omega), \tag{17}$$

the Hamiltonian functional (total energy) can be written

$$H = \int_\Omega \frac{1}{2\chi_s} x_1 \wedge {*}x_1 + \frac{1}{2\mu_0} x_2 \wedge {*}x_2, \tag{18}$$

and its first variation

$$\begin{aligned}\delta H &= \int_\Omega \frac{1}{2\chi_s}(x_1 \wedge *\delta x_1 + \delta x_1 \wedge *x_1) + \frac{1}{2\mu_0}(x_2 \wedge *\delta x_2 + \delta x_2 \wedge *x_2) \\ &= \int_\Omega \frac{1}{\chi_s}\delta x_1 \wedge *x_1 + \frac{1}{\mu_0}\delta x_2 \wedge *x_2\end{aligned} \tag{19}$$

To obtain the variational derivatives, which define the co-states (efforts) according to [8], we have to bring the variations δx_1 and δx_2 to the right of the wedge product:

$$\begin{aligned}\delta H &= \int_\Omega \frac{(-1)^{n(n-n)}}{\chi_s} *x_1 \wedge \delta x_1 + \frac{(-1)^{1(n-1)}}{\mu_0} *x_2 \wedge \delta x_2 \\ &= \int_\Omega \frac{1}{\chi_s} *x_1 \wedge \delta x_1 + (-1)^{n-1}\frac{1}{\mu_0} *x_2 \wedge \delta x_2.\end{aligned} \tag{20}$$

The efforts are the expressions in front of the variations δx_1 and δx_2:

$$e_1 = \frac{1}{\chi_s} *x_1 = *u_1, \qquad e_2 = (-1)^{n-1}\frac{1}{\mu_0} *x_2 = (-1)^{n-1}u_2. \tag{21}$$

We now recast (15) in terms of states and efforts. We rewrite the first equation:

$$\begin{aligned}-\frac{\partial}{\partial t}(\underbrace{\chi_s u_1}_{x_1}) &= (-1)^{n-1}\mathrm{d}{*}{*}u_2 \\ &= (-1)^{n-1}\mathrm{d}(\underbrace{(-1)^{n-1}u_2}_{e_2}).\end{aligned} \tag{22}$$

Second equation, after application of the Hodge star and multiplication with $(-1)^{n-1}$ (note that $\mathrm{d}{*}u_1 \in \Lambda^1$):

$$\begin{aligned}-\frac{\partial}{\partial t}(\underbrace{(-1)^{n-1}\mu_0 {*}u_2}_{x_2}) &= (-1)^{n-1}{*}{*}\mathrm{d}{*}u_1 \\ &= (-1)^{n-1}(-1)^{n-1}\mathrm{d}{*}u_1 \\ &= \mathrm{d}\underbrace{{*}u_1}_{e_1}.\end{aligned} \tag{23}$$

Eq. (15) can now be expressed as

$$-\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & (-1)^{n-1}\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{24}$$

which is the canonical form of a system of two conservation laws according to [8].

2 Finite element approximation

We will rewrite the system in weak form and derive finite-dimensional approximations with a *Galerkin* approach based on

$$\begin{aligned} u_1 &\approx u_1^h = \mathbf{u}_1^T \boldsymbol{\alpha} \in V_n^h \\ u_2 &\approx u_2^h = \mathbf{u}_2^T \boldsymbol{\beta} \in V_{n-1}^h,\end{aligned} \tag{25}$$

with $\mathbf{u}_1 \in \mathbb{R}^N$, $\mathbf{u}_2 \in \mathbb{R}^M$ the vectors of coefficients (degrees of freedom) and $\boldsymbol{\alpha} \in L^2\Lambda^n(\Omega, \mathbb{R}^N)$, $\boldsymbol{\beta} \in H^1\Lambda^{n-1}(\Omega, \mathbb{R}^M)$ the N- and M- dimensional vectors of basis n- and $(n-1)$-forms

$$\boldsymbol{\alpha} = \begin{bmatrix}\alpha_1 & \dots & \alpha_N\end{bmatrix}^T, \qquad \boldsymbol{\alpha} = \begin{bmatrix}\beta_1 & \dots & \beta_M\end{bmatrix}^T. \tag{26}$$

The sequence of spaces

$$
\begin{aligned}
V_{n-1}^h &= \operatorname{span}\{\beta_1, \dots, \beta_{n_\beta}\} \subset V_{n-1} \\
V_n^h &= \operatorname{span}\{\alpha_1, \dots, \alpha_{n_\alpha}\} \subset V_n
\end{aligned}
\tag{27}
$$

forms a finite-dimensional *subcomplex* of the de Rham complex, as $\mathrm{d}V_{n-1}^h \subset V_n^h$ holds. This relation of the two approximation subspaces can be expressed by the existence of a matrix $\mathbf{D} \in \mathbb{R}^{M\times N}$ such that

$$
\mathrm{d}\boldsymbol{\beta} = \mathbf{D}\boldsymbol{\alpha}. \tag{28}
$$

2.1 Weak formulation

We obtain the weak formulation of (15) by building the L^2 inner product with

$$
(\cdot,\cdot)_{L^2\Lambda^k(\Omega)} : L^2\Lambda^k(\Omega) \times L^2\Lambda^k(\Omega) \to \mathbb{R} \tag{29}
$$

of each term with appropriate test forms

$$
v_1 \in L^2\Lambda^n(\Omega) \qquad \text{and} \qquad v_2 \in L^2\Lambda^{n-1}(\Omega). \tag{30}
$$

Note that L^2 inner product (short notation $(\cdot,\cdot) := (\cdot,\cdot)_{L^2\Lambda^k(\Omega)}$) and the natural pairing

$$
\langle \lambda | \mu \rangle_\Omega = \int_\Omega \lambda \wedge \mu \tag{31}
$$

of $\lambda \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^{n-k}(\Omega)$ are related via the *Hodge star* operator $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$. For $\lambda, \nu \in \Lambda^k(\Omega)$,

$$
(\lambda, \nu) = \langle \lambda | {*\nu} \rangle_\Omega. \tag{32}
$$

We follow a Galerkin approach i. e. we choose the same bases for trial and test forms:

$$
u_1^h, v_1^h \in V_n^h, \qquad u_2^h, v_2^h \in V_{n-1}^h. \tag{33}
$$

According to [4] or [3], we discretize the PDE model after integration by parts to either the first or the second equation. The result are two versions of the discretized wave equation with different explicit occurrence of boundary degrees of freedom. In the formulation as a port-Hamiltonian system, this corresponds to a different choice of inputs on the boundary (often referred to as causality).

2.2 First version

We write both equations in weak form (expressed as the natural pairing of differential forms according to (29) and apply integration by parts to the first equation, i. e. the differential equation for u_1.

2.2.1 First equation: discretization after integration by parts

We apply the integration by parts formula given in the appendix.

Add to appendix.

$$
\begin{aligned}
-\chi_s \frac{\partial}{\partial t} \langle *v_1 | u_1 \rangle_\Omega &= (-1)^{n-1} \langle *v_1 | \mathrm{d}{*}{*}u_2 \rangle_\Omega \\
&= (-1)^{n-1} \langle *v_1 | (-1)^{n-1} \mathrm{d}u_2 \rangle_\Omega \\
&= \langle *v_1 | \mathrm{d}u_2 \rangle_\Omega \\
&= \langle \mathrm{d}u_2 | *v_1 \rangle_\Omega \\
&= \langle \operatorname{tr} u_2 | \operatorname{tr} *v_1 \rangle_{\partial\Omega} - (-1)^{n-1} \langle u_2 | \mathrm{d}{*v_1} \rangle_\Omega
\end{aligned}
. \tag{34}
$$

The codifferential of a k-form ω on an n-dimensional space, see Eq. (50) in [1], is defined by

$$*\delta\omega = (-1)^k \mathrm{d}{*\omega}. \tag{35}$$

We can therefore write

$$\begin{aligned} -\chi_s \frac{\partial}{\partial t}\langle *v_1|u_1\rangle_\Omega &= \langle \operatorname{tr} u_2|\operatorname{tr} *v_1\rangle_{\partial\Omega} - (-1)^{n-1}\langle u_2|(-1)^n *\delta v_1\rangle_\Omega \\ &= \langle \operatorname{tr} *v_1|\operatorname{tr} u_2\rangle_{\partial\Omega} + \langle \delta v_1|*u_2\rangle_\Omega. \end{aligned} \tag{36}$$

Remark 2. In the reformulation of the right hand side with integration by parts, we obtained

$$\langle \mathrm{d}u_2|*v_1\rangle_\Omega = \langle \delta v_1|*u_2\rangle_\Omega + \langle \operatorname{tr} *v_1|\operatorname{tr} u_2\rangle_{\partial\Omega}. \tag{37}$$

Expressing the first two terms as L^2 inner products,

$$(\mathrm{d}u_2, v_1) = (u_2, \delta v_1) + \langle \operatorname{tr} *v_1|\operatorname{tr} u_2\rangle_{\partial\Omega}, \tag{38}$$

this corresponds exactly to Eq. (53) in [1]:

$$(\mathrm{d}\omega, \mu) = (\omega, \delta\mu) + \langle \operatorname{tr} \omega|\operatorname{tr} *\mu\rangle_{\partial\omega}. \tag{39}$$

Remark 3. Note that, unlike the discussion in the appendix on the adjoint of d on a subcomplex, at this point there is no assumption on $\operatorname{tr}_{\partial\Omega} *v_1$, therefore we cannot assume the boundary term to be zero. Moreover, at this stage, v_1 is an *arbitrary* test form, and not (yet) restricted to the finite-dimensional subspace V_n^h.

We now write the latter equation in terms of the finite-dimensional approximations as indicated above, and with

$$v_1^h = \boldsymbol{\alpha}^T \mathbf{v}_1, \qquad v_2^h = \boldsymbol{\beta}^T \mathbf{v}_2. \tag{40}$$

Because we now deal with specific, given differential forms, we omit the trace operator in the boundary term, and write the corresponding differential forms as restrictions to the boundary.

$$-\chi_s \frac{\partial}{\partial t}\langle *v_1^h|u_1^h\rangle_\Omega = \langle *v_1^h|u_2^h\rangle_{\partial\Omega} + \langle \delta v_1^h|*u_2^h\rangle_\Omega. \tag{41}$$

We express each term in terms of the vectors $\mathbf{v}_1 \in \mathbb{R}^{n_\alpha}$, $\mathbf{u}_1 \in \mathbb{R}^{n_\alpha}$ and $\mathbf{u}_2 \in \mathbb{R}^{n_\beta}$. First term:

$$\begin{aligned} \langle *v_1^h|u_1^h\rangle_\Omega &= \langle u_1^h|*v_1^h\rangle_\Omega \\ &= \mathbf{u}_1^T \langle \boldsymbol{\alpha}|*\boldsymbol{\alpha}^T\rangle_\Omega \mathbf{v}_1 \\ &= \mathbf{v}_1^T \langle \boldsymbol{\alpha}|*\boldsymbol{\alpha}^T\rangle_\Omega \mathbf{u}_1 \end{aligned} \tag{42}$$

Third term with $\mathrm{d}\boldsymbol{\beta} = \mathbf{D}\boldsymbol{\alpha}$ and exploiting that δ is the adjoint operator to d on the finite-dimensional subcomplex (the codifferential of the 3-form v_1 in $\mathbb{R}^3$ is $\delta v_1 = *\mathrm{d}{*v_1}$):

$$\begin{aligned} \langle \delta v_1^h|*u_2^h\rangle_\Omega &= (\delta v_1^h, u_2^h) \\ &= (v_1^h, \mathrm{d}u_2^h) \\ &= \langle v_1^h|*\mathrm{d}u_2^h\rangle_\Omega \\ &= \mathbf{v}_1^T \langle \boldsymbol{\alpha}|*\mathrm{d}\boldsymbol{\beta}^T\rangle_\Omega \mathbf{u}_2 \\ &= \mathbf{v}_1^T \langle \boldsymbol{\alpha}|*\boldsymbol{\alpha}^T\rangle_\Omega \mathbf{D}^T \mathbf{u}_2 \end{aligned} \tag{43}$$

Second term:

$$\begin{aligned} \langle u_2^h|*v_1^h\rangle_{\partial\Omega} &= \langle v_1^h|*u_2^h\rangle_{\partial\Omega} \\ &= \mathbf{v}_1^T \langle \boldsymbol{\alpha}|*\boldsymbol{\beta}^T\rangle_{\partial\Omega} \mathbf{u}_2 \end{aligned} \tag{44}$$

Substituting these three expressions in (41), we obtain as discretized version of the weak formulation

$$-\chi_s \frac{d}{dt}\mathbf{v}_1^T \mathbf{T}_\alpha \mathbf{u}_1 = \mathbf{v}_1^T \mathbf{T}_\alpha \mathbf{D}^T \mathbf{u}_2 + \mathbf{v}_1^T \mathbf{B}_{\alpha\beta} \mathbf{u}_2, \qquad \forall\ \mathbf{v}_1 \in \mathbb{R}^N \tag{45}$$

with

$$\mathbf{T}_\alpha = \langle \boldsymbol{\alpha}|*\boldsymbol{\alpha}^T\rangle_\Omega \quad \text{and} \quad \mathbf{B}_{\alpha\beta} = \langle \boldsymbol{\alpha}|*\boldsymbol{\beta}^T\rangle_{\partial\Omega}. \tag{46}$$

2.2.2 Second equation: direct discretization

Multiplication of the left hand side of the second equation with test forms $*v_2$ gives

$$\begin{aligned}-\mu_0\frac{\partial}{\partial t}\langle *v_2|u_2\rangle_\Omega &= -(-1)^{n-1}\mu_0\frac{\partial}{\partial t}\langle u_2|*v_2\rangle_\Omega \\ &= -(-1)^{n-1}\mu_0\frac{\partial}{\partial t}\langle v_2|*u_2\rangle_\Omega\end{aligned} \tag{47}$$

For the right hand side:

$$\begin{aligned}\langle *v_2|*\mathrm{d}*u_1\rangle_\Omega &= \langle *v_2|*(-1)^n*\delta u_1\rangle_\Omega \\ &= (-1)^{n-1}\langle(-1)^n**\delta u_1|*v_2\rangle_\Omega \\ &= (-1)^{n-1}\langle(-1)^n(-1)^{n-1}\delta u_1|*v_2\rangle_\Omega \\ &= (-1)^n\langle\delta u_1|*v_2\rangle_\Omega.\end{aligned} \tag{48}$$

In the finite-dimensional subspaces (division by $(-1)^{n-1}$ of both sides), the equation reads

$$\begin{aligned}-\mu_0\frac{\partial}{\partial t}\langle v_2^h|*u_2^h\rangle_\Omega &= -\langle\delta u_1^h|*v_2^h\rangle_\Omega \\ &= -(\delta u_1^h, v_2^h) \\ &= -(\mathrm{d}v_2^h, u_1^h) \\ &= -\langle\mathrm{d}v_2^h|*u_1^h\rangle_\Omega.\end{aligned} \tag{49}$$

Expressed in terms of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}_2$, the first natural pairing is

$$\langle v_2^h|*u_2^h\rangle_\Omega = \mathbf{v}_2^T\langle\boldsymbol{\beta}|*\boldsymbol{\beta}^T\rangle_\Omega\mathbf{u}_2 \tag{50}$$

and the second one

$$\begin{aligned}\langle\mathrm{d}v_2^h|*u_1^h\rangle_\Omega &= \langle\mathrm{d}\mathbf{v}_2^T\boldsymbol{\beta}|*\boldsymbol{\alpha}^T\mathbf{u}_1\rangle_\Omega \\ &= \mathbf{v}_2^T\langle\mathrm{d}\boldsymbol{\beta}|*\boldsymbol{\alpha}^T\rangle_\Omega\mathbf{u}_1 \\ &= \mathbf{v}_2^T\mathbf{D}\langle\boldsymbol{\alpha}|*\boldsymbol{\alpha}^T\rangle_\Omega\mathbf{u}_1.\end{aligned} \tag{51}$$

Substitution of the two latter expressions in (49) yields

$$-\mu_0\frac{d}{dt}\mathbf{v}_2^T\mathbf{T}_\beta\mathbf{u}_2 = -\mathbf{v}_2^T\mathbf{D}\mathbf{T}_\alpha\mathbf{u}_1, \qquad \forall\ \mathbf{v}_2\in\mathbb{R}^M \tag{52}$$

with

$$\mathbf{T}_\beta = \langle\boldsymbol{\beta}|*\boldsymbol{\beta}^T\rangle_\Omega. \tag{53}$$

2.2.3 Finite-dimensional approximation: first version

By the validity of Eqs. (45) and (52) for all $\mathbf{v}_1\in\mathbb{R}^N$ and $\mathbf{v}_2\in\mathbb{R}^M$, we can remove $\mathbf{v}_1^T$ and $\mathbf{v}_2^T$ and obtain the system of equations

$$-\begin{bmatrix}\mathbf{T}_\alpha & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_\beta\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\chi_s\mathbf{u}_1\\ \mu_0\mathbf{u}_2\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{T}_\alpha\mathbf{D}^T\\ -\mathbf{D}\mathbf{T}_\alpha & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} + \begin{bmatrix}\mathbf{B}_{\alpha\beta}\\ \mathbf{0}\end{bmatrix}\mathbf{u}_2, \tag{54}$$

or, rearranged:

$$-\frac{d}{dt}\begin{bmatrix}\chi_s\mathbf{u}_1\\ \mu_0\mathbf{T}_\beta\mathbf{u}_2\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{D}^T\\ -\mathbf{D} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{T}_\alpha\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} + \begin{bmatrix}\mathbf{T}_\alpha^{-1}\mathbf{B}_{\alpha\beta}\\ \mathbf{0}\end{bmatrix}\mathbf{u}_2, \tag{55}$$

Denoting

$$\mathbf{x}_1 = \chi_s\mathbf{u}_1\in\mathbb{R}^N, \qquad \mathbf{x}_2 = \mu_0\mathbf{T}_\beta\mathbf{u}_2\in\mathbb{R}^M \tag{56}$$

and

$$\mathbf{e}_1 = \mathbf{T}_\alpha \mathbf{u}_1 = \frac{1}{\chi_s}\mathbf{T}_\alpha \mathbf{x}_1 \in \mathbb{R}^N, \qquad \mathbf{e}_2 = \mathbf{u}_2 = \frac{1}{\mu_0}\mathbf{T}_\beta^{-1}\mathbf{x}_2 \in \mathbb{R}^M \tag{57}$$

the vectors of discrete states and efforts, and defining the vector of input boundary efforts $\mathbf{e}_2^b \in \mathbb{R}^{M_b}$ by

$$\mathbf{T}_\alpha^{-1}\mathbf{B}_{\alpha\beta}\mathbf{e}_2 =: \mathbf{G}_1\mathbf{e}_2^b, \tag{58}$$

with M_b the number of non-zero columns of $\mathbf{B}_{\alpha\beta}$, the differential equations can be written in the form

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{D}^T\\ \mathbf{D} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{1}{\chi_s}\mathbf{T}_\alpha & \mathbf{0}\\ \mathbf{0} & \frac{1}{\mu_0}\mathbf{T}_\beta^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix} + \begin{bmatrix}\mathbf{G}_1\\ \mathbf{0}\end{bmatrix}\mathbf{e}_2^b. \tag{59}$$

Note that the definitions of discrete states and efforts (56) and (57) are discretized versions of (16) and (21).

Complete by energy approx.

2.3 Second version

2.3.1 First equation: direct discretization

The weak form of the differential equation for u_1, see (34), is

$$\begin{aligned} -\chi_s\frac{\partial}{\partial t}\langle *v_1|u_1\rangle_\Omega &= (-1)^{n-1}\langle *v_1|\mathrm{d}{*}{*}u_2\rangle_\Omega \\ &= \langle *v_1|\mathrm{d}u_2\rangle_\Omega. \end{aligned} \tag{60}$$

Written in the approximation subspaces, the left hand side is discretized according to (42). For the right hand side, we write, exploiting the last steps in (43),

$$\begin{aligned} \langle *v_1^h|\mathrm{d}u_2^h\rangle_\Omega &= \langle \mathrm{d}u_2^h|*v_1^h\rangle_\Omega \\ &= (\mathrm{d}u_2^h, v_1^h) \\ &= (v_1^h, \mathrm{d}u_2^h) \\ &= \mathbf{v}_1^T\langle\boldsymbol{\alpha}|*\boldsymbol{\alpha}^T\rangle_\Omega\mathbf{D}^T\mathbf{u}_2. \end{aligned} \tag{61}$$

The discretized weak form of the first equation is therefore

$$-\chi_s\frac{d}{dt}\mathbf{v}_1^T\mathbf{T}_\alpha\mathbf{u}_1 = \mathbf{v}_1^T\mathbf{T}_\alpha\mathbf{D}^T\mathbf{u}_2, \qquad \forall\ \mathbf{v}_1 \in \mathbb{R}^N. \tag{62}$$

2.3.2 Second equation: discretization after integration by parts

The weak form of the left hand side can be written, see (47),

$$-\mu_0\frac{\partial}{\partial t}\langle *v_2|u_2\rangle_\Omega = -(-1)^{n-1}\mu_0\frac{\partial}{\partial t}\langle v_2|*u_2\rangle_\Omega \tag{63}$$

Integration by parts, applied to the right hand side, yields

$$\begin{aligned} \langle *v_2|*\mathrm{d}*u_1\rangle_\Omega &= \langle \mathrm{d}*u_1|{*}{*}v_2\rangle_\Omega \\ &= (-1)^{n-1}\langle \mathrm{d}*u_1|v_2\rangle_\Omega \\ &= (-1)^{n-1}(\langle \operatorname{tr}*u_1|\operatorname{tr}v_2\rangle_{\partial\Omega} - (-1)^0\langle *u_1|\mathrm{d}v_2\rangle_\Omega) \\ &= (-1)^{n-1}\langle \operatorname{tr}v_2|\operatorname{tr}*u_1\rangle_{\partial\Omega} - (-1)^{n-1}\langle \mathrm{d}v_2|*u_1\rangle_\Omega \end{aligned} \tag{64}$$

Dividing left and right hand side by $(-1)^{n-1}$, and substituting the finite-dimensional approximations, we get

$$-\mu_0\frac{\partial}{\partial t}\langle v_2^h|*u_2^h\rangle_\Omega = \langle \operatorname{tr}v_2^h|\operatorname{tr}*u_1^h\rangle_{\partial\Omega} - \langle \mathrm{d}v_2^h|*u_1^h\rangle_\Omega. \tag{65}$$

The discretization of the natural pairing on the left is given in (50). The terms on the right hand side, expressed in terms of the basis forms, are

$$\langle v_2^h | {*u_1^h} \rangle_{\partial\Omega} = \mathbf{v}_2^T \langle \boldsymbol{\beta} | {*\boldsymbol{\alpha}^T} \rangle_{\partial\Omega} \mathbf{u}_1 \tag{66}$$

and, see (51)

$$-\langle \mathrm{d}v_2^h | {*u_1^h} \rangle_{\Omega} = -\mathbf{v}_2^T \mathbf{D} \langle \boldsymbol{\alpha} | {*\boldsymbol{\alpha}^T} \rangle_{\Omega} \mathbf{u}_1. \tag{67}$$

Together, the discretized weak form of the second equation is

$$-\mu_0 \frac{d}{dt} \mathbf{v}_2^T \mathbf{T}_\beta \mathbf{u}_2 = -\mathbf{v}_2^T \mathbf{D} \mathbf{T}_\alpha \mathbf{u}_1 + \mathbf{v}_2^T \mathbf{B}_{\beta\alpha} \mathbf{u}_1, \qquad \forall\, \mathbf{v}_2 \in \mathbb{R}^M \tag{68}$$

with

$$\mathbf{B}_{\beta_\alpha} = \langle \boldsymbol{\beta} | {*\boldsymbol{\alpha}^T} \rangle_{\partial\Omega} = \mathbf{B}_{\alpha\beta}^T. \tag{69}$$

Remark 4. We can easily verify that $\mathbf{B}_{\beta\alpha} = \mathbf{B}_{\alpha\beta}^T$ by the element-wise identities $\langle \beta_i | {*\alpha_j} \rangle_{\partial\Omega} = \langle \alpha_j | {*\beta_i} \rangle_{\partial\Omega}$.

2.3.3 Finite-dimensional approximation: second version

We obtain for all $\mathbf{v}_1 \in \mathbb{R}^N$ and $\mathbf{v}_2 \in \mathbb{R}^M$, the following set of equations:

$$-\begin{bmatrix} \mathbf{T}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_\beta \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \chi_s \mathbf{u}_1 \\ \mu_0 \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{T}_\alpha \mathbf{D}^T \\ -\mathbf{D}\mathbf{T}_\alpha & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{\alpha\beta}^T \end{bmatrix} \mathbf{u}_1, \tag{70}$$

or, rearranged:

$$-\frac{d}{dt} \begin{bmatrix} \chi_s \mathbf{u}_1 \\ \mu_0 \mathbf{T}_\beta \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}^T \\ -\mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{T}_\alpha \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{\alpha\beta}^T \end{bmatrix} \mathbf{u}_1. \tag{71}$$

We observe that this second version differs from the first one only in the boundary terms With the definitions of states and efforts in (56) and (57) and introducing the matrix $\mathbf{G}_2$, which contains the nonzero columns of $\mathbf{B}_{\alpha\beta}^T \mathbf{T}_\alpha^{-1}$ by

$$\mathbf{B}_{\alpha\beta}^T \mathbf{u}_1 = \mathbf{B}_{\alpha\beta}^T \mathbf{T}_\alpha^{-1} \mathbf{e}_1 =: \mathbf{G}_2 \mathbf{e}_1^b, \tag{72}$$

the port-Hamiltonian representation of the second discretized version is

$$\frac{d}{dt} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{1}{\chi_s} \mathbf{T}_\alpha & \mathbf{0} \\ \mathbf{0} & \frac{1}{\mu_0} \mathbf{T}_\beta^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_2 \end{bmatrix} \mathbf{e}_1^b. \tag{73}$$

3 Mixed boundary conditions

Comparing (59) and (73), we observe that the only difference is the input matrix. Can we arrive at a finite-dimensional approximation model with *non-uniform* causality, i. e. different efforts imposed as inputs on the boundary, *directly*, i. e. by an appropriate careful application of integration by parts to *parts* of the domain Ω? Does the result correspond directly to a *domain decomposition approach*?

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