A port-Hamiltonian formulation for the full von-Kármán plate model

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Summary. In this contribution, a port-Hamiltonian reformulation of the full von-Kármán dynamical model for geometrically non-linear plates is detailed, including the collocated boundary control and observation. Starting from the canonical equations, a set of variables is chosen so as to make the total energy quadratic. The model, reformulated in these variables, highlights a port-Hamiltonian structure ruled by a state-modulated interconnection operator.

Classical model

The classical full von-Kármán dynamical model is presented in Bilbao et al. [2015]. The problem, defined on an open connected set $\Omega \subset \mathbb{R}^2$, takes the dimensionless form

$$\begin{aligned} \ddot{\boldsymbol{u}} &= \operatorname{Div} \boldsymbol{N}, & \boldsymbol{N} &= \boldsymbol{\Phi}(\boldsymbol{\varepsilon}), & \boldsymbol{\varepsilon} &= \operatorname{Grad} \boldsymbol{u} + 1/2 \operatorname{grad} w \otimes \operatorname{grad} w, \\ \ddot{w} &= -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}(\boldsymbol{N} \operatorname{grad} w), & \boldsymbol{M} &= \boldsymbol{\Phi}(\boldsymbol{\kappa}), & \boldsymbol{\kappa} &= \operatorname{Grad} \operatorname{grad} w, \end{aligned} \tag{1}$$

where $\boldsymbol{u} \in \mathbb{R}^2$ is the in-plane displacement, w is the vertical displacement, $\boldsymbol{\varepsilon}$ is the in-plane strain tensor, $\boldsymbol{\kappa}$ is the curvature tensor, $\boldsymbol{N}$ is the in-plane stress resultant and $\boldsymbol{M}$ is the bending stress resultant. The notation $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top$ denotes the dyadic product of two vectors. The div operator is the divergence of a vector field, and grad the gradient of a scalar field. The operator $\operatorname{Grad} = \frac{1}{2}\left(\nabla + \nabla^\top\right)$ designates the symmetric part of the gradient (i. e. the deformation gradient in continuum mechanics). For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{R}^{2\times 2}$, with components U_{ij}, the divergence $\operatorname{Div}(\boldsymbol{U})$ is a vector, defined column-wise as

$$\operatorname{Div}(\boldsymbol{U}) := \sum_{i=1}^{2} \partial_{x_i} U_{ij}, \qquad \forall j = \{1, 2\}.$$

The linear tensor mapping $\boldsymbol{\Phi}$ is positive and preserves symmetry:

$$\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\mathbf{1} + (1-\nu)\boldsymbol{A}, \qquad \boldsymbol{A} = \boldsymbol{A}^\top \implies \boldsymbol{\Phi}(\boldsymbol{A}) = \boldsymbol{\Phi}(\boldsymbol{A})^\top, \qquad \text{where} \qquad \mathbf{1} = \operatorname{Diag}(1,1).$$

The total energy of the model (Hamiltonian functional)

$$H = \frac{1}{2}\int_\Omega \left\{ \|\dot{\boldsymbol{u}}\|^2 + \dot{w}^2 + \boldsymbol{N} : \boldsymbol{\varepsilon} + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \mathrm{d}\Omega, \qquad \text{where} \qquad \boldsymbol{A} : \boldsymbol{B} = \operatorname{Tr}(\boldsymbol{A}^\top \boldsymbol{B}) \tag{2}$$

consists of the kinetic energy and both membrane and bending deformation energies. This model proves conservative, see Bilbao et al. [2015]. Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

The equivalent port-Hamiltonian system (pHs)

To find a suitable port-Hamiltonian system, we first select a set of new energy variables to make the Hamiltonian functional quadratic. The selection is the same as for both the linear plate problems in Brugnoli et al. [2019a,b]:

$$\boldsymbol{\alpha}_u = \dot{\boldsymbol{u}}, \qquad \alpha_w = \dot{w}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}. \tag{3}$$

The energy is quadratic in these variables

$$H = \frac{1}{2}\int_\Omega \left\{ \|\boldsymbol{\alpha}_u\|^2 + \alpha_w^2 + \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon + \boldsymbol{\Phi}(\boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa \right\}. \tag{4}$$

By computing the variational derivative of the Hamiltonian, one obtains the so-called co-energy variables:

$$\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad e_w := \delta_{\alpha_w} H = \dot{w}, \qquad \boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon), \qquad \boldsymbol{E}_\kappa := \delta_{\boldsymbol{A}_\kappa} H = \boldsymbol{\Phi}(\boldsymbol{A}_\kappa). \tag{5}$$

Before stating the final formulation, consider the operator $\mathcal{C}(w)(\cdot) : L^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}}) \to L^2(\Omega)$ acting on symmetric tensors

$$\mathcal{C}(w)(\boldsymbol{T}) = \operatorname{div}(\boldsymbol{T} \operatorname{grad} w). \tag{6}$$

Proposition 1 *The formal adjoint of the $\mathcal{C}(w)(\cdot)$ is given by*

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\operatorname{grad}(\cdot) \otimes \operatorname{grad}(w) + \operatorname{grad}(w) \otimes \operatorname{grad}(\cdot)\right]. \tag{7}$$

Proof 1 *Consider a smooth scalar field $v \in C_0^\infty(\Omega)$ and a smooth symmetric tensor field $\boldsymbol{U} \in C_0^\infty(\Omega, \mathbb{R}^{2\times 2}_{sym})$ with compact support. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation*

$$\langle v,\, \mathcal{C}(w)(\boldsymbol{U})\rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)(v)^*,\, \boldsymbol{U}\rangle_{L^2(\Omega,\mathbb{R}^{2\times 2}_{sym})}\,. \tag{8}$$

The proof follows from the computation

$$\begin{aligned}
\langle v,\, \mathcal{C}(w)(\boldsymbol{U})\rangle_{L^2(\Omega)} &= \langle v,\, \operatorname{div}(\boldsymbol{U}\operatorname{grad} w)\rangle_{L^2(\Omega)}\,, && \textit{Integration by parts,}\\
&= \langle -\operatorname{grad} v,\, \boldsymbol{U}\operatorname{grad} w\rangle_{L^2(\Omega,\mathbb{R}^2)}\,, && \textit{Dyadic product properties,}\\
&= \langle -\operatorname{grad} v\otimes\operatorname{grad} w,\, \boldsymbol{U}\rangle_{L^2(\Omega,\mathbb{R}^{2\times 2}_{sym})}\,, && \textit{Symmetry of } \boldsymbol{U},\\
&= \langle -1/2(\operatorname{grad} v\otimes\operatorname{grad} w + \operatorname{grad} w\otimes\operatorname{grad} v),\, \boldsymbol{U}\rangle_{L^2(\Omega,\mathbb{R}^{2\times 2}_{sym})}\,.
\end{aligned} \tag{9}$$

This means

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\operatorname{grad}(\cdot)\otimes\operatorname{grad}(w) + \operatorname{grad}(w)\otimes\operatorname{grad}(\cdot)\right], \tag{10}$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u\\ \boldsymbol{A}_\varepsilon\\ \alpha_w\\ \boldsymbol{A}_\kappa\end{pmatrix} = \begin{bmatrix}\mathbf{0} & \operatorname{Div} & \mathbf{0} & \mathbf{0}\\ \operatorname{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0}\\ 0 & \mathcal{C}(w) & 0 & -\operatorname{div}\operatorname{Div}\\ \mathbf{0} & \mathbf{0} & \operatorname{Grad}\operatorname{grad} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\delta_{\boldsymbol{\alpha}_u}H\\ \delta_{\boldsymbol{A}_\varepsilon}H\\ \delta_{\boldsymbol{\alpha}_w}H\\ \delta_{\boldsymbol{A}_\kappa}H\end{pmatrix}, \tag{11}$$

The second line of system (11) represents the time derivative of the membrane strain tensor. To close the system, variable w has to be accessible. For this reason, its dynamics has to be included. The augmented system reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u\\ \boldsymbol{A}_\varepsilon\\ w\\ \alpha_w\\ \boldsymbol{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix}\mathbf{0} & \operatorname{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \operatorname{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0}\\ 0 & 0 & 0 & 1 & 0\\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div}\operatorname{Div}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \operatorname{Grad}\operatorname{grad} & \mathbf{0}\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}\delta_{\boldsymbol{\alpha}_u}H\\ \delta_{\boldsymbol{A}_\varepsilon}H\\ \delta_{w}H\\ \delta_{\alpha_w}H\\ \delta_{\boldsymbol{A}_\kappa}H\end{pmatrix}. \tag{12}$$

Given the results in Brugnoli et al. [2019a,b] and Proposition 1, the operator $\mathcal{J}$ is formally skew-adjoint. If only the kinetic and deformation energies are considered, it holds $\delta_w H = 0$. In general this terms allows accommodating other potentials, for example the gravitational one. Suitable boundary variables are then obtained considering the power balance

$$\dot{H} = \langle \gamma_0\boldsymbol{e}_u,\, \gamma_\perp\boldsymbol{E}_\varepsilon\rangle_{\partial\Omega} + \langle \gamma_0\boldsymbol{e}_w,\, \gamma_{\perp\perp,1}\boldsymbol{E}_\kappa + \gamma_0(\boldsymbol{E}_\varepsilon\boldsymbol{n}\cdot\operatorname{grad} w)\rangle_{\partial\Omega} + \langle \gamma_1\boldsymbol{e}_w,\, \gamma_{\perp\perp}\boldsymbol{E}_\kappa\rangle_{\partial\Omega}\,, \tag{13}$$

where $\gamma_0\boldsymbol{e}_u = \boldsymbol{e}_u|_{\partial\Omega}$ is the Dirichlet trace, $\gamma_\perp\boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon\boldsymbol{n}|_{\partial\Omega}$ is the normal trace ($\boldsymbol{n}$ is the outward normal vector), $\gamma_{\perp\perp,1}\boldsymbol{E}_\kappa = -\boldsymbol{n}\cdot\operatorname{Div}\boldsymbol{E}_\kappa - \partial_{\boldsymbol{s}}(\boldsymbol{n}^\top\boldsymbol{E}_\kappa\boldsymbol{s})|_{\partial\Omega}$ is the effective shear force at the boundary ($\boldsymbol{s}$ is the tangent versor at the boundary), $\gamma_1\boldsymbol{e}_w = \partial_{\boldsymbol{n}}\boldsymbol{e}_w|_{\partial\Omega}$ is the normal derivative trace and $\gamma_{\perp\perp}\boldsymbol{E}_\kappa = \boldsymbol{n}^\top\boldsymbol{E}_\kappa\boldsymbol{n}$ is the normal to normal trace. The boundary conditions are consistent with the ones assumed in Puel and Tucsnak [1996] for deriving a global existence result for this model.

Conclusions

We have presented a pHs formulation of the full von-Kármán model. The dynamics of the system exhibits a state modulated interconnection operator, while the energy remains quadratic in the chosen variables. Of particular interest is the discretization of such a model for simulation and control purposes. The Partitioned Finite Element Method (PFEM), an extension of mixed finite elements to pHs, seems to be particularly suitable to achieve a structure-preserving discretization of this model, as in Cardoso-Ribeiro et al. [2020] for the 2D Shallow Water Equation, which exhibits the same kind of polynomial nonlinearity.

References

S. Bilbao, O. Thomas, C. Touzé, and M. Ducceschi. Conservative numerical methods for the full von kármán plate equations. *Numerical Methods for Partial Differential Equations*, 31(6):1948–1970, 2015. doi: 10.1002/num.21974.

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-Hamiltonian formulation and symplectic discretization of plate models Part I: Mindlin model for thick plates. *Applied Mathematical Modelling*, 75:940–960, 2019a.

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-Hamiltonian formulation and symplectic discretization of plate models Part II: Kirchhoff model for thin plates. *Applied Mathematical Modelling*, 75:961–981, 2019b.

J. P. Puel and M. Tucsnak. Global existence for the full von kármán system. *Applied Mathematics and Optimization*, 34(2):139–160, Sep 1996. ISSN 1432-0606. doi: 10.1007/BF01182621.

F.L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A partitioned finite element method for power-preserving discretization of open systems of conservation laws. *IMA Journal of Mathematical Control and Information*, pages 1 – 41, 2020. doi: https://doi:10.1093/imamci/xxx000. In Press.