THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Institut Supérieur de l'Aéronautique et de l'Espace (ISAE)*

Présentée et soutenue le *30 Octobre 2020* **par :**

Andrea BRUGNOLI

A port-Hamiltonian formulation of flexible structures
Modelling and symplectic finite element discretization

JURY

Daniel ALAZARD	ISAE-Supaéro, Toulouse	Directeur
Valérie P. BUDINGER	ISAE-Supaéro, Toulouse	Co-directeur
Yann LE GORREC	Institut FEMTO-ST	Rapporteur
Alessandro MACCHELLI	Universitá di Bologna	Rapporteur
Thomas Hélie	Directeur de Recherches CNRS	Examinateur
Luc DUGARD	GIPSA-LAB, Grenoble	Président

École doctorale et spécialité :
EDSYS : Automatique

Unité de Recherche :
CSDV - Commande des Systèmes et Dynamique du Vol - ONERA - ISAE

Directeur de Thèse :
Daniel ALAZARD et *Valérie POMMIER-BUDINGER*

Rapporteurs :
Yann LE GORREC et *Alessandro MACCHELLI*

Abstract

This thesis aims at extending the port-Hamiltonian (pH) approach to continuum mechanics in higher geometrical dimensions (particularly in 2D). The pH formalism has a strong multiphysics character and represents a unified framework to model, analyze and control both finite- and infinite-dimensional systems. Despite the large literature on this topic, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an intrinsic, i.e. coordinate free, and equivalent pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems in port-Hamiltonian form requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

Résumé

Cette thèse vise à étendre l'approche port-hamiltonienne (pH) à la mécanique des milieux continus dans des dimensions géométriques plus élevées (en particulier on se focalise sur la dimension deux). Le formalisme pH, avec son fort caractère multiphysique, représente un cadre unifié pour modéliser, analyser et contrôler les systèmes de dimension finie et infinie. Malgré l'abondante littérature sur ce sujet, les problèmes d'élasticité en deux ou trois dimensions géométriques n'ont presque jamais été considérés. Dans ce travail de thèse la connexion entre problèmes d'élasticité et systèmes distribués port-Hamiltoniens est établie. L'originalité apportée réside dans trois contributions majeures. Tout d'abord, la nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d'élasticité écrits en forme port-Hamiltonienne nécessite l'utilisation d'éléments finis non standard. Néanmoins, l'implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

Aknowledgements

Remerciements

Ringraziamenti

Alla mia famiglia

Contents

List of Figures

List of Tables

List of Acronyms

DAE	*Differential-Algebraic Equation*
dpHs	*distributed port-Hamiltonian systems*
FEM	*Finite Element Method*
IDA-PBC	*Interconnection and Damping Assignment Passivity Based Control*
PDE	*Partial Differential Equation*
PFEM	*Partitioned Finite Element Method*
pH	*port-Hamiltonian*
pHs	*port-Hamiltonian systems*
pHDAE	*port-Hamiltonian Descriptor System*

Part I

Introduction and state of the art

Chapter 1

Introduction

Je n'ai cherché de rien prouver, mais de bien peindre et d'éclairer bien ma peinture.

André Gide
Préface de L'Immoraliste

Contents

1.1 Motivation and context

1.2 Overview of chapters

1.3 Contributions

Chapter 2

Literature review

Whereof one cannot speak, thereof one must be silent.

Ludwig Wittgenstein
Tractatus Logico-Philosophicus

2.1 Port-Hamiltonian distributed systems

For 1D linear PH systems with a generalized skew-adjoint system operator, [LGZM05] gives conditions on the assignment of boundary inputs and outputs for the system operator to generate a contraction semigroup. The latter is instrumental to show well-posedness of a linear PH system, see [JZ12]. Essentially, at most half the number of boundary port variables can be imposed as control inputs for a well-posed PH system in 1D.

2.2 Structure-preserving discretization

2.3 Mixed finite element for elasticity

2.4 Multibody dynamics

Part II

Port-Hamiltonian elasticity and thermoelasticity

Chapter 3

Elasticity in port-Hamiltonian form

I try not to break the rules but merely to test their elasticity.

Bill Veeck

Contents

Continuum mechanics is the mathematical description of how materials behave kinematically under external excitations. In this framework, the microscopic structure of a material body is neglected and a macroscopic viewpoint, that describes the body as a continuum, is adopted. This leads to a PDE describing the evolution of the displacement field. In this chapter, the general linear elastodynamics problem is recalled. A suitable port-Hamiltonian realization is then derived.

3.1 Deformation, strain and stress

In this section, the main concepts behind a deformable continuum are briefly recalled following [Lee12]. For a detailed discussion on this topic, the reader may consult [Abe12, LPKL12].

The bounded region of $\mathbb{R}^d$ $(d = 2, 3)$ occupied by a solid is called configuration. The reference configuration Ω is the domain that a bodies occupies at the initial state. To describe how the body deforms in time the deformation map $\mathbf{\Phi} : \Omega \times [0, T_f] \rightarrow \Omega' \subset \mathbb{R}^d$ is introduced. This map is differentiable and orientation preserving and the image of Ω under $\mathbf{\Phi}(\cdot, t)$ $\forall t \in [0, T_f]$ is called the deformed configuration Ω_t. Given a specific point in the reference frame is image is denoted by $\boldsymbol{y} = \mathbf{\Phi}(\boldsymbol{x}, t)$. The gradient of the deformation map is called the deformation gradient $\boldsymbol{F} := \nabla_x \boldsymbol{\Phi} = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}$. A rigid deformation maps a point $\boldsymbol{x} \in \Omega \to \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{b}(t)$, where $\boldsymbol{A}(t)$ is an orthogonal matrix and $\boldsymbol{b}(t)$ a $\mathbb{R}^d$ vector. A differentiable deformation map $\boldsymbol{\Phi}$ is a rigid deformation iff $\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I} = 0$, where $\boldsymbol{I}$ is the identity in $\mathbb{R}^{d \times d}$ (for the proof see [Cia88], page 44). For this reason, a suitable measure of the deformation is the Green-St.Venant strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I})$.

A quantity of interest is the displacement $\boldsymbol{u} : \Omega \times [0, T_f] \to \mathbb{R}^d$ with respect to the reference configuration. It is defined as $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{\Phi}(\boldsymbol{x}, t) - \boldsymbol{x}$. The gradient of the displacement verifies $\operatorname{grad} \boldsymbol{u} = \boldsymbol{F} - \boldsymbol{I}$. The strain tensor can now be written in terms of the displacement

$$\begin{aligned}
\boldsymbol{E} &= \frac{1}{2}\left[(\nabla_x \boldsymbol{u} + \boldsymbol{I})^\top (\nabla_x \boldsymbol{u} + \boldsymbol{I}) - \boldsymbol{I}\right] \\
&= \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top + (\nabla_x \boldsymbol{u})^\top (\nabla_x \boldsymbol{u})\right],
\end{aligned}$$

or in components

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right).$$

To state the balance laws the actual deformed configuration is considered. The linear and angular momentum in a subdomain $\omega_t \subset \Omega_t$ are computed as

$$\int_{\omega_t} \rho\, \boldsymbol{v} \, \mathrm{d}\omega_t, \qquad \int_{\omega_t} \rho\, \boldsymbol{y} \times \boldsymbol{v} \, \mathrm{d}\omega,$$

where ρ is the mass density and the velocity $\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt}(\boldsymbol{y}, t)$ is material time derivative of the displacement (see [Abe12, Chapter 1]). Let $\omega_{t,1}$, $\omega_{t,2}$ be two subregions in a deformed continuum Ω_t with contacting surface S_{12}. There is a force acting on this surface for a continuum that is called stress vector or traction. If $\boldsymbol{n}$ is the outward normal at $\boldsymbol{y}$ on S_{12} with respect to $\omega_{t,1}$, then the surface force that $\omega_{t,1}$ exerts on $\omega_{t,2}$ is denoted by $\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) \in \mathbb{R}^d$. By the Newton third law, the surface force that $\omega_{t,1}$ applies on $\omega_{t,2}$ is given by $\boldsymbol{t}(\boldsymbol{y}, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n})$. It is assumed that the linear and angular momentum balance hold for any subregion $\omega \in \Omega_t$

$$\begin{aligned}
\frac{d}{dt}\int_{\omega_t} \rho \boldsymbol{v} \, \mathrm{d}\omega_t &= \int_{\partial\omega_t} \boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) \, \mathrm{d}S + \int_{\omega_t} \boldsymbol{f} \, \mathrm{d}\omega_t, \\
\frac{d}{dt}\int_{\omega_t} \rho \boldsymbol{y} \times \boldsymbol{v} \, \mathrm{d}\omega_t &= \int_{\partial\omega_t} \boldsymbol{y} \times \boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) \, \mathrm{d}S + \int_{\omega_t} \boldsymbol{y} \times \boldsymbol{f} \, \mathrm{d}\omega_t,
\end{aligned}$$

where $\boldsymbol{n}$ is the outward normal to the surface $\partial\omega_t$. The following theorem characterizes the stress vector (see [Cia88, Chapter 2]):

Theorem 1 (Cauchy's theorem)
If the linear and angular momenta balance hold, then there exists a matrix valued function $\boldsymbol{\Sigma}$ *from* Ω_t *to* $\mathbb{S}$ *such that* $\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) = \boldsymbol{\Sigma}(\boldsymbol{y})\boldsymbol{n}$, $\forall \boldsymbol{y} \in \Omega_t$ *where the right-hand side is the matrix-vector multiplication.*

The set $\mathbb{S} = \mathbb{R}^{d\times d}_{\text{sym}}$ denotes the field of symmetric matrices in $\mathbb{R}^{d\times d}$. The symmetric of the stress tensor $\boldsymbol{\Sigma}$ is due to the balance of angular momentum. The divergence theorem can then be applied

$$\int_{\partial\omega} \boldsymbol{\Sigma}\,\boldsymbol{n}\;\mathrm{d}S = \int_{\omega} \nabla_y \cdot \boldsymbol{\Sigma}\;\mathrm{d}\omega,$$

where $\nabla_y \cdot$ is the tensor divergence with respect to the deformed configuration, $\nabla_y \cdot \boldsymbol{\Sigma} = \sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial y_i}$. Because the considered subregion ω is arbitrary, using the linear balance momentum and the conservation of mass the following PDE is found

$$\rho\frac{D\boldsymbol{v}}{Dt} - \nabla_y \cdot \boldsymbol{\Sigma} = \boldsymbol{f}, \qquad \boldsymbol{y} \in \Omega_t.$$

This equation is written with respect to the deformed configuration Ω_t. For a detailed derivation of this equation the reader may consult [Abe12, Chapter 4].

3.2 The linear elastodynamics problem

Whenever deformations are small, $\nabla_x \boldsymbol{u} \ll 1$, there the reference and deformed configuration are almost indistinguishable $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u} = \boldsymbol{x} + O(\nabla_x \boldsymbol{u}) \approx \boldsymbol{x}$. This allows to write the linear momentum balance in the reference configuration

$$\rho\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x}, \boldsymbol{t}) - \mathrm{Div}(\boldsymbol{\Sigma}(\boldsymbol{x}, t)) = \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega.$$

The material derivative simplifies to a partial one. The operator Div is the divergence of a tensor field with respect to the reference configuration

$$\mathrm{Div}(\boldsymbol{\Sigma}(\boldsymbol{x}, t)) = \nabla_x \cdot \boldsymbol{\Sigma}(\boldsymbol{x}, t) = \sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial x_i}.$$

Furthermore, the non-linear terms in the Green-St. Venant strain tensor can be dropped

$$\boldsymbol{E} = \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top + (\nabla_x \boldsymbol{u})^\top (\nabla_x \boldsymbol{u})\right] \approx \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top\right].$$

The linearized strain tensor (also called infinitesimal strain tensor) is the symmetric gradient of the displacement

$$\boldsymbol{\varepsilon} := \mathrm{Grad}\,\boldsymbol{u}, \qquad \text{where} \qquad \mathrm{Grad}\,\boldsymbol{u} = \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top\right].$$

To obtain a closed system of equations, it is now necessary to characterize the relation between stress and strain. This relation is normally called *constitutive law*. In the following, the particular case of elastic materials is considered. For this class of materials, the stress tensor is solely determined by the deformed configuration at a given time. An elastic material is able to resist distorting excitations and return to its original size and shape when these are removed. A linear elastic material satisfies the Hooke's law

$$\boldsymbol{\Sigma}(\boldsymbol{x}) = \boldsymbol{\mathcal{D}}(\boldsymbol{x})\, \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})).$$

The *stiffness tensor* or *elasticity tensor* $\boldsymbol{\mathcal{D}} : \mathbb{S} \to \mathbb{S}$ is a rank 4 tensor that is symmetric positive definite and uniformly bounded above and below. Because of symmetry, its components satisfy

$$\mathcal{D}_{ijkl} = \mathcal{D}_{jikl} = \mathcal{D}_{klij}.$$

From the uniform boundedness of $\boldsymbol{\mathcal{D}}$, the map $\boldsymbol{\mathcal{D}} : L^2(\Omega; \mathbb{S}) \to L^2(\Omega; \mathbb{S})$ is a symmetric positive definite bounded linear operator ($L^2(\Omega; \mathbb{S})$ is the space of square integrable symmetric tensor valued functions). The compliance tensor $\boldsymbol{\mathcal{A}}$ is defined by $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{D}}^{-1}$. Thus $\boldsymbol{\mathcal{A}} : \mathbb{S} \to \mathbb{S}$ is as well symmetric positive definite and uniformly bounded above and below. An isotropic elastic medium has the same kinematic properties in any direction and at each point. If an elastic medium is isotropic, then the stiffness and compliance tensors assume the form

$$\boldsymbol{\mathcal{D}}(\cdot) = 2\mu(\cdot) + \lambda \operatorname{Tr}(\cdot)\boldsymbol{I}, \qquad \boldsymbol{\mathcal{A}}(\cdot) = \frac{1}{2\mu}\left[(\cdot) - \frac{\lambda}{2\mu + d\lambda}\operatorname{Tr}(\cdot)\boldsymbol{I}\right], \qquad d = \{2, 3\},$$

where Tr is the trace operator and the positive scalar functions μ, λ, defined on Ω, are called the Lamé coefficients. In engineering applications it is easier to compute experimentally two other parameters: the Young modulus E and Poisson's ratio ν. Those are expressed in terms of the Lamé coefficients as

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},$$

and inversely

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \qquad \mu = \frac{E}{2(1 + \nu)}.$$

The stiffness and compliant tensor assume the expressions

$$\boldsymbol{\mathcal{D}}(\cdot) = \frac{E}{1 + \nu}\left[(\cdot) + \frac{\nu}{1 - 2\nu}\operatorname{Tr}(\cdot)\boldsymbol{I}\right], \qquad \boldsymbol{\mathcal{A}}(\cdot) = \frac{1 + \nu}{E}\left[(\cdot) - \frac{\nu}{1 + \nu(d - 2)}\operatorname{Tr}(\cdot)\boldsymbol{I}\right].$$

The linear elastodynamics problem is formulated through a vector-valued PDE

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \operatorname{Div}(\boldsymbol{\mathcal{D}} \operatorname{Grad} \boldsymbol{u}) = \boldsymbol{f}. \tag{3.1}$$

The classical elastodynamics problem is expressed in terms of the displacement as the unknown.

3.3 Port-Hamiltonian formulation

In this section a port-Hamiltonian formulation for elasticity is deduced from the classical elastodynamics problem. It must be appointed that already in the seventies a purely hyperbolic formulation for elasticity was detailed [HM78]. The missing point is the clear connection with the theory of Hamiltonian PDEs. An Hamiltonian formulation can be found in [Gri15], but without any connection to the concept of Stokes-Dirac structure induced by the underlying geometry.

First, the concept of Stokes-Dirac is recalled presented. This is normally introduced by making use of a differential geometry approach. The interested reader may consult [Kot19, Chapter 2]. Despite being really insightful in terms of geometrical structure, this approach does not encompass the case of higher-order differential operators. An extension in this sense is still an open question. Since bending problems in elasticity introduce higher-order differential operators, the language of PDE will be privileged over the one of differential forms. To have the most suitable definition of Stokes-Dirac structure for flexible systems, the approach adopted in [MvdSM05] is here recovered.

Second, distributed port-Hamiltonian systems will be introduced, in connection with the underlying Stokes-Dirac structure. Then, the linear elastodynamics problem is recast in pH form. Few reference deal with the higher-dimensional case, with the notable exceptions of [KZ15, Skr19]. The former demonstrate the well-posedness of the linear wave equations in arbitrary geometrical dimensions. The latter generalizes results of the former to treat the case of generic first order linear pHs in arbitrary geometrical dimensions. Linear elasticity falls within the assumption of [Skr19]. Therefore, it is a well posed boundary controlled pH systems.

3.3.1 Reminder on the Stokes-Dirac structure

To define the Stokes-Dirac structure for higher differential and geometrical pHs, it is important to define and characterize the differential operators that come into play.

3.3.1.1 Constant matrix differential operators

Let Ω denote a compact subset of $\mathbb{R}^d$ representing the spatial domain of the distributed parameter system. Then, let U and V denote two sets of smooth functions from Ω to $\mathbb{R}^{q_u}$ and $\mathbb{R}^{q_v}$ respectively.

Definition 1
A constant matrix differential operator of order n is a map $\mathcal{L}$ from U to V such that, given

$\boldsymbol{u} = (u_1, \dots, u_{q_u}) \in U$ *and* $\boldsymbol{v} = (v_1, ..., v_{q_v}) \in V$*:*

$$\boldsymbol{v} = \mathcal{L}\boldsymbol{u} \iff \boldsymbol{v} := \sum_{|\alpha|=0}^{n} \boldsymbol{P}_\alpha \partial^\alpha \boldsymbol{u}, \tag{3.2}$$

where $\alpha := (\alpha_1, \dots, \alpha_d)$ *is a multi-index of order* $|\alpha| := \sum_{i=1}^{d} \alpha_i$*,* $\boldsymbol{P}_\alpha$ *are a set of constant real* $q_v \times q_u$ *matrices and* $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ *is a differential operator of order* $|\alpha|$ *resulting from a combination of spatial derivatives.*

Definition 2
Consider the constant matrix differential operator (3.2). Its formal adjoint is the map $\mathcal{L}^*$ *from* $\mathcal{V}$ *to* $\mathcal{U}$ *such that:*

$$\boldsymbol{u} = \mathcal{L}^*\boldsymbol{v} \iff \boldsymbol{u} := \sum_{|\alpha|=0}^{N} (-1)^{|\alpha|} \boldsymbol{P}_\alpha^\top \partial^\alpha \boldsymbol{v}. \tag{3.3}$$

Definition 3
Let $\mathcal{J}$ *denote a constant matrix differential operator. Then,* $\mathcal{J}$ *is skew-symmetric if and only if* $\mathcal{J} = -\mathcal{J}^*$*. This corresponds to the condition:*

$$\boldsymbol{P}_\alpha = (-1)^{|\alpha|+1} \boldsymbol{P}_\alpha^\top, \quad \forall \alpha. \tag{3.4}$$

An important relation between a differential operator and its adjoint is expressed by the following theorem.

Theorem 2 ([RR04], Chapter 9, theorem 9.37)
Consider a matrix differential operator $\mathcal{L}$ *and let* $\mathcal{L}^*$ *denote its formal adjoint. Then, for each function* $\boldsymbol{u} \in U$ *and* $\boldsymbol{v} \in V$*:*

$$\int_\Omega \left(\boldsymbol{v}^\top \mathcal{L}\boldsymbol{u} - \boldsymbol{u}^\top \mathcal{L}^* \boldsymbol{v} \right) \, \mathrm{d}\Omega = \int_{\partial\Omega} \widetilde{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}) \, \mathrm{d}A, \tag{3.5}$$

where $\widetilde{\mathcal{B}}_{\mathcal{L}}$ *is a differential operator induced on the boundary* $\partial\Omega$ *by* $\mathcal{L}$*, or equivalently:*

$$\boldsymbol{v}^\top \mathcal{L}\boldsymbol{u} - \boldsymbol{u}^\top \mathcal{L}^* \boldsymbol{v} = \operatorname{div} \widetilde{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}). \tag{3.6}$$

It is important to note that $\widetilde{\mathcal{B}}_{\mathcal{L}}$ is a constant differential operator. The quantity $\widetilde{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v})$ is a constant linear combination of the functions $\boldsymbol{u}$ and $\boldsymbol{v}$ together with their spatial derivatives up to a certain order and depending on $\mathcal{L}$.

Corollary 1
Consider a skew-symmetric differential operator $\mathcal{J}$*. Then, for each function* $\boldsymbol{u} \in U$ *and* $\boldsymbol{v} \in V$ *with* $q_u = q_v = q$*:*

$$\int_\Omega \left(\boldsymbol{v}^\top \mathcal{J}\boldsymbol{u} + \boldsymbol{u}^\top \mathcal{J} \boldsymbol{v} \right) \, \mathrm{d}\Omega = \int_{\partial\Omega} \widetilde{\mathcal{B}}_{\mathcal{J}}(\boldsymbol{u}, \boldsymbol{v}) \, \mathrm{d}A, \tag{3.7}$$

where $\widetilde{\mathcal{B}}_{\mathcal{J}}$ is a symmetric differential operator on $\partial\Omega$ depending on the differential operator $\mathcal{J}$.

3.3.1.2 Constant Stokes-Dirac structures

Following [MvdSM05], let F denote the space of flows, i.e. the space of smooth functions from the compact set $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^q$. For simplicity assume that the space of efforts is $E \equiv F$ (generally speaking these spaces are Hilbert spaces linked by duality, as in [Vil07]). Given $\boldsymbol{f} = (f_1, \ldots, f_q) \in F$ and $\boldsymbol{e} = (e_1, \ldots, e_q) \in E$. Let $\boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e})$ denote the boundary terms, where $\mathcal{B}_\partial$ provides the restriction on $\partial\Omega$ of the effort e and of its spatial derivatives of proper order. Then it can be written:

$$\int_{\partial\Omega} \widetilde{\mathcal{B}}_{\mathcal{J}}(\boldsymbol{e}_1, \boldsymbol{e}_2) \, \mathrm{d}S = \int_{\partial\Omega} \mathcal{B}_{\mathcal{J}}(\boldsymbol{z}_1, \boldsymbol{z}_2) \, \mathrm{d}S, \quad \text{with} \quad \widetilde{\mathcal{B}}_{\mathcal{J}}(\cdot,\cdot) = \mathcal{B}_{\mathcal{J}}(\mathcal{B}_\partial(\cdot), \, \mathcal{B}_\partial(\cdot)). \tag{3.8}$$

Define the set

$$Z := \{\boldsymbol{z} | \; \boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e})\}. \tag{3.9}$$

The following theorem characterizes Stokes-Dirac structures for pHs of arbitrary geometrical dimension and differential order.

Theorem 3 ([MvdSM05])
Consider the space of power variables $B = F \times E \times Z$. The linear subspace $D \subset B$

$$D = \{(\boldsymbol{f}, \boldsymbol{e}, \boldsymbol{z}) \in F \times E \times Z \mid \boldsymbol{f} = -\mathcal{J}\boldsymbol{e}, \; \boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e})\}, \tag{3.10}$$

is a Stokes–Dirac structure on $\mathcal{B}$ with respect to the pairing

$$\langle\langle(\boldsymbol{f}_1, \boldsymbol{e}_1, \boldsymbol{z}_1), (\boldsymbol{f}_2, \boldsymbol{e}_2, \boldsymbol{z}_2)\rangle\rangle := \int_\Omega \left(\boldsymbol{e}_1^\top \boldsymbol{f}_2 + \boldsymbol{e}_2^\top \boldsymbol{f}_1\right) \, \mathrm{d}\Omega + \int_{\partial\Omega} \mathcal{B}_{\mathcal{J}}(\boldsymbol{z}_1, \boldsymbol{z}_2) \, \mathrm{d}S. \tag{3.11}$$

Remark 1
*The constant Stokes-Dirac structure has been defined in case of smooth vector valued functions for simplicity. In this context the pairing has been defined as the L^2 inner product of vector-valued function. The definition is indeed more general and encompasses the case of more complex functional spaces. Linear elasticity for example is defined on a mixed function space of vector- and tensor- valued functions. The result presented here remains valid provided that the proper pairing is being chosen. Furthermore, the constant differential operator may contain intrinsic operators (*Div, Grad*) as it will be shown in §3.4.*

3.3.2 Reminder of distributed port-Hamiltonian systems

A distributed conservative port-Hamiltonian system is defined by a set of variables that describes the unknowns, by a formally skew-adjoint differential operator, an energy functional and a set of boundary inputs and corresponding conjugated outputs. Such a system is described by the following set of equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{\alpha}}{\partial t} &= \mathcal{J}\boldsymbol{e}, \\
\boldsymbol{u}_\partial &= \mathcal{B}\boldsymbol{e}, \\
\boldsymbol{y}_\partial &= \mathcal{C}\boldsymbol{e}, \\
\boldsymbol{e} &= \frac{\delta H}{\delta \boldsymbol{\alpha}}.
\end{aligned}
\tag{3.12}
$$

The unknowns $\boldsymbol{\alpha}$ are called energy variables in the port-Hamiltonian framework, the formally skew-adjoint operator $\mathcal{J}$ is named interconnection operator (see appendix A for a precise definition of formal skew adjointness). $\mathcal{B}, \mathcal{C}$ are boundary operator, that provide the boundary input $\boldsymbol{u}_\partial$ and output $\boldsymbol{y}_\partial$. Vector $\boldsymbol{e}$ contains the coenergy variables. These correspond to the variational derivative of the energy functional $H(\boldsymbol{\alpha})$.

Definition 4 (Variational derivative, Def. 4.1 in [Olv93])
Consider a functional $H(\boldsymbol{\alpha})$

$$
H(\boldsymbol{\alpha}) = \int_\Omega \mathcal{H}(\boldsymbol{\alpha}) \, \mathrm{d}\Omega.
$$

Given a variation $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}} + \epsilon \boldsymbol{\delta\alpha}$ *the variational derivative* $\frac{\delta H}{\delta \boldsymbol{\alpha}}$ *is defined as*

$$
H(\bar{\boldsymbol{\alpha}} + \epsilon \boldsymbol{\delta\alpha}) = H(\bar{\boldsymbol{\alpha}}) + \epsilon \int_\Omega \frac{\delta H}{\delta \boldsymbol{\alpha}} \boldsymbol{\delta\alpha} \, \mathrm{d}\Omega + O(\epsilon^2).
$$

Remark 2
If the integrand does not contain derivative of the argument $\boldsymbol{\alpha}$ *then the variational derivative is equal to the partial derivative of the Hamiltonian density* $\mathcal{H}$

$$
\frac{\delta H}{\delta \boldsymbol{\alpha}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\alpha}}.
$$

Conservative port-Hamiltonian systems possess a peculiar property. The energy rate is given by the power due to the boundary ports $\boldsymbol{u}_\partial, \boldsymbol{y}_\partial$

$$
\begin{aligned}
\dot{H} &= \int_\Omega \frac{\partial \mathcal{H}}{\partial \boldsymbol{\alpha}} \cdot \frac{\partial \boldsymbol{\alpha}}{\partial t} \, \mathrm{d}\Omega = \langle \delta_{\boldsymbol{\alpha}} H, \partial_t \boldsymbol{\alpha} \rangle_\Omega \\
&= \int_{\partial\Omega} \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial \, \mathrm{d}S = \langle \boldsymbol{u}_\partial, \boldsymbol{y}_\partial \rangle_{\partial\Omega}
\end{aligned}
\tag{3.13}
$$

Port-Hamiltonian system is one geometrical dimensional have been the main object of study and re

3.4 Port-Hamiltonian formulation of linear elasticity

Chapter 4

Port-Hamiltonian plate (and shell?) theory

4.1 Mindlin-Reissner model

4.1.1 Lagrangian formulation

4.1.2 Port-Hamiltonian formulation

4.2 Kirchhoff-Love model

4.2.1 Lagrangian formulation

4.2.2 Port-Hamiltonian formulation

4.3 Laminated anisotropic plates

4.3.1 Thin plate assumption

4.3.2 Thick plate assumption

4.4 The membrane shell problem ?

Chapter 5

Thermoelasticity in port-Hamiltonian form

5.1 Linear coupled thermoelasticity

5.2 Thermoelastic Euler-Bernoulli beam

5.3 Thermoelastic Kirchhoff plate

Part III

Finite element structure preserving discretization

Chapter 6

Partitioned finite element method

6.1 General procedure

6.1.1 Non-linear case

6.1.2 Linear case

6.1.3 Examples

6.2 Connection with mixed finite elements

6.3 Inhomogeneous boundary conditions

6.3.1 Solution using Lagrange multipliers

6.3.2 Virtual domain decomposition

Chapter 7

Convergence numerical study

7.1 Plate problems using known mixed finite elements

7.2 Non-standard discretization of flexible structures

Chapter 8

Numerical applications

8.1 Boundary stabilization

8.2 Thermoelastic wave propagation

8.3 Mixed boundary conditions

8.3.1 Trajectory tracking of a thin beam

8.3.2 Vibroacoustic under mixed boundary conditions

8.4 Modal analysis of plates

Part IV

Port-Hamiltonian flexible multibody dynamics

Chapter 9

Modular multibody systems in port-Hamiltonian form

9.1 Reminder of the rigid case

9.2 Flexible floating body

9.3 Modular construction of multibody systems

Chapter 10

Validation

10.1 Beam systems

10.1.1 Modal analysis of a flexible mechanism

10.1.2 Non-linear crank slider

10.1.3 Hinged beam

10.2 Plate systems

10.2.1 Boundary interconnection with a rigid element

10.2.2 Actuated plate

Conclusion

Conclusions and future directions

Appendix A

Mathematical tools

ARTICLE PAUL Formal differential operator J is defined without boundary conditions (see e. g. [39], Sect. III.3). Formal skew-symmetry is verified by () ei under zero boundary conditions, where () is the inner product on the appropriate functional space.

A.1 Differential operators

The space of all, symmetric and skew-symmetric $d \times d$ matrices are denoted by $\mathbb{M}, \mathbb{S}, \mathbb{K}$ respectively. The space of $\mathbb{R}^d$ vectors is denoted by $\mathbb{V}$. $\Omega \subset \mathbb{R}^d$ is an open connected set. For a scalar field $u : \Omega \to \mathbb{R}$ the gradient is defined as

$$\operatorname{grad}(u) = \nabla u := \left(\partial_{x_1} u \dots \partial_{x_d} u\right)^\top.$$

For a vector field $\boldsymbol{u} : \Omega \to \mathbb{V}$, with components u_j, the gradient (Jacobian) is defined as

$$\operatorname{grad}(\boldsymbol{u})_{ij} := (\nabla \boldsymbol{u})_{ij} = \partial_{x_j} u_i.$$

The symmetric part of the gradient operator Grad (i. e. the deformation gradient in continuum mechanics) is thus given by

$$\operatorname{Grad}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla^\top \boldsymbol{u}\right).$$

The Hessian operator of u is then computed as follows

$$\operatorname{Hess}(u) = \nabla^2 u = \operatorname{Grad}(\operatorname{grad}(u)),$$

For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{M}$, with components u_{ij}, the divergence is a vector, defined column-wise as

$$\operatorname{Div}(\boldsymbol{U}) = \nabla \cdot \boldsymbol{U} := \left(\sum_{i=1}^d \partial_{x_i} u_{ij}\right)_{j=1,\dots,d}.$$

The double divergence of a tensor field $\boldsymbol{U}$ is then a scalar field defined as

$$\operatorname{div}(\operatorname{Div}(\boldsymbol{U})) := \sum_{i,j=1}^d \partial_{x_i}\partial_{x_j} u_{ij}.$$

Appendix B

Finite elements gallery

Appendix C

Implementation using FEniCS and Firedrake

Bibliography

[Abe12] R. Abeyaratne. *Lecture Notes on the Mechanics of Elastic Solids. Volume II: Continuum Mechanics*. Cambridge, MA and Singapore, 1st edition, 2012.

[Cia88] P. G. Ciarlet. *Mathematical Elasticity: Three-Dimensional Elasticity*. Studies in mathematics and its applications. North-Holland, 1988.

[Gri15] M. Grinfeld. *Mathematical Tools for Physicists*. John Wiley & Sons Inc, 2nd edition, jan 2015.

[HM78] Thomas J.R. Hughes and Jerrold E. Marsden. Classical elastodynamics as a linear symmetric hyperbolic system. *Journal of Elasticity*, 8(1):97–110, 1978.

[JZ12] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Number 223 in Operator Theory: Advances and Applications. Springer Verlag, Germany, 2012. https://doi.org/10.1007/978-3-0348-0399-1.

[Kot19] P. Kotyczka. *Numerical Methods for Distributed Parameter Port-Hamiltonian Systems*. TUM University Press, 2019.

[KZ15] M. Kurula and H. Zwart. Linear wave systems on n-d spatial domains. *International Journal of Control*, 88(5):1063–1077, 2015. https://www.tandfonline.com/doi/abs/10.1080/00207179.2014.993337.

[Lee12] J. Lee. *Mixed methods with weak symmetry for time dependent problems of elasticity and viscoelasticity*. PhD thesis, University of Minnesota, 2012.

[LGZM05] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and Boundary Control Systems associated with Skew-Symmetric Differential Operators. *SIAM Journal on Control and Optimization*, 44(5):1864–1892, 2005. https://doi.org/10.1137/040611677.

[LPKL12] L. D. Landau, L. P. Pitaevskii, A. M. Kosevich, and E. M. Lifshitz. *Theory of Elasticity*. Butterworth Heinemann, third edition, Dec 2012.

[MvdSM05] A. Macchelli, A. J. van der Schaft, and C. Melchiorri. Port Hamiltonian formulation of infinite dimensional systems I. Modeling. In *Proceedings of the 44th IEEE Conference on Decision and Control*, volume 4, pages 3762 – 3767. IEEE, 01 2005. https://doi.org/10.1109/CDC.2004.1429324.

[Olv93] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate texts in mathematics*. Springer-Verlag New York, 2 edition, 1993.

[RR04] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*. Number 13 in Texts in Applied Mathematics. Springer-Verlag New York, 2 edition, 2004.

[Skr19] Nathanael Skrepek. Well-posedness of linear first order port-hamiltonian systems on multidimensional spatial domains, 2019.

[Vil07] J.A. Villegas. *A Port-Hamiltonian Approach to Distributed Parameter Systems.* PhD thesis, University of Twente, May 2007.

Résumé — Malgré l'abondante littérature sur le formalisme pH, les problèmes d'élasticité en deux ou trois dimensions géométriques n'ont presque jamais été considérés. Cette thèse vise à étendre l'approche port-Hamiltonienne (pH) à la mécanique des milieux continus. L'originalité apportée réside dans trois contributions majeures. Tout d'abord, la nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d'élasticité nécessite l'utilisation d'éléments finis non standard. Néanmoins, l'implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

Mots clés : Systèmes port-Hamiltonien, méchanique des solides, discretisation symplectique, méthode des éléments finis, dynamique multicorps

---

Abstract — Despite the large literature on pH formalism, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an intrinsic, i.e. coordinate free, and equivalent pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

Keywords: Port-Hamiltonian systems, continuum mechanics, structure preserving discretization, finite element method, multibody dynamics.
