Université Fédérale
Toulouse Midi-Pyrénées

THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Institut Supérieur de l'Aéronautique et de l'Espace (ISAE)*

Présentée et soutenue le *30 Octobre 2020* **par :**

ANDREA BRUGNOLI

A port-Hamiltonian formulation of flexible structures
Modelling and symplectic finite element discretization

JURY

DANIEL ALAZARD	ISAE-Supaéro, Toulouse	Directeur
VALÉRIE P. BUDINGER	ISAE-Supaéro, Toulouse	Co-directeur
YANN LE GORREC	Institut FEMTO-ST	Rapporteur
ALESSANDRO MACCHELLI	Universitá di Bologna	Rapporteur
THOMAS HÉLIE	Directeur de Recherches CNRS	Examinateur
LUC DUGARD	GIPSA-LAB, Grenoble	Président

École doctorale et spécialité :
EDSYS : Automatique
Unité de Recherche :
CSDV - Commande des Systèmes et Dynamique du Vol - ONERA - ISAE
Directeur de Thèse :
Daniel ALAZARD et *Valérie POMMIER-BUDINGER*
Rapporteurs :
Yann LE GORREC et *Alessandro MACCHELLI*

Abstract

This thesis aims at extending the port-Hamiltonian (pH) approach to continuum mechanics in higher geometrical dimensions (particularly in 2D). The pH formalism has a strong multiphysics character and represents a unified framework to model, analyze and control both finite- and infinite-dimensional systems. Despite the large literature on this topic, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an intrinsic, i.e. coordinate free, and equivalent pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems in port-Hamiltonian form requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

Résumé

Cette thèse vise à étendre l'approche port-hamiltonienne (pH) à la mécanique des milieux continus dans des dimensions géométriques plus élevées (en particulier on se focalise sur la dimension deux). Le formalisme pH, avec son fort caractère multiphysique, représente un cadre unifié pour modéliser, analyser et contrôler les systèmes de dimension finie et infinie. Malgré l'abondante littérature sur ce sujet, les problèmes d'élasticité en deux ou trois dimensions géométriques n'ont presque jamais été considérés. Dans ce travail de thèse la connexion entre problèmes d'élasticité et systèmes distribués port-Hamiltoniens est établie. L'originalité apportée réside dans trois contributions majeures. Tout d'abord, la nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d'élasticité écrits en forme port-Hamiltonienne nécessite l'utilisation d'éléments finis non standard. Néanmoins, l'implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

Aknowledgements

Remerciements

Ringraziamenti

Alla mia famiglia

Contents

List of Figures

List of Tables

List of Acronyms

DAE	*Differential-Algebraic Equation*
dpHs	*distributed port-Hamiltonian systems*
FEM	*Finite Element Method*
IDA-PBC	*Interconnection and Damping Assignment Passivity Based Control*
PDE	*Partial Differential Equation*
PFEM	*Partitioned Finite Element Method*
pH	*port-Hamiltonian*
pHs	*port-Hamiltonian systems*
pHDAE	*port-Hamiltonian Descriptor System*

Part I

Introduction and state of the art

Chapter 1

Introduction

Je n'ai cherché de rien prouver, mais de bien peindre et d'éclairer bien ma peinture.

André Gide
Préface de L'Immoraliste

Contents

1.1 Motivation and context

1.2 Overview of chapters

1.3 Contributions

Chapter 2

Literature review

Whereof one cannot speak, thereof one must be silent.

Ludwig Wittgenstein
Tractatus Logico-Philosophicus

2.1 Port-Hamiltonian distributed systems

For 1D linear PH systems with a generalized skew-adjoint system operator, [LGZM05] gives conditions on the assignment of boundary inputs and outputs for the system operator to generate a contraction semigroup. The latter is instrumental to show well-posedness of a linear PH system, see [JZ12]. Essentially, at most half the number of boundary port variables can be imposed as control inputs for a well-posed PH system in 1D.

2.2 Structure-preserving discretization

2.3 Mixed finite element for elasticity

2.4 Multibody dynamics

Part II

Port-Hamiltonian elasticity and thermoelasticity

Chapter 3

Elasticity in port-Hamiltonian form

I try not to break the rules but merely to test their elasticity.

Bill Veeck

Contents

Continuum mechanics is the mathematical description of how materials behave kinematically under external excitations. In this framework, the microscopic structure of a material body is neglected and a macroscopic viewpoint, that describes the body as a continuum, is adopted. This leads to a PDE based model. In this chapter, the general linear elastodynamics problem is recalled. A suitable port-Hamiltonian formulation is then derived.

3.1 Continuum mechanics

In this section, the main concepts behind a deformable continuum are briefly recalled following [Lee12]. For a detailed discussion on this topic, the reader may consult [Abe12, LPKL12].

3.1.1 Non linear formulation of elasticity

The bounded region of $\mathbb{R}^d$ ($d = 2, 3$) occupied by a solid is called configuration. The reference configuration Ω is the domain that a bodies occupies at the initial state. To describe how the body deforms in time the deformation map $\boldsymbol{\Phi} : \Omega \times [0, T_f] \to \Omega' \subset \mathbb{R}^d$ is introduced. This map is differentiable and orientation preserving and the image of Ω under $\boldsymbol{\Phi}(\cdot, t)\ \forall t \in [0, T_f]$ is called the deformed configuration Ω_t. Given a specific point in the reference frame its image is denoted by $\boldsymbol{y} = \boldsymbol{\Phi}(\boldsymbol{x}, t)$. The gradient of the deformation map is called the deformation gradient $\boldsymbol{F} := \nabla_x \boldsymbol{\Phi} = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}$. A rigid deformation maps a point $\boldsymbol{x} \in \Omega \to \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{b}(t)$, where $\boldsymbol{A}(t)$ is an orthogonal matrix and $\boldsymbol{b}(t)$ an $\mathbb{R}^d$ vector. A differentiable deformation map $\boldsymbol{\Phi}$ is a rigid deformation iff $\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I} = 0$, where $\boldsymbol{I}$ is the identity in $\mathbb{R}^{d\times d}$ (for the proof see [Cia88], page 44). For this reason, a suitable measure of the deformation is the Green-St.Venant strain tensor $\frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I})$.
A quantity of interest is the displacement $\boldsymbol{u} : \Omega \times [0, T_f] \to \mathbb{R}^d$ with respect to the reference configuration. It is defined as $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{\Phi}(\boldsymbol{x}, t) - \boldsymbol{x}$. The gradient of the displacement verifies $\nabla_x \boldsymbol{u} = \boldsymbol{F} - \boldsymbol{I}$. The strain tensor can now be written in terms of the displacement

$$\begin{aligned}\frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}) &= \frac{1}{2}\left[(\nabla_x \boldsymbol{u} + \boldsymbol{I})^\top (\nabla_x \boldsymbol{u} + \boldsymbol{I}) - \boldsymbol{I}\right] \\ &= \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top + (\nabla_x \boldsymbol{u})^\top (\nabla_x \boldsymbol{u})\right],\end{aligned}$$

or in components

$$\frac{1}{2}(F_{ik}^\top F_{kj} - I_{ij}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right).$$

To state the balance laws the actual deformed configuration is considered. The linear and angular momentum in a subdomain $\omega_t \subset \Omega_t$ are computed as

$$\int_{\omega_t} \rho\, \boldsymbol{v}\ \mathrm{d}\omega_t, \qquad \int_{\omega_t} \rho\, \boldsymbol{y} \times \boldsymbol{v}\ \mathrm{d}\omega,$$

where ρ is the mass density and the velocity $\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt}(\boldsymbol{y}, t) = \frac{\partial \boldsymbol{u}}{\partial t} + (\nabla_x \boldsymbol{u}) \cdot \boldsymbol{v}$ is material time derivative of the displacement (see [Abe12, Chapter 1]). Let $\omega_{t,1}$, $\omega_{t,2}$ be two subregions in a deformed continuum Ω_t with contacting surface S_{12}. There is a force acting on this surface for a continuum that is called stress vector or traction. If $\boldsymbol{n}$ is the outward normal at $\boldsymbol{y}$ on S_{12} with respect to $\omega_{t,1}$, then the surface force that $\omega_{t,1}$ exerts on $\omega_{t,2}$ is denoted by $\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) \in \mathbb{R}^d$. By the Newton third law, the surface force that $\omega_{t,2}$ applies on $\omega_{t,1}$ is given by $\boldsymbol{t}(\boldsymbol{y}, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n})$. It is assumed that the linear and angular momentum balance hold for any subregion $\omega_t \in \Omega_t$

$$\begin{aligned}\frac{d}{dt}\int_{\omega_t} \rho \boldsymbol{v}\ \mathrm{d}\omega_t &= \int_{\partial\omega_t} \boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n})\ \mathrm{d}S + \int_{\omega_t} \boldsymbol{f}\ \mathrm{d}\omega_t, \\ \frac{d}{dt}\int_{\omega_t} \rho \boldsymbol{y} \times \boldsymbol{v}\ \mathrm{d}\omega_t &= \int_{\partial\omega_t} \boldsymbol{y} \times \boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n})\ \mathrm{d}S + \int_{\omega_t} \boldsymbol{y} \times \boldsymbol{f}\ \mathrm{d}\omega_t,\end{aligned}$$

where $\partial\omega_t$ stands for the boundary surface of the subdomain ω_t, $\boldsymbol{n}$ is the outward normal to the surface $\partial\omega_t$ and $\boldsymbol{f}$ represents an exterior body force. The following theorem characterizes the stress vector (see [Cia88, Chapter 2]):

Theorem 1 (Cauchy's theorem)
If the linear and angular momenta balance hold, then there exists a matrix-valued function $\boldsymbol{\Sigma}$ *from* Ω_t *to* $\mathbb{S}$ *such that* $\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) = \boldsymbol{\Sigma}(\boldsymbol{y})\boldsymbol{n}$, $\forall \boldsymbol{y} \in \Omega_t$ *where the right-hand side is the matrix-vector multiplication.*

The set $\mathbb{S} = \mathbb{R}^{d\times d}_{\text{sym}}$ denotes the field of symmetric matrices in $\mathbb{R}^{d\times d}$. The symmetry of the stress tensor $\boldsymbol{\Sigma}$ is due to the balance of angular momentum. The divergence theorem can then be applied

$$\int_{\partial\omega_t} \boldsymbol{\Sigma}\,\boldsymbol{n}\,\mathrm{d}S = \int_{\omega_t} \nabla_y \cdot \boldsymbol{\Sigma}\,\mathrm{d}\omega,$$

where $\nabla_y\cdot$ is the tensor divergence with respect to the deformed configuration, $\nabla_y \cdot \boldsymbol{\Sigma} = \sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial y_i}$. Because the considered subregion ω_t is arbitrary, using the linear balance momentum and the conservation of mass the following PDE is found

$$\rho\frac{D\boldsymbol{v}}{Dt} - \nabla_y \cdot \boldsymbol{\Sigma} = \boldsymbol{f}, \qquad \boldsymbol{y} \in \Omega_t.$$

This equation is written with respect to the deformed configuration Ω_t. For a detailed derivation of this equation the reader may consult [Abe12, Chapter 4]. To obtain a closed formulation, the constitutive law, namely the link between $\boldsymbol{\Sigma}$ and the strain tensor $\frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{I})$, has to be introduced. In the next such relation will be investigated for the case of linear elasticity.

3.1.2 The linear elastodynamics problem

Whenever deformations are small, $\|\nabla_x \boldsymbol{u}\| \ll 1$, then the reference and deformed configurations are almost indistinguishable $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u} = \boldsymbol{x} + O(\nabla_x\boldsymbol{u}) \approx \boldsymbol{x}$. This allows to write the linear momentum balance in the reference configuration

$$\rho\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x}, \boldsymbol{t}) - \mathrm{Div}(\boldsymbol{\Sigma}(\boldsymbol{x}, t)) = \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega.$$

The material derivative simplifies to a partial one. The operator Div is the divergence of a tensor field with respect to the reference configuration

$$\mathrm{Div}(\boldsymbol{\Sigma}(\boldsymbol{x}, t)) = \nabla_x \cdot \boldsymbol{\Sigma}(\boldsymbol{x}, t) = \left(\sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial x_i}\right)_{1\le j\le d}.$$

Furthermore, the non-linear terms in the Green-St. Venant strain tensor can be dropped

$$\frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\left[\nabla_x\boldsymbol{u} + (\nabla_x\boldsymbol{u})^\top + (\nabla_x\boldsymbol{u})^\top(\nabla_x\boldsymbol{u})\right] \approx \frac{1}{2}\left[\nabla_x\boldsymbol{u} + (\nabla_x\boldsymbol{u})^\top\right].$$

The linearized strain tensor (also called infinitesimal strain tensor) is the symmetric gradient of the displacement

$$\boldsymbol{\varepsilon} := \operatorname{Grad} \boldsymbol{u}, \qquad \text{where} \qquad \operatorname{Grad} \boldsymbol{u} = \frac{1}{2}\left[\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^\top\right]. \tag{3.1}$$

To obtain a closed system of equations, it is now necessary to characterize the relation between stress and strain. This relation is normally called *constitutive law*. In the following, the particular case of elastic materials is considered. These are able to resist distorting excitations and return to its original size and shape when these are removed. For this class of materials, the stress tensor is solely determined by the deformed configuration at a given time (Hooke's law)

$$\boldsymbol{\Sigma}(\boldsymbol{x}) = \boldsymbol{\mathcal{D}}(\boldsymbol{x})\, \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x})).$$

The *stiffness tensor* or *elasticity tensor* $\boldsymbol{\mathcal{D}} : \mathbb{S} \to \mathbb{S}$ is a rank 4 tensor that is symmetric positive definite and uniformly bounded above and below. Because of symmetry, its components satisfy

$$\mathcal{D}_{ijkl} = \mathcal{D}_{jikl} = \mathcal{D}_{klij}.$$

From the uniform boundedness of $\boldsymbol{\mathcal{D}}$, the map $\boldsymbol{\mathcal{D}} : L^2(\Omega;\mathbb{S}) \to L^2(\Omega;\mathbb{S})$ is a symmetric positive definite bounded linear operator ($L^2(\Omega;\mathbb{S})$ is the space of square integrable symmetric tensor-valued functions). The compliance tensor $\boldsymbol{\mathcal{C}}$ is defined by $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{D}}^{-1}$. Thus $\boldsymbol{\mathcal{C}} : \mathbb{S} \to \mathbb{S}$ is as well symmetric positive definite and uniformly bounded above and below. An isotropic elastic medium has the same kinematic properties in any direction and at each point. If an elastic medium is isotropic, then the stiffness and compliance tensors assume the form

$$\boldsymbol{\mathcal{D}}(\cdot) = 2\mu(\cdot) + \lambda \operatorname{Tr}(\cdot)\boldsymbol{I}, \qquad \boldsymbol{\mathcal{C}}(\cdot) = \frac{1}{2\mu}\left[(\cdot) - \frac{\lambda}{2\mu + d\lambda}\operatorname{Tr}(\cdot)\boldsymbol{I}\right], \qquad d = \{2, 3\}, \tag{3.2}$$

where Tr is the trace operator and the positive scalar functions μ, λ, defined on Ω, are called the Lamé coefficients. In engineering applications it is easier to compute experimentally two other parameters: the Young modulus E and Poisson's ratio ν. Those are expressed in terms of the Lamé coefficients as

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \tag{3.3}$$

and conversely

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{3.4}$$

The stiffness and compliant tensor assume the expressions

$$\boldsymbol{\mathcal{D}}(\cdot) = \frac{E}{1+\nu}\left[(\cdot) + \frac{\nu}{1-2\nu}\operatorname{Tr}(\cdot)\boldsymbol{I}\right], \tag{3.5}$$

$$\boldsymbol{\mathcal{C}}(\cdot) = \frac{1+\nu}{E}\left[(\cdot) - \frac{\nu}{1+\nu(d-2)}\operatorname{Tr}(\cdot)\boldsymbol{I}\right]. \tag{3.6}$$

The linear elastodynamics problem is formulated through a vector-valued PDE

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{Div}(\boldsymbol{\mathcal{D}} \,\mathrm{Grad}\, \boldsymbol{u}) = \boldsymbol{f}. \tag{3.7}$$

The classical elastodynamics problem is expressed in terms of the displacement as the unknown. This PDE goes together with appropriate boundary conditions that will be specified in 3.3.

3.2 Port-Hamiltonian systems

Before introducing the pH formulation of the elastodynamics problem, the main concept behind this formalism are recalled. First, the concept of Stokes-Dirac structure is presented. This is normally introduced by making use of a differential geometry approach. The interested reader may consult [Kot19, Chapter 2]. Despite being really insightful in terms of geometrical structure, this approach does not encompass the case of higher-order differential operators. An extension in this sense is still an open question. Since bending problems in elasticity introduce higher-order differential operators, the language of PDE will be privileged over the one of differential forms. To have the most suitable definition of Stokes-Dirac structure for flexible systems, the approach adopted in [MvdSM05] is here recovered.

Second, distributed port-Hamiltonian systems are introduced, in connection with the underlying Stokes-Dirac structure. PHs as boundary control systems have been analyzed deeply in one geometrical dimension [JZ12, LGZM05]. Here, a more general definition is given. The complete characterization of pH in arbitrary dimension is still an open research field. Two notable exceptions ([KZ15, Skr19]) provide partial answers to this problem. The first demonstrate the well-posedness of the linear wave equation in arbitrary geometrical dimensions. The second generalizes this result to treat the case of generic first order linear pHs in arbitrary geometrical dimensions.

3.2.1 The Stokes-Dirac structure

In the section the concept of Stokes-Dirac structure for distributed, i.e. infinite-dimensional, pHs is introduced. First, the finite-dimensional case is considered. Then, to introduce the infinite-dimensional extension of Dirac structure, namely the Stokes-Dirac structure, the differential operators that come into play are characterized.

3.2.1.1 Dirac Structures

Consider a finite dimensional space F over the field $\mathbb{R}$ and $E \equiv F'$ its dual, i.e. the space of linear operator $\mathbf{e} : F \to \mathbb{R}$. The elements of F are called flows, while the elements of E are

called efforts. Those are port variables and their combination gives the power flowing inside the system. The space $B = F \times E$ is called the bond space of power variables. Therefore the power is defined as $\langle \mathbf{e}, \mathbf{f} \rangle = \boldsymbol{e}(\mathbf{f})$, where $\langle \mathbf{e}, \mathbf{f} \rangle$ is the dual product between $\mathbf{f}$ and $\mathbf{e}$.

Definition 1 ([Cou90], Def. 1.1.1)
Given the finite-dimensional space F and its dual E with respect to the inner product $\langle \cdot, \cdot \rangle : F \times E \to \mathbb{R}$, define the symmetric bilinear form:

$$\langle\langle\, (\mathbf{f}_1, \mathbf{e}_1), (\mathbf{f}_2, \mathbf{e}_2)\, \rangle\rangle := \langle \mathbf{e}_1, \mathbf{f}_2 \rangle + \langle \mathbf{e}_2, \mathbf{f}_1 \rangle\,, \qquad \textit{where} \qquad (\mathbf{f}_i, \mathbf{e}_i) \in B,\ i = 1, 2 \tag{3.8}$$

A Dirac structure on $B := F \times E$ is a subspace $D \subset B$, which is maximally isotropic under $\langle\langle \cdot, \cdot \rangle\rangle$. Equivalently, a Dirac structure on $B := F \times E$ is a subspace $D \subset B$ which equals its orthogonal complement with respect to $\langle\langle \cdot, \cdot \rangle\rangle : D = D^{\perp}$.

This definition can be extended to consider distributed forces and dissipation [Vil07].

Proposition 1
Consider the space of power variables $F \times E$ and let X denote an n-dimensional space, the space of energy variables. Suppose that $F := (F_s,\ F_e)$ and that $E := (E_s,\ E_e)$, with $\dim F_s = \dim E_s = n$ and $\dim F_e = \dim E_e = m$. Moreover, let $\mathbf{J}(\mathbf{x})$ denote a skew-symmetric matrix of dimension n and $\mathbf{B}(\mathbf{x})$ a matrix of dimension $n \times m$. Then, the set

$$D := \left\{ (\mathbf{f}_s, \mathbf{f}_e, \mathbf{e}_s, \mathbf{e}_e) \in F \times E |\quad \mathbf{f}_s = -\mathbf{J}(\mathbf{x})\mathbf{e}_s - \mathbf{B}(\mathbf{x})\mathbf{f}_e,\ \mathbf{e}_e = \mathbf{B}(\mathbf{x})^{\top}\mathbf{e}_s \right\} \tag{3.9}$$

is a Dirac structure.

3.2.1.2 Finite-dimensional port-Hamiltonian systems

Consider the time-invariant dynamical system:

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{J}(\mathbf{x})\nabla H(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}, \\ \mathbf{y} &= \mathbf{B}(\mathbf{x})^{\top}\nabla H(\mathbf{x}), \end{cases} \tag{3.10}$$

where $H(\mathbf{x}) : X \to \mathbb{R}$, the Hamiltonian, is a real-valued function bounded from below. Such a system is called port-Hamiltonian, as it arises from the Hamiltonian modelling of a physical system and it interacts with the environment through the input $\mathbf{u}$, included in the formulation. The connection with the concept of Dirac structure is achieved by considering the following port behavior:

$$\begin{aligned} \mathbf{f}_s &= -\dot{\mathbf{x}}, & \mathbf{e}_s &= \nabla H(\mathbf{x}), \\ \mathbf{f}_e &= \mathbf{u}, & \mathbf{e}_e &= \mathbf{y}. \end{aligned} \tag{3.11}$$

With this choice of the port variables system (3.10) defines, by Proposition 1, a Dirac structure. Dissipation and distributed forces can be included and the corresponding system defines an extended Dirac structure, once the proper port variables have been introduced.

3.2.1.3 Constant matrix differential operators

Let Ω denote a compact subset of $\mathbb{R}^d$ representing the spatial domain of the distributed parameter system. Then, let $U = C^\infty(\Omega, \mathbb{R}^{q_u})$ and $V = C^\infty(\Omega, \mathbb{R}^{q_v})$ denote the sets of smooth functions from Ω to $\mathbb{R}^{q_u}$ and $\mathbb{R}^{q_v}$ respectively.

Definition 2
A constant matrix differential operator of order n is a map $\mathcal{L} : U \to V$ such that, given $\boldsymbol{u} = (u_1, \ldots, u_{q_u}) \in U$ and $\boldsymbol{v} = (v_1, ..., v_{q_v}) \in V$:

$$\boldsymbol{v} = \mathcal{L}\boldsymbol{u} \iff \boldsymbol{v} := \sum_{|\alpha|=0}^{n} \boldsymbol{P}_\alpha \partial^\alpha \boldsymbol{u}, \tag{3.12}$$

where $\alpha := (\alpha_1, \ldots, \alpha_d)$ is a multi-index of order $|\alpha| := \sum_{i=1}^d \alpha_i$, $\boldsymbol{P}_\alpha$ is a set of constant real $q_v \times q_u$ matrices and $\partial^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$ is a differential operator of order $|\alpha|$ resulting from a combination of spatial derivatives.

The following definition, instrumental for the case of dpHs, is a simplified version of (6).

Definition 3
Consider the constant matrix differential operator (3.12). Its formal adjoint is the map $\mathcal{L}^*$ from V to U such that:

$$\boldsymbol{u} = \mathcal{L}^*\boldsymbol{v} \iff \boldsymbol{u} := \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} \boldsymbol{P}_\alpha^\top \partial^\alpha \boldsymbol{v}. \tag{3.13}$$

Remark 1 (Differences between adjoint and formal adjoint)
The definition of formal adjoint is such that the integration by parts formula is respected

$$\int_\Omega \boldsymbol{a} \cdot (\mathcal{L}\boldsymbol{b}) \, \mathrm{d}\Omega = \int_\Omega (\mathcal{L}^*\boldsymbol{a}) \cdot \boldsymbol{b} \, \mathrm{d}\Omega,$$

where $\boldsymbol{a} \in C_0^\infty(\Omega, \mathbb{R}^{q_u})$, $\boldsymbol{b} \in C_0^\infty(\Omega, \mathbb{R}^{q_v})$ are smooth functions with compact support. This corresponds to the adjoint definition for an operator between L^2 spaces of square integrable functions

$$\langle \boldsymbol{a}, \mathcal{L}\boldsymbol{b} \rangle_{L^2(\Omega, \mathbb{R}^{q_v})} = \langle \mathcal{L}^*\boldsymbol{a}, \boldsymbol{b} \rangle_{L^2(\Omega, \mathbb{R}^{q_u})}.$$

That means that, contrarily to the adjoint of an operator, the formal adjoint definition does not regard the actual domain of the operator nor the boundary conditions. For example, the differential operators div, grad *are unbounded in the L^2 topology. Whenever unbounded operators are considered, it is important to define their domain. To avoid the need of specifying domains, the notion of formal adjoint can be evoked. The formal adjoint respects the integration by parts formula and is defined only for sufficiently smooth functions with compact support. In this sense* div, grad *are formally skew-adjoint, since for smooth functions with*

compact support, it holds

$$\langle \boldsymbol{y},\, \mathrm{grad}(x)\rangle_{L^2(\Omega,\mathbb{R}^3)} \underbrace{=}_{I.B.P.} -\langle \mathrm{div}(\boldsymbol{y}),\, x\rangle_{L^2(\Omega,\mathbb{R})}\,,$$

for $\boldsymbol{Y} \in C_0^\infty(\Omega,\mathbb{R}^n)$, $x \in C_0^\infty(\Omega)$ *(I.B.P. stands for integration by parts). The definition of the domain of the operators, that requires the knowledge of the boundary conditions, has not been specified.*

Definition 4
Let $W = C^\infty(\Omega,\mathbb{R}^q)$ *be the space of vector-valued smooth functions and* $\mathcal{J} : W \to W$ *a constant matrix differential operator. Then,* $\mathcal{J}$ *is formally skew-adjoint (or skew-symmetric) if and only if* $\mathcal{J} = -\mathcal{J}^*$. *This corresponds to the algebraic condition*

$$\boldsymbol{P}_\alpha = (-1)^{|\alpha|+1}\boldsymbol{P}_\alpha^\top, \quad \forall \alpha. \tag{3.14}$$

An important relation between a differential operator and its adjoint is expressed by the following theorem.

Theorem 2 ([RR04], Chapter 9, theorem 9.37)
Consider a matrix differential operator $\mathcal{L}$ *and let* $\mathcal{L}^*$ *denote its formal adjoint. Then, for each function* $\boldsymbol{u} \in U$ *and* $\boldsymbol{v} \in V$*:*

$$\int_\Omega \left(\boldsymbol{v}^\top \mathcal{L}\boldsymbol{u} - \boldsymbol{u}^\top \mathcal{L}^*\boldsymbol{v}\right)\, \mathrm{d}\Omega = \int_{\partial\Omega} \widetilde{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{v})\, \mathrm{d}A, \tag{3.15}$$

where $\widetilde{\mathcal{B}}_{\mathcal{L}}$ *is a differential operator induced on the boundary* $\partial\Omega$ *by* $\mathcal{L}$*, or equivalently:*

$$\boldsymbol{v}^\top \mathcal{L}\boldsymbol{u} - \boldsymbol{u}^\top \mathcal{L}^*\boldsymbol{v} = \mathrm{div}\, \widetilde{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{v}). \tag{3.16}$$

It is important to note that $\widetilde{\mathcal{B}}_{\mathcal{L}}$ is a constant differential operator. The quantity $\widetilde{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{u},\boldsymbol{v})$ is a constant linear combination of the functions $\boldsymbol{u}$ and $\boldsymbol{v}$ together with their spatial derivatives up to a certain order and depending on $\mathcal{L}$.

Corollary 1
Consider a skew-symmetric differential operator $\mathcal{J}$. *Then, for each function* $\boldsymbol{u} \in U$ *and* $\boldsymbol{v} \in V$ *with* $q_u = q_v = q$*:*

$$\int_\Omega \left(\boldsymbol{v}^\top \mathcal{J}\boldsymbol{u} + \boldsymbol{u}^\top \mathcal{J}\boldsymbol{v}\right)\, \mathrm{d}\Omega = \int_{\partial\Omega} \widetilde{\mathcal{B}}_{\mathcal{J}}(\boldsymbol{u},\boldsymbol{v})\, \mathrm{d}A, \tag{3.17}$$

where $\widetilde{\mathcal{B}}_{\mathcal{J}}$ *is a symmetric differential operator on* $\partial\Omega$ *depending on the differential operator* $\mathcal{J}$.

3.2.1.4 Constant Stokes-Dirac structures

Following [MvdSM05], let F denote the space of flows, i.e. the space of smooth functions from the compact set $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^q$. For simplicity assume that the space of efforts is $E \equiv F$

(generally speaking these spaces are Hilbert spaces linked by duality, as in [Vil07]). Given $\boldsymbol{f} = (f_1, \dots, f_q) \in F$ and $\boldsymbol{e} = (e_1, \dots, e_q) \in E$. Let $\boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e})$ denote the boundary terms, where $\mathcal{B}_\partial$ provides the restriction on $\partial\Omega$ of the effort variables $\boldsymbol{e}$ and of their spatial derivatives of proper order. The associated boundary space is $Z := \{\boldsymbol{z} |\ \boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e})\}$. Then, it holds

$$\int_{\partial\Omega} \widetilde{\mathcal{B}}_{\mathcal{J}}(\boldsymbol{e}_1, \boldsymbol{e}_2)\, \mathrm{d}S = \int_{\partial\Omega} \mathcal{B}_{\mathcal{J}}(\boldsymbol{z}_1, \boldsymbol{z}_2)\, \mathrm{d}S, \quad \text{with} \quad \widetilde{\mathcal{B}}_{\mathcal{J}}(\cdot,\cdot) = \mathcal{B}_{\mathcal{J}}(\mathcal{B}_\partial(\cdot), \mathcal{B}_\partial(\cdot)). \tag{3.18}$$

The following theorem characterizes Stokes-Dirac structures for pHs of arbitrary geometrical dimension and differential order.

Proposition 2 (Proposition 3.3 [MvdSM05])
Consider the space of power variables $B = F \times E \times Z$. The linear subspace $D \subset B$

$$D_{\mathcal{J}} = \{(\boldsymbol{f}, \boldsymbol{e}, \boldsymbol{z}) \in F \times E \times Z \mid \boldsymbol{f} = -\mathcal{J}\boldsymbol{e},\ \boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e})\}, \tag{3.19}$$

is a Stokes–Dirac structure on $\mathcal{B}$ with respect to the pairing

$$\left\langle\!\left\langle\, (\boldsymbol{f}^1, \boldsymbol{e}^1, \boldsymbol{z}^1), (\boldsymbol{f}^2, \boldsymbol{e}^2, \boldsymbol{z}^2) \,\right\rangle\!\right\rangle := \int_\Omega \left(\boldsymbol{e}^{1\top}\boldsymbol{f}^2 + \boldsymbol{e}^{2\top}\boldsymbol{f}^1\right) \mathrm{d}\Omega + \int_{\partial\Omega} \mathcal{B}_{\mathcal{J}}(\boldsymbol{z}^1, \boldsymbol{z}^2)\, \mathrm{d}S. \tag{3.20}$$

From this proposition, if $(\boldsymbol{f}, \boldsymbol{e}, \boldsymbol{z}) \in D_{\mathcal{J}}$, then $\langle\!\langle\, (\boldsymbol{f}, \boldsymbol{e}, \boldsymbol{z}), (\boldsymbol{f}, \boldsymbol{e}, \boldsymbol{z}) \,\rangle\!\rangle = 0$, that is

$$\int_\Omega \boldsymbol{e}^\top \boldsymbol{f}\, \mathrm{d}\Omega + \frac{1}{2}\int_{\partial\Omega} \mathcal{B}_{\mathcal{J}}(\boldsymbol{z}, \boldsymbol{z})\, \mathrm{d}S = 0. \tag{3.21}$$

This relation expresses the power conservation property of the Stokes–Dirac structure. It states the relation between the variation of internal energy (the integral on the domain Ω) with the power flowing through the boundary (the integral over $\partial\Omega$). Thanks to the power conservation property dpHs always dispose of an associated Stokes Dirac structure. This concept can be extended to consider dissipation or distributed forces. To this aim, it is necessary to include additional ports to account for the power exchange due to these effects (see Theorem 3.4 [MvdSM05]).

Remark 2
*The constant Stokes-Dirac structure has been defined in case of smooth vector-valued functions for simplicity. The definition is indeed more general and encompasses the case of more complex functional spaces, in particular the L^2 space of square integrable functions. Linear elasticity for example is defined on a mixed function space of vector- and tensor-valued functions. The constant differential operator may contain intrinsic operators (*Div, Grad*) as it will be shown in §3.3. The result presented here remains valid provided that the proper pairing is being chosen.*

3.2.2 Distributed port-Hamiltonian systems

A distributed conservative port-Hamiltonian system is defined by a set of variables that describes the unknowns, by a formally skew-adjoint differential operator, an energy functional and a set of boundary inputs and corresponding conjugated outputs. Such a system is described by the following set of equations

$$\begin{aligned} \frac{\partial \boldsymbol{\alpha}}{\partial t} &= \mathcal{J}\boldsymbol{e}, \\ \boldsymbol{u}_\partial &= \mathcal{B}\boldsymbol{e}, \\ \boldsymbol{y}_\partial &= \mathcal{C}\boldsymbol{e}, \\ \boldsymbol{e} &:= \frac{\delta H}{\delta \boldsymbol{\alpha}}. \end{aligned} \tag{3.22}$$

The unknowns $\boldsymbol{\alpha}$ are called energy variables in the port-Hamiltonian framework, the formally skew-adjoint operator $\mathcal{J}$ is named interconnection operator (see appendix A, Def. 6 for a precise definition of formal skew adjointness). $\mathcal{B}, \mathcal{C}$ are boundary operator, that provide the boundary input $\boldsymbol{u}_\partial$ and output $\boldsymbol{y}_\partial$. The variational derivative of the Hamiltonian define the so-called coenergy variables $\boldsymbol{e}$.

Remark 3
It will become clear in this section that the effort variable of the Stokes-Dirac structure are indeed equivalent to the co-energy variables of the pH system. This justifies using the same notation for both.

Definition 5 (Variational derivative, Def. 4.1 in [Olv93])
Consider a functional $H(\boldsymbol{\alpha})$

$$H(\boldsymbol{\alpha}) = \int_\Omega \mathcal{H}(\boldsymbol{\alpha}) \, \mathrm{d}\Omega.$$

Given a variation $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}} + \eta \boldsymbol{\delta\alpha}$ *the variational derivative* $\frac{\delta H}{\delta \boldsymbol{\alpha}}$ *is defined as*

$$H(\bar{\boldsymbol{\alpha}} + \eta \boldsymbol{\delta\alpha}) = H(\bar{\boldsymbol{\alpha}}) + \eta \int_\Omega \frac{\delta H}{\delta \boldsymbol{\alpha}} \cdot \boldsymbol{\delta\alpha} \, \mathrm{d}\Omega + O(\eta^2).$$

Remark 4
If the integrand does not contain derivative of the argument $\boldsymbol{\alpha}$ *then the variational derivative is equal to the partial derivative of the Hamiltonian density* $\mathcal{H}$

$$\frac{\delta H}{\delta \boldsymbol{\alpha}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\alpha}}.$$

Conservative port-Hamiltonian systems possess a peculiar property. The energy rate is

given by the power due to the boundary ports $\boldsymbol{u}_\partial, \boldsymbol{y}_\partial$

$$\begin{aligned} \dot{H} &= \int_\Omega \frac{\delta H}{\delta \boldsymbol{\alpha}} \cdot \frac{\partial \boldsymbol{\alpha}}{\partial t} \, \mathrm{d}\Omega = \langle \delta_{\boldsymbol{\alpha}} H, \partial_t \boldsymbol{\alpha} \rangle_\Omega \,, \qquad \text{Stokes theorem} \\ &= \int_{\partial\Omega} \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial \, \mathrm{d}S = \langle \boldsymbol{u}_\partial, \boldsymbol{y}_\partial \rangle_{\partial\Omega} \,, \end{aligned} \tag{3.23}$$

From the energy rate, the structural power balance is obtained

$$- \langle \delta_{\boldsymbol{\alpha}} H, \partial_t \boldsymbol{\alpha} \rangle_\Omega + \langle \boldsymbol{u}_\partial, \boldsymbol{y}_\partial \rangle_{\partial\Omega} = 0 \tag{3.24}$$

From (3.21), it is clear by identification that $\mathcal{B}_{\mathcal{J}}(\boldsymbol{z}, \boldsymbol{z}) = 2\, \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial$. This means that the boundary space can be split into boundary input and output

$$Z := \{ \boldsymbol{z} |\ \boldsymbol{z} = \mathcal{B}_\partial(\boldsymbol{e}) = (\boldsymbol{u}_\partial,\, \boldsymbol{y}_\partial) \}$$

If the flow, effort and boundary variables are chosen to be

$$\boldsymbol{f} := -\partial_t \boldsymbol{\alpha}, \quad \boldsymbol{e} := \delta_{\boldsymbol{\alpha}} H, \quad \boldsymbol{z} := (\boldsymbol{u}_\partial,\, \boldsymbol{y}_\partial), \tag{3.25}$$

then system (3.22) defines a Stokes Dirac structure by Proposition 2. In this rather informal treatment of dpHs, no rigorous characterization whatsoever has been introduced for operators $\mathcal{B}, \mathcal{C}$ in system (3.22). A formal characterization of these operators has been given in [LGZM05] for pH of generic order only in one geometrical dimensional. In the following examples it is shown that from the power balance appropriate boundary variables can be defined.

3.2.2.1 Wave equation

Given an in open bounded connected set $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, the vibrations of a drum can be described by the following model

$$\rho \partial_{tt} w(\boldsymbol{x}, t) = T \Delta w(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \Omega, \quad \Delta = \operatorname{div} \operatorname{grad}, \tag{3.26}$$

where the scalar fields ρ, T are the mass density and the drum tension respectively. The scalar field $w \in \mathbb{R}$ represents the drum vertical deflection . The Hamiltonian (total energy) reads

$$H = \frac{1}{2} \int_\Omega \left\{ \rho (\partial_t w)^2 + \| T \nabla w \|^2 \right\} \, \mathrm{d}\Omega.$$

To recast (3.26) in pH form the energy variables has to be introduced $\boldsymbol{\alpha} = [\alpha_p,\, \boldsymbol{\alpha}_v]^\top$

$$\alpha_p := \rho \partial_t w, \qquad \boldsymbol{\alpha}_v := \nabla w.$$

The Hamiltonian is now a quadratic function of the energy variable

$$H = \frac{1}{2}\int_\Omega \left\{ \frac{1}{\rho}\alpha_p^2 + \|T\boldsymbol{\alpha}_v\|^2 \right\} \mathrm{d}\Omega.$$

By definition, the co-energy are

$$e_p = \frac{\delta H}{\delta \alpha_p} = \frac{1}{\rho}\alpha_p, \qquad \boldsymbol{e}_v = \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = T\boldsymbol{\alpha}_v.$$

Equation (3.26) can be recast in port-Hamiltonian form

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_p \\ \boldsymbol{\alpha}_v\end{pmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & \mathbf{0}\end{bmatrix}\begin{pmatrix} e_p \\ \boldsymbol{e}_v\end{pmatrix}.$$

From the energy rate it is possible to identify the boundary variables.

$$\begin{aligned}
\dot{H} &= \int_\Omega \{e_p\, \partial_t \alpha_p + \boldsymbol{e}_v \cdot \partial_t \boldsymbol{\alpha}_v\}\ \mathrm{d}\Omega, \\
&= \int_\Omega \{e_p\ \mathrm{div}\, \boldsymbol{e}_v + \boldsymbol{e}_v \cdot \mathrm{grad}\, e_p\}\ \mathrm{d}\Omega, && \text{Chain rule,} \\
&= \int_\Omega \mathrm{div}(e_p\, \boldsymbol{e}_v)\ \mathrm{d}\Omega, && \text{Stokes theorem,} \\
&= \int_{\partial\Omega} e_p\, \boldsymbol{e}_v \cdot \boldsymbol{n}\ \mathrm{d}S = \langle e_p, \boldsymbol{e}_v \cdot \boldsymbol{n}\rangle_{\partial\Omega}\,.
\end{aligned}$$

The boundary term $\langle e_p, \boldsymbol{e}_v\rangle_{\partial\Omega}$ pairs two power variables. One is taken as control input, the other plays the role of power-conjugated output. The assignment of these roles to the boundary power variables is referred to as causality of the boundary port [Kot19, Chapter 2]. Under uniform causality assumption, either e_p or $\boldsymbol{e}_v$ can assume the role of (distributed) boundary input, but not both. This leads to two possible selection:

- First case $u_\partial = e_p, \quad y_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}$.
  This imposes the variable $e_p := \partial_t w$ as boundary input and corresponds to a classical Dirichlet condition.

- Second case $u_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}, \quad y_\partial = e_p$.
  This imposes the variable $\boldsymbol{e}_v \cdot \boldsymbol{n} := \partial_{\boldsymbol{n}} w$ as boundary input and corresponds to a classical Neumann condition.

3.2.2.2 2D shallow water equations

This formulation may be found in [CR16, Section 6.2.]. This model describes a thin fluid layer of constant density in hydrostatic balance, like the propagation of a tsunami wave far from shore. Consider an open bounded connected set $\Omega \subset \mathbb{R}^2$ and a constant bed profile. The mass conservation implies

$$\frac{\partial h}{\partial t} + \mathrm{div}(h\boldsymbol{v}) = 0,$$

where $h(x, y, t) \in \mathbb{R}$ is a scalar field representing the fluid height, $\boldsymbol{v}(x, y, t) \in \mathbb{R}^2$ is the fluid velocity field. The conservation of linear momentum reads

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla(\rho g h) = 0,$$

where ρ is the mass density and g the gravitational acceleration constant. Using the identity

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla(\|\boldsymbol{v}\|^2) + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v},$$

where $\nabla\times$ is the rotational of $\boldsymbol{v}$ (also denoted $\operatorname{curl} \boldsymbol{v}$), the momentum is rearranged as follows

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = -\nabla\left(\frac{1}{2}\rho \|\boldsymbol{v}\|^2 + \rho g h\right) - \rho(\nabla \times \boldsymbol{v}) \times \boldsymbol{v}.$$

The last term on the right side can be rewritten

$$\rho(\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = \begin{bmatrix} 0 & -\rho\omega \\ \rho\omega & 0 \end{bmatrix} \boldsymbol{v},$$

with $\omega = \partial_x v_y - \partial_y v_x$ the local vorticity term. To derive a suitable pH formulation the energy, made up of kinetic and potential contribution, has to be invoked

$$H = \frac{1}{2}\int_\Omega \left\{\rho h \|\boldsymbol{v}\|^2 + \rho g h^2\right\} \, \mathrm{d}\Omega.$$

As energy variable the fluid height and the linear momentum are chosen

$$\alpha_h = h, \qquad \boldsymbol{\alpha}_v = \rho \boldsymbol{v}.$$

The Hamiltonian is a non separable functional of the energy variables

$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2}\int_\Omega \frac{1}{\rho}\alpha_h \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h^2 \, \mathrm{d}\Omega.$$

The co-energy variables are given by

$$e_h := \frac{\delta H}{\delta \alpha_h} = \frac{1}{2\rho}\|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h, \qquad \boldsymbol{e}_v := \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \frac{1}{\rho}\alpha_h \boldsymbol{\alpha}_v.$$

The mass and momentum conservation are then rewritten as follows

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \mathcal{G} \end{bmatrix} \begin{pmatrix} e_h \\ \boldsymbol{e}_v \end{pmatrix},$$

The gyroscopic skew-symmetric term $\mathcal{G}$ introduces a non-linearity as it depends on the energy variables

$$\mathcal{G}(\alpha_h, \boldsymbol{\alpha}_v) = \frac{\omega}{\alpha_h}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \omega = \partial_x \alpha_{v,y} - \partial_y \alpha_{v,x}.$$

Despite the non-standard formulation, the energy rate provides anyway the boundary variables

$$\begin{aligned}
\dot{H} &= + \int_\Omega \{e_h\, \partial_t \alpha_h + \boldsymbol{e}_v \cdot \partial_t \boldsymbol{\alpha}_v\}\ \mathrm{d}\Omega, \\
&= - \int_\Omega \{e_h\ \mathrm{div}\, \boldsymbol{e}_v + \boldsymbol{e}_v \cdot (\mathrm{grad}\, e_h - \mathcal{G}\boldsymbol{e}_v)\}\ \mathrm{d}\Omega, && \text{skew-symmetry of } \mathcal{G}, \\
&= - \int_\Omega \{e_h\ \mathrm{div}\, \boldsymbol{e}_v + \boldsymbol{e}_v \cdot \mathrm{grad}\, e_h\}\ \mathrm{d}\Omega, && \text{Chain rule}, \\
&= - \int_\Omega \mathrm{div}(e_h\, \boldsymbol{e}_v)\ \mathrm{d}\Omega, && \text{Stokes theorem}, \\
&= - \int_{\partial\Omega} e_h\, \boldsymbol{e}_v \cdot \boldsymbol{n}\ \mathrm{d}S = - \langle e_h, \boldsymbol{e}_v \cdot \boldsymbol{n} \rangle_{\partial\Omega}\,.
\end{aligned}$$

Again two possible cases of homogeneous boundary causality arise:

- First case $u_\partial = e_h, \quad y_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}$.
  This imposes the variable $e_h := h$ as boundary input ans corresponds to a given water level for a fluid boundary.

- Second case $u_\partial = \boldsymbol{e}_v \cdot \boldsymbol{n}, \quad y_\partial = e_p$.
  This imposes the variable $\boldsymbol{e}_v \cdot \boldsymbol{n} := h\boldsymbol{v} \cdot \boldsymbol{n}$ as boundary input and corresponds to a given volumetric flow rate.

3.3 Port-Hamiltonian formulation of linear elasticity

In this section a port-Hamiltonian formulation for elasticity is deduced from the classical elastodynamics problem. It must be appointed that already in the seventies a purely hyperbolic formulation for elasticity was detailed [HM78]. The missing point is the clear connection with the theory of Hamiltonian PDEs. An Hamiltonian formulation can be found in [Gri15, Chapter 16], but without any connection to the concept of Stokes-Dirac structure induced by the underlying geometry.

3.3.1 Energy and co-energy variables

Consider an open connected set $\Omega \subset \mathbb{R}^d,\ d = (2, 3)$. The displacement within a deformable continuum are given by Eq. (3.7).

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{Div}(\boldsymbol{\mathcal{D}}\, \mathrm{Grad}\, \boldsymbol{u}) = 0, \qquad \boldsymbol{x} \in \Omega. \tag{3.27}$$

413 The contribution of the body force $\boldsymbol{f}$ has been removed for ease of presentation. To derive a
414 pH formulation, the total energy, that includes the kinetic and deformation energy, is needed

$$H = \frac{1}{2}\int_\Omega \left\{ \rho \left\| \partial_t \boldsymbol{u} \right\|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} \, \mathrm{d}\Omega. \tag{3.28}$$

The notation $\boldsymbol{A} : \boldsymbol{B} = \mathrm{Tr}(\boldsymbol{A}^\top \boldsymbol{B}) = A_{ij} B_{ij}$ denotes the tensor contraction. Recall that $\boldsymbol{\varepsilon} = \mathrm{Grad}\, \boldsymbol{u}$ and $\boldsymbol{\Sigma} = \boldsymbol{\mathcal{D}}\boldsymbol{\varepsilon}$. The energy variables are then the linear momentum and the deformation field

$$\boldsymbol{\alpha}_v = \rho \boldsymbol{v}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon},$$

415 where $\boldsymbol{v} := \partial_t \boldsymbol{u}$. The Hamiltonian can be rewritten as a quadratic functional in the energy
416 variables

$$H = \frac{1}{2}\int_\Omega \left\{ \frac{1}{\rho}\boldsymbol{\alpha}_v^2 + (\boldsymbol{\mathcal{D}}\boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon \right\} \, \mathrm{d}\Omega. \tag{3.29}$$

417 The co-energy variables are given by

$$\boldsymbol{e}_v := \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \boldsymbol{v}, \qquad \boldsymbol{E}_\varepsilon := \frac{\delta H}{\delta \boldsymbol{A}_\varepsilon} = \boldsymbol{\Sigma}. \tag{3.30}$$

418 The tensor-valued co-energy $\boldsymbol{E}_\varepsilon$ is obtained by taking the variational derivative with
419 respect to a tensor.

420 **Proposition 3**
421 *The variational derivative of the Hamiltonian with respect to the strain tensor is the stress*
422 *tensor* $\delta_{\boldsymbol{A}_\varepsilon} H = \boldsymbol{\Sigma}$.

Proof. The contribution due to the deformation part in Hamiltonian is given by:

$$H_{\mathrm{def}}(\boldsymbol{A}_\varepsilon) = \frac{1}{2}\int_\Omega (\boldsymbol{\mathcal{D}}\boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon \, \mathrm{d}\Omega.$$

A variation $\boldsymbol{\Delta A}_\varepsilon$ of the strain tensor with respect to a given value $\bar{\boldsymbol{A}}_\varepsilon$ leads to:

$$\begin{aligned} H_{\mathrm{def}}(\bar{\boldsymbol{A}}_\varepsilon + \eta \boldsymbol{\Delta A}_\varepsilon) = &+ \frac{1}{2}\int_\Omega (\boldsymbol{\mathcal{D}}\bar{\boldsymbol{A}}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon \, \mathrm{d}\Omega \\ &+ \eta \frac{1}{2}\int_\Omega \left\{ (\boldsymbol{\mathcal{D}}\bar{\boldsymbol{A}}_\varepsilon) : \boldsymbol{\Delta A}_\varepsilon + (\boldsymbol{\mathcal{D}}\boldsymbol{\Delta A}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon \right\} \, \mathrm{d}\Omega + O(\eta^2). \end{aligned}$$

The term $(\boldsymbol{\mathcal{D}}\boldsymbol{\Delta A}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon$ can be further rearranged using the symmetry of $\boldsymbol{\mathcal{D}}$ and the commutativity of the tensor contraction

$$(\boldsymbol{\mathcal{D}}\boldsymbol{\Delta A}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon = (\boldsymbol{\mathcal{D}}\bar{\boldsymbol{A}}_\varepsilon) : \boldsymbol{\Delta A}_\varepsilon,$$

so that

$$H_{\mathrm{def}}(\bar{\boldsymbol{A}}_\varepsilon + \eta \boldsymbol{\Delta A}_\varepsilon) = \frac{1}{2}\int_\Omega (\boldsymbol{\mathcal{D}}\bar{\boldsymbol{A}}_\varepsilon) : \bar{\boldsymbol{A}}_\varepsilon \, \mathrm{d}\Omega + \eta \int_\Omega (\boldsymbol{\mathcal{D}}\bar{\boldsymbol{A}}_\varepsilon) : \boldsymbol{\Delta A}_\varepsilon \, \mathrm{d}\Omega + O(\eta^2).$$

By definition of variational derivative it can be written:

$$H_{\text{def}}(\bar{\boldsymbol{A}}_\varepsilon + \eta\boldsymbol{\Delta A}_\varepsilon) = H_{\text{def}}(\bar{\boldsymbol{A}}_\varepsilon) + \eta\left\langle \frac{\delta H}{\delta \boldsymbol{A}_\varepsilon}, \boldsymbol{\Delta A}_\varepsilon \right\rangle_{L^2(\Omega,\mathbb{S})} + O(\eta^2),$$

where $L^2(\Omega,\mathbb{S})$ is the space of the square integrable $\mathbb{R}^{d\times d}$ symmetric tensors endowed with the tensor contraction as inner product

$$\langle \boldsymbol{A}, \boldsymbol{B}\rangle_{L^2(\Omega,\mathbb{S})} = \int_\Omega \boldsymbol{A} : \boldsymbol{B}\ \mathrm{d}\Omega. \tag{3.31}$$

Then, by identification

$$\frac{\delta H_{\text{def}}}{\delta \boldsymbol{A}_\varepsilon} = \boldsymbol{\mathcal{D}}\bar{\boldsymbol{A}}_\varepsilon = \boldsymbol{\Sigma}.$$

Since the Hamiltonian is separable then $\delta_{\boldsymbol{A}_\varepsilon} H_{\text{def}} = \delta_{\boldsymbol{A}_\varepsilon} H$, leading to the final result. □

3.3.2 Final system and associated Stokes-Dirac structure

It is now possible to state the final pH form

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon\end{pmatrix} = \begin{bmatrix}\boldsymbol{0} & \mathrm{Div} \\ \mathrm{Grad} & \boldsymbol{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon\end{pmatrix}. \tag{3.32}$$

The first equation of the system is the conservation of linear momentum. The second represents a compatibility condition

$$\begin{aligned}\partial_t \boldsymbol{A}_\varepsilon &= \mathrm{Grad}(\boldsymbol{e}_v),\\ \partial_t \boldsymbol{\varepsilon} &= \mathrm{Grad}(\boldsymbol{v}),\\ \partial_t \,\mathrm{Grad}\,\boldsymbol{u} &= \mathrm{Grad}(\partial_t \boldsymbol{u}).\end{aligned} \tag{3.33}$$

Assuming that $\boldsymbol{u} \in C^2$, higher order derivatives commute (Schwarz theorem). Hence, the equation is verified. The following theorem ensures the differential operator is formally skew-adjoint (one can also find this result in [PZ20, Lemma 3.3]).

Theorem 3
The formal adjoint of the tensor divergence Div *is* −Grad*, the opposite of the symmetric gradient.*

Proof. The space of symmetric tensor field in $\mathbb{R}^{d\times d}$ is denoted by $\mathbb{S}$, whereas the space of vector field in $\mathbb{R}^d$ is denoted by $\mathbb{V}$. Let us consider the Hilbert space of the square integrable symmetric tensors $L^2(\Omega,\mathbb{S})$ with scalar product is defined in (3.31). Moreover consider the Hilbert space of the square integrable vector function $L^2(\Omega,\mathbb{V})$, endowed with the usual scalar product:

$$\langle \boldsymbol{a}, \boldsymbol{b}\rangle_{L^2(\Omega,\mathbb{V})} = \int_\Omega \boldsymbol{a}\cdot\boldsymbol{b}\ \ \mathrm{d}\Omega = \int_\Omega \boldsymbol{a}^\top\boldsymbol{b}\ \ \mathrm{d}\Omega, \quad \forall \boldsymbol{a},\boldsymbol{b} \in L^2(\Omega,\mathbb{V}).$$

Let us consider the tensor divergence operator defined as:

$$\begin{aligned} \mathrm{Div}:\ L^2(\Omega,\mathbb{S}) &\to L^2(\Omega,\mathbb{V}),\\ \boldsymbol{\Psi} &\to \mathrm{Div}\,\boldsymbol{\Psi}=\boldsymbol{\psi}, \end{aligned} \qquad \text{with } \psi_j=\mathrm{div}(\Psi_{ij})=\sum_{i=1}^{d}\frac{\partial\Psi_{ij}}{\partial x_i}.$$

We try to identify Div^*

$$\begin{aligned} \mathrm{Div}^*:\ L^2(\Omega,\mathbb{V}) &\to L^2(\Omega,\mathbb{S}),\\ \boldsymbol{\phi} &\to \mathrm{Div}^*\boldsymbol{\phi}=\boldsymbol{\Phi}, \end{aligned}$$

such that

$$\langle \mathrm{Div}\,\boldsymbol{\Psi},\boldsymbol{\phi}\rangle_{L^2(\Omega,\mathbb{V})}=\langle \boldsymbol{\Psi},\mathrm{Div}^*\boldsymbol{\phi}\rangle_{L^2(\Omega,\mathbb{S})}\,, \qquad \begin{aligned}&\forall\boldsymbol{\Psi}\in\mathrm{Domain}(\mathrm{Div})\subset L^2(\Omega,\mathbb{S})\\ &\forall\boldsymbol{\phi}\in\mathrm{Domain}(\mathrm{Div}^*)\subset L^2(\Omega,\mathbb{V})\end{aligned}$$

Now let us take $\boldsymbol{\Psi}\in C_0^1(\Omega,\mathbb{S})\subset\mathrm{Domain}(\mathrm{Div})$ the space of differentiable symmetric tensors with compact support in Ω. Additionally $\boldsymbol{\phi}$ will belong to $C_0^1(\Omega,\mathbb{V})\subset\mathrm{Domain}(\mathrm{Div}^*)$, the space of differentiable vector functions with compact support in Ω. Then

$$\begin{aligned} \langle \mathrm{Div}\,\boldsymbol{\Psi},\boldsymbol{\phi}\rangle_{L^2(\Omega,\mathbb{V})} &= \int_\Omega \boldsymbol{\psi}\cdot\boldsymbol{\phi}\ \mathrm{d}\Omega,\\ &= \int_\Omega \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial\Psi_{ij}}{\partial x_i}\phi_j\ \mathrm{d}\Omega,\\ &= -\int_\Omega \sum_{i=1}^{d}\sum_{j=1}^{d}\Psi_{ij}\frac{\partial\phi_j}{\partial x_i}\ \mathrm{d}\Omega\,, && \text{since the functions vanish at the boundary,}\\ &= -\int_\Omega \sum_{i=1}^{d}\sum_{j=1}^{d}\Psi_{ij}F_{ij}\ \mathrm{d}\Omega, && \text{where}\quad F_{ij}=\frac{\partial\phi_j}{\partial x_i},\\ &= -\langle\boldsymbol{\Psi},\boldsymbol{F}\rangle_{L^2(\Omega,\mathbb{S})}\,, && \boldsymbol{F}=(\nabla\boldsymbol{\phi})^\top. \end{aligned}$$

But in this latter case, it could not be stated that $\boldsymbol{F}\in L^2(\Omega,\mathbb{S})$. Now, since $\boldsymbol{\Psi}\in L^2(\Omega,\mathbb{S})$, $\Psi_{ji}=\Psi_{ij}$, thus we are allowed to further decompose the last equality as

$$\sum_{i,j}\Psi_{ij}\frac{\partial\phi_j}{\partial x_i}=\sum_{i,j}\Psi_{ij}\frac{1}{2}\left(\frac{\partial\phi_i}{\partial x_j}+\frac{\partial\phi_j}{\partial x_i}\right)=\sum_{i,j}\Psi_{ij}\Phi_{ij}, \qquad \text{with } \Phi_{ij}:=\frac{1}{2}\left(\frac{\partial\phi_i}{\partial x_j}+\frac{\partial\phi_j}{\partial x_i}\right).$$

Thus $\boldsymbol{\Phi}=\mathrm{Grad}\,\boldsymbol{\phi}\in L^2(\Omega,\mathbb{S})$ and it can be stated that:

$$\begin{aligned} \langle \mathrm{Div}\,\boldsymbol{\Psi},\boldsymbol{\phi}\rangle_{L^2(\Omega,\mathbb{V})} &= -\int_\Omega \sum_{i,j}\Psi_{ij}\frac{1}{2}\left(\frac{\partial\phi_i}{\partial x_j}+\frac{\partial\phi_j}{\partial x_i}\right)\ \mathrm{d}\Omega\\ &= -\int_\Omega \sum_{i,j}\Psi_{ij}\Phi_{ij}\ \mathrm{d}\Omega=\langle\boldsymbol{\Psi},-\mathrm{Grad}\,\boldsymbol{\phi}\rangle_{L^2(\Omega,\mathbb{S})}\,. \end{aligned}$$

It can be concluded that the formal adjoint of Div is $\mathrm{Div}^*=-\mathrm{Grad}$. □

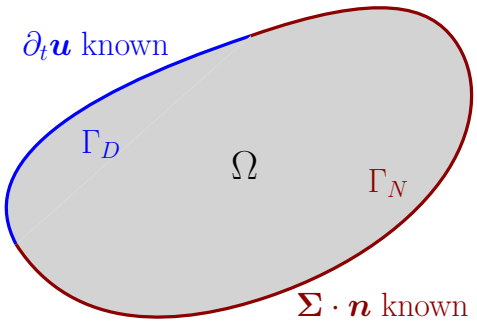

Figure 3.1: A 2D continuum with Neumann and Dirichlet boundary conditions

439 The boundary values are then found by evaluating the energy rate

$$\begin{aligned}
\dot{H} &= \int_\Omega \{\boldsymbol{e}_v \cdot \partial_t \boldsymbol{\alpha}_v + \boldsymbol{E}_\varepsilon : \partial_t \boldsymbol{A}_\varepsilon\} \, \mathrm{d}\Omega, \\
&= \int_\Omega \{\boldsymbol{e}_v \cdot \operatorname{Div} \boldsymbol{E}_\varepsilon + \boldsymbol{E}_\varepsilon : \operatorname{Grad} \boldsymbol{e}_v\} \, \mathrm{d}\Omega, && \text{Chain rule,} \\
&= \int_\Omega \operatorname{div}(\boldsymbol{E}_\varepsilon \cdot \boldsymbol{e}_v) \, \mathrm{d}\Omega, && \text{Stokes theorem (see [BBF+13, Chapter 1]),} \\
&= \int_{\partial\Omega} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n}) \, \mathrm{d}S = \langle \boldsymbol{e}_v, \boldsymbol{E}_\varepsilon \cdot \boldsymbol{n} \rangle_{\partial\Omega} .
\end{aligned} \tag{3.34}$$

440 The imposition of the velocity field along the boundary $\boldsymbol{e}_v = \partial_t \boldsymbol{u}$ corresponds to a Dirichlet
441 condition. Setting $\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n} = \boldsymbol{\Sigma} \cdot \boldsymbol{n} = \boldsymbol{t}$ corresponds to a Neumann condition. Consider a
442 partition of the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ and $\Gamma_N \cap \Gamma_D = \{\emptyset\}$, where a Dirichlet and a
443 Neumann condition applies on the subset Γ_D and Γ_N respectively (see Fig. 3.1). Then the
444 final pH formulation reads

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon \end{pmatrix} &= \underbrace{\begin{bmatrix} \boldsymbol{0} & \operatorname{Div} \\ \operatorname{Grad} & \boldsymbol{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \end{pmatrix}, \\
\boldsymbol{u}_\partial &= \underbrace{\begin{bmatrix} \boldsymbol{\gamma}_0^{\Gamma_D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\gamma}_n^{\Gamma_N} \end{bmatrix}}_{\mathcal{B}} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \end{pmatrix}, \\
\boldsymbol{y}_\partial &= \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{\gamma}_n^{\Gamma_D} \\ \boldsymbol{\gamma}_0^{\Gamma_N} & \boldsymbol{0} \end{bmatrix}}_{\mathcal{C}} \begin{pmatrix} \boldsymbol{e}_v \\ \boldsymbol{E}_\varepsilon \end{pmatrix},
\end{aligned} \tag{3.35}$$

445 where $\boldsymbol{\gamma}_0^{\Gamma_*}$ denotes the trace over the set Γ_*, namely $\boldsymbol{\gamma}_0^{\Gamma_*} \boldsymbol{e}_v = \boldsymbol{e}_v|_{\Gamma_*}$. Furthermore, $\boldsymbol{\gamma}_n^{\Gamma_*}$ denotes
446 the normal trace over the set Γ_*, namely $\boldsymbol{\gamma}_n^{\Gamma_*} \boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon \cdot \boldsymbol{n}|_{\Gamma_*}$.

Theorem 4 (Stokes-Dirac structure for elastodynamics)
Let $H^{\mathrm{Grad}}(\Omega, \mathbb{V})$ *the space of vectors with symmetric gradient in* $L^2(\Omega, \mathbb{S})$ *and* $H^{\mathrm{Div}}(\Omega, \mathbb{S})$

denote the space of symmetric tensors with divergence in $L^2(\Omega, \mathbb{V})$. *Consider the following definitions*

$$\begin{aligned} H &:= H^{\mathrm{Grad}}(\Omega, \mathbb{V}) \times H^{\mathrm{Div}}(\Omega, \mathbb{S}), \\ F &:= L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{S}), \\ F_\partial &:= L^2(\Gamma_D, \mathbb{V}) \times L^2(\Gamma_N, \mathbb{V}). \end{aligned}$$

447 *The set*

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{f}_\partial \\ \boldsymbol{e} \\ \boldsymbol{e}_\partial \end{pmatrix} \mid \boldsymbol{e} \in H,\ \boldsymbol{f} = -\mathcal{J}\boldsymbol{e},\ \boldsymbol{f}_\partial = \mathcal{B}\boldsymbol{e},\ \boldsymbol{e}_\partial = \mathcal{C}\boldsymbol{e} \right\}, \tag{3.36}$$

448 *where* $\boldsymbol{e} = (\boldsymbol{e}_v,\ \boldsymbol{E}_\varepsilon)$ *and* $\mathcal{J}, \mathcal{B}, \mathcal{C}$ *are defined in* (3.35), *is a Stokes–Dirac structure with respect*
449 *to the pairing*

$$\left\langle\!\left\langle (\boldsymbol{f}^1, \boldsymbol{f}^1_\partial, \boldsymbol{e}^1, \boldsymbol{e}^1_\partial), (\boldsymbol{f}^2, \boldsymbol{f}^2_\partial, \boldsymbol{e}^2, \boldsymbol{e}^2_\partial) \right\rangle\!\right\rangle := \left\langle \boldsymbol{e}^1, \boldsymbol{f}^2 \right\rangle_F + \left\langle \boldsymbol{e}^2, \boldsymbol{f}^1 \right\rangle_F + \left\langle \boldsymbol{e}^1_\partial, \boldsymbol{f}^2_\partial \right\rangle_{F_\partial} + \left\langle \boldsymbol{e}^2_\partial, \boldsymbol{f}^1_\partial \right\rangle_{F_\partial}, \tag{3.37}$$

where

$$\langle (\boldsymbol{a},\ \boldsymbol{b}), (\boldsymbol{c},\ \boldsymbol{d}) \rangle_{F_\partial} = \int_{\Gamma_D} \boldsymbol{a} \cdot \boldsymbol{c}\, \mathrm{d}S + \int_{\Gamma_N} \boldsymbol{b} \cdot \boldsymbol{d}\, \mathrm{d}S, \quad \boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c},\ \boldsymbol{d} \in \mathbb{V}.$$

450 *Proof.* A Stokes-Dirac is characterized by the fact that $D_{\mathcal{J}} = D_{\mathcal{J}}^\perp$. Then one has to show
451 that $D_{\mathcal{J}} \subset D_{\mathcal{J}}^\perp$ and $D_{\mathcal{J}}^\perp \subset D_{\mathcal{J}}$. The proof is found by employing the integration by parts
452 formula already used for (3.34). The main steps of Theorem 3.6 in [LGZM05] are followed here.
453

Step 1. To show that $D_{\mathcal{J}} \subset D_{\mathcal{J}}^\perp$, take $(\boldsymbol{f},\ \boldsymbol{f}_\partial,\ \boldsymbol{e},\ \boldsymbol{e}_\partial) \in D_{\mathcal{J}}$. Then

$$\begin{aligned} \langle\!\langle\, (\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial), (\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial) \,\rangle\!\rangle &= 2 \langle \boldsymbol{e}, \boldsymbol{f} \rangle_F + 2 \langle \boldsymbol{e}_\partial, \boldsymbol{f}_\partial \rangle_{F_\partial}, \\ &= 2 \langle \boldsymbol{e}, -\mathcal{J}\boldsymbol{e} \rangle_F + 2 \langle \boldsymbol{e}_\partial, \boldsymbol{f}_\partial \rangle_{F_\partial}, \\ &= -2 \int_\Omega \{ \boldsymbol{e}_v \cdot \operatorname{Div} \boldsymbol{E}_\varepsilon + \boldsymbol{E}_\varepsilon : \operatorname{Grad} \boldsymbol{e}_v \}\, \mathrm{d}\Omega \\ &\quad + 2 \int_{\Gamma_D} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n})\, \mathrm{d}S + 2 \int_{\Gamma_N} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n})\, \mathrm{d}S, \\ &= -2 \int_\Omega \{ \boldsymbol{e}_v \cdot \operatorname{Div} \boldsymbol{E}_\varepsilon + \boldsymbol{E}_\varepsilon : \operatorname{Grad} \boldsymbol{e}_v \}\, \mathrm{d}\Omega \\ &\quad + 2 \int_{\partial\Omega} \boldsymbol{e}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n})\, \mathrm{d}S, = 0, \quad \text{from (3.34)}. \end{aligned}$$

454 This implies $D_{\mathcal{J}} \subset D_{\mathcal{J}}^\perp$.

Step 2. Take $(\boldsymbol{\phi},\ \boldsymbol{\phi}_\partial,\ \boldsymbol{\epsilon},\ \boldsymbol{\epsilon}_\partial) \in D_{\mathcal{J}}^\perp$ and $\boldsymbol{e}_0 \in H$ with compact support on Ω. This implies $\mathcal{B}\boldsymbol{e}_0 = (\boldsymbol{0},\ \boldsymbol{0})$ and $\mathcal{C}\boldsymbol{e}_0 = (\boldsymbol{0},\ \boldsymbol{0})$. Taking $(-\mathcal{J}\boldsymbol{e}_0, \boldsymbol{0}, \boldsymbol{e}_0, \boldsymbol{0}) \in D_{\mathcal{J}}$ then

$$\langle\!\langle\, (\boldsymbol{\phi}, \boldsymbol{\phi}_\partial, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_\partial), (\mathcal{J}\boldsymbol{e}_0, \boldsymbol{0}, \boldsymbol{e}_0, \boldsymbol{0}) \,\rangle\!\rangle = \langle \boldsymbol{\epsilon}, -\mathcal{J}\boldsymbol{e}_0 \rangle_F + \langle \boldsymbol{e}_0, \boldsymbol{\phi} \rangle_F = 0, \quad \forall \boldsymbol{e}_0 \in H.$$

It follows that $\boldsymbol{\epsilon} \in H$ and $\boldsymbol{\phi} = -\mathcal{J}\boldsymbol{\epsilon}$.

Step 3. Take $(\boldsymbol{\phi}, \boldsymbol{\phi}_\partial, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_\partial) \in D_{\mathcal{J}}^{\perp}$ and $(\boldsymbol{f}, \boldsymbol{f}_\partial, \boldsymbol{e}, \boldsymbol{e}_\partial) \in D_{\mathcal{J}}$. Variables $\boldsymbol{e}, \boldsymbol{\epsilon}$ are indeed tuples containing a vector and a tensor, namely $\boldsymbol{e} = (\boldsymbol{e}_v, \boldsymbol{E}_\varepsilon)$, $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_v, \boldsymbol{\mathcal{E}}_\varepsilon)$. From step 2 and (3.37)

$$\begin{aligned} 0 &= -\langle \boldsymbol{e}, \mathcal{J}\boldsymbol{\epsilon}\rangle_F - \langle \mathcal{J}\boldsymbol{e}, \boldsymbol{\epsilon}\rangle_F + \langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial\rangle_{F_\partial} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial\rangle_{F_\partial}, \\ &= -\int_{\partial\Omega} \{\boldsymbol{e}_v \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n})\} \, \mathrm{d}S + \langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial\rangle_{F_\partial} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial\rangle_{F_\partial} \end{aligned}$$

Consider the splitting of the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$

$$\begin{aligned} \int_{\partial\Omega} \{\boldsymbol{e}_v \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot (\boldsymbol{E}_\varepsilon \cdot \boldsymbol{n})\} \, \mathrm{d}S = &+ \int_{\Gamma_N} \{\boldsymbol{e}_{\partial,2} \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot \boldsymbol{f}_{\partial,2}\} \, \mathrm{d}S, \\ &+ \int_{\Gamma_D} \{\boldsymbol{f}_{\partial,1} \cdot (\boldsymbol{\mathcal{E}}_\varepsilon \cdot \boldsymbol{n}) + \boldsymbol{\epsilon}_v \cdot \boldsymbol{e}_{\partial,1}\} \, \mathrm{d}S, \end{aligned}$$

where the elements of the vectors $\boldsymbol{f}_\partial = (\boldsymbol{f}_{\partial,1}, \boldsymbol{f}_{\partial,2})$, $\boldsymbol{e}_\partial = (\boldsymbol{e}_{\partial,1}, \boldsymbol{e}_{\partial,2})$ have been considering. By expanding of the terms $\langle \boldsymbol{e}_\partial, \boldsymbol{\phi}_\partial\rangle_{F_\partial} + \langle \boldsymbol{\epsilon}_\partial, \boldsymbol{f}_\partial\rangle_{F_\partial}$ and given the fact that $\boldsymbol{e}_\partial$, $\boldsymbol{f}_\partial$ have arbitrary values then

$$\boldsymbol{\phi}_\partial = \begin{bmatrix} \boldsymbol{\gamma}_0^{\Gamma_D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\gamma}_n^{\Gamma_N} \end{bmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_v \\ \boldsymbol{\mathcal{E}}_\varepsilon \end{pmatrix}, \qquad \boldsymbol{\epsilon}_\partial = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{\gamma}_n^{\Gamma_D} \\ \boldsymbol{\gamma}_0^{\Gamma_N} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_v \\ \boldsymbol{\mathcal{E}}_\varepsilon \end{pmatrix},$$

meaning that $D_{\mathcal{J}}^{\perp} \subset D_{\mathcal{J}}$. This concludes the proof. □

Linear elasticity falls within the assumption of [Skr19]. Therefore, it is a well posed boundary control pH system. A question that naturally arises is how to reformulate this system using the language of differential geometry. This is possible through the usage of vector-valued differential forms. The interested reader may consult [Bre08].

3.4 Conclusion

In this chapter, the pH formulation of elasticity have been obtained. This model represents a generalization of the wave equation to higher dimensional variables. This leads to the introduction of symmetric tensorial quantities describing the state of stress and deformation within the body.
For a plane continuum with moderate thickness, it is possible to reduce the general three-dimensional mode to two uncoupled system: one representing the in plane behavior ruled by 2D elasticity and one representing the out-of-plane deflection. This will be the object of the next chapter dedicated to the study of a pH formulation of plate bending. It is important to remember that plate models are just particular cases of three-dimensional elasticity.

Chapter 4

Port-Hamiltonian plate theory

You get tragedy where the tree, instead of bending, breaks.

Culture and Value
Ludwig Wittgenstein

Contents

Plates are plane structural element with a small thickness compared to the planar dimension. Thanks to this feature, it is not necessary to model plate structures using three-dimensional elasticity. Dimensional reduction strategies are employed to describe plate structures as two-dimensional problems. These strategies rely on a educated guess of the displacement field. For beams ans plates the displacement field is expressed in terms of unknown functions $\phi_i^j(x, y, t)$ that solely depends on the midplane coordinates (x, y)

$$u_i(x, y, z, t) = \sum_{j=0}^{m} (z)^j \phi_i^j(x, y, t).$$

where u_i, $i = \{x, y, z\}$ are the component of the displacement field. A first-order approximation is commonly used, meaning that a linear dependence on z is considered. Two main models arise from such a framework:

- the Mindlin-Reissner model for thick plates;

- the Kirchhoff-Love model for thin plates.

In this chapter it its shown how to formulate first-order plate models as pHs.

4.1 First order plate theory

As previously stated, first order theories assume a linear dependence on the vertical coordinate (cf. [Red06])

$$u_i(x,y,z,t) = \phi_i^0(x,y,t) + z\phi_i^1(x,y,t).$$

This hypothesis implies that the fibers, i.e. segments perpendicular to the mid-plane before deformation, remain straight after deformation. Additionally, for plate with moderate thickness the fibers are considered inextensible, meaning that $\phi_z^1 = 0$. These assumption lead to the following displacement field

$$\begin{aligned} u_x(x,y,z,t) &= u_x^0(x,y,t) - z\theta_x(x,y,t), \\ u_y(x,y,z,t) &= u_y^0(x,y,t) - z\theta_y(x,y,t), \\ u_z(x,y,z,t) &= u_z^0(x,y,t), \end{aligned} \tag{4.1}$$

where $u_i(x,y,t) = \phi_i^0(x,y,t)$, $\theta_i(x,y,t) = -\phi_i^1(x,y,t)$. Assuming a linear elastic behavior, the 3D strain tensor for such a displacement field takes the form

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right) - z\frac{1}{2}\left(\partial_\beta \theta_\alpha + \partial_\alpha \theta_\beta\right) = \varepsilon_{\alpha\beta}^0 - z\kappa_{\alpha\beta}, \tag{4.2}$$

$$\varepsilon_{\alpha z} = \frac{1}{2}\left(\partial_a u_z - \theta_\alpha\right) = \frac{1}{2}\gamma_\alpha, \tag{4.3}$$

where $\alpha = \{x,y\}$, $\beta = \{x,y\}$. The tensors $\boldsymbol{\varepsilon}^0$, $\boldsymbol{\kappa}$, $\boldsymbol{\gamma}$ are called membrane, bending (or curvature) and shear strain tensor

$$\boldsymbol{\varepsilon}^0 = \operatorname{Grad} \boldsymbol{u}^0, \tag{4.4}$$

$$\boldsymbol{\kappa} = \operatorname{Grad} \boldsymbol{\theta}, \tag{4.5}$$

$$\boldsymbol{\gamma} = \operatorname{grad} u_z - \boldsymbol{\theta}. \tag{4.6}$$

where $\boldsymbol{u}^0 = (u_x, u_y)^\top$, $\boldsymbol{\theta} = (\theta_x, \theta_y)^\top$. For now, it is assumed that the material is isotropic, linear elastic (in Section §4.3 this hypothesis is removed). Recall the Hooke's law for 3D continua (see Eq. (3.5))

$$\boldsymbol{\Sigma} = \frac{E}{1+\nu}\left[\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\operatorname{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I}_{3\times 3}\right].$$

where E, ν are the Young modulus and Poisson ratio. The hypothesis of inextensible fibers implies $\varepsilon_{zz} = 0$. However, imposing a plane strain condition provides a model that is too stiff. Rather than a plain strain assumption, a plain stress hypothesis is used to derive the constitutive law for plates. The displacement field (4.1) is left unchanged, but, instead of ε_{zz},

Σ_{zz} is set to zero. If $\Sigma_{zz} = 0$, one gets

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).$$

Consequently, it is computed

$$\mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{1-2\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).$$

The constitutive law for the in-plane stress takes the form

$$\boldsymbol{\Sigma}_{2D} = \boldsymbol{\mathcal{D}}_{2D}\, \boldsymbol{\varepsilon}_{2D},$$

where $\boldsymbol{\Sigma}_{2D} = \Sigma_{\alpha\beta}$, $\boldsymbol{\varepsilon}_{2D} = \varepsilon_{\alpha\beta}$ and

$$\boldsymbol{\mathcal{D}}_{2D} = \frac{E}{1-\nu^2}\left[(1-\nu)(\cdot) + \nu\,\mathrm{Tr}(\cdot)\boldsymbol{I}_{2\times 2}\right]. \tag{4.7}$$

Concerning the shear deformation, the constitutive law reduces to

$$\boldsymbol{\sigma}_s = G\boldsymbol{\gamma}, \tag{4.8}$$

where $\boldsymbol{\sigma}_\alpha := \boldsymbol{\Sigma}_{\alpha,3}$ and $G = \frac{E}{2(1+\nu)}$ is the shear modulus. In the following sections the most common plate models will be presented and discussed.

4.1.1 Mindlin-Reissner model

The Mindlin-Reissner model [Rei47, Min51] represents a first-order shear deformation theory for describing the bending of plate. The in-plane midplane displacement are zero $\boldsymbol{u}^0(x,y) = \boldsymbol{0}$ for an isotropic plate that experiences only bending. Hence, the displacement field reduces to

$$\begin{aligned} u_x(x,y,z) &= -z\partial_x\theta_x, \\ u_y(x,y,z) &= -z\partial_y\theta_y, \\ u_z(x,y,z) &= u_z^0(x,y). \end{aligned} \tag{4.9}$$

In pure bending, the strain tensor is given by

$$\boldsymbol{\varepsilon}_b := \boldsymbol{\varepsilon}_{2D}(\boldsymbol{u}^0 = \boldsymbol{0}) = -z\boldsymbol{\kappa},$$

with $\boldsymbol{\kappa}$ given by (4.5). Consequently, the stress tensor reads

$$\boldsymbol{\Sigma}_b := \boldsymbol{\Sigma}_{2D}(\boldsymbol{u}^0 = \boldsymbol{0}) = -z\boldsymbol{\mathcal{D}}_{2D}\boldsymbol{\kappa},$$

where $\boldsymbol{\mathcal{D}}_{2D}$ is defined in Eq. (4.7).

Consider an undeformed middle plane of the plate is denoted by Ω. The total domain of

the plate is the product $\Omega \times (-h/2, h/2)$, where h is the constant thickness. To effectively reduce the problem from three- to two-dimensional, the stresses have to be integrated along the fibers. Since the stress varies linearly across the thickness, one has to multiply the stress by z before the integration to get a non null contribution. Such a quantity is called bending momenta tensor and is given by

$$\boldsymbol{M} := -\int_{-h/2}^{h/2} z\boldsymbol{\Sigma}_b \, \mathrm{d}z = \boldsymbol{\mathcal{D}}_b \, \boldsymbol{\kappa}, \tag{4.10}$$

where

$$\boldsymbol{\mathcal{D}}_b = D_b \left[(1-\nu)(\cdot) + \nu \operatorname{Tr}(\cdot)\boldsymbol{I}_{2\times 2}\right], \qquad \text{where} \qquad D_b = \frac{Eh^3}{12(1-\nu^2)}. \tag{4.11}$$

The shear stress has to be integrated along the fibers as well. Given the excessive rigidity of the shear contribution, a correction factor $k = 5/6$ [Red06, Chapter 10] is introduced

$$\boldsymbol{q} = \int_{-h/2}^{h/2} k\boldsymbol{\sigma_s} = kGh\boldsymbol{\gamma}, \tag{4.12}$$

where $\boldsymbol{\gamma}$ is defined in Eq. (4.6). The equations of motion can be obtained using the Hamilton principle. It consists in minimizing the total Lagrangian, given by $L = E_{\text{def}} - E_{\text{kin}}$, where E_{def}, E_{kin} are the deformation and kinetic energy

$$E_{\text{def}} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}\Omega \, \mathrm{d}z = \frac{1}{2}\int_\Omega \{\boldsymbol{M} : \boldsymbol{\kappa} + \boldsymbol{q}\cdot\boldsymbol{\gamma}\} \, \mathrm{d}\Omega, \tag{4.13}$$

$$E_{\text{kin}} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \rho \, \|\partial_t \boldsymbol{u}\|^2 \, \mathrm{d}\Omega \, \mathrm{d}z = \frac{1}{2}\int_\Omega \left\{\frac{\rho h^3}{12}\|\partial_t \boldsymbol{\theta}\|^2 + \rho h(\partial_t u_z)^2\right\} \mathrm{d}\Omega, \tag{4.14}$$

where ρ is the mass density. The Hamilton principle states that

$$\int_0^T \delta L \, \mathrm{d}t = \int_0^T \{\delta E_{\text{def}} - \delta E_{\text{kin}}\} \, \mathrm{d}t = 0.$$

The final result is the following system of PDEs (for the detailed computations see [Red06, Chapter 10])

$$\begin{aligned} \rho\frac{\partial^2 u_z}{\partial t^2} &= \operatorname{div} \boldsymbol{q}, \qquad (x,y) \in \Omega, \\ \frac{\rho h^3}{12}\frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} &= \operatorname{Div} \boldsymbol{M} + \boldsymbol{q}, \end{aligned} \tag{4.15}$$

with $\boldsymbol{M} = \boldsymbol{\mathcal{D}}_b \operatorname{Grad}\boldsymbol{\theta}$ and $\boldsymbol{q} = kGh\,(\operatorname{grad} u_z - \boldsymbol{\theta})$. This PDE goes together with specified boundary conditions. Those will be detailed in 4.2.1.

4.1.2 Kirchhoff-Love model

The Kirchhoff model was formulated around 1850 and it is referred to as classical plate theory. The hypotheses on the displacement field consist of the following three points (see Fig. 4.1):

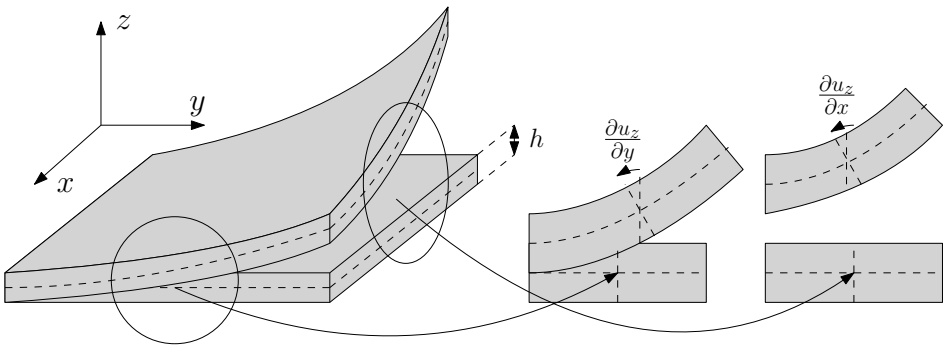

Figure 4.1: Kinematic assumption for the Kirchhoff plate

1. The fibers, segments perpendicular to the mid-plane before deformation, remain straight after deformation.

2. The fibers are inextensible.

3. While rotating, fibers remain perpendicular to the middle surface after deformation.

While the first two points are valid also for the Mindlin plate, the third assumption is peculiar to the Kirchhoff-Love model. Such an approximation is valid for plates having span-to-thickness ratio of the order of $L/h \approx 100-1000$ and implies zero transverse shear deformation

$$\boldsymbol{\gamma} = 0 \implies \varepsilon_{xz} = -\theta_x + \frac{\partial u_z}{\partial x} = 0, \qquad \varepsilon_{yz} = -\theta_y + \frac{\partial u_z}{\partial y} = 0.$$

The rotation vector is then related to the vertical displacement $\boldsymbol{\theta} = \operatorname{grad} u_z$. Plugging this into (4.5), it is found

$$\boldsymbol{\kappa} = \operatorname{Grad}\operatorname{grad} u_z = \operatorname{Hess} u_z. \tag{4.16}$$

The focus being on bending behavior, the in-plane displacement of the mid-plane are assumed to be zero $\boldsymbol{u}^0(x,y) = \boldsymbol{0}$. Hence, the displacement field assumes the form

$$\begin{aligned} u_x(x,y,z) &= -z\partial_x u_z, \\ u_y(x,y,z) &= -z\partial_y u_z, \\ u_z(x,y,z) &= u_z^0(x,y). \end{aligned} \tag{4.17}$$

For the Kirchhoff plate, the same link between the momenta and bending tensor holds

$$\boldsymbol{M} = \boldsymbol{\mathcal{D}}_b \boldsymbol{\kappa},$$

where $\boldsymbol{\mathcal{D}}_b$ and $\boldsymbol{\kappa}$ are given in (4.11), (4.16) respectively. The equations of motion can be obtained using the Hamilton principle [Red06, Chapter 2]. The deformation energy, kinetic

energy and external work read

$$E_{\text{def}} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}\Omega \, \mathrm{d}z = \frac{1}{2}\int_\Omega \{\boldsymbol{M} : \boldsymbol{\kappa}\} \, \mathrm{d}\Omega, \tag{4.18}$$

$$E_{\text{kin}} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \rho \, \|\partial_t \boldsymbol{u}\|^2 \, \mathrm{d}\Omega \, \mathrm{d}z \approx \frac{1}{2}\int_\Omega \rho h (\partial_t u_z)^2 \, \mathrm{d}\Omega. \tag{4.19}$$

Remark 5 (Rotational energy)
For the kinetic energy the rotational contribution

$$E_{rot} = \frac{1}{2}\int_\Omega \int_{-h/2}^{h/2} \left\{\rho \left(\partial_t u_x\right)^2 + \left(\partial_t u_y\right)^2\right\} \, \mathrm{d}\Omega \, \mathrm{d}z = \frac{h^3}{24}\int_\Omega \left\{\rho \left(\partial_{tx} u_z\right)^2 + \left(\partial_{ty} u_z\right)^2\right\} \, \mathrm{d}\Omega = O(h^3),$$

is neglected given the small thickness assumption.

The final result from the Hamilton's principle is the following PDE (for the detailed computations the reader may consult [Red06, Chapter 3])

$$\rho \frac{\partial^2 u_z}{\partial t^2} = -\operatorname{div}\operatorname{Div} \boldsymbol{M}, \qquad (x, y) \in \Omega. \tag{4.20}$$

Considering that $\boldsymbol{M} = \boldsymbol{\mathcal{D}}_b \operatorname{Hess} u_z$ one obtains

$$\rho \frac{\partial^2 u_z}{\partial t^2} = -D_b \Delta^2 u_z, \qquad (x, y) \in \Omega,$$

where $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4}$ is the bilaplacian. Appropriate boundary conditions for this problem will be detailed in 4.2.2.

4.2 Port-Hamiltonian formulation of plates

In this section the pH formulation of the Mindlin and Kirchhoff plate models is detailed. In [MMB05], the Mindlin plate model was put in pH form by appropriate selection of the energy variables. However, the final system does not consider the nature of the different variables that come into play, leading to a non intrinsic final formulation. Additionally, this model was presented using the jet bundle formalism in [SS17]. The Kirchhoff model was never explored in the pH framework and represents an original contribution of this thesis. The interested reader can find in [RZ18] a rigorous mathematical treatment of the biharmonic problem and its decomposition in 2D geometries (the 3D, that does not relate to plate bending, is treated in [PZ18]), but only for the static case.

4.2.1 Port-Hamiltonian Mindlin plate

Let $w := u_z$ denote the vertical displacement of the plate. Consider a bounded, connected domain $\Omega \subset \mathbb{R}^2$ and the Hamiltonian (total energy)

$$H = \frac{1}{2}\int_\Omega \left\{ \rho h \left(\frac{\partial w}{\partial t}\right)^2 + \frac{\rho h^3}{12}\left\|\frac{\partial \boldsymbol{\theta}}{\partial t}\right\|^2 + \boldsymbol{M} : \boldsymbol{\kappa} + \boldsymbol{q}\cdot\boldsymbol{\gamma} \right\} \mathrm{d}\Omega, \tag{4.21}$$

where $\boldsymbol{M}$, $\boldsymbol{\kappa}$, $\boldsymbol{q}$, $\boldsymbol{\gamma}$ are defined in Eqs. (4.10), (4.5), (4.12), (4.6) respectively. The choice of the energy variables is the same as in [MMB05] but here scalar-, vector- and tensor-valued variables are gathered together:

$$\begin{aligned} &\alpha_w = \rho h \frac{\partial w}{\partial t}, && \text{Linear momentum,} && \boldsymbol{\alpha}_\theta = \frac{\rho h^3}{12}\frac{\partial \boldsymbol{\theta}}{\partial t}, && \text{Angular momentum,} \\ &\boldsymbol{A}_\kappa = \boldsymbol{\kappa}, && \text{Curvature tensor,} && \boldsymbol{\alpha}_\gamma = \boldsymbol{\gamma}. && \text{Shear deformation.} \end{aligned} \tag{4.22}$$

The energy is now a quadratic function of the energy variables

$$H = \frac{1}{2}\int_\Omega \left\{ \frac{1}{\rho h}\alpha_w^2 + \frac{12}{\rho h^3}\left\|\boldsymbol{\alpha}_\theta\right\|^2 + (\boldsymbol{\mathcal{D}}_b \boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa + (\boldsymbol{\mathcal{D}}_s \boldsymbol{\alpha}_\gamma)\cdot\boldsymbol{\alpha}_\gamma \right\} \mathrm{d}\Omega, \tag{4.23}$$

where $\boldsymbol{\mathcal{D}}_s := Ghk\boldsymbol{I}_{2\times 2}$ and G is the shear modulus k the correction factor. The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned} &e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, && \text{Linear velocity,} && \boldsymbol{e}_\theta := \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t}, && \text{Angular velocity,} \\ &\boldsymbol{E}_\kappa := \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, && \text{Momenta tensor,} && \boldsymbol{e}_\gamma := \frac{\delta H}{\delta \boldsymbol{\alpha}_\gamma} = \boldsymbol{q} && \text{Shear stress.} \end{aligned} \tag{4.24}$$

Proposition 4
The variational derivative of the Hamiltonian with respect to the curvature tensor is the momenta tensor $\frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}$.

Proof. The proof is analogous to one already detailed in Prop. 3 □

Once the variables are concatenated together, the pH system is expressed as follows

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ \boldsymbol{0} & \boldsymbol{0} & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ \boldsymbol{0} & \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}. \tag{4.25}$$

The first two equations are equivalent to (4.15). The last two equations, like (3.33) for 3D elasticity, represent the fact the higher order derivatives commute.

We shall now establish the total energy balance in terms of boundary variables. Those will then be part of the underlying Stokes-Dirac structure of this model. The energy rate

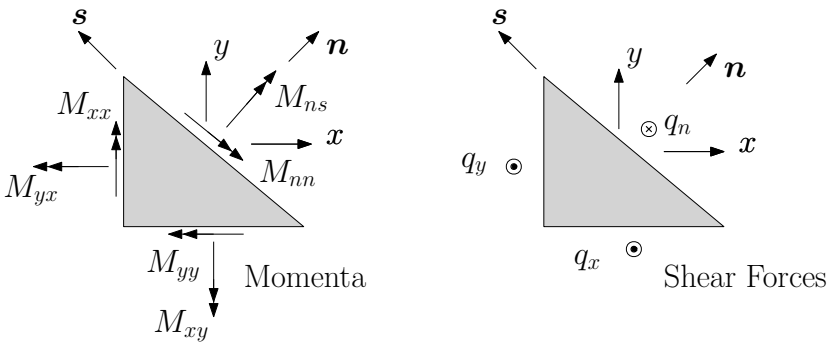

Figure 4.2: Cauchy law for momenta and forces at the boundary.

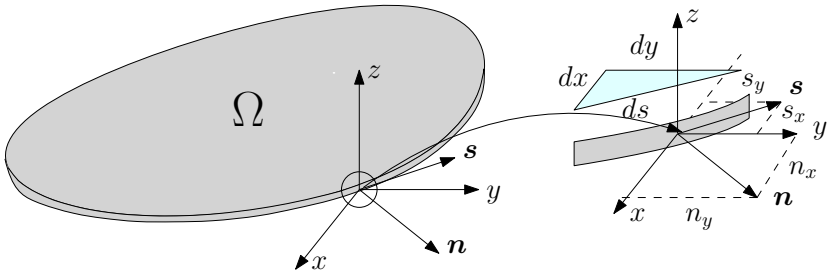

Figure 4.3: Reference frames and notations.

reads

$$\begin{aligned}\dot{H} &= \int_\Omega \left\{ \frac{\partial \alpha_w}{\partial t} e_w + \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \cdot \boldsymbol{e}_\theta + \frac{\partial \boldsymbol{A}_\kappa}{\partial t} : \boldsymbol{E}_\kappa + \frac{\partial \boldsymbol{\alpha}_\gamma}{\partial t} \cdot \boldsymbol{e}_\gamma \right\} \, \mathrm{d}\Omega \\ &= \int_\Omega \{ \mathrm{div}(\boldsymbol{e}_\gamma) e_w + \mathrm{Div}(\boldsymbol{E}_\kappa) \cdot \boldsymbol{e}_\theta + \ \mathrm{Grad}(\boldsymbol{e}_\theta) : \boldsymbol{E}_\kappa + \mathrm{grad}(e_w) \cdot \boldsymbol{e}_\gamma \} \, \mathrm{d}\Omega \qquad \text{Stokes theorem,} \\ &= \int_{\partial\Omega} \{ w_t \, q_n + \omega_n \, M_{nn} + \omega_s \, M_{ns} \} \, \mathrm{d}s,\end{aligned} \tag{4.26}$$

where s is the curvilinear abscissa. The last integral is obtained by applying the Stokes theorem. The boundary variables appearing in the last line of (4.26) and illustrated in Fig. 4.2 are defined as follows:

$$\begin{aligned} \text{Shear force} \qquad & q_n := \boldsymbol{q} \cdot \boldsymbol{n} = \boldsymbol{e}_\gamma \cdot \boldsymbol{n}, \\ \text{Flexural momentum} \qquad & M_{nn} := \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}) = \boldsymbol{E}_\kappa : (\boldsymbol{n} \otimes \boldsymbol{n}), \\ \text{Torsional momentum} \qquad & M_{ns} := \boldsymbol{M} : (\boldsymbol{s} \otimes \boldsymbol{n}) = \boldsymbol{E}_\kappa : (\boldsymbol{s} \otimes \boldsymbol{n}), \end{aligned} \tag{4.27}$$

Vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ designate the normal and tangential unit vectors to the boundary, as shown in Fig. 4.3. Given two vectors $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^m$, the notation $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top \in \mathbb{R}^{n \times m}$ denotes the

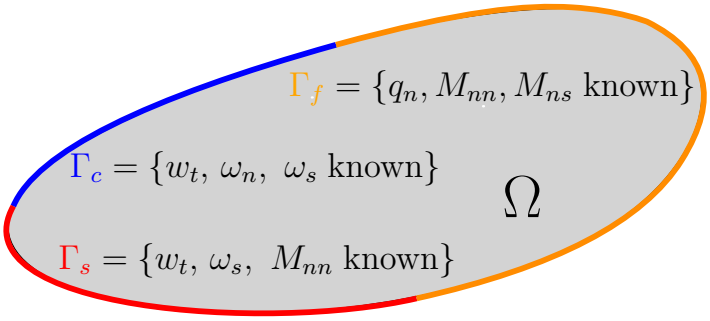

Figure 4.4: Boundary conditions for the Mindlin plate.

outer (or dyadic) product of two vectors. The corresponding power conjugated variables are

$$\begin{aligned} \text{Vertical velocity} \quad & w_t := \frac{\partial w}{\partial t} = e_w, \\ \text{Flexural rotation} \quad & \omega_n := \frac{\partial \boldsymbol{\theta}}{\partial t} \cdot \boldsymbol{n} = \boldsymbol{e}_\theta \cdot \boldsymbol{n}, \\ \text{Torsional rotation} \quad & \omega_s := \frac{\partial \boldsymbol{\theta}}{\partial t} \cdot \boldsymbol{s} = \boldsymbol{e}_\theta \cdot \boldsymbol{s}. \end{aligned} \tag{4.28}$$

Consider a partition of the boundary $\partial\Omega = \Gamma_C \cup \Gamma_S \cup \Gamma_F$, $\Gamma_C \cap \Gamma_S \cap \Gamma_F = \{\emptyset\}$. The set Γ_C, Γ_S, Γ_F could be empty. Given definitions (4.27), (4.28), the boundary conditions for the Mindlin plate [DHNLS99] (see Fig. 4.4) that are considered are:

- Clamped (C) on $\Gamma_C \subseteq \partial\Omega$: $w_t = 0,\ \omega_n = 0,\ \omega_s = 0$;
- Simply supported hard (S) on $\Gamma_S \subseteq \partial\Omega$: $w_t = 0,\ \omega_s = 0,\ M_{nn} = 0$;
- Free (F) on $\Gamma_F \subseteq \partial\Omega$: $M_{nn} = 0,\ M_{ns} = 0,\ q_n = 0$.

Then the final pH formulation reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w\\ \boldsymbol{\alpha}_\theta\\ \boldsymbol{A}_\kappa\\ \boldsymbol{\alpha}_\gamma\end{pmatrix} = \underbrace{\begin{bmatrix}0 & 0 & 0 & \operatorname{div}\\ \mathbf{0} & \mathbf{0} & \operatorname{Div} & \boldsymbol{I}_{2\times 2}\\ \mathbf{0} & \operatorname{Grad} & \mathbf{0} & \mathbf{0}\\ \operatorname{grad} & -\boldsymbol{I}_{2\times 2} & \mathbf{0} & \mathbf{0}\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}e_w\\ \boldsymbol{e}_\theta\\ \boldsymbol{E}_\kappa\\ \boldsymbol{e}_\gamma\end{pmatrix},$$

$$\boldsymbol{u}_\partial = \underbrace{\begin{bmatrix}\gamma_0^{\Gamma_C} & 0 & 0 & 0\\ 0 & \gamma_n^{\Gamma_C} & 0 & 0\\ 0 & \gamma_s^{\Gamma_C} & 0 & 0\\ \gamma_0^{\Gamma_S} & 0 & 0 & 0\\ 0 & \gamma_s^{\Gamma_S} & 0 & 0\\ 0 & 0 & \gamma_{nn}^{\Gamma_S} & 0\\ 0 & 0 & \gamma_{nn}^{\Gamma_F} & 0\\ 0 & 0 & \gamma_{ns}^{\Gamma_F} & 0\\ 0 & 0 & 0 & \gamma_n^{\Gamma_F}\end{bmatrix}}_{\mathcal{B}}\begin{pmatrix}e_w\\ \boldsymbol{e}_\theta\\ \boldsymbol{E}_\kappa\\ \boldsymbol{e}_\gamma\end{pmatrix},$$

$$\boldsymbol{y}_\partial = \underbrace{\begin{bmatrix}0 & 0 & 0 & \gamma_n^{\Gamma_C}\\ 0 & 0 & \gamma_{nn}^{\Gamma_C} & 0\\ 0 & 0 & \gamma_{ns}^{\Gamma_C} & 0\\ 0 & 0 & 0 & \gamma_n^{\Gamma_S}\\ 0 & 0 & \gamma_{ns}^{\Gamma_S} & 0\\ 0 & \gamma_n^{\Gamma_S} & 0 & 0\\ 0 & \gamma_n^{\Gamma_F} & 0 & 0\\ 0 & \gamma_s^{\Gamma_F} & 0 & 0\\ \gamma_0^{\Gamma_F} & 0 & 0 & 0\end{bmatrix}}_{\mathcal{C}}\begin{pmatrix}e_w\\ \boldsymbol{e}_\theta\\ \boldsymbol{E}_\kappa\\ \boldsymbol{e}_\gamma\end{pmatrix}, \tag{4.29}$$

where $\boldsymbol{\gamma}_0^{\Gamma_*} a = a|_{\Gamma_*}$ denotes the trace over the set Γ_*. Furthermore, notations $\gamma_n^{\Gamma_*}\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{n}|_{\Gamma_*}$, $\gamma_s^{\Gamma_*}\boldsymbol{a} = \boldsymbol{a}\cdot\boldsymbol{s}|_{\Gamma_*}$ indicate the normal and tangential trace over the set Γ_* respectively. Symbols $\gamma_{nn}^{\Gamma_*}, \gamma_{ns}^{\Gamma_*}$ denote the normal-normal trace and the normal-tangential trace of tensor-valued functions, $\gamma_{nn}^{\Gamma_*}\boldsymbol{A} = \boldsymbol{A} : (\boldsymbol{n}\otimes\boldsymbol{n})|_{\Gamma_*}$, $\gamma_{ns}^{\Gamma_*}\boldsymbol{A} = \boldsymbol{A} : (\boldsymbol{n}\otimes\boldsymbol{s})|_{\Gamma_*}$.

Remark 6

It can be observed that the interconnection structure given by $\mathcal{J}$ in (4.29) mimics that of the Timoshenko beam [JZ12, Chapter 7].

Theorem 5 (Stokes-Dirac structure for the Mindlin plate)

Consider $\mathbb{V} = \mathbb{R}^2$, $\mathbb{S} = \mathbb{R}^{2\times 2}_{sym}$ and let $H^{\operatorname{grad}}(\Omega)$ be the space of functions with gradient in $L^2(\Omega,\mathbb{V})$ and $H^{\operatorname{div}}(\Omega,\mathbb{V})$ the space of vector-valued functions with divergence in $L^2(\Omega)$. Furthermore, $H^1(\Omega,\mathbb{V})$ is the space of vectors with symmetric gradient in $L^2(\Omega,\mathbb{S})$ and $H^{\operatorname{Div}}(\Omega,\mathbb{S})$

denote the space of symmetric tensors with divergence in $L^2(\Omega,\mathbb{V})$. *Consider the definitions*

$$\begin{aligned} H &:= H^1(\Omega)\times H^{\mathrm{Grad}}(\Omega,\mathbb{V})\times H^{\mathrm{Div}}(\Omega,\mathbb{S})\times H^{\mathrm{div}}(\Omega,\mathbb{V}),\\ F &:= L^2(\Omega)\times L^2(\Omega,\mathbb{V})\times L^2(\Omega,\mathbb{S})\times L^2(\Omega,\mathbb{V}),\\ F_\partial &:= L^2(\Gamma_C,\mathbb{R}^3)\times L^2(\Gamma_S,\mathbb{R}^3)\times L^2(\Gamma_F,\mathbb{R}^3). \end{aligned}$$

593 *The set*

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \boldsymbol{f}\\ \boldsymbol{f}_\partial\\ \boldsymbol{e}\\ \boldsymbol{e}_\partial \end{pmatrix} \mid \boldsymbol{e}\in H,\ \boldsymbol{f}=-\mathcal{J}\boldsymbol{e},\ \boldsymbol{f}_\partial=\mathcal{B}\boldsymbol{e},\ \boldsymbol{e}_\partial=\mathcal{C}\boldsymbol{e} \right\}, \tag{4.30}$$

594 *where* $\boldsymbol{e}=(e_w,\,\boldsymbol{e}_\theta,\,\boldsymbol{E}_\kappa,\,\boldsymbol{e}_\gamma)$ *and* $\mathcal{J},\mathcal{B},\mathcal{C}$ *are defined in* (4.29), *is a Stokes–Dirac structure with*
595 *respect to the pairing*

$$\left\langle\!\left\langle (\boldsymbol{f}^1,\boldsymbol{f}^1_\partial,\boldsymbol{e}^1,\boldsymbol{e}^1_\partial),(\boldsymbol{f}^2,\boldsymbol{f}^2_\partial,\boldsymbol{e}^2,\boldsymbol{e}^2_\partial)\right\rangle\!\right\rangle := \left\langle \boldsymbol{e}^1,\boldsymbol{f}^2\right\rangle_F + \left\langle \boldsymbol{e}^2,\boldsymbol{f}^1\right\rangle_F + \left\langle \boldsymbol{e}^1_\partial,\boldsymbol{f}^2_\partial\right\rangle_{F_\partial} + \left\langle \boldsymbol{e}^2_\partial,\boldsymbol{f}^1_\partial\right\rangle_{F_\partial}, \tag{4.31}$$

where $\boldsymbol{e}^i_\partial=(\boldsymbol{e}^i_{\partial,1},\ \boldsymbol{e}^i_{\partial,2},\ \boldsymbol{e}^i_{\partial,3})$, $\boldsymbol{f}^i_\partial=(\boldsymbol{f}^i_{\partial,1},\ \boldsymbol{f}^i_{\partial,2},\ \boldsymbol{f}^i_{\partial,3})$ *and*

$$\langle(\boldsymbol{a},\,\boldsymbol{b},\,\boldsymbol{c}),(\boldsymbol{d},\,\boldsymbol{e},\,\boldsymbol{f})\rangle_{F_\partial} = \int_{\Gamma_C}\boldsymbol{a}\cdot\boldsymbol{d}\,\mathrm{d}S + \int_{\Gamma_S}\boldsymbol{b}\cdot\boldsymbol{e}\,\mathrm{d}S + \int_{\Gamma_F}\boldsymbol{c}\cdot\boldsymbol{f}\,\mathrm{d}S,\quad \boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c},\ \boldsymbol{d},\ \boldsymbol{e},\ \boldsymbol{f}\in\mathbb{R}^3.$$

596 *Proof.* The proof is analogous to the one of Th. 4. The integration by parts has to be carried
597 as in Eq. (4.26). □

598 The Mindlin plate falls within the assumption of [Skr19], hence it is a well posed boundary
599 control pH systems.

600 4.2.2 Port-Hamiltonian Kirchhoff plate

601 Again the starting point is the Hamiltonian (total energy)

$$H = \frac{1}{2}\int_\Omega \left\{ \rho h\left(\frac{\partial w}{\partial t}\right)^2 + \boldsymbol{M}:\boldsymbol{\kappa}\right\}\,\mathrm{d}\Omega, \tag{4.32}$$

602 where $\boldsymbol{M}$, $\boldsymbol{\kappa}$ are defined in Eqs. (4.10), (4.16). For what concerns the choice of the energy
603 variables, a scalar and a tensor variable are considered:

$$\alpha_w = \rho h\frac{\partial w}{\partial t},\quad \text{Linear momentum,}\qquad \boldsymbol{A}_\kappa=\boldsymbol{\kappa},\quad \text{Curvature tensor.} \tag{4.33}$$

604 The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$e_w := \frac{\delta H}{\delta\alpha_w} = \frac{\partial w}{\partial t},\quad \text{Linear velocity,}\qquad \boldsymbol{E}_\kappa := \frac{\delta H}{\delta\boldsymbol{A}_\kappa}=\boldsymbol{M},\quad \text{Curvature tensor.} \tag{4.34}$$

605 The port-Hamiltonian system is then written as

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w\\ \boldsymbol{A}_\kappa\end{pmatrix} = \begin{bmatrix}0 & -\operatorname{div}\circ\operatorname{Div}\\ \operatorname{Grad}\circ\operatorname{grad} & \mathbf{0}\end{bmatrix}\begin{pmatrix}e_w\\ \boldsymbol{E}_\kappa\end{pmatrix}. \tag{4.35}$$

The first equation is equivalent to (4.20). The last equation represent the fact the higher order derivatives commute

$$\begin{aligned}\partial_t \boldsymbol{A}_\kappa &= \operatorname{Grad}\operatorname{grad} e_w,\\ \partial_t \boldsymbol{\kappa} &= \operatorname{Grad}\operatorname{grad}\partial_t w,\\ \partial_t \operatorname{Grad}\operatorname{grad} w &= \operatorname{Grad}\operatorname{grad}\partial_t w,\end{aligned}$$

606 The last equation holds for $w \in C^3(\Omega)$.

607 **Theorem 6**

608 *The operator* $\operatorname{Grad}\circ\operatorname{grad}$*, corresponding to the Hessian operator, is the adjoint of the double*
609 *divergence* $\operatorname{div}\circ\operatorname{Div}$*.*

Proof. Let $\mathbb{S} = \mathbb{R}^{d\times d}_{\text{sym}}$ and consider the Hilbert space of the square integrable symmetric square tensors $L^2(\Omega,\mathbb{S})$ over an open connected set Ω (its inner product is defined in (3.31)). Consider the Hilbert space $L^2(\Omega)$ of scalar square integrable functions, endowed with the standard inner product. Consider the double divergence operator defined as:

$$\begin{aligned}A:\ L^2(\Omega,\mathbb{S}) &\to L^2(\Omega),\\ \boldsymbol{\Psi} &\to \operatorname{div}\operatorname{Div}\boldsymbol{\Psi} = \psi,\end{aligned} \qquad \text{with } \psi = \operatorname{div}\operatorname{Div}\boldsymbol{\Psi} = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2 \boldsymbol{\Psi}_{ij}}{\partial x_i \partial x_j}.$$

We shall identify A^*

$$\begin{aligned}A^*:\ L^2(\Omega) &\to L^2(\Omega,\mathbb{S}),\\ f &\to A^* f = \boldsymbol{F},\end{aligned}$$

such that

$$\langle A\boldsymbol{\Psi}, f\rangle_{L^2(\Omega)} = \langle \boldsymbol{\Psi}, A^* f\rangle_{L^2(\Omega,\mathbb{S})}, \qquad \begin{aligned}&\forall\, \boldsymbol{\Psi} \in \operatorname{Domain}(A) \subset L^2(\Omega,\mathbb{S})\\ &\forall\, f \in \operatorname{Domain}(A^*) \subset L^2(\Omega)\end{aligned}$$

The function have to belong to the operator domain, so for instance $f \in \mathcal{C}_0^2(\Omega) \in \operatorname{Domain}(A^*)$ the space of twice differentiable scalar functions with compact support and $\boldsymbol{\Psi}$ can be chosen in the set $\mathcal{C}_0^2(\Omega,\mathbb{S}) \in \operatorname{Domain}(A)$, the space of twice differentiable 2×2 symmetric tensors with compact support on Ω. A classical result is the fact that the adjoint of the vector divergence is $\operatorname{div}^* = -\operatorname{grad}$ as stated in [KZ15]. By theorem 3, it holds $\operatorname{Div}^* = -\operatorname{Grad}$. Considering that A is the composition of two different operators $A = \operatorname{div}\circ\operatorname{Div}$ and that the adjoint of a composed operator is the adjoint of each operator in reverse order,

i.e. $(B \circ C)^* = C^* \circ B^*$, then it can be stated

$$A^* = (\mathrm{div} \circ \mathrm{Div})^* = \mathrm{Div}^* \circ \mathrm{div}^* = \mathrm{Grad} \circ \mathrm{grad}.$$

Since only formal adjoints are being looked for, this concludes the proof. □

The energy rate provides the boundary port variables

$$\begin{aligned}
\dot{H} &= \int_\Omega \{\partial_t \alpha_w \, e_w + \partial_t \boldsymbol{A}_\kappa : \boldsymbol{E}_\kappa\} \, \mathrm{d}\Omega \\
&= \int_\Omega \{-\mathrm{div}\,\mathrm{Div}\,\boldsymbol{E}_\kappa \, e_w + \mathrm{Grad}\,\mathrm{grad}\, e_w : \boldsymbol{E}_\kappa\} \, \mathrm{d}\Omega, && \text{Stokes theorem} \\
&= \int_{\partial\Omega} \{-\boldsymbol{n} \cdot \mathrm{Div}\,\boldsymbol{E}_\kappa \, e_w + (\boldsymbol{n} \otimes \mathrm{grad}\, e_w) : \boldsymbol{E}_\kappa\} \, \mathrm{d}s, \\
&= \int_{\partial\Omega} \{-\boldsymbol{n} \cdot \mathrm{Div}\,\boldsymbol{E}_\kappa \, e_w + \partial_{\boldsymbol{n}} e_w \, (\boldsymbol{n} \otimes \boldsymbol{n}) : \boldsymbol{E}_\kappa + \partial_{\boldsymbol{s}} e_w \, (\boldsymbol{n} \otimes \boldsymbol{s}) : \boldsymbol{E}_\kappa\} \, \mathrm{d}s, && \text{Dyadic properties} \\
&= \int_{\partial\Omega} \{\widehat{q}_n \, w_t + \partial_{\boldsymbol{n}} w_t \, M_{nn} + \partial_{\boldsymbol{s}} w_t \, M_{ns}\} \, \mathrm{d}s.
\end{aligned} \tag{4.36}$$

where s is the curvilinear abscissa, $w_t := \partial_t w$ and $\partial_{\boldsymbol{s}} w_t$ denotes the directional derivative along the tangential versor at the boundary. Additionally, the following definitions have been introduced

$$\widehat{q}_n := -\boldsymbol{n} \cdot \mathrm{Div}(\boldsymbol{E}_\kappa), \quad M_{nn} := (\boldsymbol{n} \otimes \boldsymbol{n}) : \boldsymbol{E}_\kappa, \quad M_{ns} := (\boldsymbol{n} \otimes \boldsymbol{s}) : \boldsymbol{E}_\kappa. \tag{4.37}$$

Variables w_t and $\partial_{\boldsymbol{s}} w_t$ are not independent as they are differentially related with respect to derivation along $\boldsymbol{s}$ (see for instance [TWK59, Chapter 4]). The tangential derivative has to be moved on the torsional momentum M_{ns}. For sake of simplicity, $\partial\Omega$ is supposed to be regular. Then the integration by parts provides

$$\int_{\partial\Omega} \partial_{\boldsymbol{s}} w_t \, M_{ns} \, \mathrm{d}s = -\int_{\partial\Omega} \partial_{\boldsymbol{s}} M_{ns} \, w_t \, \mathrm{d}s. \tag{4.38}$$

The final energy balance reads

$$\dot{H} = \int_{\partial\Omega} \{w_t \, \widetilde{q}_n + \partial_{\boldsymbol{n}} w_t \, M_{nn}\} \, \mathrm{d}s, \tag{4.39}$$

where the boundary variables are

$$\begin{aligned}
&\text{Effective shear force} && \widetilde{q}_n := \widehat{q}_n - \partial_{\boldsymbol{s}} M_{ns}, \\
&\text{Flexural momentum} && M_{nn} := \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}) = \boldsymbol{E}_\kappa : (\boldsymbol{n} \otimes \boldsymbol{n}),
\end{aligned} \tag{4.40}$$

and $\widehat{q}_n$ is defined in (4.37). The corresponding power conjugated variables are:

$$\begin{aligned}
&\text{Vertical velocity} && w_t := \frac{\partial w}{\partial t} = e_w, \\
&\text{Flexural rotation} && \partial_{\boldsymbol{n}} w_t := \nabla e_w \cdot \boldsymbol{n}.
\end{aligned}, \tag{4.41}$$

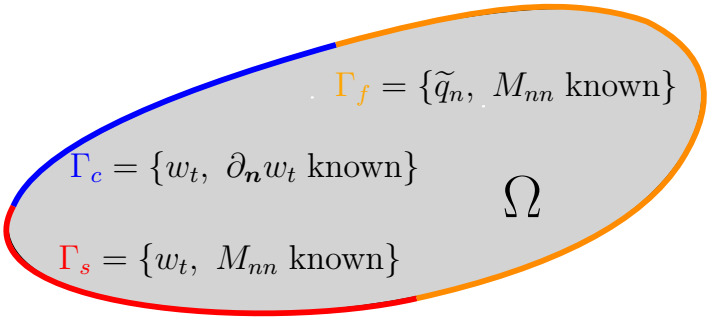

Figure 4.5: Boundary conditions for the Kirchhoff plate.

Consider a partition of the boundary $\partial\Omega = \Gamma_C \cup \Gamma_S \cup \Gamma_F$, $\Gamma_C \cap \Gamma_S \cap \Gamma_F = \{\emptyset\}$. Given definitions (4.40), (4.41), the boundary conditions for the Kirchhoff plate [GSV18] are the following:

- Clamped (C) on $\Gamma_C \subseteq \partial\Omega$: $w_t = 0,\ \partial_{\boldsymbol{n}} w_t = 0$;
- Simply supported (S) on $\Gamma_S \subseteq \partial\Omega$: $w_t = 0,\ M_{nn} = 0$;
- Free (F) on $\Gamma_F \subseteq \partial\Omega$: $\widetilde{q}_n = 0,\ M_{nn} = 0$.

Then the final pH formulation reads

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \boldsymbol{A}_\kappa\end{pmatrix} &= \underbrace{\begin{bmatrix} 0 & -\operatorname{div}\circ\operatorname{Div} \\ \operatorname{Grad}\circ\operatorname{grad} & \boldsymbol{0}\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa\end{pmatrix}, \\
\boldsymbol{u}_\partial &= \underbrace{\begin{bmatrix} \gamma_0^{\Gamma_C} & 0 \\ \gamma_1^{\Gamma_C} & 0 \\ \gamma_0^{\Gamma_S} & 0 \\ 0 & \gamma_{nn}^{\Gamma_S} \\ 0 & \gamma_{nn,1}^{\Gamma_F} \\ 0 & \gamma_{nn}^{\Gamma_F}\end{bmatrix}}_{\mathcal{B}}\begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa\end{pmatrix}, \\
\boldsymbol{y}_\partial &= \underbrace{\begin{bmatrix} 0 & \gamma_{nn,1}^{\Gamma_C} \\ 0 & \gamma_{nn}^{\Gamma_C} \\ 0 & \gamma_{nn,1}^{\Gamma_S} \\ \gamma_1^{\Gamma_S} & 0 \\ \gamma_0^{\Gamma_F} & 0 \\ \gamma_1^{\Gamma_F} & 0\end{bmatrix}}_{\mathcal{C}}\begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa\end{pmatrix},
\end{aligned}
\tag{4.42}
$$

where $\boldsymbol{\gamma}_0^{\Gamma_*} a = a|_{\Gamma_*}$ and $\gamma_1^{\Gamma_*} a = \partial_{\boldsymbol{n}} a|_{\Gamma_*}$ denote the standard and the normal derivative trace over the set Γ_* respectively. The symbol $\gamma_{nn,1}^{\Gamma_*}$ denotes the map $\gamma_{nn,1}^{\Gamma_*}\boldsymbol{A} = -\boldsymbol{n}\cdot\operatorname{Div}\boldsymbol{A} - \partial_{\boldsymbol{s}}(\boldsymbol{A}:$

$(\boldsymbol{n}\otimes\boldsymbol{s}))|_{\Gamma_*},$, while $\gamma^{\Gamma_*}_{nn}\boldsymbol{A} = \boldsymbol{A}\!:\!(\boldsymbol{n}\otimes\boldsymbol{n})|_{\Gamma_*}$ indicates the normal-normal trace of a tensor-valued function.

Remark 7
The interconnection structure $\mathcal{J}$ in (4.42) resembles that of the Bernoulli beam [CRMPB17]. The double divergence and the double gradient coincide, in dimension one, with the second derivative.

Theorem 7 (Stokes-Dirac structure for the Kirchhoff plate)
Consider $\mathbb{S} = \mathbb{R}^{2\times 2}_{sym}$ and let $H^2(\Omega)$ be the space of functions with Hessian in $L^2(\Omega,\mathbb{S})$ and $H^{\operatorname{div}\operatorname{Div}}(\Omega,\mathbb{S})$ the space of vector-valued functions with double divergence in $L^2(\Omega)$. Consider the definitions

$$\begin{aligned} H &:= H^2(\Omega)\times H^{\operatorname{div}\operatorname{Div}}(\Omega,\mathbb{S}),\\ F &:= L^2(\Omega)\times L^2(\Omega,\mathbb{S}),\\ F_\partial &:= L^2(\Gamma_C,\mathbb{R}^2)\times L^2(\Gamma_S,\mathbb{R}^2)\times L^2(\Gamma_F,\mathbb{R}^2). \end{aligned}$$

The set

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \boldsymbol{f}\\ \boldsymbol{f}_\partial\\ \boldsymbol{e}\\ \boldsymbol{e}_\partial \end{pmatrix} \mid \boldsymbol{e}\in H,\ \boldsymbol{f} = -\mathcal{J}\boldsymbol{e},\ \boldsymbol{f}_\partial = \mathcal{B}\boldsymbol{e},\ \boldsymbol{e}_\partial = \mathcal{C}\boldsymbol{e} \right\}, \tag{4.43}$$

where $\boldsymbol{e} = (e_w,\ \boldsymbol{E}_\kappa)$ and $\mathcal{J},\mathcal{B},\mathcal{C}$ are defined in (4.42), is a Stokes–Dirac structure with respect to the pairing

$$\left\langle\!\left\langle (\boldsymbol{f}^1,\boldsymbol{f}^1_\partial,\boldsymbol{e}^1,\boldsymbol{e}^1_\partial),(\boldsymbol{f}^2,\boldsymbol{f}^2_\partial,\boldsymbol{e}^2,\boldsymbol{e}^2_\partial) \right\rangle\!\right\rangle := \left\langle \boldsymbol{e}^1,\boldsymbol{f}^2\right\rangle_F + \left\langle \boldsymbol{e}^2,\boldsymbol{f}^1\right\rangle_F + \left\langle \boldsymbol{e}^1_\partial,\boldsymbol{f}^2_\partial\right\rangle_{F_\partial} + \left\langle \boldsymbol{e}^2_\partial,\boldsymbol{f}^1_\partial\right\rangle_{F_\partial}, \tag{4.44}$$

where $\boldsymbol{e}^i_\partial = (\boldsymbol{e}^i_{\partial,1},\ \boldsymbol{e}^i_{\partial,2})$, $\boldsymbol{f}^i_\partial = (\boldsymbol{f}^i_{\partial,1},\ \boldsymbol{f}^i_{\partial,2})$ and

$$\langle(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}),(\boldsymbol{d},\boldsymbol{e},\boldsymbol{f})\rangle_{F_\partial} = \int_{\Gamma_C}\boldsymbol{a}\cdot\boldsymbol{d}\,\mathrm{d}S + \int_{\Gamma_S}\boldsymbol{b}\cdot\boldsymbol{e}\,\mathrm{d}S + \int_{\Gamma_F}\boldsymbol{c}\cdot\boldsymbol{f}\,\mathrm{d}S,\quad \boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c},\ \boldsymbol{d},\ \boldsymbol{e},\ \boldsymbol{f}\in\mathbb{R}^2.$$

Proof. The proof is analogous to the one of Th. 4. The integration by parts has to be carried as in Eq. (4.36). □

4.3 Laminated anisotropic plates

Until now homogeneous isotropic materials have been considered. For this class of materials, the membrane and bending problems are decoupled. In aeronautical applications, structure are made up of laminae of different materials to enhance the mechanical properties of the resulting structure. In some cases, a certain coupling is desired, to increase the aerodynamical performance of the wing as it deforms.

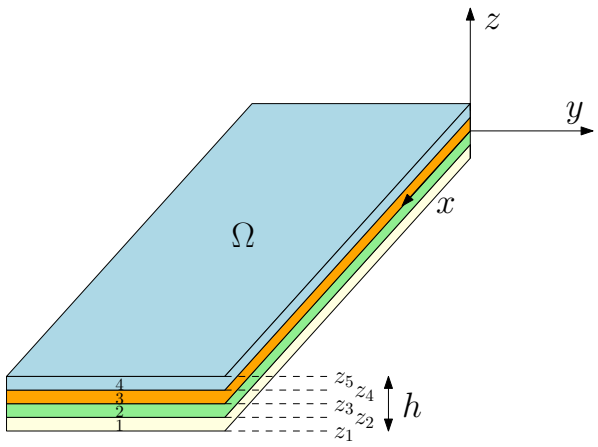

Figure 4.6: Laminated plate with 4 layers.

Consider again the deformation field given by (4.1)

$$\begin{aligned} \boldsymbol{u}(x,y,z,t) &= \boldsymbol{u}^0(x,y,t) - z\boldsymbol{\theta}(x,y,t), \\ u_z(x,y,z,t) &= u_z^0(x,y,t), \end{aligned}$$

where $\boldsymbol{u} = (u_x, u_y)$. The link between in-plane deformation (4.2) and the membrane and bending contribution (4.4), (4.5).

$$\boldsymbol{\varepsilon}_{2D} = \boldsymbol{\varepsilon}^0 - z\boldsymbol{\kappa} \qquad \text{where} \qquad \boldsymbol{\varepsilon}^0 = \operatorname{Grad} \boldsymbol{u}^0, \quad \boldsymbol{\kappa} = \operatorname{Grad} \boldsymbol{\theta}. \tag{4.45}$$

Assume that each layer is an anisotropic material under plane stress condition. Then, it holds (see [Red03, Chapter 1] for details)

$$\boldsymbol{\Sigma}_{2D}^i = \boldsymbol{\mathcal{D}}_{2D}^i \boldsymbol{\varepsilon}_{2D}^i,$$

where i indicates the layer under consideration. The matrix $\boldsymbol{\mathcal{D}}_{2D}^i$ depends on the properties of each material. To reduce the problem to bi-dimensional, the stresses have to be integrated along the thickness. Differently from isotropic plate, for laminated anisotropic plates the membrane and bending behavior are coupled. To see this consider the membrane and bending resultant of the stress

$$\boldsymbol{N} := \int_{-h/2}^{h/2} \boldsymbol{\Sigma}_{2D} \, \mathrm{d}z, \qquad \boldsymbol{M} := \int_{-h/2}^{h/2} -z\boldsymbol{\Sigma}_{2D} \, \mathrm{d}z. \tag{4.46}$$

Since the stress are discontinuous due to the change of constitutive law along the thickness, the integration has to be performed lamina-wise. Once the computations are carried out, it is found

$$\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{D}}_m & \boldsymbol{\mathcal{D}}_c \\ \boldsymbol{\mathcal{D}}_c & \boldsymbol{\mathcal{D}}_b \end{bmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{pmatrix}, \tag{4.47}$$

where

$$\boldsymbol{\mathcal{D}}_m = \sum_{i=1}^{n_{\text{layer}}} \boldsymbol{\mathcal{D}}^i_{2D}(z_{i+1} - z_i), \quad \boldsymbol{\mathcal{D}}_c = -\frac{1}{2}\sum_{i=1}^{n_{\text{layer}}} \boldsymbol{\mathcal{D}}^i_{2D}(z^2_{i+1} - z^2_i), \quad \boldsymbol{\mathcal{D}}_b = \frac{1}{3}\sum_{i=1}^{n_{\text{layer}}} \boldsymbol{\mathcal{D}}^i_{2D}(z^3_{i+1} - z^3_i), \tag{4.48}$$

and n_{layer} is the number of layers and z_i represents the height of the i^{th} layer (see Fig. 4.6). The coupling term $\boldsymbol{\mathcal{D}}_c$ disappears if a symmetric configuration is considered. For the shear contribution it is obtained

$$\boldsymbol{q} := \int_{-h/2}^{h/2} \boldsymbol{\sigma}_s \, \mathrm{d}z = \boldsymbol{\mathcal{D}}_s \boldsymbol{\gamma}, \qquad \text{where} \qquad \boldsymbol{\gamma} = \operatorname{grad} u_z - \boldsymbol{\theta}. \tag{4.49}$$

The tensor $\boldsymbol{\mathcal{D}}_s$ is not diagonal as in the isotropic case, cf.§4.2.1.

In the following section it is shown how anisotropic laminated plates can be formulated as pHs.

4.3.1 Port-Hamiltonian Mindlin laminated plate

For a shear deformable laminated plate the kinetic and deformation energy read

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}\int_\Omega \left\{ \rho h \left\| \frac{\partial \boldsymbol{u}^0}{\partial t} \right\|^2 + \rho h \left(\frac{\partial u_z}{\partial t} \right)^2 + \frac{\rho h^3}{12} \left\| \frac{\partial \boldsymbol{\theta}}{\partial t} \right\|^2 \right\} \mathrm{d}\Omega, \\ E_{\text{def}} &= \frac{1}{2}\int_\Omega \left\{ \boldsymbol{N} : \boldsymbol{\varepsilon}^0 + \boldsymbol{M} : \boldsymbol{\kappa} + \boldsymbol{q} \cdot \boldsymbol{\gamma} \right\} \mathrm{d}\Omega. \end{aligned}$$

By using the Hamilton principle the equations of motion are retrieved (see [Red03, Chapter 3] for an exhaustive explanation)

$$\begin{aligned} \frac{\partial^2 \boldsymbol{u}^0}{\partial t^2} &= \operatorname{Div} \boldsymbol{N}, \\ \frac{\partial^2 u_z}{\partial t^2} &= \operatorname{div} \boldsymbol{q}, \\ \frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} &= \operatorname{Div} \boldsymbol{M} + \boldsymbol{q}, \end{aligned} \tag{4.50}$$

where $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{q}$ are defined in Eqs. (4.47), (4.49). To get a port-Hamiltonian formulation, the following energy variable are chosen

$$\begin{aligned} \boldsymbol{\alpha}_u &= \rho h \frac{\partial \boldsymbol{u}^0}{\partial t}, & \alpha_w &= \rho h \frac{\partial \boldsymbol{u}_z}{\partial t}, & \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12}\frac{\partial \boldsymbol{\theta}}{\partial t}, \\ \boldsymbol{A}_{\varepsilon^0} &= \boldsymbol{\varepsilon}^0, & \boldsymbol{A}_\kappa &= \boldsymbol{\kappa}, & \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}. \end{aligned} \tag{4.51}$$

This choice highlights the nature of the problem in which the membrane part (equivalent to a 2D elasticity problem) and the bending part interact. The total energy $H = E_{\text{kin}} + E_{\text{def}}$ is

now a quadratic function of the energy variables

$$E_{\text{kin}} = \frac{1}{2}\int_\Omega \left\{ \frac{1}{\rho h}\left\|\frac{\partial \boldsymbol{\alpha}_u}{\partial t}\right\|^2 + \frac{1}{\rho h}\left(\frac{\partial \alpha_w}{\partial t}\right)^2 + \frac{12}{\rho h^3}\left\|\frac{\partial \boldsymbol{\alpha}_\theta}{\partial t}\right\|^2 \right\} \mathrm{d}\Omega,$$
$$E_{\text{def}} = \frac{1}{2}\int_\Omega \left\{ (\boldsymbol{\mathcal{D}}_m \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{\mathcal{D}}_c \boldsymbol{A}_\kappa) : \boldsymbol{A}_{\varepsilon^0} + (\boldsymbol{\mathcal{D}}_c \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{\mathcal{D}}_b \boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa + (\boldsymbol{\mathcal{D}}_s \boldsymbol{\alpha}_\gamma)\cdot \boldsymbol{\alpha}_\gamma \right\} \mathrm{d}\Omega,$$

674 The co-energies are equal to

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_u} = \frac{\partial \boldsymbol{u}^0}{\partial t}, & e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, & \boldsymbol{e}_\theta &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t}, \\ \boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_{\varepsilon^0}} = \boldsymbol{N}, & \boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, & \boldsymbol{e}_\gamma &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\gamma} = \boldsymbol{q} \end{aligned} \tag{4.52}$$

675 The final pH formulation is found as usual considering the dynamics (4.50) and fact that
676 higher derivatives commute

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\alpha}_u \\ \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_{\varepsilon^0} \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathrm{Div} & \boldsymbol{0} & \boldsymbol{0} \\ 0 & 0 & 0 & 0 & 0 & \mathrm{div} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_u \\ e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_{\varepsilon^0} \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix}. \tag{4.53}$$

677 The coupling between the membrane and bending part is clear when considering the link
678 between energy and co-energy variables

$$\begin{pmatrix} \boldsymbol{e}_u \\ e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_{\varepsilon^0} \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix} = \begin{bmatrix} \frac{1}{\rho h}\boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ 0 & \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \frac{12}{\rho h^3}\boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{D}}_m & \boldsymbol{\mathcal{D}}_c & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{D}}_c & \boldsymbol{\mathcal{D}}_b & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{D}}_s \end{bmatrix} \begin{pmatrix} \boldsymbol{\alpha}_u \\ \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_{\varepsilon^0} \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma \end{pmatrix} \tag{4.54}$$

679 Again appropriate boundary variables and a suitable Stokes-Dirac structure can be found for
680 this model. The final formulation is just a superposition of systems (3.35) and (4.29).

681 4.3.2 Port-Hamiltonian Kirchhoff laminated plate

According to the Kirchhoff hypotheses the kinetic and deformation energies reduce to

$$E_{\text{kin}} = \frac{1}{2}\int_\Omega \left\{ \rho h \left\|\frac{\partial \boldsymbol{u}^0}{\partial t}\right\|^2 + \rho h \left(\frac{\partial u_z}{\partial t}\right)^2 \right\} \mathrm{d}\Omega,$$
$$E_{\text{def}} = \frac{1}{2}\int_\Omega \left\{ \boldsymbol{N} : \boldsymbol{\varepsilon}^0 + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \mathrm{d}\Omega,$$

where $\boldsymbol{\kappa}$ is defined in Eq. (4.5). Furthermore, as stated in Remark 5, the rotational contribution in the kinetic energy has been neglected. The equations of motion are (see [Red03, Chapter 3] for an exhaustive explanation)

$$\begin{aligned} \frac{\partial^2 \boldsymbol{u}^0}{\partial t^2} &= \operatorname{Div} \boldsymbol{N}, \\ \frac{\partial^2 u_z}{\partial t^2} &= -\operatorname{div} \operatorname{Div} \boldsymbol{M}, \end{aligned} \tag{4.55}$$

where $\boldsymbol{N}$, $\boldsymbol{M}$ are defined in Eqs. (4.47). To get a port-Hamiltonian formulation, the following energy variable are chosen

$$\begin{aligned} \boldsymbol{\alpha}_u &= \rho h \frac{\partial \boldsymbol{u}^0}{\partial t}, & \alpha_w &= \rho h \frac{\partial \boldsymbol{u}_z}{\partial t}, \\ \boldsymbol{A}_{\varepsilon^0} &= \boldsymbol{\varepsilon}^0, & \boldsymbol{A}_\kappa &= \boldsymbol{\kappa}. \end{aligned} \tag{4.56}$$

The total energy $H = E_{\text{kin}} + E_{\text{def}}$ is now a quadratic function of the energy variables

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho h} \left\| \frac{\partial \boldsymbol{\alpha}_u}{\partial t} \right\|^2 + \frac{1}{\rho h} \left(\frac{\partial \alpha_w}{\partial t} \right)^2 \right\} \mathrm{d}\Omega, \\ E_{\text{def}} &= \frac{1}{2} \int_\Omega \left\{ (\boldsymbol{\mathcal{D}}_m \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{\mathcal{D}}_c \boldsymbol{A}_\kappa) : \boldsymbol{A}_{\varepsilon^0} + (\boldsymbol{\mathcal{D}}_c \boldsymbol{A}_{\varepsilon^0} + \boldsymbol{\mathcal{D}}_b \boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa \right\} \mathrm{d}\Omega, \end{aligned}$$

The co-energies are equal to

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_u} = \frac{\partial \boldsymbol{u}^0}{\partial t}, & e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \\ \boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_{\varepsilon^0}} = \boldsymbol{N}, & \boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, \end{aligned} \tag{4.57}$$

The final pH formulation is found as usual considering the dynamics (4.55) and fact that higher derivatives commute

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_u \\ \alpha_w \\ \boldsymbol{A}_{\varepsilon^0} \\ \boldsymbol{A}_\kappa \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \operatorname{Div} & \boldsymbol{0} \\ 0 & 0 & 0 & -\operatorname{div} \circ \operatorname{Div} \\ \operatorname{Grad} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \operatorname{Grad} \circ \operatorname{grad} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_u \\ e_w \\ \boldsymbol{E}_{\varepsilon^0} \\ \boldsymbol{E}_\kappa \end{pmatrix}. \tag{4.58}$$

Again, the coupling appears when considering the link between energy and co-energy variables

$$\begin{pmatrix} \boldsymbol{e}_u \\ e_w \\ \boldsymbol{E}_{\varepsilon^0} \\ \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} \frac{1}{\rho h} \boldsymbol{I}_{2\times 2} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ 0 & \frac{1}{\rho h} & 0 & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{D}}_m & \boldsymbol{\mathcal{D}}_c \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\mathcal{D}}_c & \boldsymbol{\mathcal{D}}_b \end{bmatrix} \begin{pmatrix} \boldsymbol{\alpha}_u \\ \alpha_w \\ \boldsymbol{A}_{\varepsilon^0} \\ \boldsymbol{A}_\kappa \end{pmatrix} \tag{4.59}$$

The energy rate provides the appropriate boundary conditions from which one can construct the Stokes-Dirac structure. The necessary computations are not performed here as the final result is just a juxtaposition of systems (3.35), (4.42).

4.4 Conclusion

In this chapter, a pH formulation for the most commonly used plate models has been detailed. Many open questions remain. In particular, how to generalize the results to shell problems, for which the domain is a surface embedded in the three dimensional space (a manifold). Computations get more involved in this case since the usage of differential geometry concepts is unavoidable. These models are important since they are widely used in the aerospace industry and ubiquitous in nature.

The reformulation of plate models using the language of differential geometry is another open research topic. Indeed, while for the Mindlin plate it should be possible to use vector-valued forms to obtain an equivalent system, for the Kirchhoff plate the task appears more involved. An interesting reference that can be provide some ideas in this direction is [Yao11].

Chapter 5

Thermoelasticity in port-Hamiltonian form

Eh bien, mon ami, la terre sera un jour ce cadavre refroidi. Elle deviendra inhabitable et sera inhabitée comme la lune, qui depuis longtemps a perdu sa chaleur vitale.

Vingt mille lieues sous les mers
Jules Verne

Contents

[Duh37]

5.1 Linear coupled thermoelasticity

5.2 Thermoelastic Euler-Bernoulli beam

5.3 Thermoelastic Kirchhoff plate

Part III

Finite element structure preserving discretization

Chapter 6

Partitioned finite element method

6.1 General procedure

6.1.1 Non-linear case

6.1.2 Linear case

6.1.3 Examples

6.2 Connection with mixed finite elements

6.3 Inhomogeneous boundary conditions

6.3.1 Solution using Lagrange multipliers

6.3.2 Virtual domain decomposition

Chapter 7

Convergence numerical study

7.1 Plate problems using known mixed finite elements

7.2 Non-standard discretization of flexible structures

Chapter 8

Numerical applications

8.1 Boundary stabilization

8.2 Thermoelastic wave propagation

8.3 Mixed boundary conditions

8.3.1 Trajectory tracking of a thin beam

8.3.2 Vibroacoustic under mixed boundary conditions

8.4 Modal analysis of plates

Part IV

Port-Hamiltonian flexible multibody dynamics

Chapter 9

Modular multibody systems in port-Hamiltonian form

9.1 Reminder of the rigid case

9.2 Flexible floating body

9.3 Modular construction of multibody systems

Chapter 10

Validation

10.1 Beam systems

10.1.1 Modal analysis of a flexible mechanism

10.1.2 Non-linear crank slider

10.1.3 Hinged beam

10.2 Plate systems

10.2.1 Boundary interconnection with a rigid element

10.2.2 Actuated plate

Conclusion

Conclusions and future directions

Appendix A

Mathematical tools

A.1 Differential operators

The space of all, symmetric and skew-symmetric $d \times d$ matrices are denoted by $\mathbb{M}, \mathbb{S}, \mathbb{K}$ respectively. The space of $\mathbb{R}^d$ vectors is denoted by $\mathbb{V}$. $\Omega \subset \mathbb{R}^d$ is an open connected set. For a scalar field $u : \Omega \to \mathbb{R}$ the gradient is defined as

$$\operatorname{grad}(u) = \nabla u := \left(\partial_{x_1} u \dots \partial_{x_d} u\right)^\top.$$

For a vector field $\boldsymbol{u} : \Omega \to \mathbb{V}$, with components u_i, the gradient (Jacobian) is defined as

$$\operatorname{grad}(\boldsymbol{u})_{ij} := (\nabla \boldsymbol{u})_{ij} = \partial_{x_j} u_i.$$

The symmetric part of the gradient operator Grad (i. e. the deformation gradient in continuum mechanics) is thus given by

$$\operatorname{Grad}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\right).$$

The Hessian operator of u is then computed as follows

$$\operatorname{Hess}(u) = \nabla^2 u = \operatorname{Grad}(\operatorname{grad}(u)),$$

For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{M}$, with components u_{ij}, the divergence is a vector, defined column-wise as

$$\operatorname{Div}(\boldsymbol{U}) = \nabla \cdot \boldsymbol{U} := \left(\sum_{i=1}^{d} \partial_{x_i} u_{ij}\right)_{j=1,\dots,d}.$$

The double divergence of a tensor field $\boldsymbol{U}$ is then a scalar field defined as

$$\operatorname{div}(\operatorname{Div}(\boldsymbol{U})) := \sum_{i,j=1}^{d} \partial_{x_i} \partial_{x_j} u_{ij}.$$

Definition 6 (Formal adjoint, Def. 5.80 [RR04])
Consider the differential operator defined on Ω

$$\mathcal{L}(\boldsymbol{x}, \partial) = \sum_{|\alpha| \le k} a_\alpha(\boldsymbol{x}) \partial^\alpha, \tag{A.1}$$

where $\alpha := (\alpha_1, \ldots, \alpha_d)$ *is a multi-index of order* $|\alpha| := \sum_{i=1}^{d} \alpha_i$, a_α *are a set of real scalars and* $\partial^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$ *is a differential operator of order* $|\alpha|$ *resulting from a combination of spatial derivatives. The formal adjoint of* $\mathcal{L}$ *is the operator defined by*

$$\mathcal{L}^*(\boldsymbol{x}, \partial)u = \sum_{|\alpha| \leq k} (-1)^\alpha \partial^\alpha (a_\alpha(\boldsymbol{x}) u(\boldsymbol{x})). \tag{A.2}$$

The importance of this definition lies in the fact that

$$\langle \phi, \mathcal{L}(\boldsymbol{x}, \partial)\psi \rangle \, \Omega = \langle \mathcal{L}^*(\boldsymbol{x}, \partial)\phi, \psi \rangle \, \Omega \tag{A.3}$$

for every $\phi, \psi \in C_0^\infty(\Omega)$. If the assumption of compact support is removed, then (A.3) no longer holds; instead the integration by parts yields additional terms involving integrals over the boundary $\partial\Omega$. However, these boundary terms vanish if ϕ and ψ satisfy certain restrictions on the boundary.

Appendix B

Finite elements gallery

Appendix C

Implementation using FEniCS and Firedrake

Bibliography

[Abe12] R. Abeyaratne. *Lecture Notes on the Mechanics of Elastic Solids. Volume II: Continuum Mechanics*. Cambridge, MA and Singapore, 1st edition, 2012.

[BBF+13] Daniele Boffi, Franco Brezzi, Michel Fortin, et al. *Mixed finite element methods and applications*, volume 44. Springer, 2013.

[Bre08] F. Brezzi. *Mixed finite elements, compatibility conditions, and applications*. Springer, 2008.

[Cia88] P. G. Ciarlet. *Mathematical Elasticity: Three-Dimensional Elasticity*. Studies in mathematics and its applications. North-Holland, 1988.

[Cou90] T.J. Courant. Dirac manifolds. *Transactions of the American Mathematical Society*, 319(2):631–661, 1990.

[CR16] F.L. Cardoso Ribeiro. *Port-Hamiltonian modeling and control of fluid-structure system*. PhD thesis, Université de Toulouse, Dec. 2016.

[CRMPB17] F. L. Cardoso-Ribeiro, D. Matignon, and V. Pommier-Budinger. A port-Hamiltonian model of liquid sloshing in moving containers and application to a fluid-structure system. *Journal of Fluids and Structures*, 69:402–427, February 2017.

[DHNLS99] R. Durán, L. Hervella-Nieto, E. Liberman, and J. Solomin. Approximation of the vibration modes of a plate by Reissner-Mindlin equations. *Mathematics of Computation of the American Mathematical Society*, 68(228):1447–1463, 1999.

[Duh37] J.M.C. Duhamel. Second mémoire sur les phénomeènes thermomécaniques. *Journal de l'École Polytechnique*, 15, cahier 25:1–57, 1837.

[Gri15] M. Grinfeld. *Mathematical Tools for Physicists*. John Wiley & Sons Inc, 2nd edition, jan 2015.

[GSV18] T. Gustafsson, R. Stenberg, and J. Videman. A posteriori estimates for conforming kirchhoff plate elements. *SIAM Journal on Scientific Computing*, 40(3):A1386–A1407, 2018.

[HM78] T. J.R. Hughes and J.E. Marsden. Classical elastodynamics as a linear symmetric hyperbolic system. *Journal of Elasticity*, 8(1):97–110, 1978.

[JZ12] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Number 223 in Operator Theory: Advances and Applications. Springer Verlag, Germany, 2012. https://doi.org/10.1007/978-3-0348-0399-1.

[Kot19] P. Kotyczka. *Numerical Methods for Distributed Parameter Port-Hamiltonian Systems*. TUM University Press, 2019.

[KZ15] M. Kurula and H. Zwart. Linear wave systems on n-d spatial domains. *International Journal of Control*, 88(5):1063–1077, 2015. https://www.tandfonline.com/doi/abs/10.1080/00207179.2014.993337.

[Lee12] J. Lee. *Mixed methods with weak symmetry for time dependent problems of elasticity and viscoelasticity*. PhD thesis, University of Minnesota, 2012.

[LGZM05] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and Boundary Control Systems associated with Skew-Symmetric Differential Operators. *SIAM Journal on Control and Optimization*, 44(5):1864–1892, 2005. https://doi.org/10.1137/040611677.

[LPKL12] L. D. Landau, L. P. Pitaevskii, A. M. Kosevich, and E. M. Lifshitz. *Theory of Elasticity*. Butterworth Heinemann, third edition, Dec 2012.

[Min51] R. D. Mindlin. Influence of rotatory inertia and shear on flexural motions of isotropic elastic Plates. *Journal of Applied Mechanics*, 18:31–38, March 1951.

[MMB05] A. Macchelli, C. Melchiorri, and L. Bassi. Port-based modelling and control of the Mindlin plate. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 5989–5994, Dec. 2005. https://doi.org/10.1109/CDC.2005.1583120.

[MvdSM05] A. Macchelli, A. J. van der Schaft, and C. Melchiorri. Port Hamiltonian formulation of infinite dimensional systems I. Modeling. In *Proceedings of the 44th IEEE Conference on Decision and Control*, volume 4, pages 3762–3767. IEEE, Jan. 2005.

[Olv93] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate texts in mathematics*. Springer-Verlag New York, 2 edition, 1993.

[PZ18] Dirk Pauly and Walter Zulehner. The divdiv-complex and applications to biharmonic equations. *Applicable Analysis*, 0(0):1–52, 2018.

[PZ20] D. Pauly and W. Zulehner. The elasticity complex, 2020. arXiv preprint arXiv:2001.11007.

[Red03] J. N. Reddy. *Mechanics of laminated composite plates and shells: theory and analysis*. CRC press, 2003.

[Red06] J. N. Reddy. *Theory and analysis of elastic plates and shells*. CRC press, 2006.

[Rei47] Eric Reissner. On bending of elastic plates. *Quarterly of Applied Mathematics*, 5(1):55–68, 1947.

[RR04] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*. Number 13 in Texts in Applied Mathematics. Springer-Verlag New York, 2 edition, 2004.

[RZ18] K. Rafetseder and W. Zulehner. A decomposition result for kirchhoff plate bending problems and a new discretization approach. *SIAM Journal on Numerical Analysis*, 56(3):1961–1986, 2018.

[Skr19] N. Skrepek. Well-posedness of linear first order port-Hamiltonian systems on multidimensional spatial domains, 2019. arXiv preprint arXiv:1910.09847.

[SS17] M. Schöberl and K. Schlacher. Variational Principles for Different Representations of Lagrangian and Hamiltonian Systems. In Hans Irschik, Alexander Belyaev, and Michael Krommer, editors, *Dynamics and Control of Advanced Structures and Machines*, pages 65–73. Springer International Publishing, 2017.

[TWK59] S. Timoshenko and S. Woinowsky-Krieger. *Theory of plates and shells*. Engineering societies monographs. McGraw-Hill, 1959.

[Vil07] J.A. Villegas. *A Port-Hamiltonian Approach to Distributed Parameter Systems*. PhD thesis, University of Twente, May 2007.

[Yao11] P.F. Yao. *Modeling and Control in Vibrational and Structural Dynamics: A Differential Geometric Approach*. Chapman & Hall/CRC Applied Mathematics & Nonlinear Science. Taylor & Francis, 2011.

Résumé — Malgré l'abondante littérature sur le formalisme pH, les problèmes d'élasticité en deux ou trois dimensions géométriques n'ont presque jamais été considérés. Cette thèse vise à étendre l'approche port-Hamiltonienne (pH) à la mécanique des milieux continus. L'originalité apportée réside dans trois contributions majeures. Tout d'abord, la nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d'élasticité nécessite l'utilisation d'éléments finis non standard. Néanmoins, l'implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

Mots clés : Systèmes port-Hamiltonien, méchanique des solides, discretisation symplectique, méthode des éléments finis, dynamique multicorps

---

Abstract — Despite the large literature on pH formalism, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an intrinsic, i.e. coordinate free, and equivalent pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

Keywords: Port-Hamiltonian systems, continuum mechanics, structure preserving discretization, finite element method, multibody dynamics.
