

# Dissipative Dynamical Systems

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# Outline

Introduction

Definition and characterization of dissipativity

Stability of dissipative systems

Interconnections of dissipative systems

Conclusions

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## Introduction

Definition and characterization of dissipativity

Stability of dissipative systems

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# Why dissipative dynamical systems?

All engineering systems exhibit dissipation.

- ▶ Electrical networks with resistors;
- ▶ Mechanical systems (viscoelastic or Coulomb friction);
- ▶ Thermodynamic systems: dissipation leads to an increase in entropy.

The notion of dissipativity establishes a natural link between the properties of input-output and state-space models. Many modern computational tools for the analysis and synthesis of control systems are based on it.

Jan C. Willems. "Dissipative dynamical systems Part I: General theory". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 321–351

Jan C. Willems. "Dissipative dynamical systems Part II: Linear systems with quadratic supply rates". In: *Archive for Rational Mechanics and Analysis* 45.5 (1972), pp. 352–393

Arjan van der Schaft. *L2-gain and passivity in nonlinear control*. Springer-Verlag, 1999

## Some mathematical notation

$\mathbb{R}_+ = [0, \infty)$  denotes the set of positive reals.

$\mathbb{R}_+^2 := \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 \geq t_1\}$  (causal triangular sector of  $\mathbb{R}^2$ ).

Let  $V$  be a finite dimensional normed linear space with norm  $\|\cdot\|_V$ .

(If  $V = \mathbb{R}^n$  then the Euclidean norm is denoted by  $\|x\|_2 = \sqrt{x^\top x}$ )

### Definition (Local $L_{\text{loc}}^p$ Banach spaces)

For each positive integer  $p \in 1, 2, \dots$ , the set  $L_{\text{loc}}^p(\mathbb{R}, V)$  consists of all functions  $f : \mathbb{R} \rightarrow V$ , which are measurable and satisfy

$$\int_a^b \|f(t)\|_V^p dt < \infty, \quad \forall a, b \in \mathbb{R}.$$

The case  $p = \infty$  consists of all bounded measurable functions on compact intervals, i.e.  $\sup_{t \in [a, b]} \|f(t)\|_V < \infty$ .

## General setting

Consider the state-space system with inputs and outputs

$$\Sigma : \quad \begin{aligned} \dot{x} &= f(x, u), & u(t) &\in U, \\ y &= h(x, u), & y(t) &\in Y, \end{aligned}$$

where  $x(t) \in \mathcal{X}$ . In general  $\mathcal{X}$  is a manifold and  $U, Y$  vector spaces.

For sake simplicity, assume  $\mathcal{X} \subseteq \mathbb{R}^n$ ,  $U = \mathbb{R}^m$ ,  $Y = \mathbb{R}^p$ .

### Theorem

*Suppose  $f, h$  to be Lipschitz continuous in  $x$  and  $u$  jointly.*

*Then system  $\Sigma$  has a unique solution  $\forall x(t_0) \in \mathcal{X}$ ,  $u(\cdot) \in L^2_{loc}(\mathbb{R}, U)$  with  $x(\cdot) \in L^2_{loc}(\mathbb{R}, \mathcal{X})$ ,  $y(\cdot) \in L^2_{loc}(\mathbb{R}, Y)$ .*

# Reachability and controllability

## Definition (State transition function)

Given the system  $\Sigma$ , the state transition function  $\phi$  is the map

$$\phi(t_1, t_0, x(t_0), u(\cdot)) : \mathbb{R}_+^2 \times \mathcal{X} \times L_{\text{loc}}^2(\mathbb{R}, U) \rightarrow \mathbb{R}^n$$

such that  $x(t_1) = \phi(t_1, t_0, x(t_0), u(\cdot))$ .

## Definition (Reachability and controllability)

The state space  $\mathcal{X}$  of system  $\Sigma$  is said to be **reachable** from  $x_{-1}$  if

$$\forall x \in \mathcal{X}, \exists t_{-1} \leq 0, \exists u(\cdot) \in L_{\text{loc}}^2(\mathbb{R}, U) \text{ such that } x = \phi(0, t_{-1}, x_{-1}, u(\cdot)).$$

It is said to be **controllable** to  $x_1$  if

$$\forall x \in \mathcal{X}, \exists t_1 > 0, \exists u(\cdot) \in L_{\text{loc}}^2(\mathbb{R}, U) \text{ such that } x_1 = \phi(t_1, 0, x, u(\cdot)).$$

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# The mathematical definition of dissipativity

On the combined space  $U \times Y$  consider the supply rate function  $s : U \times Y \rightarrow \mathbb{R}$ .

## Definition (Dissipative state space system)

A state space system  $\Sigma$  is said to be dissipative w.r.t. the supply rate  $s$  if there exists a function  $S : \mathcal{X} \rightarrow \mathbb{R}_+$  (the storage function), such that  $\forall x(t_0) \in \mathcal{X}$  at any time  $t_0$ ,  $\forall u(\cdot)$  and  $\forall t_1 \geq t_0$ , the following inequality holds

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t)) \, dt, \quad \text{Dissipation Inequality.} \quad (1)$$

If equality holds then the system is called conservative (w.r.t. the supply rate  $s$ ).

## Corollary (Convexity of the storage functions set)

*Given two storage functions  $S_1$  and  $S_2$  then any convex combination  $\alpha S_1 + (1 - \alpha) S_2$ ,  $\alpha = [0, 1]$  is also a storage function.*

## Passive systems and $L^2$ finite gain

Two important class of supply rate functions:

- ▶ passive systems  $s(u, y) = u^\top y$ ;
- ▶ finite  $L^2$  gain  $s(u, y) = \frac{1}{2}\gamma\|u\|_2^2 - \frac{1}{2}\|y\|_2^2, \quad \gamma \geq 0$ .

### Definition (Passive system)

$\Sigma$  with  $U = Y = \mathbb{R}^m$  is **passive** if it is dissipative w.r.t.

$$s(u, y) = u^\top y.$$

$\Sigma$  is **input strictly passive** if  $\exists \delta > 0$  such that  $\Sigma$  is dissipative w.r.t.

$$s(u, y) = u^\top y - \delta\|u\|_2^2.$$

$\Sigma$  is **output strictly passive** if  $\exists \varepsilon > 0$  such that  $\Sigma$  is dissipative w.r.t.

$$s(u, y) = u^\top y - \varepsilon\|y\|_2^2$$

$\Sigma$  is **lossless** if it is conservative with respect to  $s(u, y) = u^\top y$ .

## Passive systems and $L^2$ finite gain

Two important class of supply rate functions:

- ▶ passive systems  $s(u, y) = u^\top y$ ;
- ▶ finite  $L^2$  gain  $s(u, y) = \frac{1}{2}\gamma\|u\|_2^2 - \frac{1}{2}\|y\|_2^2, \quad \gamma \geq 0$ .

### Definition ( $L^2$ finite gain)

A system  $\Sigma$  with  $U = \mathbb{R}^m$ ,  $Y = \mathbb{R}^p$  has  $L^2$ -gain  $\leq \gamma$  ( $\gamma \geq 0$ ) if it is dissipative w.r.t.

$$s(u, y) = \frac{1}{2}\gamma\|u\|_2^2 - \frac{1}{2}\|y\|_2^2.$$

The  $L^2$ -gain of  $\Sigma$  is defined as

$$\gamma(\Sigma) := \inf\{\gamma \mid \Sigma \text{ has } L^2\text{-gain} \leq \gamma\}.$$

$\Sigma$  is said to have  $L^2$ -gain  $< \gamma$  if  $\exists \tilde{\gamma} \leq \gamma$  such that  $\Sigma$  has  $L^2$ -gain  $\leq \tilde{\gamma}$ .

$\Sigma$  is called inner if it is conservative with respect to  $s(u, y) = \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\|y\|_2^2$ .

## How to establish dissipativity? The available storage

### Theorem (Necessary and sufficient conditions for dissipativity)

Consider system  $\Sigma$  and supply rate  $s(u, y)$ .  $\Sigma$  is dissipative with respect to  $s$  iff

$$S_a(x) := \sup_{\substack{u(\cdot) \\ T \geq 0}} - \int_0^T s(u(t), y(t)) \, dt, \quad x(0) = x, \quad (2)$$

is finite  $\forall x \in \mathcal{X}$ .

Furthermore, if  $S_a$  is finite  $\forall x \in \mathcal{X}$  then  $S_a$  is a storage function, called the **available storage**, and all other possible storage functions  $S$  satisfy

$$S_a(x) \leq S(x) - \inf_x S(x), \quad \forall x \in \mathcal{X}$$

Moreover  $\inf_x S_a(x) = 0$ .

The available storage is the minimal storage function.

## Proof

- (If) Suppose  $S_a$  is finite. Then  $S_a \geq 0$  (sup of a set that contains 0). Compare  $S(x(t_0))$  and  $S(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t)) \, dt$  with  $s(u, y)$  evaluated on a trajectory generated by  $u : [t_0, t_1] \rightarrow \mathbb{R}^m$  that drives  $x(t_0)$  at  $t_0$  to  $x(t_1)$  at  $t_1$ .

Since  $S_a$  is the supremum over all  $u(\cdot)$  it follows

$$S_a(x(t_0)) \geq S_a(x(t_1)) - \int_{t_0}^{t_1} s(u(t), y(t)) \, dt \implies S_a \text{ is a storage function.}$$

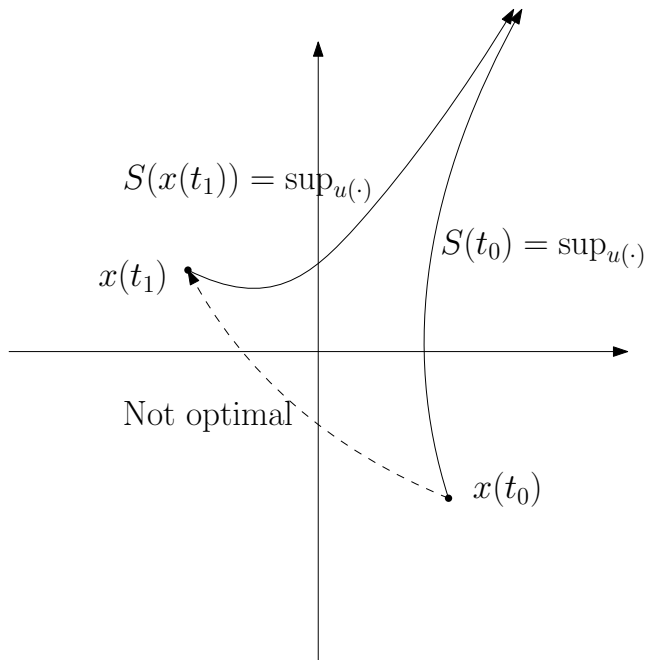
- (Only if) Suppose  $\Sigma$  dissipative. Then  $\exists S \geq 0$  such that  $\forall u(\cdot)$

$$S(x(0)) + \int_0^T s(u(t), y(t)) \, dt \geq S(x(T)) \geq 0.$$

This implies that

$$S(x(0)) \geq \sup_{\substack{u(\cdot) \\ T \geq 0}} - \int_0^T s(u(t), y(t)) \, dt = S_a(x(0)) \implies S_a(x(0)) < \infty$$

Then  $S' = S - \inf_x S(x)$  satisfy the dissipation inequality so  $S'(x) \geq S_a(x), \forall x$  and  $\inf_x S'(x) = 0$  (and hence  $\inf S_a(x) = 0$ ).



## Reachability and Storage functions

If the system is reachable from  $x^*$ , the finiteness of  $S_a$  needs to be checked only in  $x^*$

### Proposition

*Assume that  $\Sigma$  is reachable from  $x^* \in \mathcal{X}$ . Then  $\Sigma$  is dissipative iff  $S_a(x^*) < \infty$ .*

### Proof

(If) Suppose there exists  $x \in \mathcal{X}$  such that  $S_a(x) = \infty$ . Since by reachability  $x$  can be reached from  $x^*$  in finite time, this would imply (by time invariance) that also  $S_a(x^*) = \infty$ .

## The maximal storage: the required supply

If  $\Sigma$  is reachable from  $x^*$ , there exists another canonically defined storage function.

### Theorem

*Assume that  $\Sigma$  is reachable from  $x^* \in \mathcal{X}$ .*

*Define the required supply (from  $x^*$ )  $S_r : \mathcal{X} \rightarrow \mathbb{R} \cup \{-\infty\}$  as*

$$S_r(x) := \inf_{\substack{u(\cdot) \\ T \geq 0}} \int_{-T}^0 s(u(t), y(t)) \, dt, \quad x(-T) = x^*, \quad x(0) = x. \quad (3)$$

*Then the following holds:*

1.  $S_r$  satisfies the dissipation inequality.
2.  $\Sigma$  is dissipative iff  $\exists K > -\infty$  such that  $S_r(x) \geq K, \forall x \in \mathcal{X}$ .
3. If  $S$  is a storage function for  $\Sigma$ , then

$$S(x) \leq S_r(x) + S(x^*), \quad x \in \mathcal{X},$$

*and  $S_r(x) + S(x^*)$  is itself a storage function (and in particular  $S_r(x) + S_a(x^*)$ ).*



## Proof

1. To steer the system from  $x^*$  at  $-T$  to  $x(t_1)$  consider  $u(\cdot) : [-T, t_1] \rightarrow U$  which first take  $x^*$  to  $x(t_0)$  at time  $t_0 \leq t_1$ , and then equal to a given input  $u(\cdot) : [t_0, t_1] \rightarrow U$  transferring  $x(t_0)$  to  $x(t_1)$ . This is a suboptimal policy, so

$$S_r(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t)) \, dt \geq S_r(x(t_1)).$$

2. For the second claim, by definition of  $S_a$  and  $S_r$

$$S_a(x^*) = \sup_x -S_r(x),$$

then  $\Sigma$  is dissipative iff  $\exists K > -\infty$  such that  $S_r(x) \geq -K$ ,  $\forall x$ .

3. Let  $S$  satisfy the dissipation inequality.

Then for any  $u(\cdot) : [-T, 0] \rightarrow U$  such that  $x(-T) = x^*$  to  $x(0) = x$  it holds

$$S(x) - S(x^*) \leq \int_{-T}^0 s(u(t), y(t)) \, dt.$$

Taking the infimum on the right-hand side over all  $u(\cdot)$  proves the claim.

If  $S \geq 0$ , then  $S_r + S(x^*) \geq 0$  is a storage function.

# The a priori bounds

## The available storage

It is the amount of internal storage which may be recovered from the system.

## The required supply

It is the amount of supply which has to be delivered to the system in order to transfer it from a state of minimum storage to a given state.

## Alternative definition of dissipativity

If  $\Sigma$  is dissipative with a storage function  $S$  for which  $x^* = \arg \min_x S(x)$ , then also  $S - S(x^*)$  is a storage function, which is zero at  $x^*$ . Motions starting from  $x^*$  verify

$$\int_0^T s(u(t), y(t)) \geq 0, \quad x(0) = x^*, \quad \forall T \geq 0. \quad (4)$$

### Definition (Dissipativity from $x^*$ )

A system  $\Sigma$  with supply rate  $s$  is called dissipative from  $x^*$  if (4) holds.

### Proposition

*A dissipative system  $\Sigma$  is dissipative from  $x^0$  iff its storage function satisfies  $S(x^*) = 0$ . If additionally the system is reachable from  $x^*$  then the system is dissipative and its required supply satisfies  $S_r(x^*) = 0$ .*

**Proof** (Only if) Assume  $\Sigma$  is dissipative from  $x^*$ . By definition of  $S_a$  it holds  $S_a(x^*) = 0$ . If  $\Sigma$  is reachable from  $x^*$  then by the previous proposition the system is dissipative, and  $S_r(x^*) = 0$ .

## Theorem

Let  $\Sigma$  be dissipative and dissipative from  $x^*$ . Suppose that  $s$  is such that

$$\exists u(x) \text{ such that } s(u(x), h(x, u(x))) \leq 0, \quad x \in \mathcal{X}. \quad (5)$$

for which  $x^*$  is a globally asymptotically equilibrium for the closed-loop system  $\dot{x} = f(x, u(x))$ . Then any storage function  $S$  attains its minimum at  $x^*$  and

$$S_a(x) \leq S(x) - S(x^*), \quad \forall x \in \mathcal{X}.$$

**Proof** Consider the dissipation inequality for any  $S$ , rewritten as

$$-\int_0^T s(u(t), y(t)) \, dt \leq S(x) - S(x(T)), \quad x(0) = x.$$

Extend  $u(\cdot) : [0, T] \rightarrow U$  to the infinite time interval  $[0, \infty)$  by considering on  $(T, \infty)$  a feedback  $u(x)$  verifying (5) such that  $x^*$  is a globally asymptotical equilibrium. Since  $s(u(x), h(x, u(x))) \leq 0$  and convergence of  $x(t)$  to  $x^*$  for  $t \rightarrow \infty$  that

$$-\int_0^T s(u(t), y(t)) \, dt \leq S(x) - S(x^*)$$

Taking the supremum at the left-hand side for  $u(\cdot) : [0, T] \rightarrow U$  concludes the proof.

## Corollary

*Consider a system  $\Sigma$  that is dissipative and reachable from  $x^*$ , and for which  $s$  verifies (5), such that  $x^*$  is a global asymptotical equilibrium for  $\dot{x} = f(x, u(x))$ .*

*Then any storage function  $S$  attains its minimum at  $x^*$  and the storage function  $S'(x) := S(x) - S(x^*)$  satisfies*

$$S_a(x) \leq S'(x) \leq S_r(x), \quad \forall x \in \mathcal{X},$$

*where  $S_a(x^*) = S_r(x^*) = 0$ .*

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## Reminder on Lyapunov stability

Consider  $\dot{x} = f(x)$ ,  $x \in \mathcal{X}$  with  $f$  locally Lipschitz continuous.

Denote  $x(t; x_0)$  the solution for  $x(0) = x_0$  with  $t \in [0, T(x_0))$  and  $T(x_0) > 0$  maximal.

### Definition (Stability)

Let  $x^*$  be an equilibrium  $f(x^*) = 0$ , and thus  $x(t; x^*) = x^*$ ,  $\forall t$ . The equilibrium  $x^*$  is

1. **stable**, if for each  $\varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0$  such that

$$\|x_0 - x^*\| \leq \delta(\varepsilon) \implies \|x(t; x_0) - x^*\| < \varepsilon, \quad \forall t \geq 0.$$

2. **asymptotically stable**, if it is stable and additionally there exists  $\hat{\delta}$  such that

$$\|x_0 - x^*\| \leq \hat{\delta} \implies \lim_{t \rightarrow \infty} x(t; x_0) = x^*$$

3. **globally asymptotically stable**, if it is stable and

$$\lim_{t \rightarrow \infty} x(t; x_0) = x^*, \quad \forall x_0 \in \mathcal{X}.$$

4. **unstable**, if it is not stable.

## Reminder on Lyapunov stability

### Definition (Lyapunov Functions)

Let  $x^*$  be an equilibrium of  $\dot{x} = f(x)$ . A  $C^1$  function  $V : \mathcal{X} \rightarrow \mathbb{R}_+$  satisfying

$$V(x^*) = 0, \quad V(x) > 0, \quad x \neq x^*,$$

that is  $V$  is positive definite at  $x^*$ , and

$$\dot{V}(x) := \nabla V(x) \cdot f(x) \leq 0, \quad x \in \mathcal{X},$$

is called a Lyapunov function for the equilibrium  $x^*$

### Theorem

*Let  $x^*$  be an equilibrium. If there exists a Lyapunov function  $V$  for the equilibrium  $x^*$ , then  $x^*$  is a stable equilibrium. If moreover*

$$\dot{V}(x) < 0, \quad \forall x \in \mathcal{X}, \quad x \neq x^*,$$

*then  $x^*$  is an asymptotically stable equilibrium, which is globally asymptotically stable if  $V$  is proper (that is, the sets  $\{x \in \mathcal{X} \mid 0 \leq V(x) \leq c\}$  are compact for every  $c \in \mathbb{R}_+$ , equivalent to  $V$  is radially unbounded if  $\mathcal{X} = \mathbb{R}^n$ ).*



## First stability result

Assume  $S(x) \in C^1(\mathcal{X}, \mathbb{R}_+)$ . Then it holds

$$\nabla S(x) \cdot f(x, u) \leq s(u, h(x, u)), \quad \forall x, u.$$

### Proposition

*Let  $s(u, y)$  be a supply rate, and  $S : \mathcal{X} \rightarrow \mathbb{R}_+$  be a  $C^1$  storage function for  $\Sigma$ . Assume that  $s$  satisfies*

$$s(0, y) \leq 0, \quad \forall y \in Y,$$

*Assume that  $x^* \in \mathcal{X}$  is an equilibrium for the unforced system  $\dot{x} = f(x, 0)$ . Then  $x^*$  is a stable equilibrium of the unforced system with Lyapunov function*

*$V(x) := S(x) - S(x^*)$  for  $x$  around  $x^*$ , while  $s(0, h(x^*, 0)) = 0$ . If additionally,  $\dot{S}(x) < 0$ ,  $\forall x \neq x^*$ , then  $x^*$  is an asymptotically stable equilibrium*

**Proof** Since  $\nabla S(x) \cdot f(x, 0) \leq s(0, h(x, 0)) \leq 0$ ,  $S$  is nonincreasing along solutions of  $\dot{x} = f(x, 0)$ . Since  $f(x^*, 0) = 0$ , it holds  $s(0, h(x^*, 0)) = 0$ . The rest follows from Lyapunov stability theorem.

## Refinement via LaSalle

The condition  $\dot{S} < 0$  can be relaxed by using the LaSalle invariance principle.

### Definition (Invariant set)

A set  $\mathcal{N} \subset \mathcal{X}$  is invariant for  $\dot{x} = f(x)$  if  $x(t; x_0) \in \mathcal{N}$ ,  $\forall x_0 \in \mathcal{N}$ ,  $\forall t \in \mathbb{R}$ , and is positively invariant if this holds  $\forall t \geq 0$

### Theorem (LaSalle's invariance principle)

*Let  $V : X \rightarrow \mathbb{R}$  be a  $C^1$  function for which  $\dot{V}(x) := \nabla V(x) \cdot f(x) \leq 0$ ,  $\forall x \in \mathcal{X}$ . Suppose there exists a compact set  $\mathcal{C}$  which is positively invariant for  $\dot{x} = f(x)$ . Then for any  $x_0 \in \mathcal{C}$  the solution  $x(t; x_0)$  converges for  $t \rightarrow \infty$  to the largest subset  $\mathcal{I}$  of  $\mathcal{A} = \{x \in \mathcal{X} \mid \dot{V}(x) = 0\} \cap \mathcal{C}$  that is invariant for  $\dot{x} = f(x)$ .*

# The pendulum example

Dynamics:

$$\dot{x}_1 = x_2,$$

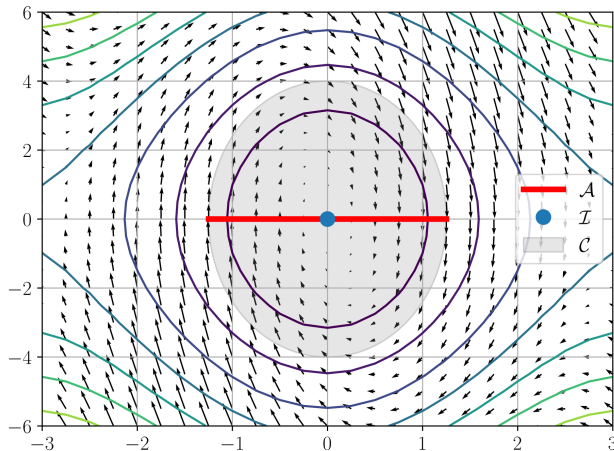
$$\dot{x}_2 = -\frac{g}{l} \sin(x_1) - \frac{r}{ml^2} x_2$$

Sets:

$$\mathcal{C} = \{x \in \mathcal{X} \mid V(x_1, x_2) \leq k\},$$

$$\begin{aligned} \mathcal{A} &= \{x \in \mathcal{X} \mid \dot{V} = 0\} \cap \mathcal{C}, \\ &= \{x_2 = 0\} \cap \mathcal{C}, \end{aligned}$$

$$\mathcal{I} = \{(0, 0)\}$$



$$V(x_1, x_2) = mgl(1 - \cos x_1) + \frac{1}{2} ml^2 x_2^2$$

## Proposition

Let  $S : \mathcal{X} \rightarrow R_+$  be a  $C^1$  storage function for  $\Sigma$ . Assume that  $s$  satisfies

$$s(0, y) \leq 0, \quad \forall y \in Y$$

Assume that  $x^* \in \mathcal{X}$  is a strict local minimum for  $S$ . Assume also that no solution of  $\dot{x} = f(x, 0)$  other than  $x(t) \equiv x^*$  remains in  $\{x \in \mathcal{X} \mid s(0, h(x, 0)) = 0\}$ ,  $\forall t$ . Then  $x^*$  is an asymptotically stable equilibrium of  $\dot{x} = f(x, 0)$ , which is globally asymptotically stable if  $V(x) := S(x) - S(x^*) \geq 0$  is proper.

**Proof**  $\dot{S}(x) = 0 \implies s(0, h(x, 0)) = 0$ . The statement now directly follows from LaSalle's Invariance principle.

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## The open character of dissipativity theory

Consider  $k$  systems  $\Sigma_i$  with input, state, and output spaces  $U_i, \mathcal{X}_i, Y_i, i = 1, \dots, k$ . Suppose  $\Sigma_i$  are dissipative with respect to the supply rates

$$s_i(u_i, y_i), \quad u_i \in U_i, y_i \in Y_i, i = 1, \dots, k,$$

and storage functions  $S_i(x_i), i = 1, \dots, k$ .

Now consider an interconnection of  $\Sigma_i, i = 1, \dots, k$ , defined through

$$I \subset U_1 \times Y_1 \times \dots \times U_k \times Y_k \times U_e \times Y_e,$$

where  $U_e, Y_e$  are spaces of external input and output.

### Proposition

*Suppose the supply rates  $s_1, \dots, s_k$  and the interconnection subset  $I$  are such that  $\exists s_e : U_e \times Y_e \rightarrow \mathbb{R}$  for which*

$$\begin{aligned} s_1(u_1, y_1) + \dots + s_k(u_k, y_k) &\leq s_e(u_e, y_e), \\ \forall ((u_1, y_1), \dots, (u_k, y_k), (u_e, y_e)) &\in I. \end{aligned}$$

*Then the interconnected system  $\Sigma_I$  is dissipative with respect to the supply rate  $s_e$ , with storage function  $S(x_1, \dots, x_k) := S_1(x_1) + \dots + S_k(x_k)$*

# The Lyapunov function of interconnected systems

For simplicity the spaces of external inputs and outputs are removed.

## Proposition

*Suppose the supply rates  $s_1, \dots, s_k$  and the interconnection subset  $I$  are such that there exist positive constants  $\alpha_1, \dots, \alpha_k$  for which*

$$\begin{aligned}\alpha_1 s_1(u_1, y_1) + \dots + \alpha_k s_k(u_k, y_k) &\leq 0, \\ \forall ((u_1, y_1), \dots, (u_k, y_k)) &\in I.\end{aligned}\tag{6}$$

*Then the function*

$$S_\alpha(x_1, \dots, x_k) := \alpha_1 S_1(x_1) + \dots + \alpha_k S_k(x_k)$$

*satisfies  $\dot{S}_\alpha \leq 0$  along all solutions of the interconnected system  $\Sigma_I$ .*

**Proof** It suffices to multiply each dissipation inequality by  $\alpha_1$ , add them and use the inequality (6).

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# Conclusions

Some important considerations:

- ▶ The definition of a dissipative dynamical system postulates the existence of a storage function. The dynamical equations are insufficient to specify the storage function uniquely.
- ▶ The storage function satisfies an a priori bound. It is bounded from below by the available storage and from above by the required supply. These bounds possess a variational characterization.
- ▶ In dissipative systems states for which the storage function attains a local minimum are locally stable and the storage function is a suitable Lyapunov function.
- ▶ Immediate extension to interconnected systems: the sum of the storage functions of the individual subsystems is a storage function for the interconnected system.

# Bibliography

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