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Portwings Internal Meeting
Challenges and outlook for the numerics

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April 29, 2021

Overview

1. What numerics for portwings? (2 years)
   - Split discretization
   - Algebraic topology: from continuous to discrete
   - Application to portwings
2. What about mechanics? (6 months)
3. Combining numerics and control theory (4 months)

Outline

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What numerics for the portwings project?

Methods should preserve the continuous structure at the discrete level. Which structure?

1. Cohomology: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$;
2. Variational structure $\delta \int l = 0$, (l Lagrangian density);
3. Hamiltonian structure $\dot{\mathcal{F}} = \{F, H\}$, $\{\cdot, \cdot\}$ Poisson brackets.
4. ...

Recent developments:

- splitting of topological and metric operators (Bauer and Behrens 2018);
- Lie group structure and underlying variational formulation (Gawlik and Gay-Balmaz 2020);
- connection with algebraic topology, i.e. de Rham complex and more general Hilbert complexes, e.g. elasticity (Bochev and Hyman 2006; Arnold, Falk, and Winther 2006; Palha et al. 2014);

Principle behind split discretization

- Fluid equation written in covariant form (exterior calculus);
- Split Hamiltonian form $\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}$.
  - Topological braket depending on d (exterior derivative) or ι_v (interior product).
  - Metric dependent $\mathcal{H}$, since it depends on $*$

Linear shallow water waves in Hamiltonian form

- $\mathcal{H} = \frac{1}{2}\int_M \{\bar{h}\,\|\boldsymbol{u}\|^2 + gh^2\}\,\mathrm{d}x$, with $\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} = \bar{h}\boldsymbol{u}$, $\frac{\delta\mathcal{H}}{\delta h} = gh$, where g gravity acc. and $\bar{h}$ equilibrium fluid height.
- $\{\mathcal{F}, \mathcal{G}\} = -(\frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}, \nabla\frac{\delta\mathcal{G}}{\delta h})_{L^2} - (\frac{\delta\mathcal{F}}{\delta h}, \nabla\cdot\frac{\delta\mathcal{G}}{\delta\boldsymbol{u}})_{L^2}$.

$$\mathcal{F} = \int \boldsymbol{u}\,\mathrm{d}\Omega : \ \dot{\mathcal{F}} = -\left(\frac{\delta\mathcal{F}}{\delta\boldsymbol{u}}, \nabla\frac{\delta\mathcal{H}}{\delta h}\right) \rightarrow \partial_t\boldsymbol{u} = -g\nabla h.$$

$$\mathcal{F} = \int h\,\mathrm{d}\Omega : \ \dot{\mathcal{F}} = -\left(\frac{\delta\mathcal{F}}{\delta h}, \nabla\cdot\frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\right) \rightarrow \partial_t h = -\bar{h}\nabla\cdot\boldsymbol{u}.$$

Split and weak (or mixed) form

De Rham complex: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$.

Split form

$$\begin{tikzcd} h \in V^0 \arrow[r, "\nabla"] \arrow[d, "\tilde{h}=\tilde{*}h"'] & V^1 \ni \boldsymbol{u} \arrow[d, "\boldsymbol{u}=\tilde{*}\boldsymbol{u}"] \\ \tilde{h} \in V^3 & V^2 \ni \tilde{\boldsymbol{u}} \arrow[l, "\nabla\cdot"'] \end{tikzcd}$$

- Assume full $\tilde{*}$ in metric eqs.
- Both $\nabla, \nabla\cdot$ are imposed strongly.
- Both diff. eqs exact

Weak form (Mixed FE)

$$\begin{tikzcd} h \in V^3 \arrow[r, "\widehat{\nabla}"] \arrow[d, "\tilde{h}=h"'] & V^2 \ni \boldsymbol{u} \arrow[d, "\tilde{\boldsymbol{u}}=\boldsymbol{u}"] \\ \tilde{h} \in V^3 & V^2 \ni \tilde{\boldsymbol{u}} \arrow[l, "\nabla\cdot"'] \end{tikzcd}$$

- Assume $\tilde{*} = \mathrm{Id}$, i.e. $\tilde{\boldsymbol{u}} = \boldsymbol{u}$, $\tilde{h} = h$.
- Weak gradient $\widehat{\nabla}$.
- Moment Eq. weak

Split decomposition in practice

1. First projection of the strong form. For the 1D case:

$$\partial_t \mathbf{u}_e^{(1)} + g\mathbf{D}^{en}\mathbf{h}_n^{(0)} = \mathbf{0}, \qquad \mathbf{u}_e^{(1)} : 1 \text{ form}, \quad \mathbf{h}_n^{(0)} : 0 \text{ form},$$

$$\partial_t \widetilde{\mathbf{h}}_e^{(1)} + \bar{h}\mathbf{D}^{en}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{0}, \qquad \widetilde{\mathbf{h}}_e^{(1)} : 1 \text{ form}, \quad \widetilde{\mathbf{u}}_n^{(0)} : 0 \text{ form}.$$

   $\mathbf{D}^{en}$ metric free approximation of exterior derivative.

2. Project the metric closure relations:
   - High accuracy $CG_1^u - CG_1^h$ spaces:

$$\mathbf{M}^{nn}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{P}^{ne}\mathbf{u}_e^{(1)}, \qquad \mathbf{M}^{nn}\mathbf{h}_n^{(0)} = \mathbf{P}^{ne}\widetilde{\mathbf{h}}_e^{(1)};$$

   - Low accuracy $DG_0^u - DG_0^h$ spaces:

$$\mathbf{M}^{en}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{I}^{ee}\mathbf{u}_e^{(1)}, \qquad \mathbf{M}^{en}\mathbf{h}_n^{(0)} = \mathbf{I}^{ee}\widetilde{\mathbf{h}}_e^{(1)};$$

   - Medium accuracy $CG_1^u - DG_0^h$ spaces:

$$\mathbf{M}^{nn}\widetilde{\mathbf{u}}_n^{(0)} = \mathbf{P}^{ne}\mathbf{u}_e^{(1)}, \qquad \mathbf{M}^{en}\mathbf{h}_n^{(0)} = \mathbf{I}^{ee}\widetilde{\mathbf{h}}_e^{(1)};$$

   $\mathbf{M}^{nn}$, $\mathbf{M}^{en}$ metric dependent, $\mathbf{P}^{ne}$ metric free averaging op.

Weak (mixed) form

Weak formulation: find $\boldsymbol{u} \in V^2(\mathbb{R}^3),\ h \in V^3(\mathbb{R})$

$$
\begin{aligned}
(\boldsymbol{v}_u, \partial_t \boldsymbol{u})_{L^2} &= -(\boldsymbol{v}_u, g\widehat{\nabla} h)_{L^2} = (\nabla \cdot \boldsymbol{v}_u, gh)_{L^2}, && \forall \boldsymbol{v}_u \in V^2, \\
(v_h, \partial_t h)_{L^2} &= -(v_h, \bar{h}\nabla \cdot \boldsymbol{u})_{L^2}, && \forall v_h \in V^3.
\end{aligned}
$$

After projection

$$
\begin{bmatrix} \mathbf{M}^{nn} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{ee} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{h}_e \end{pmatrix} = \begin{bmatrix} \mathbf{0} & g\mathbf{D}^{ne} \\ -\bar{h}\mathbf{D}^{en} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{h}_e \end{pmatrix}, \qquad \mathbf{D}^{ne} = \mathbf{D}^{en\top}.
$$

Another formulation with weak divergence: find $\boldsymbol{u} \in V^1(\mathbb{R}^3),\ h \in V^0(\mathbb{R})$

$$
\begin{aligned}
(\boldsymbol{v}_u, \partial_t \boldsymbol{u})_{L^2} &= -(\boldsymbol{v}_u, g\nabla h)_{L^2}, && \forall \boldsymbol{v}_u \in V^1, \\
(v_h, \partial_t h)_{L^2} &= -(v_h, \bar{h}\widehat{\nabla \cdot} \boldsymbol{u})_{L^2} = (\nabla v_h, \bar{h}\boldsymbol{u})_{L^2}, && \forall v_h \in V^0.
\end{aligned}
$$

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Connection with algebraic topology (dim M=2)

$$
\begin{tikzcd}
 & C^0(D) \arrow[rr, "\delta"] \arrow[dl, bend right, "\mathcal{I}^0"'] \arrow[ddd, leftrightarrow, "*_h"] & & C^1(D) \arrow[rr, "\delta"] \arrow[dl, bend right, "\mathcal{I}^1"'] \arrow[ddd, leftrightarrow, "*_h"] & & C^2(D) \arrow[dl, bend right, "\mathcal{I}^2"'] \arrow[ddd, leftrightarrow, "*_h"] \\
\Lambda^0(M) \arrow[ur, bend right, "\mathcal{R}^0"'] \arrow[rr, "d"] \arrow[ddd, leftrightarrow, "*"] & & \Lambda^1(M) \arrow[ur, bend right, "\mathcal{R}^1"'] \arrow[rr, "d"] \arrow[ddd, leftrightarrow, "*"] & & \Lambda^2(M) \arrow[ur, bend right, "\mathcal{R}^2"'] \arrow[ddd, leftrightarrow, "*"] & \\
 & & & & & \\
 & \widetilde{C}^2(D) \arrow[dl, bend right, "\widetilde{\mathcal{I}}^2"'] & & \widetilde{C}^1(D) \arrow[ll, "\delta"] \arrow[dl, bend right, "\widetilde{\mathcal{I}}^1"'] & & \widetilde{C}^0(D) \arrow[ll, "\delta"] \arrow[dl, bend right, "\widetilde{\mathcal{I}}^0"'] \\
\widetilde{\Lambda}^2(M) \arrow[ur, bend right, "\widetilde{\mathcal{R}}^2"'] & & \widetilde{\Lambda}^1(M) \arrow[ll, "d"] \arrow[ur, bend right, "\widetilde{\mathcal{R}}^1"'] & & \widetilde{\Lambda}^0(M) \arrow[ll, "d"] \arrow[ur, bend right, "\widetilde{\mathcal{R}}^0"'] &
\end{tikzcd}
$$

Diagram taken from Palha et al. 2014.

- $C^k(D)$ space of cochains on the primal grid D;
- $\widetilde{C}^k$ space of cochains on the dual grid $\widetilde{D}$;
- δ coboundary operator (dual of ∂ on chains);
- $\mathcal{I}^k, \widetilde{\mathcal{I}}^k$ interpolation operators;
- $\mathcal{R}^k, \widetilde{\mathcal{R}}^k$ reduction operators (dof.);

Connection with Finite Elements and Finite Volumes

How to discretize the Hodge Laplacian $\Delta = dd^* + d^*d$?
The exterior derivative can be discretize exactly on one grid, but not the codifferential $d^* = (-1)^{n(k+1)+1} * d *$.

Two different solutions:

- Dual grid discretization: the Hodge star is treated numerically by considering a dual grid and leads to the explicit construction of the Hodge matrix (analogously to finite volumes);
- Single grid discretization: the Hodge operator is incorporated implicitly by using an inner product. The codifferential is converted into an exterior derivative by using the integration by parts (equivalent to mixed finite element method).

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How to dicretize Navier Stokes?

Combine both approaches:

- Use the split discretization for the topological part (ideal fluid).
- Use a weak formulation to discretize the Hodge Laplancian (due to the stress tensor).

Timeline: 1 year to look into discretization of ideal fluid.
$+$ 0.5y for Navier-Stokes, 0.5y for the fluid structure interaction.

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Continuum mechanics and differential geometry

In Arnold, Falk, and Winther 2006 the authors present linear elasticity using vector-valued forms and its associated complex.

Thanks to the Navier-Stokes equation we already have the answer

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_v \\ \boldsymbol{A}_\varepsilon\end{pmatrix} = \begin{bmatrix}\mathbf{0} & *^2 \circ \mathrm{d}_\nabla \\ \nabla \circ \sharp \circ * & \mathbf{0}\end{bmatrix}\begin{pmatrix}\delta_{\boldsymbol{\alpha}_v} H \\ \delta_{\boldsymbol{A}_\varepsilon} H\end{pmatrix}$$

- $\nabla,\ \mathrm{d}_\nabla$ covariant derivative and exterior covariant derivative;
- $\boldsymbol{\alpha}_v \in \Lambda^1(M)$ momentum one form;
- $\delta_{\boldsymbol{\alpha}_v} H \in \Lambda^{n-1}(M)$ velocity $n-1$ form;
- $\boldsymbol{A}_\varepsilon \in \mathfrak{X}(M) \otimes \Lambda^1(M)$ strain tensor;
- $\delta_{\boldsymbol{A}_\varepsilon} H \in \Lambda^1(M) \otimes \Lambda^{n-1}(M)$ stress tensor;

Continuum mechanics and differential geometry

The wing can be modeled as a thin shell structure dim(M)=2.
Higher order differential operator:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \boldsymbol{A}_\varepsilon\end{pmatrix} = \begin{bmatrix}\mathbf{0} & -d \circ *^2 \circ \mathrm{d}_\nabla \\ \nabla \circ \sharp \circ d & \mathbf{0}\end{bmatrix}\begin{pmatrix}\delta_{\alpha_w} H \\ \delta_{\boldsymbol{A}_\varepsilon} H\end{pmatrix}$$

- $\alpha_w \in \Lambda^2(M)$ momentum top form;
- $\delta_{\alpha_w} H \in \Lambda^0(M)$ velocity 0 form;
- $\boldsymbol{A}_\varepsilon \in \mathfrak{X}(M) \otimes \Lambda^1(M)$ bending strain tensor;
- $\delta_{\boldsymbol{A}_\varepsilon} H \in \Lambda^1(M) \otimes \Lambda^{n-1}(M)$ bending stress tensor;

Timeline: 6 months for elaborate the theoretical models.
$+$ 6 months for the numerics.

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Energy shaping via neural network approximators

Main idea: combine passivity based control (energy shaping + damping injection) with optimization thanks to neural networks (NNs).

- The damping coefficient and added energy are parametrized by a NN.
- A numerical model of the system is fed to an otpimization procedure whose solution gives the NN's parameters

Timeline: first validated results by September on simpler PDEs.

Tools available

Firedrake: `https://www.firedrakeproject.org/`.

FEniCS: `https://fenicsproject.org/`.

Fluid Structure Interaction in Fenics: Bergersen et al. 2020.

Mesh morphing in FEniCS: `https://bitbucket.org/Epoxid/femorph/src/c7317791c8f00d70fe16d593344cb164a53cad9b/?at=dokken%2Frestructuring`

PyDec: `https://github.com/hirani/pydec`

Learning Python for scientific computing `https://faculty.math.illinois.edu/~hirani/cbmg/index.html`

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