



# THÈSE

En vue de l'obtention du

**DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE**

Délivré par : *l'Institut Supérieur de l'Aéronautique et de l'Espace (ISAE)*

---

---

Présentée et soutenue le *30 Octobre 2020* par :

**ANDREA BRUGNOLI**

---

**A port-Hamiltonian formulation of flexible structures  
Modelling and symplectic finite element discretization**

---

---

## JURY

|                      |                              |              |
|----------------------|------------------------------|--------------|
| DANIEL ALAZARD       | ISAE-Supaéro, Toulouse       | Directeur    |
| VALÉRIE P. BUDINGER  | ISAE-Supaéro, Toulouse       | Co-directeur |
| DENIS MATIGNON       | ISAE-Supaéro, Toulouse       | Examineur    |
| THOMAS HÉLIE         | Directeur de Recherches CNRS | Examineur    |
| YANN LE GORREC       | Institut FEMTO-ST            | Rapporteur   |
| ALESSANDRO MACCHELLI | Università di Bologna        | Rapporteur   |

---

**École doctorale et spécialité :**

*EDSYS : Automatique*

**Unité de Recherche :**

*CSDV - Commande des Systèmes et Dynamique du Vol - ONERA - ISAE*

**Directeur de Thèse :**

*Daniel ALAZARD et Valérie POMMIER-BUDINGER*

**Rapporteurs :**

*Yann LE GORREC et Alessandro MACCHELLI*



# Abstract

3 This thesis aims at extending the port-Hamiltonian (pH) approach to continuum mechanics  
 4 in higher geometrical dimensions (particularly in 2D). The pH formalism has a strong mul-  
 5 tiphysics character and represents a unified framework to model, analyze and control both  
 6 finite- and infinite-dimensional systems. Despite the large literature on this topic, elasticity  
 7 problems in higher geometrical dimensions have almost never been considered. This work  
 8 establishes the connection between port-Hamiltonian distributed systems and elasticity prob-  
 9 lems. The originality resides in three major contributions. First, the novel pH formulation  
 10 of plate models and coupled thermoelastic phenomena is presented. The use of tensor cal-  
 11 culus is mandatory for continuum mechanical models and the inclusion of tensor variables is  
 12 necessary to obtain an intrinsic, i.e. coordinate free, and equivalent pH description. Second,  
 13 a finite element based discretization technique, capable of preserving the structure of the  
 14 infinite-dimensional problem at a discrete level, is developed and validated. The discretiza-  
 15 tion of elasticity problems in port-Hamiltonian form requires the use of non-standard finite  
 16 elements. Nevertheless, the numerical implementation is performed thanks to well-established  
 17 open-source libraries, providing external users with an easy to use tool for simulating flexible  
 18 systems in pH form. Third, flexible multibody systems are recast in pH form by making use of  
 19 a floating frame description valid under small deformations assumptions. This reformulation  
 20 include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.



# Résumé

22 Cette thèse vise à étendre l'approche port-Hamiltonienne (pH) à la mécanique des milieux  
 23 continus dans des dimensions géométriques plus élevées (en particulier on se focalise sur la  
 24 dimension deux). Le formalisme pH, avec son fort caractère multiphysique, représente un  
 25 cadre unifié pour modéliser, analyser et contrôler les systèmes de dimension finie et infinie.  
 26 Malgré l'abondante littérature sur ce sujet, les problèmes d'élasticité en deux ou trois dimen-  
 27 sions géométriques n'ont presque jamais été considérés. Dans ce travail de thèse la connexion  
 28 entre problèmes d'élasticité et systèmes distribués port-Hamiltoniens est établie. L'originalité  
 29 apportée réside dans trois contributions majeures. Tout d'abord, une nouvelle formula-  
 30 tion pH des modèles de plaques et des phénomènes thermoélastiques couplés est présen-  
 31 tée. L'utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et  
 32 l'introduction de variables tensorielles est nécessaire pour obtenir une description pH équiva-  
 33 lente qui soit intrinsèque, c'est-à-dire indépendante des coordonnées choisies. Deuxièmement,  
 34 une technique de discrétisation basée sur les éléments finis et capable de préserver la structure  
 35 du problème de la dimension infinie au niveau discret est développée et validée. La discrétis-  
 36 sation des problèmes d'élasticité écrits en forme port-Hamiltonienne nécessite l'utilisation  
 37 d'éléments finis non standards. Néanmoins, l'implémentation numérique est réalisée grâce  
 38 à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil  
 39 facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nou-  
 40 velle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation,  
 41 valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques  
 42 linéaires et exploite la modularité intrinsèque des systèmes pH.



# Acknowledgments





# Remerciements



# Ringraziamenti



# Contents

|    |  |            |
|----|--|------------|
| 48 | <b>Abstract</b>                                    | <b>i</b>   |
| 49 | <b>Résumé</b>                                      | <b>iii</b> |
| 50 | <b>Acknowledgments</b>                             | <b>v</b>   |
| 51 | <b>Remerciements</b>                               | <b>vii</b> |
| 52 | <b>Ringraziamenti</b>                              | <b>ix</b>  |
| 53 | <b>List of Acronyms</b>                            | <b>xxi</b> |
| 54 | <b>I Introduction and state of the art</b>         | <b>1</b>   |
| 55 | <b>1 Introduction</b>                              | <b>3</b>   |
| 56 | 1.1 Motivation and context . . . . .               | 3          |
| 57 | 1.2 Overview of chapters . . . . .                 | 3          |
| 58 | 1.3 Contributions . . . . .                        | 3          |
| 59 | <b>2 Literature review</b>                         | <b>5</b>   |
| 60 | 2.1 Port-Hamiltonian distributed systems . . . . . | 5          |
| 61 | 2.2 Structure-preserving discretization . . . . .  | 5          |
| 62 | 2.3 Mixed finite element for elasticity . . . . .  | 5          |
| 63 | 2.4 Multibody dynamics . . . . .                   | 6          |
| 64 | <b>3 Reminder on port-Hamiltonian systems</b>      | <b>7</b>   |
| 65 | 3.1 Finite dimensional setting . . . . .           | 8          |
| 66 | 3.1.1 Dirac structure . . . . .                    | 8          |

|    |           |  |           |
|----|-----------|--|-----------|
| 67 | 3.1.2     | Finite dimensional port-Hamiltonian systems . . . . .        | 9         |
| 68 | 3.2       | Infinite dimensional setting . . . . .                       | 9         |
| 69 | 3.2.1     | Linear differential operators . . . . .                      | 10        |
| 70 | 3.2.2     | Constant Stokes-Dirac structures . . . . .                   | 11        |
| 71 | 3.2.3     | Distributed port-Hamiltonian systems . . . . .               | 14        |
| 72 | 3.3       | Conclusion . . . . .   | 20        |
| 73 | <b>II</b> | <b>Port-Hamiltonian elasticity and thermoelasticity</b>      | <b>21</b> |
| 74 | <b>4</b>  | <b>Elasticity in port-Hamiltonian form</b>                   | <b>23</b> |
| 75 | 4.1       | Continuum mechanics . . . . .                                | 23        |
| 76 | 4.1.1     | Non linear formulation of elasticity . . . . .               | 23        |
| 77 | 4.1.2     | The linear elastodynamics problem . . . . .                  | 25        |
| 78 | 4.2       | Port-Hamiltonian formulation of linear elasticity . . . . .  | 27        |
| 79 | 4.2.1     | Energy and co-energy variables . . . . .                     | 27        |
| 80 | 4.2.2     | Final system and associated Stokes-Dirac structure . . . . . | 29        |
| 81 | 4.3       | Conclusion . . . . .   | 33        |
| 82 | <b>5</b>  | <b>Port-Hamiltonian plate theory</b>                         | <b>35</b> |
| 83 | 5.1       | First order plate theory . . . . .                           | 36        |
| 84 | 5.1.1     | Mindlin-Reissner model . . . . .                             | 37        |
| 85 | 5.1.2     | Kirchhoff-Love model . . . . .                               | 38        |
| 86 | 5.2       | Port-Hamiltonian formulation of isotropic plates . . . . .   | 40        |
| 87 | 5.2.1     | Port-Hamiltonian Mindlin plate . . . . .                     | 41        |
| 88 | 5.2.2     | Port-Hamiltonian Kirchhoff plate . . . . .                   | 45        |
| 89 | 5.3       | Laminated anisotropic plates . . . . .                       | 50        |
| 90 | 5.3.1     | Port-Hamiltonian laminated Mindlin plate . . . . .           | 52        |
| 91 | 5.3.2     | Port-Hamiltonian laminated Kirchhoff plate . . . . .         | 53        |

|     |  |            |
|-----|--|------------|
| 92  | 5.4 Conclusion . . . . .   | 54         |
| 93  | <b>6 Thermoelasticity in port-Hamiltonian form</b>                         | <b>57</b>  |
| 94  | 6.1 Port-Hamiltonian linear coupled thermoelasticity . . . . .             | 57         |
| 95  | 6.1.1 The heat equation as a pH descriptor system . . . . .                | 58         |
| 96  | 6.1.2 Classical thermoelasticity . . . . .                                 | 59         |
| 97  | 6.1.3 Thermoelasticity as two coupled pHs . . . . .                        | 61         |
| 98  | 6.2 Thermoelastic port-Hamiltonian bending . . . . .                       | 63         |
| 99  | 6.2.1 Thermoelastic Euler-Bernoulli beam . . . . .                         | 63         |
| 100 | 6.2.2 Thermoelastic Kirchhoff plate . . . . .                              | 65         |
| 101 | 6.3 Conclusion . . . . .   | 67         |
| 102 | <b>III Finite element structure preserving discretization</b>              | <b>69</b>  |
| 103 | <b>7 Partitioned finite element method</b>                                 | <b>71</b>  |
| 104 | 7.1 Discretization under uniform boundary condition . . . . .              | 71         |
| 105 | 7.1.1 General procedure . . . . .  | 73         |
| 106 | 7.1.2 Linear case . . . . .  | 82         |
| 107 | 7.1.3 Linear flexible structures . . . . .                                 | 84         |
| 108 | 7.2 Mixed boundary conditions . . . . .                                    | 93         |
| 109 | 7.2.1 Solution using Lagrange multipliers . . . . .                        | 95         |
| 110 | 7.2.2 Virtual domain decomposition . . . . .                               | 97         |
| 111 | 7.3 Conclusion . . . . .   | 101        |
| 112 | <b>8 Convergence numerical study</b>                                       | <b>103</b> |
| 113 | 8.1 Plate problems using known mixed finite elements . . . . .             | 103        |
| 114 | 8.1.1 Mindlin plate mixed discretization . . . . .                         | 104        |
| 115 | 8.1.2 The Hellan-Herrmann-Johnson scheme for the Kirchhoff plate . . . . . | 107        |
| 116 | 8.2 Dual mixed discretization of plate problems . . . . .                  | 108        |

|     |           |   |            |
|-----|-----------|---|------------|
| 117 | 8.2.1     | Dual mixed discretization of the Mindlin plate . . . . .  | 108        |
| 118 | 8.3       | Numerical experiments . . . . .                           | 109        |
| 119 | 8.3.1     | Numerical test for the Mindlin plate . . . . .            | 110        |
| 120 | 8.3.2     | Numerical test for the Kirchhoff plate . . . . .          | 116        |
| 121 | 8.4       | Conclusion . . . . .                                      | 119        |
| 122 | <b>9</b>  | <b>Numerical applications</b>                             | <b>123</b> |
| 123 | 9.1       | Boundary stabilization . . . . .                          | 123        |
| 124 | 9.2       | Thermoelastic wave propagation . . . . .                  | 123        |
| 125 | 9.3       | Mixed boundary conditions . . . . .                       | 123        |
| 126 | 9.3.1     | Trajectory tracking of a thin beam . . . . .              | 123        |
| 127 | 9.3.2     | Vibroacoustic under mixed boundary conditions . . . . .   | 123        |
| 128 | 9.4       | Modal analysis of plates . . . . .                        | 123        |
| 129 | <b>IV</b> | <b>Port-Hamiltonian flexible multibody dynamics</b>       | <b>125</b> |
| 130 | <b>10</b> | <b>Modular multibody systems in port-Hamiltonian form</b> | <b>127</b> |
| 131 | 10.1      | Reminder of the rigid case . . . . .                      | 127        |
| 132 | 10.2      | Flexible floating body . . . . .                          | 127        |
| 133 | 10.3      | Modular construction of multibody systems . . . . .       | 127        |
| 134 | <b>11</b> | <b>Validation</b>   | <b>129</b> |
| 135 | 11.1      | Beam systems . . . . .                                    | 129        |
| 136 | 11.1.1    | Modal analysis of a flexible mechanism . . . . .          | 129        |
| 137 | 11.1.2    | Non-linear crank slider . . . . .                         | 129        |
| 138 | 11.1.3    | Hinged beam . . . . .                                     | 129        |
| 139 | 11.2      | Plate systems . . . . .                                   | 129        |
| 140 | 11.2.1    | Boundary interconnection with a rigid element . . . . .   | 129        |
| 141 | 11.2.2    | Actuated plate . . . . .                                  | 129        |



|     |  |            |
|-----|--|------------|
| 142 | <b>Conclusions and future directions</b>           | <b>133</b> |
| 143 | <b>A Mathematical tools</b>                        | <b>135</b> |
| 144 | A.1 Differential operators . . . . .               | 135        |
| 145 | A.2 Integration by parts . . . . .                 | 136        |
| 146 | A.3 Bilinear forms . . . . .                       | 137        |
| 147 | <b>B Finite elements gallery</b>                   | <b>139</b> |
| 148 | <b>C Implementation using FEniCS and Firedrake</b> | <b>141</b> |
| 149 | <b>Bibliography</b>                                | <b>143</b> |



# List of Figures

150

|     |     |   |     |
|-----|-----|---|-----|
| 151 | 4.1 | A 2D continuum with Neumann and Dirichlet boundary conditions . . . . . | 31  |
| 152 | 5.1 | Kinematic assumption for the Kirchhoff plate . . . . .                  | 39  |
| 153 | 5.2 | Cauchy law for momenta and forces at the boundary. . . . .              | 42  |
| 154 | 5.3 | Reference frames and notations. . . . .                                 | 42  |
| 155 | 5.4 | Boundary conditions for the Mindlin plate. . . . .                      | 43  |
| 156 | 5.5 | Boundary conditions for the Kirchhoff plate. . . . .                    | 48  |
| 157 | 5.6 | Laminated plate with 4 layers. . . . .                                  | 50  |
| 158 | 6.1 | Boundary conditions for the thermoelastic problem. . . . .              | 60  |
| 159 | 7.1 | Partition of boundary into two connected sets. . . . .                  | 94  |
| 160 | 7.2 | Splitting of the domain. . . . .  | 97  |
| 161 | 7.3 | Interconnection at the interface $\Gamma_{12}$ . . . . .                | 97  |
| 162 | 8.1 | Error for the Mindlin plate using the BJT elements . . . . .            | 113 |
| 163 | 8.2 | Error for the Mindlin plate using the AFW elements . . . . .            | 115 |
| 164 | 8.3 | Error for the Mindlin plate using the CGDG elements . . . . .           | 116 |
| 165 | 8.4 | Error for the simply supported Kirchhoff plate (HHJ elements) . . . . . | 117 |
| 166 | 8.5 | Error for the CSFS Kirchhoff plate (HHJ elements) . . . . .             | 119 |



# List of Tables

167

|     |      |   |     |
|-----|------|---|-----|
| 168 | 8.1  | Physical parameters for the Mindlin plate. . . . .                                    | 111 |
| 169 | 8.2  | Mindlin plate convergence result for the BJT scheme $k = 1$ . . . . .                 | 112 |
| 170 | 8.3  | Mindlin plate convergence result for the BJT scheme $k = 2$ . . . . .                 | 112 |
| 171 | 8.4  | Mindlin plate convergence result for the BJT scheme $k = 3$ . . . . .                 | 112 |
| 172 | 8.5  | Mindlin plate convergence result for the AFW scheme $k = 1$ . . . . .                 | 114 |
| 173 | 8.6  | Mindlin plate convergence result for the AFW scheme $k = 2$ . . . . .                 | 114 |
| 174 | 8.7  | Mindlin plate convergence result for the AFW scheme $k = 3$ . . . . .                 | 114 |
| 175 | 8.8  | Mindlin plate convergence result for the Lagrange multiplier $\mathbf{E}_r$ . . . . . | 114 |
| 176 | 8.9  | Mindlin plate convergence result for the CGDG scheme $k = 1$ . . . . .                | 114 |
| 177 | 8.10 | Mindlin plate convergence result for the CGDG scheme $k = 2$ . . . . .                | 120 |
| 178 | 8.11 | Mindlin plate convergence result for the CGDG scheme $k = 3$ . . . . .                | 120 |
| 179 | 8.12 | Physical parameters for the Kirchhoff plate. . . . .                                  | 120 |
| 180 | 8.13 | Kirchoff plate convergence result for SSSS test $k = 1$ . . . . .                     | 120 |
| 181 | 8.14 | Kirchoff plate convergence result for SSSS test $k = 2$ . . . . .                     | 120 |
| 182 | 8.15 | Kirchoff plate convergence result for SSSS test $k = 3$ . . . . .                     | 121 |
| 183 | 8.16 | Kirchoff plate convergence result for CSFS test $k = 1$ . . . . .                     | 121 |
| 184 | 8.17 | Kirchoff plate convergence result for CSFS test $k = 2$ . . . . .                     | 121 |
| 185 | 8.18 | Kirchoff plate convergence result for CSFS test $k = 3$ . . . . .                     | 121 |



# List of Acronyms

186

|     |                |   |
|-----|----------------|---|
| 187 | <b>DAE</b>     | <i>Differential-Algebraic Equation</i>                                |
| 188 | <b>dpHs</b>    | <i>distributed port-Hamiltonian systems</i>                           |
| 189 | <b>FEM</b>     | <i>Finite Element Method</i>  |
| 190 | <b>IDA-PBC</b> | <i>Interconnection and Damping Assignment Passivity Based Control</i> |
| 191 | <b>PDE</b>     | <i>Partial Differential Equation</i>                                  |
| 192 | <b>PFEM</b>    | <i>Partitioned Finite Element Method</i>                              |
| 193 | <b>pH</b>      | <i>port-Hamiltonian</i>   |
| 194 | <b>pHs</b>     | <i>port-Hamiltonian systems</i>                                       |
| 195 | <b>pHDAE</b>   | <i>port-Hamiltonian Descriptor System</i>                             |

196





## Part I

# Introduction and state of the art



199

200

# Introduction

201

202

I was born not knowing and have had only a little time to change that  
here and there.

*Richard Feynman*  
*Letter to Armando Garcia J.*

203

## Contents

204

205

206

207

208

210

|            |   |          |
|------------|---|----------|
| <b>1.1</b> | <b>Motivation and context . . . . .</b> | <b>3</b> |
| <b>1.2</b> | <b>Overview of chapters . . . . .</b>   | <b>3</b> |
| <b>1.3</b> | <b>Contributions . . . . .</b>          | <b>3</b> |

211

## 1.1 Motivation and context

212

## 1.2 Overview of chapters

213

## 1.3 Contributions



# Literature review

---

Books serve to show a man that those original thoughts of his aren't very new after all.

---

*Abraham Lincoln*

## 2.1 Port-Hamiltonian distributed systems

**Differential geometry** An interesting reference that can provide some ideas in this direction is [Yao11, NY04].

For 1D linear PH systems with a generalized skew-adjoint system operator, [LGZM05] gives conditions on the assignment of boundary inputs and outputs for the system operator to generate a contraction semigroup. The latter is instrumental to show well-posedness of a linear PH system, see [JZ12]. Essentially, at most half the number of boundary port variables can be imposed as control inputs for a well-posed PH system in one-dimensional domains. The complete characterization of pH in arbitrary dimension is still an open research field. Two notable exceptions [KZ15, Skr19] provide partial answers to this problem. The first demonstrate the well-posedness of the linear wave equation in arbitrary geometrical dimensions. The second generalizes this result to treat the case of generic first order linear pHs in arbitrary geometrical dimensions.

## 2.2 Structure-preserving discretization

## 2.3 Mixed finite element for elasticity

Thanks to [?], it has become evident that there is a strict link between discretization of port-Hamiltonian (pH) systems and mixed finite elements. Velocity-stress formulation for the wave dynamics and elastodynamics problems are indeed Hamiltonian and their mixed discretization preserves such a structure. For instance in [KK15] the authors employed mixed finite elements to obtain a symplectic semi-discretization for the wave equation. This allows using known finite element scheme to preserve the pH structure at the discrete level.

240 Mixed finite elements for the wave equation have been studied in [Gev88, BJT00]. For  
241 elastodynamics the construction of stable elements gets more complicated because of the  
242 presence of the symmetric stress tensor. Existing elements enforce symmetry either strongly  
243 [BJT01] or weakly [AL14].

## 244 2.4 Multibody dynamics

---

# Reminder on port-Hamiltonian systems

## Contents

|            |   |           |
|------------|---|-----------|
| <b>3.1</b> | <b>Finite dimensional setting</b>           | <b>8</b>  |
| 3.1.1      | Dirac structure                             | 8         |
| 3.1.2      | Finite dimensional port-Hamiltonian systems | 9         |
| <b>3.2</b> | <b>Infinite dimensional setting</b>         | <b>9</b>  |
| 3.2.1      | Linear differential operators               | 10        |
| 3.2.2      | Constant Stokes-Dirac structures            | 11        |
| 3.2.3      | Distributed port-Hamiltonian systems        | 14        |
| <b>3.3</b> | <b>Conclusion</b>                           | <b>20</b> |



The main mathematical aspects behind the pH formalism are recalled in this chapter. First, the finite dimensional case is considered. The geometric concept of Dirac structure [Cou90] is first presented. Finite dimensional port-Hamiltonian system are then introduced by making clear their intimate connection with the concept of Dirac structure. Second, the infinite dimensional case is recalled. The equivalent of Dirac structures for the infinite-dimensional case is the concept of Stokes-Dirac structure. Analogously to what happens in the finite-dimensional case, infinite-dimensional (or distributed) port-Hamiltonian systems are intimately related to the concept of Stokes-Dirac structure.

This notion of Stokes-Dirac structure was first introduced in the literature by making use of a differential geometry approach [vdSM02]. Despite being really insightful in terms of geometrical structure, this approach does not encompass the case of higher-order differential operators. An extension in this sense is still an open question. Since bending problems in elasticity introduce higher-order differential operators, the language of PDE will be privileged over the one of differential forms.

### 3.1 Finite dimensional setting

Finite dimensional port-Hamiltonian are characterized by geometrical structures called Dirac structures. It is important to define this geometric concept and see how pHs relate to it.

#### 3.1.1 Dirac structure

Consider a finite dimensional space  $F$  over the field  $\mathbb{R}$  and  $E \equiv F'$  its dual, i.e. the space of linear operator  $\mathbf{e} : F \rightarrow \mathbb{R}$ . The elements of  $F$  are called flows, while the elements of  $E$  are called efforts. Those are port variables and their combination gives the power flowing inside the system. The space  $B = F \times E$  is called the bond space of power variables. Therefore the power is defined as  $\langle \mathbf{e}, \mathbf{f} \rangle = e(\mathbf{f})$ , where  $\langle \mathbf{e}, \mathbf{f} \rangle$  is the dual product between  $\mathbf{f}$  and  $\mathbf{e}$ .

**Definition 1** (Dirac Structure [Cou90], Def. 1.1.1)

Given the finite-dimensional space  $F$  and its dual  $E$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{E \times F} : F \times E \rightarrow \mathbb{R}$ , consider the symmetric bilinear form:

$$\langle \langle (\mathbf{f}_1, \mathbf{e}_1), (\mathbf{f}_2, \mathbf{e}_2) \rangle \rangle := \langle \mathbf{e}_1, \mathbf{f}_2 \rangle_{E \times F} + \langle \mathbf{e}_2, \mathbf{f}_1 \rangle_{E \times F}, \quad \text{where} \quad \mathbf{f}_i, \mathbf{e}_i \in B, \quad i = 1, 2 \quad (3.1)$$

A Dirac structure on  $B := F \times E$  is a subspace  $D \subset B$ , which is maximally isotropic under  $\langle \langle \cdot, \cdot \rangle \rangle$ . Equivalently, a Dirac structure on  $B := F \times E$  is a subspace  $D \subset B$  which equals its orthogonal complement with respect to  $\langle \langle \cdot, \cdot \rangle \rangle : D = D^\perp$ .

This definition can be extended to consider distributed forces and dissipation [Vil07].

**Proposition 1** (Characterization of Dirac structures)

Consider the space of power variables  $F \times E$  and let  $X$  denote an  $n$ -dimensional space, the space of energy variables. Suppose that  $F := F_s \times F_e$  and that  $E := E_s \times E_e$ , with  $\dim F_s = \dim E_s = n$  and  $\dim F_e = \dim E_e = m$ . Moreover, let  $\mathbf{J}(\mathbf{x})$  denote a skew-symmetric matrix of dimension  $n$  and  $\mathbf{B}(\mathbf{x})$  a matrix of dimension  $n \times m$ . Then, the set

$$D := \left\{ (\mathbf{f}_s, \mathbf{f}_e, \mathbf{e}_s, \mathbf{e}_e) \in F \times E \mid \mathbf{f}_s = \mathbf{J}(\mathbf{x})\mathbf{e}_s + \mathbf{B}(\mathbf{x})\mathbf{f}_e, \mathbf{e}_e = -\mathbf{B}(\mathbf{x})^\top \mathbf{e}_s \right\} \quad (3.2)$$

is a Dirac structure.

It is now possible to make the connection between Dirac structures and pH system explicit.



---

### 3.1.2 Finite dimensional port-Hamiltonian systems

Consider the time-invariant dynamical system:

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{J}(\mathbf{x})\nabla H(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}, \\ \mathbf{y} &= \mathbf{B}(\mathbf{x})^\top \nabla H(\mathbf{x}), \end{cases} \quad (3.3)$$

where  $H(\mathbf{x}) : X \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , the Hamiltonian, is a real-valued function bounded from below. Such a system is called port-Hamiltonian, as it arises from the Hamiltonian modelling of a physical system and it interacts with the environment through the input  $\mathbf{u}$ , included in the formulation. The connection with the concept of Dirac structure is achieved by considering the following port behavior:

$$\begin{aligned} \mathbf{f}_s &= \dot{\mathbf{x}}, & \mathbf{e}_s &= \nabla H(\mathbf{x}), \\ \mathbf{f}_e &= \mathbf{u}, & \mathbf{e}_e &= -\mathbf{y}. \end{aligned} \quad (3.4)$$

With this choice of the port variables, system (3.3) defines, by Proposition 1, a Dirac structure. Dissipation and distributed forces can be included and the corresponding system defines an extended Dirac structure, once the proper port variables have been introduced.

System 3.3 is a pH system in canonical form. Recently, finite dimensional differential algebraic port-Hamiltonian systems (pHDAE) have been introduced both for linear [BMXZ18] and non linear systems [MM19]. This enriched description share all the crucial features of ordinary pHs, but easily account for algebraic constraints, time-dependent transformations and explicit dependence on time in the Hamiltonian. The application of the proposed discretization method lead naturally to pHDAE systems.

## 3.2 Infinite dimensional setting

Infinite dimensional spaces appears whenever differential operators have to be considered. In this section we first explain what defines a differential operator. Then Stokes-Dirac structures, characterized by a skew-symmetric differential operator, are introduced. Finally distributed port-Hamiltonian systems and their connection to the concept of Stokes-Dirac structure are illustrated.

Before starting we recall how inner products of square integrable function are computed. Let  $\Omega$  denote a compact subset of  $\mathbb{R}^d$  and let  $L^2(\Omega, \mathbb{A})$  be the space of square integrable functions over the set  $\mathbb{A}$  in  $\Omega$ , with inner product denoted by  $\langle \cdot, \cdot \rangle_{L^2(\Omega, \mathbb{A})}$ . The set  $\mathbb{A}$  can either denote scalars  $\mathbb{R}$ , vectors  $\mathbb{R}^d$ , tensors  $\mathbb{R}^{d \times d}$  or a Cartesian product of those. For scalars  $(a, b) \in L^2(\Omega)$ , vectors  $(\mathbf{a}, \mathbf{b}) \in L^2(\Omega, \mathbb{R}^d)$  and tensors  $(\mathbf{A}, \mathbf{B}) \in L^2(\Omega, \mathbb{R}^{d \times d})$  the  $L^2$  inner

---

product is given by

$$\langle a, b \rangle_{L^2(\Omega)} = \int_{\Omega} ab \, d\Omega, \quad \langle \mathbf{a}, \mathbf{b} \rangle_{L^2(\Omega, \mathbb{R}^d)} = \int_{\Omega} \mathbf{a} \cdot \mathbf{b} \, d\Omega, \quad \langle \mathbf{A}, \mathbf{B} \rangle_{L^2(\Omega, \mathbb{R}^{d \times d})} = \int_{\Omega} \mathbf{A} : \mathbf{B} \, d\Omega. \quad (3.5)$$

The notation  $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$  denotes the tensor contraction. Furthermore, the space of square integrable vector-valued functions over the boundary of  $\Omega$  is indicated by  $L^2(\partial\Omega, \mathbb{R}^m)$ .

This space is endowed with the inner product

$$\langle \mathbf{a}_{\partial}, \mathbf{b}_{\partial} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} = \int_{\partial\Omega} \mathbf{a}_{\partial} \cdot \mathbf{b}_{\partial} \, dS, \quad \mathbf{a}_{\partial}, \mathbf{b}_{\partial} \in \mathbb{R}^m. \quad (3.6)$$

### 3.2.1 Linear differential operators

Let  $\Omega$  denote a compact subset of  $\mathbb{R}^d$  representing the spatial domain of the distributed parameter system. Consider two function space  $F_1, F_2$  over the sets  $\mathbb{A}, \mathbb{B}$  defined on  $\Omega \subset \mathbb{R}^d$  and a map  $\mathcal{L}$  relating the two

$$\begin{aligned} \mathcal{L} : F_1(\Omega, \mathbb{A}) &\longrightarrow F_2(\Omega, \mathbb{B}), \\ \mathbf{u} &\longrightarrow \mathbf{v}. \end{aligned} \quad (3.7)$$

Sets  $\mathbb{A}, \mathbb{B}$  can either denote scalars  $\mathbb{R}$ , vectors  $\mathbb{R}^d$ , tensors  $\mathbb{R}^{d \times d}$  or a Cartesian product of those. Given  $\mathbf{u} \in F_1$ ,  $\mathbf{v} \in F_2$  The map  $\mathcal{L}$  is a linear differential operator if it can be represented by a linear combination of derivatives of  $\mathbf{u}$

$$\mathbf{v} = \mathcal{L}\mathbf{u} \iff \mathbf{v} := \sum_{|\alpha|=0}^n \mathcal{P}_{\alpha} \partial^{\alpha} \mathbf{u}, \quad (3.8)$$

where  $\alpha := (\alpha_1, \dots, \alpha_d)$  is a multi-index of order  $|\alpha| := \sum_{i=1}^d \alpha_i$  and  $\partial^{\alpha} := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$  is a differential operator of order  $|\alpha|$  resulting from a combination of spatial derivatives.  $\mathcal{P}_{\alpha} : \mathbb{A} \rightarrow \mathbb{B}$  is a constant algebraic operator from set  $\mathbb{A}$  to  $\mathbb{B}$ .

**Example 1** (Divergence operator in  $\mathbb{R}^d$ )

Given  $\mathbf{u} \in C^{\infty}(\Omega, \mathbb{R}^d)$ ,  $v \in C^{\infty}(\Omega)$ , where  $C^{\infty}(\Omega, \mathbb{R}^d)$ ,  $C^{\infty}(\Omega)$  denotes the set of smooth vector- and scalar-valued function defined on  $\Omega$ , the divergence operator in Cartesian coordinate is expressed as

$$v = \operatorname{div} \mathbf{u} = \sum_{i=1}^d \mathbf{e}_i \cdot \partial_{x_i} \mathbf{u}, \quad (3.9)$$

where  $\mathbf{e}_i$  is the  $i$ -th element of the canonical basis in  $\mathbb{R}^d$ .

The differential operators employed in this thesis are reported in Appendix A. A very important notion related to a differential operator is the one of formal adjoint.

**Definition 2** (Formal Adjoint)

Let  $\mathcal{L} = L^2(\Omega, \mathbb{A}) \rightarrow L^2(\Omega, \mathbb{B})$  be a differential operator and  $\mathbf{u} \in C_0^\infty(\Omega, \mathbb{A})$ ,  $\mathbf{v}(\Omega, \mathbb{B})$  be smooth variables with compact support on  $\Omega$ . The formal adjoint of the differential operator  $\mathcal{L}$ , denoted by  $\mathcal{L}^* = L^2(\Omega, \mathbb{B}) \rightarrow L^2(\Omega, \mathbb{A})$ , is defined by the relation

$$\langle \mathcal{L}\mathbf{u}, \mathbf{v} \rangle_{L^2(\Omega, \mathbb{B})} = \langle \mathbf{u}, \mathcal{L}^*\mathbf{v} \rangle_{L^2(\Omega, \mathbb{A})}. \quad (3.10)$$

This definition represent an extension to generic sets  $\mathbb{A}$ ,  $\mathbb{B}$  of Def. 5.80 in [RR04] (reported in Appendix A).

**Remark 1** (Differences between adjoint and formal adjoint)

The definition of formal adjoint is such that the integration by parts formula is respected. Contrarily to the adjoint of an operator, the formal adjoint definition does not regard the actual domain of the operator nor the boundary conditions. For example, the differential operators  $\text{div}$ ,  $\text{grad}$  are unbounded in the  $L^2$  topology. Whenever unbounded operators are considered, it is important to define their domain. To avoid the need of specifying domains, the notion of formal adjoint is used. The formal adjoint respects the integration by parts formula and is defined only for sufficiently smooth functions with compact support. In this sense the formal adjoint of  $\text{div}$  is  $-\text{grad}$ , since for smooth functions with compact support, it holds

$$\langle \text{div } \mathbf{y}, x \rangle_{L^2(\Omega, \mathbb{R})} \underbrace{=}_{I.B.P.} - \langle \mathbf{y}, \text{grad } x \rangle_{L^2(\Omega, \mathbb{R}^d)},$$

for  $\mathbf{y} \in C_0^\infty(\Omega, \mathbb{R}^d)$ ,  $x \in C_0^\infty(\Omega)$  (I.B.P. stands for integration by parts). The definition of the domain of the operators, that requires the knowledge of the boundary conditions, has not been specified.

For pHs formally skew-adjoint operators (or simply skew-symmetric) plays a fundamental role.

**Definition 3** (Formally skew-adjoint operator)

Let  $\mathcal{J} : L^2(\Omega, \mathbb{F}) \rightarrow L^2(\Omega, \mathbb{F})$  be a linear differential operator. Notice that the set  $\mathbb{F}$  in the domain and co-domain is the same. Then,  $\mathcal{J}$  is formally skew-adjoint (or skew-symmetric) if and only if  $\mathcal{J} = -\mathcal{J}^*$ .

If functions with compact support are considered, i.e.  $\mathbf{u}_1, \mathbf{u}_2 \in C_0^\infty(\Omega, \mathbb{F})$  a formally skew-adjoint operator is characterized by the relation

$$\langle \mathcal{J}\mathbf{u}_1, \mathbf{u}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{u}_1, \mathcal{J}\mathbf{u}_2 \rangle_{L^2(\Omega, \mathbb{A})} = 0. \quad (3.11)$$

### 3.2.2 Constant Stokes-Dirac structures

Constant Stokes-Dirac structures are the infinite-dimensional generalization of constant Dirac structures (i.e. Dirac structures for which the matrices  $\mathbf{J}$ ,  $\mathbf{B}$  in (3.3) are constant). Stokes-

Dirac structure are characterized by the fact that they are equal their orthogonal complement with respect to a bilinear product. So we recall the definition of orthogonal companion for the case of smooth functions.

**Definition 4** (Orthogonal complement)

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$  be an open connected set and  $C^\infty(\partial\Omega, \mathbb{R}^m)$  the space of smooth functions over its boundary. Consider the space

$$B = C^\infty(\Omega, \mathbb{F}) \times C^\infty(\partial\Omega, \mathbb{R}^m) \times C^\infty(\Omega, \mathbb{F}) \times C^\infty(\partial\Omega, \mathbb{R}^m) \quad (3.12)$$

and the bilinear pairing defined by

$$\begin{aligned} \langle \langle \cdot, \cdot \rangle \rangle : B \times B &\longrightarrow \mathbb{R}, \\ (\mathbf{a}, \mathbf{a}_\partial, \mathbf{b}, \mathbf{b}_\partial) \times (\mathbf{c}, \mathbf{c}_\partial, \mathbf{d}, \mathbf{d}_\partial) &\longrightarrow \langle \mathbf{a}, \mathbf{d} \rangle_{L^2(\Omega, \mathbb{F})} + \langle \mathbf{b}, \mathbf{c} \rangle_{L^2(\Omega, \mathbb{F})} + \\ &\quad \langle \mathbf{a}_\partial, \mathbf{d}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \mathbf{b}_\partial, \mathbf{c}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \quad (3.13)$$

Given a linear subspace  $W \subset B$ , its orthogonal complement is the set

$$W^\perp = \{\mathbf{v} \in B \mid \langle \langle \mathbf{v}, \mathbf{w} \rangle \rangle = 0, \forall \mathbf{w} \in W\} \quad (3.14)$$

We can now define what a Stokes-Dirac structure is.

**Definition 5** (Stokes-Dirac structure)

A subset  $D \subset B$ , with  $B$  defined in (3.12), is a Stokes-Dirac structure iff

$$D = D^\perp, \quad (3.15)$$

where the orthogonal complement has been defined in Eq. (3.14)

For a subset to be a Stokes-Dirac structures a link between flow and effort variables must hold. Consider  $\mathbf{f} \in C^\infty(\Omega, \mathbb{F})$  and  $\mathbf{e} \in C^\infty(\Omega, \mathbb{F})$  and the following relation between the two

$$\mathbf{f} = \mathcal{J}\mathbf{e}, \quad \mathcal{J} = -\mathcal{J}^*, \quad (3.16)$$

where  $\mathcal{J}$  is a formally skew-adjoint operator. A Stokes-Dirac structure requires the specification of boundary variables in order to express a general power conservation property for open physical systems. We make therefore the following assumption, over the existence of appropriate boundary operators.

**Assumption 1** (Existence of boundary operators)

Assume that exist two linear boundary operators  $\mathcal{B}_\partial, \mathcal{C}_\partial$  such that for  $\mathbf{u}_1, \mathbf{u}_2 \in C^\infty(\Omega, \mathbb{F})$  the following integration by parts formula holds

$$\langle \mathcal{J}\mathbf{u}_1, \mathbf{u}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{u}_1, \mathcal{J}\mathbf{u}_2 \rangle_{L^2(\Omega, \mathbb{A})} = \langle \mathcal{B}_\partial\mathbf{u}_1, \mathcal{C}_\partial\mathbf{u}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \mathcal{B}_\partial\mathbf{u}_2, \mathcal{C}_\partial\mathbf{u}_1 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \quad (3.17)$$

This assumption is necessary to appropriately define a Stokes-Dirac structure. Only few particular cases, like the transport equation, do not verify it. We can now characterize generic Stokes-Dirac structure for smooth functions spaces.

**Proposition 2** (Characterization of Stokes-Dirac structures)

Let  $B$  be defined as in Eq. (3.12) and  $\mathcal{J}$  be a formally skew adjoint operator verifying Assumption 1. The set

$$D_{\mathcal{J}} = \{(\mathbf{f}, \mathbf{f}_{\partial}, \mathbf{e}, \mathbf{e}_{\partial}) \in B \mid \mathbf{f} = \mathcal{J}\mathbf{e}, \mathbf{f}_{\partial} = \mathcal{B}_{\partial}\mathbf{e}, \mathbf{e}_{\partial} = -\mathcal{C}_{\partial}\mathbf{e}\} \quad (3.18)$$

is a Stokes-Dirac structure with respect to the bilinear pairing (3.13).

*Proof.* A Stokes-Dirac is characterized by the fact that  $D_{\mathcal{J}} = D_{\mathcal{J}}^{\perp}$ . Then one has to show that  $D_{\mathcal{J}} \subset D_{\mathcal{J}}^{\perp}$  and  $D_{\mathcal{J}}^{\perp} \subset D_{\mathcal{J}}$ . Following [LGZM05], the proof is obtained following three steps.

*Step 1.* To show that  $D_{\mathcal{J}} \subset D_{\mathcal{J}}^{\perp}$ , take  $(\mathbf{f}, \mathbf{f}_{\partial}, \mathbf{e}, \mathbf{e}_{\partial}) \in D_{\mathcal{J}}$ . Then

$$\begin{aligned} \langle (\mathbf{f}, \mathbf{f}_{\partial}, \mathbf{e}, \mathbf{e}_{\partial}), (\mathbf{f}, \mathbf{f}_{\partial}, \mathbf{e}, \mathbf{e}_{\partial}) \rangle &= 2 \langle \mathbf{e}, \mathbf{f} \rangle_{L^2(\Omega, \mathbb{F})} + 2 \langle \mathbf{e}_{\partial}, \mathbf{f}_{\partial} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &= 2 \langle \mathbf{e}, \mathcal{J}\mathbf{e} \rangle_{L^2(\Omega, \mathbb{F})} + 2 \langle \mathbf{e}_{\partial}, \mathbf{f}_{\partial} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &\stackrel{\text{Eq. (3.17)}}{=} 2 \langle \mathcal{B}_{\partial}\mathbf{e}, \mathcal{C}_{\partial}\mathbf{e} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + 2 \langle \mathbf{e}_{\partial}, \mathbf{f}_{\partial} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &\stackrel{\text{Eq. (3.18)}}{=} 2 \langle \mathcal{B}_{\partial}\mathbf{e}, \mathcal{C}_{\partial}\mathbf{e} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} - 2 \langle \mathcal{B}_{\partial}\mathbf{e}, \mathcal{C}_{\partial}\mathbf{e} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &= 0. \end{aligned}$$

This implies  $D_{\mathcal{J}} \subset D_{\mathcal{J}}^{\perp}$ .

*Step 2.* Take  $(\phi, \phi_{\partial}, \epsilon, \epsilon_{\partial}) \in D_{\mathcal{J}}^{\perp}$  and  $\mathbf{e}_0 \in C_0^{\infty}(\Omega, \mathbb{F})$ . This implies  $\mathcal{B}_{\partial}\mathbf{e}_0 = (\mathbf{0}, \mathbf{0})$  and  $\mathcal{C}_{\partial}\mathbf{e}_0 = (\mathbf{0}, \mathbf{0})$ . Taking  $(\mathcal{J}\mathbf{e}_0, \mathbf{0}, \mathbf{e}_0, \mathbf{0}) \in D_{\mathcal{J}}$  then

$$\langle (\phi, \phi_{\partial}, \epsilon, \epsilon_{\partial}), (\mathcal{J}\mathbf{e}_0, \mathbf{0}, \mathbf{e}_0, \mathbf{0}) \rangle = \langle \epsilon, \mathcal{J}\mathbf{e}_0 \rangle_{L^2(\Omega, \mathbb{F})} + \langle \mathbf{e}_0, \phi \rangle_{L^2(\Omega, \mathbb{F})} = 0, \quad \forall \mathbf{e}_0 \in C_0^{\infty}(\Omega, \mathbb{F}).$$

It follows that  $\epsilon \in C_0^{\infty}(\Omega, \mathbb{F})$  and  $\phi = \mathcal{J}\epsilon$ .

*Step 3.* Take  $(\phi, \phi_{\partial}, \epsilon, \epsilon_{\partial}) \in D_{\mathcal{J}}^{\perp}$  and  $(\mathbf{f}, \mathbf{f}_{\partial}, \mathbf{e}, \mathbf{e}_{\partial}) \in D_{\mathcal{J}}$ . From step 2 and (3.17)

$$\begin{aligned}
0 &= \langle \mathcal{J}e, \epsilon \rangle_{L^2(\Omega, \mathbb{F})} + \langle e, \mathcal{J}\epsilon \rangle_{L^2(\Omega, \mathbb{F})} + \langle e_\partial, \phi_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \epsilon_\partial, f_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
&\stackrel{\text{Eq. (3.17)}}{=} \langle \mathcal{B}_\partial e, \mathcal{C}_\partial \epsilon \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \mathcal{B}_\partial \epsilon, \mathcal{C}_\partial e \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle e_\partial, \phi_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \epsilon_\partial, f_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
&= \langle \mathcal{B}_\partial e, \mathcal{C}_\partial \epsilon \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \mathcal{B}_\partial \epsilon, \mathcal{C}_\partial e \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle -\mathcal{C}_\partial e, \phi_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \epsilon_\partial, \mathcal{B}_\partial e \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
&= \langle \mathcal{B}_\partial e, \mathcal{C}_\partial \epsilon + \epsilon_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} + \langle \mathcal{B}_\partial \epsilon - \phi_\partial, \mathcal{C}_\partial e \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \quad \text{By linearity,} \\
&= \langle e_\partial, \mathcal{C}_\partial \epsilon + \epsilon_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} - \langle \mathcal{B}_\partial \epsilon - \phi_\partial, f_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}.
\end{aligned}$$

Given the fact that  $e_\partial, f_\partial$  are arbitrary then

$$\phi_\partial = \mathcal{B}_\partial \epsilon, \quad \epsilon_\partial = -\mathcal{C}_\partial \epsilon,$$

408 meaning that  $D_{\mathcal{J}}^\perp \subset D_{\mathcal{J}}$ . This concludes the proof.  $\square$

### 409 3.2.3 Distributed port-Hamiltonian systems

410 A distributed lossless port-Hamiltonian system is defined by a set of variables that describes  
 411 the unknowns, by a formally skew-adjoint differential operator, an energy functional and a  
 412 set of boundary inputs and corresponding conjugated outputs. Such a system is described by  
 413 the following set of equations, defined on an open connected set  $\Omega \subset \mathbb{R}^d$

$$\begin{aligned}
\partial_t \alpha &= \mathcal{J} \delta_\alpha H, & \alpha &\in C^\infty(\Omega, \mathbb{F}), \\
u_\partial &= \mathcal{B}_\partial \delta_\alpha H, & u_\partial &\in \mathbb{R}^m, \\
y_\partial &= \mathcal{C}_\partial \delta_\alpha H, & y_\partial &\in \mathbb{R}^m.
\end{aligned} \tag{3.19}$$

414 The unknowns  $\alpha$  are called energy variables in the port-Hamiltonian framework, the formally  
 415 skew-adjoint operator  $\mathcal{J}$  is named interconnection operator (see Def. 3 for a precise definition  
 416 of formal skew adjointness).  $\mathcal{B}_\partial, \mathcal{C}_\partial$  are boundary operators, that provide the boundary input  
 417  $u_\partial$  and output  $y_\partial$  [TW09, Chapter 4]. The functional  $H(\alpha) : C^\infty(\Omega, \mathbb{F}) \rightarrow \mathbb{R}$  corresponds to  
 418 the Hamiltonian functional and in all the examples considered in this thesis coincide with the  
 419 total energy of the system. Notation  $\delta_\alpha H$  indicates the variational derivative of  $H$ .

**Definition 6** (Variational derivative, Def. 4.1 in [Olv93])

Consider a functional  $H(\alpha) : C^\infty(\Omega, \mathbb{F}) \rightarrow \mathbb{R}$

$$H(\alpha) = \int_\Omega \mathcal{H}(\alpha) \, d\Omega.$$

Given a variation  $\alpha = \bar{\alpha} + \eta \delta \alpha$  the variational derivative  $\frac{\delta H}{\delta \alpha}$  is defined as

$$H(\bar{\alpha} + \eta \delta \alpha) = H(\bar{\alpha}) + \eta \langle \delta_\alpha H, \delta \alpha \rangle_{L^2(\Omega, \mathbb{F})} + O(\eta^2).$$

**Remark 2**

---

If the integrand does not contain derivative of the argument  $\alpha$  then the variational derivative is equal to the partial derivative of the Hamiltonian density  $\mathcal{H}$

$$\frac{\delta H}{\delta \alpha} = \frac{\partial \mathcal{H}}{\partial \alpha}.$$

**Remark 3** (Co-energy variables)

The variational derivative of the Hamiltonian defines the co-energy variables  $\mathbf{e} := \delta_\alpha H$ . These are equivalent to the effort variables of the Stokes-Dirac structure as we will immediately show.

Suppose that operators  $\mathcal{J}$ ,  $\mathcal{B}_\partial$ ,  $\mathcal{C}_\partial$  in Eq. 3.19 verify Ass. 1. Then, System (3.19) is lossless since the energy rate is given by

$$\begin{aligned} \dot{H} &= \langle \delta_\alpha H, \partial_t \alpha \rangle_{L^2(\Omega, \mathbb{F})}, \\ &\stackrel{\text{Eq. (3.17)}}{=} \langle \mathcal{B}_\partial \delta_\alpha H, \mathcal{C}_\partial \delta_\alpha H \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &= \langle \mathbf{u}_\partial, \mathbf{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \tag{3.20}$$

The connection between the concept of Stokes-Dirac structure and dpHs becomes clear if the following port behavior is considered

$$\begin{aligned} \mathbf{f} &= \partial_t \alpha, & \mathbf{e} &= \delta_\alpha H, \\ \mathbf{f}_\partial &= \mathbf{u}_\partial, & \mathbf{e}_\partial &= -\mathbf{y}_\partial. \end{aligned} \tag{3.21}$$

By proposition (2) System (3.19) under the port behavior (3.21) defines a Stokes-Dirac structure. No rigorous characterization has been given so far for operators  $\mathcal{J}$ ,  $\mathcal{B}_\partial$ ,  $\mathcal{C}_\partial$  in system (3.19). A formal characterization of these operators has been given in [LGZM05] for pH of generic order only in one geometrical dimensional. In Chapter 7 the operator  $\mathcal{J}$  will be better characterize using an appropriate partition. By applying a general integration by parts formula, the operators  $\mathcal{B}_\partial$ ,  $\mathcal{C}_\partial$  associated to  $\mathcal{J}$  can be defined as well. The following examples clarifies this assertion for some known pHs.

### 3.2.3.1 Wave equation

Given an open bounded connected set  $\Omega \subset \mathbb{R}^2$  with Lipschitz continuous boundary  $\partial\Omega$ , the propagation of sound in air can be described by the following model [TRLGK18]

$$\begin{aligned} \chi_s \partial_t p(\mathbf{x}, t) &= -\operatorname{div} \mathbf{v}, \\ \mu_0 \partial_t \mathbf{v}(\mathbf{x}, t) &= -\operatorname{grad} p, \end{aligned} \tag{3.22}$$

where the scalars  $\chi_s$ ,  $\mu_0$  are the constant adiabatic compressibility factor and the steady state mass density respectively. The scalar field  $p \in \mathbb{R}$  and vector field  $\mathbf{v} \in \mathbb{R}^2$  represents the variation of pressure and velocity from the steady state. The Hamiltonian (total energy) reads

$$H = \frac{1}{2} \int_{\Omega} \left\{ \chi_s p^2 + \mu_0 \|\mathbf{v}\|^2 \right\} d\Omega.$$


---

To recast (3.22) in pH form the energy variables has to be introduced  $\boldsymbol{\alpha} = [\alpha_p, \boldsymbol{\alpha}_v]^\top$

$$\alpha_p := \chi_s p, \quad \boldsymbol{\alpha}_v := \mu_0 \mathbf{v}.$$

The Hamiltonian is rewritten as

$$H = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\chi_s} \alpha_p^2 + \frac{1}{\mu_0} \|\boldsymbol{\alpha}_v\|^2 \right\} d\Omega.$$

By definition, the co-energy are

$$e_p = \frac{\delta H}{\delta \alpha_p} = \frac{1}{\chi_s} \alpha_p = p, \quad \mathbf{e}_v = \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \frac{1}{\mu_0} \boldsymbol{\alpha}_v = \mathbf{v}.$$

Equation (3.22) can be recast in port-Hamiltonian form

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_p \\ \boldsymbol{\alpha}_v \end{pmatrix} = - \begin{bmatrix} 0 & \text{div} \\ \text{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_p \\ \mathbf{e}_v \end{pmatrix}.$$

From the energy rate it is possible to identify the boundary variables.

$$\begin{aligned} \dot{H} &= + \int_{\Omega} \{e_p \partial_t \alpha_p + \mathbf{e}_v \cdot \partial_t \boldsymbol{\alpha}_v\} d\Omega, \\ &= - \int_{\Omega} \{e_p \text{div} \mathbf{e}_v + \mathbf{e}_v \cdot \text{grad} e_p\} d\Omega, && \text{Chain rule,} \\ &= - \int_{\Omega} \text{div}(e_p \mathbf{e}_v) d\Omega, && \text{Stokes theorem,} \\ &= - \int_{\partial\Omega} e_p \mathbf{e}_v \cdot \mathbf{n} dS = - \langle e_p, \mathbf{e}_v \cdot \mathbf{n} \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}. \end{aligned}$$

437 The boundary term  $\langle e_p, \mathbf{e}_v \cdot \mathbf{n} \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$  pairs two power variables. One is taken as control  
 438 input, the other plays the role of power-conjugated output. The assignment of these roles to  
 439 the boundary power variables is referred to as causality of the boundary port [KML18],[Kot19,  
 440 Chapter 2]. Under uniform causality assumption, either  $e_p$  or  $\mathbf{e}_v$  can assume the role of  
 441 (distributed) boundary input, but not both. This leads to two possible selections:

- 442 • First case  $u_{\partial} = e_p, \quad y_{\partial} = \mathbf{e}_v \cdot \mathbf{n}$ .  
 443 This imposes the variable  $e_p := p$  as boundary input and corresponds to a classical  
 444 Dirichlet condition. The boundary operator for this case are given by

$$\mathcal{B}_{\partial} \begin{pmatrix} e_p \\ \mathbf{e}_v \end{pmatrix} = e_p|_{\partial\Omega}, \quad \mathcal{C}_{\partial} \begin{pmatrix} e_p \\ \mathbf{e}_v \end{pmatrix} = \mathbf{e}_v \cdot \mathbf{n}|_{\partial\Omega},$$

445 corresponding to the standard trace and normal trace operators.

- 446 • Second case  $u_{\partial} = \mathbf{e}_v \cdot \mathbf{n}, \quad y_{\partial} = e_p$ .  
 447 This imposes the variable  $\mathbf{e}_v \cdot \mathbf{n} := \mathbf{v} \cdot \mathbf{n}$  as boundary input and corresponds to a  
 448 Neumann condition. The boundary operators are therefore switched with respect to



the previous case

$$\mathcal{B}_\partial \begin{pmatrix} e_p \\ e_v \end{pmatrix} = e_v \cdot \mathbf{n}|_{\partial\Omega}, \quad \mathcal{C}_\partial \begin{pmatrix} e_p \\ e_v \end{pmatrix} = e_p|_{\partial\Omega}.$$

### 3.2.3.2 Euler Bernoulli beam

The Euler-Bernoulli beam model consists of one PDE, describing the vertical displacement along the beam length:

$$\rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t) + \frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 w}{\partial x^2} \right) = 0, \quad x \in \Omega = \{0, L\}, \quad (3.23)$$

where  $w(x, t)$  is the transverse displacement of the beam. The coefficients  $\rho(x)$ ,  $A(x)$ ,  $E(x)$  and  $I(x)$  are the mass density, cross section, Young's modulus of elasticity and the moment of inertia of a cross section. The energy variables are then chosen as follows:

$$\alpha_w = \rho A(x) \frac{\partial w}{\partial t}(x, t), \quad \text{Linear Momentum}, \quad \alpha_\kappa = \frac{\partial^2 w}{\partial x^2}(x, t), \quad \text{Curvature}. \quad (3.24)$$

Those variables are collected in the vector  $\alpha = (\alpha_w, \alpha_\kappa)^T$ , so that the Hamiltonian can be written as a quadratic functional in the energy variables:

$$H = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho A} \alpha_w^2 + EI \alpha_\kappa^2 \right\} d\Omega \quad (3.25)$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}(x, t), & \text{Vertical velocity,} \\ e_\kappa &:= \frac{\delta H}{\delta \alpha_\kappa} = EI(x) \frac{\partial^2 w}{\partial x^2}(x, t), & \text{Flexural momentum.} \end{aligned} \quad (3.26)$$

The underlying interconnection structure is then found to be:

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}. \quad (3.27)$$

The power flow gives access to the boundary variables:

$$\begin{aligned} \dot{H} &= \int_\Omega \{ e_w \partial_t \alpha_w + e_\kappa \partial_t \alpha_\kappa \} d\Omega, \\ &= \int_\Omega \{ -e_w \partial_{xx} e_\kappa + e_\kappa \partial_{xx} e_w \} d\Omega, & \text{Integration by parts,} \\ &= \int_{\partial\Omega} \{ -e_w \partial_x e_\kappa + e_\kappa \partial_x e_w \} ds = \langle -e_w|_{\partial\Omega}, \partial_x e_\kappa|_{\partial\Omega} \rangle_{\mathbb{R}^4} + \langle e_\kappa|_{\partial\Omega}, \partial_x e_w|_{\partial\Omega} \rangle_{\mathbb{R}^4} \end{aligned} \quad (3.28)$$

Since the system is of differential order two, two pairing appears, giving rise to four combination of uniform boundary causality

- First case  $u_{\partial,1} = e_w$ ,  $u_{\partial,2} = \partial_x e_w$ ,  $y_{\partial,1} = -\partial_x e_\kappa$ ,  $y_{\partial,2} = e_\kappa$ .  
This imposes the vertical  $e_w := \partial_t w$  and angular velocity  $\partial_x e_w := \partial_{xt} w$  as boundary inputs

$$\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \quad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \quad (3.29)$$

If the inputs are null a clamped boundary condition is obtained.

- Second case  $u_{\partial,1} = e_w$ ,  $u_{\partial,2} = e_\kappa$ ,  $y_{\partial,1} = -\partial_x e_\kappa$ ,  $y_{\partial,2} = \partial_x e_w$ .  
This imposes the vertical velocity and flexural momentum  $e_\kappa := EI \partial_{xx} w$  as boundary inputs

$$\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \quad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \quad (3.30)$$

Zero inputs lead to a simply supported condition.

- Third case  $u_{\partial,1} = -\partial_x e_\kappa$ ,  $u_{\partial,2} = e_\kappa$ ,  $y_{\partial,1} = e_w$ ,  $y_{\partial,2} = \partial_x e_w$ .  
This imposes the shear force  $\partial_x e_\kappa := \partial_x (EI \partial_{xx} w)$  and flexural momentum as boundary inputs

$$\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \quad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \quad (3.31)$$

Null inputs correspond to a free condition.

- Fourth case  $u_{\partial,1} = -\partial_x e_\kappa$ ,  $u_{\partial,2} = \partial_x e_w$ ,  $y_{\partial,1} = e_w$ ,  $y_{\partial,2} = e_\kappa$ .  
This imposes the shear force and angular velocity as boundary inputs

$$\mathcal{B}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} -\partial_x e_\kappa(L) \\ \partial_x e_\kappa(0) \\ \partial_x e_w(L) \\ -\partial_x e_w(0) \end{pmatrix} \in \mathbb{R}^4 \quad \mathcal{C}_\partial \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{pmatrix} e_w(L) \\ -e_w(0) \\ e_\kappa(L) \\ -e_\kappa(0) \end{pmatrix} \in \mathbb{R}^4 \quad (3.32)$$

### 3.2.3.3 2D shallow water equations

This formulation may be found in [CR16, Section 6.2]. This model describes a thin fluid layer of constant density in hydrostatic balance, like the propagation of a tsunami wave far from

shore. Consider an open bounded connected set  $\Omega \subset \mathbb{R}^2$  and a constant bed profile. The mass conservation implies

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\mathbf{v}) = 0,$$

where  $h(x, y, t) \in \mathbb{R}$  is a scalar field representing the fluid height,  $\mathbf{v}(x, y, t) \in \mathbb{R}^2$  is the fluid velocity field. The conservation of linear momentum reads

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla(\rho g h) = 0,$$

where  $\rho$  is the mass density and  $g$  the gravitational acceleration constant. Using the identity

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{2} \nabla(\|\mathbf{v}\|^2) + (\nabla \times \mathbf{v}) \times \mathbf{v},$$

where  $\nabla \times$  is the rotational of  $\mathbf{v}$  (also denoted  $\operatorname{curl} \mathbf{v}$ ), the momentum is rearranged as follows

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \left( \frac{1}{2} \rho \|\mathbf{v}\|^2 + \rho g h \right) - \rho(\nabla \times \mathbf{v}) \times \mathbf{v}.$$

The last term on the right-hand side can be rewritten

$$\rho(\nabla \times \mathbf{v}) \times \mathbf{v} = \begin{bmatrix} 0 & -\rho\omega \\ \rho\omega & 0 \end{bmatrix} \mathbf{v},$$

with  $\omega = \partial_x v_y - \partial_y v_x$  the local vorticity term. To derive a suitable pH formulation, the total energy, made up of kinetic and potential contribution, has to be invoked

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho h \|\mathbf{v}\|^2 + \rho g h^2 \right\} d\Omega.$$

479 As energy variable the fluid height and the linear momentum are chosen

$$\alpha_h = h, \quad \boldsymbol{\alpha}_v = \rho \mathbf{v}. \quad (3.33)$$

480 The Hamiltonian is a non separable functional of the energy variables

$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho} \alpha_h \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h^2 \right\} d\Omega. \quad (3.34)$$

481 The co-energy variables are given by

$$e_h := \frac{\delta H}{\delta \alpha_h} = \frac{1}{2\rho} \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h, \quad e_v := \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v. \quad (3.35)$$

482 The mass and momentum conservation are then rewritten as follows

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \boldsymbol{\mathcal{G}} \end{bmatrix} \begin{pmatrix} e_h \\ e_v \end{pmatrix}, \quad (3.36)$$

The gyroscopic skew-symmetric term  $\mathcal{G}$  introduces a non-linearity as it depends on the energy variables

$$\mathcal{G}(\alpha_h, \alpha_v) = \frac{\omega}{\alpha_h} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \omega = \partial_x \alpha_{v,y} - \partial_y \alpha_{v,x}.$$

483 Despite the non-standard formulation, the energy rate provides anyway the boundary vari-  
484 ables

$$\begin{aligned} \dot{H} &= + \int_{\Omega} \{e_h \partial_t \alpha_h + \mathbf{e}_v \cdot \partial_t \alpha_v\} \, d\Omega, \\ &= - \int_{\Omega} \{e_h \operatorname{div} \mathbf{e}_v + \mathbf{e}_v \cdot (\operatorname{grad} e_h - \mathcal{G} \mathbf{e}_v)\} \, d\Omega, && \text{skew-symmetry of } \mathcal{G}, \\ &= - \int_{\Omega} \{e_h \operatorname{div} \mathbf{e}_v + \mathbf{e}_v \cdot \operatorname{grad} e_h\} \, d\Omega, && \text{Chain rule,} \\ &= - \int_{\Omega} \operatorname{div}(e_h \mathbf{e}_v) \, d\Omega, && \text{Stokes theorem,} \\ &= - \int_{\partial\Omega} e_h \mathbf{e}_v \cdot \mathbf{n} \, dS = - \langle e_h, \mathbf{e}_v \cdot \mathbf{n} \rangle_{\partial\Omega}. \end{aligned} \tag{3.37}$$

485 Again two possible cases of uniform boundary causality arise:

- 486 • First case  $u_{\partial} = e_h, \quad y_{\partial} = \mathbf{e}_v \cdot \mathbf{n}$ .  
487 This imposes the variable  $e_h := h$  as boundary input and corresponds to a given water  
488 level for a fluid boundary.
- 489 • Second case  $u_{\partial} = \mathbf{e}_v \cdot \mathbf{n}, \quad y_{\partial} = e_p$ .  
490 This imposes the variable  $\mathbf{e}_v \cdot \mathbf{n} := h\mathbf{v} \cdot \mathbf{n}$  as boundary input and corresponds to a given  
491 volumetric flow rate.

### 492 3.3 Conclusion

493 In this chapter, the main mathematical tools needed to understand infinite-dimensional pHs  
494 were recalled. A general characterization of the underlying operators behind a boundary  
495 control pH system is still an open topic. In Chapter 7, these operators are characterized, in  
496 connection to the discretization method developed.

497

## Part II

498

# Port-Hamiltonian elasticity and thermoelasticity

499



# Elasticity in port-Hamiltonian form

I try not to break the rules but merely to test their elasticity.

*Bill Veeck*

## Contents

|            |  |           |
|------------|--|-----------|
| <b>4.1</b> | <b>Continuum mechanics . . . . .</b>                               | <b>23</b> |
| 4.1.1      | Non linear formulation of elasticity . . . . .                     | 23        |
| 4.1.2      | The linear elastodynamics problem . . . . .                        | 25        |
| <b>4.2</b> | <b>Port-Hamiltonian formulation of linear elasticity . . . . .</b> | <b>27</b> |
| 4.2.1      | Energy and co-energy variables . . . . .                           | 27        |
| 4.2.2      | Final system and associated Stokes-Dirac structure . . . . .       | 29        |
| <b>4.3</b> | <b>Conclusion . . . . .</b>  | <b>33</b> |



Continuum mechanics is the mathematical description of how materials behave kinematically under external excitations. In this framework, the microscopic structure of a material body is neglected and a macroscopic viewpoint, that describes the body as a continuum, is adopted. This leads to a PDE based model. In this chapter, the general linear elastodynamics problem is recalled. A suitable port-Hamiltonian formulation is then derived.

## 4.1 Continuum mechanics

In this section, the main concepts behind a deformable continuum are briefly recalled following [Lee12]. For a detailed discussion on this topic, the reader may consult [Abe12, LPKL12].

### 4.1.1 Non linear formulation of elasticity

The bounded region of  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$  occupied by a solid is called configuration. The reference configuration  $\Omega$  is the domain that a body occupies at the initial state. To describe how the

body deforms in time the deformation map  $\Phi : \Omega \times [0, T_f] \rightarrow \Omega' \subset \mathbb{R}^d$  is introduced. This map is differentiable and orientation preserving, and the image of  $\Omega$  under  $\Phi(\cdot, t) \forall t \in [0, T_f]$  is called the deformed configuration  $\Omega_t$ . Given a specific point in the reference frame its image is denoted by  $\mathbf{y} = \Phi(\mathbf{x}, t)$ . The gradient of the deformation map is called the deformation gradient  $\mathbf{F} := \nabla_{\mathbf{x}} \Phi = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ . A rigid deformation maps a point  $\mathbf{x} \in \Omega \rightarrow \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$ , where  $\mathbf{A}(t)$  is an orthogonal matrix and  $\mathbf{b}(t) \in \mathbb{R}^d$  a vector. A differentiable deformation map  $\Phi$  is a rigid deformation iff  $\mathbf{F}^\top \mathbf{F} - \mathbf{I} = 0$ , where  $\mathbf{I}$  is the identity in  $\mathbb{R}^{d \times d}$  (for the proof see [Cia88], page 44). For this reason, a suitable measure of the deformation is the Green-St.Venant strain tensor  $\frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I})$ .

A quantity of interest is the displacement  $\mathbf{u} : \Omega \times [0, T_f] \rightarrow \mathbb{R}^d$  with respect to the reference configuration. It is defined as  $\mathbf{u}(\mathbf{x}, t) = \Phi(\mathbf{x}, t) - \mathbf{x}$ . The gradient of the displacement verifies  $\nabla_{\mathbf{x}} \mathbf{u} = \mathbf{F} - \mathbf{I}$ . The strain tensor can now be written in terms of the displacement

$$\begin{aligned} \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}) &= \frac{1}{2} \left[ (\nabla_{\mathbf{x}} \mathbf{u} + \mathbf{I})^\top (\nabla_{\mathbf{x}} \mathbf{u} + \mathbf{I}) - \mathbf{I} \right] \\ &= \frac{1}{2} \left[ \nabla_{\mathbf{x}} \mathbf{u} + (\nabla_{\mathbf{x}} \mathbf{u})^\top + (\nabla_{\mathbf{x}} \mathbf{u})^\top (\nabla_{\mathbf{x}} \mathbf{u}) \right], \end{aligned}$$

or in components

$$\frac{1}{2}(F_{ik}^\top F_{kj} - I_{ij}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right).$$

To state the balance laws the actual deformed configuration is considered. The linear and angular momenta in a subdomain  $\omega_t \subset \Omega_t$  are computed as

$$\int_{\omega_t} \rho \mathbf{v} \, d\omega_t, \quad \text{and} \quad \int_{\omega_t} \rho \mathbf{y} \times \mathbf{v} \, d\omega_t,$$

where  $\rho$  is the mass density and the velocity  $\mathbf{v} = \frac{D\mathbf{u}}{Dt}(\mathbf{y}, t) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t)$  is the material time derivative of the displacement (see [Abe12, Chapter 1]). Let  $\omega_{t,1}, \omega_{t,2}$  be two subregions in a deformed continuum  $\Omega_t$  with contacting surface  $S_{12}$ . There is a force acting on this surface for a continuum that is called stress vector. If  $\mathbf{n}$  is the outward normal at  $\mathbf{y}$  on  $S_{12}$  with respect to  $\omega_{t,1}$ , then the surface force that  $\omega_{t,1}$  exerts on  $\omega_{t,2}$  is denoted by  $\mathbf{t}(\mathbf{y}, \mathbf{n}) \in \mathbb{R}^d$ . By the Newton third law, the surface force that  $\omega_{t,2}$  applies on  $\omega_{t,1}$  is given by  $\mathbf{t}(\mathbf{y}, -\mathbf{n}) = -\mathbf{t}(\mathbf{y}, \mathbf{n})$ . It is assumed that the linear and angular momentum balances hold for any subregion  $\omega_t \in \Omega_t$

$$\begin{aligned} \frac{d}{dt} \int_{\omega_t} \rho \mathbf{v} \, d\omega_t &= \int_{\partial\omega_t} \mathbf{t}(\mathbf{y}, \mathbf{n}) \, dS + \int_{\omega_t} \mathbf{f} \, d\omega_t, \\ \frac{d}{dt} \int_{\omega_t} \rho \mathbf{y} \times \mathbf{v} \, d\omega_t &= \int_{\partial\omega_t} \mathbf{y} \times \mathbf{t}(\mathbf{y}, \mathbf{n}) \, dS + \int_{\omega_t} \mathbf{y} \times \mathbf{f} \, d\omega_t, \end{aligned}$$

526 where  $\partial\omega_t$  stands for the boundary surface of the subdomain  $\omega_t$ ,  $\mathbf{n}$  is the outward normal to  
 527 the surface  $\partial\omega_t$  and  $\mathbf{f}$  represents an exterior body force. The following theorem characterizes  
 528 the stress vector (see [Cia88, Chapter 2]):

529 **Theorem 1** (Cauchy's theorem)



530 *If the linear and angular momenta balances hold, then there exists a matrix-valued function*  
 531  *$\Sigma$  from  $\Omega_t$  to  $\mathbb{S}$  such that  $\mathbf{t}(\mathbf{y}, \mathbf{n}) = \Sigma(\mathbf{y})\mathbf{n}$ ,  $\forall \mathbf{y} \in \Omega_t$  where the right-hand side is the matrix-*  
 532 *vector multiplication.*

The set  $\mathbb{S} = \mathbb{R}_{\text{sym}}^{d \times d}$  denotes the field of symmetric matrices in  $\mathbb{R}^{d \times d}$ . The symmetry of the stress tensor  $\Sigma$  is due to the balance of angular momentum. The divergence theorem can then be applied

$$\int_{\partial\omega_t} \Sigma \mathbf{n} \, dS = \int_{\omega_t} \nabla_y \cdot \Sigma \, d\omega_t,$$

where  $\nabla_y \cdot$  is the tensor divergence with respect to the deformed configuration,  $\nabla_y \cdot \Sigma = \sum_{i=1}^d \frac{\partial \Sigma_{ij}}{\partial y_i}$ . Because the considered subregion  $\omega_t$  is arbitrary, using the linear balance momentum and the conservation of mass, the following PDE is found

$$\rho \frac{D\mathbf{v}}{Dt} - \nabla_y \cdot \Sigma = \mathbf{f}, \quad \mathbf{y} \in \Omega_t.$$

533 This equation is written with respect to the deformed configuration  $\Omega_t$ . For a detailed deriva-  
 534 tion of this equation the reader may consult [Abe12, Chapter 4]. To obtain a closed formula-  
 535 tion, the constitutive law, namely the link between the stress tensor  $\Sigma$  and the strain tensor  
 536  $\frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I})$ , has to be introduced. In the next section such relation will be discussed for the  
 537 case of linear elasticity.

#### 538 4.1.2 The linear elastodynamics problem

Whenever deformations are small,  $\|\nabla_x \mathbf{u}\| \ll 1$ , then the reference and deformed configurations are almost indistinguishable  $\mathbf{y} = \mathbf{x} + \mathbf{u} = \mathbf{x} + O(\nabla_x \mathbf{u}) \approx \mathbf{x}$ . This allows writing the linear momentum balance in the reference configuration

$$\rho \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t) - \text{Div } \Sigma(\mathbf{x}, t) = \mathbf{f}, \quad \mathbf{x} \in \Omega.$$

The material derivative simplifies to a partial one. The operator Div is the divergence of a tensor field with respect to the reference configuration (see Appendix A for a description of the differential operators)

$$\text{Div } \Sigma(\mathbf{x}, t) = \nabla_x \cdot \Sigma(\mathbf{x}, t) = \left( \sum_{i=1}^d \frac{\partial \Sigma_{ij}}{\partial x_i} \right)_{1 \leq j \leq d}.$$

Furthermore, the non-linear terms in the Green-St. Venant strain tensor can be dropped

$$\frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}) = \frac{1}{2} \left[ \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^\top + (\nabla_x \mathbf{u})^\top (\nabla_x \mathbf{u}) \right] \approx \frac{1}{2} \left[ \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^\top \right].$$

539 The linearized strain tensor (also called infinitesimal strain tensor) is the symmetric gradient  
 540 of the displacement

$$\boldsymbol{\varepsilon} := \text{Grad } \mathbf{u}, \quad \text{where} \quad \text{Grad } \mathbf{u} = \frac{1}{2} \left[ \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^\top \right]. \quad (4.1)$$

To obtain a closed system of equations, it is now necessary to characterize the relation between stress and strain. This relation is normally called *constitutive law*. In the following, the particular case of elastic materials is considered. These materials are able to return back to their original size and shapes after forces are removed. For this class of materials, the stress tensor is solely determined from the deformed configuration at a given time (Hooke's law)

$$\boldsymbol{\Sigma}(\mathbf{x}) = \boldsymbol{\mathcal{D}}(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x})).$$

The *stiffness tensor* or *elasticity tensor*  $\boldsymbol{\mathcal{D}} : \mathbb{S} \rightarrow \mathbb{S}$  is a rank 4 tensor that is symmetric positive definite and uniformly bounded above and below. Because of symmetry, its components satisfy

$$\mathcal{D}_{ijkl} = \mathcal{D}_{jikl} = \mathcal{D}_{klij}.$$

541 From the uniform boundedness of  $\boldsymbol{\mathcal{D}}$ , the map  $\boldsymbol{\mathcal{D}} : L^2(\Omega, \mathbb{S}) \rightarrow L^2(\Omega, \mathbb{S})$  is a symmetric positive  
 542 definite bounded linear operator ( $L^2(\Omega, \mathbb{S})$  is the space of square integrable symmetric tensor-  
 543 valued functions). The compliance tensor  $\boldsymbol{\mathcal{C}}$  is defined by  $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{D}}^{-1}$ . Thus  $\boldsymbol{\mathcal{C}} : \mathbb{S} \rightarrow \mathbb{S}$  is as  
 544 well symmetric positive definite and uniformly bounded above and below. An isotropic elastic  
 545 medium has the same kinematic properties in any direction and at each point. If an elastic  
 546 medium is isotropic, then the stiffness and compliance tensors assume the form

$$\boldsymbol{\mathcal{D}}(\cdot) = 2\mu(\cdot) \mathbf{I} + \lambda \text{Tr}(\cdot) \mathbf{I}, \quad \boldsymbol{\mathcal{C}}(\cdot) = \frac{1}{2\mu} \left[ (\cdot) - \frac{\lambda}{2\mu + d\lambda} \text{Tr}(\cdot) \mathbf{I} \right], \quad d = \{2, 3\}, \quad (4.2)$$

547 where  $\text{Tr}$  is the trace operator and the positive scalar functions  $\mu, \lambda$ , defined on  $\Omega$ , are called  
 548 the Lamé coefficients. In engineering applications it is easier to compute experimentally two  
 549 other parameters: the Young modulus  $E$  and Poisson's ratio  $\nu$ . Those are expressed in terms  
 550 of the Lamé coefficients as

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad (4.3)$$

551 and conversely

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}. \quad (4.4)$$

The stiffness and compliant tensor are expressed as

$$\boldsymbol{\mathcal{D}}(\cdot) = \frac{E}{1 + \nu} \left[ (\cdot) + \frac{\nu}{1 - 2\nu} \text{Tr}(\cdot) \mathbf{I} \right], \quad (4.5)$$

$$\boldsymbol{\mathcal{C}}(\cdot) = \frac{1 + \nu}{E} \left[ (\cdot) - \frac{\nu}{1 + \nu(d - 2)} \text{Tr}(\cdot) \mathbf{I} \right]. \quad (4.6)$$

The linear elastodynamics problem is formulated through a vector-valued PDE

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \text{Div}(\mathcal{D} \text{Grad } \mathbf{u}) = \mathbf{f}. \quad (4.7)$$

The classical elastodynamics problem is expressed considering the displacement  $\mathbf{u}$  as the unknown. This PDE goes together with appropriate boundary conditions that will be specified in 4.2.

## 4.2 Port-Hamiltonian formulation of linear elasticity

In this section a port-Hamiltonian formulation for elasticity is deduced from the classical elastodynamics problem. It must be highlighted that already in the seventies a purely hyperbolic formulation for elasticity was detailed [HM78]. The missing point is the clear connection with the theory of Hamiltonian PDEs. An Hamiltonian formulation can be found in [Gri15, Chapter 16], but without any connection to the concept of Stokes-Dirac structure induced by the underlying geometry.

### 4.2.1 Energy and co-energy variables

Consider an open connected set  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ . The displacement within a deformable continuum is given by Eq. (4.7).

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \text{Div}(\mathcal{D} \text{Grad } \mathbf{u}) = 0, \quad \mathbf{x} \in \Omega. \quad (4.8)$$

The contribution of the body force  $\mathbf{f}$  has been removed for ease of presentation. To derive a pH formulation, the total energy, that includes the kinetic and deformation energy, is needed

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho \|\partial_t \mathbf{u}\|^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} d\Omega. \quad (4.9)$$

The notation  $\mathbf{A} : \mathbf{B} = \text{Tr}(\mathbf{A}^\top \mathbf{B}) = \sum_{i,j} A_{ij} B_{ij}$  denotes the tensor contraction. Recall that  $\boldsymbol{\varepsilon} = \text{Grad } \mathbf{u}$  and  $\boldsymbol{\Sigma} = \mathcal{D} \boldsymbol{\varepsilon}$ . The energy variables are then the linear momentum and the deformation field

$$\boldsymbol{\alpha}_v = \rho \mathbf{v}, \quad \mathbf{A}_\varepsilon = \boldsymbol{\varepsilon},$$

where  $\mathbf{v} := \partial_t \mathbf{u}$ . The Hamiltonian can be rewritten as a quadratic functional in the energy variables

$$H = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho} \boldsymbol{\alpha}_v^2 + (\mathcal{D} \mathbf{A}_\varepsilon) : \mathbf{A}_\varepsilon \right\} d\Omega. \quad (4.10)$$

The co-energy variables are given by

$$\mathbf{e}_v := \frac{\delta H}{\delta \boldsymbol{\alpha}_v} = \mathbf{v}, \quad \mathbf{E}_\varepsilon := \frac{\delta H}{\delta \mathbf{A}_\varepsilon} = \boldsymbol{\Sigma}. \quad (4.11)$$

The tensor-valued co-energy  $\mathbf{E}_\varepsilon$  is obtained by taking the variational derivative with respect to a tensor.

**Proposition 3**

*The variational derivative of the Hamiltonian with respect to the strain tensor is the stress tensor  $\delta_{\mathbf{A}_\varepsilon} H = \boldsymbol{\Sigma}$ .*

*Proof.* Let  $\mathbb{S} : \mathbb{R}_{\text{sym}}^{d \times d}$  be the space of symmetric tensor and  $L^2(\Omega, \mathbb{S})$  the space of the square integrable symmetric tensors endowed with the tensor contraction as inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle_{L^2(\Omega, \mathbb{S})} = \int_{\Omega} \mathbf{A} : \mathbf{B} \, d\Omega. \quad (4.12)$$

The contribution due to the deformation part in Hamiltonian is given by:

$$H_{\text{def}}(\mathbf{A}_\varepsilon) = \frac{1}{2} \int_{\Omega} (\mathcal{D} \mathbf{A}_\varepsilon) : \mathbf{A}_\varepsilon \, d\Omega.$$

A variation  $\Delta \mathbf{A}_\varepsilon$  of the strain tensor with respect to a given value  $\bar{\mathbf{A}}_\varepsilon$  leads to:

$$\begin{aligned} H_{\text{def}}(\bar{\mathbf{A}}_\varepsilon + \eta \Delta \mathbf{A}_\varepsilon) &= + \frac{1}{2} \int_{\Omega} (\mathcal{D} \bar{\mathbf{A}}_\varepsilon) : \bar{\mathbf{A}}_\varepsilon \, d\Omega \\ &+ \eta \frac{1}{2} \int_{\Omega} \left\{ (\mathcal{D} \bar{\mathbf{A}}_\varepsilon) : \Delta \mathbf{A}_\varepsilon + (\mathcal{D} \Delta \mathbf{A}_\varepsilon) : \bar{\mathbf{A}}_\varepsilon \right\} \, d\Omega + O(\eta^2). \end{aligned}$$

The term  $(\mathcal{D} \Delta \mathbf{A}_\varepsilon) : \bar{\mathbf{A}}_\varepsilon$  can be further rearranged using the symmetry of  $\mathcal{D}$  and the commutativity of the tensor contraction

$$(\mathcal{D} \Delta \mathbf{A}_\varepsilon) : \bar{\mathbf{A}}_\varepsilon = (\mathcal{D} \bar{\mathbf{A}}_\varepsilon) : \Delta \mathbf{A}_\varepsilon,$$

so that

$$H_{\text{def}}(\bar{\mathbf{A}}_\varepsilon + \eta \Delta \mathbf{A}_\varepsilon) = \frac{1}{2} \int_{\Omega} (\mathcal{D} \bar{\mathbf{A}}_\varepsilon) : \bar{\mathbf{A}}_\varepsilon \, d\Omega + \eta \int_{\Omega} (\mathcal{D} \bar{\mathbf{A}}_\varepsilon) : \Delta \mathbf{A}_\varepsilon \, d\Omega + O(\eta^2).$$

By definition of variational derivative it can be written:

$$H_{\text{def}}(\bar{\mathbf{A}}_\varepsilon + \eta \Delta \mathbf{A}_\varepsilon) = H_{\text{def}}(\bar{\mathbf{A}}_\varepsilon) + \eta \left\langle \frac{\delta H}{\delta \mathbf{A}_\varepsilon}, \Delta \mathbf{A}_\varepsilon \right\rangle_{L^2(\Omega, \mathbb{S})} + O(\eta^2),$$

Then, by identification

$$\frac{\delta H_{\text{def}}}{\delta \mathbf{A}_\varepsilon} = \mathcal{D} \bar{\mathbf{A}}_\varepsilon = \boldsymbol{\Sigma}.$$

Since the Hamiltonian is separable then  $\delta_{\mathbf{A}_\varepsilon} H_{\text{def}} = \delta_{\mathbf{A}_\varepsilon} H$ , leading to the final result.  $\square$

---

### 4.2.2 Final system and associated Stokes-Dirac structure

It is now possible to state the final pH form

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_v \\ \mathbf{A}_\varepsilon \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \text{Div} \\ \text{Grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \end{pmatrix}. \quad (4.13)$$

The first equation of the system is the conservation of linear momentum. The second represents a compatibility condition

$$\begin{aligned} \partial_t \mathbf{A}_\varepsilon &= \text{Grad}(\mathbf{e}_v), \\ \partial_t \boldsymbol{\varepsilon} &= \text{Grad}(\mathbf{v}), \\ \partial_t \text{Grad } \mathbf{u} &= \text{Grad}(\partial_t \mathbf{u}). \end{aligned} \quad (4.14)$$

Assuming that  $\mathbf{u} \in C^2$ , higher order derivatives commute (Clairaut's theorem). Hence, the equation is verified. The following theorem ensures the differential operator is formally skew-adjoint (one can also find this result in the recent article [PZ20, Lemma 3.3], available as arXiv preprint).

#### Theorem 2

*The formal adjoint of the tensor divergence Div is  $-\text{Grad}$ , the opposite of the symmetric gradient.*

*Proof.* We denote by  $\mathbb{V} = \mathbb{R}^d$  the space of vector field in  $\mathbb{R}^d$  and by  $\mathbb{S} = \mathbb{R}^{d \times d}$  the space of symmetric tensor field in  $\mathbb{R}^{d \times d}$ . Let us consider the Hilbert space of the square integrable symmetric tensors  $L^2(\Omega, \mathbb{S})$  with scalar product defined in (4.12). Moreover consider the Hilbert space of the square integrable vector function  $L^2(\Omega, \mathbb{V})$ , endowed with the usual scalar product

$$\langle \mathbf{a}, \mathbf{b} \rangle_{L^2(\Omega, \mathbb{V})} = \int_{\Omega} \mathbf{a} \cdot \mathbf{b} \, d\Omega.$$

Let us consider the tensor divergence operator defined as:

$$\begin{aligned} \text{Div} : L^2(\Omega, \mathbb{S}) &\rightarrow L^2(\Omega, \mathbb{V}), \\ \boldsymbol{\Psi} &\rightarrow \text{Div } \boldsymbol{\Psi} = \boldsymbol{\psi}, \end{aligned} \quad \text{with } \psi_j = \text{div}(\Psi_{ij}) = \sum_{i=1}^d \frac{\partial \Psi_{ij}}{\partial x_i}.$$

We try to identify  $\text{Div}^*$

$$\begin{aligned} \text{Div}^* : L^2(\Omega, \mathbb{V}) &\rightarrow L^2(\Omega, \mathbb{S}), \\ \boldsymbol{\phi} &\rightarrow \text{Div}^* \boldsymbol{\phi} = \boldsymbol{\Phi}, \end{aligned}$$

such that

$$\begin{aligned} \langle \text{Div } \boldsymbol{\Psi}, \boldsymbol{\phi} \rangle_{L^2(\Omega, \mathbb{V})} &= \langle \boldsymbol{\Psi}, \text{Div}^* \boldsymbol{\phi} \rangle_{L^2(\Omega, \mathbb{S})}, & \forall \boldsymbol{\Psi} \in \text{Dom}(\text{Div}) \subset L^2(\Omega, \mathbb{S}), \\ & & \forall \boldsymbol{\phi} \in \text{Dom}(\text{Div}^*) \subset L^2(\Omega, \mathbb{V}). \end{aligned}$$

Now let us take  $\boldsymbol{\Psi} \in C_0^1(\Omega, \mathbb{S}) \subset \text{Domain}(\text{Div})$  the space of differentiable symmetric tensors

---

with compact support in  $\Omega$ . Additionally  $\phi$  will belong to  $C_0^1(\Omega, \mathbb{V}) \subset \text{Dom}(\text{Div}^*)$ , the space of differentiable vector functions with compact support in  $\Omega$ . Then

$$\begin{aligned}
 \langle \text{Div } \Psi, \phi \rangle_{L^2(\Omega, \mathbb{V})} &= \int_{\Omega} \psi \cdot \phi \, d\Omega, \\
 &= \int_{\Omega} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial \Psi_{ij}}{\partial x_i} \phi_j \, d\Omega, \\
 &= - \int_{\Omega} \sum_{i=1}^d \sum_{j=1}^d \Psi_{ij} \frac{\partial \phi_j}{\partial x_i} \, d\Omega, \quad \text{since the functions vanish at the boundary,} \\
 &= - \int_{\Omega} \sum_{i=1}^d \sum_{j=1}^d \Psi_{ij} F_{ij} \, d\Omega, \quad \text{where } F_{ij} = \frac{\partial \phi_j}{\partial x_i}, \\
 &= - \langle \Psi, \mathbf{F} \rangle_{L^2(\Omega, \mathbb{S})}, \quad \mathbf{F} = \text{grad } \phi.
 \end{aligned}$$

591 But in this latter case, it could not be stated that  $\mathbf{F} \in L^2(\Omega, \mathbb{S})$ . Now, since  $\Psi \in L^2(\Omega, \mathbb{S})$ ,  
 592  $\Psi_{ji} = \Psi_{ij}$ , thus the last equality can be further decomposed as

$$\sum_{i,j} \Psi_{ij} \frac{\partial \phi_j}{\partial x_i} = \sum_{i,j} \Psi_{ij} \frac{1}{2} \left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right) = \sum_{i,j} \Psi_{ij} \Phi_{ij}, \quad \text{with } \Phi_{ij} := \frac{1}{2} \left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right).$$

Thus  $\Phi = \text{Grad } \phi \in L^2(\Omega, \mathbb{S})$  and it can be stated that:

$$\begin{aligned}
 \langle \text{Div } \Psi, \phi \rangle_{L^2(\Omega, \mathbb{V})} &= - \int_{\Omega} \sum_{i,j} \Psi_{ij} \frac{1}{2} \left( \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right) \, d\Omega \\
 &= - \int_{\Omega} \sum_{i,j} \Psi_{ij} \Phi_{ij} \, d\Omega = \langle \Psi, -\text{Grad } \phi \rangle_{L^2(\Omega, \mathbb{S})}.
 \end{aligned}$$

593 It can be concluded that the formal adjoint of  $\text{Div}$  is  $\text{Div}^* = -\text{Grad}$ . □

594 The boundary values are then found by evaluating the energy rate

$$\begin{aligned}
 \dot{H} &= \int_{\Omega} \{ \mathbf{e}_v \cdot \partial_t \boldsymbol{\alpha}_v + \mathbf{E}_{\varepsilon} : \partial_t \mathbf{A}_{\varepsilon} \} \, d\Omega, \\
 &= \int_{\Omega} \{ \mathbf{e}_v \cdot \text{Div } \mathbf{E}_{\varepsilon} + \mathbf{E}_{\varepsilon} : \text{Grad } \mathbf{e}_v \} \, d\Omega, \\
 &= \int_{\Omega} \text{div}(\mathbf{E}_{\varepsilon} \mathbf{e}_v) \, d\Omega, \quad \text{Stokes theorem (see Appendix A Eq. (A.6)),} \\
 &= \int_{\partial\Omega} \mathbf{e}_v \cdot (\mathbf{E}_{\varepsilon} \mathbf{n}) \, dS = \langle \mathbf{e}_v, \mathbf{E}_{\varepsilon} \mathbf{n} \rangle_{L^2(\partial\Omega, \mathbb{R}^d)}.
 \end{aligned} \tag{4.15}$$

595 The imposition of the velocity field along the boundary  $\mathbf{e}_v = \partial_t \mathbf{u}$  corresponds to a Dirichlet  
 596 condition. Setting  $\mathbf{E}_{\varepsilon} \mathbf{n} = \boldsymbol{\Sigma} \mathbf{n} = \mathbf{t}$  (the traction) corresponds to a Neumann condition.

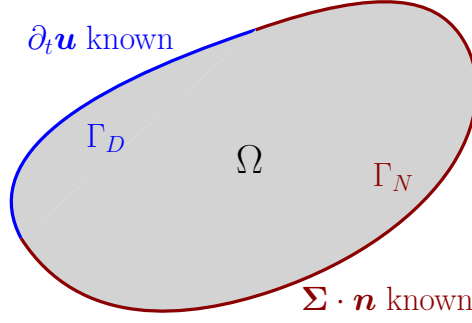


Figure 4.1: A 2D continuum with Neumann and Dirichlet boundary conditions

597 Consider a partition of the boundary  $\partial\Omega = \bar{\Gamma}_N \cup \bar{\Gamma}_D$  and  $\Gamma_N \cap \Gamma_D = \{\emptyset\}$ , where a Dirichlet  
 598 and a Neumann condition applies on the open subset  $\Gamma_D$  and  $\Gamma_N$  respectively (see Fig. 4.1).  
 599 Then the final pH formulation reads

$$\begin{aligned}
 \frac{\partial}{\partial t} \begin{pmatrix} \alpha_v \\ \mathbf{A}_\varepsilon \end{pmatrix} &= \underbrace{\begin{bmatrix} \mathbf{0} & \text{Div} \\ \text{Grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \end{pmatrix}, \\
 \mathbf{u}_\partial &= \underbrace{\begin{bmatrix} \gamma_0^{\Gamma_D} & \mathbf{0} \\ \mathbf{0} & \gamma_n^{\Gamma_N} \end{bmatrix}}_{\mathcal{B}_\partial} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \end{pmatrix}, \\
 \mathbf{y}_\partial &= \underbrace{\begin{bmatrix} \mathbf{0} & \gamma_n^{\Gamma_D} \\ \gamma_0^{\Gamma_N} & \mathbf{0} \end{bmatrix}}_{\mathcal{C}_\partial} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \end{pmatrix},
 \end{aligned} \tag{4.16}$$

600 where  $\gamma_0^{\Gamma_*}$  denotes the trace over the set  $\Gamma_*$ , namely  $\gamma_0^{\Gamma_*} \mathbf{e}_v = \mathbf{e}_v|_{\Gamma_*}$ . Furthermore,  $\gamma_n^{\Gamma_*}$  denotes  
 601 the normal trace over the set  $\Gamma_*$ , namely  $\gamma_n^{\Gamma_*} \mathbf{E}_\varepsilon = \mathbf{E}_\varepsilon \mathbf{n}|_{\Gamma_*}$ .

**Conjecture 1** (Stokes-Dirac structure for elastodynamics)

Let  $H^{\text{Grad}}(\Omega, \mathbb{V})$  denote the space of vectors with symmetric gradient in  $L^2(\Omega, \mathbb{S})$  and  $H^{\text{Div}}(\Omega, \mathbb{S})$  be the space of symmetric tensors with divergence in  $L^2(\Omega, \mathbb{V})$ . Consider the following definitions

$$\begin{aligned}
 H &:= H^{\text{Grad}}(\Omega, \mathbb{V}) \times H^{\text{Div}}(\Omega, \mathbb{S}), \\
 F &:= L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{S}), \\
 F_\partial &:= L^2(\Gamma_D, \mathbb{V}) \times L^2(\Gamma_N, \mathbb{V}).
 \end{aligned}$$

602 The set

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \mathbf{f} \\ \mathbf{f}_\partial \\ \mathbf{e} \\ \mathbf{e}_\partial \end{pmatrix} \mid \mathbf{e} \in H, \mathbf{f} = \mathcal{J}\mathbf{e}, \mathbf{f}_\partial = \mathcal{B}_\partial \mathbf{e}, \mathbf{e}_\partial = -\mathcal{C}_\partial \mathbf{e} \right\}, \tag{4.17}$$

where  $\mathbf{e} = (\mathbf{e}_v, \mathbf{E}_\varepsilon)$  and  $\mathcal{J}, \mathcal{B}_\partial, \mathcal{C}_\partial$  are defined in (4.16), is a Stokes–Dirac structure with respect to the pairing

$$\langle\langle (\mathbf{f}^1, \mathbf{f}_\partial^1, \mathbf{e}^1, \mathbf{e}_\partial^1), (\mathbf{f}^2, \mathbf{f}_\partial^2, \mathbf{e}^2, \mathbf{e}_\partial^2) \rangle\rangle := \langle \mathbf{e}^1, \mathbf{f}^2 \rangle_F + \langle \mathbf{e}^2, \mathbf{f}^1 \rangle_F + \langle \mathbf{e}_\partial^1, \mathbf{f}_\partial^2 \rangle_{F_\partial} + \langle \mathbf{e}_\partial^2, \mathbf{f}_\partial^1 \rangle_{F_\partial}, \quad (4.18)$$

where

$$\langle\langle (\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d}) \rangle\rangle_{F_\partial} = \int_{\Gamma_D} \mathbf{a} \cdot \mathbf{c} \, dS + \int_{\Gamma_N} \mathbf{b} \cdot \mathbf{d} \, dS, \quad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{V}.$$

**Crucial points to obtain a rigorous proof** The crucial point that needs to be elucidated is where the boundary variables live. These variables belong to the fractional Sobolev spaces  $H^{\frac{1}{2}}(\partial\Omega, \mathbb{V})$ ,  $H^{-\frac{1}{2}}(\partial\Omega, \mathbb{V})$  linked by duality with respect to the pivot space  $L^2(\partial\Omega, \mathbb{V})$ . This is why a  $L^2$  inner product has been assumed as boundary inner product. Furthermore, the partition of the boundary due to the non uniform boundary control complicates the proof, since one has to properly connect the two partitions at their interconnection.

**Elements to support the conjecture** A Stokes–Dirac is characterized by the fact that  $D_{\mathcal{J}} = D_{\mathcal{J}}^\perp$ . Then one has to show that  $D_{\mathcal{J}} \subset D_{\mathcal{J}}^\perp$  and  $D_{\mathcal{J}}^\perp \subset D_{\mathcal{J}}$ . The main steps of Theorem 3.6 in [LGZM05] are followed here to support the substantiation of the conjecture. The integration by parts formula is applied as in (4.15).

Step 1. To show that  $D_{\mathcal{J}} \subset D_{\mathcal{J}}^\perp$ , take  $(\mathbf{f}, \mathbf{f}_\partial, \mathbf{e}, \mathbf{e}_\partial) \in D_{\mathcal{J}}$ . Then

$$\begin{aligned} \langle\langle (\mathbf{f}, \mathbf{f}_\partial, \mathbf{e}, \mathbf{e}_\partial), (\mathbf{f}, \mathbf{f}_\partial, \mathbf{e}, \mathbf{e}_\partial) \rangle\rangle &= 2 \langle \mathbf{e}, \mathbf{f} \rangle_F + 2 \langle \mathbf{e}_\partial, \mathbf{f}_\partial \rangle_{F_\partial}, \\ &= 2 \langle \mathbf{e}, \mathcal{J}\mathbf{e} \rangle_F + 2 \langle \mathbf{e}_\partial, \mathbf{f}_\partial \rangle_{F_\partial}, \\ &= 2 \int_{\Omega} \{ \mathbf{e}_v \cdot \text{Div } \mathbf{E}_\varepsilon + \mathbf{E}_\varepsilon : \text{Grad } \mathbf{e}_v \} \, d\Omega \\ &\quad - 2 \int_{\Gamma_D} \mathbf{e}_v \cdot (\mathbf{E}_\varepsilon \mathbf{n}) \, dS - 2 \int_{\Gamma_N} \mathbf{e}_v \cdot (\mathbf{E}_\varepsilon \mathbf{n}) \, dS, \\ &= 2 \int_{\Omega} \{ \mathbf{e}_v \cdot \text{Div } \mathbf{E}_\varepsilon + \mathbf{E}_\varepsilon : \text{Grad } \mathbf{e}_v \} \, d\Omega \\ &\quad - 2 \int_{\partial\Omega} \mathbf{e}_v \cdot (\mathbf{E}_\varepsilon \mathbf{n}) \, dS = 0, \quad \text{from (4.15)}. \end{aligned}$$

This implies  $D_{\mathcal{J}} \subset D_{\mathcal{J}}^\perp$ .

Step 2. Take  $(\phi, \phi_\partial, \epsilon, \epsilon_\partial) \in D_{\mathcal{J}}^\perp$  and  $\mathbf{e}_0 \in H$  with compact support on  $\Omega$ . This implies  $\mathcal{B}_\partial \mathbf{e}_0 = (\mathbf{0}, \mathbf{0})$  and  $\mathcal{C}_\partial \mathbf{e}_0 = (\mathbf{0}, \mathbf{0})$ . Taking  $(\mathcal{J}\mathbf{e}_0, \mathbf{0}, \mathbf{e}_0, \mathbf{0}) \in D_{\mathcal{J}}$  then

$$\langle\langle (\phi, \phi_\partial, \epsilon, \epsilon_\partial), (\mathcal{J}\mathbf{e}_0, \mathbf{0}, \mathbf{e}_0, \mathbf{0}) \rangle\rangle = \langle \epsilon, \mathcal{J}\mathbf{e}_0 \rangle_F + \langle \mathbf{e}_0, \phi \rangle_F = 0, \quad \forall \mathbf{e}_0 \in H.$$

It follows that  $\epsilon \in H$  and  $\phi = \mathcal{J}\epsilon$ .



Step 3. Take  $(\phi, \phi_\partial, \epsilon, \epsilon_\partial) \in D_{\mathcal{J}}^\perp$  and  $(\mathbf{f}, \mathbf{f}_\partial, \mathbf{e}, \mathbf{e}_\partial) \in D_{\mathcal{J}}$ . Variables  $\mathbf{e}, \epsilon$  are indeed tuples containing a vector and a tensor, namely  $\mathbf{e} = (\mathbf{e}_v, \mathbf{E}_\epsilon)$ ,  $\epsilon = (\epsilon_v, \mathcal{E}_\epsilon)$ . From step 2 and (4.18)

$$\begin{aligned} 0 &= \langle \mathbf{e}, \mathcal{J}\epsilon \rangle_F + \langle \mathcal{J}\mathbf{e}, \epsilon \rangle_F + \langle \mathbf{e}_\partial, \phi_\partial \rangle_{F_\partial} + \langle \epsilon_\partial, \mathbf{f}_\partial \rangle_{F_\partial}, \\ &= \int_{\partial\Omega} \{ \mathbf{e}_v \cdot (\mathcal{E}_\epsilon \cdot \mathbf{n}) + \epsilon_v \cdot (\mathbf{E}_\epsilon \cdot \mathbf{n}) \} \, dS + \langle -\mathcal{C}_\partial \mathbf{e}, \phi_\partial \rangle_{F_\partial} + \langle \epsilon_\partial, \mathcal{B}_\partial \mathbf{e} \rangle_{F_\partial} \end{aligned}$$

Consider the splitting of the boundary  $\partial\Omega = \bar{\Gamma}_N \cup \bar{\Gamma}_D$

$$\begin{aligned} \int_{\partial\Omega} \{ \mathbf{e}_v \cdot (\mathcal{E}_\epsilon \cdot \mathbf{n}) + \epsilon_v \cdot (\mathbf{E}_\epsilon \cdot \mathbf{n}) \} \, dS &= + \int_{\Gamma_N} \{ \mathbf{e}_{\partial,2} \cdot (\mathcal{E}_\epsilon \cdot \mathbf{n}) + \epsilon_v \cdot \mathbf{f}_{\partial,2} \} \, dS, \\ &+ \int_{\Gamma_D} \{ \mathbf{f}_{\partial,1} \cdot (\mathcal{E}_\epsilon \cdot \mathbf{n}) + \epsilon_v \cdot \mathbf{e}_{\partial,1} \} \, dS, \end{aligned}$$

where the elements of the vectors  $\mathbf{f}_\partial = (\mathbf{f}_{\partial,1}, \mathbf{f}_{\partial,2})$ ,  $\mathbf{e}_\partial = (\mathbf{e}_{\partial,1}, \mathbf{e}_{\partial,2})$  have been considered. By expanding of the terms  $\langle \mathbf{e}_\partial, \phi_\partial \rangle_{F_\partial} + \langle \epsilon_\partial, \mathbf{f}_\partial \rangle_{F_\partial}$  and given the fact that  $\mathbf{e}$  is arbitrary then

$$\phi_\partial = \begin{bmatrix} \gamma_0^{\Gamma_D} & \mathbf{0} \\ \mathbf{0} & \gamma_n^{\Gamma_N} \end{bmatrix} \begin{pmatrix} \epsilon_v \\ \mathcal{E}_\epsilon \end{pmatrix}, \quad \epsilon_\partial = - \begin{bmatrix} \mathbf{0} & \gamma_n^{\Gamma_D} \\ \gamma_0^{\Gamma_N} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \epsilon_v \\ \mathcal{E}_\epsilon \end{pmatrix},$$

meaning that  $D_{\mathcal{J}}^\perp \subset D_{\mathcal{J}}$ .

Linear elasticity falls within the assumption of [Skr19]. Therefore, it is a well posed boundary control pH system. A question that naturally arises is how to reformulate this system using the language of differential geometry. This is possible through the usage of vector-valued differential forms. The interested reader may consult [Bre08].

## 4.3 Conclusion

In this chapter, the pH formulation of elasticity has been obtained. This model represents a generalization of the wave equation to higher dimensional variables. This leads to the introduction of symmetric tensorial quantities describing the state of stress and deformation within the body.

For a plane continuum with moderate thickness, it is possible to reduce the general three-dimensional mode to two uncoupled systems: one representing the in-plane behavior ruled by 2D elasticity and one representing the out-of-plane deflection. This will be the object of the next chapter dedicated to the study of a pH formulation of plate bending. It is important to remember that plate models are just particular cases of three-dimensional elasticity.



# Port-Hamiltonian plate theory

You get tragedy where the tree, instead of bending, breaks.

*Culture and Value*  
Ludwig Wittgenstein

## Contents

|            |   |           |
|------------|---|-----------|
| <b>5.1</b> | <b>First order plate theory . . . . .</b>                         | <b>36</b> |
| 5.1.1      | Mindlin-Reissner model . . . . .                                  | 37        |
| 5.1.2      | Kirchhoff-Love model . . . . .                                    | 38        |
| <b>5.2</b> | <b>Port-Hamiltonian formulation of isotropic plates . . . . .</b> | <b>40</b> |
| 5.2.1      | Port-Hamiltonian Mindlin plate . . . . .                          | 41        |
| 5.2.2      | Port-Hamiltonian Kirchhoff plate . . . . .                        | 45        |
| <b>5.3</b> | <b>Laminated anisotropic plates . . . . .</b>                     | <b>50</b> |
| 5.3.1      | Port-Hamiltonian laminated Mindlin plate . . . . .                | 52        |
| 5.3.2      | Port-Hamiltonian laminated Kirchhoff plate . . . . .              | 53        |
| <b>5.4</b> | <b>Conclusion . . . . .</b>                                       | <b>54</b> |



lates are plane structural elements with a small thickness compared to the planar dimensions. Thanks to this feature, it is not necessary to model plate structures using three-dimensional elasticity. Dimensional reduction strategies are employed to describe plate structures as two-dimensional problems. These strategies rely on an educated guess of the displacement field. For beams and plates this field is expressed in terms of unknown functions  $\phi_i^j(x, y, t)$  that solely depends on the midplane coordinates  $(x, y)$

$$u_i(x, y, z, t) = \sum_{j=0}^m (z)^j \phi_i^j(x, y, t).$$

where  $u_i$ ,  $i = \{x, y, z\}$  are the components of the displacement field. A first-order approximation is commonly used, meaning that a linear dependence on  $z$  is considered. Two main models arise from such a framework:

- the Mindlin-Reissner model for thick plates;

- the Kirchhoff-Love model for thin plates.

In this chapter it is shown how to formulate first-order plate models as pHs.

## 5.1 First order plate theory

As previously stated, first order theories assume a linear dependence on the vertical coordinate (cf. [Red06])

$$u_i(x, y, z, t) = \phi_i^0(x, y, t) + z\phi_i^1(x, y, t).$$

This hypothesis implies that the fibers, i.e. segments perpendicular to the mid-plane before deformation, remain straight after deformation. Additionally, for plate with moderate thickness the fibers are considered inextensible, meaning that  $\phi_z^1 = 0$ . These assumptions lead to the following displacement field

$$\begin{aligned} u_x(x, y, z, t) &= u_x^0(x, y, t) - z\theta_x(x, y, t), \\ u_y(x, y, z, t) &= u_y^0(x, y, t) - z\theta_y(x, y, t), \\ u_z(x, y, z, t) &= u_z^0(x, y, t), \end{aligned} \tag{5.1}$$

where  $u_i^0(x, y, t) = \phi_i^0(x, y, t)$ ,  $\theta_i(x, y, t) = -\phi_i^1(x, y, t)$ . Assuming a linear elastic behavior, the 3D strain tensors for such a displacement field takes the form

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta) - z\frac{1}{2}(\partial_\beta \theta_\alpha + \partial_\alpha \theta_\beta) = \varepsilon_{\alpha\beta}^0 - z\kappa_{\alpha\beta}, \tag{5.2}$$

$$\varepsilon_{\alpha z} = \frac{1}{2}(\partial_\alpha u_z - \theta_\alpha) = \frac{1}{2}\gamma_\alpha, \tag{5.3}$$

where  $\alpha = \{x, y\}$ ,  $\beta = \{x, y\}$ . The tensors  $\varepsilon^0$ ,  $\kappa$ ,  $\gamma$  are called membrane, bending (or curvature) and shear strain tensor

$$\varepsilon^0 = \text{Grad } \mathbf{u}^0, \tag{5.4}$$

$$\kappa = \text{Grad } \boldsymbol{\theta}, \tag{5.5}$$

$$\gamma = \text{grad } u_z - \boldsymbol{\theta}. \tag{5.6}$$

where  $\mathbf{u}^0 = (u_x, u_y)^\top$ ,  $\boldsymbol{\theta} = (\theta_x, \theta_y)^\top$ . For now, it is assumed that the material is isotropic, linear elastic (in Section §5.3 this hypothesis is removed). Recall the Hooke's law for 3D continua (see Eq. (4.5))

$$\boldsymbol{\Sigma} = \frac{E}{1+\nu} \left[ \boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu} \text{Tr}(\boldsymbol{\varepsilon}) \mathbf{I}_{3 \times 3} \right].$$

where  $E$ ,  $\nu$  are the Young modulus and Poisson ratio. The hypothesis of inextensible fibers implies  $\varepsilon_{zz} = 0$ . However, imposing a plane strain condition provides a model that is too stiff. Rather than a plain strain assumption, a plain stress hypothesis is used to derive the constitutive law for plates. The displacement field (5.1) is left unchanged, but, instead of  $\varepsilon_{zz}$ ,

$\Sigma_{zz}$  is set to zero. If  $\Sigma_{zz} = 0$ , one gets

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).$$

Consequently, it is computed

$$\text{Tr}(\boldsymbol{\varepsilon}) = \frac{1-2\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}).$$

The constitutive law for the in-plane stress takes the form

$$\boldsymbol{\Sigma}_{2D} = \boldsymbol{\mathcal{D}}_{2D} \boldsymbol{\varepsilon}_{2D},$$

where  $\boldsymbol{\Sigma}_{2D} = \Sigma_{\alpha\beta}$ ,  $\boldsymbol{\varepsilon}_{2D} = \varepsilon_{\alpha\beta}$  and

$$\boldsymbol{\mathcal{D}}_{2D} = \frac{E}{1-\nu^2} [(1-\nu)(\cdot) + \nu \text{Tr}(\cdot) \mathbf{I}_{2 \times 2}]. \quad (5.7)$$

Concerning the shear deformation, the constitutive law reduces to

$$\boldsymbol{\sigma}_s = G\boldsymbol{\gamma}, \quad (5.8)$$

where  $\boldsymbol{\sigma}_s := \boldsymbol{\Sigma}_{\alpha,3}$  and  $G = \frac{E}{2(1+\nu)}$  is the shear modulus. In the following sections, the most common plate models will be presented.

### 5.1.1 Mindlin-Reissner model

The Mindlin-Reissner model [Rei47, Min51] represents a first-order shear deformation theory for describing the bending of plate. The in-plane midplane displacement are zero  $\mathbf{u}^0(x, y) = \mathbf{0}$  for an isotropic plate that experiences only bending. Hence, the displacement field reduces to

$$\begin{aligned} u_x(x, y, z) &= -z\partial_x\theta_x, \\ u_y(x, y, z) &= -z\partial_y\theta_y, \\ u_z(x, y, z) &= u_z^0(x, y). \end{aligned} \quad (5.9)$$

In pure bending, the strain tensor is given by

$$\boldsymbol{\varepsilon}_b := \boldsymbol{\varepsilon}_{2D}(\mathbf{u}^0 = \mathbf{0}) = -z\boldsymbol{\kappa},$$

with  $\boldsymbol{\kappa}$  given by (5.5). Consequently, the stress tensor reads

$$\boldsymbol{\Sigma}_b := \boldsymbol{\Sigma}_{2D}(\mathbf{u}^0 = \mathbf{0}) = -z\boldsymbol{\mathcal{D}}_{2D}\boldsymbol{\kappa},$$

where  $\boldsymbol{\mathcal{D}}_{2D}$  is defined in Eq. (5.7).

The undeformed middle plane of the plate is denoted by  $\Omega$ . The total domain of the

plate is the product  $\Omega \times (-h/2, h/2)$ , where  $h$  is the constant thickness. To effectively reduce the problem from three- to two-dimensional, the stresses have to be integrated along the fibers. Since the stress varies linearly across the thickness, the stress has to be multiplied by  $z$  before the integration to get a non null contribution. The resulting quantity is called bending momenta tensor and is given by

$$\mathbf{M} := - \int_{-h/2}^{h/2} z \boldsymbol{\Sigma}_b \, dz = \mathcal{D}_b \boldsymbol{\kappa}, \quad (5.10)$$

where

$$\mathcal{D}_b = D_b [(1 - \nu)(\cdot) + \nu \operatorname{Tr}(\cdot) \mathbf{I}_{2 \times 2}], \quad \text{where} \quad D_b = \frac{Eh^3}{12(1 - \nu^2)}. \quad (5.11)$$

The shear stress has to be integrated along the fibers as well. Given the excessive rigidity of the shear contribution, a correction factor  $K_{\text{sh}} = 5/6$  [Red06, Chapter 10] is introduced

$$\mathbf{q} = \int_{-h/2}^{h/2} K_{\text{sh}} \boldsymbol{\sigma}_s \, dz = K_{\text{sh}} Gh \boldsymbol{\gamma}, \quad (5.12)$$

where  $\boldsymbol{\gamma}$  is defined in Eq. (5.6). The equations of motion can be obtained using Hamilton's principle. It consists in minimizing the total Lagrangian, given by  $L = E_{\text{def}} - E_{\text{kin}}$ , where  $E_{\text{def}}$  and  $E_{\text{kin}}$  are the deformation and kinetic energies

$$E_{\text{def}} = \frac{1}{2} \int_{\Omega} \int_{-h/2}^{h/2} \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \, d\Omega \, dz = \frac{1}{2} \int_{\Omega} \{ \mathbf{M} : \boldsymbol{\kappa} + \mathbf{q} \cdot \boldsymbol{\gamma} \} \, d\Omega, \quad (5.13)$$

$$E_{\text{kin}} = \frac{1}{2} \int_{\Omega} \int_{-h/2}^{h/2} \rho \|\partial_t \mathbf{u}\|^2 \, d\Omega \, dz = \frac{1}{2} \int_{\Omega} \left\{ \frac{\rho h^3}{12} \|\partial_t \boldsymbol{\theta}\|^2 + \rho h (\partial_t u_z)^2 \right\} \, d\Omega, \quad (5.14)$$

where  $\rho$  is the mass density. The Hamilton principle states that

$$\int_0^T \delta L \, dt = \int_0^T \{ \delta E_{\text{def}} - \delta E_{\text{kin}} \} \, dt = 0.$$

The final result is the following system of PDEs (for the detailed computations see [Red06, Chapter 10])

$$\begin{aligned} \rho h \frac{\partial^2 u_z}{\partial t^2} &= \operatorname{div} \mathbf{q}, & (x, y) \in \Omega, \\ \frac{\rho h^3}{12} \frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} &= \operatorname{Div} \mathbf{M} + \mathbf{q}, \end{aligned} \quad (5.15)$$

with  $\mathbf{M} = \mathcal{D}_b \operatorname{Grad} \boldsymbol{\theta}$  and  $\mathbf{q} = K_{\text{sh}} Gh (\operatorname{grad} u_z - \boldsymbol{\theta})$ . This PDE goes together with specified boundary conditions. Those will be detailed in 5.2.1.

### 5.1.2 Kirchhoff-Love model

The Kirchhoff model was formulated around 1850 and it is referred to as classical plate theory. The hypotheses on the displacement field consist of the following three points (see Fig. 5.1):

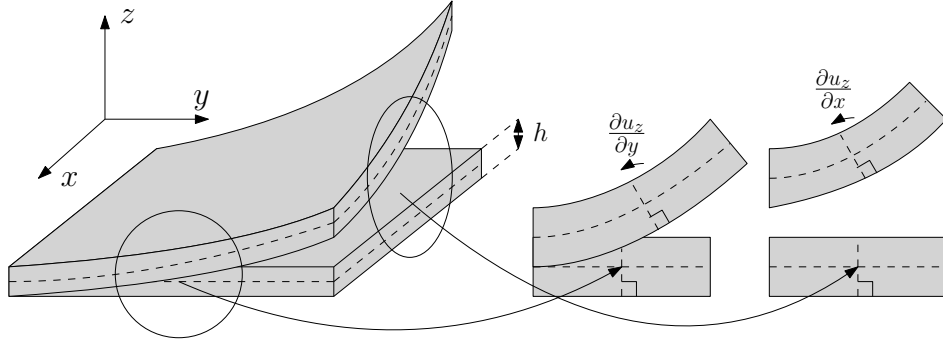


Figure 5.1: Kinematic assumption for the Kirchhoff plate

1. The fibers, segments perpendicular to the mid-plane before deformation, remain straight after deformation.
2. The fibers are inextensible.
3. While rotating, fibers remain perpendicular to the middle surface after deformation.

While the first two points are valid also for the Mindlin plate, the third assumption is specific to the Kirchhoff-Love model. Such an approximation is valid for plates having span-to-thickness ratio of the order of  $L/h \approx 100 - 1000$  and implies zero transverse shear deformation

$$\gamma = 0 \implies \varepsilon_{xz} = -\theta_x + \frac{\partial u_z}{\partial x} = 0, \quad \varepsilon_{yz} = -\theta_y + \frac{\partial u_z}{\partial y} = 0.$$

The rotation vector is then related to the vertical displacement  $\boldsymbol{\theta} = \text{grad } u_z$ . Plugging this into (5.5), it is found

$$\boldsymbol{\kappa} = \text{Grad grad } u_z = \text{Hess } u_z. \quad (5.16)$$

Since the focus is on bending behavior, the in-plane displacement of the mid-plane are assumed to be zero  $\mathbf{u}^0(x, y) = \mathbf{0}$ . Hence, the displacement field assumes the form

$$\begin{aligned} u_x(x, y, z) &= -z \partial_x u_z, \\ u_y(x, y, z) &= -z \partial_y u_z, \\ u_z(x, y, z) &= u_z^0(x, y). \end{aligned} \quad (5.17)$$

For the Kirchhoff plate, the same link between the momenta and bending tensor holds

$$\mathbf{M} = \mathcal{D}_b \boldsymbol{\kappa},$$

where  $\mathcal{D}_b$  and  $\boldsymbol{\kappa}$  are given in (5.11), (5.16) respectively. The equations of motion can be obtained using Hamilton's principle [Red06, Chapter 2]. The deformation energy, kinetic

energy and external work read

$$E_{\text{def}} = \frac{1}{2} \int_{\Omega} \int_{-h/2}^{h/2} \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \, d\Omega \, dz = \frac{1}{2} \int_{\Omega} \{ \mathbf{M} : \boldsymbol{\kappa} \} \, d\Omega, \quad (5.18)$$

$$E_{\text{kin}} = \frac{1}{2} \int_{\Omega} \int_{-h/2}^{h/2} \rho \, \|\partial_t \mathbf{u}\|^2 \, d\Omega \, dz \approx \frac{1}{2} \int_{\Omega} \rho h (\partial_t u_z)^2 \, d\Omega. \quad (5.19)$$

**Remark 4** (Rotational energy)

*For the kinetic energy the rotational contribution*

$$E_{\text{rot}} = \frac{1}{2} \int_{\Omega} \int_{-h/2}^{h/2} \left\{ \rho (\partial_t u_x)^2 + (\partial_t u_y)^2 \right\} \, d\Omega \, dz = \frac{h^3}{24} \int_{\Omega} \rho \left\{ (\partial_{tx} u_z)^2 + (\partial_{ty} u_z)^2 \right\} \, d\Omega = O(h^3),$$

701 *is neglected given the small thickness assumption.*

702 The final result from the Hamilton's principle is the following PDE (for the detailed  
703 computations the reader may consult [Red06, Chapter 3])

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = -\operatorname{div} \operatorname{Div}(\mathcal{D}_b \operatorname{Grad} \operatorname{grad} u_z), \quad (x, y) \in \Omega. \quad (5.20)$$

Developing the calculations, one obtains

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = -D_b \Delta^2 u_z, \quad (x, y) \in \Omega,$$

704 where  $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4}$  is the bi-Laplacian. Appropriate boundary conditions for  
705 this problem will be detailed in 5.2.2.

## 706 5.2 Port-Hamiltonian formulation of isotropic plates

707 In this section the pH formulation of the isotropic Mindlin and Kirchhoff plate models is  
708 detailed. In [MMB05], the Mindlin plate model was put in pH form by appropriate selection  
709 of the energy variables. However, the final system does not consider the nature of the different  
710 variables that come into play, leading to a non intrinsic final formulation. Additionally, this  
711 model was presented using the jet bundle formalism in [SS17]. The Kirchhoff model was never  
712 explored in the pH framework and represents an original contribution of this thesis. The  
713 interested reader can find in [RZ18] a rigorous mathematical treatment of the biharmonic  
714 problem and its decomposition in 2D geometries, but only for the static case (the 3D case,  
715 that does not relate to plate bending, is treated in [DZ18]).



### 5.2.1 Port-Hamiltonian Mindlin plate

Let  $w := u_z$  denote the vertical displacement of the plate. Consider a bounded, connected domain  $\Omega \subset \mathbb{R}^2$  and the Hamiltonian (total energy)

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + \frac{\rho h^3}{12} \left\| \frac{\partial \boldsymbol{\theta}}{\partial t} \right\|^2 + \mathbf{M} : \boldsymbol{\kappa} + \mathbf{q} \cdot \boldsymbol{\gamma} \right\} d\Omega, \quad (5.21)$$

where  $\mathbf{M}$ ,  $\boldsymbol{\kappa}$ ,  $\mathbf{q}$ ,  $\boldsymbol{\gamma}$  are defined in Eqs. (5.10), (5.5), (5.12), (5.6) respectively. The choice of the energy variables is the same as in [MMB05] but here scalar-, vector- and tensor-valued variables are gathered together:

$$\begin{aligned} \alpha_w &= \rho h \frac{\partial w}{\partial t}, & \text{Linear momentum,} & & \alpha_{\theta} &= \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t}, & \text{Angular momentum,} \\ \mathbf{A}_{\kappa} &= \boldsymbol{\kappa}, & \text{Curvature tensor,} & & \boldsymbol{\alpha}_{\gamma} &= \boldsymbol{\gamma}. & \text{Shear deformation.} \end{aligned} \quad (5.22)$$

The energy is now a quadratic function of the energy variables

$$H = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho h} \alpha_w^2 + \frac{12}{\rho h^3} \|\alpha_{\theta}\|^2 + (\mathcal{D}_b \mathbf{A}_{\kappa}) : \mathbf{A}_{\kappa} + (\mathcal{D}_s \boldsymbol{\alpha}_{\gamma}) \cdot \boldsymbol{\alpha}_{\gamma} \right\} d\Omega, \quad (5.23)$$

where  $\mathcal{D}_s := G h K_{\text{sh}} \mathbf{I}_{2 \times 2}$ ,  $G$  is the shear modulus and  $K_{\text{sh}}$  the correction factor. The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, & \text{Linear velocity,} & & e_{\theta} &:= \frac{\delta H}{\delta \alpha_{\theta}} = \frac{\partial \boldsymbol{\theta}}{\partial t}, & \text{Angular velocity,} \\ \mathbf{E}_{\kappa} &:= \frac{\delta H}{\delta \mathbf{A}_{\kappa}} = \mathbf{M}, & \text{Momenta tensor,} & & \mathbf{e}_{\gamma} &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_{\gamma}} = \mathbf{q} & \text{Shear stress.} \end{aligned} \quad (5.24)$$

#### Proposition 4

The variational derivative of the Hamiltonian with respect to the curvature tensor is the momenta tensor  $\frac{\delta H}{\delta \mathbf{A}_{\kappa}} = \mathbf{M}$ .

*Proof.* The proof is analogous to the one already detailed in Prop. 3 □

Once the variables are concatenated together, the pH system is expressed as follows

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_{\theta} \\ \mathbf{A}_{\kappa} \\ \boldsymbol{\alpha}_{\gamma} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \text{div} \\ \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{I}_{2 \times 2} \\ \mathbf{0} & \text{Grad} & \mathbf{0} & \mathbf{0} \\ \text{grad} & -\mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ e_{\theta} \\ \mathbf{E}_{\kappa} \\ e_{\gamma} \end{pmatrix}. \quad (5.25)$$

The first two equations are equivalent to (5.15). The last two equations, like (4.14) for 3D elasticity, represent the fact that the higher order derivatives commute. We shall now establish the total energy balance in terms of boundary variables as they will be part of the

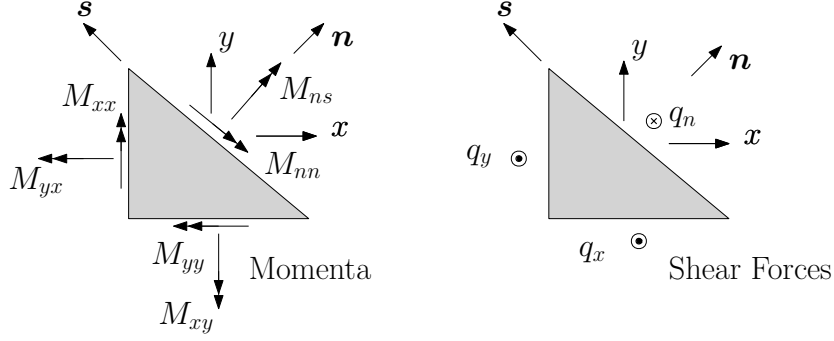


Figure 5.2: Cauchy law for momenta and forces at the boundary.

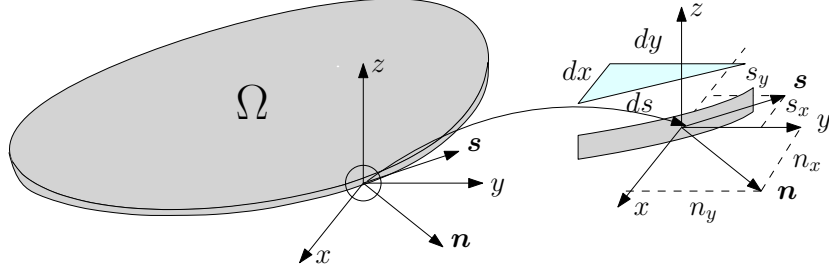


Figure 5.3: Reference frames and notations.

underlying Stokes-Dirac structure of this model. The energy rate reads

$$\begin{aligned}
 \dot{H} &= \int_{\Omega} \left\{ \frac{\partial \alpha_w}{\partial t} e_w + \frac{\partial \alpha_\theta}{\partial t} \cdot \mathbf{e}_\theta + \frac{\partial \mathbf{A}_\kappa}{\partial t} : \mathbf{E}_\kappa + \frac{\partial \alpha_\gamma}{\partial t} \cdot \mathbf{e}_\gamma \right\} d\Omega \\
 &= \int_{\Omega} \{ \operatorname{div}(\mathbf{e}_\gamma) e_w + \operatorname{Div}(\mathbf{E}_\kappa) \cdot \mathbf{e}_\theta + \operatorname{Grad}(\mathbf{e}_\theta) : \mathbf{E}_\kappa + \operatorname{grad}(e_w) \cdot \mathbf{e}_\gamma \} d\Omega \quad \text{Stokes theorem,} \\
 &= \int_{\partial\Omega} \{ w_t q_n + \omega_n M_{nn} + \omega_s M_{ns} \} ds,
 \end{aligned} \tag{5.26}$$

where  $s$  is the curvilinear abscissa. The last integral is obtained by applying the Stokes theorem. The boundary variables appearing in the last line of (5.26) and illustrated in Fig. 5.2 are defined as follows:

$$\begin{aligned}
 \text{Shear force} \quad q_n &:= \mathbf{q} \cdot \mathbf{n} = \mathbf{e}_\gamma \cdot \mathbf{n}, \\
 \text{Flexural momentum} \quad M_{nn} &:= \mathbf{M} : (\mathbf{n} \otimes \mathbf{n}) = \mathbf{E}_\kappa : (\mathbf{n} \otimes \mathbf{n}), \\
 \text{Torsional momentum} \quad M_{ns} &:= \mathbf{M} : (\mathbf{s} \otimes \mathbf{n}) = \mathbf{E}_\kappa : (\mathbf{s} \otimes \mathbf{n}),
 \end{aligned} \tag{5.27}$$

Given two vectors  $\mathbf{a} \in \mathbb{R}^n$ ,  $\mathbf{b} \in \mathbb{R}^m$ , the notation  $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^\top \in \mathbb{R}^{n \times m}$  denotes the outer (or dyadic) product of two vectors. Vectors  $\mathbf{n}$  and  $\mathbf{s}$  designate the normal and tangential unit vectors to the boundary, as shown in Fig. 5.3. The corresponding power conjugated

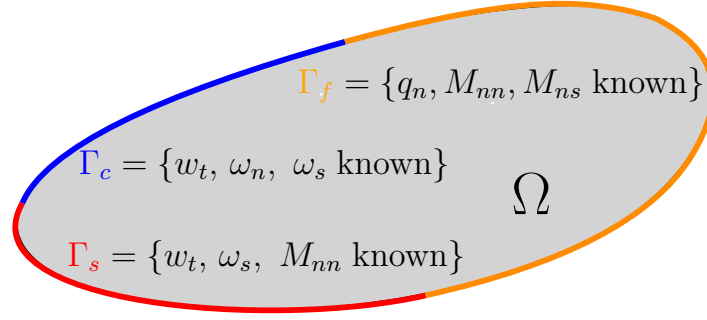


Figure 5.4: Boundary conditions for the Mindlin plate.

740 variables are

$$\begin{aligned}
 \text{Vertical velocity} \quad w_t &:= \frac{\partial w}{\partial t} = e_w, \\
 \text{Flexural rotation} \quad \omega_n &:= \frac{\partial \theta}{\partial t} \cdot \mathbf{n} = \mathbf{e}_\theta \cdot \mathbf{n}, \\
 \text{Torsional rotation} \quad \omega_s &:= \frac{\partial \theta}{\partial t} \cdot \mathbf{s} = \mathbf{e}_\theta \cdot \mathbf{s}.
 \end{aligned} \tag{5.28}$$

741 Consider a partition of the boundary  $\partial\Omega = \bar{\Gamma}_C \cup \bar{\Gamma}_S \cup \bar{\Gamma}_F$ ,  $\Gamma_C \cap \Gamma_S \cap \Gamma_F = \{\emptyset\}$ . The open  
 742 subset  $\Gamma_C, \Gamma_S, \Gamma_F$  could be empty. Given definitions (5.27), (5.28), the boundary conditions  
 743 for the Mindlin plate [DHNLS99] (see Fig. 5.4) that are considered are:

744 • Clamped (C) on  $\Gamma_C \subseteq \partial\Omega$ :  $w_t, \omega_n, \omega_s$  known;

745 • Simply supported hard (S) on  $\Gamma_S \subseteq \partial\Omega$ :  $w_t, \omega_s, M_{nn}$  known;

746 • Free (F) on  $\Gamma_F \subseteq \partial\Omega$ :  $M_{nn}, M_{ns}, q_n$  known.

747 Then the final pH formulation reads

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \boldsymbol{\alpha}_\theta \\ \mathbf{A}_\kappa \\ \boldsymbol{\alpha}_\gamma \end{pmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & \text{div} \\ \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{I}_{2 \times 2} \\ \mathbf{0} & \text{Grad} & \mathbf{0} & \mathbf{0} \\ \text{grad} & -\mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ e_\gamma \end{pmatrix}, \\
\mathbf{u}_\partial &= \underbrace{\begin{bmatrix} \gamma_0^{\Gamma^C} & 0 & 0 & 0 \\ 0 & \gamma_n^{\Gamma^C} & 0 & 0 \\ 0 & \gamma_s^{\Gamma^C} & 0 & 0 \\ \gamma_0^{\Gamma^S} & 0 & 0 & 0 \\ 0 & \gamma_s^{\Gamma^S} & 0 & 0 \\ 0 & 0 & \gamma_{nn}^{\Gamma^S} & 0 \\ 0 & 0 & \gamma_{nn}^{\Gamma^F} & 0 \\ 0 & 0 & \gamma_{ns}^{\Gamma^F} & 0 \\ 0 & 0 & 0 & \gamma_n^{\Gamma^F} \end{bmatrix}}_{\mathcal{B}_\partial} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ e_\gamma \end{pmatrix}, \\
\mathbf{y}_\partial &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & \gamma_n^{\Gamma^C} \\ 0 & 0 & \gamma_{nn}^{\Gamma^C} & 0 \\ 0 & 0 & \gamma_{ns}^{\Gamma^C} & 0 \\ 0 & 0 & 0 & \gamma_n^{\Gamma^S} \\ 0 & 0 & \gamma_{ns}^{\Gamma^S} & 0 \\ 0 & \gamma_n^{\Gamma^S} & 0 & 0 \\ 0 & \gamma_n^{\Gamma^F} & 0 & 0 \\ 0 & \gamma_s^{\Gamma^F} & 0 & 0 \\ \gamma_0^{\Gamma^F} & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{C}_\partial} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ e_\gamma \end{pmatrix},
\end{aligned} \tag{5.29}$$

where  $\gamma_0^{\Gamma^*} a = a|_{\Gamma^*}$  denotes the trace over the set  $\Gamma^*$ . Furthermore, notations  $\gamma_n^{\Gamma^*} \mathbf{a} = \mathbf{a} \cdot \mathbf{n}|_{\Gamma^*}$ ,  $\gamma_s^{\Gamma^*} \mathbf{a} = \mathbf{a} \cdot \mathbf{s}|_{\Gamma^*}$  indicate respectively the normal and tangential traces over the set  $\Gamma^*$ . Symbols  $\gamma_{nn}^{\Gamma^*}, \gamma_{ns}^{\Gamma^*}$  denote the normal-normal trace and the normal-tangential trace of tensor-valued functions and  $\gamma_{nn}^{\Gamma^*} \mathbf{A} = \mathbf{A} : (\mathbf{n} \otimes \mathbf{n})|_{\Gamma^*}$ ,  $\gamma_{ns}^{\Gamma^*} \mathbf{A} = \mathbf{A} : (\mathbf{n} \otimes \mathbf{s})|_{\Gamma^*}$ .

#### Remark 5

It can be observed that the interconnection structure given by  $\mathcal{J}$  in (5.29) mimics that of the Timoshenko beam [JZ12, Chapter 7].

#### Conjecture 2 (Stokes-Dirac structure for the Mindlin plate)

Consider  $\mathbb{V} = \mathbb{R}^2$ ,  $\mathbb{S} = \mathbb{R}_{sym}^{2 \times 2}$  and let  $H^1(\Omega)$  be the space of functions with gradient in  $L^2(\Omega, \mathbb{V})$  and  $H^{\text{div}}(\Omega, \mathbb{V})$  the space of vector-valued functions with divergence in  $L^2(\Omega)$ . Furthermore,  $H^1(\Omega, \mathbb{V})$  is the space of vectors with symmetric gradient in  $L^2(\Omega, \mathbb{S})$  and  $H^{\text{Div}}(\Omega, \mathbb{S})$  denotes

the space of symmetric tensors with divergence in  $L^2(\Omega, \mathbb{V})$ . Consider the definitions

$$\begin{aligned} H &:= H^1(\Omega) \times H^{\text{Grad}}(\Omega, \mathbb{V}) \times H^{\text{Div}}(\Omega, \mathbb{S}) \times H^{\text{div}}(\Omega, \mathbb{V}), \\ F &:= L^2(\Omega) \times L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{V}), \\ F_\partial &:= L^2(\Gamma_C, \mathbb{R}^3) \times L^2(\Gamma_S, \mathbb{R}^3) \times L^2(\Gamma_F, \mathbb{R}^3). \end{aligned}$$

The set

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \mathbf{f} \\ \mathbf{f}_\partial \\ \mathbf{e} \\ \mathbf{e}_\partial \end{pmatrix} \mid \mathbf{e} \in H, \mathbf{f} = \mathcal{J}\mathbf{e}, \mathbf{f}_\partial = \mathcal{B}_\partial \mathbf{e}, \mathbf{e}_\partial = -\mathcal{C}_\partial \mathbf{e} \right\}, \quad (5.30)$$

where  $\mathbf{e} = (e_w, \mathbf{e}_\theta, \mathbf{E}_\kappa, \mathbf{e}_\gamma)$  and  $\mathcal{J}, \mathcal{B}_\partial, \mathcal{C}_\partial$  are defined in (5.29), is a Stokes–Dirac structure with respect to the pairing

$$\langle \langle (\mathbf{f}^1, \mathbf{f}_\partial^1, \mathbf{e}^1, \mathbf{e}_\partial^1), (\mathbf{f}^2, \mathbf{f}_\partial^2, \mathbf{e}^2, \mathbf{e}_\partial^2) \rangle \rangle := \langle \mathbf{e}^1, \mathbf{f}^2 \rangle_F + \langle \mathbf{e}^2, \mathbf{f}^1 \rangle_F + \langle \mathbf{e}_\partial^1, \mathbf{f}_\partial^2 \rangle_{F_\partial} + \langle \mathbf{e}_\partial^2, \mathbf{f}_\partial^1 \rangle_{F_\partial}, \quad (5.31)$$

where  $\mathbf{e}_\partial^i = (e_{\partial,1}^i, e_{\partial,2}^i, e_{\partial,3}^i)$ ,  $\mathbf{f}_\partial^i = (f_{\partial,1}^i, f_{\partial,2}^i, f_{\partial,3}^i)$  and

$$\langle (\mathbf{a}, \mathbf{b}, \mathbf{c}), (\mathbf{d}, \mathbf{e}, \mathbf{f}) \rangle_{F_\partial} = \int_{\Gamma_C} \mathbf{a} \cdot \mathbf{d} \, dS + \int_{\Gamma_S} \mathbf{b} \cdot \mathbf{e} \, dS + \int_{\Gamma_F} \mathbf{c} \cdot \mathbf{f} \, dS, \quad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathbb{R}^3.$$

**Crucial points and elements in favor of the conjecture** Analogously to what was stated in Conjecture 1, the boundary spaces have to properly defined. If the integration by parts is carried out as in Eq. (5.26), one can follow the same lines of Conjecture 1 to support the present Conjecture.

The Mindlin plate falls within the assumption of [Skr19], hence it is a well posed boundary control pH systems.

### 5.2.2 Port-Hamiltonian Kirchhoff plate

Again the starting point is the Hamiltonian (total energy)

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + \mathbf{M} : \boldsymbol{\kappa} \right\} \, d\Omega, \quad (5.32)$$

where  $\mathbf{M}$ ,  $\boldsymbol{\kappa}$  are defined in Eqs. (5.10), (5.16). For what concerns the choice of the energy variables, a scalar and a tensor variable are considered:

$$\alpha_w = \rho h \frac{\partial w}{\partial t}, \quad \text{Linear momentum}, \quad \mathbf{A}_\kappa = \boldsymbol{\kappa}, \quad \text{Curvature tensor.} \quad (5.33)$$

768 The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \quad \text{Linear velocity}, \quad \mathbf{E}_\kappa := \frac{\delta H}{\delta \mathbf{A}_\kappa} = \mathbf{M}, \quad \text{Curvature tensor.} \quad (5.34)$$

769 The port-Hamiltonian system is then written as

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \mathbf{A}_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\text{div} \circ \text{Div} \\ \text{Grad} \circ \text{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix}. \quad (5.35)$$

The first equation is equivalent to (5.20). The last equation represents the fact that higher order derivatives commute

$$\begin{aligned} \partial_t \mathbf{A}_\kappa &= \text{Grad grad } e_w, \\ \partial_t \kappa &= \text{Grad grad } \partial_t w, \\ \partial_t \text{Grad grad } w &= \text{Grad grad } \partial_t w. \end{aligned}$$

770 The last equation holds for  $w \in C^3(\Omega)$ .

### 771 Theorem 3

772 The operator  $\text{Grad} \circ \text{grad}$ , corresponding to the Hessian operator, is the adjoint of the double  
773 divergence  $\text{div} \circ \text{Div}$ .

*Proof.* Let  $\mathbb{S} = \mathbb{R}_{\text{sym}}^{d \times d}$  and consider the Hilbert space of the square integrable symmetric square tensors  $L^2(\Omega, \mathbb{S})$  over an open connected set  $\Omega$  (its inner product is defined in (4.12)). Consider the Hilbert space  $L^2(\Omega)$  of scalar square integrable functions, endowed with the standard inner product. Consider the double divergence operator defined as:

$$\begin{aligned} \text{div Div} : L^2(\Omega, \mathbb{S}) &\rightarrow L^2(\Omega), \\ \Psi &\rightarrow \text{div Div } \Psi = \psi, \end{aligned} \quad \text{with } \psi = \text{div Div } \Psi = \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 \Psi_{ij}}{\partial x_i \partial x_j}.$$

We shall identify  $\text{div Div}^*$

$$\begin{aligned} \text{div Div}^* : L^2(\Omega) &\rightarrow L^2(\Omega, \mathbb{S}), \\ f &\rightarrow \text{div Div}^* f = \mathbf{F}, \end{aligned}$$

such that

$$\begin{aligned} \langle \text{div Div } \Psi, f \rangle_{L^2(\Omega)} &= \langle \Psi, \text{div Div}^* f \rangle_{L^2(\Omega, \mathbb{S})}, & \forall \Psi \in \text{Dom}(\text{div Div}) \subset L^2(\Omega, \mathbb{S}) \\ & & \forall f \in \text{Dom}(\text{div Div}^*) \subset L^2(\Omega) \end{aligned}$$

The function has to belong to the operator domain, so for instance  $f \in C_0^2(\Omega) \in \text{Dom}(\text{div Div}^*)$  the space of twice differentiable scalar functions with compact support and  $\Psi$  can be chosen in the set  $C_0^2(\Omega, \mathbb{S}) \in \text{Dom}(\text{div Div})$ , the space of twice differentiable symmetric tensors with

compact support on  $\Omega$ . A classical result is the fact that the adjoint of the vector divergence is  $\operatorname{div}^* = -\operatorname{grad}$  as stated in [KZ15]. By theorem 2, it holds  $\operatorname{Div}^* = -\operatorname{Grad}$ . Considering that  $\operatorname{div} \operatorname{Div} = \operatorname{div} \circ \operatorname{Div}$  is the composition of two different operators and that the adjoint of a composed operator is the adjoint of each operator in reverse order, i.e.  $(B \circ C)^* = C^* \circ B^*$ , then it can be stated

$$(\operatorname{div} \circ \operatorname{Div})^* = \operatorname{Div}^* \circ \operatorname{div}^* = \operatorname{Grad} \circ \operatorname{grad}.$$

774 Since only formal adjoints are being looked for, this concludes the proof.  $\square$

775 The energy rate provides the boundary port variables

$$\begin{aligned} \dot{H} &= \int_{\Omega} \{ \partial_t \alpha_w e_w + \partial_t \mathbf{A}_{\kappa} : \mathbf{E}_{\kappa} \} \, d\Omega \\ &= \int_{\Omega} \{ -\operatorname{div} \operatorname{Div} \mathbf{E}_{\kappa} e_w + \operatorname{Grad} \operatorname{grad} e_w : \mathbf{E}_{\kappa} \} \, d\Omega, && \text{Stokes theorem} \\ &= \int_{\partial\Omega} \{ -\mathbf{n} \cdot \operatorname{Div} \mathbf{E}_{\kappa} e_w + (\mathbf{n} \otimes \operatorname{grad} e_w) : \mathbf{E}_{\kappa} \} \, ds, \\ &= \int_{\partial\Omega} \{ -\mathbf{n} \cdot \operatorname{Div} \mathbf{E}_{\kappa} e_w + \partial_{\mathbf{n}} e_w (\mathbf{n} \otimes \mathbf{n}) : \mathbf{E}_{\kappa} + \partial_s e_w (\mathbf{n} \otimes \mathbf{s}) : \mathbf{E}_{\kappa} \} \, ds, && \text{Dyadic properties} \\ &= \int_{\partial\Omega} \{ \hat{q}_n w_t + \partial_{\mathbf{n}} w_t M_{nn} + \partial_s w_t M_{ns} \} \, ds. \end{aligned} \tag{5.36}$$

776 where  $s$  is the curvilinear abscissa,  $w_t := \partial_t w$  and  $\partial_s w_t$  denotes the directional derivative  
777 along the tangential versor at the boundary. Additionally, the following definitions have been  
778 introduced

$$\hat{q}_n := -\mathbf{n} \cdot \operatorname{Div}(\mathbf{E}_{\kappa}), \quad M_{nn} := (\mathbf{n} \otimes \mathbf{n}) : \mathbf{E}_{\kappa}, \quad M_{ns} := (\mathbf{n} \otimes \mathbf{s}) : \mathbf{E}_{\kappa}. \tag{5.37}$$

779 Variables  $w_t$  and  $\partial_s w_t$  are not independent as they are differentially related with respect to  
780 derivation along  $\mathbf{s}$  (see for instance [TWK59, Chapter 4]). The tangential derivative has to be  
781 moved on the torsional momentum  $M_{ns}$ . For sake of simplicity,  $\partial\Omega$  is supposed to be regular.  
782 Then the integration by parts provides

$$\int_{\partial\Omega} \partial_s w_t M_{ns} \, ds = - \int_{\partial\Omega} \partial_s M_{ns} w_t \, ds. \tag{5.38}$$

783 The final energy balance reads

$$\dot{H} = \int_{\partial\Omega} \{ w_t \tilde{q}_n + \partial_{\mathbf{n}} w_t M_{nn} \} \, ds, \tag{5.39}$$

784 where the boundary variables are

$$\begin{aligned} \text{Effective shear force} \quad \tilde{q}_n &:= \hat{q}_n - \partial_s M_{ns}, \\ \text{Flexural momentum} \quad M_{nn} &:= \mathbf{M} : (\mathbf{n} \otimes \mathbf{n}) = \mathbf{E}_{\kappa} : (\mathbf{n} \otimes \mathbf{n}), \end{aligned} \tag{5.40}$$

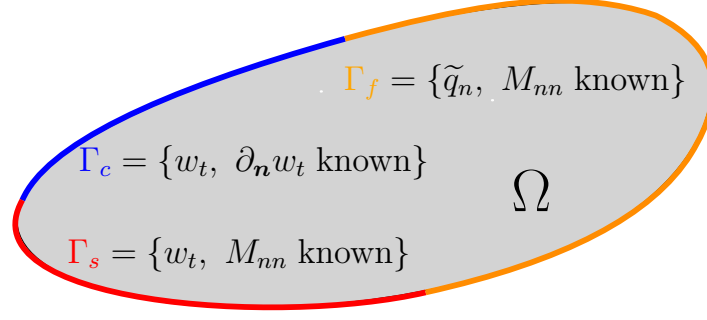


Figure 5.5: Boundary conditions for the Kirchhoff plate.

785 and  $\hat{q}_n$  is defined in (5.37). The corresponding power conjugated variables are:

$$\begin{aligned} \text{Vertical velocity} \quad w_t &:= \frac{\partial w}{\partial t} = e_w, \\ \text{Flexural rotation} \quad \partial_n w_t &:= \nabla e_w \cdot \mathbf{n}. \end{aligned} \tag{5.41}$$

786 Consider a partition of the boundary  $\partial\Omega = \bar{\Gamma}_C \cup \bar{\Gamma}_S \cup \bar{\Gamma}_F$ ,  $\Gamma_C \cap \Gamma_S \cap \Gamma_F = \{\emptyset\}$ , where  
 787  $\Gamma_C, \Gamma_S, \Gamma_F$  are open subset of  $\partial\Omega$ . Given definitions (5.40), (5.41), the boundary conditions  
 788 for the Kirchhoff plate [GSV18] are the following (see Fig. 5.5):

789 • Clamped (C) on  $\Gamma_C \subseteq \partial\Omega$ :  $w_t, \partial_n w_t$  known;

790 • Simply supported (S) on  $\Gamma_S \subseteq \partial\Omega$ :  $w_t, M_{nn}$  known;

791 • Free (F) on  $\Gamma_F \subseteq \partial\Omega$ :  $\tilde{q}_n, M_{nn}$  known.

792 Then the final pH formulation reads



$$\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \mathbf{A}_\kappa \end{pmatrix} &= \underbrace{\begin{bmatrix} 0 & -\operatorname{div} \circ \operatorname{Div} \\ \operatorname{Grad} \circ \operatorname{grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix}, \\
\mathbf{u}_\partial &= \underbrace{\begin{bmatrix} \gamma_0^{\Gamma_C} & 0 \\ \gamma_1^{\Gamma_C} & 0 \\ \gamma_0^{\Gamma_S} & 0 \\ 0 & \gamma_{nn}^{\Gamma_S} \\ 0 & \gamma_{nn,1}^{\Gamma_F} \\ 0 & \gamma_{nn}^{\Gamma_F} \end{bmatrix}}_{\mathcal{B}_\partial} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix}, \\
\mathbf{y}_\partial &= \underbrace{\begin{bmatrix} 0 & \gamma_{nn,1}^{\Gamma_C} \\ 0 & \gamma_{nn}^{\Gamma_C} \\ 0 & \gamma_{nn,1}^{\Gamma_S} \\ \gamma_1^{\Gamma_S} & 0 \\ \gamma_0^{\Gamma_F} & 0 \\ \gamma_1^{\Gamma_F} & 0 \end{bmatrix}}_{\mathcal{C}_\partial} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix},
\end{aligned} \tag{5.42}$$

where  $\gamma_0^{\Gamma_*} a = a|_{\Gamma_*}$  and  $\gamma_1^{\Gamma_*} a = \partial_{\mathbf{n}} a|_{\Gamma_*}$  denote the standard and the normal derivative trace over the set  $\Gamma_*$  respectively. The symbol  $\gamma_{nn,1}^{\Gamma_*}$  denotes the map  $\gamma_{nn,1}^{\Gamma_*} \mathbf{A} = -\mathbf{n} \cdot \operatorname{Div} \mathbf{A} - \partial_s(\mathbf{A} : (\mathbf{n} \otimes \mathbf{s}))|_{\Gamma_*}$ , while  $\gamma_{nn}^{\Gamma_*} \mathbf{A} = \mathbf{A} : (\mathbf{n} \otimes \mathbf{n})|_{\Gamma_*}$  indicates the normal-normal trace of a tensor-valued function.

#### Remark 6

The interconnection structure  $\mathcal{J}$  in (5.42) mimics that of the Bernoulli beam [CRMPB17]. The double divergence and the Hessian coincide, in dimension one, with the second derivative.

#### Conjecture 3 (Stokes-Dirac structure for the Kirchhoff plate)

Consider  $\mathbb{S} = \mathbb{R}_{sym}^{2 \times 2}$  and let  $H^2(\Omega)$  be the space of functions with Hessian in  $L^2(\Omega, \mathbb{S})$  and  $H^{\operatorname{div} \operatorname{Div}}(\Omega, \mathbb{S})$  the space of vector-valued functions with double divergence in  $L^2(\Omega)$ . Consider the definitions

$$\begin{aligned}
H &:= H^2(\Omega) \times H^{\operatorname{div} \operatorname{Div}}(\Omega, \mathbb{S}), \\
F &:= L^2(\Omega) \times L^2(\Omega, \mathbb{S}), \\
F_\partial &:= L^2(\Gamma_C, \mathbb{R}^2) \times L^2(\Gamma_S, \mathbb{R}^2) \times L^2(\Gamma_F, \mathbb{R}^2).
\end{aligned}$$

The set

$$D_{\mathcal{J}} = \left\{ \begin{pmatrix} \mathbf{f} \\ \mathbf{f}_\partial \\ \mathbf{e} \\ \mathbf{e}_\partial \end{pmatrix} \mid \mathbf{e} \in H, \mathbf{f} = \mathcal{J}\mathbf{e}, \mathbf{f}_\partial = \mathcal{B}_\partial \mathbf{e}, \mathbf{e}_\partial = -\mathcal{C}_\partial \mathbf{e} \right\}, \tag{5.43}$$

where  $\mathbf{e} = (e_w, \mathbf{E}_\kappa)$  and  $\mathcal{J}, \mathcal{B}_\partial, \mathcal{C}_\partial$  are defined in (5.42), is a Stokes-Dirac structure with

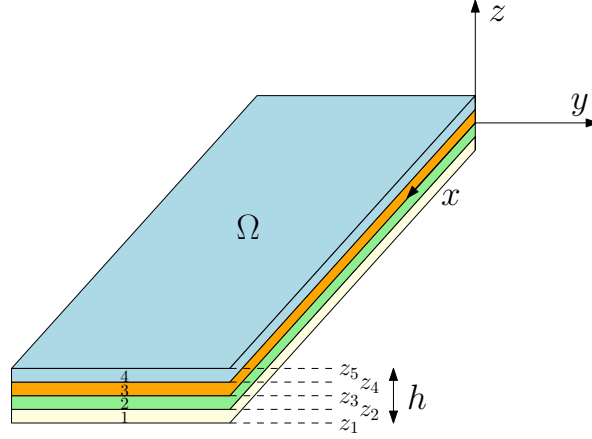


Figure 5.6: Laminated plate with 4 layers.

802 respect to the pairing

$$\langle\langle (\mathbf{f}^1, \mathbf{f}_\partial^1, \mathbf{e}^1, \mathbf{e}_\partial^1), (\mathbf{f}^2, \mathbf{f}_\partial^2, \mathbf{e}^2, \mathbf{e}_\partial^2) \rangle\rangle := \langle \mathbf{e}^1, \mathbf{f}^2 \rangle_F + \langle \mathbf{e}^2, \mathbf{f}^1 \rangle_F + \langle \mathbf{e}_\partial^1, \mathbf{f}_\partial^2 \rangle_{F_\partial} + \langle \mathbf{e}_\partial^2, \mathbf{f}_\partial^1 \rangle_{F_\partial}, \quad (5.44)$$

where  $\mathbf{e}_\partial^i = (\mathbf{e}_{\partial,1}^i, \mathbf{e}_{\partial,2}^i)$ ,  $\mathbf{f}_\partial^i = (\mathbf{f}_{\partial,1}^i, \mathbf{f}_{\partial,2}^i)$  and

$$\langle (\mathbf{a}, \mathbf{b}, \mathbf{c}), (\mathbf{d}, \mathbf{e}, \mathbf{f}) \rangle_{F_\partial} = \int_{\Gamma_C} \mathbf{a} \cdot \mathbf{d} \, dS + \int_{\Gamma_S} \mathbf{b} \cdot \mathbf{e} \, dS + \int_{\Gamma_F} \mathbf{c} \cdot \mathbf{f} \, dS, \quad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathbb{R}^2.$$

803 **Validity of the conjecture** The integration by parts has to be carried as in Eq. (5.36) to  
 804 retrieve a similar discussion to the one in Conjecture 1.

### 805 5.3 Laminated anisotropic plates

806 Until now homogeneous isotropic materials have been considered. For this class of materials,  
 807 the membrane and bending problems are decoupled. In aeronautical applications, structure  
 808 are made up of laminae of different materials to enhance the mechanical properties of the  
 809 resulting structure. In some cases, a certain coupling is desired, to increase the aerodynamical  
 810 performance of the wing as it deforms.

811 Consider again the deformation field given by (5.1)

$$\begin{aligned} \mathbf{u}(x, y, z, t) &= \mathbf{u}^0(x, y, t) - z\boldsymbol{\theta}(x, y, t), \\ u_z(x, y, z, t) &= u_z^0(x, y, t), \end{aligned}$$

812 where  $\mathbf{u} = (u_x, u_y)$ . The link between in-plane deformation (5.2) and the membrane and

bending contribution (5.4), (5.5).

$$\varepsilon_{2D} = \varepsilon^0 - z\kappa \quad \text{where} \quad \varepsilon^0 = \text{Grad } \mathbf{u}^0, \quad \kappa = \text{Grad } \boldsymbol{\theta}. \quad (5.45)$$

Assume that each layer is an anisotropic material under plane stress condition. Then, it holds (see [Red03, Chapter 1] for details)

$$\boldsymbol{\Sigma}_{2D}^i = \mathcal{D}_{2D}^i \boldsymbol{\varepsilon}_{2D}^i,$$

where  $i$  indicates the layer under consideration. The matrix  $\mathcal{D}_{2D}^i$  depends on the properties of each material. To reduce the problem to bi-dimensional, the stresses have to be integrated along the thickness. Consider the membrane and bending resultant of the stress

$$\mathbf{N} := \sum_{i=1}^{n_{\text{layer}}} \int_{z_i}^{z_{i+1}} \boldsymbol{\Sigma}_{2D}^i \, dz, \quad \mathbf{M} := \sum_{i=1}^{n_{\text{layer}}} \int_{z_i}^{z_{i+1}} -z \boldsymbol{\Sigma}_{2D}^i \, dz. \quad (5.46)$$

where  $n_{\text{layer}}$  is the number of layers and  $z_i$  represents the height of the  $i^{\text{th}}$  layer (see Fig. 5.6). Since the stress are discontinuous due to the change of constitutive law along the thickness, the integration has to be performed lamina-wise. Once the computations are carried out, it is found

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \end{pmatrix} = \begin{bmatrix} \mathcal{D}_m & \mathcal{D}_c \\ \mathcal{D}_c & \mathcal{D}_b \end{bmatrix} \begin{pmatrix} \varepsilon^0 \\ \kappa \end{pmatrix}, \quad (5.47)$$

where

$$\mathcal{D}_m = \sum_{i=1}^{n_{\text{layer}}} \mathcal{D}_{2D}^i (z_{i+1} - z_i), \quad \mathcal{D}_c = -\frac{1}{2} \sum_{i=1}^{n_{\text{layer}}} \mathcal{D}_{2D}^i (z_{i+1}^2 - z_i^2), \quad \mathcal{D}_b = \frac{1}{3} \sum_{i=1}^{n_{\text{layer}}} \mathcal{D}_{2D}^i (z_{i+1}^3 - z_i^3), \quad (5.48)$$

Differently from isotropic plate, for laminated anisotropic plates the membrane and bending behavior are coupled. The coupling term  $\mathcal{D}_c$  disappears if a symmetric configuration is considered. For the shear contribution it is obtained

$$\mathbf{q} := \int_{-h/2}^{h/2} \boldsymbol{\sigma}_s \, dz = \mathcal{D}_s \boldsymbol{\gamma}, \quad \text{where} \quad \boldsymbol{\gamma} = \text{grad } u_z - \boldsymbol{\theta}. \quad (5.49)$$

The tensor  $\mathcal{D}_s$  is not diagonal as in the isotropic case, cf. §5.2.1.

In the following section it is shown how anisotropic laminated plates can be formulated as pHs.

### 5.3.1 Port-Hamiltonian laminated Mindlin plate

For a shear deformable laminated plate the kinetic and deformation energy read

$$E_{\text{kin}} = \frac{1}{2} \int_{\Omega} \left\{ \rho h \left\| \frac{\partial \mathbf{u}^0}{\partial t} \right\|^2 + \rho h \left( \frac{\partial u_z}{\partial t} \right)^2 + \frac{\rho h^3}{12} \left\| \frac{\partial \boldsymbol{\theta}}{\partial t} \right\|^2 \right\} d\Omega,$$

$$E_{\text{def}} = \frac{1}{2} \int_{\Omega} \left\{ \mathbf{N} : \boldsymbol{\varepsilon}^0 + \mathbf{M} : \boldsymbol{\kappa} + \mathbf{q} \cdot \boldsymbol{\gamma} \right\} d\Omega.$$

By using Hamilton's principle the equations of motion are retrieved (see [Red03, Chapter 3] for an exhaustive explanation)

$$\begin{aligned} \rho h \frac{\partial^2 \mathbf{u}^0}{\partial t^2} &= \text{Div } \mathbf{N}, \\ \rho h \frac{\partial^2 u_z}{\partial t^2} &= \text{div } \mathbf{q}, \\ \frac{\rho h^3}{12} \frac{\partial^2 \boldsymbol{\theta}}{\partial t^2} &= \text{Div } \mathbf{M} + \mathbf{q}, \end{aligned} \tag{5.50}$$

where  $\mathbf{N}$ ,  $\mathbf{M}$ ,  $\mathbf{q}$  are defined in Eqs. (5.47), (5.49). To get a port-Hamiltonian formulation, the following energy variables are chosen

$$\begin{aligned} \boldsymbol{\alpha}_u &= \rho h \frac{\partial \mathbf{u}^0}{\partial t}, & \alpha_w &= \rho h \frac{\partial u_z}{\partial t}, & \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t}, \\ \mathbf{A}_{\varepsilon^0} &= \boldsymbol{\varepsilon}^0, & \mathbf{A}_\kappa &= \boldsymbol{\kappa}, & \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}. \end{aligned} \tag{5.51}$$

This choice highlights the nature of the problem in which the membrane part (equivalent to a 2D elasticity problem) and the bending part interact. The total energy  $H = E_{\text{kin}} + E_{\text{def}}$  is now a quadratic function of the energy variables

$$E_{\text{kin}} = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho h} \left\| \frac{\partial \boldsymbol{\alpha}_u}{\partial t} \right\|^2 + \frac{1}{\rho h} \left( \frac{\partial \alpha_w}{\partial t} \right)^2 + \frac{12}{\rho h^3} \left\| \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} \right\|^2 \right\} d\Omega,$$

$$E_{\text{def}} = \frac{1}{2} \int_{\Omega} \{ (\mathcal{D}_m \mathbf{A}_{\varepsilon^0} + \mathcal{D}_c \mathbf{A}_\kappa) : \mathbf{A}_{\varepsilon^0} + (\mathcal{D}_c \mathbf{A}_{\varepsilon^0} + \mathcal{D}_b \mathbf{A}_\kappa) : \mathbf{A}_\kappa + (\mathcal{D}_s \boldsymbol{\alpha}_\gamma) \cdot \boldsymbol{\alpha}_\gamma \} d\Omega,$$

The co-energies are equal to

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial u_z}{\partial t}, & e_\theta &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t}, \\ \mathbf{E}_\kappa &:= \frac{\delta H}{\delta \mathbf{A}_\kappa} = \mathbf{M}, & e_\gamma &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\gamma} = \mathbf{q} \end{aligned} \tag{5.52}$$

The final pH formulation is found as usual considering the dynamics (5.50) and Clairaut's theorem

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_u \\ \alpha_w \\ \alpha_\theta \\ \mathbf{A}_{\varepsilon^0} \\ \mathbf{A}_\kappa \\ \alpha_\gamma \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & \text{div} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{I}_{2 \times 2} \\ \text{Grad} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \text{Grad} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{grad} & -\mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_u \\ e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_{\varepsilon^0} \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}. \quad (5.53)$$

The coupling between the membrane and bending part is clear when considering the link between energy and co-energy variables

$$\begin{pmatrix} \mathbf{e}_u \\ e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_{\varepsilon^0} \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix} = \begin{bmatrix} \frac{1}{\rho h} \mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \frac{12}{\rho h^3} \mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{D}_m & \mathcal{D}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{D}_c & \mathcal{D}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{D}_s \end{bmatrix} \begin{pmatrix} \alpha_u \\ \alpha_w \\ \alpha_\theta \\ \mathbf{A}_{\varepsilon^0} \\ \mathbf{A}_\kappa \\ \alpha_\gamma \end{pmatrix}. \quad (5.54)$$

Again appropriate boundary variables and a suitable Stokes-Dirac structure can be found for this model. The final formulation is just a superposition of systems (4.16) and (5.29).

### 5.3.2 Port-Hamiltonian laminated Kirchhoff plate

According to the Kirchhoff hypotheses the kinetic and deformation energies reduce to

$$E_{\text{kin}} = \frac{1}{2} \int_{\Omega} \left\{ \rho h \left\| \frac{\partial \mathbf{u}^0}{\partial t} \right\|^2 + \rho h \left( \frac{\partial u_z}{\partial t} \right)^2 \right\} d\Omega,$$

$$E_{\text{def}} = \frac{1}{2} \int_{\Omega} \left\{ \mathbf{N} : \varepsilon^0 + \mathbf{M} : \kappa \right\} d\Omega,$$

where  $\kappa$  is defined in Eq. (5.5). Furthermore, as stated in Remark 4, the rotational contribution in the kinetic energy has been neglected. The equations of motion are (see [Red03, Chapter 3] for an exhaustive explanation)

$$\rho h \frac{\partial^2 \mathbf{u}^0}{\partial t^2} = \text{Div } \mathbf{N},$$

$$\rho h \frac{\partial^2 u_z}{\partial t^2} = -\text{div Div } \mathbf{M},$$

where  $\mathbf{N}$ ,  $\mathbf{M}$  are defined in Eqs. (5.47). To get a port-Hamiltonian formulation, the following energy variables are chosen

$$\begin{aligned}\alpha_u &= \rho h \frac{\partial \mathbf{u}^0}{\partial t}, & \alpha_w &= \rho h \frac{\partial u_z}{\partial t}, \\ \mathbf{A}_{\varepsilon^0} &= \boldsymbol{\varepsilon}^0, & \mathbf{A}_\kappa &= \boldsymbol{\kappa}.\end{aligned}\tag{5.56}$$

The total energy  $H = E_{\text{kin}} + E_{\text{def}}$  is now a quadratic function of the energy variables

$$\begin{aligned}E_{\text{kin}} &= \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho h} \left\| \frac{\partial \alpha_u}{\partial t} \right\|^2 + \frac{1}{\rho h} \left( \frac{\partial \alpha_w}{\partial t} \right)^2 \right\} d\Omega, \\ E_{\text{def}} &= \frac{1}{2} \int_{\Omega} \{ (\mathcal{D}_m \mathbf{A}_{\varepsilon^0} + \mathcal{D}_c \mathbf{A}_\kappa) : \mathbf{A}_{\varepsilon^0} + (\mathcal{D}_c \mathbf{A}_{\varepsilon^0} + \mathcal{D}_b \mathbf{A}_\kappa) : \mathbf{A}_\kappa \} d\Omega,\end{aligned}$$

The co-energies are equal to

$$\begin{aligned}e_u &:= \frac{\delta H}{\delta \alpha_u} = \frac{\partial \mathbf{u}^0}{\partial t}, & e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial u_z}{\partial t}, \\ \mathbf{E}_\kappa &:= \frac{\delta H}{\delta \mathbf{A}_{\varepsilon^0}} = \mathbf{N}, & \mathbf{E}_\kappa &:= \frac{\delta H}{\delta \mathbf{A}_\kappa} = \mathbf{M},\end{aligned}\tag{5.57}$$

The final pH formulation is found to be

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_u \\ \alpha_w \\ \mathbf{A}_{\varepsilon^0} \\ \mathbf{A}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{0} \\ 0 & 0 & 0 & -\text{div} \circ \text{Div} \\ \text{Grad} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \text{Grad} \circ \text{grad} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_u \\ e_w \\ \mathbf{E}_{\varepsilon^0} \\ \mathbf{E}_\kappa \end{pmatrix}.\tag{5.58}$$

Again, the coupling appears when considering the link between energy and co-energy variables

$$\begin{pmatrix} e_u \\ e_w \\ \mathbf{E}_{\varepsilon^0} \\ \mathbf{E}_\kappa \end{pmatrix} = \begin{bmatrix} \frac{1}{\rho h} \mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \frac{1}{\rho h} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_m & \mathcal{D}_c \\ \mathbf{0} & \mathbf{0} & \mathcal{D}_c & \mathcal{D}_b \end{bmatrix} \begin{pmatrix} \alpha_u \\ \alpha_w \\ \mathbf{A}_{\varepsilon^0} \\ \mathbf{A}_\kappa \end{pmatrix}.\tag{5.59}$$

The energy rate provides the appropriate boundary conditions from which one can construct the Stokes-Dirac structure. The necessary computations are not performed here as the final result is just a juxtaposition of systems (4.16), (5.42).

## 5.4 Conclusion

In this chapter, a pH formulation for the most commonly used plate models has been detailed. Many open questions remain. In particular, how to generalize the results to shell problems, for which the domain is a surface embedded in the three dimensional space (a manifold). Computations get more involved in this case since the usage of differential geometry concepts is unavoidable. These models are important since they are widely used in the aerospace in-

---

dustry and ubiquitous in nature.

The reformulation of plate models using the language of differential geometry is another open research topic. Indeed, while for the Mindlin plate it should be possible to use vector-valued forms to obtain an equivalent system, for the Kirchhoff plate higher order Stokes-Dirac structure are needed [\[NY04\]](#).

---





# Thermoelasticity in port-Hamiltonian form

Eh bien, mon ami, la terre sera un jour ce cadavre refroidi. Elle deviendra inhabitable et sera inhabitée comme la lune, qui depuis longtemps a perdu sa chaleur vitale.

*Vingt mille lieues sous les mers*  
Jules Verne

## Contents

|            |   |           |
|------------|---|-----------|
| <b>6.1</b> | <b>Port-Hamiltonian linear coupled thermoelasticity</b> | <b>57</b> |
| 6.1.1      | The heat equation as a pH descriptor system             | 58        |
| 6.1.2      | Classical thermoelasticity                              | 59        |
| 6.1.3      | Thermoelasticity as two coupled pHs                     | 61        |
| <b>6.2</b> | <b>Thermoelastic port-Hamiltonian bending</b>           | <b>63</b> |
| 6.2.1      | Thermoelastic Euler-Bernoulli beam                      | 63        |
| 6.2.2      | Thermoelastic Kirchhoff plate                           | 65        |
| <b>6.3</b> | <b>Conclusion</b>                                       | <b>67</b> |



Thermoelasticity is the study of deformable bodies undergoing thermal excitations. It is a clear example of a multiphysics phenomenon since the heat transfer and elastic vibrations within the body mutually interact. In this chapter, a linear model of thermoelasticity is obtained under the pH formalism. Each physics is described separately and the final system is obtained considering a power-preserving interconnection of two pHs.

## 6.1 Port-Hamiltonian linear coupled thermoelasticity

In this section, a pH formulation of heat transfer is first introduced. The classical model of thermoelasticity is then recalled. The same model is found by interconnecting the heat equation and the linear elastodynamics problem seen as pHs. It is shown that the interconnection

---

preserves a quadratic functional that plays the role of a fictitious energy. The resulting system is dissipative with respect to this functional. The construction makes use of the intrinsic modularity of pHs [KZvdSB10].

### 6.1.1 The heat equation as a pH descriptor system

Consider the heat equation in a bounded connected set  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$ , describing the evolution of the temperature field  $T(\mathbf{x}, t)$

$$\rho c_\epsilon \frac{\partial T}{\partial t} = k \Delta T + r_Q, \quad \mathbf{x} \in \Omega, \quad (6.1)$$

where  $\rho$ ,  $c_\epsilon$ ,  $k$ ,  $r_Q$  are the mass density, the specific heat density at constant strain, the thermal diffusivity and an heat source. Symbol  $\Delta$  denotes the Laplacian in  $\mathbb{R}^d$ . The Dirichlet and Neumann condition of this problem are

$$\begin{aligned} T \text{ known on } \Gamma_D^T, & \quad \text{Dirichlet condition,} \\ -k \text{ grad } T \cdot \mathbf{n} \text{ known on } \Gamma_N^T, & \quad \text{Neumann condition,} \end{aligned}$$

where a partition of the boundary  $\partial\Omega = \Gamma_D^T \cup \Gamma_N^T$  has been considered. This model can be put in pH form by means of a canonical interconnection structure. An algebraic relationship that describes the Fourier law has to be incorporated in the model (cf. [Kot19, Chapter 2]). Here, a differential-algebraic formulation is exploited to obtain the same system.

Let  $T_0$  be a constant reference temperature (the introduction of this variables is instrumental for coupled thermoelasticity). The functional

$$H_T = \frac{1}{2} \int_{\Omega} \rho c_\epsilon T_0 \left( \frac{T - T_0}{T_0} \right)^2 d\Omega$$

has the physical dimension of an energy and represents a Lyapunov functional of this system. Even though it does not represent the internal energy, it has some important properties. Select as energy variable

$$\alpha_T := \rho c_\epsilon (T - T_0),$$

whose corresponding co-energy is

$$e_T := \frac{\delta H_T}{\delta \alpha_T} = \frac{\alpha_T}{\rho c_\epsilon T_0} = \frac{T - T_0}{T_0} =: \theta.$$

Introducing the heat flux  $\mathbf{j}_Q := -k \text{ grad } T$  as additional variable, the heat equation (6.1) is

---

equivalently reformulated as

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_T \\ \mathbf{j}_Q \end{pmatrix} &= \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & -(T_0 k)^{-1} \end{bmatrix} \begin{pmatrix} e_T \\ \mathbf{j}_Q \end{pmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} u_T, \\ y_T &= \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_T \\ \mathbf{j}_Q \end{pmatrix}. \end{aligned} \quad (6.2)$$

with  $u_T := r_Q$  and  $y_T$  represents the corresponding power-conjugated variable. In matrix notation, it is obtained

$$\begin{aligned} \mathcal{E}_T \partial_t \boldsymbol{\alpha}_T &= (\mathcal{J}_T - \mathcal{R}_T) \mathbf{e}_T + \mathcal{B}_T u_T, \\ y_d &= \mathcal{B}_T^* \mathbf{e}_T \end{aligned} \quad (6.3)$$

where  $\boldsymbol{\alpha}_T = (\alpha_T, \mathbf{j}_Q)$ ,  $\mathbf{e}_T = (e_T, \mathbf{j}_Q)$  and

$$\mathcal{E}_T = \begin{bmatrix} 1 & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathcal{J}_T = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \mathbf{0} \end{bmatrix}, \quad \mathcal{R}_T = \begin{bmatrix} 0 & 0 \\ \mathbf{0} & (T_0 k)^{-1} \end{bmatrix}, \quad \mathcal{B}_T = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}.$$

The system is an example of pH descriptor system (cf. [BMXZ18] for the finite dimensional case). The Hamiltonian reads

$$H_T = \frac{1}{2} \int_{\Omega} \mathbf{e}_T \cdot \mathcal{E}_T \boldsymbol{\alpha}_T \, d\Omega. \quad (6.4)$$

The power rate is then deduced

$$\begin{aligned} \dot{H}_T &= \int_{\Omega} \mathbf{e}_T \cdot \mathcal{E}_T \partial_t \boldsymbol{\alpha}_T \, d\Omega, \\ &= \int_{\Omega} \mathbf{e}_T \cdot \{(\mathcal{J}_T - \mathcal{R}_T) \mathbf{e} + \mathcal{B}_T u_T\} \, d\Omega, \\ &= \int_{\Omega} u_T y_T \, d\Omega - \int_{\Omega} \left( e_T \operatorname{div} \mathbf{j}_Q + \mathbf{j}_Q \operatorname{grad} e_T + \frac{\|\mathbf{j}_Q\|^2}{k T_0} \right) \, d\Omega, \\ &\leq \int_{\Omega} u_T y_T \, d\Omega - \int_{\partial\Omega} e_T \mathbf{j}_Q \cdot \mathbf{n} \, dS. \end{aligned} \quad (6.5)$$

This choice of Hamiltonian allows retrieving the classical boundary conditions and leads to a dissipative system. Other formulations, based on an entropy or internal energy functionals, are possible for the heat equation [DMSB09, SHM19a]. These provide an accrescent or a lossless system. Unfortunately these formulations are non linear and their discretization is a difficult task [SHM19b].

### 6.1.2 Classical thermoelasticity

The derivation of the classical theory of thermoelasticity is not carried out here. The reader may consult in [HE09, Chapter 1] or [Abe12, Chapter 8] for a detailed discussion on this topic.

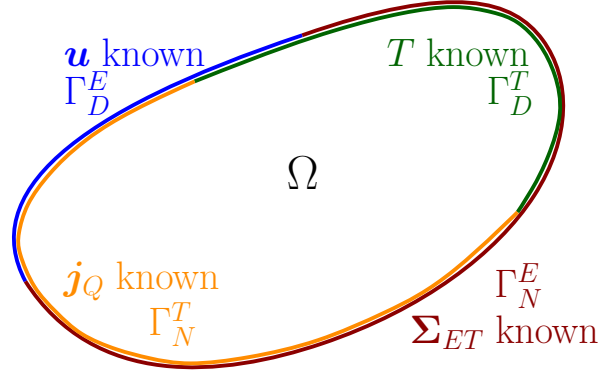


Figure 6.1: Boundary conditions for the thermoelastic problem.

922 Consider a bounded connected set  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$ . The classical equations for linear  
 923 fully-coupled thermoelasticity for an isotropic thermoelastic material are [Bio56, Car73]

$$\begin{aligned}
 \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} &= \text{Div}(\boldsymbol{\Sigma}_{ET}), \\
 \rho c_\epsilon \frac{\partial T}{\partial t} &= -\text{div}(\mathbf{j}_Q) - \mathcal{C}_\beta : \frac{\partial \boldsymbol{\varepsilon}}{\partial t}, \\
 \boldsymbol{\Sigma}_{ET} &= \boldsymbol{\Sigma}_E + \boldsymbol{\Sigma}_T, \\
 \boldsymbol{\Sigma}_E &= 2\mu \boldsymbol{\varepsilon} + \lambda \text{Tr}(\boldsymbol{\varepsilon}) \mathbf{I}_{d \times d}, \\
 \boldsymbol{\Sigma}_T &= -\mathcal{C}_\beta \theta, \\
 \boldsymbol{\varepsilon} &= \text{Grad}(\mathbf{u}), \\
 \mathbf{j}_Q &= -k \text{grad } T.
 \end{aligned} \tag{6.6}$$

924 For simplicity the coupling term

$$\mathcal{C}_\beta := T_0 \beta (2\mu + d\lambda) \mathbf{I}_{d \times d}$$

925 has been introduced. Field  $\mathbf{u}$  is the displacement,  $\boldsymbol{\varepsilon}$  is the infinitesimal strain tensor,  $\boldsymbol{\Sigma}_E, \boldsymbol{\Sigma}_T$   
 926 are the stress tensor contribution due to mechanical deformation and a thermal field. Co-  
 927 efficients  $\lambda, \mu$  are the Lamé parameters, and  $\beta$  the thermal expansion coefficient. Given a  
 928 partition of the boundary  $\partial\Omega = \Gamma_D^E \cup \Gamma_N^E = \Gamma_D^T \cup \Gamma_N^T$  for the elastic and thermal domain. The  
 929 general boundary conditions read (see Fig. 6.1)

$$\begin{aligned}
 \mathbf{u} \text{ known on } \Gamma_D^E \times (0, +\infty), & \quad T \text{ known on } \Gamma_D^T \times (0, +\infty), \\
 \boldsymbol{\Sigma}_{ET} \cdot \mathbf{n} \text{ known on } \Gamma_N^E \times (0, +\infty), & \quad \mathbf{j}_Q \cdot \mathbf{n} \text{ known on } \Gamma_N^T \times (0, +\infty).
 \end{aligned} \tag{6.7}$$

930 In the following section an equivalent system is constructed by interconnecting the heat  
 931 equation and the elastodynamics system in a structured manner.

---

### 6.1.3 Thermoelasticity as two coupled pHs

Consider again the equation of elasticity on  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$  (cf. Eq. (4.16)), together with a distributed input  $\mathbf{u}_E$  that plays the role of a distributed force

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_v \\ \mathbf{A}_\varepsilon \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \text{Div} \\ \text{Grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \end{pmatrix} + \begin{bmatrix} \mathbf{I}_{d \times d} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_E, \\ \mathbf{y}_E &= \begin{bmatrix} \mathbf{I}_{d \times d} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \end{pmatrix}, \end{aligned} \quad (6.8)$$

with Hamiltonian

$$H_E = \frac{1}{2} \int_{\Omega} \{ \boldsymbol{\alpha}_v \cdot \mathbf{e}_v + \mathbf{A}_\varepsilon : \mathbf{E}_\varepsilon \} \, d\Omega.$$

Recall the pH formulation of the heat equation (6.2)

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_T \\ \mathbf{j}_Q \end{pmatrix} &= \begin{bmatrix} 0 & -\text{div} \\ -\text{grad} & -(T_0 k)^{-1} \end{bmatrix} \begin{pmatrix} e_T \\ \mathbf{j}_Q \end{pmatrix} + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} u_T, \\ \mathbf{y}_T &= \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_T \\ \mathbf{j}_Q \end{pmatrix}, \end{aligned} \quad (6.9)$$

with Hamiltonian  $H_T$  defined in (6.4). The linear thermoelastic problem can be expressed as a coupled port-Hamiltonian system. Consider the following interconnection

$$\mathbf{u}_E = -\text{Div}(\mathcal{C}_\beta \mathbf{y}_T), \quad u_T = -\mathcal{C}_\beta : \text{Grad}(\mathbf{y}_E). \quad (6.10)$$

The interconnection is power preserving as it can be compactly written as

$$\mathbf{u}_E = \mathcal{A}_\beta(\mathbf{y}_T), \quad u_T = -\mathcal{A}_\beta^*(\mathbf{y}_E).$$

where  $\mathcal{A}_\beta^*$  denotes the formal adjoint. The assertion is justified by the following proposition.

#### Proposition 5

Let  $C_0^\infty(\Omega)$ ,  $C_0^\infty(\Omega, \mathbb{R}^d)$  be the space of smooth functions and vector-valued functions respectively. Given  $y_T \in C_0^\infty(\Omega)$ ,  $\mathbf{y}_E \in C_0^\infty(\Omega, \mathbb{R}^d)$ , the coupling operator

$$\begin{aligned} \mathcal{A}_\beta : C_0^\infty(\Omega) &\rightarrow C_0^\infty(\Omega, \mathbb{R}^d), \\ y_T &\rightarrow -\text{Div}(\mathcal{C}_\beta y_T) \end{aligned} \quad (6.11)$$

has formal adjoint

$$\begin{aligned} \mathcal{A}_\beta^* : C_0^\infty(\Omega, \mathbb{R}^d) &\rightarrow C_0^\infty(\Omega) \\ \mathbf{y}_E &\rightarrow +\mathcal{C}_\beta : \text{Grad}(\mathbf{y}_E) \end{aligned} \quad (6.12)$$

*Proof.* It is necessary to show

$$\langle \mathbf{y}_E, \mathcal{A}_\beta y_T \rangle_{L^2(\Omega, \mathbb{R}^d)} = \langle \mathcal{A}_\beta^* \mathbf{y}_E, y_T \rangle_{L^2(\Omega)}, \quad (6.13)$$


---

945 where for  $\mathbf{u}_E, \mathbf{y}_E \in C_0^\infty(\Omega)$ ,  $u_T, y_T \in C_0^\infty(\Omega)$

$$\langle \mathbf{u}_E, \mathbf{y}_E \rangle_{L^2(\Omega, \mathbb{R}^d)} = \int_{\Omega_E} \mathbf{u}_E \cdot \mathbf{y}_E \, d\Omega, \quad \langle u_T, y_T \rangle_{L^2(\Omega)} = \int_{\Omega_T} u_T y_T \, d\Omega. \quad (6.14)$$

946 The proof is a simple application of Theorem 5

$$\begin{aligned} \langle \mathbf{y}_E, \mathcal{A}_\beta y_T \rangle_{L^2(\Omega, \mathbb{R}^d)} &= - \int_{\Omega} \mathbf{y}_E \cdot \text{Div}(\mathcal{C}_\beta y_T) \, d\Omega, \\ &= \int_{\Omega} \text{Grad}(\mathbf{y}_E) : \mathcal{C}_\beta y_T \, d\Omega, \\ &= \int_{\Omega} \mathcal{A}_\beta^*(\mathbf{y}_E) y_T \, d\Omega, \\ &= \langle \mathcal{A}_\beta^* \mathbf{y}_E, y_T \rangle_{L^2(\Omega)}. \end{aligned} \quad (6.15)$$

947 This concludes the proof. □

948 If the compact support assumption is removed, it is obtained

$$\begin{aligned} \langle u_T, y_T \rangle_{L^2(\Omega)} + \langle \mathbf{u}_E, \mathbf{y}_E \rangle_{L^2(\Omega, \mathbb{R}^3)} &= - \int_{\Omega} \{ (\mathcal{C}_\beta : \text{Grad} \, \mathbf{e}_v) e_T + \text{Div}(\mathcal{C}_\beta e_T) \cdot \mathbf{e}_v \} \, d\Omega, \\ &= - \int_{\Omega} \text{div}(e_T \mathcal{C}_\beta \cdot \mathbf{e}_v) \, d\Omega, \\ &= - \int_{\partial\Omega} (e_T \mathcal{C}_\beta \cdot \mathbf{n}) \cdot \mathbf{e}_v \, dS. \end{aligned} \quad (6.16)$$

Using the expression of  $y_T, \mathbf{y}_E$ , considering that  $T_0$  is constant and applying Schwarz theorem for smooth function, the inputs are equal to

$$\mathbf{u}_E = \text{Div}(\boldsymbol{\Sigma}_T), \quad u_T = -\mathcal{C}_\beta : \text{Grad}(\mathbf{v}) = -\mathcal{C}_\beta : \frac{\partial \boldsymbol{\varepsilon}}{\partial t}.$$

949 The coupled thermoelastic problem can now be written as

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_v \\ \mathbf{A}_\varepsilon \\ \alpha_T \\ \mathbf{j}_Q \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \text{Div} & \mathcal{A}_\beta & \mathbf{0} \\ \text{Grad} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_\beta^* & 0 & 0 & -\text{div} \\ \mathbf{0} & \mathbf{0} & -\text{grad} & -(T_0 k)^{-1} \end{bmatrix} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \\ e_T \\ \mathbf{j}_Q \end{pmatrix}, \quad (6.17)$$

with total energy given by  $H = H_E + H_T$ . The power balance for each subsystem is given by

$$\dot{H}_E = \int_{\Omega} \mathbf{u}_E \cdot \mathbf{y}_E \, d\Omega + \int_{\partial\Omega} \mathbf{e}_v \cdot (\mathbf{E}_\varepsilon \cdot \mathbf{n}) \, dS, \quad (6.18)$$

$$\dot{H}_T \leq \int_{\Omega} u_T y_T \, d\Omega - \int_{\partial\Omega} \theta \mathbf{j}_Q \cdot \mathbf{n} \, dS, \quad (6.19)$$

The overall power balance is easily computed considering Eqs. (6.18) (6.19) and (6.16)

$$\dot{H} = \dot{H}_E + \dot{H}_T \leq \int_{\partial\Omega} \{[\mathbf{E}_\varepsilon - e_T \mathcal{C}_\beta] \cdot \mathbf{n}\} \cdot \mathbf{e}_v \, dS - \int_{\partial\Omega} \theta \, \mathbf{j}_Q \cdot \mathbf{n} \, dS. \quad (6.20)$$

This result is the same stated in [Car73], page 332. From the power balance the classical boundary conditions are retrieved. This allows defining appropriate boundary operators for the thermoelastic problem

$$\mathbf{u}_\partial = \underbrace{\begin{bmatrix} \gamma_0^{\Gamma_D^E} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_n^{\Gamma_N^E} & -\gamma_n^{\Gamma_N^E}(\mathcal{C}_\beta \cdot) & \mathbf{0} \\ 0 & 0 & \gamma_0^{\Gamma_D^T} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma_n^{\Gamma_N^T} \end{bmatrix}}_{\mathcal{B}_\partial} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \\ e_T \\ \mathbf{j}_Q \end{pmatrix}, \quad \mathbf{y}_\partial = \underbrace{\begin{bmatrix} \mathbf{0} & \gamma_n^{\Gamma_D^E} & -\gamma_n^{\Gamma_D^E}(\mathcal{C}_\beta \cdot) & \mathbf{0} \\ \gamma_0^{\Gamma_N^E} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma_n^{\Gamma_D^T} \\ 0 & 0 & \gamma_0^{\Gamma_N^T} & 0 \end{bmatrix}}_{\mathcal{C}_\partial} \begin{pmatrix} \mathbf{e}_v \\ \mathbf{E}_\varepsilon \\ e_T \\ \mathbf{j}_Q \end{pmatrix}. \quad (6.21)$$

System (6.17) together with (6.21) is a pH system with boundary control and observation. Indeed, the classical thermoelastic problem can be modeled as two coupled systems, demonstrating the modularity of the pH paradigm.

## 6.2 Thermoelastic port-Hamiltonian bending

In this section, the thermoelastic bending of thin beam and plate structures is described as coupled interconnection of pHs. Starting from classical thermoelastic models a suitable pH formulation can be obtained. This couples a mechanical system defined on a reduced domain (uni-dimensional for beams, bi-dimensional for plates), to a thermal domain defined in the three-dimensional space.

### 6.2.1 Thermoelastic Euler-Bernoulli beam

The model for the linear thermoelastic vibrations of an isotropic thin rod is detailed in [Cha62, LR00]. The domain of the beam is uni-dimensional  $\Omega_E = \{0, L\}$ , while the thermal domain is three-dimensional  $\Omega_T = \{0, L\} \times S$ , where  $S$  is the set representing the beam cross section. The set  $S$  is assumed to be constant along the axis for simplicity. The ruling equations are

$$\begin{aligned} \rho A \frac{\partial^2 w}{\partial t^2} &= -EI \frac{\partial^4 w}{\partial x^4} - \beta E T_0 \frac{\partial^2}{\partial x^2} \int_S z \theta \, dx \, dy, & x \in \{0, L\} &= \Omega_E, \\ \rho c_{\epsilon, B} T_0 \frac{\partial \theta}{\partial t} &= k T_0 \Delta \theta + \beta T_0 E z \frac{\partial^3 w}{\partial x^2 \partial t}, & (x, y, z) \in \Omega_E \times S &= \Omega_T, \end{aligned} \quad (6.22)$$

where  $w(x, t)$  is the vertical displacement of the beam  $I = \int_S z^2 \, dx \, dy$  the second moment of area,  $E$  the Young modulus and  $A$  the cross section. The constant  $c_{\epsilon, B}$  is due to the thermoelastic coupling (cf. [Cha62, LR00] for a detailed explanation). The other terms have

meaning than in Section §6.1. Since the normalized temperature  $\theta(x, y, z, t)$  depends on all spatial coordinates, the symbol  $\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$  is the Laplacian in three dimensions. The physical constants are assumed to be constant for simplicity.

The coupling operator is defined as

$$\mathcal{A}_{\beta,B}(y_T) := -\beta ET_0 \partial_{xx} \left( \int_S z y_T \, dx \, dy \right). \quad (6.23)$$

To unveil an interconnection that is power with respect to a certain function, the formal adjoint of the coupling operator is needed.

**Proposition 6**

Let  $C_0^\infty(\Omega_T)$ ,  $C_0^\infty(\Omega_E)$  be the space of smooth functions with compact support defined on  $\Omega_T$  and  $\Omega_E$  respectively. Given  $y_T \in C_0^\infty(\Omega_T)$ ,  $y_E \in C_0^\infty(\Omega_E)$  the formal adjoint of the coupling operator is

$$\mathcal{A}_{\beta,B}^*(y_E) = -\beta ET_0 z \partial_{xx} y_E. \quad (6.24)$$

*Proof.* The formal adjoint is defined by the relation

$$\langle y_E, \mathcal{A}_{\beta,B} y_T \rangle_{L^2(\Omega_E)} = \langle \mathcal{A}_{\beta,B}^* y_E, y_T \rangle_{L^2(\Omega_T)}, \quad (6.25)$$

where for  $u_E, y_E \in C_0^\infty(\Omega_E)$ ,  $u_T, y_T \in C_0^\infty(\Omega_T)$

$$\langle u_E, y_E \rangle_{L^2(\Omega_E)} = \int_{\Omega_E} u_E y_E \, dx, \quad \langle u_T, y_T \rangle_{L^2(\Omega_T)} = \int_{\Omega_T} y_T y_T \, dx \, dy \, dz. \quad (6.26)$$

Using Def. (6.23) and the integration by parts, one finds

$$\begin{aligned} \langle y_E, \mathcal{A}_{\beta,B} y_T \rangle_{L^2(\Omega_E)} &= \int_{\Omega_E} y_E \mathcal{A}_{\beta,B} y_T \, dx, \\ &= - \int_{\Omega_E} y_E \beta ET_0 \partial_{xx} \left( \int_S z y_T \, dx \, dy \right) \, dx, \\ &= - \int_{\Omega_E} (\partial_{xx} y_E) \beta ET_0 \left( \int_S z y_T \, dx \, dy \right) \, dx, \end{aligned} \quad (6.27)$$

Since  $\Omega_T = \Omega_E \times S$  and from the properties of multiple integrals, it is found

$$\begin{aligned} - \int_{\Omega_E} \partial_{xx} (y_E) \beta ET_0 \left( \int_S z y_T \, dx \, dy \right) \, dx &= - \int_{\Omega_E} \int_S (\partial_{xx} y_E) \beta ET_0 z y_T \, dx \, dx \, dy, \\ &= - \int_{\Omega_T} (\partial_{xx} y_E) \beta ET_0 z y_T \, dx \, dx \, dy, \\ &= \langle \mathcal{A}_{\beta,B}^* y_E, y_T \rangle_{L^2(\Omega_T)}. \end{aligned} \quad (6.28)$$

This concludes the proof. □



Using Eqs. (6.23) and (6.24), System (6.22), is rewritten as

$$\begin{aligned}\rho A \frac{\partial^2 w}{\partial t^2} &= -EI \frac{\partial^4 w}{\partial x^4} + \mathcal{A}_{\beta,B} \theta, \\ \rho c_{\epsilon,B} T_0 \frac{\partial \theta}{\partial t} &= k T_0 \Delta \theta - \mathcal{A}_{\beta,B}^* \frac{\partial w}{\partial t}.\end{aligned}\quad (6.29)$$

Consider the Hamiltonian functional

$$H = H_E + H_T = \frac{1}{2} \int_{\Omega_E} \left\{ \rho A \left( \frac{\partial w}{\partial t} \right)^2 + EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right\} dx + \frac{1}{2} \int_{\Omega_T} \rho c_{\epsilon,B} T_0 \theta^2 dx dy dz. \quad (6.30)$$

The energy variables are chosen to make the Hamiltonian functional quadratic

$$\alpha_w = \rho A \partial_t w, \quad \alpha_\kappa = \partial_{xx} w, \quad \alpha_T = \rho c_{\epsilon,B} T_0 \theta. \quad (6.31)$$

The corresponding co-energy variables evaluate to

$$e_w := \frac{\delta H}{\delta \alpha_w} = \partial_t w, \quad e_\kappa := \frac{\delta H}{\delta \alpha_\kappa} = EI \partial_{xx} w, \quad e_T := \frac{\delta H}{\delta \alpha_T} = \theta. \quad (6.32)$$

System (6.29) can now be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_\kappa \\ \alpha_T \\ j_Q \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} & \mathcal{A}_{\beta,B} & 0 \\ \partial_{xx} & 0 & 0 & 0 \\ -\mathcal{A}_{\beta,B}^* & 0 & 0 & -\text{div} \\ 0 & 0 & -\text{grad} & -(kT_0)^{-1} \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \\ e_T \\ j_Q \end{pmatrix}, \quad (6.33)$$

This system is the equivalent of (6.17) for bending of beams. Hence, following the same reasoning, it can be obtained starting from each subsystem in pH form by means of an appropriate interconnection.

### 6.2.2 Thermoelastic Kirchhoff plate

For the bending of thin plate, several different models have been proposed [Cha62, Lag89, Sim99, Nor06]. Here, the Chadwick model [Cha62] is considered. The thin plate occupies the open connected set  $\Omega_E \times \left\{ -\frac{h}{2}, \frac{h}{2} \right\}$ , where  $h$  is the plate thickness. The system of equations describe the midplane vertical displacement and the evolution of the temperature in the 3D domain

$$\begin{aligned}\rho h \frac{\partial^2 w}{\partial t^2} &= -D_b \Delta_{2D}^2 w - \frac{\beta T_0 E}{1-\nu} \Delta_{2D} \left( \int_{-h/2}^{h/2} z \theta dz \right), & (x, y) \in \Omega_E, \\ \rho c_{\epsilon,P} T_0 \frac{\partial \theta}{\partial t} &= -k T_0 \Delta_{3D} + \frac{\beta T_0 E z}{1-\nu} \Delta_{2D} \left( \frac{\partial w}{\partial t} \right), & (x, y, z) \in \Omega_E \times \left\{ -\frac{h}{2}, \frac{h}{2} \right\} = \Omega_T,\end{aligned}\quad (6.34)$$

where  $w(x, y, t)$  is the vertical deflection,  $D_b = \frac{E h^3}{12(1-\nu^2)}$  the bending rigidity (cf. Eq. (5.11)),  $\nu$  the Poisson modulus and  $c_{\epsilon,P}$  a constant (depending on the heat capacity at constant strain

and other coupling parameters, cf. [Cha62]). Symbols  $\Delta_{2D} = \partial_{xx} + \partial_{yy}$ ,  $\Delta_{3D} = \partial_{xx} + \partial_{yy} + \partial_{zz}$  are the two- and three-dimensional Laplacian.

The coupling operator is here defined as

$$\mathcal{A}_{\beta,P}(y_T) := -\frac{\beta T_0 E}{1-\nu} \Delta_{2D} \left( \int_{-h/2}^{h/2} z y_T \, dz \right). \quad (6.35)$$

Analouglsy with respect to the Euler-Bernoulli beam its formal adjoint is sought for.

### Proposition 7

Let  $C_0^\infty(\Omega_T)$ ,  $C_0^\infty(\Omega_E)$  be the space of smooth functions with compact support defined on  $\Omega_T$  and  $\Omega_E$  respectively. Given  $y_T \in C_0^\infty(\Omega_T)$ ,  $y_E \in C_0^\infty(\Omega_E)$  the formal adjoint of the coupling operator is

$$\mathcal{A}_{\beta,B}^*(y_E) = -\frac{\beta T_0 E z}{1-\nu} \Delta_{2D} y_E. \quad (6.36)$$

*Proof.* The proof is completely identical to Prop. 6. □

System 6.34 is rewritten as

$$\begin{aligned} \rho h \frac{\partial^2 w}{\partial t^2} &= -D_b \Delta_{2D}^2 w + \mathcal{A}_{\beta,P} \theta, \\ \rho c_{\epsilon,P} T_0 \frac{\partial \theta}{\partial t} &= -k T_0 \Delta_{3D} \theta - \mathcal{A}_{\beta,P}^* \left( \frac{\partial w}{\partial t} \right), \end{aligned} \quad (6.37)$$

The Hamiltonian functional equals

$$\begin{aligned} H = H_E + H_T &= \frac{1}{2} \int_{\Omega_E} \left\{ \rho h \left( \frac{\partial w}{\partial t} \right)^2 + (\mathcal{D}_b \text{Hess}_{2D} w) : \text{Hess}_{2D} w \right\} \, dx \, dy \\ &+ \frac{1}{2} \int_{\Omega_T} \rho c_{\epsilon,P} T_0 \theta^2 \, dx \, dy \, dz, \end{aligned} \quad (6.38)$$

where  $\text{Hess}_{2D}$  is the Hessian in two dimensions and  $\mathcal{D}_b$  was defined in (5.11) (cf. Sec. §5.1.1).

The energy and co-energy variables are

$$\begin{aligned} \alpha_w &= \rho h \partial_t w, & \mathbf{A}_\kappa &= \text{Hess}_{2D} w, & \alpha_T &= \rho c_{\epsilon,P} T_0 \theta, \\ e_w &= \partial_t w, & \mathbf{E}_\kappa &= \mathcal{D}_b \text{Hess}_{2D} w, & e_T &= \theta. \end{aligned} \quad (6.39)$$

System (6.37) is rewritten as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \mathbf{A}_\kappa \\ \alpha_T \\ j_Q \end{pmatrix} = \begin{bmatrix} 0 & -\text{div Div}_{2D} & \mathcal{A}_{\beta,P} & 0 \\ \text{Hess}_{2D} & \mathbf{0} & \mathbf{0} & 0 \\ -\mathcal{A}_{\beta,P}^* & 0 & 0 & -\text{div}_{3D} \\ \mathbf{0} & \mathbf{0} & -\text{grad}_{3D} & -(kT_0)^{-1} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \\ e_T \\ j_Q \end{pmatrix}, \quad (6.40)$$

---

1018 The subscript  $2D$ ,  $3D$  refers to two- and three-dimensional operators respectively. The final  
 1019 system reproduces the same structured coupling already observed for (6.17), (6.33).

1020 **Remark 7**

1021 *The thermoelastic bending can be reduced to two problems defined on the same domain (cf.*  
 1022 *[HZ97] for beams and [AL00] for plates) by introducing the following approximation of the*  
 1023 *temperature field*

$$\theta(x, y, z) = \theta_0 + z\theta_1, \quad (6.41)$$

1024 *where  $\theta_0 = \theta_0(x)$ ,  $\theta_1 = \theta_1(x)$  for beams and  $\theta_0 = \theta_0(x, y)$ ,  $\theta_1 = \theta_1(x, y)$  for plates. However,*  
 1025 *this introducing a strong simplification as the thermal phenomena typically occur in the whole*  
 1026 *three-dimensional space.*

1027 **6.3 Conclusion**

1028 In this chapter, it was shown classical linear thermoelastic problem are equivalent to two  
 1029 coupled port-Hamiltonian systems. This is especially interesting for the simulation of ther-  
 1030 moelastic phenomena: each subsystem can be discretized separately and then coupled to the  
 1031 other using the discretized coupling operator. This allows to track easily how the energy flows  
 1032 within the two physics.

---



1033

## Part III

1034

# Finite element structure preserving discretization

1035



# Partitioned finite element method

Every truth is simple... is that not doubly a lie?

*Twilight of the Idols*  
*Friedrich Nietzsche*

## Contents

|            |  |            |
|------------|--|------------|
| <b>7.1</b> | <b>Discretization under uniform boundary condition</b> | <b>71</b>  |
| 7.1.1      | General procedure                                      | 73         |
| 7.1.2      | Linear case  | 82         |
| 7.1.3      | Linear flexible structures                             | 84         |
| <b>7.2</b> | <b>Mixed boundary conditions</b>                       | <b>93</b>  |
| 7.2.1      | Solution using Lagrange multipliers                    | 95         |
| 7.2.2      | Virtual domain decomposition                           | 97         |
| <b>7.3</b> | <b>Conclusion</b>                                      | <b>101</b> |



Discretization is the process of transferring continuous models into discrete counterparts. The discrete model should be faithful to the continuous one. To this aim, it is usually essential that the main properties of the continuous system are preserved at the discrete level. An algorithm that is capable of conserving properties at the discrete level is called structure-preserving [CMKO11]. In this chapter, a method to spatially discretize infinite-dimensional pHs into finite-dimensional ones in a structure preserving manner is illustrated.

## 7.1 Discretization under uniform boundary condition

A discrete version of a infinite-dimensional pH system is meant to preserve the underlying properties related to power continuity. To achieve this purpose, the discretization procedure consists of two steps [KML18]:

- Finite-dimensional approximation of the Stokes-Dirac structure, i.e. the formally skew symmetric differential operator that defines the structure. The duality of the power

variables has to be mapped onto the finite approximation. The subspace of the discrete variables will be represented by a Dirac structure.

- The Hamiltonian requires as well a suitable discretization, which gives rise to a discrete Hamiltonian.

A structure-preserving discretization is able to construct an equivalent pH system that possess the structural properties of the original model:

| Infinite dimensional pH system   | Structure-preserving discretization  |
|--|--|
| <p>PDE with distributed inputs:</p> $\frac{\partial \alpha}{\partial t}(\mathbf{x}, t) = \mathcal{J} \frac{\delta H}{\delta \alpha} + \mathcal{B} \mathbf{u}_\Omega(\mathbf{x}, t),$ $\mathbf{y}_\Omega(\mathbf{x}, t) = \mathcal{B}^* \frac{\delta H}{\delta \alpha}.$ <p>Boundary conditions:</p> $\mathbf{u}_\partial = \mathcal{B}_\partial \frac{\delta H}{\delta \alpha}, \quad \mathbf{y}_\partial = \mathcal{C}_\partial \frac{\delta H}{\delta \alpha}.$ <p>Power balance (Stokes Theorem):</p> $\dot{H} = \int_{\partial\Omega} \mathbf{u}_\partial \cdot \mathbf{y}_\partial \, dS + \int_{\Omega} \mathbf{u}_\Omega \cdot \mathbf{y}_\Omega \, d\Omega.$ | <p>Resulting ODE:</p> $\dot{\alpha}_d = \mathbf{J} \nabla H_d + \mathbf{B}_\Omega \mathbf{u}_\Omega + \mathbf{B}_\partial \mathbf{u}_\partial,$ $\mathbf{y}_\Omega = \mathbf{B}_\Omega^\top \nabla H_d,$ $\mathbf{y}_\partial = \mathbf{B}_\partial^\top \nabla H_d.$ <p>Discretized Hamiltonian:</p> $H_d := H(\alpha \equiv \alpha_d).$ <p>Power balance:</p> $\dot{H} = \mathbf{u}_\partial^\top \mathbf{y}_\partial + \mathbf{u}_\Omega^\top \mathbf{y}_\Omega.$ |

In this thesis the Partitioned Finite Element Method (PFEM), originally presented in [CRML18, CRML19], is chosen to obtain discretized models of dpHs. This procedure boils down to three simple steps

1. The system is written in weak form;
2. An integration by parts is applied to highlight the appropriate boundary control;
3. A Galerkin method is employed to obtain a finite-dimensional system. For the approximation basis the finite element method is here employed but spectral methods can be used as well.

Once the system has been put into weak form, a subset of the equations is integrated by parts, so that boundary variables are naturally included into the formulation and appear as control inputs, the collocated outputs being defined accordingly. The discretization of energy and co-energy variables (and the associated test functions) leads directly to a full rank representation for the finite-dimensional pH system. This approach makes possible the usage of FEM software, like FEniCS [LMW<sup>+</sup>12], or Firedrake [RHM<sup>+</sup>17]. The procedure is universal, as it relies on a general integration by parts formula that characterizes multi-dimensional pHs. This is why the methodology is illustrated in all its generality and then detailed for



some particular examples.

This methodology is easily applicable under a uniform causality assumption. The case of mixed boundary conditions requires additional care and will be treated in the subsequent Section §7.2.

### 7.1.1 General procedure

Given an open connected set  $\Omega \in \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$ , consider a generic pH system defined on  $\Omega$

$$\partial_t \boldsymbol{\alpha} = \mathcal{J} \mathbf{e}, \quad \boldsymbol{\alpha} \in L^2(\Omega, \mathbb{F}), \quad \mathcal{J} : L^2(\Omega, \mathbb{F}) \rightarrow L^2(\Omega, \mathbb{F}) \mid \mathcal{J} = -\mathcal{J}^*, \quad (7.1a)$$

$$\mathbf{e} := \delta_{\boldsymbol{\alpha}} H, \quad \mathbf{e} \in H^{\mathcal{J}} := \left\{ \mathbf{e} \in L^2(\Omega, \mathbb{F}) \mid \mathcal{J} \mathbf{e} \in L^2(\Omega, \mathbb{F}) \right\}, \quad (7.1b)$$

$$\mathbf{u}_{\partial} = \mathcal{B}_{\partial} \mathbf{e}, \quad \mathbf{u}_{\partial} \in \mathbb{R}^m, \quad (7.1c)$$

$$\mathbf{y}_{\partial} = \mathcal{C}_{\partial} \mathbf{e}, \quad \mathbf{y}_{\partial} \in \mathbb{R}^m. \quad (7.1d)$$

The operator  $\mathcal{J} : L^2(\Omega, \mathbb{F}) \rightarrow L^2(\Omega, \mathbb{F})$  is a differential, formally skew adjoint operator  $\mathcal{J} = -\mathcal{J}^*$  over the space  $L^2(\Omega, \mathbb{F})$ . The set  $\mathbb{F}$  is an appropriate Cartesian product of either scalar, vectorial or tensorial quantities. Its precise definition depends on the example upon consideration. For scalars  $(a, b) \in L^2(\Omega)$ , vectors  $(\mathbf{a}, \mathbf{b}) \in L^2(\Omega, \mathbb{R}^d)$  and tensors  $(\mathbf{A}, \mathbf{B}) \in L^2(\Omega, \mathbb{R}^{d \times d})$  the  $L^2$  inner product is given by

$$\langle a, b \rangle_{L^2(\Omega)} = \int_{\Omega} ab \, d\Omega, \quad \langle \mathbf{a}, \mathbf{b} \rangle_{L^2(\Omega, \mathbb{R}^d)} = \int_{\Omega} \mathbf{a} \cdot \mathbf{b} \, d\Omega, \quad \langle \mathbf{A}, \mathbf{B} \rangle_{L^2(\Omega, \mathbb{R}^{d \times d})} = \int_{\Omega} \mathbf{A} : \mathbf{B} \, d\Omega. \quad (7.2)$$

For scalars  $a_{\partial}, b_{\partial} \in L^2(\partial\Omega)$  and vectors  $\mathbf{a}_{\partial}, \mathbf{b}_{\partial} \in L^2(\partial\Omega, \mathbb{R}^m)$  defined on the boundary the inner product is defined as

$$\langle a_{\partial}, b_{\partial} \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} a_{\partial} b_{\partial} \, dS, \quad \langle \mathbf{a}_{\partial}, \mathbf{b}_{\partial} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} = \int_{\partial\Omega} \mathbf{a}_{\partial} \cdot \mathbf{b}_{\partial} \, dS. \quad (7.3)$$

The Hamiltonian functional of Eq. (7.1b) is allowed to be non linear in the energy variables

$$H = \int_{\Omega} \mathcal{H}(\boldsymbol{\alpha}) \, d\Omega,$$

where  $\mathcal{H}(\boldsymbol{\alpha}) : L^2(\Omega, \mathbb{F}) \rightarrow \mathbb{R}$  is a non linear function.

To applied this methodology the non linearities are restricted to the Hamiltonian and a uniform causality condition is supposed to characterize the system. It is required as well that the system admits a partition of the variables. This requirement is always encounter in the following examples. These hypotheses are resumed in the following assumptions.

**Assumption 2** (Partitioning of the system)

1109 Consider system (7.1a). It is assumed that the Hilbert space  $L^2(\Omega, \mathbb{F}) := L^2(\Omega, \mathbb{F})$  admits the  
 1110 splitting  $L^2(\Omega, \mathbb{F}) = L^2(\Omega, \mathbb{A}) \times L^2(\Omega, \mathbb{B})$ . This means that  $\mathbb{F} = \mathbb{A} \times \mathbb{B}$ .  
 1111

1112 The operator  $\mathcal{J}$  is assumed to be skew-symmetric (or formally skew-adjoint) on  $L^2(\Omega, \mathbb{F})$   
 1113 and linear:

$$\mathcal{J} = \mathcal{J}_a + \mathcal{J}_d, \quad (7.4)$$

1114 where  $\mathcal{J}_a$  is the algebraic contribution (a skew-symmetric matrix) and  $\mathcal{J}_d$  the differential  
 1115 contribution. The algebraic part is assumed to take the form

$$\mathcal{J}_a = \begin{bmatrix} 0 & -\mathbf{L}^\top \\ \mathbf{L} & 0 \end{bmatrix}, \quad \begin{array}{l} \mathbf{L}^\top : L^2(\Omega, \mathbb{B}) \rightarrow L^2(\Omega, \mathbb{A}), \\ \mathbf{L} : L^2(\Omega, \mathbb{A}) \rightarrow L^2(\Omega, \mathbb{B}), \end{array} \quad (7.5)$$

1116 where  $\mathbf{L}$  is a bounded operator. Analogously, the linear differential operator  $\mathcal{J}_d$  is assumed to  
 1117 be of the form

$$\mathcal{J}_d = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix}, \quad \begin{array}{l} \mathcal{L}^* : L^2(\Omega, \mathbb{B}) \rightarrow L^2(\Omega, \mathbb{A}), \\ \mathcal{L} : L^2(\Omega, \mathbb{A}) \rightarrow L^2(\Omega, \mathbb{B}), \end{array} \quad (7.6)$$

1118 where  $\mathcal{L}^*$  denotes the formal adjoint of the linear differential operator  $\mathcal{L}$ . The operator  $\mathcal{L}$  is  
 1119 unbounded and can be either a first or a second order differential operator (in the latter case  
 1120 it can be expressed as  $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2$ ). Given the splitting  $L^2(\Omega, \mathbb{A}) \times L^2(\Omega, \mathbb{B}) = L^2(\Omega, \mathbb{F})$  the  
 1121 Hilbert space  $H^\mathcal{J}$  can be split as well as

$$H^\mathcal{J} = H^\mathcal{L} \times H^{-\mathcal{L}^*}, \quad \begin{array}{l} H^\mathcal{L} := \{ \mathbf{u}_1 \in L^2(\Omega, \mathbb{A}) \mid \mathcal{L}\mathbf{u}_1 \in L^2(\Omega, \mathbb{B}) \}, \\ H^{-\mathcal{L}^*} := \{ \mathbf{u}_2 \in L^2(\Omega, \mathbb{B}) \mid -\mathcal{L}^*\mathbf{u}_2 \in L^2(\Omega, \mathbb{A}) \} \end{array} \quad (7.7)$$

1122 The boundary operators are then supposed to fulfill the following assumption, that guar-  
 1123 antees a uniform causality condition.

1124 **Assumption 3** (General integration by parts formula)

1125 Assume that there exist two boundary operators  $\mathcal{N}_{\partial,1}$ ,  $\mathcal{N}_{\partial,2}$  such that for  $(\mathbf{u}_1, \mathbf{u}_2) \in H^\mathcal{L} \times H^{-\mathcal{L}^*}$   
 1126 a general integration by parts formula holds

$$\langle \mathbf{u}_2, \mathcal{L}\mathbf{u}_1 \rangle_{L^2(\Omega, \mathbb{B})} - \langle \mathcal{L}^*\mathbf{u}_2, \mathbf{u}_1 \rangle_{L^2(\Omega, \mathbb{A})} = \langle \mathcal{N}_{\partial,1}\mathbf{u}_1, \mathcal{N}_{\partial,2}\mathbf{u}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \quad (7.8)$$

1127 The boundary operators  $\mathcal{B}_\partial, \mathcal{C}_\partial$  of Eqs. (7.1c), (7.1d), are then assumed to verify, in an  
 1128 exclusive manner, either

$$\mathcal{B}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}, \quad \mathcal{C}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix}, \quad (7.9)$$

1129 or

$$\mathcal{B}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix}, \quad \mathcal{C}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}. \quad (7.10)$$

1130 **Remark 8** (Duality pairing for rigged Hilbert spaces)

1131 The integration by part formula establishes a duality pairing between Sobolev spaces. This

duality pairing is then compatible with an  $L^2$  inner product in presence of a rigged Hilbert space (Gelfand triple). Without entering into technical details, we shall always use this equivalence of representation. Therefore, the boundary integrals are expressed as  $L^2$  inner product over the boundary.

Thanks to Assumption 2, System (7.1) is rewritten as

$$\partial_t \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{L}^\top - \mathcal{L}^* \\ \mathbf{L} + \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \begin{aligned} \alpha_1 &\in L^2(\Omega, \mathbb{A}), \\ \alpha_2 &\in L^2(\Omega, \mathbb{B}), \end{aligned} \quad (7.11a)$$

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} := \begin{pmatrix} \delta_{\alpha_1} H \\ \delta_{\alpha_2} H \end{pmatrix}, \quad \begin{aligned} e_1 &\in H^\mathcal{L}, \\ e_2 &\in H^{-\mathcal{L}^*}. \end{aligned} \quad (7.11b)$$

In light of Assumption 3, if Eq. (7.9) holds the boundary variables are given by

$$\mathbf{u}_\partial = \mathcal{N}_2 \mathbf{e}_2, \quad \mathbf{y}_\partial = \mathcal{N}_1 \mathbf{e}_1, \quad \mathbf{u}_\partial, \mathbf{y}_\partial \in \mathbb{R}^m. \quad (7.12)$$

Otherwise, if Eq. (7.10) applies, then

$$\mathbf{u}_\partial = \mathcal{N}_1 \mathbf{e}_1, \quad \mathbf{y}_\partial = \mathcal{N}_2 \mathbf{e}_2, \quad \mathbf{u}_\partial, \mathbf{y}_\partial \in \mathbb{R}^m. \quad (7.13)$$

In both cases, the power balance reads

$$\begin{aligned} \dot{H} &= \langle \mathbf{e}_1, \partial_t \alpha_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathbf{e}_2, \partial_t \alpha_2 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= \langle \mathbf{e}_1, -\mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathbf{e}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= \langle \mathcal{N}_{\partial,1} \mathbf{e}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &= \langle \mathbf{y}_\partial, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \quad (7.14)$$

We are now in a position to illustrate the methodology.

**Step 1** First consider the weak form of system (7.11a), obtained by taking the  $L^2$  inner product introducing an appropriate test function  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{A} \times \mathbb{B} = \mathbb{F}$  and integrating over the domain  $\Omega$

$$\begin{aligned} \langle \mathbf{v}_1, \partial_t \alpha_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}, \\ \langle \mathbf{v}_2, \partial_t \alpha_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}. \end{aligned} \quad (7.15)$$

To obtain a closed system, the constitutive law (7.11b) and the output variables (7.1d) are put in weak form

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= \langle \mathbf{v}_1, \delta_{\alpha_1} H \rangle_{L^2(\Omega, \mathbb{A})}, \\ \langle \mathbf{v}_2, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \delta_{\alpha_2} H \rangle_{L^2(\Omega, \mathbb{B})}, \\ \langle \mathbf{v}_\partial, \mathbf{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathbf{v}_\partial, \mathcal{C}_\partial \mathbf{e} \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \end{aligned} \quad (7.16)$$

where the test function  $\mathbf{v}_\partial \in L^2(\partial\Omega, \mathbb{R}^m)$  is defined on the boundary  $\partial\Omega$  and  $\mathcal{C}_\partial$  is defined either by Eq. (7.9) or (7.10).

**Step 2** Next the integration by part has to be carried out. The choice is dictated by the boundary control to be imposed on the system. Consider again Eq. (7.15). The integration by parts can be carried out either on term  $-\langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}$ , or on term  $\langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}$ . Depending on which line undergoes the integration by parts (this is why the name Partitioned Finite Element method), two structure preserving weak forms are obtained. These differ by the boundary causality imposed to the system.

**Integration by parts of the term  $-\langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}$**  In this case case, using Eq. (7.8), it is obtained

$$-\langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} = -\langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \quad (7.17)$$

Then the weak form of the system dynamics reads

$$\begin{aligned} \langle \mathbf{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ \langle \mathbf{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \end{aligned} \quad (7.18)$$

The following proposition is crucial as the lossless character of the infinite-dimensional system (due to the formally skew-adjoint operator) translates into an equivalent property for the corresponding bilinear form in the weak form.

### Proposition 8

Given the Hilbert space  $H_2^\mathcal{L} := H^\mathcal{L} \times L^2(\Omega, \mathbb{B})$  and variables  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in H_2^\mathcal{L}$ ,  $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in H_2^\mathcal{L}$ , the bilinear form

$$\begin{aligned} j_\mathcal{L} : H_2^\mathcal{L} \times H_2^\mathcal{L} &\longrightarrow \mathbb{R}, \\ (\mathbf{v}, \mathbf{e}) &\longrightarrow -\langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} \end{aligned}$$

is skew-symmetric.

*Proof.* The proof is obtained by the following computation

$$\begin{aligned} j_\mathcal{L}(\mathbf{v}, \mathbf{e}) &= -\langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= -\left( -\langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} \right), \\ &= -\left( -\langle \mathcal{L} \mathbf{e}_1, \mathbf{v}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{e}_2, \mathcal{L} \mathbf{v}_1 \rangle_{L^2(\Omega, \mathbb{B})} \right) = -j_\mathcal{L}(\mathbf{e}, \mathbf{v}). \end{aligned}$$

□

Now assume that the system satisfies the boundary causality condition 7.12. Then, this

choice of the integration by parts lead to the following weak formulation

$$\begin{aligned}
\langle \mathbf{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
\langle \mathbf{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\
\langle \mathbf{v}_1, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= \langle \mathbf{v}_1, \delta_{\alpha_1} H \rangle_{L^2(\Omega, \mathbb{A})}, \\
\langle \mathbf{v}_2, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \delta_{\alpha_2} H \rangle_{L^2(\Omega, \mathbb{B})}, \\
\langle \mathbf{v}_\partial, \mathbf{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathbf{v}_\partial, \mathcal{N}_{\partial,1} \mathbf{e}_1 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}.
\end{aligned} \tag{7.19}$$

**Integration by parts of the term  $\langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}$**  Using Eq. (7.8), it is obtained

$$\langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} = \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathcal{N}_{\partial,1} \mathbf{e}_1 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \tag{7.20}$$

Then the weak form of the system dynamics reads

$$\begin{aligned}
\langle \mathbf{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}, \\
\langle \mathbf{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)},
\end{aligned} \tag{7.21}$$

Again the bilinear form arising from the formally skew-adjoint operator is skew-symmetric.

### Proposition 9

Given the Hilbert space  $H_1^{-\mathcal{L}^*} = L^2(\Omega, \mathbb{A}) \times H^{-\mathcal{L}^*}$  and variables  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in H_1^{-\mathcal{L}^*}$ ,  $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2) \in H_1^{-\mathcal{L}^*}$ , the bilinear form

$$\begin{aligned}
j_{-\mathcal{L}^*} : H_1^{-\mathcal{L}^*} \times H_1^{-\mathcal{L}^*} &\longrightarrow \mathbb{R}, \\
(\mathbf{v}, \mathbf{e}) &\longrightarrow -\langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})}
\end{aligned}$$

is skew-symmetric.

*Proof.* The proof follows from the computation

$$\begin{aligned}
j_{-\mathcal{L}^*}(\mathbf{v}, \mathbf{e}) &= -\langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})}, \\
&= -\left( -\langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} \right), \\
&= -\left( -\langle \mathbf{e}_1, \mathcal{L}^* \mathbf{v}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{L}^* \mathbf{e}_2, \mathbf{v}_1 \rangle_{L^2(\Omega, \mathbb{A})} \right) = -j_{-\mathcal{L}^*}(\mathbf{e}, \mathbf{v}).
\end{aligned}$$

□

Now assume that the system satisfies the boundary causality condition (7.13). Then, the

1169 final weak formulation reads

$$\begin{aligned}
\langle \mathbf{v}_1, \partial_t \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}, \\
\langle \mathbf{v}_2, \partial_t \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial, 2} \mathbf{v}_2, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
\langle \mathbf{v}_1, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= \langle \mathbf{v}_1, \delta_{\boldsymbol{\alpha}_1} H \rangle_{L^2(\Omega, \mathbb{A})}, \\
\langle \mathbf{v}_2, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \delta_{\boldsymbol{\alpha}_2} H \rangle_{L^2(\Omega, \mathbb{B})}, \\
\langle \mathbf{v}_\partial, \mathbf{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathbf{v}_\partial, \mathcal{N}_{\partial, 2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}.
\end{aligned} \tag{7.22}$$

1170 **Galerkin discretization** To conclude the illustration of this methodology, a Galerkin dis-  
 1171 cretization is introduced. This means that test, energy and co-energy functions are discretized  
 1172 using the same basis. Furthermore the boundary variables are discretized as well using bases  
 1173 defined over the boundary

$$\begin{aligned}
\mathbf{v}_1 &\approx \sum_{i=1}^{n_1} \phi_1^i(\mathbf{x}) v_1^i, & \boldsymbol{\alpha}_1 &\approx \sum_{i=1}^{n_1} \phi_1^i(\mathbf{x}) \alpha_1^i(t), & \mathbf{e}_1 &\approx \sum_{i=1}^{n_1} \phi_1^i(\mathbf{x}) e_1^i(t), & \mathbf{x} &\in \Omega, \\
\mathbf{v}_2 &\approx \sum_{i=1}^{n_2} \phi_2^i(\mathbf{x}) v_2^i, & \boldsymbol{\alpha}_2 &\approx \sum_{i=1}^{n_2} \phi_2^i(\mathbf{x}) \alpha_2^i(t), & \mathbf{e}_2 &\approx \sum_{i=1}^{n_2} \phi_2^i(\mathbf{x}) e_2^i(t), & \mathbf{x} &\in \Omega, \\
\mathbf{v}_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(\mathbf{s}) v_\partial^i, & \mathbf{u}_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(\mathbf{s}) u_\partial^i(t), & \mathbf{y}_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(\mathbf{s}) y_\partial^i(t), & \mathbf{s} &\in \partial\Omega,
\end{aligned} \tag{7.23}$$

1174 where  $\phi_1^i \in \mathbb{A}$ ,  $\phi_2^i \in \mathbb{B}$ ,  $\phi_\partial^i \in \mathbb{R}^m$ .

1175 **Discretization of the weak form (7.19)** Plugging the approximation into the weak  
 1176 form (7.19) and consider that the resulting equation holds  $\forall v_1^i, v_2^j, v_\partial^k$  ( $i \in \{1, n_1\}$ ,  $j \in$   
 1177  $\{1, n_2\}$ ,  $k \in \{1, n_\partial\}$ ), the finite dimensional system is obtained

$$\begin{aligned}
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_\mathcal{L}^\top \\ \mathbf{D}_0 + \mathbf{D}_\mathcal{L} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} &= \begin{bmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{bmatrix}, \\
\mathbf{M}_{\partial} \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.
\end{aligned} \tag{7.24}$$

1178 Vectors  $\boldsymbol{\alpha}_{d,1}$ ,  $\boldsymbol{\alpha}_{d,2}$ ,  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $\mathbf{u}_\partial$ ,  $\mathbf{y}_\partial$  are given by the column-wise concatenation of their respec-  
 1179 tive degrees of freedom. The matrices are defined as follows

$$\begin{aligned}
M_1^{ij} &= \langle \phi_1^i, \phi_1^j \rangle_{L^2(\Omega, \mathbb{A})}, & D_0^{mi} &= \langle \phi_2^m, \mathbf{L} \phi_1^i \rangle_{L^2(\Omega, \mathbb{B})}, & B_1^{ik} &= \langle \mathcal{N}_{\partial, 1} \phi_1^i, \phi_\partial^k \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\
M_2^{mn} &= \langle \phi_2^m, \phi_2^n \rangle_{L^2(\Omega, \mathbb{B})}, & D_\mathcal{L}^{mi} &= \langle \phi_2^m, \mathcal{L} \phi_1^i \rangle_{L^2(\Omega, \mathbb{B})}, & M_\partial^{lk} &= \langle \phi_\partial^l, \phi_\partial^k \rangle_{L^2(\partial\Omega, \mathbb{R}^m)},
\end{aligned} \tag{7.25}$$

where  $i, j \in \{1, n_1\}$ ,  $m, n \in \{1, n_2\}$ ,  $l, k \in \{1, n_\partial\}$ . Introducing the definitions

$$\begin{aligned}\delta_{\alpha_{d,1}} H_d &:= \delta_{\alpha_1} H \left( \alpha_1 = \sum_{i=1}^{n_1} \phi_1^i \alpha_1^i, \alpha_2 = \sum_{i=1}^{n_1} \phi_2^i \alpha_2^i \right), \\ \delta_{\alpha_{d,2}} H_d &:= \delta_{\alpha_2} H \left( \alpha_1 = \sum_{i=1}^{n_1} \phi_1^i \alpha_1^i, \alpha_2 = \sum_{i=1}^{n_1} \phi_2^i \alpha_2^i \right),\end{aligned}$$

the discretized gradient of the Hamiltonian read

$$\begin{aligned}\partial_{\alpha_{d,1}^i} H_d(\alpha_d) &= \left\langle \phi_1^i, \delta_{\alpha_{d,1}} H_d \right\rangle_{L^2(\Omega, \mathbb{A})}, \quad i \in \{1, n_1\}, \\ \partial_{\alpha_{d,2}^j} H_d(\alpha_d) &= \left\langle \phi_2^j, \delta_{\alpha_{d,2}} H_d \right\rangle_{L^2(\Omega, \mathbb{B})}, \quad j \in \{1, n_2\}.\end{aligned}\tag{7.26}$$

A pH system in canonical form is found observing that Sys. (7.24) is compactly rewritten as

$$\mathbf{M} \dot{\alpha}_d = \mathbf{J}_{\mathcal{L}} \mathbf{e} + \mathbf{B} \mathbf{u}_\partial, \tag{7.27}$$

$$\mathbf{M} \mathbf{e} = \nabla H_d(\alpha_d), \tag{7.28}$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \mathbf{B}^\top \mathbf{e}, \tag{7.29}$$

where  $\alpha_d = (\alpha_{d,1}^\top \ \alpha_{d,2}^\top)^\top$ ,  $\mathbf{e} = (\mathbf{e}_1^\top \ \mathbf{e}_2^\top)^\top$ ,  $\nabla H_d(\alpha_d) = (\partial_{\alpha_{d,1}}^\top H_d(\alpha_d) \ \partial_{\alpha_{d,2}}^\top H_d(\alpha_d))^\top$  and

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix}, \quad \mathbf{J}_{\mathcal{L}} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}. \tag{7.30}$$

Plugging (7.28) into (7.27), a pH system in canonical form is obtained

$$\begin{aligned}\dot{\alpha}_d &= \mathbf{J} \nabla H_d(\alpha_d) + \mathbf{B} \mathbf{u}_\partial, & \text{where} \quad \mathbf{J} &= \mathbf{M}^{-1} \mathbf{J}_{\mathcal{L}} \mathbf{M}^{-1}, \\ \hat{\mathbf{y}}_\partial &= \mathbf{B}^\top \nabla H_d(\alpha_d), & \text{where} \quad \hat{\mathbf{y}}_\partial &= \mathbf{M}_\partial \mathbf{y}_\partial.\end{aligned}\tag{7.31}$$

The structure preserving character of the method is evident from the preservation at the discrete level of the power balance. The finite dimensional counterpart of the energy rate is given by

$$\begin{aligned}\dot{H}_d &= \nabla^\top H_d(\alpha_d) \dot{\alpha}_d, \\ &= \nabla^\top H_d(\alpha_d) \mathbf{J} \nabla H_d(\alpha_d) + \nabla^\top H_d(\alpha_d) \mathbf{B} \mathbf{u}_\partial, & \text{Skew-symmetry of } \mathbf{J} \\ &= \hat{\mathbf{y}}_\partial^\top \mathbf{u}_\partial.\end{aligned}\tag{7.32}$$

This result mimics its infinite dimensional equivalent (7.14).

**Discretization of the weak form (7.22)** Plugging the approximation into the weak form (7.22) a finite dimensional system with a different causality is obtained

$$\begin{aligned}
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top + \mathbf{D}_{-\mathcal{L}^*} \\ \mathbf{D}_0 - \mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial, \\
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} &= \begin{pmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{pmatrix}, \\
\mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.
\end{aligned} \tag{7.33}$$

1189 The differences with respect to formulation (7.24) reside in matrices  $\mathbf{D}_{-\mathcal{L}^*}$ ,  $\mathbf{B}_2$ , whose defi-  
 1190 nitions are

$$D_{-\mathcal{L}^*}^{im} = \langle \phi_1^i, -\mathcal{L}^* \phi_2^m \rangle_{L^2(\Omega, \mathbb{A})}, \quad B_2^{mk} = \langle \mathcal{N}_{\partial,2} \phi_2^m, \phi_\partial^k \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \tag{7.34}$$

1191 where  $i \in \{1, n_1\}$ ,  $m \in \{1, n_2\}$ ,  $k \in \{1, n_\partial\}$ . System (7.33) can be put in canonical form by  
 1192 replacing the co-energy variables by the discretized gradient.

1193 **Example: the irrotational shallow water equations** Consider as an example the  
 1194 shallow water equations detailed in Sec. §3.2.3.3. The flow is assumed to be irrotational  
 1195 ( $\nabla \times \mathbf{v} = 0$ ). As a consequence the term  $\mathcal{G}$  in Eq. (3.36) vanishes. To fulfill Assumption 3,  
 1196 the incoming volumetric flow is known at the boundary, so that a uniform Neumann condition  
 1197 is imposed. This lead to the following boundary control system, defined on an open connected  
 1198 set  $\Omega \subset \mathbb{R}^2$

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} &= - \begin{bmatrix} 0 & \text{div} \\ \text{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_h \\ \mathbf{e}_v \end{pmatrix}, & \alpha_h &\in L^2(\Omega), \\
& & \boldsymbol{\alpha}_v &\in L^2(\Omega, \mathbb{R}^2), \\
\begin{pmatrix} e_h \\ \mathbf{e}_v \end{pmatrix} &:= \begin{pmatrix} \delta_{\alpha_h} H \\ \delta_{\boldsymbol{\alpha}_v} H \end{pmatrix} = \begin{pmatrix} \frac{1}{2\rho} \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h \\ \frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v \end{pmatrix}, & e_h &\in H^1(\Omega), \\
& & \mathbf{e}_v &\in H^{\text{div}}(\Omega, \mathbb{R}^2), \\
u_\partial &= -\mathbf{e}_v \cdot \mathbf{n}, & u_\partial &\in \mathbb{R}, \\
y_\partial &= e_h, & y_\partial &\in \mathbb{R},
\end{aligned} \tag{7.35}$$

where the Hamiltonian is a non linear functional in the energy variables

$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho} \alpha_h \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h^2 \right\} d\Omega.$$

1199 The energy and co-energy variables are related to the physical variables (fluid height and  
 1200 velocity) through Eqs. (3.33), (3.35). In this case  $\mathbb{A} = \mathbb{R}$ ,  $\mathbb{B} = \mathbb{R}^2$  and  $\mathcal{L} = \text{grad}$ ,  $-\mathcal{L}^* = \text{div}$ .  
 1201 This implies  $H^\mathcal{L} = H^1(\Omega)$ ,  $H^{-\mathcal{L}^*} = H^{\text{div}}(\Omega, \mathbb{R}^2)$ . As shown in (3.37), the energy rate equals

$$\dot{H} = -\langle \mathbf{e}_v, \text{grad } e_h \rangle_{L^2(\Omega, \mathbb{R}^2)} - \langle \text{div } \mathbf{e}_v, e_h \rangle_{L^2(\Omega)} = \langle -\mathbf{e}_v \cdot \mathbf{n}, e_h \rangle_{L^2(\partial\Omega)}. \tag{7.36}$$

1202 The boundary operators are therefore given by

$$\begin{aligned}
u_\partial &= \mathcal{N}_{\partial,2} \mathbf{e}_v = -\gamma_n \mathbf{e}_v = -\mathbf{e}_v \cdot \mathbf{n}|_{\partial\Omega}, \\
y_\partial &= \mathcal{N}_{\partial,1} e_h = \gamma_0 e_h = e_h|_{\partial\Omega}.
\end{aligned} \tag{7.37}$$



1203 This system represents a particular example of the general formulation of the general frame-  
 1204 work (7.11), together with boundary conditions (7.12). To obtain a finite dimensional system,  
 1205 the test variables  $v_h$ ,  $\mathbf{v}_v$  are introduced and the integration by parts is performed on the div  
 1206 operator, leading to the weak form

$$\begin{aligned}
 \langle v_h, \partial_t \alpha_h \rangle_{L^2(\Omega)} &= \langle \text{grad } v_h, \mathbf{e}_v \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \gamma_0 v_h, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega)}, \\
 \langle \mathbf{v}_v, \partial_t \boldsymbol{\alpha}_v \rangle_{L^2(\Omega, \mathbb{R}^2)} &= - \langle \mathbf{v}_v, \text{grad } e_h \rangle_{L^2(\Omega, \mathbb{R}^2)}, \\
 \langle v_h, e_h \rangle_{L^2(\Omega)} &= \left\langle v_h, \frac{1}{2\rho} \|\boldsymbol{\alpha}_v\|^2 + \rho g \alpha_h \right\rangle_{L^2(\Omega)}, \\
 \langle \mathbf{v}_v, \mathbf{e}_v \rangle_{L^2(\Omega, \mathbb{R}^2)} &= \left\langle \mathbf{v}_v, \frac{1}{\rho} \boldsymbol{\alpha}_h \boldsymbol{\alpha}_v \right\rangle_{L^2(\Omega, \mathbb{R}^2)}, \\
 \langle v_\partial, y_\partial \rangle_{L^2(\partial\Omega)} &= \langle v_\partial, \gamma_0 e_h \rangle_{L^2(\partial\Omega)}.
 \end{aligned} \tag{7.38}$$

1207 Introducing a Galerkin approximation as in (7.23)

$$\begin{aligned}
 v_h &\approx \sum_{i=1}^{n_h} \phi_h^i(\mathbf{x}) v_h^i, & \alpha_h &\approx \sum_{i=1}^{n_h} \phi_h^i(\mathbf{x}) \alpha_h^i(t), & e_h &\approx \sum_{i=1}^{n_h} \phi_h^i(\mathbf{x}) e_h^i(t), & \mathbf{x} \in \Omega, \\
 \mathbf{v}_v &\approx \sum_{i=1}^{n_v} \phi_v^i(\mathbf{x}) \mathbf{v}_v^i, & \boldsymbol{\alpha}_v &\approx \sum_{i=1}^{n_v} \phi_v^i(\mathbf{x}) \boldsymbol{\alpha}_v^i(t), & \mathbf{e}_v &\approx \sum_{i=1}^{n_v} \phi_v^i(\mathbf{x}) \mathbf{e}_v^i(t), & \mathbf{x} \in \Omega, \\
 v_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(s) v_\partial^i, & u_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(s) u_\partial^i(t), & y_\partial &\approx \sum_{i=1}^{n_\partial} \phi_\partial^i(s) y_\partial^i(t), & s \in \partial\Omega,
 \end{aligned} \tag{7.39}$$

1208 the finite dimensional system is obtained

$$\begin{aligned}
 \begin{bmatrix} \mathbf{M}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,h} \\ \dot{\boldsymbol{\alpha}}_{d,v} \end{pmatrix} &= - \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\text{grad}}^\top \\ \mathbf{D}_{\text{grad}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_h \\ \mathbf{e}_v \end{pmatrix} + \begin{bmatrix} \mathbf{B}_h \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\
 \begin{bmatrix} \mathbf{M}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v \end{bmatrix} \begin{pmatrix} \mathbf{e}_h \\ \mathbf{e}_v \end{pmatrix} &= \begin{bmatrix} \partial_{\boldsymbol{\alpha}_{d,h}} H_d(\boldsymbol{\alpha}_{d,h}, \boldsymbol{\alpha}_{d,v}) \\ \partial_{\boldsymbol{\alpha}_{d,v}} H_d(\boldsymbol{\alpha}_{d,h}, \boldsymbol{\alpha}_{d,v}) \end{bmatrix}, \\
 \mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_h^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_h \\ \mathbf{e}_v \end{pmatrix}.
 \end{aligned} \tag{7.40}$$

1209 The matrices are defined as follows

$$\begin{aligned}
 M_h^{ij} &= \langle \phi_h^i, \phi_h^j \rangle_{L^2(\Omega)}, & D_{\text{grad}}^{mi} &= \langle \phi_v^m, \text{grad } \phi_h^i \rangle_{L^2(\Omega, \mathbb{R}^2)}, \\
 M_v^{mn} &= \langle \phi_v^m, \phi_v^n \rangle_{L^2(\Omega, \mathbb{R}^2)}, & B_h^{ik} &= \langle \gamma_0 \phi_h^i, \phi_\partial^k \rangle_{L^2(\partial\Omega)}, \\
 M_\partial^{lk} &= \langle \phi_\partial^l, \phi_\partial^k \rangle_{L^2(\partial\Omega)},
 \end{aligned} \tag{7.41}$$

where  $i, j \in \{1, n_h\}$ ,  $m, n \in \{1, n_v\}$ ,  $l, k \in \{1, n_\partial\}$ . The discretized gradient of the Hamiltonian read

$$\begin{aligned} \partial_{\alpha_{d,h}^i} H_d(\alpha_{d,h}, \alpha_{d,v}) &= \left\langle \phi_h^i, \frac{1}{2\rho} \left\| \sum_{r=1}^{n_2} \phi_v^r \alpha_v^r \right\|^2 + \rho g \sum_{r=1}^{n_1} \phi_h^r \alpha_h^r \right\rangle_{L^2(\Omega)}, \quad i \in \{1, n_h\}, \\ \partial_{\alpha_{d,v}^m} H_d(\alpha_{d,h}, \alpha_{d,v}) &= \left\langle \phi_v^m, \frac{1}{\rho} \left( \sum_{r=1}^{n_1} \phi_h^r \alpha_h^r \right) \left( \sum_{r=1}^{n_2} \phi_v^r \alpha_v^r \right) \right\rangle_{L^2(\Omega, \mathbb{R}^2)}, \quad m \in \{1, n_v\}. \end{aligned} \quad (7.42)$$

One possible finite element discretization for this problem can be found in [Pir89]. The non linear nature of the problem strongly complicates the analysis. The presence of shocks has to be accounted for in the numerical discretization. The proposed methodology has to cope with finite time shocks to become a valid alternative to already well established strategies.

### 7.1.2 Linear case

The general framework detailed in Sec. 7.1.1 is valid for both linear and non linear system. However, in the linear case a major simplification occurs since the constitutive law connecting energy and co-energy variables is easily invertible. This allows a description based on co-energy variables only.

To make the system linear, the additional assumption is introduced.

#### Assumption 4

The Hamiltonian is assumed to be a positive quadratic functional in the energy variables  $\alpha_1, \alpha_2$ . Furthermore, the Hamiltonian is considered to be separable with respect to  $\alpha_1, \alpha_2$  (this hypothesis is always met for the systems under consideration). Therefore, it can be expressed as

$$H = \frac{1}{2} \langle \alpha_1, \mathcal{Q}_1 \alpha_1 \rangle_{L^2(\Omega, \mathbb{A})} + \frac{1}{2} \langle \alpha_2, \mathcal{Q}_2 \alpha_2 \rangle_{L^2(\Omega, \mathbb{B})}, \quad (7.43)$$

where  $\mathcal{Q}_1, \mathcal{Q}_2$  are positive symmetric operators, bounded from below and above

$$m_1 \mathbf{I}_{\mathbb{A}} \leq \mathcal{Q}_1 \leq M_1 \mathbf{I}_{\mathbb{A}}, \quad m_2 \mathbf{I}_{\mathbb{B}} \leq \mathcal{Q}_2 \leq M_2 \mathbf{I}_{\mathbb{B}}, \quad m_1 > 0, m_2 > 0, M_1 > 0, M_2 > 0,$$

where  $\mathbf{I}_{\mathbb{A}}, \mathbf{I}_{\mathbb{B}}$  are the identity operator in  $\mathbb{A}, \mathbb{B}$  respectively. Because of Assumption 4, the co-energy variables are given by

$$e_1 := \delta_{\alpha_1} H = \mathcal{Q}_1 \alpha_1, \quad e_2 := \delta_{\alpha_2} H = \mathcal{Q}_2 \alpha_2 \quad (7.44)$$

Since  $\mathcal{Q}_1, \mathcal{Q}_2$  are positive bounded from below and above, it is possible to invert them to obtain

$$\alpha_1 = \mathcal{Q}_1^{-1} e_1 = \mathcal{M}_1 e_1, \quad \alpha_2 = \mathcal{Q}_2^{-1} e_2 = \mathcal{M}_2 e_2, \quad \mathcal{M}_1 := \mathcal{Q}_1^{-1}, \mathcal{M}_2 := \mathcal{Q}_2^{-1}. \quad (7.45)$$

1232 The Hamiltonian is then written in terms of co-energy variables as

$$H = \frac{1}{2} \langle \mathbf{e}_1, \mathcal{M}_1 \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \frac{1}{2} \langle \mathbf{e}_2, \mathcal{M}_2 \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})}. \quad (7.46)$$

1233 Under assumptions 2, 3, 4, a pH linear system is expressed as

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{L}^\top - \mathcal{L}^* \\ \mathbf{L} + \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{matrix} \mathbf{e}_1 \in H^\mathcal{L}, \\ \mathbf{e}_2 \in H^{-\mathcal{L}^*}. \end{matrix} \quad (7.47)$$

1234 If Eq. (7.9) holds the boundary variables equal

$$\mathbf{u}_\partial = \mathcal{N}_2 \mathbf{e}_2, \quad \mathbf{y}_\partial = \mathcal{N}_1 \mathbf{e}_1, \quad \mathbf{u}_\partial, \mathbf{y}_\partial \in \mathbb{R}^m. \quad (7.48)$$

1235 Whereas if Eq. (7.10) holds, then

$$\mathbf{u}_\partial = \mathcal{N}_1 \mathbf{e}_1, \quad \mathbf{y}_\partial = \mathcal{N}_2 \mathbf{e}_2, \quad \mathbf{u}_\partial, \mathbf{y}_\partial \in \mathbb{R}^m. \quad (7.49)$$

1236 From equation (7.46), the power balance reads

$$\begin{aligned} \dot{H} &= \langle \mathbf{e}_1, \mathcal{M}_1 \partial_t \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathbf{e}_2, \mathcal{M}_2 \partial_t \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= \langle \mathbf{e}_1, -\mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathbf{e}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\ &= \langle \mathcal{N}_{\partial,1} \mathbf{e}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ &= \langle \mathbf{y}_\partial, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \quad (7.50)$$

1237 To get a finite dimensional approximation the same procedure detailed in Sec. §7.1.1 is  
1238 followed. The only difference is that there is no need to discretize the constitutive relations  
1239 as those are already incorporated in the dynamics.

1240 Once the system is put into weak form, if the operator  $-\mathcal{L}^*$  is integrated by parts, one  
1241 obtains the weak form

$$\begin{aligned} \langle \mathbf{v}_1, \mathcal{M}_1 \partial_t \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ \langle \mathbf{v}_2, \mathcal{M}_2 \partial_t \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}, \\ \langle \mathbf{v}_\partial, \mathbf{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathbf{v}_\partial, \mathcal{N}_{\partial,1} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \quad (7.51)$$

1242 Otherwise, if operator  $\mathcal{L}$  is integrated by parts, it is computed

$$\begin{aligned} \langle \mathbf{v}_1, \mathcal{M}_1 \partial_t \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})}, \\ \langle \mathbf{v}_2, \mathcal{M}_2 \partial_t \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ \langle \mathbf{v}_\partial, \mathbf{y}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathbf{v}_\partial, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}. \end{aligned} \quad (7.52)$$

1243 After introducing a Galerkin approximation as in (7.23), the discretized version of the weak  
1244 form (7.51) reads

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\ \mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \end{aligned} \quad (7.53)$$

1245 The only difference with respect to Eq. (7.24) concerns the mass matrices

$$M_{\mathcal{M}_1}^{ij} = \langle \phi_1^i, \mathcal{M}_1 \phi_1^j \rangle_{L^2(\Omega, \mathbb{A})}, \quad M_{\mathcal{M}_2}^{mn} = \langle \phi_2^m, \mathcal{M}_2 \phi_2^n \rangle_{L^2(\Omega, \mathbb{B})} \quad i, j \in \{1, n_1\}, \quad m, n \in \{1, n_2\}. \quad (7.54)$$

1246 If the Galerkin approximation is applied to the weak form (7.52), it is obtained

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top + \mathbf{D}_{-\mathcal{L}^*}^\top \\ \mathbf{D}_0 - \mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial, \\ \mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \end{aligned} \quad (7.55)$$

1247 In both cases, it is easy to verify that the Hamiltonian

$$H_d = \frac{1}{2} \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \mathbf{e}_2, \quad (7.56)$$

1248 once differentiated in time, provides the energy rate

$$\dot{H}_d = \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial = \hat{\mathbf{y}}_\partial^\top \mathbf{u}_\partial, \quad \text{where} \quad \hat{\mathbf{y}}_\partial := \mathbf{M}_\partial \mathbf{y}_\partial. \quad (7.57)$$

1249 This result mimics its finite dimensional counterpart (7.50).

### 1250 7.1.3 Linear flexible structures

1251 In this section, some linear example from the elasticity realms are considered. We restrict  
1252 the discussion to linear problems. This case is anyway significant, as these examples are  
1253 frequently encountered in engineering applications.

#### 1254 7.1.3.1 Euler-Bernoulli beam

1255 We reconsider the example discussed in Sec. §3.2.3.2. The relation between energy and  
1256 co-energy variables is given by Eqs. (3.24), (3.26)

$$\alpha_w = \rho A e_w, \quad \alpha_\kappa = \frac{1}{EI} e_\kappa \quad (7.58)$$

1257 The coefficients  $\rho, A, E$  and  $I$  are the mass density, the cross section area, Young's modulus  
1258 of elasticity and the moment of inertia of the cross section.

1259 **Control through forces and torques** Given an interval  $\Omega = (0, L)$ , a thin beam under  
 1260 free boundary condition (forces and torques imposed at the boundary) can be modeled in  
 1261 terms of co-energy variables by the following system

$$\begin{bmatrix} \rho A & 0 \\ 0 & (EI)^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \quad \begin{matrix} e_w \in H^2(\Omega), \\ e_\kappa \in H^2(\Omega), \end{matrix} \quad (7.59a)$$

$$\mathbf{u}_\partial = \begin{bmatrix} 0 & \gamma_0 \\ 0 & -\gamma_1 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \quad \mathbf{u}_\partial \in \mathbb{R}^4, \quad (7.59b)$$

$$\mathbf{y}_\partial = \begin{bmatrix} \gamma_1 & 0 \\ \gamma_0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \quad \mathbf{y}_\partial \in \mathbb{R}^4. \quad (7.59c)$$

1262 The boundary operator  $\gamma_0, \gamma_1$  denote the trace and the first derivative trace along the bound-  
 1263 ary. In a one-dimensional domain the boundary degenerates to two single points

$$\gamma_0 a = a|_{\partial\Omega} = \begin{pmatrix} -a(0) \\ +a(L) \end{pmatrix}, \quad \gamma_1 a = \partial_x a|_{\partial\Omega} = \begin{pmatrix} -\partial_x a(0) \\ +\partial_x a(L) \end{pmatrix}. \quad (7.60)$$

1264 In this case  $\mathbb{A} = \mathbb{B} = \mathbb{R}$ . The operators  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{L}, N_{\partial,1}, N_{\partial,2}$  read

$$\mathcal{M}_1 = \rho A, \quad \mathcal{M}_2 = (EI)^{-1}, \quad \mathcal{L} = \partial_{xx}, \quad N_{\partial,1} = \begin{bmatrix} \gamma_1 \\ \gamma_0 \end{bmatrix}, \quad N_{\partial,2} = \begin{bmatrix} \gamma_0 \\ -\gamma_1 \end{bmatrix}. \quad (7.61)$$

1265 The Hamiltonian is given by

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho A e_w^2 + (EI)^{-1} e_\kappa^2 \right\} d\Omega. \quad (7.62)$$

1266 Applying twice the integration by parts formula, one obtains the power balance

$$\begin{aligned} \dot{H} &= \langle e_w, \rho A \partial_t e_w \rangle_{L^2(\Omega)} + \langle e_\kappa, (EI)^{-1} \partial_t e_\kappa \rangle_{L^2(\Omega)}, \\ &= \langle e_w, -\partial_{xx} e_\kappa \rangle_{L^2(\Omega)} + \langle e_\kappa, \partial_{xx} e_w \rangle_{L^2(\Omega)}, \\ &= \langle \gamma_1 e_w, \gamma_0 e_\kappa \rangle_{\mathbb{R}^2} + \langle \gamma_0 e_w, -\gamma_1 e_\kappa \rangle_{\mathbb{R}^2}, \\ &= \langle \mathbf{y}_\partial, \mathbf{u}_\partial \rangle_{\mathbb{R}^4}. \end{aligned} \quad (7.63)$$

1267 Given the test functions  $v_w, v_\kappa$ , the weak form is readily obtained as

$$\begin{aligned} \langle v_w, \rho A \partial_t e_w \rangle_{L^2(\Omega)} &= \langle v_w, -\partial_{xx} e_\kappa \rangle_{L^2(\Omega)}, \\ \langle v_\kappa, (EI)^{-1} \partial_t e_\kappa \rangle_{L^2(\Omega)} &= \langle v_\kappa, \partial_{xx} e_w \rangle_{L^2(\Omega)}. \end{aligned} \quad (7.64)$$

1268 If the integration by parts is applied twice to the first line of Eq. (7.59a), it is obtained

$$\begin{aligned} \langle v_w, \rho A \partial_t e_w \rangle_{L^2(\Omega)} &= -\langle \partial_{xx} v_w, e_\kappa \rangle_{L^2(\Omega)} + \langle \gamma_1 v_w, (u_{\partial,1}, u_{\partial,2}) \rangle_{\mathbb{R}^2} + \langle \gamma_0 v_w, (u_{\partial,3}, u_{\partial,4}) \rangle_{\mathbb{R}^2}, \\ \langle v_\kappa, (EI)^{-1} \partial_t e_\kappa \rangle_{L^2(\Omega)} &= \langle v_\kappa, \partial_{xx} e_w \rangle_{L^2(\Omega)}. \end{aligned} \quad (7.65)$$

1269 Introducing a Galerkin discretization for test and efforts functions

$$v_w = \sum_{i=1}^{n_w} \phi_w^i v_w^i, \quad e_w = \sum_{i=1}^{n_w} \phi_w^i e_w^i(t), \quad v_\kappa = \sum_{i=1}^{n_\kappa} \phi_\kappa^i v_\kappa^i, \quad e_\kappa = \sum_{i=1}^{n_\kappa} \phi_\kappa^i e_\kappa^i(t), \quad (7.66)$$

1270 and considering that  $\mathbf{u}_\partial \in \mathbb{R}^4$ ,  $\mathbf{y}_\partial \in \mathbb{R}^4$ , the following is obtained

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{\rho A} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{EI^{-1}} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\kappa \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\partial xx}^\top \\ \mathbf{D}_{\partial xx} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix} + \begin{bmatrix} \mathbf{B}_w \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\ \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_w^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix}. \end{aligned} \quad (7.67)$$

1271 The matrices  $\mathbf{M}_{\rho A}$ ,  $\mathbf{M}_{EI^{-1}}$ ,  $\mathbf{D}_{\partial xx}$  are defined as ( $i, j \in \{1, n_w\}$ ,  $m, n \in \{1, n_\kappa\}$ )

$$M_{\rho A}^{ij} = \langle \phi_w^i, \rho A \phi_w^j \rangle_{L^2(\Omega)}, \quad M_{EI^{-1}}^{mn} = \langle \phi_\kappa^m, (EI)^{-1} \phi_\kappa^n \rangle_{L^2(\Omega)}, \quad D_{\partial xx}^{mi} = \langle \phi_\kappa^m, \partial_{xx} \phi_w^i \rangle_{L^2(\Omega)}. \quad (7.68)$$

1272 The  $\mathbf{B}_w$  is composed of four column vectors  $\mathbf{B}_w = [\mathbf{b}_w^1 \ \mathbf{b}_w^2 \ \mathbf{b}_w^3 \ \mathbf{b}_w^4]$

$$b_w^{1,i} = -\partial_x \phi_w^i(0), \quad b_w^{2,i} = \partial_x \phi_w^i(L), \quad b_w^{3,i} = -\phi_w^i(0), \quad b_w^{4,i} = \phi_w^i(L), \quad i \in \{1, n_w\}. \quad (7.69)$$

**Control through linear and angular velocities** Equivalently, the second line of Eq. (7.59a) could have been integrated by parts to control through the linear and angular velocities at the extremities. Consider the system with known forces and torques at the extremities

$$\begin{bmatrix} \rho A & 0 \\ 0 & (EI)^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \quad \begin{aligned} e_w &\in H^2(\Omega), \\ e_\kappa &\in H^2(\Omega), \end{aligned} \quad (7.70a)$$

$$\mathbf{u}_\partial = \begin{bmatrix} \gamma_1 & 0 \\ \gamma_0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \quad \mathbf{u}_\partial \in \mathbb{R}^4, \quad (7.70b)$$

$$\mathbf{y}_\partial = \begin{bmatrix} 0 & \gamma_0 \\ 0 & -\gamma_1 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}, \quad \mathbf{y}_\partial \in \mathbb{R}^4. \quad (7.70c)$$

1273 Once the system is put into weak form and the second line of Eq. (7.70a) is integrated twice,  
1274 it is computed

$$\begin{aligned} \langle v_w, \rho A \partial_t e_w \rangle_{L^2(\Omega)} &= \langle v_w, -\partial_{xx} e_\kappa \rangle_{L^2(\Omega)}, \\ \langle v_\kappa, (EI)^{-1} \partial_t e_\kappa \rangle_{L^2(\Omega)} &= \langle \partial_{xx} v_\kappa, e_w \rangle_{L^2(\Omega)} + \langle \gamma_0 v_\kappa, (u_{\partial,1}, u_{\partial,2}) \rangle_{\mathbb{R}^2} + \langle -\gamma_1 v_\kappa, (u_{\partial,3}, u_{\partial,4}) \rangle_{\mathbb{R}^2}. \end{aligned} \quad (7.71)$$

1275 Replacing a Galerkin approximation, it is obtained

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{\rho A} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{EI^{-1}} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\kappa \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\partial_{xx}} \\ -\mathbf{D}_{-\partial_{xx}}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_\kappa \end{bmatrix} \mathbf{u}_\partial, \\ \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{0} & \mathbf{B}_\kappa^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix}. \end{aligned} \quad (7.72)$$

1276 The matrix  $\mathbf{D}_{-\partial_{xx}}$  is defined as

$$D_{-\partial_{xx}}^{im} = \left\langle \phi_w^i, -\partial_{xx}\phi_\kappa^m \right\rangle_{L^2(\Omega)}, \quad i, \in \{1, n_w\}, \quad m \in \{1, n_\kappa\}. \quad (7.73)$$

1277 The  $\mathbf{B}_\kappa$  is composed of four column vectors  $\mathbf{B}_\kappa = [\mathbf{b}_\kappa^1 \mathbf{b}_\kappa^2 \mathbf{b}_\kappa^3 \mathbf{b}_\kappa^4]$

$$b_\kappa^{1,m} = -\phi_\kappa^m(0), \quad b_\kappa^{2,m} = \phi_\kappa^m(L), \quad b_\kappa^{3,m} = \partial_x \phi_\kappa^m(0), \quad b_\kappa^{4,m} = -\partial_x \phi_\kappa^m(L), \quad m \in \{1, n_\kappa\}. \quad (7.74)$$

1278 Both discretization require the use of Hermite polynomials to meet the regularity require-  
1279 ment. Indeed, to lower the regularity requirement for the finite elements employed in the  
1280 discretization, both lines can be integrated by parts. This will be discussed in Chap. 8.

### 1281 7.1.3.2 Kirchhoff plate

1282 The link between the energy and co-energy variables for the isotropic Kirchhoff model is the  
1283 following (5.33)

$$\alpha_w = \rho h e_w, \quad \mathbf{A}_\kappa = \mathbf{C}_b \mathbf{E}_\kappa, \quad \text{where} \quad \mathbf{C}_b := \mathbf{D}_b^{-1} \quad (7.75)$$

1284 where  $\rho$  is the mass density,  $h$  the plate thickness and  $\mathbf{D}_b$ , the bending rigidity tensor, cf. Eq.  
1285 (5.11). The bending compliance is given by

$$\mathbf{C}_b = \frac{12}{Eh^3} [(1 + \nu)(\cdot) - \nu \text{Tr}(\cdot) \mathbf{I}_{2 \times 2}]. \quad (7.76)$$

Given an open connected set  $\Omega \subset \mathbb{R}^2$ , the Kirchhoff plate model (5.42) in co-energy form controlled by forces and momenta is then expressed as

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbf{C}_b \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\text{div Div} \\ \text{Hess} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix}, \quad \begin{aligned} e_w &\in H^2(\Omega), \\ \mathbf{E}_\kappa &\in H^{\text{div Div}}(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2}), \end{aligned} \quad (7.77a)$$

$$\mathbf{u}_\partial = \begin{bmatrix} 0 & \gamma_{nn,1} \\ 0 & \gamma_{nn} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix}, \quad \mathbf{u}_\partial \in \mathbb{R}^2, \quad (7.77b)$$

$$\mathbf{y}_\partial = \begin{bmatrix} \gamma_0 & 0 \\ \gamma_1 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{E}_\kappa \end{pmatrix}, \quad \mathbf{y}_\partial \in \mathbb{R}^2, \quad (7.77c)$$

1286 We recall the expressions of the trace maps

$$\begin{aligned}\gamma_0 a &= a|_{\partial\Omega}, & \gamma_{nn,1} \mathbf{A} &= -\mathbf{n} \cdot \text{Div } \mathbf{A} - \partial_s(\mathbf{A} : (\mathbf{n} \otimes \mathbf{s}))|_{\partial\Omega}, \\ \gamma_1 a &= \partial_{\mathbf{n}} a|_{\partial\Omega}, & \gamma_{nn} \mathbf{A} &= \mathbf{A} : (\mathbf{n} \otimes \mathbf{n})|_{\partial\Omega}.\end{aligned}\quad (7.78)$$

1287 In this case, the sets are  $\mathbb{A} = \mathbb{R}$ ,  $\mathbb{B} = \mathbb{R}_{\text{sym}}^{2 \times 2}$ . The operators  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{L}, N_{\partial,1}, N_{\partial,2}$  are

$$\mathcal{M}_1 = \rho h, \quad \mathcal{M}_2 = \mathbf{C}_b, \quad \mathcal{L} = \text{Hess}, \quad N_{\partial,1} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}, \quad N_{\partial,2} = \begin{bmatrix} \gamma_{nn,1} \\ \gamma_{nn} \end{bmatrix}. \quad (7.79)$$

1288 The energy rate from Eq. (5.39) equals  $\dot{H} = \langle \mathbf{y}_{\partial}, \mathbf{u}_{\partial} \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$ . Introducing the test  
1289 functions  $(v_w, \mathbf{V}_{\kappa})$  and integrating by parts twice the first line of (7.77a) one gets

$$\begin{aligned}\langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} &= -\langle \text{Hess } v_w, \mathbf{E}_{\kappa} \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})} + \langle \gamma_0 v_w, u_{\partial,1} \rangle_{L^2(\partial\Omega)} + \langle \gamma_1 v_w, u_{\partial,2} \rangle_{L^2(\partial\Omega)}, \\ \langle \mathbf{V}_{\kappa}, \mathbf{C}_b \partial_t \mathbf{V}_{\kappa} \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})} &= \langle \mathbf{V}_{\kappa}, \text{Hess } e_w \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})}.\end{aligned}\quad (7.80)$$

1290 Introducing a Galerkin discretization for test and efforts functions

$$\begin{aligned}v_w &= \sum_{i=1}^{n_w} \phi_w^i v_w^i, & \mathbf{V}_{\kappa} &= \sum_{i=1}^{n_{\kappa}} \Phi_{\kappa}^i v_{\kappa}^i, & v_{\partial} &= \sum_{i=1}^{n_{\partial}} \phi_{\partial}^i v_{\partial}^i, & \mathbf{y}_{\partial} &= \sum_{i=1}^{n_{\partial}} \phi_{\partial}^i y_{\partial}^i. \\ e_w &= \sum_{i=1}^{n_w} \phi_w^i e_w^i, & \mathbf{E}_{\kappa} &= \sum_{i=1}^{n_{\kappa}} \Phi_{\kappa}^i e_{\kappa}^i, & \mathbf{u}_{\partial} &= \sum_{i=1}^{n_{\partial}} \phi_{\partial}^i u_{\partial}^i,\end{aligned}\quad (7.81)$$

1291 the following finite dimensional system is obtained

$$\begin{aligned}\begin{bmatrix} \mathbf{M}_{\rho h} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathbf{C}_b} \end{bmatrix} \begin{pmatrix} \dot{e}_w \\ \dot{e}_{\kappa} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\text{Hess}}^{\top} \\ \mathbf{D}_{\text{Hess}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ e_{\kappa} \end{pmatrix} + \begin{bmatrix} \mathbf{B}_w & \mathbf{B}_{\partial_n w} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{u}_{\partial}, \\ \mathbf{M}_{\partial} \mathbf{y}_{\partial} &= \begin{bmatrix} \mathbf{B}_w^{\top} & \mathbf{0} \\ \mathbf{B}_{\partial_n w}^{\top} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ e_{\kappa} \end{pmatrix}.\end{aligned}\quad (7.82)$$

1292 The matrices  $\mathbf{M}_{\rho h}, \mathbf{M}_{\mathbf{C}_b}, \mathbf{D}_{\text{Hess}}$  are defined as  $(i, j \in \{1, n_w\}, m, n \in \{1, n_{\kappa}\})$

$$M_{\rho h}^{ij} = \langle \phi_w^i, \rho h \phi_w^j \rangle_{L^2(\Omega)}, \quad M_{\mathbf{C}_b}^{mn} = \langle \Phi_{\kappa}^m, \mathbf{C}_b \Phi_{\kappa}^n \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})}, \quad D_{\text{Hess}}^{mi} = \langle \Phi_{\kappa}^m, \text{Hess } \phi_w^i \rangle_{L^2(\Omega)}. \quad (7.83)$$

1293 Matrices  $\mathbf{B}_w, \mathbf{B}_{\partial_n w}$  are given by

$$B_w^{il} = \langle \gamma_0 \phi_w^i, \phi_{\partial,1}^l \rangle_{L^2(\partial\Omega)}, \quad B_{\partial_n w}^{il} = \langle \gamma_1 \phi_w^i, \phi_{\partial,2}^l \rangle_{L^2(\partial\Omega)}, \quad l \in \{1, n_{\partial}\}. \quad (7.84)$$

1294 This kind of discretization requires  $H^2$  conforming element. The construction of those is  
1295 rather involved [AFS68, Bel69] and they are computationally expensive. Nevertheless, this  
1296 kind of discretization is able to handle generic boundary conditions [GSV18]. For this reason,  
1297 it is the most adapted for the pH framework.



---

To lower the regularity requirement for the finite elements many non conforming discretization have been proposed. The most employed is the Hellan-Herrmann-Johnson element [AB85, BR90]. However, this method does not handle generic non homogeneous boundary conditions. Given the unavailability of the boundary for interconnections, the modularity feature of pHs cannot be fully exploited.

**Remark 9** (On the  $H^{\text{div Div}}$  space)

*Equivalently, the second line of Eq. (7.77a) can be integrated by parts twice to obtain a discretized system whose input are the linear velocity and the angular velocity at the boundary. However, while for the  $H^2$  space conforming finite elements are available, for the  $H^{\text{div Div}}$  no conforming finite elements have been proposed. This makes the discretization unfeasible.*

### 7.1.3.3 Mindlin plate

Using Eqs. (5.22) and (5.24), the relation between co-energy and energy variables for the isotropic Mindlin plate is found to be

$$\begin{aligned} \alpha_w &= \rho h e_w, & \alpha_\theta &= I_\theta e_\theta, & I_\theta &:= \rho h^3 / 12, \\ \mathbf{A}_\kappa &= \mathbf{C}_b \mathbf{E}_\kappa, & \alpha_\gamma &= C_s e_\gamma, & C_s &:= 1 / (K_{\text{sh}} G h), \end{aligned} \tag{7.85}$$

where  $K_{\text{sh}}$  is the shear correction factor,  $G$  the shear modulus. The other variables have the same meaning as in Sec. §7.1.3.2.

**Control through forces and torques** A pH representation in co-energy variables with known forces and momenta at the boundary is given by the system

---

$$\begin{bmatrix} \rho h & 0 & 0 & 0 \\ \mathbf{0} & I_\theta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C_s \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \text{div} \\ \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{I}_{2 \times 2} \\ \mathbf{0} & \text{Grad} & \mathbf{0} & \mathbf{0} \\ \text{grad} & -\mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}, \quad \begin{aligned} e_w &\in H^1(\Omega), \\ \mathbf{e}_\theta &\in H^{\text{Grad}}(\Omega, \mathbb{R}^2), \\ \mathbf{E}_\kappa &\in H^{\text{Div}}(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2}), \\ \mathbf{e}_\gamma &\in H^{\text{div}}(\Omega, \mathbb{R}^2), \end{aligned} \quad (7.86a)$$

$$\mathbf{u}_\partial = \begin{bmatrix} 0 & 0 & 0 & \gamma_n \\ 0 & 0 & \gamma_{nn} & 0 \\ 0 & 0 & \gamma_{ns} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}, \quad \mathbf{u}_\partial \in \mathbb{R}^3, \quad (7.86b)$$

$$\mathbf{y}_\partial = \begin{bmatrix} \gamma_0 & 0 & 0 & 0 \\ 0 & \gamma_n & 0 & 0 \\ 0 & \gamma_s & 0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}, \quad \mathbf{y}_\partial \in \mathbb{R}^3. \quad (7.86c)$$

1317 The trace operators are defined as

$$\begin{aligned} \gamma_0 a &= a|_{\partial\Omega}, & \gamma_n \mathbf{a} &= \mathbf{a} \cdot \mathbf{n}|_{\partial\Omega}, & \gamma_{nn} \mathbf{A} &= \mathbf{A} : (\mathbf{n} \otimes \mathbf{n})|_{\partial\Omega}, \\ \gamma_s \mathbf{a} &= \mathbf{a} \cdot \mathbf{s}|_{\partial\Omega}, & \gamma_{ns} \mathbf{A} &= \mathbf{A} : (\mathbf{n} \otimes \mathbf{s})|_{\partial\Omega}. \end{aligned} \quad (7.87)$$

1318 The variables assume value in the sets  $\mathbb{A} = \mathbb{R} \times \mathbb{R}^2$ ,  $\mathbb{B} = \mathbb{R}_{\text{sym}}^{2 \times 2} \times \mathbb{R}^2$ . The mass operators  
1319 are given by

$$\mathcal{M}_1 = \begin{bmatrix} \rho h & 0 \\ \mathbf{0} & I_\theta \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} \mathbf{C}_b & \mathbf{0} \\ \mathbf{0} & C_s \end{bmatrix}. \quad (7.88)$$

1320 The  $\mathbf{L}$ ,  $\mathcal{L}$ ,  $\mathcal{N}_{\partial,1}$ ,  $\mathcal{N}_{\partial,1}$  operators are

$$\mathbf{L} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{2 \times 2} \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} \mathbf{0} & \text{Grad} \\ \text{grad} & \mathbf{0} \end{bmatrix}, \quad \mathcal{N}_{\partial,1} = \begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_n \\ 0 & \gamma_s \end{bmatrix}, \quad \mathcal{N}_{\partial,2} = \begin{bmatrix} 0 & \gamma_n \\ \gamma_{nn} & 0 \\ \gamma_{ns} & 0 \end{bmatrix}. \quad (7.89)$$

1321 The energy rate is retrieved from Eq. (5.26)  $\dot{H} = \langle \mathbf{y}_\partial, \mathbf{u}_\partial \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$ . Introducing the test  
1322 functions  $(v_w, \mathbf{v}_\theta, \mathbf{V}_\kappa, \mathbf{v}_\gamma)$  and integrating by parts the first two lines of (7.86a) one gets

$$\begin{aligned} \langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} &= -\langle \text{grad } v_w, \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \gamma_0 v_w, u_{\partial,1} \rangle_{L^2(\partial\Omega)}, \\ \langle \mathbf{v}_\theta, I_\theta \partial_t \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} &= -\langle \text{Grad } \mathbf{v}_\theta, \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})} + \langle \mathbf{v}_\theta, \mathbf{e}_\gamma \rangle_{L^2(\Omega)} + \langle \gamma_0 \mathbf{v}_\theta, \gamma_n \mathbf{E}_\kappa \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}, \\ \langle \mathbf{V}_\kappa, \mathbf{C}_b \partial_t \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})} &= \langle \mathbf{V}_\kappa, \text{Grad } \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})}, \\ \langle \mathbf{v}_\gamma, C_s \partial_t \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} &= \langle \mathbf{v}_\gamma, \text{grad } e_w \rangle_{L^2(\Omega, \mathbb{R}^2)} - \langle \mathbf{v}_\gamma, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)}. \end{aligned} \quad (7.90)$$

1323 The term  $\langle \gamma_0 \mathbf{v}_\theta, \mathbf{u}_{\partial,2} \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$  can be decomposed in its tangential and normal components

$$\langle \gamma_0 \mathbf{v}_\theta, \gamma_n \mathbf{E}_\kappa \rangle_{L^2(\partial\Omega, \mathbb{R}^2)} = \langle \gamma_n \mathbf{v}_\theta, u_{\partial,2} \rangle_{L^2(\partial\Omega)} + \langle \gamma_s \mathbf{v}_\theta, u_{\partial,3} \rangle_{L^2(\partial\Omega)} \quad (7.91)$$

1324 Introducing a Galerkin discretization for test and efforts functions

$$\begin{aligned} v_w &= \sum_{i=1}^{n_w} \phi_w^i v_w^i, & \mathbf{v}_\theta &= \sum_{i=1}^{n_\theta} \phi_\theta^i v_\theta^i, & \mathbf{V}_\kappa &= \sum_{i=1}^{n_\kappa} \Phi_\kappa^i v_\kappa^i, & \mathbf{v}_\gamma &= \sum_{i=1}^{n_\gamma} \phi_\gamma^i v_\gamma^i, & \mathbf{v}_\partial &= \sum_{i=1}^{n_\partial} \phi_\partial^i v_\partial^i, \\ e_w &= \sum_{i=1}^{n_w} \phi_w^i e_w^i, & \mathbf{e}_\theta &= \sum_{i=1}^{n_\theta} \phi_\theta^i e_\theta^i, & \mathbf{E}_\kappa &= \sum_{i=1}^{n_\kappa} \Phi_\kappa^i e_\kappa^i, & \mathbf{e}_\gamma &= \sum_{i=1}^{n_\gamma} \phi_\gamma^i e_\gamma^i, & \mathbf{u}_\partial &= \sum_{i=1}^{n_\partial} \phi_\partial^i u_\partial^i, \\ & & & & & & & & \mathbf{y}_\partial &= \sum_{i=1}^{n_\partial} \phi_\partial^i y_\partial^i. \end{aligned} \quad (7.92)$$

1325 the following finite dimensional system is obtained

$$\begin{aligned} \text{Diag} \begin{bmatrix} \mathbf{M}_{\rho h} \\ \mathbf{M}_{I_\theta} \\ \mathbf{M}_{\mathbf{C}_b} \\ \mathbf{M}_{C_s} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_w \\ \dot{\mathbf{e}}_\theta \\ \dot{\mathbf{e}}_\kappa \\ \dot{\mathbf{e}}_\gamma \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D}_{\text{grad}}^\top \\ \mathbf{0} & \mathbf{0} & -\mathbf{D}_{\text{Grad}}^\top & -\mathbf{D}_0^\top \\ \mathbf{0} & \mathbf{D}_{\text{Grad}} & \mathbf{0} & \mathbf{0} \\ \mathbf{D}_{\text{grad}} & \mathbf{D}_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix} + \begin{bmatrix} \mathbf{B}_w & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\theta_n} & \mathbf{B}_{\theta_s} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\ \mathbf{M}_\partial \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{B}_w^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\theta_n}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\theta_s}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}. \end{aligned} \quad (7.93)$$

1326 The notation  $\text{Diag}$  denotes a block diagonal matrix. The mass matrices  $\mathbf{M}_{\rho h}$ ,  $\mathbf{M}_{I_\theta}$ ,  $\mathbf{M}_{\mathbf{C}_b}$ ,  $\mathbf{M}_{C_s}$   
1327 are computed as

$$\begin{aligned} M_{\rho h}^{ij} &= \langle \phi_w^i, \rho h \phi_w^j \rangle_{L^2(\Omega)}, & M_{\mathbf{C}_b}^{pq} &= \langle \Phi_\kappa^p, \mathbf{C}_b \Phi_\kappa^q \rangle_{L^2(\Omega, \mathbb{R}^{2 \times 2})}, \\ M_{I_\theta}^{mn} &= \langle \phi_\kappa^m, I_\theta \phi_\kappa^n \rangle_{L^2(\Omega, \mathbb{R}^2)}, & M_{C_s}^{rs} &= \langle \phi_\gamma^r, C_s \phi_\gamma^s \rangle_{L^2(\Omega, \mathbb{R}^2)}, \end{aligned} \quad (7.94)$$

1328 where  $i, j \in \{1, n_w\}$ ,  $m, n \in \{1, n_\theta\}$ ,  $p, q \in \{1, n_\kappa\}$ ,  $r, s \in \{1, n_\gamma\}$ . Matrices  $\mathbf{D}_{\text{grad}}$ ,  $\mathbf{D}_{\text{Grad}}$ ,  $\mathbf{D}_0$   
1329 assume the form

$$\begin{aligned} D_{\text{grad}}^{rj} &= \langle \phi_\gamma^r, \text{grad} \phi_w^j \rangle_{L^2(\Omega, \mathbb{R}^2)}, & D_0^{rn} &= -\langle \phi_\gamma^r, \phi_\theta^n \rangle_{L^2(\Omega, \mathbb{R}^2)}, \\ D_{\text{Grad}}^{pn} &= \langle \Phi_\kappa^p, \text{Grad} \phi_\theta^n \rangle_{L^2(\Omega, \mathbb{R}^{2 \times 2})}, \end{aligned} \quad (7.95)$$

1330 Matrices  $\mathbf{B}_w$ ,  $\mathbf{B}_{\theta_n}$ ,  $\mathbf{B}_{\theta_s}$  are computed as ( $l \in \{1, n_\partial\}$ )

$$B_w^{il} = \langle \gamma_0 \phi_w^i, \phi_{\partial,1}^l \rangle_{L^2(\partial\Omega)}, \quad B_{\theta_n}^{ml} = \langle \gamma_n \phi_\theta^m, \phi_{\partial,2}^l \rangle_{L^2(\partial\Omega)}, \quad B_{\theta_s}^{ml} = \langle \gamma_s \phi_\theta^m, \phi_{\partial,3}^l \rangle_{L^2(\partial\Omega)}. \quad (7.96)$$

**Control through linear and angular velocities** If instead the opposite causality is considered, the continuous system read

$$\begin{bmatrix} \rho h & 0 & 0 & 0 \\ \mathbf{0} & I_\theta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C_s \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & \text{div} \\ \mathbf{0} & \mathbf{0} & \text{Div} & \mathbf{I}_{2 \times 2} \\ \mathbf{0} & \text{Grad} & \mathbf{0} & \mathbf{0} \\ \text{grad} & -\mathbf{I}_{2 \times 2} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}, \quad (7.97a)$$

$$\mathbf{u}_\partial = \begin{bmatrix} \gamma_0 & 0 & 0 & 0 \\ 0 & \gamma_n & 0 & 0 \\ 0 & \gamma_s & 0 & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}, \quad \mathbf{u}_\partial \in \mathbb{R}^3, \quad (7.97b)$$

$$\mathbf{y}_\partial = \begin{bmatrix} 0 & 0 & 0 & \gamma_n \\ 0 & 0 & \gamma_{nn} & 0 \\ 0 & 0 & \gamma_{ns} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{E}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}, \quad \mathbf{y}_\partial \in \mathbb{R}^3. \quad (7.97c)$$

1331 Integrating by parts the last two lines of (7.97a) one gets

$$\begin{aligned} \langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} &= \langle v_w, \text{div } \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)}, \\ \langle \mathbf{v}_\theta, I_\theta \partial_t \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} &= \langle \mathbf{v}_\theta, \text{Div } \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \mathbf{v}_\theta, \mathbf{e}_\gamma \rangle_{L^2(\Omega)}, \\ \langle \mathbf{V}_\kappa, \mathbf{C}_b \partial_t \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}_{\text{sym}}^{2 \times 2})} &= -\langle \text{Div } \mathbf{V}_\kappa, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \gamma_n \mathbf{V}_\kappa, \gamma_0 \mathbf{e}_\theta \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}, \\ \langle \mathbf{v}_\gamma, C_s \partial_t \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} &= -\langle \text{div } \mathbf{v}_\gamma, e_w \rangle_{L^2(\Omega)} - \langle \mathbf{v}_\gamma, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} + \langle \gamma_0 v_w, u_{\partial,1} \rangle_{L^2(\partial\Omega)}. \end{aligned} \quad (7.98)$$

1332 The term  $\langle \gamma_n \mathbf{V}_\kappa, \gamma_0 \mathbf{e}_\theta \rangle_{L^2(\partial\Omega, \mathbb{R}^2)}$  can be decomposed in its tangential and normal components

$$\langle \gamma_n \mathbf{V}_\kappa, \gamma_0 \mathbf{e}_\theta \rangle_{L^2(\partial\Omega, \mathbb{R}^2)} = \langle \gamma_{nn} \mathbf{V}_\kappa, u_{\partial,2} \rangle_{L^2(\partial\Omega)} + \langle \gamma_{ns} \mathbf{V}_\kappa, u_{\partial,3} \rangle_{L^2(\partial\Omega)}. \quad (7.99)$$

1333 Plugging approximation (7.92) into this system, one computes

$$\begin{aligned} \text{Diag} \begin{bmatrix} \mathbf{M}_{\rho h} \\ \mathbf{M}_{I_\theta} \\ \mathbf{M}_{\mathbf{C}_b} \\ \mathbf{M}_{C_s} \end{bmatrix} \begin{pmatrix} \dot{e}_w \\ \dot{\mathbf{e}}_\theta \\ \dot{\mathbf{E}}_\kappa \\ \dot{\mathbf{e}}_\gamma \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\text{div}} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{\text{Div}} & -\mathbf{D}_0^\top \\ \mathbf{0} & -\mathbf{D}_{\text{Div}}^\top & \mathbf{0} & \mathbf{0} \\ -\mathbf{D}_{\text{div}}^\top & \mathbf{D}_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{M_{nn}} & \mathbf{B}_{M_{ns}} \\ \mathbf{B}_{q_n} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{u}_\partial, \\ \mathbf{M}_{\partial} \mathbf{y}_\partial &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{q_n}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{M_{nn}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{M_{ns}}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \mathbf{e}_\theta \\ \mathbf{e}_\kappa \\ \mathbf{e}_\gamma \end{pmatrix}. \end{aligned} \quad (7.100)$$

1334 Matrices  $\mathbf{D}_{\text{div}}$ ,  $\mathbf{D}_{\text{Div}}$  assume the form ( $i, j \in \{1, n_w\}$ ,  $m, n \in \{1, n_\theta\}$ ,  $p, q \in \{1, n_\kappa\}$ ,  $r, s \in$   
1335  $\{1, n_\gamma\}$ )

$$D_{\text{div}}^{is} = \langle \phi_w^i, \text{div } \phi_\gamma^s \rangle_{L^2(\Omega)}, \quad D_{\text{Div}}^{mq} = \langle \phi_\theta^m, \text{Div } \Phi_\kappa^q \rangle_{L^2(\Omega, \mathbb{R}^2)}. \quad (7.101)$$

Matrix  $\mathbf{B}_{q_n}$ ,  $\mathbf{B}_{M_{nn}}$ ,  $\mathbf{B}_{M_{ns}}$  are computed as ( $l \in \{1, n_\partial\}$ )

$$B_{q_n}^{rl} = \left\langle \gamma_n \phi_\gamma^r, \phi_{\partial,1}^l \right\rangle_{L^2(\partial\Omega)}, \quad B_{M_{nn}}^{pl} = \left\langle \gamma_{nn} \Phi_\kappa^p, \phi_{\partial,2}^l \right\rangle_{L^2(\partial\Omega)}, \quad B_{M_{ns}}^{pl} = \left\langle \gamma_{ns} \Phi_\kappa^p, \phi_{\partial,3}^l \right\rangle_{L^2(\partial\Omega)}. \quad (7.102)$$

This finite dimensional system represents a purely mixed discretization of the problem and is really close to the plane elasticity system. Conforming finite elements for the plane elasticity system on simplicial meshes have been constructed in [AW02]. The resulting element is rather cumbersome and computationally expensive as the stress tensor has at least 24 degrees of freedom on a triangle. For this reason, many finite element discretization imposes the symmetry of the stress tensor weakly [AFW07]. To actually implement the discretization, in Chap. 8 the Mindlin plate problem is going to be reformulated so that the momenta tensor is only weakly symmetric.

## 7.2 Mixed boundary conditions

In this section Assumption 3 on uniform boundary condition is modified to account for general non homogeneous boundary conditions. The discretization of Stokes-Dirac structure under mixed causality has been already treated in [KML18]. However, to satisfy the power balance at a discrete level, some additional parameters are introduced. This makes the employment of this methodology not simple and dependent on the considered application. Furthermore, elasticity models do not fall within the required assumptions.

We propose here two methodologies to tackle mixed boundary conditions within the Partitioned Finite Element Method. The first introduces Lagrange multipliers, and therefore algebraic constraints, to enforce the mixed causality. Finite dimensional differential algebraic port-Hamiltonian systems (pHDAE) have been introduced in [BMXZ18] for linear systems and in [MM19] for non linear systems. This enriched description share all the crucial features of ordinary pHs, but easily account for algebraic constraints, time-dependent transformations and explicit dependence on time in the Hamiltonian. The second method employs a domain decomposition technique to interconnect systems with different causalities. For the sake of simplicity The illustration is restrained to the linear case.

The open connected set  $\Omega \subset \mathbb{R}^d$ ,  $d = \{1, 2, 3\}$ , with Lipschitz boundary  $\partial\Omega$  represent the spatial domain. The boundary is split into two partition  $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ ,  $\Gamma_1 \cap \Gamma_2 = \{\emptyset\}$ . The set  $\Gamma_1$ ,  $\Gamma_2$  are considered to be connected, cf. Fig. 7.1.

**Remark 10** (Connectedness of  $\Gamma_1, \Gamma_2$ )

*Disconnected set can be handled as well. This requires the introduction of an heavy notation and complicates the illustration. For sake of simplicity, the connectedness hypothesis applies.*

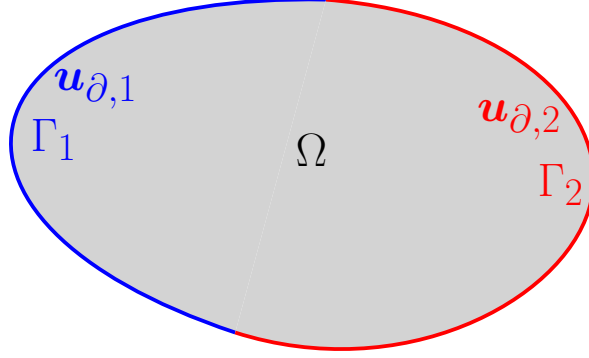


Figure 7.1: Partition of boundary into two connected sets.

1369 For scalars  $a_{\partial,*}, b_{\partial,*} \in L^2(\Gamma_*)$  and vectors  $\mathbf{a}_{\partial,*}, \mathbf{b}_{\partial,*} \in L^2(\Gamma_*, \mathbb{R}^m)$  defined on the boundary  
 1370 partition  $\Gamma_*$  the inner product is defined as

$$\langle a_{\partial,*}, b_{\partial,*} \rangle_{L^2(\Gamma_*)} = \int_{\Gamma_*} a_{\partial,*} b_{\partial,*} \, d\Gamma_*, \quad \langle \mathbf{a}_{\partial,*}, \mathbf{b}_{\partial,*} \rangle_{L^2(\Gamma_*, \mathbb{R}^m)} = \int_{\Gamma_*} \mathbf{a}_{\partial,*} \cdot \mathbf{b}_{\partial,*} \, d\Gamma_*. \quad (7.103)$$

Consider now the following boundary control linear pH system in co-energy form

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{L}^\top - \mathcal{L}^* \\ \mathbf{L} + \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{matrix} \mathbf{e}_1 \in H^\mathcal{L}, \\ \mathbf{e}_2 \in H^{-\mathcal{L}*}, \end{matrix} \quad (7.104a)$$

$$\begin{pmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{matrix} \mathbf{u}_{\partial,1} \in \mathbb{R}^m, \\ \mathbf{u}_{\partial,2} \in \mathbb{R}^m, \end{matrix} \quad (7.104b)$$

$$\begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_1} \\ \mathcal{N}_{\partial,1}^{\Gamma_2} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{matrix} \mathbf{y}_{\partial,1} \in \mathbb{R}^m, \\ \mathbf{y}_{\partial,2} \in \mathbb{R}^m. \end{matrix} \quad (7.104c)$$

1371 The operator  $\mathcal{N}_{\partial,*}^{\Gamma_\circ}$  with  $*, \circ \in \{1, 2\}$  represent now the restriction of operator  $\mathcal{N}_{\partial,*}$ , defined  
 1372 in Eq. (7.8), over the subset  $\Gamma_\circ$ . The boundary inputs and output are now vectors  $\mathbb{R}^{2m}$ . This  
 1373 does not mean that the boundary conditions have been doubled, but only that the components  
 1374 of  $\mathbf{u}_\partial, \mathbf{y}_\partial$  are only defined on the subsets  $\Gamma_1, \Gamma_2$  of the overall boundary. This corresponds to  
 1375 a slight modification of Assumption 3.

1376 Given the additive property of the integral, it is possible to write

$$\begin{aligned} \langle \mathcal{N}_{\partial,1} \mathbf{e}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{e}_1, \mathcal{N}_{\partial,2}^{\Gamma_1} \mathbf{e}_2 \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{e}_1, \mathcal{N}_{\partial,2}^{\Gamma_2} \mathbf{e}_2 \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \\ &= \langle \mathbf{u}_{\partial,1}, \mathbf{y}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathbf{y}_{\partial,2}, \mathbf{u}_{\partial,2} \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}. \end{aligned} \quad (7.105)$$

1377 The continuous power balance is obtained using Eqs. (7.50) and (7.105)

$$\dot{H} = \langle \mathbf{u}_{\partial,1}, \mathbf{y}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathbf{y}_{\partial,2}, \mathbf{u}_{\partial,2} \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}. \quad (7.106)$$

## 7.2.1 Solution using Lagrange multipliers

This solution introduces a Lagrange multiplier for the boundary control that does not arise explicitly in the weak form. To illustrate the idea, consider again the weak form 7.51 (obtained by integration by parts of the  $-\mathcal{L}^*$  partition) of Sys. 7.104

$$\begin{aligned}\langle \mathbf{v}_1, \mathcal{M}_1 \partial_t \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{A})} - \langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}, \\ \langle \mathbf{v}_2, \mathcal{M}_2 \partial_t \mathbf{e}_2 \rangle_{L^2(\Omega, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega, \mathbb{B})}.\end{aligned}\quad (7.107)$$

The term  $\langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)}$  can be split into the two boundary contributions, as in Eq. (7.105). The variable  $\mathbf{y}_{\partial,1}$  plays here the role of a Lagrange multiplier  $\mathbf{y}_{\partial,1} = \boldsymbol{\lambda}_{\partial,1}$

$$\begin{aligned}\langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega, \mathbb{R}^m)} &= \langle \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{v}_1, \mathcal{N}_{\partial,2}^{\Gamma_1} \mathbf{e}_2 \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{v}_1, \mathcal{N}_{\partial,2}^{\Gamma_2} \mathbf{e}_2 \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \\ &= \langle \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{v}_1, \mathbf{y}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{v}_1, \mathbf{u}_{\partial,2} \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \\ &= \langle \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{v}_1, \boldsymbol{\lambda}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{v}_1, \mathbf{u}_{\partial,2} \rangle_{L^2(\Gamma_2, \mathbb{R}^m)},\end{aligned}\quad (7.108)$$

If test function  $\mathbf{v}_{\partial,1}, \mathbf{v}_{\partial,2} \in \mathbb{R}^m$  are introduced, the input and outputs definitions

$$\mathbf{u}_{\partial,1} = \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{e}_1, \quad \mathbf{y}_{\partial,1} = \boldsymbol{\lambda}_{\partial,1}, \quad \mathbf{y}_{\partial,2} = \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{e}_1, \quad (7.109)$$

can be put into weak form to obtain

$$\begin{aligned}\langle \mathbf{v}_{\partial,1}, \mathbf{u}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} &= \langle \mathbf{v}_{\partial,1}, \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{e}_1 \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \\ \langle \mathbf{v}_{\partial,1}, \mathbf{y}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} &= \langle \mathbf{v}_{\partial,1}, \boldsymbol{\lambda}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \\ \langle \mathbf{v}_{\partial,2}, \mathbf{y}_{\partial,2} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} &= \langle \mathbf{v}_{\partial,2}, \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{e}_1 \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}.\end{aligned}\quad (7.110)$$

As usual, a Galerkin approximation is introduced

$$\begin{aligned}\mathbf{v}_1 &\approx \sum_{i=1}^{n_1} \phi_1^i(\mathbf{x}) v_1^i, & \mathbf{e}_1 &\approx \sum_{i=1}^{n_1} \phi_1^i(\mathbf{x}) e_1^i(t), & \Delta_{\partial,1} &\approx \sum_{i=1}^{n_{\partial,1}} \phi_{\partial,1}^i(\mathbf{s}_1) \Delta_{\partial,1}^i, & \mathbf{s}_1 &\in \Gamma_1, \\ \mathbf{v}_2 &\approx \sum_{i=1}^{n_2} \phi_2^i(\mathbf{x}) v_2^i, & \mathbf{e}_2 &\approx \sum_{i=1}^{n_2} \phi_2^i(\mathbf{x}) e_2^i(t), & \square_{\partial,2} &\approx \sum_{i=1}^{n_{\partial,2}} \phi_{\partial,2}^i(\mathbf{s}_2) \square_{\partial,2}^i(t), & \mathbf{s}_2 &\in \Gamma_2.\end{aligned}\quad (7.111)$$

where  $\triangle$  stays for  $v, u, y, \lambda$  and  $\square$  for  $v, u, y$ . Replacing the approximation 7.111 into Eqs. 7.107, 7.108, 7.110, the following differential-algebraic system is constructed

$$\begin{aligned} \text{Diag} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,1} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top - \mathbf{D}_{\mathcal{L}}^\top & \mathbf{B}_{1,\Gamma_1} \\ \mathbf{D}_0 + \mathbf{D}_{\mathcal{L}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_{1,\Gamma_1}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_{1,\Gamma_2} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{\partial,1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix}, \\ \begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,1} \\ \mathbf{B}_{1,\Gamma_2}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix}. \end{aligned} \quad (7.112)$$

Apart for matrices  $\mathbf{M}_{\partial,1}, \mathbf{M}_{\partial,2}, \mathbf{B}_{1,\Gamma_1}, \mathbf{B}_{1,\Gamma_2}$ ,

$$\begin{aligned} M_{\partial,1}^{lk} &= \langle \phi_{\partial,1}^l, \phi_{\partial,1}^k \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \quad (l, k) \in \{1, n_{\partial,1}\}, \quad B_{1,\Gamma_1}^{ik} = \langle \mathcal{N}_{\partial,1}^{\Gamma_1} \phi_1^i, \phi_{\partial,1}^k \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \quad i \in \{1, n_1\}, \\ M_{\partial,2}^{fg} &= \langle \phi_{\partial,2}^f, \phi_{\partial,2}^g \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \quad (f, g) \in \{1, n_{\partial,2}\}, \quad B_{1,\Gamma_2}^{ig} = \langle \mathcal{N}_{\partial,1}^{\Gamma_2} \phi_1^i, \phi_{\partial,2}^g \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \end{aligned} \quad (7.113)$$

the other matrices keep the same definition as in (7.53). The discrete Hamiltonian, whose expression is [BMXZ18]

$$H_d = \frac{1}{2} \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \mathbf{e}_2. \quad (7.114)$$

gives rise to the discrete power balance

$$\begin{aligned} \dot{H}_d &= \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \dot{\mathbf{e}}_1 + \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \dot{\mathbf{e}}_2, \\ &= -\mathbf{e}_1^\top (\mathbf{D}_0 + \mathbf{D}_{\mathcal{L}})^\top \mathbf{e}_2 + \mathbf{e}_2^\top (\mathbf{D}_0 + \mathbf{D}_{\mathcal{L}}) \mathbf{e}_1 + \mathbf{e}_1^\top (\mathbf{B}_{1,\Gamma_1} \boldsymbol{\lambda}_{\partial,1} + \mathbf{B}_{1,\Gamma_2} \mathbf{u}_{\partial,2}), \\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}, \\ &= \hat{\mathbf{y}}_{\partial,1}^\top \mathbf{u}_{\partial,1} + \hat{\mathbf{y}}_{\partial,2}^\top \mathbf{u}_{\partial,2}, \quad \text{where} \quad \hat{\mathbf{y}}_{\partial,1} := \mathbf{M}_{\partial,1} \mathbf{y}_{\partial,1}, \quad \hat{\mathbf{y}}_{\partial,2} := \mathbf{M}_{\partial,2} \mathbf{y}_{\partial,2}. \end{aligned} \quad (7.115)$$

This result is the finite dimensional equivalent of (7.106).

Equivalently, the weak form Eq. 7.52 may be used as a starting point. The computation follows in a completely analogous manner. The only difference is that  $\mathbf{y}_{\partial,2} = \boldsymbol{\lambda}_{\partial,2}$  plays the role of the Lagrange multiplier. The final finite dimensional system then is given by

$$\begin{aligned} \text{Diag} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^\top + \mathbf{D}_{-\mathcal{L}}^* & \mathbf{0} \\ \mathbf{D}_0 - \mathbf{D}_{-\mathcal{L}}^* & \mathbf{0} & \mathbf{B}_{2,\Gamma_2} \\ \mathbf{0} & -\mathbf{B}_{2,\Gamma_2}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{2,\Gamma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix}, \\ \begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,\Gamma_1}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix}. \end{aligned} \quad (7.116)$$

where  $\mathbf{B}_{2,\Gamma_1}, \mathbf{B}_{2,\Gamma_2}$  are given by

$$B_{2,\Gamma_1}^{mk} = \langle \mathcal{N}_{\partial,2}^{\Gamma_1} \phi_2^m, \phi_{\partial,1}^k \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, \quad B_{2,\Gamma_2}^{mg} = \langle \mathcal{N}_{\partial,2}^{\Gamma_2} \phi_2^m, \phi_{\partial,2}^g \rangle_{L^2(\Gamma_2, \mathbb{R}^m)}, \quad (7.117)$$

where  $m \in \{1, n_2\}$ ,  $k \in \{1, n_{\partial,1}\}$ ,  $g \in \{1, n_{\partial,2}\}$ . This solution can be applied to incorporate



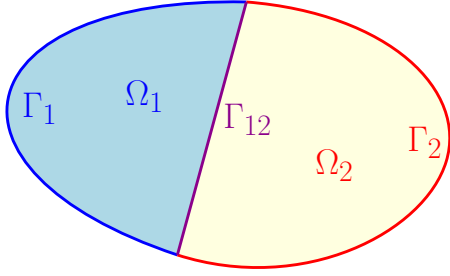
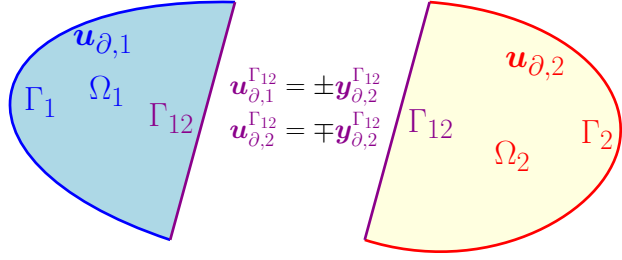


Figure 7.2: Splitting of the domain.

Figure 7.3: Interconnection at the interface  $\Gamma_{12}$ .

all possible mixed boundary conditions in a systematic manner. However the finite element discretization is required to satisfy the inf-sup condition. Simulating the resulting system is harder, since the algebraic constraints pose additional difficulties for the time integration.

### 7.2.2 Virtual domain decomposition

Since the boundary subsets  $\Gamma_1$ ,  $\Gamma_2$  are supposed to be connected set, a single interface is sufficient to decompose the system appropriately. In Fig. 7.2 the splitting of the domain is accomplished by introducing the interface  $\Gamma_{12}$ . This separation line that separates the domain is an additional degree of freedom, as it can be freely drawn. If the finite element method is used for the basis functions, the interface should be drawn so that the meshing of the subdomains does not generate excessively skewed triangles.

The idea is based on the fact that System 7.104 can be split into two systems with uniform causality. The following set of boundary variables is used for  $\Omega_1$  subdomain

$$\begin{pmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,1}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ \mathcal{N}_{\partial,1}^{\Gamma_{12}} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,1}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_1} \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_{12}} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \quad (7.118)$$

Whereas for the  $\Omega_2$  subdomain, the boundary variables are

$$\begin{pmatrix} \mathbf{u}_{\partial,2} \\ \mathbf{u}_{\partial,2}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_2} \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_{12}} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{y}_{\partial,2} \\ \mathbf{y}_{\partial,2}^{\Gamma_{12}} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ \mathcal{N}_{\partial,1}^{\Gamma_{12}} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \quad (7.119)$$

The following relations then hold (cf. Fig. 7.3)

$$\mathbf{u}_{\partial,1}^{\Gamma_{12}} = \pm \mathbf{y}_{\partial,2}^{\Gamma_{12}}, \quad \mathbf{u}_{\partial,2}^{\Gamma_{12}} = \mp \mathbf{y}_{\partial,1}^{\Gamma_{12}}. \quad (7.120)$$

The plus or minus sign is due to the fact that either  $\mathcal{N}_{\partial,1}^{\Gamma_{12}}$  or  $\mathcal{N}_{\partial,2}^{\Gamma_{12}}$  contains a scalar product with the outgoing normal (or the tangent unit vector) at  $\Gamma_{12}$  (that has opposite direction depending on which subdomain is considered). These relations are at the core of the methodology, since they state the equivalence between a problem with mixed causalities and the interconnection of two problems with uniform causality.

To obtain a final system with the desired causality, the weak form has to be carried out separately on each subdomain. In particular, on subdomain  $\Omega_1$  the  $\mathcal{L}$  operator is integrated by parts, whereas on subdomain  $\Omega_2$  the  $-\mathcal{L}^*$  operator undergoes the integration by parts. Consequently, on subdomains  $\Omega_1$  ( $\Omega_2$ ) the boundary input  $\mathbf{u}_{\partial,1}$  ( $\mathbf{u}_{\partial,2}$ ) explicitly appears. Let  $L^2(\Omega_*, \mathbb{A})$  be the  $L^2$  space restricted to the subdomain  $\Omega_*$ , and let  $L^2(\Omega_*, \mathbb{B})$  be the restriction of the  $L^2$  space to  $\Omega_*$  for  $* \in \{1, 2\}$ . The weak form of the dynamics (7.104a) for the  $\Omega_1$  contribution reads

$$\begin{aligned} \langle \mathbf{v}_1, \mathcal{M}_1 \partial_t \mathbf{e}_1 \rangle_{L^2(\Omega_1, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega_1, \mathbb{A})} - \langle \mathbf{v}_1, \mathcal{L}^* \mathbf{e}_2 \rangle_{L^2(\Omega_1, \mathbb{A})} \\ \langle \mathbf{v}_2, \mathcal{M}_2 \partial_t \mathbf{e}_2 \rangle_{L^2(\Omega_1, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega_1, \mathbb{B})} + \langle \mathcal{L}^* \mathbf{v}_2, \mathbf{e}_1 \rangle_{L^2(\Omega_1, \mathbb{A})} + \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathcal{N}_{\partial,1} \mathbf{e}_1 \rangle_{L^2(\partial\Omega_1, \mathbb{R}^m)}. \end{aligned} \quad (7.121)$$

For  $\Omega_2$ , we get

$$\begin{aligned} \langle \mathbf{v}_1, \mathcal{M}_1 \partial_t \mathbf{e}_1 \rangle_{L^2(\Omega_2, \mathbb{A})} &= -\langle \mathbf{v}_1, \mathbf{L}^\top \mathbf{e}_2 \rangle_{L^2(\Omega_2, \mathbb{A})} - \langle \mathcal{L} \mathbf{v}_1, \mathbf{e}_2 \rangle_{L^2(\Omega_2, \mathbb{B})} + \langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega_2, \mathbb{R}^m)}, \\ \langle \mathbf{v}_2, \mathcal{M}_2 \partial_t \mathbf{e}_2 \rangle_{L^2(\Omega_2, \mathbb{B})} &= \langle \mathbf{v}_2, \mathbf{L} \mathbf{e}_1 \rangle_{L^2(\Omega_2, \mathbb{B})} + \langle \mathbf{v}_2, \mathcal{L} \mathbf{e}_1 \rangle_{L^2(\Omega_2, \mathbb{B})}. \end{aligned} \quad (7.122)$$

Since  $\partial\Omega_1 = \bar{\Gamma}_1 \cup \bar{\Gamma}_{12}$  and  $\partial\Omega_2 = \bar{\Gamma}_2 \cup \bar{\Gamma}_{12}$ , the boundary terms can be decomposed

$$\begin{aligned} \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathcal{N}_{\partial,1} \mathbf{e}_1 \rangle_{L^2(\partial\Omega_1, \mathbb{R}^m)} &= \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathcal{N}_{\partial,1} \mathbf{e}_1 \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,2} \mathbf{v}_2, \mathcal{N}_{\partial,1} \mathbf{e}_1 \rangle_{L^2(\Gamma_{12}, \mathbb{R}^m)}, \\ &= \langle \mathcal{N}_{\partial,2}^{\Gamma_1} \mathbf{v}_2, \mathcal{N}_{\partial,1}^{\Gamma_1} \mathbf{e}_1 \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,2}^{\Gamma_{12}} \mathbf{v}_2, \mathcal{N}_{\partial,1}^{\Gamma_{12}} \mathbf{e}_1 \rangle_{L^2(\Gamma_{12}, \mathbb{R}^m)}, \\ &= \langle \mathcal{N}_{\partial,2}^{\Gamma_1} \mathbf{v}_2, \mathbf{u}_{\partial,1} \rangle_{L^2(\Gamma_1, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,2}^{\Gamma_{12}} \mathbf{v}_2, \mathbf{u}_{\partial,1}^{\Gamma_{12}} \rangle_{L^2(\Gamma_{12}, \mathbb{R}^m)}. \end{aligned} \quad (7.123)$$

Analogously, for the remaining boundary term we find

$$\langle \mathcal{N}_{\partial,1} \mathbf{v}_1, \mathcal{N}_{\partial,2} \mathbf{e}_2 \rangle_{L^2(\partial\Omega_2, \mathbb{R}^m)} = \langle \mathcal{N}_{\partial,1}^{\Gamma_2} \mathbf{v}_1, \mathbf{u}_{\partial,2} \rangle_{L^2(\Gamma_2, \mathbb{R}^m)} + \langle \mathcal{N}_{\partial,1}^{\Gamma_{12}} \mathbf{v}_1, \mathbf{u}_{\partial,2}^{\Gamma_{12}} \rangle_{L^2(\Gamma_{12}, \mathbb{R}^m)}. \quad (7.124)$$

A Galerkin approximation, analogous to (7.111), is used for each subdomain

$$\begin{aligned} \mathbf{v}_{1,1} &\approx \sum_{i=1}^{n_{1,1}} \phi_{1,1}^i(\mathbf{x}_1) v_{1,1}^i, & \mathbf{x}_1 \in \Omega_1, & & \mathbf{v}_{1,2} &\approx \sum_{i=1}^{n_{1,2}} \phi_{1,2}^i(\mathbf{x}_2) v_{1,2}^i, & \mathbf{x}_2 \in \Omega_2, \\ \mathbf{v}_{2,1} &\approx \sum_{i=1}^{n_{2,1}} \phi_{2,1}^i(\mathbf{x}_1) v_{2,1}^i, & & & \mathbf{v}_{2,2} &\approx \sum_{i=1}^{n_{2,2}} \phi_{2,2}^i(\mathbf{x}_2) v_{2,2}^i, \\ \mathbf{e}_{1,1} &\approx \sum_{i=1}^{n_{1,1}} \phi_{1,1}^i(\mathbf{x}_1) e_{1,1}^i(t), & & & \mathbf{e}_{1,2} &\approx \sum_{i=1}^{n_{1,2}} \phi_{1,2}^i(\mathbf{x}_2) e_{1,2}^i(t), \\ \mathbf{e}_{2,1} &\approx \sum_{i=1}^{n_{2,1}} \phi_{2,1}^i(\mathbf{x}_1) e_{2,1}^i(t), & & & \mathbf{e}_{2,2} &\approx \sum_{i=1}^{n_{2,2}} \phi_{2,2}^i(\mathbf{x}_2) e_{2,2}^i(t). \end{aligned} \quad (7.125)$$

For the boundary variables, additional terms for the common interface are needed

$$\begin{aligned} \square_{\partial,1} &\approx \sum_{i=1}^{n_{\partial,1}} \phi_{\partial,1}^i(\mathbf{s}_1) \square_{\partial,1}^i(t), \quad \mathbf{s}_1 \in \Gamma_1, & \square_{\partial,1}^{\Gamma_{12}} &\approx \sum_{i=1}^{n_{\partial,12}} \phi_{\partial,12}^i(\mathbf{s}_{12}) \square_{\partial,1}^{i,\Gamma_{12}}(t), \\ & & & \mathbf{s}_{12} \in \Gamma_{12}. \\ \square_{\partial,2} &\approx \sum_{i=1}^{n_{\partial,2}} \phi_{\partial,2}^i(\mathbf{s}_2) \square_{\partial,2}^i(t), \quad \mathbf{s}_2 \in \Gamma_2, & \square_{\partial,2}^{\Gamma_{12}} &\approx \sum_{i=1}^{n_{\partial,12}} \phi_{\partial,12}^i(\mathbf{s}_{12}) \square_{\partial,2}^{i,\Gamma_{12}}(t), \end{aligned} \quad (7.126)$$

where  $\square$  stays for  $v, u, y$ .

**Remark 11** (Choice of the interface basis functions)

Notice that the same basis functions  $\phi_{\partial,12}$  are used for both interface variables. This is necessary in order to dispose of the same degrees of freedom for the interconnection.

Replacing approximations 7.111, 7.126 into Eqs. 7.121, 7.123, 7.118, a finite dimensional system for the  $\Omega_1$  subdomain is obtained

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{\mathcal{M}_1}^{\Omega_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2}^{\Omega_1} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_{1,1} \\ \dot{\mathbf{e}}_{2,1} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^{\Omega_1 \top} + \mathbf{D}_{-\mathcal{L}^*}^{\Omega_1} \\ \mathbf{D}_0^{\Omega_1} - \mathbf{D}_{-\mathcal{L}^*}^{\Omega_1 \top} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,1} \\ \mathbf{e}_{2,1} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{2,\Gamma_1}^{\Omega_1} & \mathbf{B}_{2,\Gamma_{12}}^{\Omega_1} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,1}^{\Gamma_{12}} \end{pmatrix}, \\ \begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,12} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,1}^{\Gamma_{12}} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,\Gamma_1}^{\Omega_1 \top} \\ \mathbf{0} & \mathbf{B}_{2,\Gamma_{12}}^{\Omega_1 \top} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,1} \\ \mathbf{e}_{2,1} \end{pmatrix}. \end{aligned} \quad (7.127)$$

The mass and interconnection operator matrices are the restriction to the subdomain of the matrices given in (7.116)

$$\begin{aligned} M_{\mathcal{M}_1}^{\Omega_1,ij} &= \langle \phi_{1,1}^i, \mathcal{M}_1 \phi_{1,1}^j \rangle_{L^2(\Omega_1, \mathbb{A})}, & D_0^{\Omega_1,mj} &= \langle \phi_{2,1}^i, \mathbf{L} \phi_{1,1}^j \rangle_{L^2(\Omega_1, \mathbb{B})}, & i, j &\in \{1, n_{1,1}\}, \\ M_{\mathcal{M}_2}^{\Omega_1,mn} &= \langle \phi_{2,1}^m, \mathcal{M}_2 \phi_{2,1}^n \rangle_{L^2(\Omega_1, \mathbb{B})}, & D_{-\mathcal{L}^*}^{\Omega_1,in} &= \langle \phi_{1,1}^m, -\mathcal{L}^* \phi_{2,1}^n \rangle_{L^2(\Omega_1, \mathbb{A})}, & m, n &\in \{1, n_{2,1}\}. \end{aligned} \quad (7.128)$$

Matrices  $\mathbf{M}_{\partial,1}$  is constructed as in Eq. (7.116). Matrix  $\mathbf{M}_{\partial,12}$  is similarly built

$$M_{\partial,12}^{lk} = \langle \phi_{\partial,12}^l, \phi_{\partial,12}^k \rangle_{L^2(\Gamma_{12}, \mathbb{R}^m)}, \quad l, k \in \{1, n_{\partial,12}\}. \quad (7.129)$$

The novel matrices  $\mathbf{B}_{2,\Gamma_1}^{\Omega_1}, \mathbf{B}_{1,\Gamma_{12}}^{\Omega_1}$  have elements

$$\begin{aligned} B_{2,\Gamma_1}^{\Omega_1,mh} &= \langle \mathcal{N}_{\partial,2}^{\Gamma_1} \phi_{2,1}^m, \phi_{\partial,1}^h \rangle_{L^2(\Gamma_1, \mathbb{R}^m)}, & m &\in \{1, n_{2,1}\}, & h &\in \{1, n_{\partial,1}\}, \\ B_{2,\Gamma_{12}}^{\Omega_1,mk} &= \langle \mathcal{N}_{\partial,2}^{\Gamma_{12}} \phi_{2,1}^m, \phi_{\partial,12}^k \rangle_{L^2(\Gamma_{12}, \mathbb{R}^m)}, & & & k &\in \{1, n_{\partial,12}\}. \end{aligned} \quad (7.130)$$

If instead the approximations are plugged into Eqs. 7.122, 7.124, 7.119, a finite dimensional system for the  $\Omega_2$  subdomain is computed

$$\begin{aligned}
\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1}^{\Omega_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2}^{\Omega_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_{1,2} \\ \dot{\mathbf{e}}_{2,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{0} & -\mathbf{D}_0^{\Omega_2 \top} - \mathbf{D}_{\mathcal{L}}^{\Omega_2 \top} \\ \mathbf{D}_0^{\Omega_2} + \mathbf{D}_{\mathcal{L}}^{\Omega_2} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,2} \\ \mathbf{e}_{2,2} \end{pmatrix} + \begin{bmatrix} \mathbf{B}_{1,\Gamma_2}^{\Omega_2} & \mathbf{B}_{1,\Gamma_{12}}^{\Omega_2} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{\partial,2} \\ \mathbf{u}_{\partial,2}^{\Gamma_{12}} \end{pmatrix}, \\
\begin{bmatrix} \mathbf{M}_{\partial,2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,12} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,2} \\ \mathbf{y}_{\partial,2}^{\Gamma_{12}} \end{pmatrix} &= \begin{bmatrix} \mathbf{B}_{1,\Gamma_2}^{\Omega_2 \top} & \mathbf{0} \\ \mathbf{B}_{1,\Gamma_{12}}^{\Omega_2 \top} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{1,2} \\ \mathbf{e}_{2,2} \end{pmatrix}.
\end{aligned} \tag{7.131}$$

1443 The mass and interconnection operator matrices are the restriction to the subdomain of  
 1444 the matrices given in (7.112)

$$\begin{aligned}
M_{\mathcal{M}_1}^{\Omega_2,ij} &= \left\langle \phi_{1,2}^i, \mathcal{M}_1 \phi_{1,2}^j \right\rangle_{L^2(\Omega_2, \mathbb{A})}, \quad D_0^{\Omega_2,mj} = \left\langle \phi_{2,2}^i, \mathbf{L} \phi_{1,2}^j \right\rangle_{L^2(\Omega_2, \mathbb{B})}, \quad i, j \in \{1, n_{1,2}\}, \\
M_{\mathcal{M}_2}^{\Omega_2,mn} &= \left\langle \phi_{2,2}^m, \mathcal{M}_2 \phi_{2,2}^n \right\rangle_{L^2(\Omega_2, \mathbb{B})}, \quad D_{\mathcal{L}}^{\Omega_2,mj} = \left\langle \phi_{2,2}^m, \mathcal{L} \phi_{1,2}^j \right\rangle_{L^2(\Omega_2, \mathbb{B})}, \quad m, n \in \{1, n_{2,2}\}.
\end{aligned} \tag{7.132}$$

1445 Matrix  $\mathbf{M}_{\partial,2}$  is constructed as in (7.112). The elements of matrices  $\mathbf{B}_{1,\Gamma_2}$ ,  $\mathbf{B}_{1,\Gamma_{12}}$  are computed  
 1446 as

$$\begin{aligned}
B_{1,\Gamma_2}^{ig} &= \left\langle \mathcal{N}_{\partial,1}^{\Gamma_2} \phi_{1,2}^i, \phi_{\partial,2}^g \right\rangle_{L^2(\Gamma_2)}, \quad i \in \{1, n_{1,2}\}, \quad g \in \{1, n_{\partial,2}\}, \\
B_{1,\Gamma_{12}}^{ik} &= \left\langle \mathcal{N}_{\partial,1}^{\Gamma_{12}} \phi_{1,2}^i, \phi_{\partial,12}^k \right\rangle_{L^2(\Gamma_{12})}, \quad k \in \{1, n_{\partial,12}\}.
\end{aligned} \tag{7.133}$$

1447 Systems (7.127), (7.131) are compactly rewritten as

| System (7.127)   | System (7.131)   |
|--|--|
| $ \begin{aligned} \mathbf{M}_{\Omega_1} \dot{\mathbf{e}}_{\Omega_1} &= \mathbf{J}_{\Omega_1} \mathbf{e}_{\Omega_1} + \mathbf{B}_{\Gamma_1}^{\Omega_1} \mathbf{u}_{\partial,1} + \mathbf{B}_{\Gamma_{12}}^{\Omega_1} \mathbf{u}_{\partial,1}^{\Gamma_{12}}, \\ \mathbf{M}_{\partial,1} \mathbf{y}_{\partial,1} &= \mathbf{B}_{\Gamma_1}^{\Omega_1 \top} \mathbf{e}_{\Omega_1}, \\ \mathbf{M}_{\partial,12} \mathbf{y}_{\partial,1}^{\Gamma_{12}} &= \mathbf{B}_{\Gamma_{12}}^{\Omega_1 \top} \mathbf{e}_{\Omega_1}. \end{aligned} \tag{7.134} $ <p>with Hamiltonian <math>H_{d,1} = \frac{1}{2} \mathbf{e}_{\Omega_1}^\top \mathbf{M}_{\Omega_1} \mathbf{e}_{\Omega_1}</math></p> | $ \begin{aligned} \mathbf{M}_{\Omega_2} \dot{\mathbf{e}}_{\Omega_2} &= \mathbf{J}_{\Omega_2} \mathbf{e}_{\Omega_2} + \mathbf{B}_{\Gamma_2}^{\Omega_2} \mathbf{u}_{\partial,2} + \mathbf{B}_{\Gamma_{12}}^{\Omega_2} \mathbf{u}_{\partial,2}^{\Gamma_{12}}, \\ \mathbf{M}_{\partial,2} \mathbf{y}_{\partial,2} &= \mathbf{B}_{\Gamma_2}^{\Omega_2 \top} \mathbf{e}_{\Omega_2}, \\ \mathbf{M}_{\partial,12} \mathbf{y}_{\partial,2}^{\Gamma_{12}} &= \mathbf{B}_{\Gamma_{12}}^{\Omega_2 \top} \mathbf{e}_{\Omega_2}. \end{aligned} \tag{7.135} $ <p>with Hamiltonian <math>H_{d,2} = \frac{1}{2} \mathbf{e}_{\Omega_2}^\top \mathbf{M}_{\Omega_2} \mathbf{e}_{\Omega_2}</math></p> |

1449 To obtain a system with the desired causality, an interconnection is employed to connect  
 1450 the two Systems (7.134), (7.135) along the shared boundary  $\Gamma_{12}$ . Given (7.120), the gyrator  
 1451 interconnection is computed as

$$\begin{aligned}
\mathbf{u}_{\partial,1}^{\Gamma_{12}} &= \pm \mathbf{y}_{\partial,2}^{\Gamma_{12}} = \pm \mathbf{M}_{\partial,12}^{-1} \mathbf{B}_{\Gamma_{12}}^{\Omega_2 \top} \mathbf{e}_{\Omega_2}, \\
\mathbf{u}_{\partial,2}^{\Gamma_{12}} &= \mp \mathbf{y}_{\partial,1}^{\Gamma_{12}} = \mp \mathbf{M}_{\partial,12}^{-1} \mathbf{B}_{\Gamma_{12}}^{\Omega_1 \top} \mathbf{e}_{\Omega_1},
\end{aligned} \tag{7.136}$$

1452 The coupling matrix is then defined by

$$\mathbf{C} := \mathbf{B}_{\Gamma_{12}}^{\Omega_1} \mathbf{M}_{\partial,12}^{-1} \mathbf{B}_{\Gamma_{12}}^{\Omega_2 \top}. \tag{7.137}$$

1453 Plugging Eq. (7.136) into 7.134, 7.135, the final system with mixed causality is obtained

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_{\Omega_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\Omega_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_{\Omega_1} \\ \dot{\mathbf{e}}_{\Omega_2} \end{pmatrix} &= \begin{bmatrix} \mathbf{J}_{\Omega_1} & \pm \mathbf{C} \\ \mp \mathbf{C}^\top & \mathbf{J}_{\Omega_2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{\Omega_1} \\ \mathbf{e}_{\Omega_2} \end{pmatrix} + \begin{bmatrix} \mathbf{B}_{\Gamma_1}^{\Omega_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\Gamma_2}^{\Omega_2} \end{bmatrix} \begin{pmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{pmatrix}, \\ \begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} &= \begin{bmatrix} \mathbf{B}_{\Gamma_1}^{\Omega_1 \top} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\Gamma_2}^{\Omega_2 \top} \end{bmatrix} \begin{pmatrix} \mathbf{e}_{\Omega_1} \\ \mathbf{e}_{\Omega_2} \end{pmatrix}. \end{aligned} \quad (7.138)$$

1454 The total Hamiltonian is the sum

$$H_d = H_{d,1} + H_{d,2} = \frac{1}{2} \mathbf{e}_{\Omega_1}^\top \mathbf{M}_{\Omega_1} \mathbf{e}_{\Omega_1} + \frac{1}{2} \mathbf{e}_{\Omega_2}^\top \mathbf{M}_{\Omega_2} \mathbf{e}_{\Omega_2}. \quad (7.139)$$

1455 So, the power rate is

$$\begin{aligned} \dot{H}_d &= \mathbf{e}_{\Omega_1}^\top \mathbf{M}_{\Omega_1} \dot{\mathbf{e}}_{\Omega_1} + \mathbf{e}_{\Omega_2}^\top \mathbf{M}_{\Omega_2} \dot{\mathbf{e}}_{\Omega_2}, \\ &= \mathbf{e}_{\Omega_1}^\top \mathbf{J}_{\Omega_1} \mathbf{e}_{\Omega_1} + \mathbf{e}_{\Omega_2}^\top \mathbf{J}_{\Omega_2} \mathbf{e}_{\Omega_2} \pm \mathbf{e}_{\Omega_1}^\top \mathbf{C} \mathbf{e}_{\Omega_2} \mp \mathbf{e}_{\Omega_2}^\top \mathbf{C}^\top \mathbf{e}_{\Omega_1} + \mathbf{e}_{\Omega_1}^\top \mathbf{B}_{\Gamma_1}^{\Omega_1} \mathbf{u}_{\partial,1} + \mathbf{e}_{\Omega_2}^\top \mathbf{B}_{\Gamma_2}^{\Omega_2} \mathbf{u}_{\partial,2}, \\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}, \\ &= \hat{\mathbf{y}}_{\partial,1}^\top \mathbf{u}_{\partial,1} + \hat{\mathbf{y}}_{\partial,2}^\top \mathbf{u}_{\partial,2}, \quad \text{where} \quad \hat{\mathbf{y}}_{\partial,1} := \mathbf{M}_{\partial,1} \mathbf{y}_{\partial,1}, \quad \hat{\mathbf{y}}_{\partial,2} := \mathbf{M}_{\partial,2} \mathbf{y}_{\partial,2}. \end{aligned} \quad (7.140)$$

1456 Again this results mimics its corresponding infinite-dimensional (7.106).

1457

1458 This technique allows obtaining a system with the correct causality, but has some draw-  
 1459 backs. Suitable finite elements are required for both kind of discretization detailed in Sec.  
 1460 7.1.1, but the two are not always available (see Remark 9). A rigorous numerical convergence  
 1461 analysis of this technique appears rather involved. Some cases of mixed conditions, in par-  
 1462 ticular conditions on single components of vectors, cannot be handled by this technique. For  
 1463 example, the simply supported condition in beams and plates imposes zero normal compo-  
 1464 nent of the traction at the boundary. Furthermore two different meshes are required and the  
 1465 interconnection has to manipulate carefully the degrees of freedom. This makes the imple-  
 1466 mentation heavier than the Lagrange multiplier solution §7.2.1.

## 1467 7.3 Conclusion

1468 In this chapter a universal discretization method for multi-dimensional pHs has been detailed.  
 1469 The underlying Assumptions 2, 3 are indeed those that characterize the well-posedness of  
 1470 multi-dimensional pHs [Skr19]. For the time being, it has being shown that this technique  
 1471 is capable of constructing a finite-dimensional pHs from an infinite-dimensional one. For  
 1472 this reason, it is a structure-preserving method. The questions of numerical convergence and  
 1473 choice of approximation basis (in this thesis the focus is on the finite element method but  
 1474 spectral methods can be employed as well) are addressed in the next chapter, for the linear  
 1475 case only.



# Convergence numerical study

---

Aristotle maintained that women have fewer teeth than men; although he was twice married, it never occurred to him to verify this statement by examining his wives' mouths.

---

*The Impact of Science on Society*  
Bertrand Russell



He application of the Partitioned Finite Element leads to a finite-dimensional pH systems, that can be discretized using finite elements method. To quantify how well the numerical solution approximates the true one, it is important to estimate the rate of convergence of the finite elements.

In this first part of this chapter, pH plate problems are discretized using mixed finite elements. This means that the divergence operator explicitly appears in the weak formulation. In this second part the discretization is of dual-mixed type [A.90], meaning that the gradient operator comes out in the weak formulation.

Homogeneous boundary conditions will be always considered. This are enforced weakly or strongly depending on the specific formulation under analysis.

## 8.1 Plate problems using known mixed finite elements

First we focused on the Mindlin plate. This problem is a combination of plane wave dynamics and plane elastodynamics. A classical mixed formulation requires  $H^{\text{div}}$  conforming elements both for the wave dynamics [BJT00] and elastodynamics [BJT01, AL14]. To obtain a suitable discretization of the Mindlin problem one has to combine the two. Additional difficulties arising from the symmetry of the stress tensor that can be imposed strongly [BJT01] or weakly [AL14].

We then discuss the Kirchhoff plate problem. For this problem non-conforming finite element approximation have been studied, but for the static case [BR90].

**Notations** The space of all, symmetric and skew-symmetric  $d \times d$  matrices are denoted by  $\mathbb{M}, \mathbb{S}, \mathbb{K}$  respectively. The space of  $\mathbb{R}^d$  vectors is denoted by  $\mathbb{V}$ . The symbol  $\Omega \subset \mathbb{R}^2$  denotes an open connected set. The standard notation  $H^m(\Omega)$  denotes the Sobolev space of square integrable functions with  $m^{\text{th}}$  derivative in  $L^2$  and norm  $\|\cdot\|_m$ . In particular,  $H_0^1(\Omega)$  is the space of weakly derivable functions with vanishing trace. For  $\mathbb{X} \subseteq \mathbb{M}$ , let

$$\begin{aligned} H^{\text{div}}(\Omega, \mathbb{V}) &= \{\mathbf{u} \in L^2(\Omega, \mathbb{V}) \mid \text{div}(\mathbf{u}) \in L^2(\Omega)\}, \\ H^{\text{Div}}(\Omega, \mathbb{X}) &= \{\mathbf{U} \in L^2(\Omega, \mathbb{X}) \mid \text{Div}(\mathbf{U}) \in L^2(\Omega; \mathbb{V})\}, \end{aligned}$$

which are Hilbert spaces with the norm

$$\|\mathbf{u}\|_{\text{div}}^2 = \|\mathbf{u}\|^2 + \|\text{div}(\mathbf{u})\|^2, \quad \|\mathbf{U}\|_{\text{Div}}^2 = \|\mathbf{U}\|^2 + \|\text{Div}(\mathbf{U})\|^2.$$

1502 Let  $X$  be a Hilbert space, and  $t_f$  a positive real number. We denote by  $L^\infty([0, t_f]; X)$  or  
 1503  $L^\infty(X)$  the space of functions  $f : [0, t_f] \rightarrow X$  for which the time-space norm  $\|\cdot\|_{L^\infty([0, t_f]; X)}$   
 1504 satisfies

$$\|f\|_{L^\infty([0, t_f]; X)} = \text{ess sup}_{t \in [0, t_f]} \|f\|_X < \infty.$$

### 1505 8.1.1 Mindlin plate mixed discretization

1506 We consider the weak formulation (7.98), reported in Section §7.1.2. The boundary terms  
 1507 disappears since homogeneous boundary conditions apply. We present first a scheme that  
 1508 enforces the symmetry of the momenta tensor strongly and then a scheme in which the  
 1509 symmetry of the momenta tensor is imposed weakly.

#### 1510 8.1.1.1 Mindlin plate with strongly imposed symmetry

The weak formulation with strongly imposed symmetry seeks

$$\{e_w, \mathbf{e}_\theta, \mathbf{E}_\kappa, \mathbf{e}_\gamma\} \in L^2(\Omega) \times L^2(\Omega, \mathbb{V}) \times H^{\text{Div}}(\Omega, \mathbb{S}) \times H^{\text{div}}(\Omega, \mathbb{V})$$

1511 so that

$$\begin{aligned} \langle v_w, \rho b \dot{e}_w \rangle_{L^2(\Omega)} &= \langle v_w, \text{div } \mathbf{e}_\gamma \rangle_{L^2(\Omega)} + (v_w, f), & \forall v_w \in L^2(\Omega), \\ \langle \mathbf{v}_\theta, I_\theta \dot{\mathbf{e}}_\theta \rangle_{L^2(\Omega, \mathbb{V})} &= \langle \mathbf{v}_\theta, \text{Div } \mathbf{E}_\kappa + \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{V})} + \langle \mathbf{v}_\theta, \boldsymbol{\tau} \rangle_{L^2(\Omega, \mathbb{V})}, & \forall \mathbf{v}_\theta \in L^2(\Omega, \mathbb{V}), \\ \langle \mathbf{V}_\kappa, \mathbf{C}_b \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{S})} &= - \langle \text{Div } \mathbf{V}_\kappa, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{S})}, & \forall \mathbf{V}_\kappa \in H^{\text{Div}}(\Omega, \mathbb{S}), \\ \langle \mathbf{v}_\gamma, C_s \dot{\mathbf{e}}_\gamma \rangle_{L^2(\Omega, \mathbb{V})} &= - \langle \text{div } \mathbf{v}_\gamma, e_w \rangle_{L^2(\Omega)} + \langle \mathbf{v}_\gamma, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{V})}, & \forall \mathbf{v}_\gamma \in H^{\text{div}}(\Omega, \mathbb{V}). \end{aligned} \tag{8.1}$$

1512 The plate thickness is indicated by the  $b$  symbol, to avoid confusion with the average  
 1513 mesh size indicated by  $h$ . A distributed force  $f$  and torque  $\boldsymbol{\tau}$  are considered in order to find  
 1514 a manufactured solution for this problem.



Obtaining stable finite elements that embed the symmetry of the stress tensor for the elastodynamics problem has proven to be a difficult task [AW02]. The easiest construction is the one presented in [BJT01]. This finite element solution can be implemented in FIREDRAKE [RHM<sup>+</sup>17] thanks to the extruded mesh functionality [MBM<sup>+</sup>16]. The main disadvantage is that this scheme requires the domain to be given by a union of rectangles, as the mesh elements have to be square. However, this allows constructing a simple element for the momenta tensor. The polynomial spaces for the discretization are

$$N_k = \{p(x, y) \mid p(x, y) = \sum_{i \leq k, j \leq k} a_{ij} x^i y^j\}.$$

Given a regular mesh  $\mathcal{R}_h$  with square elements  $Q$  the following spaces are introduced as discretization spaces

$$\begin{aligned} L_{h,\text{BJT}}^2(\Omega) &= \{w_h \in L^2(\Omega) \mid \forall Q, w_h|_Q \in N_{k-1}\}, \\ L_{h,\text{BJT}}^2(\Omega, \mathbb{V}) &= \{\boldsymbol{\theta}_h \in L^2(\Omega, \mathbb{V}) \mid \forall Q, \boldsymbol{\theta}_h|_Q \in (N_{k-1})^2\}, \\ H_{h,\text{BJT}}^{\text{Div}}(\Omega, \mathbb{S}) &= \{m_{12} \in H^1(\Omega) \mid \forall Q, m_{12}|_Q \in N_k\} \\ &\quad \cup \{(m_{11}, m_{22}) \in H^{\text{div}}(\Omega, \mathbb{V}) \mid \forall Q, (m_{11}, m_{22})|_Q \in N_k\}, \\ H_{h,\text{BJT}}^{\text{div}}(\Omega, \mathbb{V}) &= \{\mathbf{q}_h \in H^{\text{div}}(\Omega, \mathbb{V}) \mid \forall Q, \mathbf{q}_h|_Q \in N_k\}, \end{aligned} \tag{8.2}$$

where BTJ stands for the initials of the authors in [BJT00, BJT01]. Combining the results of both papers, the following error estimates are conjectured.

**Conjecture 4** (Convergence rate for the BJT element)

*Assuming a smooth solution to problem (8.1), the following error estimates hold*

$$\begin{aligned} \|e_w - e_w^h\|_{L^\infty(L^2)} &\lesssim h^k, & \|\mathbf{E}_\kappa - \mathbf{E}_\kappa^h\|_{L^\infty(L^2)} &\lesssim h^k, \\ \|e_\theta - e_\theta^h\|_{L^\infty(L^2)} &\lesssim h^k, & \|e_\gamma - e_\gamma^h\|_{L^\infty(L^2)} &\lesssim h^k, \end{aligned} \tag{8.3}$$

where the notation  $a \lesssim b$  means  $a \leq Cb$ . The constant  $C$  depends only on the true solution and on the final time.

### 8.1.1.2 Mindlin plate with weakly imposed symmetry

To impose the symmetry of the momenta tensor weakly, we modify the third equation in (8.1). The symmetric gradient can be rewritten as

$$\text{Grad } \boldsymbol{\theta} = \text{grad } \boldsymbol{\theta} - \text{skw}(\text{grad } \boldsymbol{\theta}),$$

1534 where  $\text{skw}(\mathbf{A}) = (\mathbf{A} - \mathbf{A}^\top)/2$  is the skew-symmetric part of matrix  $\mathbf{A}$ . Consider the weak  
1535 form of the third equation in (8.1) before applying the integration by parts

$$\langle \mathbf{V}_\kappa, \mathbf{C}_b \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{M})} = \langle \mathbf{V}_\kappa, \text{Grad } \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{M})}.$$

Introducing the new variable  $\mathbf{E}_r = \text{skw}(\text{grad } \boldsymbol{\theta})$ , then  $\{\mathbf{e}_\theta, \mathbf{E}_\kappa, \mathbf{E}_r\} \in L^2(\Omega, \mathbb{V}) \times H^{\text{Div}}(\Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{K})$  satisfy (remind that  $\mathbf{e}_\theta = \partial_t \boldsymbol{\theta}$ )

$$\begin{aligned} \langle \mathbf{V}_\kappa, \mathbf{C}_b \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{M})} &= \langle \mathbf{V}_\kappa, \text{grad } \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{M})} - \langle \mathbf{V}_\kappa, \dot{\mathbf{E}}_r \rangle_{L^2(\Omega, \mathbb{M})}, \\ &= -\langle \text{Div } \mathbf{V}_\kappa, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{V})} - \langle \mathbf{V}_\kappa, \dot{\mathbf{E}}_r \rangle_{L^2(\Omega, \mathbb{M})}. \end{aligned}$$

The momenta tensor is weakly symmetric if

$$\langle \mathbf{V}_r, \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{M})},$$

or equivalently

$$\langle \mathbf{V}_r, \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{M})}.$$

1536 The weak formulation then consists in finding

$$\{e_w, \mathbf{e}_\theta, \mathbf{E}_\kappa, \mathbf{e}_\gamma, \mathbf{E}_r\} \in L^2(\Omega) \times L^2(\Omega, \mathbb{V}) \times H^{\text{Div}}(\Omega, \mathbb{M}) \times H^{\text{div}}(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{K}).$$

1537 so that

$$\begin{aligned} \langle v_w, \rho b \dot{e}_w \rangle_{L^2(\Omega)} &= \langle v_w, \text{div } \mathbf{e}_\gamma \rangle_{L^2(\Omega)} + (v_w, f), & \forall v_w \in L^2(\Omega), \\ \langle \mathbf{v}_\theta, I_\theta \dot{\mathbf{e}}_\theta \rangle_{L^2(\Omega, \mathbb{V})} &= \langle \mathbf{v}_\theta, \text{Div } \mathbf{E}_\kappa + \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{V})} + \langle \mathbf{v}_\theta, \boldsymbol{\tau} \rangle_{L^2(\Omega, \mathbb{V})}, & \forall \mathbf{v}_\theta \in L^2(\Omega, \mathbb{V}), \\ \langle \mathbf{V}_\kappa, \mathbf{C}_b \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{M})} &= -\langle \text{Div } \mathbf{V}_\kappa, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{V})} - \langle \mathbf{V}_\kappa, \dot{\mathbf{E}}_r \rangle_{L^2(\Omega, \mathbb{M})}, & \forall \mathbf{V}_\kappa \in H^{\text{Div}}(\Omega, \mathbb{M}), \\ \langle \mathbf{v}_\gamma, C_s \dot{\mathbf{e}}_\gamma \rangle_{L^2(\Omega, \mathbb{V})} &= -\langle \text{div } \mathbf{v}_\gamma, e_w \rangle_{L^2(\Omega)} + \langle \mathbf{v}_\gamma, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{V})}, & \forall \mathbf{v}_\gamma \in H^{\text{div}}(\Omega, \mathbb{V}), \\ \langle \mathbf{V}_r, \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{M})} &= 0 & \forall \mathbf{V}_r \in L^2(\Omega, \mathbb{K}). \end{aligned} \tag{8.4}$$

1538 Consider a regular triangulation  $\mathcal{T}_h$  with elements  $T$ . The space of polynomials of order  
1539  $k$  on a mesh cell is denoted by  $P_k$ . The following spaces are used as discretization spaces

$$\begin{aligned} L_{h, \text{AFW}}^2(\Omega) &= \{w_h \in L^2(\Omega) \mid \forall T, w_h|_T \in P_{k-1}\}, \\ L_{h, \text{AFW}}^2(\Omega, \mathbb{V}) &= \{\boldsymbol{\theta}_h \in L^2(\Omega, \mathbb{V}) \mid \forall T, \boldsymbol{\theta}_h|_T \in (P_{k-1})^2\}, \\ H_{h, \text{AFW}}^{\text{Div}}(\Omega, \mathbb{M}) &= \{(m_{11}, m_{12}) \in H^{\text{div}}(\Omega, \mathbb{V}) \mid \forall T, (m_{11}, m_{12})|_T \in \text{BDM}[k]\} \\ &\quad \cup \{(m_{21}, m_{22}) \in H^{\text{div}}(\Omega, \mathbb{V}) \mid \forall T, (m_{21}, m_{22})|_T \in \text{BDM}[k]\}, \\ H_{h, \text{AFW}}^{\text{div}}(\Omega, \mathbb{V}) &= \{\mathbf{q}_h \in H^{\text{div}}(\Omega, \mathbb{V}) \mid \forall T, \mathbf{q}_h|_T \in \text{RT}[k-1]\}, \\ L_{h, \text{AFW}}^2(\Omega, \mathbb{K}) &= \{\mathbf{R}_h \in L^2(\Omega, \mathbb{K}) \mid \forall T, \mathbf{R}_h|_T \in P_{k-1}\}, \end{aligned} \tag{8.5}$$

where  $BDM$  are the Brezzi-Douglas-Marini elements [BDM85] and  $RT$  the Raviart-Thomas elements [RT77]. The acronym AFW stands for Arnold-Falk-Winther, that proposed this kind on discretization for static elasticity [AFW07]. A convergence analysis for the general elastodynamics problem with weak symmetry in the  $L^\infty(L^2)$  norm is detailed [AL14]. A convergence study for the wave equation with mixed finite elements in the  $L^\infty(L^2)$  is presented in [Gev88]. Combining the results of the two, the following error estimates are conjectured:

**Conjecture 5** (Rate of convergence for the AFW element)

*Assuming a smooth solution to problem (8.4), the following error estimates hold*

$$\begin{aligned} \|e_w - e_w^h\|_{L^\infty(L^2)} &\lesssim h^k, & \|\mathbf{E}_\kappa - \mathbf{E}_\kappa^h\|_{L^\infty(L^2)} &\lesssim h^k, & \|\mathbf{E}_r - \mathbf{E}_r^h\|_{L^\infty(L^2)} &\lesssim h^k, \\ \|\mathbf{e}_\theta - \mathbf{e}_\theta^h\|_{L^\infty(L^2)} &\lesssim h^k, & \|\mathbf{e}_\gamma - \mathbf{e}_\gamma^h\|_{L^\infty(L^2)} &\lesssim h^k, \end{aligned} \quad (8.6)$$

### 8.1.2 The Hellan-Herrmann-Johnson scheme for the Kirchhoff plate

For the Kirchhoff plate, the Hellan-Herrmann-Johnson scheme [Hel67, Her67, Joh73] (HHJ) can be used to obtain a structure-preserving discretization. Given the non conforming nature of this scheme, it is necessary to first introduce the discrete functional spaces and state the problem directly in discrete form. The illustration of the method follows closely [AW19]. The vertical displacement is approximated using continuous Lagrange polynomials, while the momenta tensor is discretized using the HHJ element

$$\begin{aligned} W_h &= \{w_h \in H_0^1(\Omega) \mid \forall T, w_h|_T \in P_k\}, \\ S_h &= \{\mathbf{M}_h \in L^2(\Omega, \mathbb{S}) \mid \forall T, \mathbf{M}_h|_T \in (P_{k-1})_{\text{sym}}^{2 \times 2}, \\ &\quad \mathbf{M}_h \text{ is normal-normal continuous across elements}\}. \end{aligned} \quad (8.7)$$

The normal to normal continuity means that if two triangles  $T_1, T_2$  share a common edge  $E$  then  $\mathbf{n}^\top(\mathbf{M}_h|_{T_1})\mathbf{n} = \mathbf{n}^\top(\mathbf{M}_h|_{T_2})\mathbf{n}$  on  $E$ . Taking system (5.35) and multiplying the first equation by  $v_w \in W_h$  and integrating over a triangle

$$\begin{aligned} -\langle v_w, \text{div Div } \mathbf{E}_\kappa \rangle_{L^2(T)} &= \langle \nabla v_w, \text{Div } \mathbf{E}_\kappa \rangle_{L^2(T, \mathbb{V})} - \langle v_w, \text{Div } \mathbf{E}_\kappa \cdot \mathbf{n} \rangle_{L^2(\partial T)}, \\ &= -\langle \text{Hess } v_w, \mathbf{E}_\kappa \rangle_{L^2(T, \mathbb{S})} - \langle v_w, \text{Div } \mathbf{E}_\kappa \cdot \mathbf{n} \rangle_{L^2(\partial T)}, \\ &\quad + \langle \partial_s v_w, \mathbf{s}^\top \mathbf{E}_\kappa \mathbf{n} \rangle_{L^2(\partial T)} + \langle \partial_n v_w, \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n} \rangle_{L^2(\partial T)}. \end{aligned}$$

A double integration by parts is applied to get the final equation. For the last term a summation over all triangles provides

$$\sum_{T \in \mathcal{T}_h} \langle \partial_n v_w, \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n} \rangle_{L^2(\partial T)} = \sum_{E \in \mathcal{E}_h} \langle \llbracket \partial_n v_w \rrbracket, \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n} \rangle_{L^2(E)},$$

where  $\mathcal{E}_h$  is the set of all edges belonging to the mesh and  $\llbracket a \rrbracket = a|_{T_1} + a|_{T_2}$  denotes the jump of a function across shared edges. For a boundary edge it is simply the value of the function.

For the other terms, it holds

$$\langle v_w, \text{Div } \mathbf{E}_\kappa \cdot \mathbf{n} \rangle_{L^2(\partial T)} = 0, \quad \langle \partial_s v_w, \mathbf{s}^\top \mathbf{E}_\kappa \mathbf{n} \rangle_{L^2(\partial T)} = 0,$$

since  $v_w$  is continuous across the edge boundaries and the normal switches sign. We are now in a position to state the final weak form. Given the bilinear form

$$d_h(v_w, \mathbf{E}_\kappa) := - \sum_{T \in \mathcal{T}_h} \langle \text{Hess } v_w, \mathbf{E}_\kappa \rangle_{L^2(T, \mathbb{S})} + \sum_{E \in \mathcal{E}_h} \langle \llbracket \partial_n v_w \rrbracket, \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n} \rangle_{L^2(E)},$$

find  $(e_w, \mathbf{E}_\kappa) \in W_h \times S_h$  such that

$$\begin{aligned} \langle v_w, \rho b \dot{e}_w \rangle_{L^2(\Omega)} &= +d_h(v_w, \mathbf{E}_\kappa) + \langle v_w, f \rangle_{L^2(\Omega)}, & v_w &\in W_h, \\ \langle \mathbf{V}_\kappa, \mathbf{C}_b \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{S})} &= -d_h(e_w, \mathbf{V}_\kappa), & \mathbf{V}_\kappa &\in S_h. \end{aligned} \quad (8.8)$$

For the associated static problem, under the hypothesis of smooth solutions, optimal convergence of order  $O(k)$  for  $w \in H^1$  and  $\mathbf{M} \in L^2$  has been established. So, it is natural to conjecture the following result for the dynamic problem:

#### Conjecture 6

Assuming a smooth solution for problem (8.8), the following error estimates hold

$$\|e_w - e_w^h\|_{L^\infty(H^1)} \lesssim h^k, \quad \|\mathbf{E}_\kappa - \mathbf{E}_\kappa^h\|_{L^\infty(L^2)} \lesssim h^k. \quad (8.9)$$

## 8.2 Dual mixed discretization of plate problems

In this section the discretization of the Kirchhoff and Mindlin plates, together with the Euler-Bernoulli beam

### 8.2.1 Dual mixed discretization of the Mindlin plate

First of all we construct a family of finite elements capable of discretizing problem (7.90), that seeks

$$\{e_w, \mathbf{e}_\theta, \mathbf{E}_\kappa, \mathbf{e}_\gamma\} \in H^1(\Omega) \times H^{\text{Grad}}(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{V})$$

so that

$$\begin{aligned} \langle v_w, \rho h \partial_t e_w \rangle_{L^2(\Omega)} &= - \langle \text{grad } v_w, \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)}, & \forall v_w &\in H^1(\Omega), \\ \langle \mathbf{v}_\theta, I_\theta \partial_t \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)} &= - \langle \text{Grad } \mathbf{v}_\theta, \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}})} + \langle \mathbf{v}_\theta, \mathbf{e}_\gamma \rangle_{L^2(\Omega)}, & \forall \mathbf{v}_\theta &\in H^{\text{Grad}}(\Omega, \mathbb{V}), \\ \langle \mathbf{V}_\kappa, \mathbf{C}_b \partial_t \mathbf{E}_\kappa \rangle_{L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}})} &= \langle \mathbf{V}_\kappa, \text{Grad } \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}})}, & \forall \mathbf{V}_\kappa &\in L^2(\Omega, \mathbb{S}), \\ \langle \mathbf{v}_\gamma, C_s \partial_t \mathbf{e}_\gamma \rangle_{L^2(\Omega, \mathbb{R}^2)} &= \langle \mathbf{v}_\gamma, \text{grad } e_w \rangle_{L^2(\Omega, \mathbb{R}^2)} - \langle \mathbf{v}_\gamma, \mathbf{e}_\theta \rangle_{L^2(\Omega, \mathbb{R}^2)}, & \forall \mathbf{v}_\gamma &\in L^2(\Omega, \mathbb{V}). \end{aligned} \quad (8.10)$$

1572 Consider a regular triangulation  $\mathcal{T}_h$  with elements  $T$ . The space of polynomials of order  
 1573  $k$  on a mesh cell is denoted by  $P_k$ . The following family of finite elements is conforming to  
 1574 the weak formulation (7.90)

$$\begin{aligned} H_h^1(\Omega) &= \{w_h \in H^1(\Omega) \mid \forall T, w_h|_T \in P_k\}, \\ H_h^{\text{Grad}}(\Omega, \mathbb{R}^2) &= \{\boldsymbol{\theta}_h \in H^{\text{Grad}}(\Omega, \mathbb{R}^2) \mid \forall T, \boldsymbol{\theta}_h|_T \in (P_k)^2\}, \\ L_h^2(\Omega, \mathbb{S}) &= \{\mathbf{M}_h \in L^2(\Omega, \mathbb{S}) \mid \forall T, \mathbf{M}_h|_T \in (P_{k-1})_{\text{sym}}^{2 \times 2}\}, \\ L_h^2(\Omega, \mathbb{R}^2) &= \{\mathbf{q}_h \in L^2(\Omega, \mathbb{R}^2) \mid \forall T, \mathbf{q}_h|_T \in (P_{k-1})^2\}, \end{aligned} \quad (8.11)$$

1575 To approximate spaces  $H_h^1(\Omega)$ ,  $H_h^{\text{Grad}}(\Omega, \mathbb{R}^2)$  Lagrange polynomials of order  $k$  are selected.  
 1576 For spaces  $L_h^2(\Omega, \mathbb{S})$ ,  $L_h^2(\Omega, \mathbb{R}^2)$  Discontinuous Galerkin polynomials of order  $k-1$  are employed.  
 1577 We use the acronym CGDG to denote this combination of elements

$$\text{CGDG} = \text{CG} \times \text{CG}(\mathbb{R}^2) \times \text{DG}(\mathbb{S}) \times \text{DG}(\mathbb{R}^2)$$

1578 This selection of finite elements can be seen as a standard discretization of the problem  
 1579 combined with a reduced integration of the stress tensor. For this reason, the following  
 1580 conjecture on the error estimates is proposed.

#### 1581 Conjecture 7

1582 *Assuming a smooth solution to problem (7.90), the following error estimates hold*

$$\begin{aligned} \|e_w - e_w^h\|_{L^\infty(H^1(\Omega))} &\lesssim h^k, \quad \|\mathbf{E}_\kappa - \mathbf{E}_\kappa^h\|_{L^\infty(L^2(\Omega))} \lesssim h^k, \\ \|e_\theta - e_\theta^h\|_{L^\infty(H^{\text{Grad}}(\Omega, \mathbb{R}^2))} &\lesssim h^k, \quad \|e_\gamma - e_\gamma^h\|_{L^\infty(L^2(\Omega, \mathbb{S}))} \lesssim h^k, \end{aligned} \quad (8.12)$$

1583 where the notation  $a \lesssim b$  means  $a \leq Cb$ . The constant  $C$  depends only on the true solution  
 1584 and on the final time.

### 1585 8.3 Numerical experiments

In this section numerical test cases are used to verify the conjectured orders of convergence for the two problems. Upon discretization (cf. Section §7.1.2), systems (8.1), (8.4) and (7.90) assumes the form

$$\underbrace{\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{D} \\ -\mathbf{D}^\top & \mathbf{0} \end{bmatrix}}_{\mathbf{J}} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}.$$

The mass matrix  $\mathbf{M}$  is symmetric and positive definite. In case of weak enforcement of the symmetry (8.4) the final system reads

$$\underbrace{\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{A}_\lambda^\top \\ \mathbf{0} & \mathbf{A}_\lambda & \mathbf{0} \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\lambda} \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{D} & \mathbf{0} \\ -\mathbf{D}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{J}} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \lambda \end{pmatrix} + \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

1586 where  $\mathbf{A}_\lambda$  is the matrix obtained by discretization of  $\langle \mathbf{V}_r, \dot{\mathbf{E}}_\kappa \rangle_{L^2(\Omega, \mathbb{M})}$ .  
 1587 Because of the presence of the Lagrange multiplier, the mass matrix  $\mathbf{M}$  is symmetric and  
 1588 indefinite, giving rise to a saddle point problem. The numerical solution of this kind of prob-  
 1589 lems is notoriously much harder than that of positive definite ones [BGL05]. The FIREDRAKE  
 1590 library [RHM<sup>+</sup>17] is used to generate the matrices. To integrate the equations in time a  
 1591 Crank-Nicholson scheme has been used, for all simulations. The time step is set to  $\Delta t = h/10$   
 1592 to have a lower impact of the time discretization error with respect to the spatial error. The  
 1593 final time is set to one  $t_f = 1[\text{s}]$  for all simulations. To compute the  $L^\infty(X)$  space-time  
 1594 dependent norm the discrete norm  $L_{\Delta t}^\infty(X)$  is used

$$\|\cdot\|_{L^\infty(X)} \approx \|\cdot\|_{L_{\Delta t}^\infty(X)} = \max_{t \in t_i} \|\cdot\|_X,$$

1595 where  $t_i$  are the discrete simulation instants.

### 1596 8.3.1 Numerical test for the Mindlin plate

1597 To validate the method first we test a finite element combinations on an analytic solution.  
 1598 Constructing an analytical solution for a vibrating Mindlin plate is far from trivial. Therefore,  
 1599 the solution for the static case [BadVMR13] is exploited.

1600

Consider a distributed static force given by

$$\begin{aligned} f_s(x, y) = & \frac{E_Y}{12(1-\nu^2)} \{12y(y-1)(5x^2-5x+1) \\ & \times [2y^2(y-1)2 + x(x-1)(5y^2-5y+1)] + 12x(x-1) \\ & \times (5y^2-5y+1)[2x^2(x-1)2 + y(y-1)(5x^2-5x+1)]\}. \end{aligned}$$

The static displacement and rotation are given by

$$\begin{aligned} w_s(x, y) = & \frac{1}{3}x^3(x-1)^3y^3(y-1)^3 - \frac{2b^2}{5(1-\nu)}[y^3(y-1)^3x(x-1)(5x^2-5x+1). \\ \theta_s(x, y) = & \begin{pmatrix} y^3(y-1)^3 x^2(x-1)^2(2x-1) \\ x^3(x-1)^3 y^2(y-1)^2(2y-1) \end{pmatrix} \end{aligned}$$

1601 The static solution solves the following problem defined on the square domain  $\Omega = (0, 1) \times$

(0, 1) under clamped boundary condition:

$$\begin{aligned} 0 &= \operatorname{div} \mathbf{q}_s + f_s, & \mathcal{D}_b^{-1} \mathbf{M}_s &= \operatorname{Grad} \boldsymbol{\theta}_s, & w_s|_{\partial\Omega} &= 0, \\ 0 &= \operatorname{Div} \mathbf{M}_s + \mathbf{q}_s, & D_s^{-1} \mathbf{q}_s &= \operatorname{grad} w_s - \boldsymbol{\theta}_s, & \boldsymbol{\theta}_s|_{\partial\Omega} &= 0. \end{aligned} \quad (8.13)$$

Given the linear nature of the system a solution for the dynamic problem is found by multiplying the static solution by a time dependent term. For simplicity a sinus function is chosen

$$w_d(x, y, t) = w_s(x, y) \sin(t), \quad \boldsymbol{\theta}_d(x, y, t) = \boldsymbol{\theta}_s(x, y) \sin(t).$$

Appropriate forcing terms have to be introduced to compensate the inertial accelerations. The force and torque in the dynamical case become

$$f_d = f_s \sin(t) + \rho b \partial_{tt} w_d, \quad \boldsymbol{\tau}_d = \frac{\rho b^3}{12} \partial_{tt} \boldsymbol{\theta}_d.$$

For the port-Hamiltonian system the unknowns are the linear and angular velocities, the momenta tensor and the shear force. The exact solution and boundary conditions are thus given by

$$\begin{aligned} e_w^{\text{ex}} &= w_s(x, y) \cos(t), & \mathbf{E}_\kappa^{\text{ex}} &= \mathcal{D}_b \operatorname{Grad} \boldsymbol{\theta}_d, & e_w^{\text{ex}}|_{\partial\Omega} &= 0, \\ e_\theta^{\text{ex}} &= \boldsymbol{\theta}_s(x, y) \cos(t), & \mathbf{e}_\gamma^{\text{ex}} &= D_s(\operatorname{grad} w_d - \boldsymbol{\theta}_d), & e_\theta^{\text{ex}}|_{\partial\Omega} &= 0. \end{aligned} \quad (8.14)$$

Variables  $(e_w^{\text{ex}}, e_\theta^{\text{ex}}, \mathbf{E}_\kappa^{\text{ex}}, \mathbf{e}_\gamma^{\text{ex}})$  under excitations  $(f_d, \boldsymbol{\tau}_d)$  solve problem (7.86a). The solution being smooth, the conjectures 4 and 5 should hold. The numerical values of the physical parameters are reported in Table 8.1.

| Plate parameters |                        |       |                 |         |
|------------------|------------------------|-------|-----------------|---------|
| $E$              | $\rho$                 | $\nu$ | $K_{\text{sh}}$ | $b$     |
| 1 [Pa]           | 1 [kg/m <sup>3</sup> ] | 0.3   | 5/6             | 0.1 [m] |

Table 8.1: Physical parameters for the Mindlin plate.

**Results for the mixed strong symmetry formulation (BTJ elements (8.2))** The weak form (8.1) and its corresponding finite elements (8.2) was implemented using FIREDRAKE extruded mesh functionality [MBM<sup>+</sup>16]. A direct solver based on an LU preconditioner is used. In Fig. 8.1 and Tables 8.2, 8.3, 8.4 the errors for  $(e_w, e_\theta, \mathbf{E}_\kappa, \mathbf{e}_\gamma)$  are reported. As one can notice, the conjectured error estimates (8.3) are respected for all variables.

**Results for the mixed weak symmetry formulation (AFW elements (8.5))** Formulation (8.4) and its element (8.5) are considered here. A direct solver failed for high order cases (i.e.  $k = 3$ ). For this reason a generalized minimal residual method is used with restart number of iterations equal to 100. In Fig. 8.2 and Tables 8.5, 8.6, 8.7 the errors for variables  $(e_w, e_\theta, \mathbf{E}_\kappa, \mathbf{e}_\gamma)$  are reported. The errors for  $(e_w, e_\theta, \mathbf{e}_\gamma)$  respect the conjectured result (8.6). Variable  $\mathbf{E}_\kappa$  exhibit a superconvergence phenomenon for the case  $k = 1$ . In [AL14] no numerical study was carried out for the case  $k = 1$ . The BDM elements might be responsible

---

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error                                       | Order | Error   | Order | Error                                       | Order |
| 4             | 1.62e-05                          | —     | 1.51e-04                                    | —     | 4.89e-08  | —     | 5.83e-07                                    | —     |
| 8             | 6.52e-06                          | 1.31  | 4.59e-05                                    | 1.71  | 1.45e-08  | 1.75  | 2.01e-07                                    | 1.53  |
| 16            | 3.28e-06                          | 0.98  | 2.17e-05                                    | 1.07  | 5.69e-09  | 1.34  | 9.41e-08                                    | 1.09  |
| 32            | 1.64e-06                          | 0.99  | 1.07e-05                                    | 1.01  | 2.63e-09  | 1.10  | 4.64e-08                                    | 1.02  |
| 64            | 8.24e-07                          | 0.99  | 5.39e-06                                    | 1.00  | 1.29e-09  | 1.02  | 2.31e-08                                    | 1.00  |

---

Table 8.2: Mindlin plate convergence result for the BJT scheme  $k = 1$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error                                       | Order | Error   | Order | Error                                       | Order |
| 4             | 8.05e-06                          | —     | 7.22e-05                                    | —     | 1.72e-08  | —     | 2.42e-07                                    | —     |
| 8             | 2.12e-06                          | 1.92  | 1.87e-05                                    | 1.94  | 4.42e-09  | 1.96  | 6.06e-08                                    | 2.00  |
| 16            | 5.42e-07                          | 1.96  | 4.09e-06                                    | 2.19  | 1.14e-09  | 1.95  | 1.43e-08                                    | 2.07  |
| 32            | 1.36e-07                          | 1.99  | 1.04e-06                                    | 1.97  | 2.89e-10  | 1.97  | 3.56e-09                                    | 2.00  |
| 64            | 3.41e-08                          | 1.99  | 2.62e-07                                    | 1.99  | 7.26e-11  | 1.99  | 8.88e-10                                    | 2.00  |

---

Table 8.3: Mindlin plate convergence result for the BJT scheme  $k = 2$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error                                       | Order | Error   | Order | Error                                       | Order |
| 2             | 6.98e-07                          | —     | 1.42e-05                                    | —     | 1.54e-09  | —     | 3.31e-08                                    | —     |
| 4             | 1.09e-07                          | 2.67  | 2.14e-06                                    | 2.72  | 2.31e-10  | 2.73  | 4.61e-09                                    | 2.84  |
| 8             | 1.44e-08                          | 2.91  | 2.29e-07                                    | 3.22  | 2.42e-11  | 3.25  | 6.36e-10                                    | 2.85  |
| 16            | 1.83e-09                          | 2.97  | 2.05e-08                                    | 3.19  | 2.62e-12  | 3.20  | 8.44e-11                                    | 2.91  |
| 32            | 2.30e-10                          | 2.99  | 2.94e-09                                    | 3.08  | 3.00e-13  | 3.12  | 1.07e-11                                    | 2.97  |

---

Table 8.4: Mindlin plate convergence result for the BJT scheme  $k = 3$ .



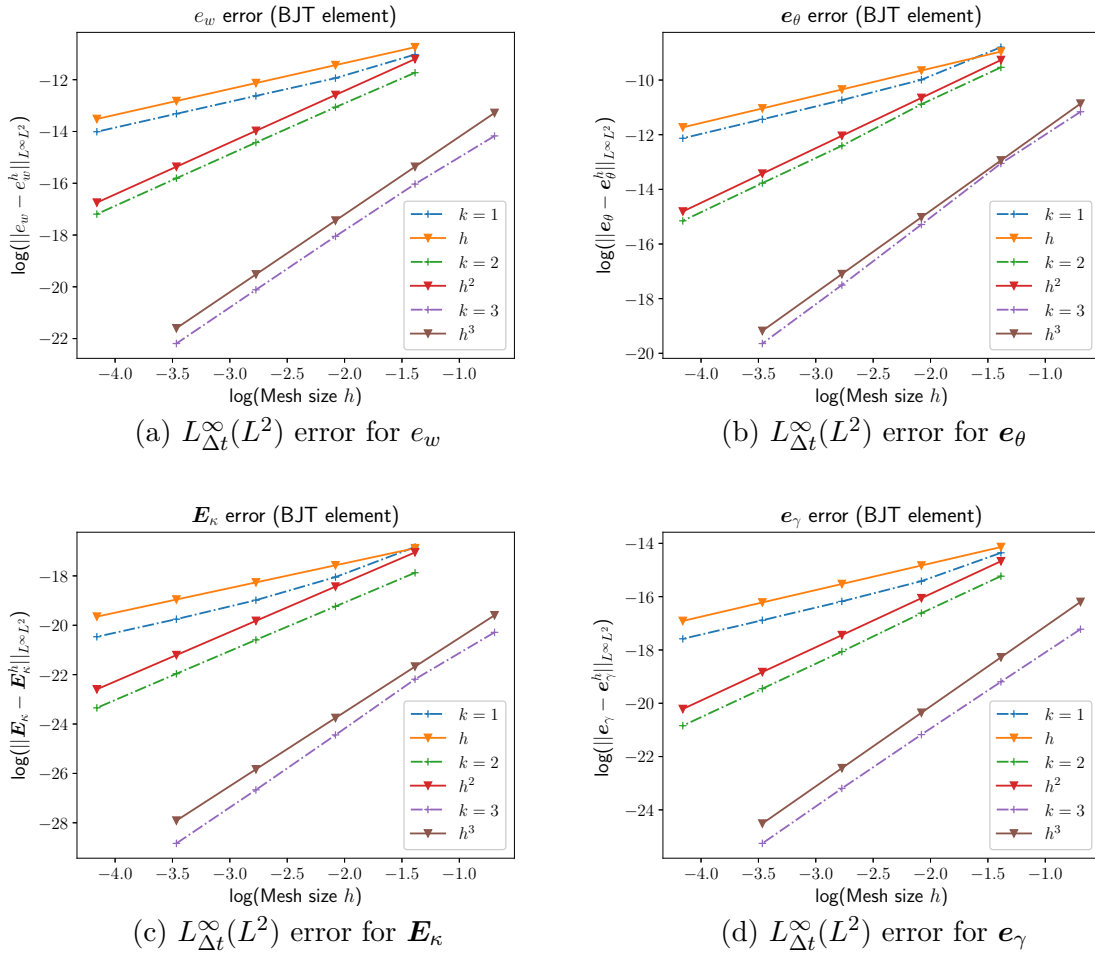


Figure 8.1: Error for the Mindlin plate using the BJT elements

for such superconvergence. The convergence order of  $(\mathbf{E}_\kappa, \mathbf{e}_\gamma)$  deteriorates for  $k = 3$  for the finest mesh. This must be linked to errors due to the underlying large saddle-point problem. Indeed in [AL14] an hybridization method is used to transform the saddle-point problem into a positive definite one. The results for the Lagrange multiplier is reported in Fig. 8.2e and Table 8.8. For this variable an order 2 of convergence is observed for all cases.

**Results for dual mixed formulation (CGDG elements (8.11))** A direct solver based on an LU preconditioner is used. In Fig. 8.3 and Tables 8.9 the errors are reported. Conjecture 7 is verified for this test.

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error                                       | Order | Error   | Order | Error                                       | Order |
| 4             | 1.05e-05                          | —     | 7.96e-05                                    | —     | 3.75e-08  | —     | 6.54e-07                                    | —     |
| 8             | 5.33e-06                          | 0.98  | 3.53e-05                                    | 1.17  | 1.15e-08  | 1.70  | 3.73e-07                                    | 0.80  |
| 16            | 2.68e-06                          | 0.99  | 1.75e-05                                    | 1.00  | 3.02e-09  | 1.92  | 1.92e-07                                    | 0.95  |
| 32            | 1.34e-06                          | 0.99  | 8.80e-06                                    | 0.99  | 7.71e-10  | 1.97  | 9.72e-08                                    | 0.98  |

Table 8.5: Mindlin plate convergence result for the AFW scheme  $k = 1$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error                                       | Order | Error   | Order | Error                                       | Order |
| 4             | 8.43e-06                          | —     | 8.10e-05                                    | —     | 1.80e-08  | —     | 2.68e-07                                    | —     |
| 8             | 2.28e-06                          | 1.88  | 1.82e-05                                    | 2.15  | 4.79e-09  | 1.90  | 6.99e-08                                    | 1.93  |
| 16            | 5.85e-07                          | 1.96  | 4.41e-06                                    | 2.04  | 1.22e-09  | 1.96  | 1.75e-08                                    | 1.99  |
| 32            | 1.47e-07                          | 1.98  | 1.12e-06                                    | 1.97  | 3.03e-10  | 2.01  | 4.47e-09                                    | 1.97  |

Table 8.6: Mindlin plate convergence result for the AFW scheme  $k = 2$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error                                       | Order | Error   | Order | Error                                       | Order |
| 2             | 1.11e-06                          | —     | 1.63e-05                                    | —     | 2.14e-09  | —     | 4.63e-08                                    | —     |
| 4             | 1.63e-07                          | 2.77  | 2.56e-06                                    | 2.67  | 2.61e-10  | 3.04  | 6.96e-09                                    | 2.73  |
| 8             | 2.13e-08                          | 2.93  | 2.63e-07                                    | 3.28  | 2.42e-11  | 3.42  | 9.90e-10                                    | 2.81  |
| 16            | 2.93e-09                          | 2.86  | 4.24e-08                                    | 2.63  | 8.99e-12  | 1.43  | 3.64e-10                                    | 1.44  |

Table 8.7: Mindlin plate convergence result for the AFW scheme  $k = 3$ .

| $\ \mathbf{E}_r - \mathbf{E}_r^h\ _{L^\infty(L^2)}$ |          |       |          |       |          |       |
|---|----------|-------|----------|-------|----------|-------|
| $\frac{1}{h}$                                       | $k = 1$  |       | $k = 2$  |       | $k = 3$  |       |
|   | Error    | Order | Error    | Order | Error    | Order |
| 4   | 2.45e-09 | —     | 1.07e-09 | —     | 1.57e-09 | —     |
| 8   | 4.98e-10 | 2.29  | 2.48e-10 | 2.11  | 3.52e-10 | 2.15  |
| 16  | 1.26e-10 | 1.97  | 6.11e-11 | 2.02  | 8.67e-11 | 2.02  |
| 32  | 3.19e-11 | 1.98  | 1.52e-11 | 1.99  | 2.16e-11 | 2.00  |

Table 8.8: Mindlin plate convergence result for the Lagrange multiplier  $\mathbf{E}_r$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(H^1)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(H^{\text{Grad}})}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error   | Order | Error   | Order | Error                                       | Order |
| 8             | 7.30e-05                          | —     | 5.52e-04  | —     | 3.99e-08  | —     | 9.02e-07                                    | —     |
| 16            | 3.13e-05                          | 1.22  | 2.26e-04  | 1.28  | 1.88e-08  | 1.08  | 5.47e-07                                    | 0.72  |
| 32            | 1.57e-05                          | 0.99  | 1.11e-04  | 1.02  | 8.84e-09  | 1.09  | 2.94e-07                                    | 0.89  |
| 64            | 7.87e-06                          | 0.99  | 5.57e-05  | 0.99  | 4.31e-09  | 1.03  | 1.50e-07                                    | 0.97  |
| 128           | 3.94e-06                          | 0.99  | 2.78e-05  | 0.99  | 2.14e-09  | 1.01  | 7.55e-08                                    | 0.99  |

Table 8.9: Mindlin plate convergence result for the CGDG scheme  $k = 1$ .

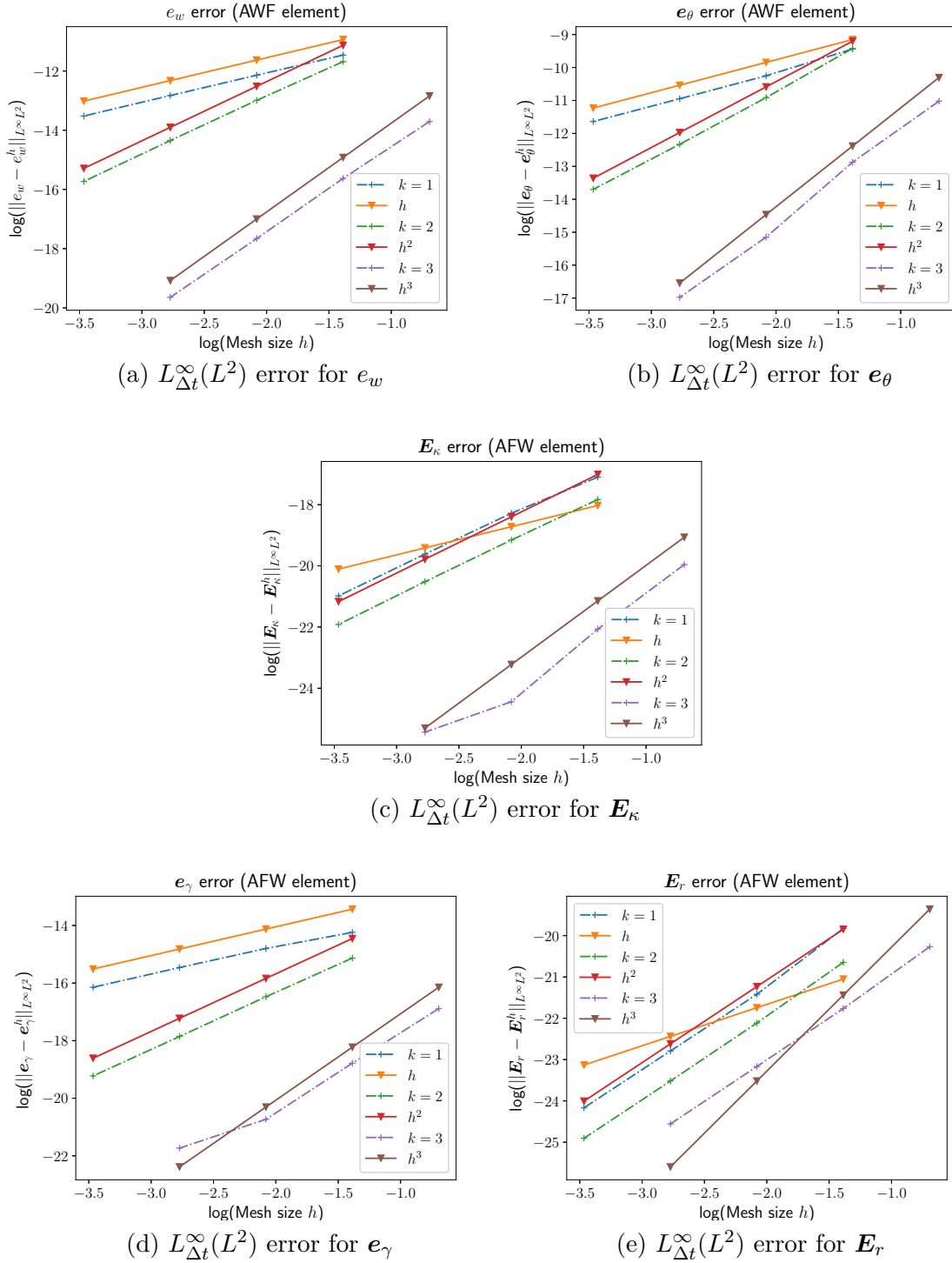


Figure 8.2: Error for the Mindlin plate using the AFW elements

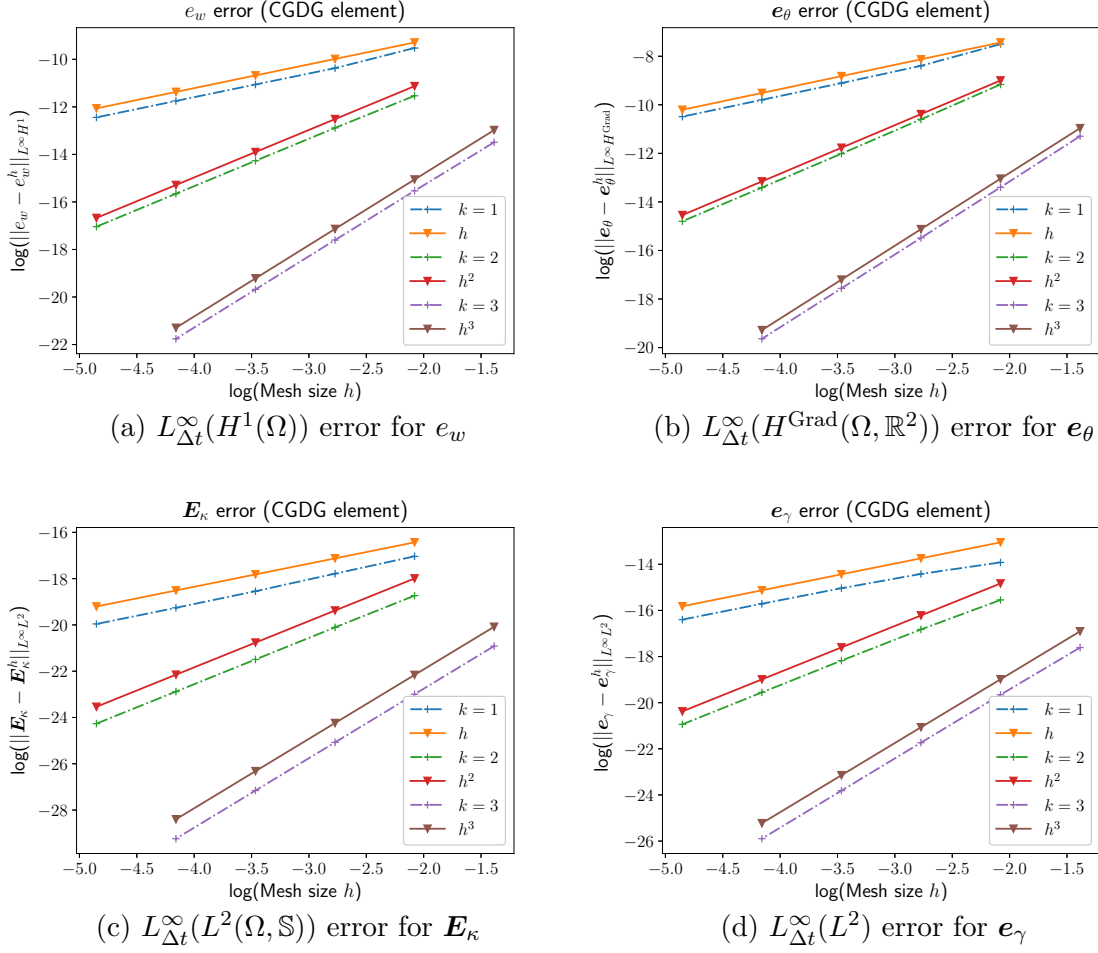


Figure 8.3: Error for the Mindlin plate using the CGDG elements

### 8.3.2 Numerical test for the Kirchhoff plate

The weak form (8.8) and the finite elements (8.7) are considered. The HHJ elements were included in FENICS and FIREDRAKE thanks to the contribution of Lizao Li [Li18]. Two numerical tests are performed to verify these elements. Both tests are solved using a direct solver with an LU preconditioner.

#### 8.3.2.1 Simply supported test

An analytical solution for simply supported Kirchhoff plates is readily available. Consider the following solution of problem (5.20) under simply supported conditions on a square unitary

1641 domain  $\Omega = (0, 1) \times (0, 1)$

$$w^{\text{ex}}(x, y, t) = \sin(\pi x) \sin(\pi y) \sin(t), \quad (x, y) \in \Omega.$$

1642 The forcing term is given by

$$f = (4D\pi^4 - \rho b) \sin(\pi x) \sin(\pi y) \sin(t), \quad D = \frac{E_Y b^3}{12(1 - \nu^2)}.$$

1643 The corresponding variables in the port-Hamiltonian frame work are

$$e_w^{\text{ex}} = \partial_t w^{\text{ex}}, \quad \mathbf{E}_\kappa^{\text{ex}} = \mathcal{D} \nabla^2 w^{\text{ex}},$$

1644 under simply supported boundary conditions

$$e_w|_{\partial\Omega} = 0, \quad \mathbf{E}_\kappa : (\mathbf{n} \otimes \mathbf{n})|_{\partial\Omega} = 0.$$

1645 Variables  $(e_w^{\text{ex}}, \mathbf{E}_\kappa^{\text{ex}})$  under excitation  $f$  solve problem (5.35). The physical parameters used  
1646 in simulation are reported in Table 8.12.

1647 **Results for the HHJ elements (8.7)** Results are shown in Fig. 8.4 and Tables 8.13, 8.14  
1648 and 8.15. The conjectured error estimates are respected.

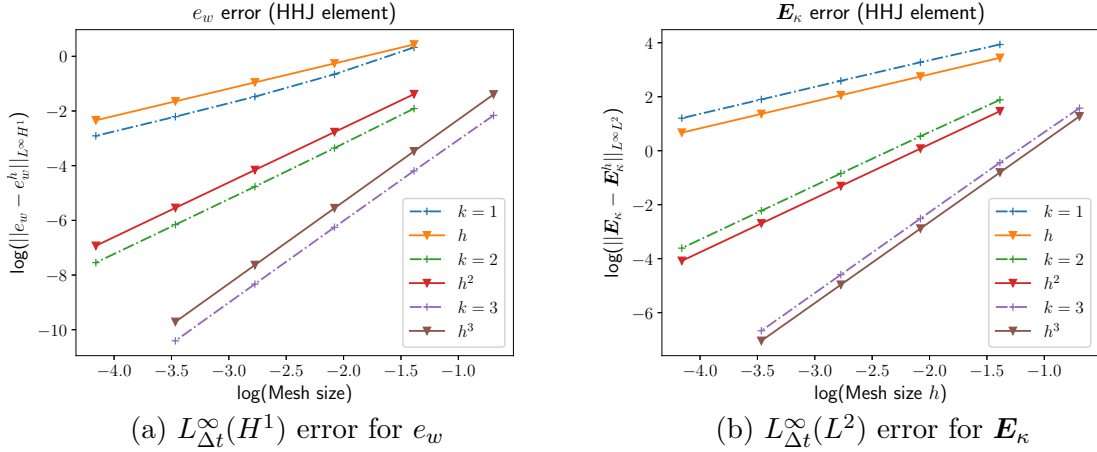


Figure 8.4: Error for the simply supported Kirchhoff plate (HHJ elements)

1649 **Results for the dual mixed formulation**

---

1650 **8.3.2.2 Mixed boundary conditions (CSFS)**

We retrieve the manufactured solution for the static case from [Raf18]. Consider a square plate  $\Omega = (-1, 1) \times (-1, 1)$  with simply supported top and bottom boundary, clamped left boundary and free right boundary. The stiffness tensor is the identity

$$\mathcal{D}_b = \text{Id}.$$

The density  $\rho$  and thickness  $b$  are the same as in 8.12. The static load is given by

$$f_s = 4\pi \sin(\pi x) \sin(\pi y).$$

The exact static solution is given by

$$w_s(x, y) = [(c_1 + c_2 x) \cosh(\pi x) + (c_3 + c_4 x) \sinh(\pi x) + \sin(\pi x)] \sin(\pi y).$$

The coefficient are then computed depending on the boundary conditions. For the considered case (CSFS) it is obtained

$$\begin{aligned} c_1 &= -2 \frac{\sinh(\pi) - 3 \sinh(3\pi) + \pi[4\pi \sinh(\pi) + 7 \cosh(\pi) - 3 \cosh(3\pi)]}{5 + 8\pi^2 + 3 \cosh(4\pi)}, \\ c_2 &= -\frac{8\pi[2\pi \sinh(\pi) + \cosh(\pi)]}{5 + 8\pi^2 + 3 \cosh(4\pi)}, \\ c_3 &= \frac{10 \cosh(\pi) + 6 \cosh(\pi) + 16\pi[\sinh(\pi) + \pi \cosh(\pi)]}{5 + 8\pi^2 + 3 \cosh(4\pi)}, \\ c_4 &= \frac{2\pi(5 \sinh(\pi) - 3 \sinh(3\pi) + 4\pi \cosh(\pi))}{5 + 8\pi^2 + 3 \cosh(4\pi)} \end{aligned}$$

The dynamical solution is constructed as in paragraph ??, meaning that a the static solution is multiplied by a sinusoidal function in time

$$w_d(x, y) = w_s(x, y) \sin(t).$$

The dynamical force is then given by

$$f_d(x, y, t) = f_s(x, y) \sin(t) + \rho b \partial_{tt} w_d$$

1651 For the port-Hamiltonian system the exact solution are thus given by

$$e_w^{\text{ex}} = w_s(x, y) \cos(t), \quad E_\kappa^{\text{ex}} = \mathcal{D}_b \text{ Grad } \theta_d. \quad (8.15)$$


---

The boundary conditions read

$$\begin{array}{cccc}
 C & S & F & S \\
 e_w^{\text{ex}}|_{x=-1} = 0, & e_w^{\text{ex}}|_{y=-1} = 0, & \partial_x E_{\kappa,xx} + \partial_y E_{\kappa,xy}|_{x=1} = 0, & e_w^{\text{ex}}|_{y=1} = 0, \\
 \partial_x e_w^{\text{ex}}|_{x=-1} = 0, & E_{\kappa,yy}|_{y=-1} = 0, & E_{\kappa,xx}|_{x=1} = 0. & E_{\kappa,yy}|_{y=1} = 0.
 \end{array} \tag{8.16}$$

Variables  $(e_w^{\text{ex}}, \mathbf{E}_{\kappa}^{\text{ex}})$  under excitations  $f_d$  solve problem (7.77a).

**Results for the HHJ elements (8.7)** The results are reported in Fig. 8.5 and Tables 8.16, 8.17, 8.18. The conjecture 6 is verified for all orders.

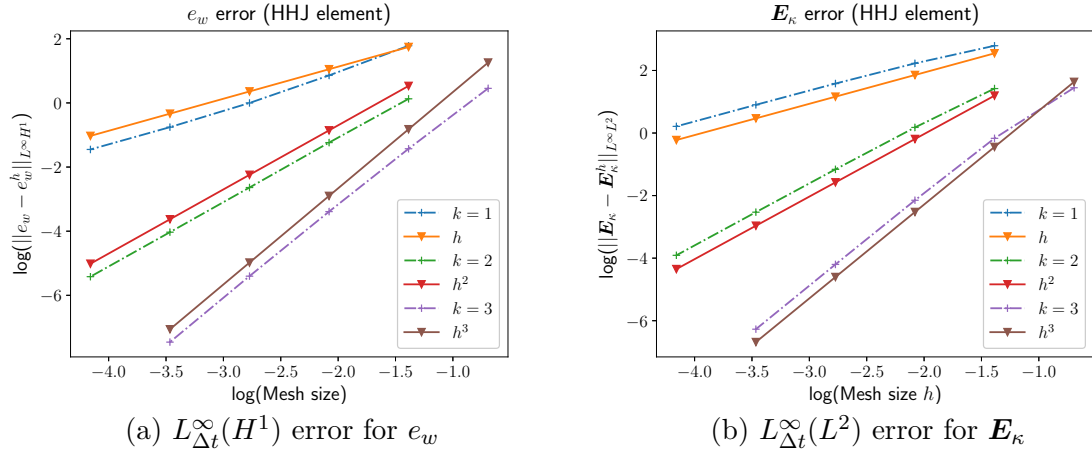


Figure 8.5: Error for the CSFS Kirchhoff plate (HHJ elements)

**Results for the dual mixed formulation**

## 8.4 Conclusion

In this paper, the link between mixed finite element method and pH plate models has been studied. It was shown that existing elements can be used to obtain structure-preserving discretization. A rigorous error analysis is still to be done but it should be easy to prove, given the available results. Since the pH framework provides a powerful description of boundary controlled systems, it is important that numerical methods be capable of handling generic boundary conditions. The methods discussed here possess this feature in the Mindlin plate case. For the Kirchhoff plate, a promising methodology is detailed in [?], but the dynamical case has not been considered yet. Future developments include the analysis and discretization of viscoelastic and thermoelastic problems in pH form.

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(H^1)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(H^{\text{Grad}})}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error   | Order | Error   | Order | Error                                       | Order |
| 8             | 9.78e-06                          | —     | 1.04e-04  | —     | 7.30e-09  | —     | 1.77e-07                                    | —     |
| 16            | 2.53e-06                          | 1.95  | 2.49e-05  | 2.07  | 1.85e-09  | 1.97  | 4.93e-08                                    | 1.84  |
| 32            | 6.35e-07                          | 1.99  | 6.06e-06  | 2.04  | 4.63e-10  | 1.99  | 1.27e-08                                    | 1.95  |
| 64            | 1.58e-07                          | 1.99  | 1.50e-06  | 2.01  | 1.15e-10  | 2.00  | 3.21e-09                                    | 1.98  |
| 128           | 3.97e-08                          | 2.00  | 3.74e-07  | 2.00  | 2.89e-11  | 2.00  | 8.06e-10                                    | 1.99  |

Table 8.10: Mindlin plate convergence result for the CGDG scheme  $k = 2$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(H^1)}$ |       | $\ e_\theta - e_\theta^h\ _{L^\infty(H^{\text{Grad}})}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       | $\ e_\gamma - e_\gamma^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|---|-------|---|-------|
|               | Error                             | Order | Error   | Order | Error   | Order | Error                                       | Order |
| 4             | 1.38e-06                          | —     | 1.24e-05  | —     | 8.24e-10  | —     | 2.24e-08                                    | —     |
| 8             | 1.79e-07                          | 2.94  | 1.51e-06  | 3.03  | 1.03e-10  | 2.99  | 2.90e-09                                    | 2.94  |
| 16            | 2.26e-08                          | 2.98  | 1.88e-07  | 3.00  | 1.28e-11  | 3.00  | 3.64e-10                                    | 2.99  |
| 32            | 2.83e-09                          | 2.99  | 2.36e-08  | 2.99  | 1.60e-12  | 3.00  | 4.54e-11                                    | 3.00  |
| 64            | 3.54e-10                          | 2.99  | 2.95e-09  | 2.99  | 2.00e-13  | 3.00  | 5.67e-12                                    | 3.00  |

Table 8.11: Mindlin plate convergence result for the CGDG scheme  $k = 3$ .

| Plate parameters |                           |       |           |
|------------------|---------------------------|-------|-----------|
| $E$              | $\rho$                    | $\nu$ | $b$       |
| 136 [GPa]        | 5600 [kg/m <sup>3</sup> ] | 0.3   | 0.001 [m] |

Table 8.12: Physical parameters for the Kirchhoff plate.

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|
|               | Error                             | Order | Error   | Order |
| 4             | 1.38e-00                          | —     | 5.11e+01  | —     |
| 8             | 5.17e-01                          | 1.41  | 2.64e+01  | 0.95  |
| 16            | 2.28e-01                          | 1.18  | 1.33e+01  | 0.98  |
| 32            | 1.09e-01                          | 1.05  | 6.68e-00  | 0.99  |
| 64            | 5.45e-02                          | 1.01  | 3.34e-00  | 0.99  |

Table 8.13: Kirchhoff plate convergence result for SSSS test  $k = 1$ .

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|
|               | Error                             | Order | Error   | Order |
| 4             | 1.47e-01                          | —     | 6.58e-00  | —     |
| 8             | 3.48e-02                          | 2.08  | 1.70e-00  | 1.94  |
| 16            | 8.51e-03                          | 2.03  | 4.31e-01  | 1.98  |
| 32            | 2.11e-03                          | 2.00  | 1.08e-01  | 1.99  |
| 64            | 5.28e-04                          | 2.00  | 2.70e-02  | 1.99  |

Table 8.14: Kirchhoff plate convergence result for SSSS test  $k = 2$ .



---

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|
|               | Error                             | Order | Error   | Order |
| 2             | 1.15e-01                          | —     | 4.85e-00  | —     |
| 4             | 1.51e-02                          | 2.92  | 6.42e-01  | 2.91  |
| 8             | 1.92e-03                          | 2.97  | 8.10e-02  | 2.98  |
| 16            | 2.41e-04                          | 2.99  | 1.01e-02  | 2.99  |
| 32            | 3.02e-05                          | 2.99  | 1.26e-03  | 3.00  |

---

Table 8.15: Kirchoff plate convergence result for SSSS test  $k = 3$ .

---

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|
|               | Error                             | Order | Error   | Order |
| 4             | 5.94e-00                          | —     | 1.62e+01  | —     |
| 8             | 2.35e-00                          | 1.33  | 9.27e-00  | 0.81  |
| 16            | 9.98e-01                          | 1.23  | 4.86e-00  | 0.93  |
| 32            | 4.69e-01                          | 1.08  | 2.46e-00  | 0.98  |
| 64            | 2.34e-01                          | 1.00  | 1.23e-00  | 0.99  |

---

Table 8.16: Kirchoff plate convergence result for CSFS test  $k = 1$ .

---

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|
|               | Error                             | Order | Error   | Order |
| 4             | 1.13e-00                          | —     | 4.14e-00  | —     |
| 8             | 2.90e-01                          | 1.96  | 1.19e-00  | 1.79  |
| 16            | 7.14e-02                          | 2.02  | 3.13e-01  | 1.93  |
| 32            | 1.77e-02                          | 2.00  | 7.96e-02  | 1.97  |
| 64            | 4.43e-03                          | 2.00  | 2.00e-02  | 1.98  |

---

Table 8.17: Kirchoff plate convergence result for CSFS test  $k = 2$ .

---

| $\frac{1}{h}$ | $\ e_w - e_w^h\ _{L^\infty(L^2)}$ |       | $\ \mathbf{E}_\kappa - \mathbf{E}_\kappa^h\ _{L^\infty(L^2)}$ |       |
|---------------|-----------------------------------|-------|---|-------|
|               | Error                             | Order | Error   | Order |
| 2             | 1.57e-00                          | —     | 4.25e-00  | —     |
| 4             | 2.39e-01                          | 2.71  | 8.44e-01  | 2.33  |
| 8             | 3.37e-02                          | 2.82  | 1.16e-01  | 2.85  |
| 16            | 4.50e-03                          | 2.90  | 1.49e-02  | 2.95  |
| 32            | 5.76e-04                          | 2.96  | 1.89e-03  | 2.98  |

---

Table 8.18: Kirchoff plate convergence result for CSFS test  $k = 3$ .



# Numerical applications

## 1670 9.1 Boundary stabilization

## 1671 9.2 Thermoelastic wave propagation

## 1672 9.3 Mixed boundary conditions

### 1673 9.3.1 Trajectory tracking of a thin beam

### 1674 9.3.2 Vibroacoustic under mixed boundary conditions

## 1675 9.4 Modal analysis of plates



1676

## Part IV

1677

# Port-Hamiltonian flexible multibody dynamics

1678



# Modular multibody systems in port-Hamiltonian form

## 10.1 Reminder of the rigid case

## 10.2 Flexible floating body

## 10.3 Modular construction of multibody systems





1686

1687

# Validation

1688

---

## 11.1 Beam systems

### 11.1.1 Modal analysis of a flexible mechanism

### 11.1.2 Non-linear crank slider

### 11.1.3 Hinged beam

## 11.2 Plate systems

### 11.2.1 Boundary interconnection with a rigid element

### 11.2.2 Actuated plate



# Conclusion



# Conclusions and future directions

Je n'ai cherché de rien prouver, mais de bien peindre et d'éclairer bien ma  
peinture.

---

*André Gide*  
*Préface de L'Immoraliste*



# Mathematical tools

## A.1 Differential operators

The space of all, symmetric and skew-symmetric  $d \times d$  matrices are denoted by  $\mathbb{M}$ ,  $\mathbb{S}$ ,  $\mathbb{K}$  respectively. The space of  $\mathbb{R}^d$  vectors is denoted by  $\mathbb{V}$ .  $\Omega \subset \mathbb{R}^d$  is an open connected set. For a scalar field  $u : \Omega \rightarrow \mathbb{R}$  the gradient is defined as

$$\text{grad}(u) = \nabla u := \left( \partial_{x_1} u \dots \partial_{x_d} u \right)^\top.$$

For a vector field  $\mathbf{u} : \Omega \rightarrow \mathbb{V}$ , with components  $u_i$ , the gradient (Jacobian) is defined as

$$\text{grad}(\mathbf{u})_{ij} := (\nabla \mathbf{u})_{ij} = \partial_{x_i} u_j.$$

The symmetric part of the gradient operator Grad (i. e. the deformation gradient in continuum mechanics) is thus given by

$$\text{Grad}(\mathbf{u}) := \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) \in \mathbb{S}.$$

The Hessian operator of  $u$  is then computed as follows

$$\text{Hess}(u) = \nabla^2 u = \text{Grad}(\text{grad}(u)).$$

For a tensor field  $\mathbf{U} : \Omega \rightarrow \mathbb{M}$ , with components  $u_{ij}$ , the divergence is a vector, defined column-wise as

$$\text{Div}(\mathbf{U}) = \nabla \cdot \mathbf{U} := \left( \sum_{i=1}^d \partial_{x_i} u_{ij} \right)_{j=1, \dots, d}.$$

The double divergence of a tensor field  $\mathbf{U}$  is then a scalar field defined as

$$\text{div}(\text{Div}(\mathbf{U})) := \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} \partial_{x_j} u_{ij}.$$

**Definition 7** (Formal adjoint, Def. 5.80 [RR04])

Consider the differential operator defined on  $\Omega$

$$\mathcal{L}(\mathbf{x}, \partial) = \sum_{|\alpha| \leq k} a_\alpha(\mathbf{x}) \partial^\alpha, \tag{A.1}$$

where  $\alpha := (\alpha_1, \dots, \alpha_d)$  is a multi-index of order  $|\alpha| := \sum_{i=1}^d \alpha_i$ ,  $a_\alpha$  are a set of real scalars and  $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$  is a differential operator of order  $|\alpha|$  resulting from a combination of spatial derivatives. The formal adjoint of  $\mathcal{L}$  is the operator defined by

$$\mathcal{L}^*(\mathbf{x}, \partial)u = \sum_{|\alpha| \leq k} (-1)^\alpha \partial^\alpha (a_\alpha(\mathbf{x})u(\mathbf{x})). \quad (\text{A.2})$$

The importance of this definition lies in the fact that

$$\langle \phi, \mathcal{L}(\mathbf{x}, \partial)\psi \rangle_\Omega = \langle \mathcal{L}^*(\mathbf{x}, \partial)\phi, \psi \rangle_\Omega \quad (\text{A.3})$$

for every  $\phi, \psi \in C_0^\infty(\Omega)$ . If the assumption of compact support is removed, then (A.3) no longer holds; instead the integration by parts yields additional terms involving integrals over the boundary  $\partial\Omega$ . However, these boundary terms vanish if  $\phi$  and  $\psi$  satisfy certain restrictions on the boundary.

## A.2 Integration by parts

**Theorem 4** (Integration by parts for tensors)

Consider a smooth tensor-valued function  $\mathbf{A} \in \mathbb{R}^{d \times d}$  and vector-valued function  $\mathbf{b} \in \mathbb{V} = \mathbb{R}^d$ . The following integration by parts formula holds

$$\int_\Omega \{\text{Div}(\mathbf{A}) \cdot \mathbf{b} + \mathbf{A} : \text{grad}(\mathbf{b})\} \, d\Omega = \int_\Omega \text{div}(\mathbf{A}\mathbf{b}) \, d\Omega = \int_{\partial\Omega} (\mathbf{A}^\top \mathbf{n}) \cdot \mathbf{b} \, dS, \quad (\text{A.4})$$

where  $\mathbf{n}$  is the outward normal at the boundary and  $dS$  the infinitesimal surface.

*Proof.* Consider the components expression of Eq. (A.4)

$$\begin{aligned} \int_\Omega \{\text{Div}(\mathbf{A}) \cdot \mathbf{b} + \mathbf{A} : \text{grad}(\mathbf{b})\} \, d\Omega &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \{(\partial_{x_i} A_{ij})b_j + A_{ij}(\partial_{x_i} b_j)\} \, d\Omega, \\ &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} (A_{ij}b_j) \, d\Omega = \int_\Omega \text{div}(\mathbf{A}\mathbf{b}) \, d\Omega, \\ &= \int_{\partial\Omega} \sum_{i=1}^d \sum_{j=1}^d (n_i A_{ij})b_j \, dS = \int_{\partial\Omega} (\mathbf{A}^\top \mathbf{n}) \cdot \mathbf{b} \, dS. \end{aligned} \quad (\text{A.5})$$

□

The previous result can be specialized for symmetric tensor field [BBF<sup>+</sup>13, Chapter 1].

**Theorem 5** (Integration by parts for symmetric tensors)

Consider a smooth tensor-valued function  $\mathbf{M} \in \mathbb{S} = \mathbb{R}_{sym}^{d \times d}$  and vector-valued function  $\mathbf{b} \in \mathbb{V} =$



1723  $\mathbb{R}^d$ . Then, it holds

$$\int_{\Omega} \{\text{Div}(\mathbf{M}) \cdot \mathbf{b} + \mathbf{M} : \text{Grad}(\mathbf{b})\} \, d\Omega = \int_{\Omega} \text{div}(\mathbf{M}\mathbf{b}) \, d\Omega = \int_{\partial\Omega} (\mathbf{M}\mathbf{n}) \cdot \mathbf{b} \, dS. \quad (\text{A.6})$$

1724 *Proof.* Consider the components expression of Eq. (A.6)

$$\int_{\Omega} \{\text{Div}(\mathbf{M}) \cdot \mathbf{b} + \mathbf{M} : \text{Grad}(\mathbf{b})\} \, d\Omega = \int_{\Omega} \sum_{i=1}^d \sum_{j=1}^d \left\{ (\partial_{x_i} M_{ij}) b_j + M_{ij} \frac{1}{2} (\partial_{x_i} b_j + \partial_{x_j} b_i) \right\} \, d\Omega, \quad (\text{A.7})$$

1725 The term  $M_{ij} \frac{1}{2} (\partial_{x_i} b_j + \partial_{x_j} b_i)$  can be manipulated exploiting the symmetry of the tensor  $\mathbf{M}$

$$\begin{aligned} \sum_{i=1}^d \sum_{j=1}^d \frac{1}{2} (M_{ij} \partial_{x_i} b_j + M_{ij} \partial_{x_j} b_i) &= \sum_{i=1}^d \sum_{j=1}^d \frac{1}{2} (M_{ij} \partial_{x_i} b_j + M_{ji} \partial_{x_i} b_j), \\ &= \sum_{i=1}^d \sum_{j=1}^d \frac{1}{2} (M_{ij} + M_{ji}) \partial_{x_i} b_j \quad \text{Since } \mathbf{M} \text{ is symmetric,} \\ &= \sum_{i=1}^d \sum_{j=1}^d M_{ij} \partial_{x_i} b_j = \mathbf{M} : \text{grad}(\mathbf{b}) \end{aligned} \quad (\text{A.8})$$

1726 Then it holds

$$\int_{\Omega} \{\text{Div}(\mathbf{M}) \cdot \mathbf{b} + \mathbf{M} : \text{Grad}(\mathbf{b})\} \, d\Omega = \int_{\Omega} \{\text{Div}(\mathbf{M}) \cdot \mathbf{b} + \mathbf{M} : \text{grad}(\mathbf{b})\} \, d\Omega \quad (\text{A.9})$$

1727 Using Eq (A.4) then

$$\begin{aligned} \int_{\Omega} \{\text{Div}(\mathbf{M}) \cdot \mathbf{b} + \mathbf{M} : \text{Grad}(\mathbf{b})\} \, d\Omega &= \int_{\Omega} \{\text{Div}(\mathbf{M}) \cdot \mathbf{b} + \mathbf{M} : \text{grad}(\mathbf{b})\} \, d\Omega, \\ &= \int_{\partial\Omega} (\mathbf{M}^{\top} \mathbf{n}) \cdot \mathbf{b} \, dS, \quad \text{Since } \mathbf{M} \text{ is symmetric,} \\ &= \int_{\partial\Omega} (\mathbf{M} \mathbf{n}) \cdot \mathbf{b} \, dS. \end{aligned} \quad (\text{A.10})$$

1728 This concludes the proof.  $\square$

## 1729 A.3 Bilinear forms

**Definition 8** (Skew-symmetric bilinear form)

A bilinear form on the Hilbert space  $H$

$$\begin{aligned} b : H \times H &\longrightarrow \mathbb{R}, \\ (\mathbf{v}, \mathbf{u}) &\longrightarrow b(\mathbf{v}, \mathbf{u}), \end{aligned}$$

*is skew-symmetric iff*

$$b(\boldsymbol{v}, \boldsymbol{u}) = -b(\boldsymbol{u}, \boldsymbol{v}).$$

# Finite elements gallery



1733

APPENDIX C

1734

# Implementation using FEniCS and Firedrake

1735

1736

---



# Bibliography

1737

- 1738 [A.90] Douglas N. A. Mixed finite element methods for elliptic problems. *Computer*  
1739 *Methods in Applied Mechanics and Engineering*, 82(1):281 – 300, 1990. Pro-  
1740 ceedings of the Workshop on Reliability in Computational Mechanics.
- 1741 [AB85] D. Arnold and F. Brezzi. Mixed and nonconforming finite element meth-  
1742 ods: implementation, postprocessing and error estimates. *ESAIM: Mathemat-*  
1743 *ical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse*  
1744 *Numérique*, 19(1):7–32, 1985.
- 1745 [Abe12] R. Abeyaratne. *Lecture Notes on the Mechanics of Elastic Solids. Volume II:*  
1746 *Continuum Mechanics*. Cambridge, MA and Singapore, 1st edition, 2012.
- 1747 [AFS68] J. H. Argyris, I. Fried, and D. W. Scharpf. The tuba family of plate ele-  
1748 ments for the matrix displacement method. *The Aeronautical Journal (1968)*,  
1749 72(692):701–709, 1968.
- 1750 [AFW07] D. Arnold, R. Falk, and R. Winther. Mixed finite element methods for lin-  
1751 ear elasticity with weakly imposed symmetry. *Mathematics of Computation*,  
1752 76(260):1699–1723, 2007.
- 1753 [AL00] G. Avalos and I. Lasiecka. Boundary controllability of thermoelastic plates via  
1754 the free boundary conditions. *SIAM Journal on Control and Optimization*,  
1755 38(2):337–383, 2000.
- 1756 [AL14] D. Arnold and J. Lee. Mixed methods for elastodynamics with weak symmetry.  
1757 *SIAM Journal on Numerical Analysis*, 52(6):2743–2769, 2014.
- 1758 [AW02] D. Arnold and R. Winther. Mixed finite elements for elasticity. *Numerische*  
1759 *Mathematik*, 92(3):401–419, 2002.
- 1760 [AW19] D. N. Arnold and S. W. Walker. The Hellan-Herrmann-Johnson method with  
1761 curved elements, 2019. arXiv preprint arXiv:1909.09687.
- 1762 [BadVMR13] L. Beirão da Veiga, D. Mora, and R. Rodríguez. Numerical analysis of a locking-  
1763 free mixed finite element method for a bending moment formulation of Reissner-  
1764 Mindlin plate model. *Numerical Methods for Partial Differential Equations*,  
1765 29(1):40–63, 2013.
- 1766 [BBF<sup>+</sup>13] D. Boffi, F. Brezzi, M. Fortin, et al. *Mixed finite element methods and applica-*  
1767 *tions*, volume 44. Springer, 2013.
- 1768 [BDM85] F. Brezzi, J. Jr. Douglas, and L.D. Marini. Two families of mixed finite elements  
1769 for second order elliptic problems. *Numerische Mathematik*, 47:217–236, 1985.

- 
- 1770 [Bel69] K. Bell. A refined triangular plate bending finite element. *International Journal*  
1771 *for Numerical Methods in Engineering*, 1(1):101–122, 1969.
- 1772 [BGL05] M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point  
1773 problems. *Acta Numerica*, 14:1–137, 2005.
- 1774 [Bio56] M. A. Biot. Thermoelasticity and irreversible thermodynamics. *Journal of*  
1775 *Applied Physics*, 27(3):240–253, 1956.
- 1776 [BJT00] E. Bécache, P. Joly, and C. Tsogka. An analysis of new mixed finite elements for  
1777 the approximation of wave propagation problems. *SIAM Journal on Numerical*  
1778 *Analysis*, 37(4):1053–1084, 2000.
- 1779 [BJT01] E. Bécache, P. Joly, and C. Tsogka. A new family of mixed finite elements  
1780 for the linear elastodynamic problem. *SIAM Journal on Numerical Analysis*,  
1781 39:2109–2132, 06 2001.
- 1782 [BMXZ18] C. Beattie, V. Mehrmann, H. Xu, and H. Zwart. Linear port-Hamiltonian  
1783 descriptor systems. *Mathematics of Control, Signals, and Systems*, 30(4):17,  
1784 2018.
- 1785 [BR90] H. Blum and R. Rannacher. On mixed finite element methods in plate bending  
1786 analysis. *Computational Mechanics*, 6(3):221–236, May 1990.
- 1787 [Bre08] F. Brezzi. *Mixed finite elements, compatibility conditions, and applications*.  
1788 Springer, 2008.
- 1789 [Car73] D. E. Carlson. Linear thermoelasticity. In C. Truesdell, editor, *Linear Theories*  
1790 *of Elasticity and Thermoelasticity: Linear and Nonlinear Theories of Rods,*  
1791 *Plates, and Shells*, pages 297–345. Springer, Berlin, Heidelberg, 1973.
- 1792 [Cha62] P. Chadwick. On the propagation of thermoelastic disturbances in thin plates  
1793 and rods. *Journal of the Mechanics and Physics of Solids*, 10(2):99–109, 1962.
- 1794 [Cia88] P. G. Ciarlet. *Mathematical Elasticity: Three-Dimensional Elasticity*. Studies  
1795 in mathematics and its applications. North-Holland, 1988.
- 1796 [CMKO11] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren. Topics in structure-  
1797 preserving discretization. *Acta Numerica*, 20:1–119, 2011.
- 1798 [Cou90] T.J. Courant. Dirac manifolds. *Transactions of the American Mathematical*  
1799 *Society*, 319(2):631–661, 1990.
- 1800 [CR16] F.L. Cardoso Ribeiro. *Port-Hamiltonian modeling and control of fluid-structure*  
1801 *system*. PhD thesis, Université de Toulouse, Dec. 2016.
- 1802 [CRML18] F.L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A structure-preserving par-  
1803 titioned finite element method for the 2d wave equation. *IFAC-PapersOnLine*,  
1804 51(3):119 – 124, 2018. 6th IFAC Workshop on Lagrangian and Hamiltonian  
1805 Methods for Nonlinear Control LHMNC 2018.
-



- 
- 1806 [CRML19] F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A partitioned finite ele-  
 1807 ment method for power-preserving discretization of open systems of conserva-  
 1808 tion laws, 2019. arXiv preprint arXiv:1906.05965.
- 1809 [CRMPB17] F. L. Cardoso-Ribeiro, D. Matignon, and V. Pommier-Budinger. A port-  
 1810 Hamiltonian model of liquid sloshing in moving containers and application to a  
 1811 fluid-structure system. *Journal of Fluids and Structures*, 69:402–427, February  
 1812 2017.
- 1813 [DHNLS99] R. Durán, L. Hervella-Nieto, E. Liberman, and J. Solomin. Approximation of  
 1814 the vibration modes of a plate by Reissner-Mindlin equations. *Mathematics of*  
 1815 *Computation of the American Mathematical Society*, 68(228):1447–1463, 1999.
- 1816 [DMSB09] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx. *Modeling and*  
 1817 *Control of Complex Physical Systems*. Springer Verlag, 2009.
- 1818 [DZ18] Pauly D. and W. Zulehner. The divdiv-complex and applications to biharmonic  
 1819 equations. *Applicable Analysis*, pages 1–52, 2018.
- 1820 [Gev88] T. Geveci. On the application of mixed finite element methods to the wave  
 1821 equations. *ESAIM: M2AN*, 22(2):243–250, 1988.
- 1822 [Gri15] M. Grinfeld. *Mathematical Tools for Physicists*. John Wiley & Sons Inc, 2nd  
 1823 edition, jan 2015.
- 1824 [GSV18] T. Gustafsson, R. Stenberg, and J. Videman. A posteriori estimates for con-  
 1825 forming kirchhoff plate elements. *SIAM Journal on Scientific Computing*,  
 1826 40(3):A1386–A1407, 2018.
- 1827 [HE09] R. B. Hetnarski and M. R. Eslami. *Thermal stresses: advanced theory and*  
 1828 *applications*, volume 158. Springer, 2009.
- 1829 [Hel67] K. Hellan. Analysis of elastic plates in flexure by a simplified finite element  
 1830 method. *Acta Polytechnica Scandinavica*, 1967.
- 1831 [Her67] L. R. Herrmann. Finite-element bending analysis for plates. *Journal of the*  
 1832 *Engineering Mechanics Division*, 93(5):13–26, 1967.
- 1833 [HM78] T. J.R. Hughes and J.E. Marsden. Classical elastodynamics as a linear sym-  
 1834 metric hyperbolic system. *Journal of Elasticity*, 8(1):97–110, 1978.
- 1835 [HZ97] S. W. Hansen and B. Y. Zhang. Boundary control of a linear thermoelastic  
 1836 beam. *Journal of Mathematical Analysis and Applications*, 210(1):182–205,  
 1837 1997.
- 1838 [Joh73] Claes Johnson. On the convergence of a mixed finite-element method for plate  
 1839 bending problems. *Numerische Mathematik*, 21(1):43–62, 1973.
-

- 
- 1840 [JZ12] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-*  
1841 *dimensional Spaces*. Number 223 in Operator Theory: Advances and Ap-  
1842 *plications*. Springer Verlag, Germany, 2012. [https://doi.org/10.1007/](https://doi.org/10.1007/978-3-0348-0399-1)  
1843 [978-3-0348-0399-1](https://doi.org/10.1007/978-3-0348-0399-1).
- 1844 [KK15] R. C. Kirby and T. T. Kieu. Symplectic-mixed finite element approximation of  
1845 linear acoustic wave equations. *Numerische Mathematik*, 130(2):257–291, Jun  
1846 2015.
- 1847 [KML18] P. Kotyczka, B. Maschke, and L. Lefèvre. Weak form of Stokes-Dirac struc-  
1848 *tures and geometric discretization of port-Hamiltonian systems*. *Journal of*  
1849 *Computational Physics*, 361:442 – 476, 2018.
- 1850 [Kot19] P. Kotyczka. *Numerical Methods for Distributed Parameter Port-Hamiltonian*  
1851 *Systems*. TUM University Press, 2019.
- 1852 [KZ15] M. Kurula and H. Zwart. Linear wave systems on n-d spatial domains. *Interna-*  
1853 *tional Journal of Control*, 88(5):1063–1077, 2015. [https://www.tandfonline.](https://www.tandfonline.com/doi/abs/10.1080/00207179.2014.993337)  
1854 [com/doi/abs/10.1080/00207179.2014.993337](https://www.tandfonline.com/doi/abs/10.1080/00207179.2014.993337).
- 1855 [KZvdSB10] M. Kurula, H. Zwart, A. J. van der Schaft, and J. Behrndt. Dirac structures  
1856 and their composition on Hilbert spaces. *Journal of mathematical analysis*  
1857 *and applications*, 372(2):402–422, 2010. [https://doi.org/10.1016/j.jmaa.](https://doi.org/10.1016/j.jmaa.2010.07.004)  
1858 [2010.07.004](https://doi.org/10.1016/j.jmaa.2010.07.004).
- 1859 [Lag89] J. E. Lagnese. *Boundary Stabilization of Thin Plates*. Society for Industrial  
1860 and Applied Mathematics, 1989.
- 1861 [Lee12] J. Lee. *Mixed methods with weak symmetry for time dependent problems of*  
1862 *elasticity and viscoelasticity*. PhD thesis, University of Minnesota, 2012.
- 1863 [LGZM05] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and Boundary  
1864 Control Systems associated with Skew-Symmetric Differential Operators. *SIAM*  
1865 *Journal on Control and Optimization*, 44(5):1864–1892, 2005. [https://doi.](https://doi.org/10.1137/040611677)  
1866 [org/10.1137/040611677](https://doi.org/10.1137/040611677).
- 1867 [Li18] L. Li. *Regge finite elements with applications in solid mechanics and relativity*.  
1868 PhD thesis, University of Minnesota, 2018.
- 1869 [LMW<sup>+</sup>12] A. Logg, K. A. Mardal, G. N. Wells, et al. *Automated Solution of Differential*  
1870 *Equations by the Finite Element Method*. Springer, 2012.
- 1871 [LPKL12] L. D. Landau, L. P. Pitaevskii, A. M. Kosevich, and E. M. Lifshitz. *Theory of*  
1872 *Elasticity*. Butterworth Heinemann, third edition, Dec 2012.
- 1873 [LR00] R. Lifshitz and M. L. Roukes. Thermoelastic damping in micro-and nanome-  
1874 *chanical systems*. *Physical review B*, 61(8):5600, 2000.
-

- 
- 1875 [MBM<sup>+</sup>16] A. T. T. McRae, G.-T. Bercea, L. Mitchell, D. A. Ham, and C. J. Cotter. Auto-  
1876 mated generation and symbolic manipulation of tensor product finite elements.  
1877 *SIAM Journal on Scientific Computing*, 38(5):S25–S47, 2016.
- 1878 [Min51] R. D. Mindlin. Influence of rotatory inertia and shear on flexural motions of  
1879 isotropic elastic Plates. *Journal of Applied Mechanics*, 18:31–38, March 1951.
- 1880 [MM19] V. Mehrmann and R. Morandin. Structure-preserving discretization for port-  
1881 hamiltonian descriptor systems. In *2019 IEEE 58th Conference on Decision*  
1882 *and Control (CDC)*, pages 6863–6868, 2019.
- 1883 [MMB05] A. Macchelli, C. Melchiorri, and L. Bassi. Port-based modelling and control of  
1884 the Mindlin plate. In *Proceedings of the 44th IEEE Conference on Decision and*  
1885 *Control*, pages 5989–5994, Dec. 2005. [https://doi.org/10.1109/CDC.2005.](https://doi.org/10.1109/CDC.2005.1583120)  
1886 [1583120](https://doi.org/10.1109/CDC.2005.1583120).
- 1887 [Nor06] A.N. Norris. Dynamics of thermoelastic thin plates: A comparison of four  
1888 theories. *Journal of Thermal Stresses*, 29(2):169–195, 2006.
- 1889 [NY04] G. Nishida and M. Yamakita. A higher order stokes-dirac structure for  
1890 distributed-parameter port-hamiltonian systems. In *Proceedings of the 2004*  
1891 *American Control Conference*, volume 6, pages 5004–5009 vol.6, 2004.
- 1892 [Olv93] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of  
1893 *Graduate texts in mathematics*. Springer-Verlag New York, 2nd edition, 1993.
- 1894 [Pir89] O. A. Pironneau. *Finite element methods for fluids*. John Wiley and Sons,  
1895 1989.
- 1896 [PZ20] D. Pauly and W. Zulehner. The elasticity complex, 2020. arXiv preprint  
1897 arXiv:2001.11007.
- 1898 [Raf18] K. Rafetseder. *A New Approach to Mixed Methods for Kirchhoff-Love Plates*  
1899 *and Shells*. PhD thesis, Johannes Kepler Universität Linz, May 2018.
- 1900 [Red03] J. N. Reddy. *Mechanics of laminated composite plates and shells: theory and*  
1901 *analysis*. CRC press, 2003.
- 1902 [Red06] J. N. Reddy. *Theory and analysis of elastic plates and shells*. CRC press, 2006.
- 1903 [Rei47] E. Reissner. On bending of elastic plates. *Quarterly of Applied Mathematics*,  
1904 5(1):55–68, 1947.
- 1905 [RHM<sup>+</sup>17] F. Rathgeber, D.A. Ham, L. Mitchell, M. Lange, F. Luporini, A. T.T. McRae,  
1906 G.T. Bercea, G. R. Markall, and P.H.J. Kelly. Firedrake: automating the finite  
1907 element method by composing abstractions. *ACM Transactions on Mathematical*  
1908 *Software (TOMS)*, 43(3):24, 2017.
- 1909 [RR04] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equa-*  
1910 *tions*. Number 13 in Texts in Applied Mathematics. Springer-Verlag New York,  
1911 2nd edition, 2004.
-

- 
- 1912 [RT77] P. A. Raviart and J. M. Thomas. A mixed finite element method for 2-nd order  
1913 elliptic problems. In Ilio Galligani and Enrico Magenes, editors, *Mathematical*  
1914 *Aspects of Finite Element Methods*, pages 292–315, Berlin, Heidelberg, 1977.  
1915 Springer Berlin Heidelberg.
- 1916 [RZ18] K. Rafetseder and W. Zulehner. A decomposition result for Kirchhoff plate  
1917 bending problems and a new discretization approach. *SIAM Journal on Nu-*  
1918 *merical Analysis*, 56(3):1961–1986, 2018.
- 1919 [SHM19a] A. Serhani, G. Haine, and D. Matignon. Anisotropic heterogeneous n-D  
1920 heat equation with boundary control and observation: I. Modeling as port-  
1921 Hamiltonian system. *IFAC-PapersOnLine*, 52(7):51 – 56, 2019. 3rd IFAC  
1922 Workshop on Thermodynamic Foundations for a Mathematical Systems The-  
1923 ory TFMST 2019.
- 1924 [SHM19b] A. Serhani, G. Haine, and D. Matignon. Anisotropic heterogeneous n-D heat  
1925 equation with boundary control and observation: II. Structure-preserving dis-  
1926 cretization. *IFAC-PapersOnLine*, 52(7):57 – 62, 2019. 3rd IFAC Workshop  
1927 on Thermodynamic Foundations for a Mathematical Systems Theory TFMST  
1928 2019.
- 1929 [Sim99] J. G. Simmonds. Major simplifications in a current linear model for the motion  
1930 of a thermoelastic plate. *Quarterly of Applied Mathematics*, 57(4):673–679,  
1931 1999.
- 1932 [Skr19] N. Skrepek. Well-posedness of linear first order port-Hamiltonian systems on  
1933 multidimensional spatial domains, 2019. arXiv preprint arXiv:1910.09847.
- 1934 [SS17] M. Schöberl and K. Schlacher. Variational Principles for Different Represen-  
1935 tations of Lagrangian and Hamiltonian Systems. In Hans Irschik, Alexander  
1936 Belyaev, and Michael Krommer, editors, *Dynamics and Control of Advanced*  
1937 *Structures and Machines*, pages 65–73. Springer International Publishing, 2017.
- 1938 [TRLGK18] V. Trenchant, H. Ramírez, Y. Le Gorrec, and P. Kotyczka. Finite differences  
1939 on staggered grids preserving the port-Hamiltonian structure with application  
1940 to an acoustic duct. *Journal of Computational Physics*, 373, 06 2018.
- 1941 [TW09] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*.  
1942 Springer Science & Business Media, 2009.
- 1943 [TWK59] S. Timoshenko and S. Woinowsky-Krieger. *Theory of plates and shells*. Engi-  
1944 neering societies monographs. McGraw-Hill, 1959.
- 1945 [vdSM02] A.J. van der Schaft and B. Maschke. Hamiltonian formulation of distributed-  
1946 parameter systems with boundary energy flow. *Journal of Geometry and*  
1947 *Physics*, 42(1):166 – 194, 2002.
- 1948 [Vil07] J.A. Villegas. *A Port-Hamiltonian Approach to Distributed Parameter Systems*.  
1949 PhD thesis, University of Twente, May 2007.
-

- 
- 1950 [Yao11] P.F. Yao. *Modeling and Control in Vibrational and Structural Dynamics: A*  
1951 *Differential Geometric Approach*. Chapman & Hall/CRC Applied Mathematics  
1952 & Nonlinear Science. Taylor & Francis, 2011.
-



**Résumé** — Malgré l’abondante littérature sur le formalisme pH, les problèmes d’élasticité en deux ou trois dimensions géométriques n’ont presque jamais été considérés. Cette thèse vise à étendre l’approche port-Hamiltonienne (pH) à la mécanique des milieux continus. L’originalité apportée réside dans trois contributions majeures. Tout d’abord, la nouvelle formulation pH des modèles de plaques et des phénomènes thermoélastiques couplés est présentée. L’utilisation du calcul tensoriel est obligatoire pour modéliser les milieux continus et l’introduction de variables tensorielles est nécessaire pour obtenir une description pH équivalente qui soit intrinsèque, c’est-à-dire indépendante des coordonnées choisies. Deuxièmement, une technique de discrétisation basée sur les éléments finis et capable de préserver la structure du problème de la dimension infinie au niveau discret est développée et validée. La discrétisation des problèmes d’élasticité nécessite l’utilisation d’éléments finis non standard. Néanmoins, l’implémentation numérique est réalisée grâce à des bibliothèques open source bien établies, fournissant aux utilisateurs externes un outil facile à utiliser pour simuler des systèmes flexibles sous forme pH. Troisièmement, une nouvelle formulation pH de la dynamique multicorps flexible est dérivée. Cette reformulation, valable sous de petites hypothèses de déformations, inclut toutes sortes de modèles élastiques linéaires et exploite la modularité intrinsèque des systèmes pH.

**Mots clés :** Systèmes port-Hamiltonien, mécanique des solides, discretisation symplectique, méthode des éléments finis, dynamique multicorps

**Abstract** — Despite the large literature on pH formalism, elasticity problems in higher geometrical dimensions have almost never been considered. This work establishes the connection between port-Hamiltonian distributed systems and elasticity problems. The originality resides in three major contributions. First, the novel pH formulation of plate models and coupled thermoelastic phenomena is presented. The use of tensor calculus is mandatory for continuum mechanical models and the inclusion of tensor variables is necessary to obtain an equivalent and intrinsic, i.e. coordinate free, pH description. Second, a finite element based discretization technique, capable of preserving the structure of the infinite-dimensional problem at a discrete level, is developed and validated. The discretization of elasticity problems requires the use of non-standard finite elements. Nevertheless, the numerical implementation is performed thanks to well-established open-source libraries, providing external users with an easy to use tool for simulating flexible systems in pH form. Third, flexible multibody systems are recast in pH form by making use of a floating frame description valid under small deformations assumptions. This reformulation include all kinds of linear elastic models and exploits the intrinsic modularity of pH systems.

**Keywords:** Port-Hamiltonian systems, continuum mechanics, structure preserving discretization, finite element method, multibody dynamics.