

A port-Hamiltonian formulation for the full von-Kármán plate model

Andrea Brugnoli and Denis Matignon

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UNIVERSITY
OF TWENTE.



Outline

Why port-Hamiltonian systems?

Von-Kármán theory of thin beams in pH form

Numerical discretization

Numerical convergence study

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A unified language for multiphysics in engineering

The port-Hamiltonian (pH) paradigm provides a language to understand multiphysics:

- ▶ **Physics** is at the core: pH systems are **passive** with respect to the **energy storage function**.
- ▶ The **topological** and **metrical** structure of the equations is clearly separated (mimetic discretization).
- ▶ PH systems are **closed under interconnection**.



Finite dimensional pH systems

A theory still under development

There is **not a unique definition** of pH systems, even in finite dimension.

Definition (Finite dimensional pH system)

The following time-invariant dynamical system is a pH system

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u},$$

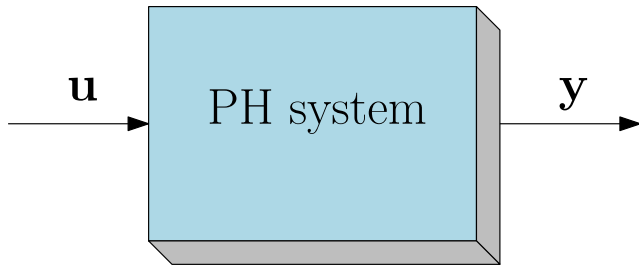
$$\mathbf{y} = \mathbf{B}^\top \mathbf{x}.$$

$\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$ the input and output and

- ▶ $\mathbf{J}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})^\top \in \mathbb{R}^{n \times n}$ the interconnection operator
- ▶ $\mathbf{B} \in \mathbb{R}^{n \times m}$ the control operator.
- ▶ $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x} : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\mathbf{M} > 0$, the Hamiltonian.

Finite dimensional pH systems

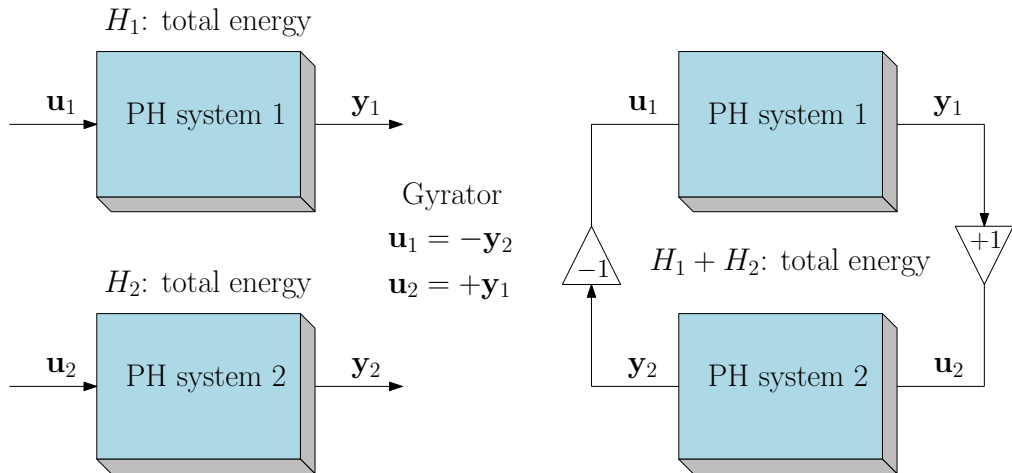
H : total energy



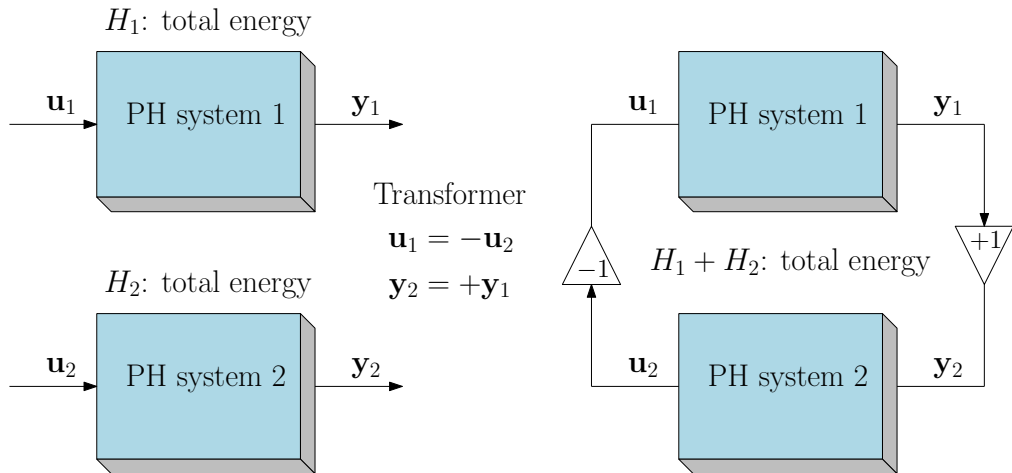
Lossless: $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

Interconnection of pH systems



Interconnection of pH systems



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Why port-Hamiltonian systems?

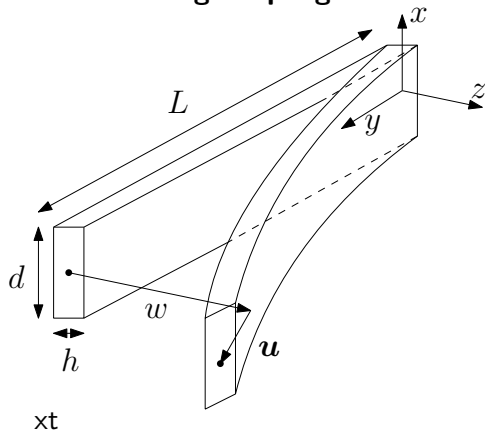
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The von-Kármán assumption

Second-order approximation of geometrically exact beam/plate theory **capturing the axial bending coupling.**

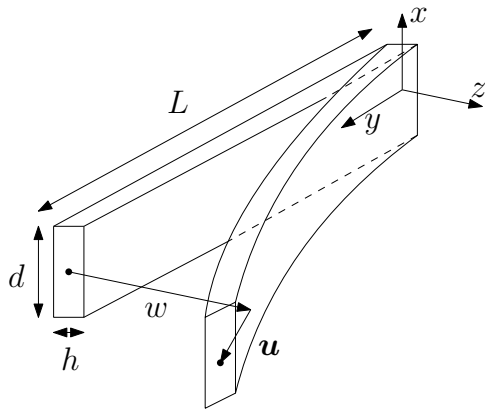


Basic geometric assumptions

- ▶ Out of plane deflection comparable to the thickness: $w/h = \mathcal{O}(1)$.
- ▶ The squares of the in-plane stretching terms are negligible compared to the square of the rotations.

The von-Kármán assumption

Second-order approximation of geometrically exact beam/plate theory **capturing the axial bending coupling**.



Geometrical non-linearities allow describing bifurcations (i.e. buckling).

Linear isotropic plates ($\Omega \subset \mathbb{R}^2$)

The axial and bending behavior are uncoupled if $w/h \ll 1$:

Membrane displacement
(2D elastodynamics)

$$\rho h \partial_{tt} \mathbf{u} = \operatorname{Div} \mathbf{N},$$

$$\mathbf{N} = D_m \Phi(\boldsymbol{\varepsilon}_m),$$

$$\boldsymbol{\varepsilon}_m = \operatorname{Sym}(\nabla \mathbf{u}) = \operatorname{Grad} \mathbf{u}$$

Total membrane energy

$$H_m = \frac{1}{2} \int_{\Omega} \rho h \|\partial_t \mathbf{u}\|^2 + D_m \Phi(\boldsymbol{\varepsilon}_m) : \mathbf{N} \, d\Omega.$$

Vertical displacement
(Kirchhoff plate)

$$\rho h \partial_{tt} w = -\operatorname{div} \operatorname{Div} \mathbf{M},$$

$$\mathbf{M} = D_b \Phi(\boldsymbol{\kappa}),$$

$$\boldsymbol{\kappa} = \operatorname{Hess} w = \operatorname{Grad} \operatorname{grad} w.$$

Total bending energy

$$H_b = \frac{1}{2} \int_{\Omega} \rho h (\partial_t w)^2 + D_b \Phi(\boldsymbol{\kappa}) : \mathbf{M} \, d\Omega.$$

The linear mapping $\Phi(\mathbf{A}) = \nu \operatorname{Tr}(\mathbf{A}) \mathbf{1} + (1 - \nu) \mathbf{A}$ is positive and preserves symmetry.

Von-Kármán plates

Decomposition strain field

$$\boldsymbol{\varepsilon} = \boxed{\text{Grad } \mathbf{u}} + \boxed{1/2 \text{ grad } w \otimes \text{grad } w} - z \boxed{\text{Hess } w} = \boldsymbol{\varepsilon}_m - z \boldsymbol{\kappa}.$$

Linear membrane def. 

Quadratic membrane def. 

Linear bending def. 

Von-Kármán plate Dynamics

$$\rho h \partial_{tt} u = \text{Div } \mathbf{N},$$

$$\rho h \partial_{tt} w = -\text{div Div } \mathbf{M} + \text{div}(\mathbf{N} \text{ grad } w),$$

Total energy

$$H = \frac{1}{2} \int_{\Omega} \rho h \{ \|\partial_t \mathbf{u}\|^2 + \|\partial_t w\|^2 \} + D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m) : \mathbf{N} + D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}) : \mathbf{M} \, d\Omega$$

Port-Hamiltonian Von-Kármán plates

Energy variables

The Hamiltonian functional is quadratic in the following variables

$$\begin{array}{llll}\alpha_u = \rho h \partial_t \mathbf{u}, & \text{Axial momentum,} & \alpha_w = \rho h \partial_t w, & \text{Bending momentum,} \\ \mathbf{A}_\varepsilon = \varepsilon_m, & \text{Membrane strain,} & \mathbf{A}_\kappa = \kappa, & \text{Curvature}\end{array}$$

Co-energy variables

The variational derivative of the Hamiltonian gives the co-energy variables

$$\begin{array}{ll}e_u := \delta_{\alpha_u} H = \dot{\mathbf{u}}, & e_w := \delta_{\alpha_w} H = \dot{w}, \\ \mathbf{E}_\varepsilon := \delta_{\mathbf{A}_\varepsilon} H = D_m \Phi(\mathbf{A}_\varepsilon), & \mathbf{E}_\kappa := \delta_{\mathbf{A}_\kappa} H = D_b \Phi(\mathbf{A}_\kappa)\end{array}$$

or more compactly $e := \delta_\alpha H = \mathcal{Q}\alpha$ with

$$\mathcal{Q} = \text{Diag} [(\rho h)^{-1}, D_m \Phi, (\rho h)^{-1}, D_b \Phi].$$

The port-Hamiltonian realization

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_u \\ \mathbf{A}_\varepsilon \\ \alpha_w \\ \mathbf{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \text{Div} & \mathbf{0} & \mathbf{0} \\ \text{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ \mathbf{0} & \mathcal{C}(w) & \mathbf{0} & -\text{div Div} \\ \mathbf{0} & \mathbf{0} & \text{Grad grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \delta_{\alpha_u} H \\ \delta_{\mathbf{A}_\varepsilon} H \\ \delta_{\alpha_w} H \\ \delta_{\mathbf{A}_\kappa} H \end{pmatrix}.$$

The operator \mathcal{J} is formally skew-adjoint.

The coupling term reads

$$\begin{aligned} \mathcal{C}(w)(\cdot) : L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) &\rightarrow L^2(\Omega), \\ \mathbf{X} &\rightarrow \text{div}(\mathbf{X} \text{ grad } w), \end{aligned}$$

and its formal adjoint

$$\begin{aligned} \mathcal{C}(w)^*(\cdot) : L^2(\Omega) &\rightarrow L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}), \\ y &\rightarrow -\text{Sym} [\text{grad}(y) \otimes \text{grad}(w)]. \end{aligned}$$

Pure coenergy formulation

Incorporation of the constitutive equations

Once the \mathcal{Q} operator (matrix) is inverted, the dynamics is expressed :

$$\begin{pmatrix} \rho h \partial_t e_u \\ (D_m \Phi)^{-1} \partial_t \mathbf{E}_\varepsilon \\ \rho h \partial_t e_w \\ (D_b \Phi)^{-1} \partial_t \mathbf{E}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \text{Div} & \mathbf{0} & \mathbf{0} \\ \text{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & \mathcal{C}(w) & 0 & -\text{div Div} \\ \mathbf{0} & \mathbf{0} & \text{Grad grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_u \\ \mathbf{E}_\varepsilon \\ e_w \\ \mathbf{E}_\kappa \end{pmatrix},$$

where the vertical position is given by

$$w(t) = w_0 + \int_0^t e_w(\tau) \, d\tau.$$

Energy rate and boundary conditions

Proposition

The energy rate reads

$$\dot{H} = \langle \gamma_0 \mathbf{e}_u | \gamma_\perp \mathbf{E}_\varepsilon \rangle_{\partial\Omega} + \langle \gamma_0 \mathbf{e}_w | \gamma_{\perp\perp,1} \mathbf{E}_\kappa + \gamma_0 (\mathbf{E}_\varepsilon \mathbf{n} \cdot \text{grad } w) \rangle_{\partial\Omega} + \langle \gamma_1 \mathbf{e}_w | \gamma_{\perp\perp} \mathbf{E}_\kappa \rangle_{\partial\Omega},$$

- ▶ $\gamma_0 \mathbf{e}_u = \mathbf{e}_u|_{\partial\Omega}$ is the Dirichlet trace;
- ▶ $\gamma_\perp \mathbf{E}_\varepsilon = \mathbf{E}_\varepsilon \mathbf{n}|_{\partial\Omega}$ is the normal trace;
- ▶ $\gamma_{\perp\perp,1} \mathbf{E}_\kappa = -\mathbf{n} \cdot \text{Div } \mathbf{E}_\kappa - \partial_s(\mathbf{n}^\top \mathbf{E}_\kappa \mathbf{s})|_{\partial\Omega}$ is the effective shear force;
- ▶ $\gamma_1 \mathbf{e}_w = \partial_n \mathbf{e}_w|_{\partial\Omega}$ is the normal derivative trace;
- ▶ $\gamma_{\perp\perp} \mathbf{E}_\kappa = \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n}$ is the normal to normal trace.

Boundary conditions classification

BCs	Traction	Bending	
Kinematical/Dirichlet	$\gamma_0 \mathbf{e}_u$	$\gamma_0 \mathbf{e}_w$	$\gamma_1 \mathbf{e}_w$
Dynamical/Neumann	$\gamma_\perp \mathbf{E}_\varepsilon$	$\gamma_{\perp\perp,1} \mathbf{E}_\kappa + \gamma_0 (\mathbf{E}_\varepsilon \mathbf{n} \cdot \text{grad } w)$	$\gamma_{\perp\perp} \mathbf{E}_\kappa$

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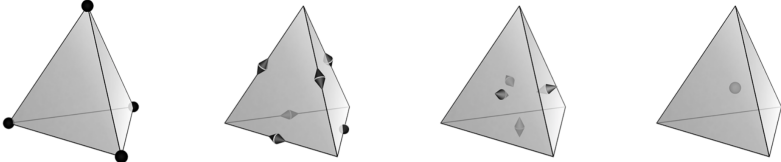
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Hilbert complexes and stable discretization of PDEs

Crucial concept to derive stable convergent approximations: **Hilbert complexes**.

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\text{grad}} & H^{\text{curl}}(\Omega) & \xrightarrow{\text{curl}} & H^{\text{div}}(\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\ \downarrow \Pi_{s,h}^{-,0} & & \downarrow \Pi_{s,h}^{-,1} & & \downarrow \Pi_{s,h}^{-,2} & & \downarrow \Pi_{s,h}^{-,1} \\ \text{CG}_s(\Omega_h) & \xrightarrow{\text{grad}} & \text{NED}_s^1(\Omega_h) & \xrightarrow{\text{curl}} & \text{RT}_s(\Omega_h) & \xrightarrow{\text{div}} & \text{DG}_{s-1}(\Omega_h) \end{array}$$


The Whitney forms $s = 0$ form a subcomplex of the de Rham complex

This framework is well developed for linear problems¹ and open source finite elements libraries implement them. Definitely less for non linear elasticity problems.

¹Douglas N. Arnold, Richard S. Falk, and Ragnar Winther (2006). "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numerica* 15, pp. 1–155.

Weak formulation for 2D Elastodynamics

Grad primal formulation

Suppose the traction (Neumann) boundary condition $\gamma_{\perp} \mathbf{E}_{\varepsilon} = \mathbf{g}_N$.

Find $\mathbf{e}_u \in H^1(\Omega; \mathbb{R}^2)$, $\mathbf{E}_{\varepsilon} \in H(\text{rot rot}, \Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$

such that $\forall \psi_u \in H^1(\Omega; \mathbb{R}^2)$, $\forall \Psi_{\varepsilon} \in H(\text{rot rot}, \Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ the following holds

$$\begin{aligned}(\psi_u, \rho h \partial_t \mathbf{e}_u)_{\Omega} &= -(\text{Grad } \psi_u, \mathbf{e}_u)_{\Omega} + \langle \gamma_0 \psi_u | \mathbf{g}_N \rangle_{\partial \Omega}, \\(\Psi_{\varepsilon}, (D_m \Phi)^{-1} \partial_t \mathbf{E}_{\varepsilon})_{\Omega} &= (\Psi_{\varepsilon}, \text{Grad } \mathbf{e}_u)_{\Omega}.\end{aligned}$$

This formulation is related to the strain elasticity complex in 2D

$$H^1(\Omega; \mathbb{R}^2) \xrightarrow{\text{Grad}} H(\text{rot rot}, \Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) \xrightarrow{\text{rot rot}} L^2(\Omega).$$

Weak formulation for 2D Elastodynamics

Div dual formulation

Suppose the kinematic (Dirichlet) boundary condition $\gamma_0 \mathbf{e}_u = \mathbf{g}_D$.

Find $\mathbf{e}_u \in L^2(\Omega; \mathbb{R}^2)$, $\mathbf{E}_\varepsilon \in H(\text{Div}, \Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$

such that $\forall \psi_u \in L^2(\Omega; \mathbb{R}^2)$, $\forall \Psi_\varepsilon \in H(\text{Div}, \Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ the following holds

$$\begin{aligned}(\psi_u, \rho h \partial_t \mathbf{e}_u)_\Omega &= (\psi_u, \text{Div } \mathbf{E}_\varepsilon)_\Omega, \\(\Psi_\varepsilon, (D_m \Phi)^{-1} \partial_t \mathbf{E}_\varepsilon)_\Omega &= -(\text{Div } \Psi_\varepsilon, \mathbf{e}_u)_\Omega + \langle \gamma_\perp \mathbf{E}_\varepsilon | \mathbf{g}_D \rangle_{\partial\Omega}.\end{aligned}$$

This formulation is related to the Elasticity complex in 2D

$$H^2(\Omega) \xrightarrow{\text{curl curl}} H(\text{Div}, \Omega; \mathbb{S}) \xrightarrow{\text{Div}} L^2(\Omega; \mathbb{R}^2).$$

The theory is not directly applicable to this problem

Semi discretization in space is obtained using finite elements forming a **subcomplex of original Hilbert complex**. But the theory is still under development² and no open source software.

So heuristic choice of the finite elements.

²Long Chen and Xuehai Huang (2022). “Finite Element Complexes in Two Dimensions”. In: *arXiv preprint arXiv:2206.00851*.

Heuristic Finite element choice and final system

For the proposed weak formulation, the following FE spaces are selected

$$\mathbf{e}_u^h \in \text{CG}_{2s-1}, \quad \mathbf{E}_\varepsilon^h \in \text{DG}_{2s-2}, \quad e_w^h \in \text{CG}_s, \quad \mathbf{E}_\kappa^h \in \text{HHJ}_{s-1}, \quad s \geq 1.$$

Combination of spectral FE for the membrane part and the non conforming Hellan-Hermann-Johnson method for bending.

$$\begin{aligned} (\psi_u^h, \rho h \partial_t \mathbf{e}_u^h)_\Omega &= -(\text{Grad } \psi_u^h, \mathbf{e}_u^h)_\Omega, \\ (\Psi_\varepsilon^h, (D_m \Phi)^{-1} \partial_t \mathbf{E}_\varepsilon^h)_\Omega &= (\Psi_\varepsilon^h, \text{Grad } \mathbf{e}_u^h)_\Omega + (\Psi_\varepsilon^h, \text{Sym}\{\text{grad } e_w^h \otimes \text{grad } w^h\})_\Omega, \\ (\psi_w^h, \rho b \partial_t e_w^h)_\Omega &= +d_h(\psi_w^h, \mathbf{E}_\kappa^h) - (\text{Sym}\{\text{grad } \psi_w^h \otimes \text{grad } w^h\}, \mathbf{E}_\varepsilon^h)_\Omega, \\ (\Psi_\kappa^h, (D_b \Phi)^{-1} \partial_t \mathbf{E}_\kappa^h)_\Omega &= -d_h(e_w^h, \Psi_\kappa^h). \end{aligned}$$

Finite dimensional system (Galerkin projection)

$$\mathbf{M} \dot{\mathbf{e}} = \mathbf{J}(\mathbf{w}) \mathbf{e} + \mathbf{B} \mathbf{u},$$

$$\dot{\mathbf{w}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{e},$$

$$\mathbf{y} = \mathbf{B}^\top \mathbf{e}.$$

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Manufactured solution

Consider $\Omega = [0, 1] \times [0, 1]$. The following manufactured solution is considered

$$\mathbf{u}^{\text{ex}} = \begin{pmatrix} x^4(1-x^4)\sin^2(\pi y) \\ \sin^2(\pi x)y^4(1-y^4) \end{pmatrix} \sin(2\pi t), \quad w^{\text{ex}} = \sin(\pi x)\sin(\pi y)\sin(2\pi t),$$

together with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad w|_{\partial\Omega} = 0, \quad \mathbf{n}^\top \mathbf{M} \mathbf{n}|_{\partial\Omega} = 0.$$

A Crank-Nicholson scheme is used for time integration.

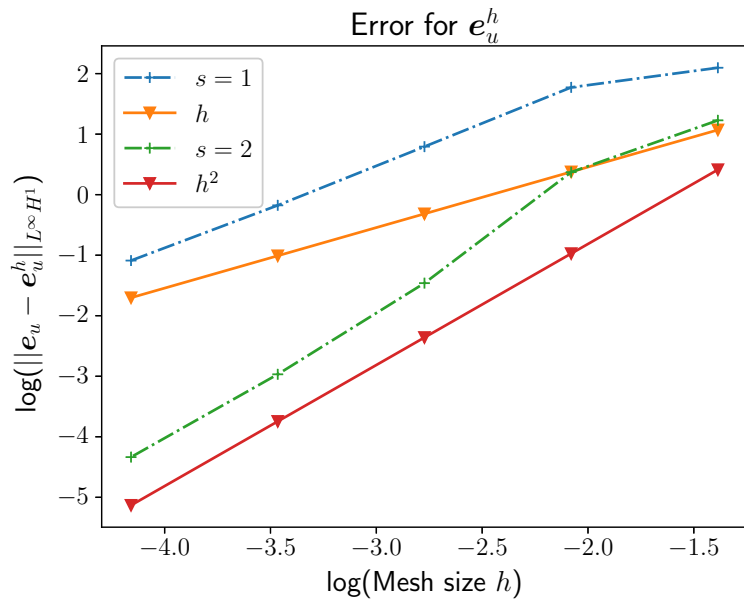
Convergence measure

The discrete time-space norm $L_{\Delta t}^\infty(\mathcal{X})$ ($\mathcal{X} = H^1$ or L^2) is used to measure convergence

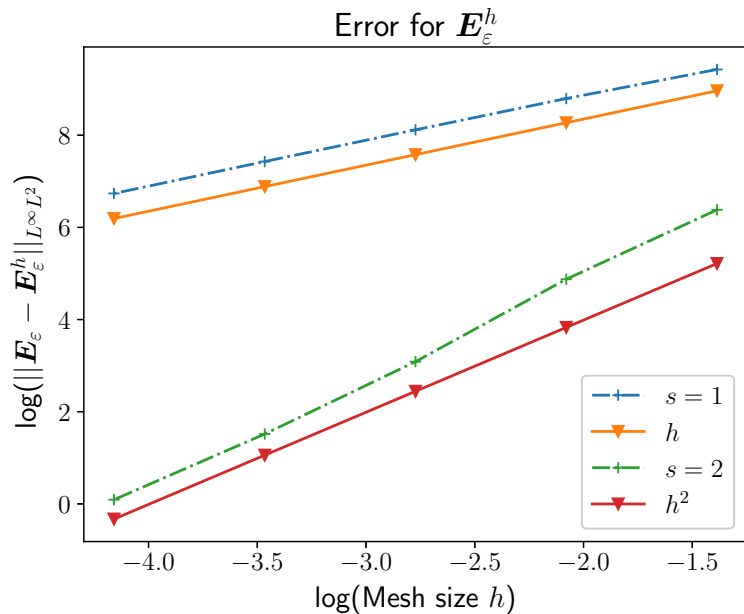
$$\|\cdot\|_{L^\infty(\mathcal{X})} \approx \|\cdot\|_{L_{\Delta t}^\infty(\mathcal{X})} = \max_{t \in t_i} \|\cdot\|_{\mathcal{X}},$$

where t_i are the discrete simulation instants.

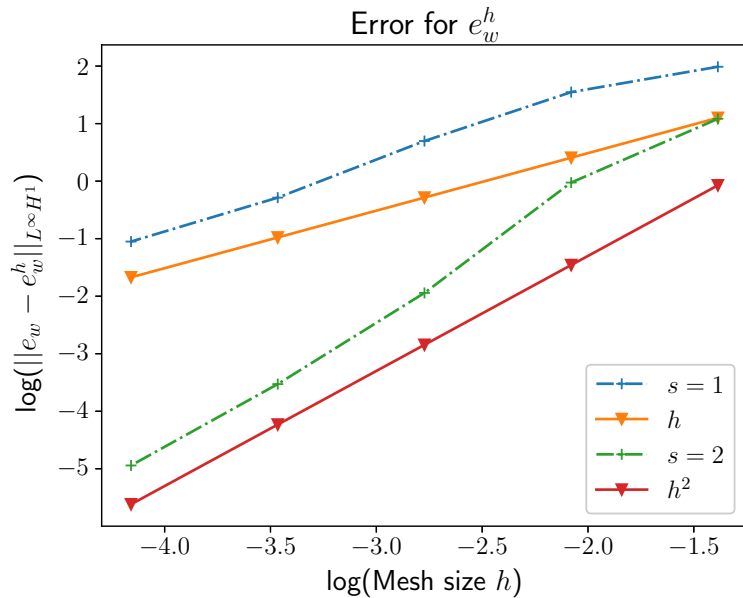
Results



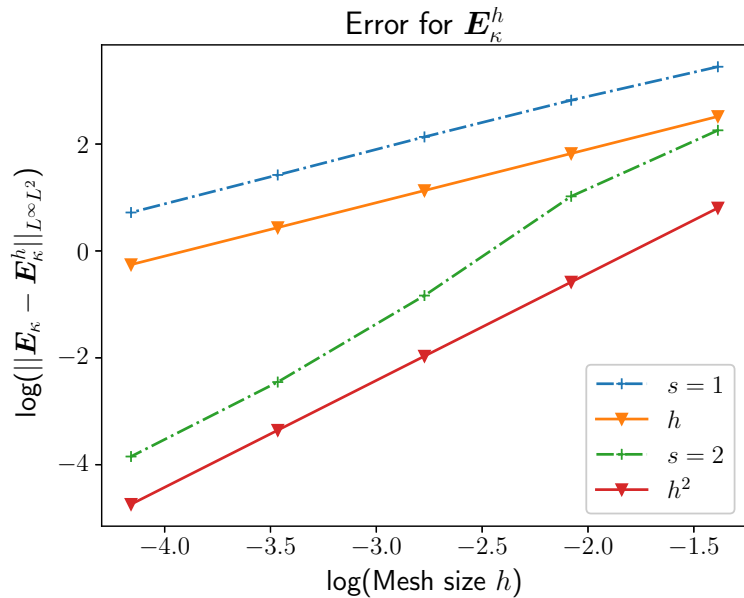
Results



Results



Results



Conclusion and Outlook

- ▶ First step into pH non linear mechanics. The **geometrical non linearities** belong to the **interconnection operator**.
- ▶ Can be used to study more complex phenomena. The discretization method guarantees **exact discrete energy conservation** when **symplectic time integration** is used.
- ▶ **System theory perspective**: incorporates **interactions** with the environment by means of the **boundary conditions**.

References I



Arnold, Douglas N., Richard S. Falk, and Ragnar Winther (2006). “Finite element exterior calculus, homological techniques, and applications”. In: *Acta Numerica* 15, pp. 1–155.



Chen, Long and Xuehai Huang (2022). “Finite Element Complexes in Two Dimensions”. In: *arXiv preprint arXiv:2206.00851*.

Mixed finite element construction: Kirchhoff plate

Find $(e_w, \mathbf{E}_\kappa) \in \text{CG}_s \times \text{HHJ}_{s-1}$ such that

$$\begin{aligned} (\psi_w, \rho b \partial_t e_w)_\Omega &= +d_h(\psi_w, \mathbf{E}_\kappa), & \forall \psi_w \in \text{CG}_s, \\ (\mathbf{\Psi}_\kappa, (D_b \Phi)^{-1} \partial_t \mathbf{E}_\kappa)_\Omega &= -d_h(e_w, \mathbf{\Psi}_\kappa), & \forall \mathbf{\Psi}_\kappa \in \text{HHJ}_{s-1}. \end{aligned}$$

This method is non-conforming and includes inter-cells terms

$$d_h(v_w, \mathbf{E}_\kappa) := - \sum_{T \in \mathcal{T}_h} (\text{Hess } v_w, \mathbf{E}_\kappa)_\Omega + \sum_{E \in \mathcal{E}_h} ([\partial_n v_w], \mathbf{n}^\top \mathbf{E}_\kappa \mathbf{n})_\Omega.$$

The div Div distributional complex in 2D and the HHJ finite element complex:

$$\begin{array}{ccccc} H^1(\Omega; \mathbb{R}^2) & \xrightarrow{\text{Sym curl}} & H^{-1}(\text{div Div}, \Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) & \xrightarrow{\text{div Div}} & L^2(\Omega) \\ \downarrow I_h & & \downarrow \Pi_h & & \downarrow Q_h \\ \text{CG}_s & \xrightarrow{\text{Sym curl}} & \text{HHJ}_{s-1} & \xrightarrow{\text{div Div}_h} & \text{CG}_s, \end{array}$$