Partitioned Finite Element Method for the Mindlin Plate as a Port-Hamiltonian system

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PH framework and elasticity

The pH framework is appealing for its modularity and for being an interdisciplinary modeling tool.
Main points of interest in this presentation:

- extend the pH framework to linear elasticity in 2D[1];
- make use of the widespread finite element method[2] [3] [4]

[1] A. Macchelli, C. Melchiorri, and L. Bassi. "Port-based Modelling and Control of the Mindlin Plate". In: *Proceedings of the 44th IEEE Conference on Decision and Control.* 2005, pp. 5989–5994.

[2] P. Kotyczka, B. Maschke, and L. Lefèvre. "Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems". In: *Journal of Computational Physics* 361 (2018), pp. 442 –476.

[3] F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. "A structure-preserving Partitioned Finite Element Method for the 2D wave equation". In: *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control.* Valparaíso, CL, 2018, pp. 1–6.

[4] D. Arnold and J. Lee. "Mixed Methods for Elastodynamics with Weak Symmetry". In: *SIAM Journal on Numerical Analysis* 52.6 (2014), pp. 2743–2769.

Plan

The Mindlin-Reissner model

The classical model is a system 3PDEs:

$$\begin{cases} \rho h \dfrac{\partial^2 w}{\partial t^2} & = \mathrm{div}(\boldsymbol{q}), \\ \rho \dfrac{h^3}{12} \dfrac{\partial^2 \boldsymbol{\theta}}{\partial t^2} & = \boldsymbol{q} + \mathrm{Div}(\boldsymbol{M}), \end{cases}$$

with the parameters and variables

- ρ the material density;
- h the plate thickness;
- the vertical displacement scalar field w;
- the cross section deflection vector field $\boldsymbol{\theta} = (\theta_x, \theta_y)$;
- the bending symmetric tensor field $\boldsymbol{M}$;
- shear stress vector field $\boldsymbol{q}$;

The divergence of a tensor field is a vector defined column-wise as

$$\mathrm{Div}(\boldsymbol{M}) := \left(\sum_{\alpha=1}^{2} \partial_{x_\alpha} m_{\alpha\beta} \right)_{\beta=1,\dots,2} .$$

Constitutive equations

For an homogeneous, isotropic material (Greek indexes equal 1,2)

$$\boldsymbol{M}_{\alpha\beta} = \boldsymbol{D}_{\alpha\beta\iota\lambda}\boldsymbol{K}_{\iota\lambda} \qquad \boldsymbol{q}_{\alpha} = \boldsymbol{C}_{\alpha\beta}\boldsymbol{\gamma}_{\beta}$$

The fourth and second order tensor $\boldsymbol{D}_{\alpha\beta\iota\lambda}$ (bending stiffness) and $\boldsymbol{C}_{\alpha\beta}$ (shear stiffness) are symmetric, positive definite.

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The fourth and second order tensor $\boldsymbol{D}_{\alpha\beta\iota\lambda}$ (bending stiffness) and $\boldsymbol{C}_{\alpha\beta}$ (shear stiffness) are symmetric, positive definite.

The variables

$$\boldsymbol{K} := \mathrm{Grad}(\boldsymbol{\theta}), \qquad \boldsymbol{\gamma} := \mathrm{grad}(w) - \boldsymbol{\theta}.$$

are the bending curvature and shear strain. The symmetric gradient of a vector field is defined as

$$\mathrm{Grad}(\boldsymbol{\theta}) := \frac{1}{2}\left(\nabla\boldsymbol{\theta} + \nabla^T\boldsymbol{\theta}\right).$$

Hamiltonian energy and pH system

The Hamiltonian (total energy) is given by

$$H = \frac{1}{2} \int_\Omega \underbrace{\rho h \left(\frac{\partial w}{\partial t}\right)^2 + \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t} \cdot \frac{\partial \boldsymbol{\theta}}{\partial t}}_{\text{Kinetic energy}} + \underbrace{\boldsymbol{M} : \boldsymbol{K} + \boldsymbol{q} \cdot \boldsymbol{\gamma}}_{\text{Potential energy}} \, \mathrm{d}\Omega,$$

where $\boldsymbol{M} : \boldsymbol{K} := \sum_{\alpha,\beta} m_{\alpha\beta} \, \kappa_{\alpha\beta}$ is the tensor contraction.

Port Hamiltonian systems

Linear port Hamiltonian system

$$
\begin{cases}
\dfrac{\partial \alpha}{\partial t} = J \dfrac{\delta H}{\delta \alpha}, \\
H = < \alpha, Q\alpha >_{\mathcal{L}^2} \\
e := \dfrac{\delta H}{\delta \alpha} = Q\alpha,
\end{cases}
$$

Jargon:

- α: energies;
- H: Hamiltonian;
- e: coenergies.

Operators:

- J: skew symmetric unbounded operator;
- Q: bounded symmetric.

How do we get there?

Energy, coenergy variables

Energy variables:

$$
\begin{aligned}
\alpha_w &= \rho h \frac{\partial w}{\partial t}, & \qquad \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12} \frac{\partial \boldsymbol{\theta}}{\partial t}, \\
\boldsymbol{A}_\kappa &= \boldsymbol{K}, & \qquad \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}.
\end{aligned}
$$

Energy, coenergy variables

Energy variables:

$$
\begin{aligned}
\alpha_w &= \rho h \frac{\partial w}{\partial t}, & \boldsymbol{\alpha}_\theta &= \frac{\rho h^3}{12}\frac{\partial \boldsymbol{\theta}}{\partial t}, \\
\boldsymbol{A}_\kappa &= \boldsymbol{K}, & \boldsymbol{\alpha}_\gamma &= \boldsymbol{\gamma}.
\end{aligned}
$$

The coenergies are given by the Hamiltonian variational derivative

$$
\begin{aligned}
e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, & \boldsymbol{e}_\theta &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_\theta} = \frac{\partial \boldsymbol{\theta}}{\partial t}, \\
\boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, & \boldsymbol{e}_{\epsilon_s} &:= \frac{\delta H}{\delta \boldsymbol{\alpha}_{\boldsymbol{\gamma}}} = \boldsymbol{q}.
\end{aligned}
$$

PH system

The classical model is rewritten as port-Hamiltonian system using the energy and coenergy variables

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ 0 & \mathrm{Grad} & 0 & 0 \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}}_{J} \begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix},$$

with J skew symmetric and

$$\begin{pmatrix} e_w \\ \boldsymbol{e}_\theta \\ \boldsymbol{E}_\kappa \\ \boldsymbol{e}_\gamma \end{pmatrix} = \underbrace{\begin{bmatrix} 1/(\rho h) & 0 & 0 & 0 \\ 0 & 12/(\rho h^3) & 0 & 0 \\ 0 & 0 & \boldsymbol{D} & 0 \\ 0 & 0 & 0 & \boldsymbol{C} \end{bmatrix}}_{Q} \begin{pmatrix}\alpha_w \\ \boldsymbol{\alpha}_\theta \\ \boldsymbol{A}_\kappa \\ \boldsymbol{\alpha}_\gamma\end{pmatrix},$$

with Q coercive.

PH system

$$J = \begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & \mathrm{Div} & \boldsymbol{I}_{2\times 2} \\ 0 & \mathrm{Grad} & 0 & 0 \\ \mathrm{grad} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1/(\rho h) & 0 & 0 & 0 \\ 0 & 12/(\rho h^3) & 0 & 0 \\ 0 & 0 & \boldsymbol{D} & 0 \\ 0 & 0 & 0 & \boldsymbol{C} \end{bmatrix}$$

Strong form for the Mindlin plate

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} & = J e, \\ e & = Q\alpha, \\ H & = < \alpha, Q\alpha >_{\mathcal{L}^2} = < \alpha, e >_{\mathcal{L}^2} \end{cases}$$

$$\alpha := (\alpha_w, \boldsymbol{\alpha}_\theta, \boldsymbol{A}_\kappa, \boldsymbol{\alpha}_\gamma), \qquad e := (e_w, \boldsymbol{e}_\theta, \boldsymbol{E}_\kappa, \boldsymbol{e}_\gamma),$$

$$\alpha, \ e \in \mathcal{L}^2 = L^2(\Omega) \times L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \times L^2(\Omega, \mathbb{R}^2)$$

Boundary variables

Taking the energy rate and applying of the Green theorem

$$\dot{H} = \int_{\partial\Omega} \{ w_t \, q_n + \omega_n \, m_{nn} + \omega_s \, m_{ns} \} \; \mathrm{d}s.$$

The dynamic boundary variable are defined as

$$\begin{aligned} \text{Shear Force} \qquad & q_n := \boldsymbol{q} \cdot \boldsymbol{n}, \\ \text{Flexural momentum} \qquad & m_{nn} := \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}), \\ \text{Torsional momentum} \qquad & m_{ns} := \boldsymbol{M} : (\boldsymbol{s} \otimes \boldsymbol{n}), \end{aligned}$$

where $\boldsymbol{u} \otimes \boldsymbol{v}$ denotes the outer product of vectors.
The corresponding power conjugated boundary variables are

$$\begin{aligned} \text{Vertical velocity} \qquad & w_t := \partial_t w, \\ \text{Flexural rotation} \qquad & \omega_n := \partial_t \boldsymbol{\theta} \cdot \boldsymbol{n}, \\ \text{Torsional rotation} \qquad & \omega_s := \partial_t \boldsymbol{\theta} \cdot \boldsymbol{s}. \end{aligned}$$

Boundary variables

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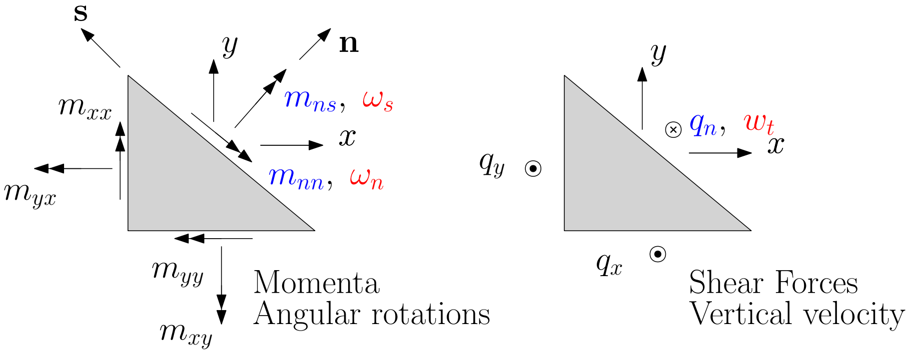
$$\dot{H} = \int_{\partial\Omega} \{ w_t \, q_n + \omega_n \, m_{nn} + \omega_s \, m_{ns} \} \, \mathrm{d}s.$$



Plan

1. Introduction
2. PH formulation of the Mindlin plate
3. Structure preserving discretization
   - Boundary control through forces and momenta
   - Boundary control through kinematic variables
4. Discretization procedure
5. Numerical simulations
6. Conclusion

Main step to follow

The structure-preserving discretization consists of three steps:

1. write the system in weak form;
2. perform integrations by parts to get the chosen boundary control;
3. select the finite element spaces to achieve a finite-dimensional system.

J decomposition

Decomposing J in operators to be integrated by parts

$$J = {\color{red}J_{\mathrm{div}}} + {\color{blue}J_{\mathrm{grad}}} + J_{\boldsymbol{I}}$$

$$J = \begin{bmatrix} 0 & 0 & 0 & {\color{red}\mathrm{div}} \\ 0 & 0 & {\color{red}\mathrm{Div}} & \boldsymbol{I}_{2\times 2} \\ 0 & {\color{blue}\mathrm{Grad}} & 0 & 0 \\ {\color{blue}\mathrm{grad}} & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}$$

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$$J_{\mathrm{div}} := \begin{bmatrix} 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & \mathrm{Div} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

$$J_{\text{grad}} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \text{Grad} & 0 & 0 \\ \text{grad} & 0 & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = J_{\text{div}} + J_{\text{grad}} + J_{\boldsymbol{I}}$$

$$J_{\boldsymbol{I}} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{I}_{2\times 2} \\ 0 & 0 & 0 & 0 \\ 0 & -\boldsymbol{I}_{2\times 2} & 0 & 0 \end{bmatrix}$$

J decomposition

Decomposing J in operators to be integrated by parts

$$J = {\color{red}J_{\mathrm{div}}} + {\color{blue}J_{\mathrm{grad}}} + J_{\boldsymbol{I}}$$

From these definitions, it holds

$${\color{red}J_{\mathrm{div}}} = -{\color{blue}J_{\mathrm{grad}}^*},$$

where A^* is the formal adjoint of operator A.

Weak form

Basic Weak form (before the integration by parts)

$$\left(v, \frac{\partial \alpha}{\partial t}\right)_{\mathcal{L}^2} = (v, Je)_{\mathcal{L}^2}.$$

In order to preserve the pH structure the bilinear form (v, Je) has to give rise to a skew symmetric matrix. Two strategies naturally achieve this goal:

Weak form

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1. integrating by parts J_{div} (gradient formulation)

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2. integrating by parts J_{grad} (divergence formulation)

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In order to preserve the pH structure the bilinear form (v, Je) has to give rise to a skew symmetric matrix. Two strategies naturally achieve this goal:

1. integrating by parts J_{div} (gradient formulation)
2. integrating by parts J_{grad} (divergence formulation)

Remark: other choices are possible but less physical.

Gradient formulation

If the operator J_{div} is integrated by parts

$$(v, Je) = j_{\mathrm{grad}}(v, e) + f_N(v),$$

the bilinear form

$$\begin{aligned} j_{\mathrm{grad}}(v, e) &= (J_{\mathrm{div}}^* v, e) + (v, J_{\mathrm{grad}}\, e) + (v, J_{\boldsymbol{I}}\, e), \\ &= (-J_{\mathrm{grad}}\, v, e) + (v, J_{\mathrm{grad}}\, e) + (v, J_{\boldsymbol{I}}\, e), \end{aligned}$$

is skew symmetric.

Gradient formulation

The functional

$$
\begin{aligned}
f_N(v) &= \int_{\partial\Omega} \left\{ v_w \, q_n + v_{\omega_n} \, m_{nn} + v_{\omega_s} \, m_{ns} \right\} \, \mathrm{d}s, \\
&= \int_{\partial\Omega} v_\partial \, u_\partial \, \mathrm{d}s.
\end{aligned}
$$

express the boundary control $\boldsymbol{u}_\partial$ in terms of forces and momenta:

$$
\boldsymbol{u}_\partial = \mathrm{Trace} \begin{pmatrix} q_n \\ m_{nn} \\ m_{ns} \end{pmatrix}, \qquad \boldsymbol{y}_\partial = \mathrm{Trace} \begin{pmatrix} w_t \\ \omega_n \\ \omega_s \end{pmatrix}.
$$

Divergence formulation

If the operator J_{grad} is integrated by parts

$$(v, Je) = j_{\text{div}}(v, e) + f_D(v), \tag{1}$$

the bilinear form

$$\begin{aligned} j_{\text{div}}(v, e) &= (v, J_{\text{div}}\, e) + (J^*_{\text{grad}}\, v, e) + (v, J_{\boldsymbol{I}}\, e), \\ &= (v, J_{\text{div}}\, e) + (-J_{\text{div}}\, v, e) + (v, J_{\boldsymbol{I}}\, e), \end{aligned}$$

is skew symmetric.

Divergence formulation

The functional

$$
\begin{aligned}
f_D(v) &= \int_{\partial\Omega} \left\{ v_{q_n} w_t + v_{m_{nn}} \omega_n + v_{m_{ns}} \omega_s \right\} \, \mathrm{d}s, \\
&= \int_{\partial\Omega} v_\partial u_\partial \, \mathrm{d}s.
\end{aligned}
$$

expresses the boundary controls u_∂ in terms of linear and angular velocities:

$$
u_\partial = \mathrm{Trace} \begin{pmatrix} w_t \\ \omega_n \\ \omega_s \end{pmatrix}, \qquad y_\partial = \mathrm{Trace} \begin{pmatrix} q_n \\ m_{nn} \\ m_{ns} \end{pmatrix}.
$$

Plan

Homogeneous boundary conditions

Homogeneous boundary conditions:

- Clamped (C): $w_t = 0,\ \omega_n = 0,\ \omega_s = 0;$
- Simply supported hard (S): $w_t = 0,\ m_{nn} = 0,\ \omega_s = 0;$
- Free (F): $q_n = 0,\ m_{nn} = 0,\ m_{ns} = 0.$

The gradient formulation is adopted to discretize the system. This implies that

- variables in Blue are imposed weakly by setting $f_N(v) = 0$
- variables in Red have to be imposed strongly (by select a functional space that incorporates those or by introducing Lagrange multipliers)

Galerkin method

Test and co-energy variables are discretized by a Galerkin Method, while energy variables are retrieve using the relation $\alpha = Q^{-1} e$.

Finite-dimensional system

Replacing the approximated variables into the weak form

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} \boldsymbol{J}_{\text{grad}} & \boldsymbol{G}_D \\ -\boldsymbol{G}_D{}^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{bmatrix} \boldsymbol{B}_N \\ 0 \end{bmatrix} \boldsymbol{u}_N$$

- $\boldsymbol{G}_D$ accounts for Dirichlet (essential) BCs;
- $\boldsymbol{B}_N$ account for inhomogeneous Neumann (natural) BCs;

Finite element (FE) choice

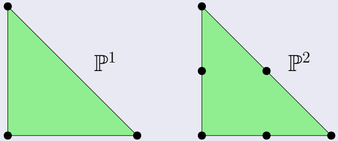
Domain of J:

$$\mathcal{D}(J) = H^1(\Omega) \times H^1(\Omega, \mathbb{R}^2) \times H^{\mathrm{Div}}(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \times H^{\mathrm{div}}(\Omega, \mathbb{R}^2) \quad + \mathrm{BCs}.$$

Heuristic for selecting stable FE

Given the symmetric structure of the problem, all variables (v, e, λ) are discretized by the same FE space (same family, same degree). The analysis were conducted using two different spaces:

1. the first order Lagrange polynomials $\mathbb{P}_1$;
2. the second order Lagrange polynomials $\mathbb{P}_2$.



Plan

Eigenvalues analysis for a square plate

Non-dimensional eigenfrequencies:

$$\widehat{\omega}^h_{mn} = \omega^h_{mn} L \left(\frac{2(1+\nu)\rho}{E} \right)^{1/2} \tag{1}$$

m and n being the numbers of half-waves occurring in the modes shapes in the x and y directions. The only parameters which influence the results are the Poisson's ratio $\nu = 0.3$ (fixed) and the thickness-to-span ratio h/L.
The error is computed by[5]

$$\varepsilon = \frac{\mathrm{abs}(\widehat{\omega}^h_{mn} - \omega^{DR}_{mn})}{\omega^{DR}_{mn}}. \tag{2}$$

The eigenproblem is solved using the QR algorithm.

---

[5] D.J. Dawe and O.L. Roufaeil. "Rayleigh-Ritz vibration analysis of Mindlin plates". In: *Journal of Sound and Vibration* 69.3 (1980), pp. 345–359.

Eigenvalues for the thick case $h/L = 0.1$

BCs	Mode	$N = 10$	$N = 20$	D-R
CCCC	$\widehat{\omega}_{11}$	1.5999	1.5917	1.594
	$\widehat{\omega}_{21}$	3.0615	3.0410	3.046
	$\widehat{\omega}_{12}$	3.0615	3.0410	3.046
	$\widehat{\omega}_{22}$	4.3161	4.2682	4.285
SSSS	$\widehat{\omega}_{11}$	0.9324	0.9324	0.930
	$\widehat{\omega}_{21}$	2.2227	2.2226	2.219
	$\widehat{\omega}_{12}$	2.2227	2.2226	2.219
	$\widehat{\omega}_{22}$	3.4142	3.3608	3.406
SCSC	$\widehat{\omega}_{11}$	1.3111	1.3013	1.302
	$\widehat{\omega}_{21}$	2.4155	2.3966	2.398
	$\widehat{\omega}_{12}$	2.9082	2.8871	2.888
	$\widehat{\omega}_{22}$	3.8906	3.8458	3.852
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	1.0855	1.0982	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7636	1.7461	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.6696	2.6575	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.2248	3.1997	3.216

Table: Eigenvalues for $h/L = 0.1$ using $\mathbb{P}_1$:
■ reference, ■ $\varepsilon < 2\%$.

Eigenvalues for the thick case $h/L = 0.1$

BCs	Mode	$N=5$	$N=10$	D-R
CCCC	$\widehat{\omega}_{11}$	1.5976	1.5914	1.594
	$\widehat{\omega}_{21}$	3.0584	3.0405	3.046
	$\widehat{\omega}_{12}$	3.0677	3.0405	3.046
	$\widehat{\omega}_{22}$	4.3109	4.2662	4.285
SSSS	$\widehat{\omega}_{11}$	0.9304	0.9302	0.930
	$\widehat{\omega}_{21}$	2.2223	2.2194	2.219
	$\widehat{\omega}_{12}$	2.2224	2.2194	2.219
	$\widehat{\omega}_{22}$	3.4128	3.4061	3.406
SCSC	$\widehat{\omega}_{11}$	1.3053	1.3004	1.302
	$\widehat{\omega}_{21}$	2.4040	2.3946	2.398
	$\widehat{\omega}_{12}$	2.9060	2.8858	2.888
	$\widehat{\omega}_{22}$	3.8721	3.8415	3.852
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	1.0845	1.0797	1.089
	$\widehat{\omega}_{\frac{3}{2}1}$	1.7559	1.7425	1.758
	$\widehat{\omega}_{\frac{1}{2}2}$	2.6762	2.6547	2.673
	$\widehat{\omega}_{\frac{5}{2}1}$	3.2186	3.1954	3.216

Table: Eigenvalues for $h/L = 0.1$ using $\mathbb{P}_2$:
reference, $\varepsilon < 2\%$, $\varepsilon < 5\%$, $\varepsilon < 15\%$.

Eigenvalues for the thin case $h/L = 0.01$

BCs	Mode	$N = 10$	$N = 20$	D-R
CCCC	$\widehat{\omega}_{11}$	0.1967	0.1765	0.1754
	$\widehat{\omega}_{21}$	0.4030	0.3604	0.3576
	$\widehat{\omega}_{12}$	0.4030	0.3604	0.3576
	$\widehat{\omega}_{22}$	0.6431	0.5358	0.5274
SSSS	$\widehat{\omega}_{11}$	0.1706	0.1128	0.0963
	$\widehat{\omega}_{21}$	0.3576	0.2660	0.2406
	$\widehat{\omega}_{12}$	0.3576	0.2660	0.2406
	$\widehat{\omega}_{22}$	0.5803	0.4442	0.3848
SCSC	$\widehat{\omega}_{11}$	0.1864	0.1487	0.1411
	$\widehat{\omega}_{21}$	0.3649	0.2829	0.2668
	$\widehat{\omega}_{12}$	0.3987	0.3485	0.3377
	$\widehat{\omega}_{22}$	0.6075	0.4933	0.4608
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	0.1238	0.1166	0.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.2207	0.1954	0.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3204	0.3078	0.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.4144	0.3751	0.3740

Table: Eigenvalues for $h/L = 0.01$ using $\mathbb{P}_1$: ■ reference, ■ $\varepsilon < 2\%$, ■ $\varepsilon < 5\%$, ■ $\varepsilon < 15\%$, ■ $\varepsilon < 30\%$, ■ $\varepsilon < 50\%$, ■ $\varepsilon < 80\%$.

Eigenvalues for the thin case $h/L = 0.01$

BCs	Mode	$N=5$	$N=10$	D-R
CCCC	$\widehat{\omega}_{11}$	0.1872	0.1762	0.1754
	$\widehat{\omega}_{21}$	0.3725	0.3598	0.3576
	$\widehat{\omega}_{12}$	0.4055	0.3598	0.3576
	$\widehat{\omega}_{22}$	0.6043	0.5335	0.5274
SSSS	$\widehat{\omega}_{11}$	0.0963	0.0963	0.0963
	$\widehat{\omega}_{21}$	0.2422	0.2406	0.2406
	$\widehat{\omega}_{12}$	0.2430	0.2406	0.2406
	$\widehat{\omega}_{22}$	0.3874	0.3848	0.3848
SCSC	$\widehat{\omega}_{11}$	0.1492	0.1418	0.1411
	$\widehat{\omega}_{21}$	0.2827	0.2683	0.2668
	$\widehat{\omega}_{12}$	0.3608	0.3394	0.3377
	$\widehat{\omega}_{22}$	0.4940	0.4654	0.4608
CCCF	$\widehat{\omega}_{\frac{1}{2}1}$	0.1197	0.1169	0.1171
	$\widehat{\omega}_{\frac{3}{2}1}$	0.2092	0.1960	0.1951
	$\widehat{\omega}_{\frac{1}{2}2}$	0.3188	0.3089	0.3093
	$\widehat{\omega}_{\frac{5}{2}1}$	0.3938	0.3757	0.3740

Table: Eigenvalues for $h/L = 0.01$ using $\mathbb{P}_2$:
reference, $\varepsilon < 2\%$, $\varepsilon < 5\%$, $\varepsilon < 15\%$.

Settings for time domain simulation

We consider a square plate under different BCs and external excitations.

- Finite element space $\mathbb{P}_2$;
- Number of finite elements 10×10;
- Integrator Störmer-Verlet;
- Integration step $1[\mu s]$;
- Total simulation time $t_{\text{fin}} = 10[ms]$.

First simulation

Boundary conditions:

- $x = 0 \rightarrow$ Clamped,
- $x = 1 \rightarrow$ Free,
- $y = 0 \rightarrow q_n = f(t),\ m_{nn} = m_{ns} = 0$,
- $y = 1 \rightarrow q_n = -f(t),\ m_{nn} = m_{ns} = 0$,

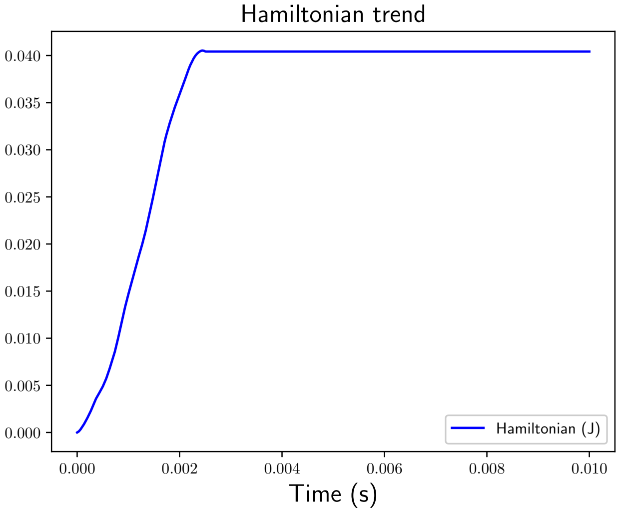
where

$$f(t) = \begin{cases} 10^6\ [Pa \cdot m], & \forall t < 0.25\ t_{\text{fin}}, \\ 0, & \forall t \geq 0.25\ t_{\text{fin}}. \end{cases} \tag{3}$$

First simulation

Simulation 1

First simulation



Second simulation

Boundary conditions: The set of BC for the second simulation is

- $x = 0 \rightarrow$ Clamped,
- $x = 1 \rightarrow q_n = g(y,t)$, $m_{nn} = m_{ns} = 0$,
- $y = 0 \rightarrow$ Clamped,
- $y = 1 \rightarrow$ Clamped,

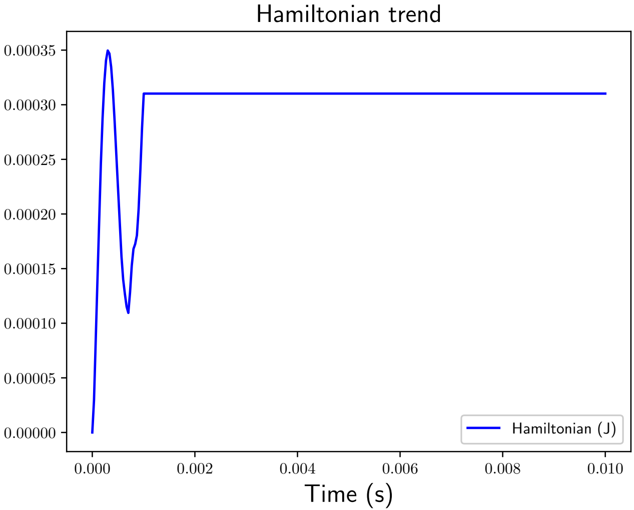
where

$$g(y,t) = \begin{cases} 10^6 \sin\left(\frac{2\pi}{L} y\right) \ [Pa \cdot m], & \forall t < 0.1\, t_{\text{fin}}, \\ 0, & \forall t \geq 0.1\, t_{\text{fin}}. \end{cases} \tag{3}$$

Second simulation

Simulation 2

Second simulation

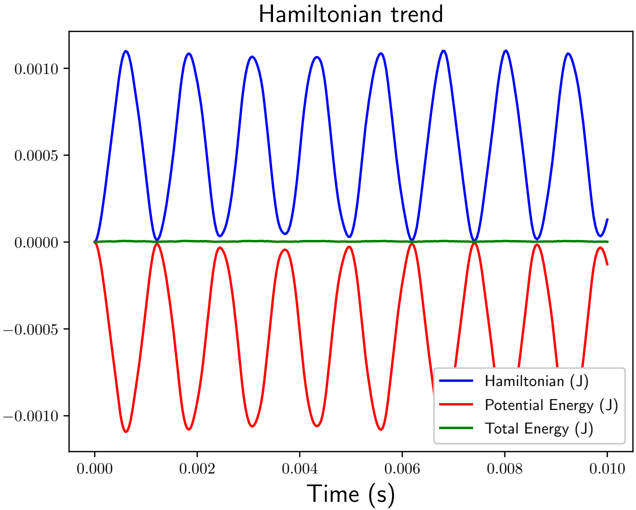


Third simulation

Clamped plate subjected to gravity

Simulation 3

Third simulation



Plan

Conclusion

Present and future developments:

- extend the port Hamiltonian formalism to thin plate[6];
- model reduction for pHDAE of second order[7];
- rigorous numerical analysis of the problem[8];

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[8] Eliane Becache, Patrick Joly, and Chrysoula Tsogka. "A New Family of Mixed Finite Elements for the Linear Elastodynamic Problem". In: *SIAM Journal on Numerical Analysis* 39 (June 2001), pp. 2109–2132. DOI: 10.1137/S0036142999359189.

Thank you for your attention. Questions?

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