

A port-Hamiltonian formulation of flexible  
structures  
Modelling and symplectic finite element discretization

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# Contents

<b>I</b>	<b>Introduction and state of the art</b>	<b>7</b>
<b>1</b>	<b>Introduction</b>	<b>9</b>
1.1	Motivation and context . . . . .	9
1.2	Overview of chapters . . . . .	9
1.3	Contributions . . . . .	9
<b>2</b>	<b>Literature review</b>	<b>11</b>
2.1	Port-Hamiltonian distributed systems . . . . .	11
2.2	Structure-preserving discretization . . . . .	11
2.3	Mixed finite elements for elasticity . . . . .	11
2.4	Multibody dynamics . . . . .	11
<b>II</b>	<b>Port-Hamiltonian elasticity and thermoelasticity</b>	<b>13</b>
<b>3</b>	<b>Elasticity in port-Hamiltonian form</b>	<b>15</b>
3.1	The linear elastodynamics problem . . . . .	15
3.2	Port-Hamiltonian formulation . . . . .	15
<b>4</b>	<b>Port-Hamiltonian plate (and shell?) theory</b>	<b>17</b>
4.1	Mindlin-Reissner model . . . . .	17
4.1.1	Lagrangian formulation . . . . .	17
4.1.2	Port-Hamiltonian formulation . . . . .	17
4.2	Kirchhoff-Love model . . . . .	17
4.2.1	Lagrangian formulation . . . . .	17
4.2.2	Port-Hamiltonian formulation . . . . .	17
4.3	Laminated anisotropic plates . . . . .	17
4.3.1	Thin plate assumption . . . . .	17
4.3.2	Thick plate assumption . . . . .	17
4.4	The membrane shell problem? . . . . .	17
4.4.1	Reminder of differential geometry . . . . .	17
4.4.2	Port-Hamiltonian formulation . . . . .	17
4.4.3	Particular geometrical cases . . . . .	17

<b>5</b>	<b>Thermoelasticity in port-Hamiltonian form</b>	<b>19</b>
5.1	Linear coupled thermoelasticity . . . . .	19
5.2	Thermoelastic Euler-Bernoulli beam . . . . .	19
5.3	Thermoelastic Kirchhoff plate . . . . .	19
<b>III</b>	<b>Finite element structure preserving discretization</b>	<b>21</b>
<b>6</b>	<b>Partitioned finite element method</b>	<b>23</b>
6.1	General procedure . . . . .	23
6.1.1	Non-linear case . . . . .	23
6.1.2	Linear case . . . . .	23
6.1.3	Examples . . . . .	23
6.2	Connection with mixed finite elements . . . . .	23
6.3	Inhomogeneous mixed boundary conditions . . . . .	23
6.3.1	Solution using Lagrange multipliers . . . . .	23
6.3.2	Virtual domain decomposition . . . . .	23
<b>7</b>	<b>Convergence numerical study</b>	<b>25</b>
7.1	Plate problems using known mixed finite elements . . . . .	25
7.2	Non-standard discretization of flexible structures . . . . .	25
<b>8</b>	<b>Numerical applications</b>	<b>27</b>
8.1	Boundary stabilization . . . . .	27
8.1.1	Kirchhoff plate . . . . .	27
8.1.2	Non-linear Saint Venant model . . . . .	27
8.2	Thermoelastic wave propagation . . . . .	27
8.3	Mixed boundary conditions . . . . .	27
8.3.1	Backstepping stabilization of thin beams . . . . .	27
8.3.2	Vibroacoustic under mixed boundary control . . . . .	27
8.3.3	Eigenfrequencies of the Mindlin plate . . . . .	27
<b>IV</b>	<b>Port-Hamiltonian flexible multibody dynamics</b>	<b>29</b>
<b>9</b>	<b>Modular multibody system in port-Hamiltonian form</b>	<b>31</b>
9.1	Reminder for the rigid case . . . . .	31
9.2	Flexible floating body . . . . .	31
9.3	Modular construction of multibody systems . . . . .	31
<b>10</b>	<b>Validation</b>	<b>33</b>
10.1	Beam systems . . . . .	33
10.1.1	Modal analysis of a flexible mechanism . . . . .	33
10.1.2	Non-linear crank slider . . . . .	33
10.1.3	Hinged beam . . . . .	33

<i>CONTENTS</i>	5
10.2 Plate systems . . . . .	33
10.2.1 Boundary interconnection with a rigid element . . . .	33
10.2.2 Actuated plate . . . . .	33
<b>Conclusions and future directions</b>	<b>37</b>
<b>A Mathematical tools</b>	<b>39</b>
<b>B Finite elements gallery</b>	<b>41</b>
<b>C Implementation guide using FEniCS and Firedrake</b>	<b>43</b>



## Part I

# Introduction and state of the art





# Chapter 1

## Introduction

1.1 Motivation and context

1.2 Overview of chapters

1.3 Contributions



## Chapter 2

# Literature review

2.1 Port-Hamiltonian distributed systems

2.2 Structure-preserving discretization

2.3 Mixed finite elements for elasticity

2.4 Multibody dynamics



## Part II

# Port-Hamiltonian elasticity and thermoelasticity



## Chapter 3

# Elasticity in port-Hamiltonian form

3.1 The linear elastodynamics problem

3.2 Port-Hamiltonian formulation





## Chapter 4

# Port-Hamiltonian plate (and shell?) theory

### 4.1 Mindlin-Reissner model

#### 4.1.1 Lagrangian formulation

#### 4.1.2 Port-Hamiltonian formulation

### 4.2 Kirchhoff-Love model

#### 4.2.1 Lagrangian formulation

#### 4.2.2 Port-Hamiltonian formulation

### 4.3 Laminated anisotropic plates

#### 4.3.1 Thin plate assumption

#### 4.3.2 Thick plate assumption

### 4.4 The membrane shell problem?

#### 4.4.1 Reminder of differential geometry

#### 4.4.2 Port-Hamiltonian formulation

#### 4.4.3 Particular geometrical cases

Cylindrical geometry

Toroidal geometry



## Chapter 5

# Thermoelasticity in port-Hamiltonian form

- 5.1 Linear coupled thermoelasticity
- 5.2 Thermoelastic Euler-Bernoulli beam
- 5.3 Thermoelastic Kirchhoff plate



## Part III

# Finite element structure preserving discretization



## Chapter 6

# Partitioned finite element method

### 6.1 General procedure

#### 6.1.1 Non-linear case

#### 6.1.2 Linear case

#### 6.1.3 Examples

### 6.2 Connection with mixed finite elements

### 6.3 Inhomogeneous mixed boundary conditions

#### 6.3.1 Solution using Lagrange multipliers

#### 6.3.2 Virtual domain decomposition





## Chapter 7

# Convergence numerical study

- 7.1 Plate problems using known mixed finite elements
- 7.2 Non-standard discretization of flexible structures



## Chapter 8

# Numerical applications

### 8.1 Boundary stabilization

#### 8.1.1 Kirchhoff plate

#### 8.1.2 Non-linear Saint Venant model

### 8.2 Thermoelastic wave propagation

### 8.3 Mixed boundary conditions

#### 8.3.1 Backstepping stabilization of thin beams

#### 8.3.2 Vibroacoustic under mixed boundary control

#### 8.3.3 Eigenfrequencies of the Mindlin plate



## Part IV

# Port-Hamiltonian flexible multibody dynamics



## Chapter 9

# Modular multibody system in port-Hamiltonian form

9.1 Reminder for the rigid case

9.2 Flexible floating body

9.3 Modular construction of multibody systems





## Chapter 10

# Validation

### 10.1 Beam systems

#### 10.1.1 Modal analysis of a flexible mechanism

#### 10.1.2 Non-linear crank slider

#### 10.1.3 Hinged beam

### 10.2 Plate systems

#### 10.2.1 Boundary interconnection with a rigid element

#### 10.2.2 Actuated plate



## Conclusions and future directions



# Conclusions and future directions



## Appendix A

# Mathematical tools





## Appendix B

### Finite elements gallery



## Appendix C

# Implementation guide using FEniCS and Firedrake