A port-Hamiltonian formulation for the full von-Kármán plate model

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Outline

Why port-Hamiltonian systems?

Von-Kármán theory of thin beams in pH form

Numerical discretization

Numerical convergence study

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A unified language for multiphysics in engineering

The port-Hamiltonian (pH) paradigm provides a language to understand multiphysics:

- **Physics** is at the core: port-Hamiltonian systems are **passive** with respect to the **energy storage function**.
- The **topological** and **metrical** structure of the equation is clearly separated (mimetic discretization).
- PH systems are **closed under interconnection**.

Finite dimensional pH systems

A theory still under developement

There is **not a unique definition** of pH systems, even in finite dimension.

Definition (Finite dimensional pH system)

The following time-invariant dynamical system is a pH system

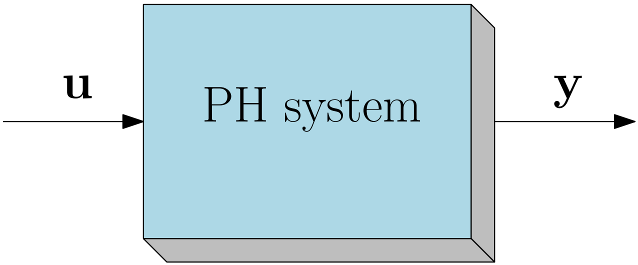
$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{B}^\top \mathbf{x}.$$

$\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$ the input and output and

- $\mathbf{J}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})^\top \in \mathbb{R}^{n\times n}$ the interconnection operator
- $\mathbf{B} \in \mathbb{R}^{n\times m}$ the control operator.
- $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{M} > 0$, the Hamiltonian.

Finite dimensional pH systems

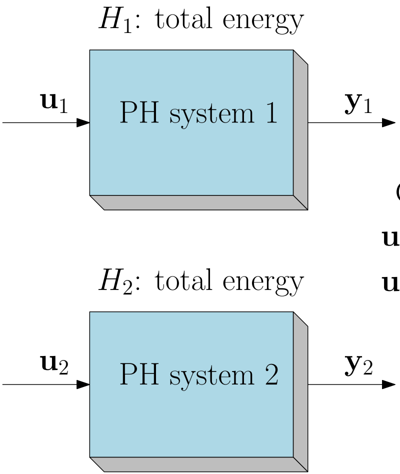
H: total energy



Lossless: $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

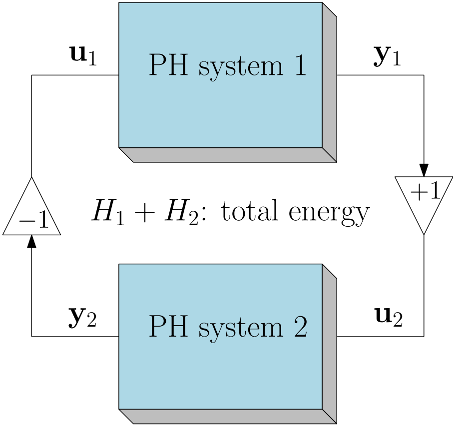
Interconnection of pH systems



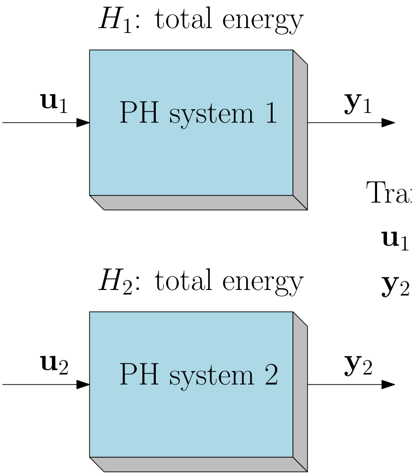
Gyrator

$\mathbf{u}_1 = -\mathbf{y}_2$

$\mathbf{u}_2 = +\mathbf{y}_1$



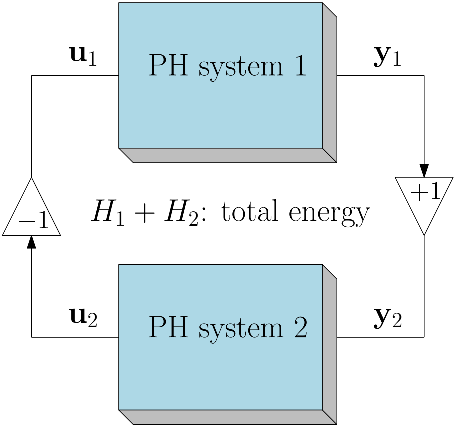
Interconnection of pH systems



Transformer

$\mathbf{u}_1 = -\mathbf{u}_2$

$\mathbf{y}_2 = +\mathbf{y}_1$



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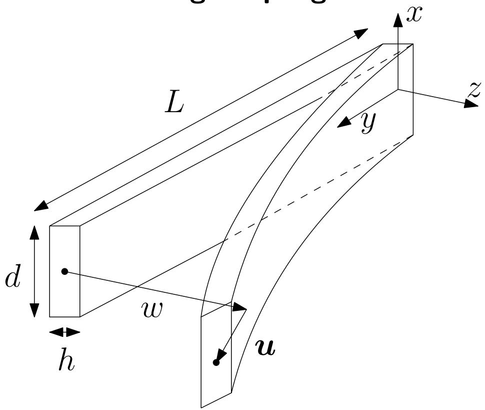
Linear vs Von-Kármán plate theory

Geometrical non-linearities allow describing bifurcations (i.e. buckling).

The von-Kármán assumption

Second-order approximation of geometrically exact beam/plate theory **capturing the axial bending coupling.**



Basic geometric assumption

- Out of plane deflection comparable to the thickness: $w/h = \mathcal{O}(1)$.
- The squares of the in-plane stretching terms are negligible compared to the square of the rotations.

Linear isotropic plates

The axial and bending behavior are uncoupled if $w/h \ll 1$:

Axial displacement (planar elastodynamics)

$$
\begin{aligned}
\rho h \partial_{tt} \boldsymbol{u} &= \operatorname{Div} \boldsymbol{N}, \\
\boldsymbol{N} &= D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m), \\
\boldsymbol{\varepsilon}_m &= \mathrm{Sym}(\nabla \boldsymbol{u}) = \operatorname{Grad} \boldsymbol{u}
\end{aligned}
$$

Vertical displacement (Kirchhoff plate)

$$
\begin{aligned}
\rho h \partial_{tt} w &= -\operatorname{div} \operatorname{Div} \boldsymbol{M}, \\
\boldsymbol{M} &= D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}), \\
\boldsymbol{\kappa} &= \operatorname{Hess} w = \operatorname{Grad} \operatorname{grad} w.
\end{aligned}
$$

The linear mapping $\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\mathbf{1} + (1-\nu)\boldsymbol{A}$ is positive and preserves symmetry.

Von-Kármán plates

Decomposition strain field

$$\varepsilon = \boxed{\operatorname{Grad} \boldsymbol{u}} + \boxed{1/2 \operatorname{grad} w \otimes \operatorname{grad} w} - z \boxed{\operatorname{Hess} w} = \varepsilon_m - z\boldsymbol{\kappa}.$$

Linear membrane def.

Quadratic membrane def.

Linear bending def.

Von-Kármán plate Dynamics

$$\rho A \partial_{tt} u = \operatorname{Div} \boxed{\boldsymbol{N}},$$
$$\rho A \partial_{tt} w = -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}(\boxed{\boldsymbol{N}} \operatorname{grad} w),$$

Total energy $H = \frac{1}{2} \int_\Omega \{ D_m \Phi(\varepsilon_m) : \boldsymbol{N} + D_b \Phi(\kappa) : \boldsymbol{M} \} \, \mathrm{d}\Omega$

Port-Hamiltonian Von-Kármán plates

Energy variables

The Hamiltonian functional is quadratic in the following variables

$\boldsymbol{\alpha}_u = \rho h \partial_t \boldsymbol{u}$, Axial momentum, $\alpha_w = \rho h \partial_t w$, Bending momentum,

$\boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}_m$, Membrane strain, $\boldsymbol{A}_\kappa = \boldsymbol{\kappa}$, Curvature

Co-energy variables

The variational derivative of the Hamiltonian gives the co-energy variables

$$\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad \boldsymbol{e}_w := \delta_{\alpha_w} H = \dot{w},$$

$$\boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = D_m \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon), \qquad \boldsymbol{E}_\kappa := \delta_{\boldsymbol{A}_\kappa} H = D_b \boldsymbol{\Phi}(\boldsymbol{A}_\kappa)$$

or more compactly $\boldsymbol{e} := \delta_\alpha H = \mathcal{Q}\boldsymbol{\alpha}$ with

$$\mathcal{Q} = \mathrm{Diag}\left[(\rho h)^{-1},\ D_m \boldsymbol{\Phi},\ (\rho h)^{-1},\ D_b \boldsymbol{\Phi}\right].$$

The port-Hamiltonian realization

To close the system, variable w has to be accessible.

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{\alpha}_u \\ \boldsymbol{A}_\varepsilon \\ w \\ \alpha_w \\ \boldsymbol{A}_\kappa\end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{C}(w) & -1 & 0 & -\mathrm{div}\,\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{Grad}\,\mathrm{grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix}\delta_{\boldsymbol{\alpha}_u} H \\ \delta_{\boldsymbol{A}_\varepsilon} H \\ \delta_w H \\ \delta_{\alpha_w} H \\ \delta_{\boldsymbol{A}_\kappa} H\end{pmatrix},$$

The operator $\mathcal{J}$ is formally skew-adjoint.
The coupling term reads

$$\begin{aligned} \mathcal{C}(w)(\cdot) : L^2(\Omega;\mathbb{R}^{2\times 2}) &\rightarrow L^2(\Omega), \\ \boldsymbol{X} &\rightarrow \mathrm{div}(\boldsymbol{X}\,\mathrm{grad}\, w), \end{aligned}$$

and its formal adjoint

$$\begin{aligned} \mathcal{C}(w)^*(\cdot) : L^2(\Omega) &\rightarrow L^2(\Omega;\mathbb{R}^{2\times 2}), \\ y &\rightarrow -\mathrm{Sym}\left[\mathrm{grad}(y) \otimes \mathrm{grad}(w)\right]. \end{aligned}$$

Pure coenergy formulation

Incorporation of the constitutive equations

Once the $\mathcal{Q}$ operator (matrix) is inverted, the dynamics is expressed :

$$\begin{pmatrix} \rho h \partial_t \boldsymbol{e}_u \\ (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon \\ \partial_t w \\ \rho h \partial_t e_w \\ (D_b \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \mathrm{Div} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \mathrm{Grad} & \boldsymbol{0} & \boldsymbol{0} & -\mathcal{C}(w)^* & \boldsymbol{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div}\mathrm{Div} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \mathrm{Grad}\,\mathrm{grad} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_u \\ \boldsymbol{E}_\varepsilon \\ \delta_w H \\ e_w \\ \boldsymbol{E}_\kappa \end{pmatrix},$$

In the sequel, the quantity $\delta_w H$ is removed as no displacement dependent potential (e.g. gravity) is considered

Energy rate and boundary conditions

Proposition

The energy rate reads

$$\dot{H} = \langle \gamma_0 \boldsymbol{e}_u | \gamma_\perp \boldsymbol{E}_\varepsilon \rangle_{\partial\Omega} + \langle \gamma_0 \boldsymbol{e}_w | \gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0 (\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w) \rangle_{\partial\Omega} + \langle \gamma_1 \boldsymbol{e}_w | \gamma_{\perp\perp} \boldsymbol{E}_\kappa \rangle_{\partial\Omega},$$

- $\gamma_0 \boldsymbol{e}_u = \boldsymbol{e}_u|_{\partial\Omega}$ *is the Dirichlet trace;*
- $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon \boldsymbol{n}|_{\partial\Omega}$ *is the normal trace;*
- $\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa = -\boldsymbol{n} \cdot \operatorname{Div} \boldsymbol{E}_\kappa - \partial_{\boldsymbol{s}} (\boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{s})|_{\partial\Omega}$ *is the effective shear force;*
- $\gamma_1 \boldsymbol{e}_w = \partial_{\boldsymbol{n}} \boldsymbol{e}_w|_{\partial\Omega}$ *is the normal derivative trace;*
- $\gamma_{\perp\perp} \boldsymbol{E}_\kappa = \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n}$ *is the normal to normal trace.*

Boundary conditions classification

BCs	Traction	Bending	
Kinematical/Dirichlet	$\gamma_0 \boldsymbol{e}_u$	$\gamma_0 \boldsymbol{e}_w$	$\gamma_1 \boldsymbol{e}_w$
Dynamical/Neumann	$\gamma_\perp \boldsymbol{E}_\varepsilon$	$\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0 (\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w)$	$\gamma_{\perp\perp} \boldsymbol{E}_\kappa$

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Mixed finite element construction

Crucial concept to derive stable convergent approximations: **Hilbert complexes**.
The most famous complex is the de Rham complex in 3D:

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

But there are many more. Strain elasticity complex in 2D (planar elastodynamics):

$$H^1(\Omega; \mathbb{R}^2) \xrightarrow{\text{Grad}} H(\text{rot rot}, \Omega; \mathbb{S}) \xrightarrow{\text{rot rot}} L^2(\Omega)$$

The div Div complex in 2D with lower regularity (Kirchhoff plate):

$$H^1(\Omega, \mathbb{R}^2) \xrightarrow{\mathrm{Sym}\,\text{curl}} H^{-1}(\text{div Div}, \Omega; \mathbb{S}) \xrightarrow{\text{div Div}} L^2(\Omega)$$

Mixed finite element construction[1]

Crucial concept: Hilbert complexes for elasticity $H^1 \xrightarrow{\partial_x} L^2$.

Key requirements for mixed Galerkin approximation

- The subspaces $H^1_h \subset H^1$, $L^2_h \subset L^2$ form a subcomplex $H^1_h \xrightarrow{\partial_x} L^2_h$ (i.e. $\partial_x H^1_h \subset L^2_h$).
- they admit bounded linear projections $\pi_h^{H^1} : H^1 \to H^1_h$ and $\pi_h^{L^2} : L^2 \to L^2_h$ which commute with ∂_x:
$$\partial_x \pi_h^{H^1} = \pi_h^{L^2} \partial_x.$$

Satisfied for $CG_k \xrightarrow{\partial_x} DG_{k-1}$

$$\mathrm{CG}_k = \{u \in H^1(\Omega) |\ \forall \text{edge in the mesh},\ u|_{\text{edge}} \in P_k\},$$

$$\mathrm{DG}_{k-1} = \{u \in L^2(\Omega) |\ \forall \text{edge in the mesh},\ u|_{\text{edge}} \in P_{k-1}\},$$

where P_k space of polynomials of degree k.

[1] Arnold, Falk, and Winther 2006.

Finite element choice and final system

For the proposed weak formulation, the FE spaces become

$$e_u^h \in \mathrm{CG}_{2k-1}, \qquad e_\varepsilon^h \in \mathrm{DG}_{2k-2}, \qquad (e_w^h,\ e_\kappa^h,\ w^h) \in CG_k, \quad k \geq 1.$$

Implications:

- Subcomplex property for the linear part: $\partial_x \mathrm{CG}_{2k-1} \subset \mathrm{DG}_{2k-2}$.
- The non linear part respects

$$\partial_x \mathrm{CG}_k \cdot \partial_x \mathrm{CG}_k \subset \mathrm{DG}_{2k-2}.$$

Finite dimensional system (Galerkin projection)

$$\begin{aligned}
\mathbf{M}\dot{\mathbf{e}} &= \mathbf{J}(\mathbf{w})\mathbf{e} + \mathbf{B}\mathbf{u}, \\
\dot{\mathbf{w}} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{e}, \\
\mathbf{y} &= \mathbf{B}^\top \mathbf{e}.
\end{aligned}$$

Outline

Manufactured solution

The following manufactured solution is considered

$$u^{\text{ex}} = x^3[1 - (x/L)^3]\sin(2\pi t), \qquad w^{\text{ex}} \quad = \sin(\pi x/L)\sin(2\pi t),$$

together with the boundary conditions

$$u|_0^L = 0, \quad w|_0^L = 0, \quad m_{xx}|_0^L = 0.$$

A Crank-Nicholson scheme is used for time integration.

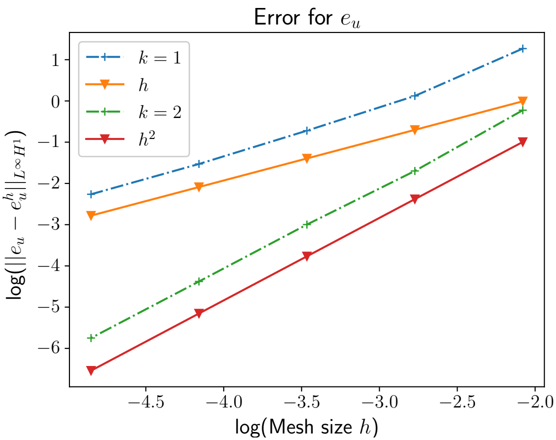
Convergence measure

The discrete time-space norm $L^\infty_{\Delta t}(\mathcal{X})(\mathcal{X} = H^1 \text{or } L^2)$ is used to measure convergence

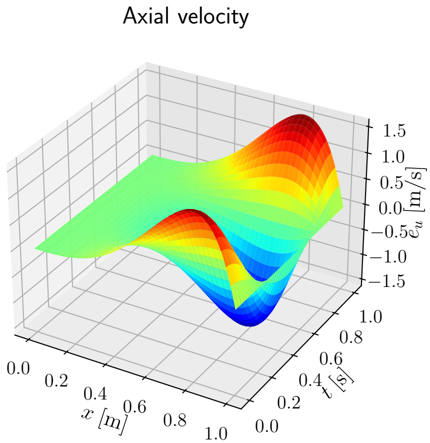
$$||\cdot||_{L^\infty(\mathcal{X})} \approx ||\cdot||_{L^\infty_{\Delta t}(\mathcal{X})} = \max_{t \in t_i} ||\cdot||_{\mathcal{X}},$$

where t_i are the discrete simulation instants.

Results

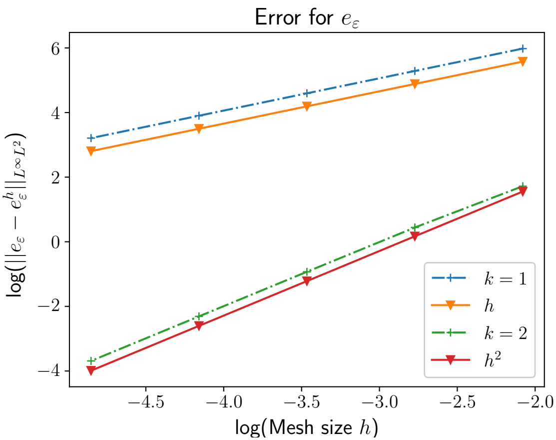


$L^\infty_{\Delta t}(H^1)$ error for e_u.

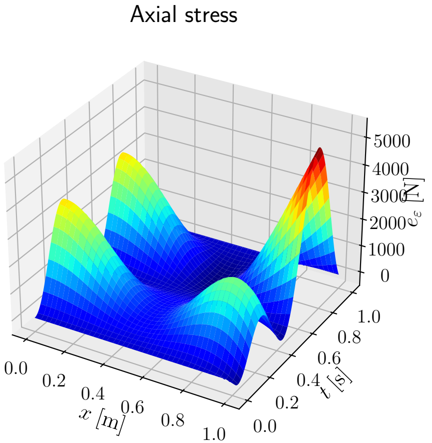


e_u^h ($h = 2^{-5}, k = 2$).

Results

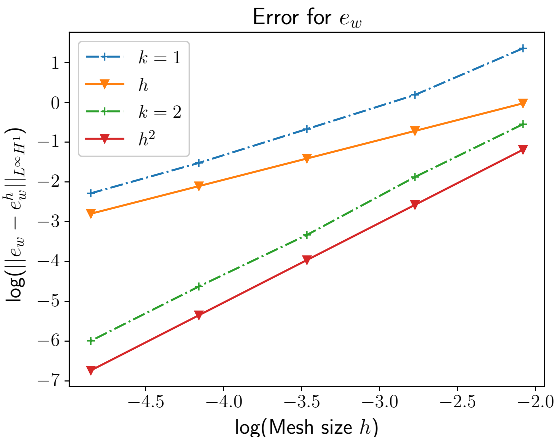


$L^\infty_{\Delta t}(L^2)$ error for e_ε.

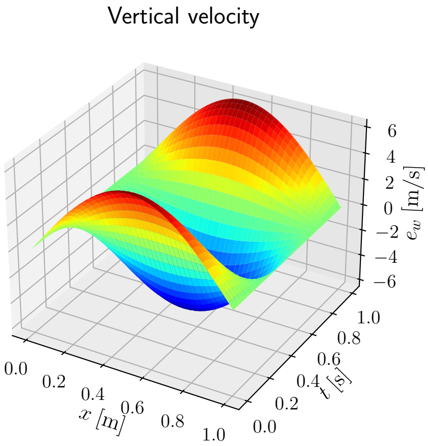


e_ε^h for $h = 2^{-5}$, $k = 2$.

Results

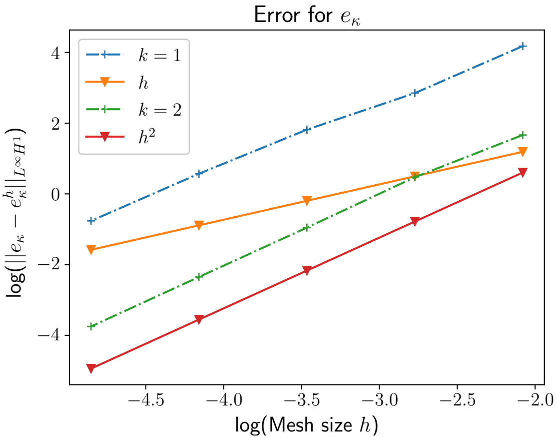


$L^\infty_{\Delta t}(H^1)$ error for e_w.

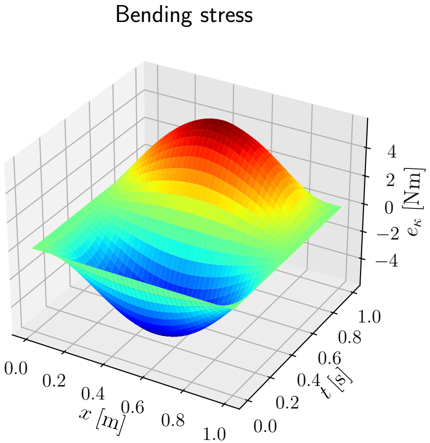


e_w^h for $h = 2^{-5}, k = 2$.

Results



$L^\infty_{\Delta t}(H^1)$ error for e_κ.



e^h_κ for $h = 2^{-5}$, $k = 2$.

Conclusion and Outlook

- First step into pH non linear mechanics. The geometrical non linearities belong to the interconnection operator.
- Natural extension for the 2D case (fancier FE).
- Can be used to study more complex phenomena. The discretization method guarantees energy conservation.

References I

Arnold, Douglas N., Richard S. Falk, and Ragnar Winther (2006). "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numerica* 15, pp. 1–155.

Port-Hamiltonian von-Kármán plates

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\alpha}_u \\ \boldsymbol{A}_\varepsilon \\ w \\ \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{C}(w) & -1 & 0 & -\operatorname{div}\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathrm{Grad}\,\mathrm{grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \delta_{\boldsymbol{\alpha}_u} H \\ \delta_{\boldsymbol{A}_\varepsilon} H \\ \delta_w H \\ \delta_{\alpha_w} H \\ \delta_{\boldsymbol{A}_\kappa} H \end{pmatrix},$$

where

$$\mathcal{C}(w)(\boldsymbol{T}) = \operatorname{div}(\boldsymbol{T}\,\mathrm{grad}\,w),$$

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\mathrm{grad}(\cdot)\otimes\mathrm{grad}(w) + \mathrm{grad}(w)\otimes\mathrm{grad}(\cdot)\right].$$