

Interconnection of the Kirchhoff plate within the port-Hamiltonian framework

Andrea Brugnoli¹, Daniel Alazard¹, Valérie Pommier-Budinger¹, Denis Matignon¹

¹ISAE-SUPAERO, Toulouse

- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
 - Classical formulation
 - Port-Hamiltonian formulation
- 3 Structure preserving discretization
 - The partitioned finite element method
 - Application to the Kirchhoff plate
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
- 3 Structure preserving discretization
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

General infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z, t) &= \mathcal{J} \frac{\delta H}{\delta x} + B u(z, t), \\ y(z, t) &= B^* \frac{\delta H}{\delta x}. \end{cases}$$

With boundary conditions

$$u_{\partial} = \mathcal{B} \frac{\delta H}{\delta x}, \quad y_{\partial} = \mathcal{C} \frac{\delta H}{\delta x}$$

Energy rate: $\dot{H} = \int_{\partial\Omega} u_{\partial} y_{\partial} \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$

- x energy variables, $e = \delta_x H =:$ co-energy variables ;
- \mathcal{J} : skew-symmetric differential operator;
- \mathcal{B}, \mathcal{C} : boundary operator;
- u, y, B : distributed input, output and control operator.

Linear infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z, t) &= \mathcal{J} \mathcal{Q} x + B u(z, t), \\ y(z, t) &= B^* \mathcal{Q} x. \end{cases}$$

With boundary conditions

$$u_{\partial} = \mathcal{B} \mathcal{Q} x, \quad y_{\partial} = \mathcal{C} \mathcal{Q} x$$

$$\text{Energy rate: } \dot{H} = \int_{\partial\Omega} u_{\partial} y_{\partial} \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$$

- x energy variables, $e = \delta_x H = \mathcal{Q} x$: co-energy variables (\mathcal{Q} symmetric positive operator);
- \mathcal{J} : skew-symmetric differential operator;
- \mathcal{B}, \mathcal{C} : boundary operator;
- u, y, B : distributed input, output and control operator.

Structure preserving discretization

The main properties of the dpH (conservative, passive system) are preserved at a discrete level.

Infinite dimensional pHs

PDE:

$$\partial_t x(z, t) = \mathcal{J} \frac{\delta H}{\delta x} + B u(z, t),$$

$$y(z, t) = B^* \frac{\delta H}{\delta x}.$$

Boundary conditions:

$$u_\partial = \mathcal{B} \frac{\delta H}{\delta x}, \quad y_\partial = \mathcal{C} \frac{\delta H}{\delta x}$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} u_\partial y_\partial \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$$

Finite dimensional pHs

ODE:

$$\dot{x} = J \partial_x H + B_d u_d + B_\partial u_\partial,$$

$$y_d = B_d^T \partial_x H,$$

$$y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

The main properties of the dpH (conservative, passive system) are preserved at a discrete level.

Infinite dimensional pHs (linear case)

PDE:

$$\partial_t x(z, t) = \mathcal{J} Q x + B u(z, t),$$

$$y(z, t) = B^* Q x.$$

Boundary conditions:

$$u_{\partial} = \mathcal{B} Q x, \quad y_{\partial} = \mathcal{C} Q x$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} u_{\partial} y_{\partial} \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$$

Finite dimensional pHs (linear case)

ODE:

$$\dot{x} = J Q x + B_d u_d + B_{\partial} u_{\partial},$$

$$y_d = B_d^T Q x,$$

$$y_{\partial} = B_{\partial}^T Q x$$

Power balance:

$$\dot{H} = u_{\partial}^T y_{\partial} + u_d^T y_d$$

Available methods

- Spectral methods (Moulla 2012):
 - ✓ Rapid spectral convergence;
 - ✗ Only for 1D problem;
- Finite differences (Trenchant 2018);
 - ✓ Valid up to 2D geometries;
 - ✗ Requires *ad hoc* implementation (staggered grids);
- Finite elements based
 - Golo 2004, Kotyczka 2018: the implementation requires exterior calculus knowledge and depends on the some parameters that ensure the preservation of power flow;
 - [Cardoso-Ribeiro 2018](#):
 - ✓ Natural extension of the mixed finite element method to pH systems;
 - ✓ Implementable using well-established libraries (Fenics, Firedrake);

Why structure preserving discretization?

A distributed pH system interconnected to a strictly positive real boundary controller is asymptotically stable¹.

This allows the construction of observed based controller based on a finite dimensional discretization of the plant² that guarantees stability of the infinite dimensional system (no spillover effect).

¹J. A. Villegas et al. "Exponential Stability of a Class of Boundary Control Systems". In: *IEEE Transactions on Automatic Control* 54.1 (2009), pp. 142–147. ISSN: 2334-3303. DOI: 10.1109/TAC.2008.2007176.

²Jesus Toledo et al. *Observer-based boundary control of distributed port-Hamiltonian systems*. 2019. arXiv: 1910.04193.

- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
 - Classical formulation
 - Port-Hamiltonian formulation
- 3 Structure preserving discretization
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

Classical bilaplacian formulation

For an homogeneous isotropic material

$$\rho h \frac{\partial^2 w}{\partial t^2} + D \Delta^2 w = p, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2.$$

$\Delta^2 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$ is the bilaplacian operator

- ρ [kg/m³] is the mass density;
- h [m] is the plate thickness;
- p [N/m²] is an external distributed force;
- D [Pa m] is the bending stiffness;

Bending moment formulation

$$\rho h \frac{\partial^2 w}{\partial t^2} + \operatorname{div} \operatorname{Div}(\mathbf{M}) = p, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2.$$

Where $\mathbf{M} = \mathbb{D} \nabla^2 w \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ is the bending moment tensor and $\nabla^2 = \operatorname{Grad} \circ \operatorname{grad}$ the Hessian.

$$\operatorname{div} \operatorname{Div}(\mathbf{M}) = \partial_{xx} M_{11} + 2\partial_{xy} M_{12} + \partial_{yy} M_{22}$$

- ρ [kg/m³] is the mass density;
- h [m] is the plate thickness;
- p [N/m²] is an external distributed force;
- \mathbb{D} is the bending rigidity tensor (symmetric, positive). For an homogeneous isotropic material

$$\mathbb{D}\mathbf{A} = D \{ (1 - \nu)\mathbf{A} + \nu \operatorname{Tr}(\mathbf{A})\mathbf{I} \};$$

Boundary conditions

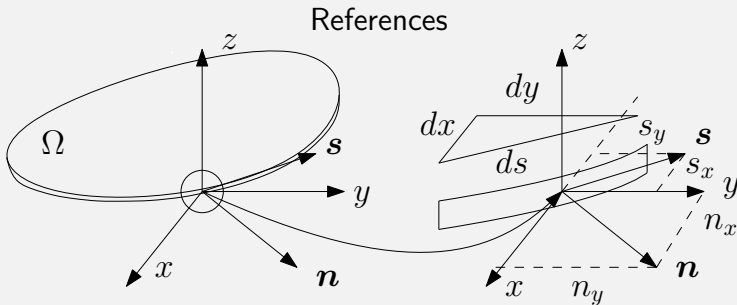
For the boundary variables consider the definitions

Flexural moment $M_{nn} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{n}),$

Shear stress $q = \text{Div } M$,

Torsional moment $M_{ns} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{s})$, Kirchhoff shear force $\tilde{q}_n = -\mathbf{q} \cdot \mathbf{n} - \partial_s M_{ns}$.

$\mathbf{A}:\mathbf{B} = \sum_{i,j} A_{ij}B_{ij}$ is the tensor contraction and $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^\top \in \mathbb{R}^{2 \times 2}$ is the dyadic product.



Boundary conditions

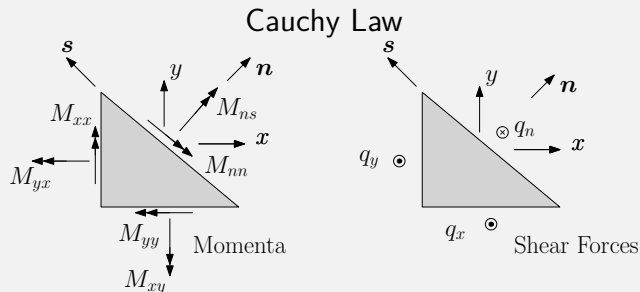
For the boundary variables consider the definitions

Flexural moment $M_{nn} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{n})$,

Shear stress $\mathbf{q} = \text{Div } \mathbf{M}$,

Torsional moment $M_{ns} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{s})$, Kirchhoff shear force $\tilde{q}_n = -\mathbf{q} \cdot \mathbf{n} - \partial_s M_{ns}$.

$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$ is the tensor contraction and $\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^\top \in \mathbb{R}^{2 \times 2}$ is the dyadic product.



Boundary conditions

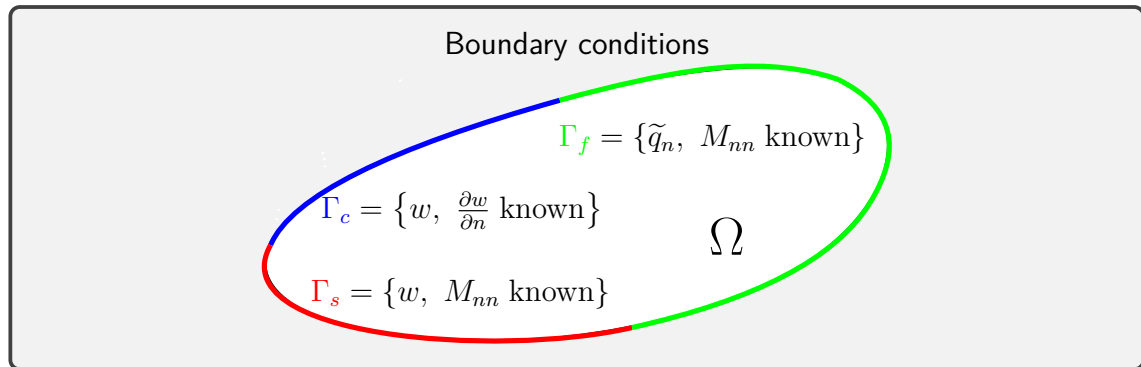
For the boundary variables consider the definitions

Flexural moment $M_{nn} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{n}),$

Shear stress $\mathbf{q} = \text{Div } \mathbf{M},$

Torsional moment $M_{ns} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{s}),$ Kirchhoff shear force $\tilde{q}_n = -\mathbf{q} \cdot \mathbf{n} - \partial_s M_{ns}.$

$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$ is the tensor contraction and $\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^\top \in \mathbb{R}^{2 \times 2}$ is the dyadic product.



- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
 - Classical formulation
 - Port-Hamiltonian formulation
- 3 Structure preserving discretization
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

The total energy of the system is given by the sum of kinetic and deformation energy

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho h (\partial_t w)^2 + \mathbb{D} \nabla^2 w : \nabla^2 w \right\} d\Omega$$

Consider the following choice for the energy variables

$$\alpha_1 := \rho h \partial_t w, \quad \text{Linear momentum}$$

$$\mathbf{A}_2 := \nabla^2 w, \quad \text{Curvature}$$

This leads to the following co-energy variables

$$e_1 := \frac{\delta H}{\delta \alpha_1} = \partial_t w, \quad \text{Vertical velocity}$$

$$\mathbf{E}_2 := \frac{\delta H}{\delta \mathbf{A}_2} = \mathbf{M}, \quad \text{Bending moment}$$

The system

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_1 \\ \mathbf{A}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_1 \\ \mathbf{E}_2 \end{pmatrix}, \quad \begin{pmatrix} e_1 \\ \mathbf{E}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} (\rho h)^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix}}_{\mathcal{Q}} \begin{pmatrix} \alpha_1 \\ \mathbf{A}_2 \end{pmatrix}$$

defines a Stokes-Dirac structure. The proof is readily obtained considering that the following holds

$$(\operatorname{div} \operatorname{Div})^* = \nabla^2$$

This means that the operator \mathcal{J} is formally skew-adjoint.

It is worth noticing that the boundary variables are defined by the power balance

$$\dot{H} = \int_{\partial\Omega} \{ \partial_t w \tilde{q}_n + \partial_n(\partial_t w) M_{nn} \} \, ds.$$

- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
- 3 Structure preserving discretization
 - The partitioned finite element method
 - Application to the Kirchhoff plate
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

The partitioned finite element method (PFEM)

General form of a linear pH system in co-energy variables

$$\mathcal{M} \frac{\partial e}{\partial t} = \mathcal{J}e, \quad \mathcal{M} = \mathcal{Q}^{-1}$$

General procedure for PFEM

- 1 Put the system into weak form:

$$\left(v, \mathcal{M} \frac{\partial e}{\partial t} \right)_{\Omega} = (v, \mathcal{J}e)_{\Omega}.$$

- 2 Apply integration by part on a partition of \mathcal{J} :

$$(v, \mathcal{J}e)_{\Omega} \stackrel{i.b.p.}{=} \widehat{j(v, e)_{\Omega} + b(v, u_{\partial})_{\partial\Omega}},$$

so that $j(v, e)_{\Omega}$ is a skew-symmetric bilinear form.

- 3 Discretization by Galerkin method (same basis function for test and co-energy variables)

- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
- 3 **Structure preserving discretization**
 - The partitioned finite element method
 - Application to the Kirchhoff plate
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the **first line of the operator \mathcal{J}** is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} \text{d}\Omega \\ &= \underbrace{\int_\Omega \left\{ -\nabla^2 v_1 : \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} \text{d}\Omega}_{j_{\text{Hess}}(v, e)} + \underbrace{\int_{\partial\Omega} \{v_1 q_n + \partial_n v_1 M_{nn}\} \text{d}s}_{b_N(v, u_\partial)_{\partial\Omega}} \end{aligned}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the second line of the operator \mathcal{J} is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega \\ &= \underbrace{\int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \text{div Div } \mathbf{V}_2 e_1 \right\} d\Omega}_{j_{\text{div Div}}(\mathbf{v}, \mathbf{e})} + \underbrace{\int_{\partial\Omega} \{ v_{q_n} \partial_t w + v_{m_n} \partial_n \partial_t w \} ds}_{b_D(\mathbf{v}, \mathbf{u}_\partial)_{\partial\Omega}}, \end{aligned}$$

where $v_{q_n} = -(\text{Div } \mathbf{V}_2) \cdot \mathbf{n} - \partial_s v_{m_s}$, $v_{m_s} = \mathbf{V}_2 : (\mathbf{n} \otimes \mathbf{s})$, $v_{m_n} = \mathbf{V}_2 : (\mathbf{n} \otimes \mathbf{n})$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

The selection depends on the control variables. For **Neumann** control the first line is integrated by parts. For **Dirichlet control** the second.

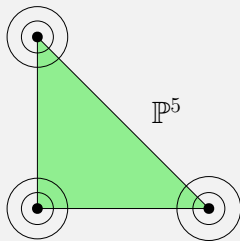
Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

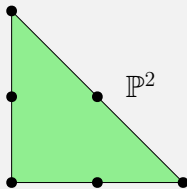
$$m(\boldsymbol{v}, \partial_t \boldsymbol{e}) = j_{\text{Hess}}(\boldsymbol{v}, \boldsymbol{e}) + b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}.$$

For both $\boldsymbol{e}_1, \boldsymbol{E}_2$ the H^2 conforming Bell elements are selected. Dirichlet boundary conditions are enforced by Lagrange multipliers, i.e. $\boldsymbol{u}_\partial|_{\Gamma_D} = \boldsymbol{\lambda}$. Those are discretized using quadratic Lagrange polynomials.

Bell element



Lagrange polynomials



Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

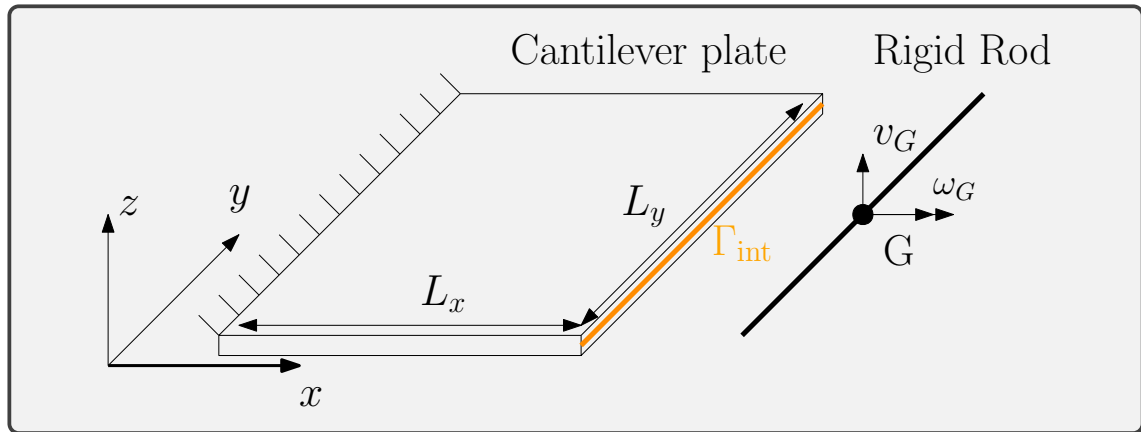
$$m(\boldsymbol{v}, \partial_t \boldsymbol{e}) = j_{\text{Hess}}(\boldsymbol{v}, \boldsymbol{e}) + b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}.$$

For both $\boldsymbol{e}_1, \boldsymbol{E}_2$ the H^2 conforming Bell elements are selected. Dirichlet boundary conditions are enforced by Lagrange multipliers, i.e. $\boldsymbol{u}_\partial|_{\Gamma_D} = \boldsymbol{\lambda}$. Those are discretized using quadratic Lagrange polynomials.

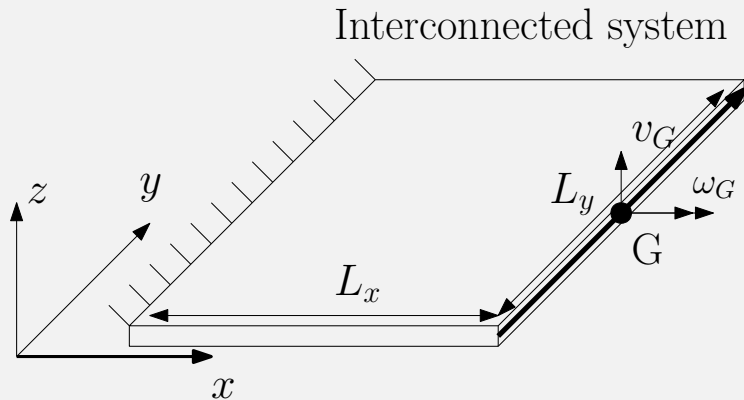
$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \boldsymbol{u},$$
$$\boldsymbol{y} = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
- 3 Structure preserving discretization
- 4 Interconnection with rigid elements**
- 5 Stabilization by boundary injection

Rigid rod welded to a cantilever plate



Rigid rod welded to a cantilever plate



Boundary interconnection of the Kirchhoff plate

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod

$$\text{dpH} \begin{cases} \frac{\partial x_1}{\partial t} = \mathcal{J} \frac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \frac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \frac{\delta H_1}{\delta x_1} \end{cases} \quad \text{pH} \begin{cases} \frac{dx_2}{dt} = J \frac{\partial H_2}{\partial x_2} + B u_2 \\ y_2 = B^T \frac{\partial H_2}{\partial x_2} + D u_2 \end{cases},$$

where $u_{\partial,1} \in \mathcal{U}$, $y_{\partial,1} \in \mathcal{Y} = \mathcal{U}'$ belong to some Hilbert spaces and $x_2 \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$. The interconnection is power-preserving if

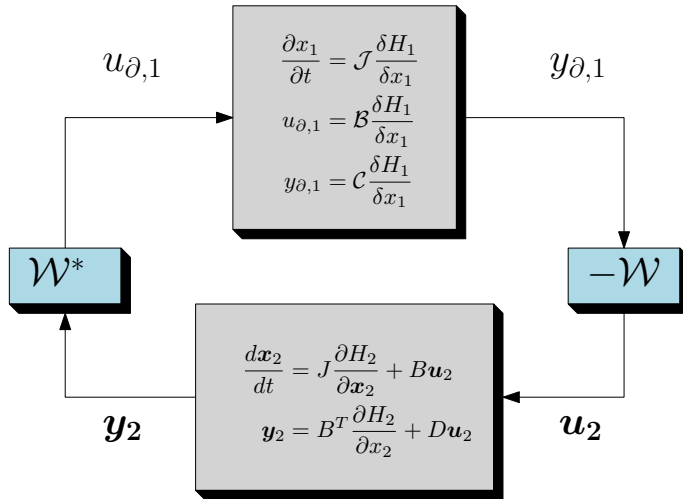
$$\langle u_{\partial,1}, y_{\partial,1} \rangle_{\mathcal{U} \times \mathcal{Y}} + \langle u_2, y_2 \rangle_{\mathbb{R}^m} = 0.$$

This is achieved by introducing a compact operator $\mathcal{W} : \mathcal{Y} \rightarrow \mathbb{R}^m$

$$u_2 = -\mathcal{W} y_{\partial,1}, \quad u_{\partial,1} = \mathcal{W}^* y_2,$$

Boundary interconnection of the Kirchhoff plate

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod



Boundary interconnection of the Kirchhoff plate

Plate ($\Omega = [0, L_x] \times [0, L_y]$)

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix}$$

$$u_{\partial, \text{pl}} = \partial_t w(x = L_x, y),$$

$$y_{\partial, \text{pl}} = \tilde{q}_n(x = L_x, y).$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \mathbf{u}_{\text{rod}},$$

$$\mathbf{y}_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space \mathcal{Y} is the space of square-integrable functions with support on $\Gamma_{\text{int}} = \{(x, y) \mid x = L_x, 0 \leq y \leq L_y\}$. The interconnection operator then provides the total force and torque acting on the rigid rod

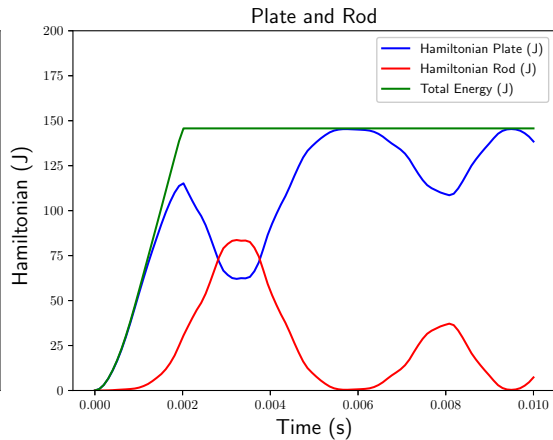
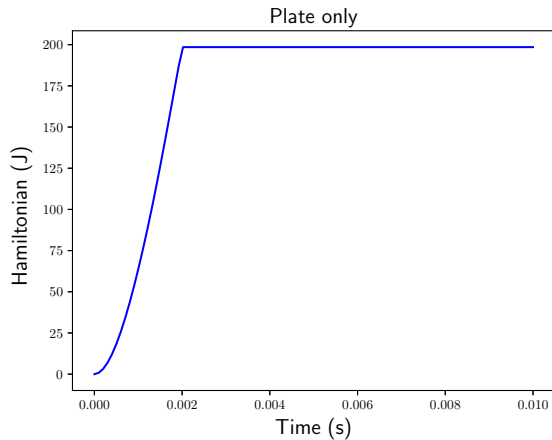
$$\mathcal{W}y_{\partial, \text{pl}} = - \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial, \text{pl}} \, ds \\ \int_{\Gamma_{\text{int}}} (y - L_y/2) y_{\partial, \text{pl}} \, ds \end{pmatrix}.$$

The adjoint operator provides a rigid movement as the plate input at Γ_{int}

$$\langle \mathcal{W}y_{\partial, \text{pl}}, \mathbf{y}_{\text{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial, \text{pl}}, \mathcal{W}^* \mathbf{y}_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})},$$

$$\mathcal{W}^* \mathbf{y}_{\text{rod}} = v_G + \omega_G (y - L_y/2).$$

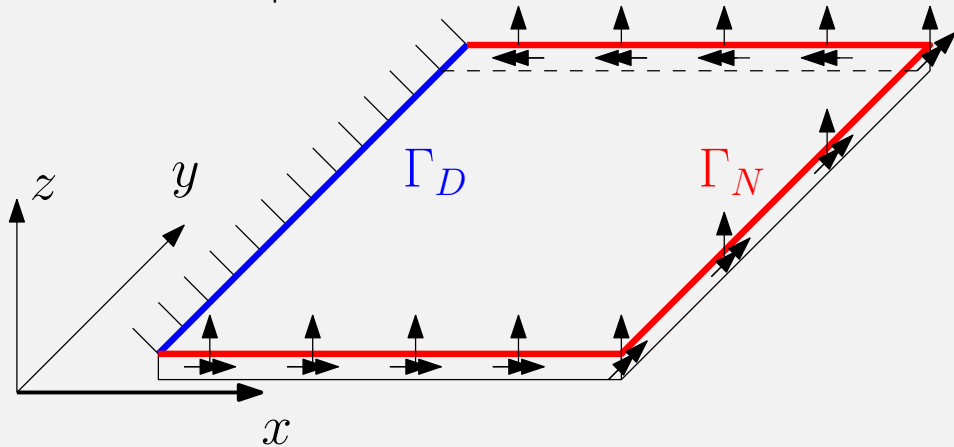
$$\begin{array}{l} \text{Distributed load } (t_{\text{end}} = 10 [\text{ms}]) \\ p = \begin{cases} 10^5 \left[y + 10 (y - L_y/2)^2 \right] [Pa], & \forall t < 2 [\text{ms}], \\ 0, & \forall t \geq 2 [\text{ms}]. \end{cases} \end{array}$$



- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
- 3 Structure preserving discretization
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

Boundary control of a cantilever plate

Cantilever plate at the left side and controlled elsewhere



Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} \quad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

subjected to the following boundary conditions

$$\begin{aligned} \partial_t w|_{\Gamma_D} &= 0, \\ \partial_x \partial_t w|_{\Gamma_D} &= 0, \\ M_{nn}|_{\Gamma_N} &= u_M, \\ \tilde{q}|_{\Gamma_N} &= u_F, \end{aligned} \quad \begin{aligned} \Gamma_D &= \{x = 0\} \\ \Gamma_N &= \{y = 0 \cup x = 1 \cup y = 1\} \end{aligned}$$

with initial conditions (compatible with the constraints):

$$\partial_t w(x, y, 0) = x^2; \quad \mathbf{M}(x, y, 0) = 0.$$

Obtain a finite-dimensional uncontrolled system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

Apply the control law $u = -Ky$, $K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

with $R = BKB^T \succeq 0$.

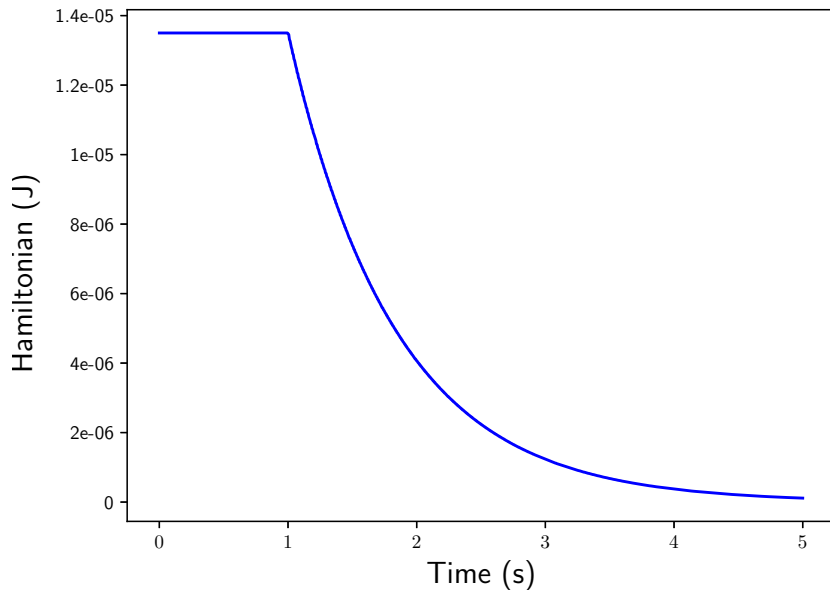
The Hamiltonian $\dot{H} = -e^T R e \leq 0$ is a non increasing function and by La Salle principle the equilibrium point $e = 0$ is asymptotically stable.

Stabilization by boundary injection

Control parameter ($t_{\text{end}} = 5[\text{s}]$)

$$K = \begin{cases} 0, & \forall t < 1 [\text{s}], \\ 100, & \forall t \geq 1 [\text{s}]. \end{cases}$$

Stabilization by boundary injection



The following has been presented:

³H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;

³H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic bcs;

³H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic bcs;
- interconnection with rigid elements (multibody framework);

³H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic bcs;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

³H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic bcs;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

³H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic bcs;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

- The Hellan-Herrmann-Johnson method³, but difficulties when dealing with inhomogeneous bcs;

³H. Blum and R. Rannacher. "On mixed finite element methods in plate bending analysis". In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. "A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach". In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic bcs;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

- The Hellan-Herrmann-Johnson method³, but difficulties when dealing with inhomogeneous bcs;
- New discretization method capable that handles inhomogeneous bcs⁴.

³H. Blum and R. Rannacher. "On mixed finite element methods in plate bending analysis". In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

⁴Katharina. Rafetseder and Walter. Zulehner. "A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach". In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

Thanks for your attention
Questions?



C. Beattie et al. “Linear port-Hamiltonian descriptor systems”. In: *Mathematics of Control, Signals, and Systems* 30.4 (2018), p. 17. URL: <https://link.springer.com/article/10.1007%2Fs00498-018-0223-3>.



K. Bell. “A refined triangular plate bending finite element”. In: *International Journal for Numerical Methods in Engineering* 1.1 (1969), pp. 101–122. DOI: 10.1002/nme.1620010108.



H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: 10.1007/BF00350239.



A. Brugnoli et al. “Port-Hamiltonian formulation and symplectic discretization of plate models. Part II: Kirchhoff model for thin plates”. In: *Applied Mathematical Modelling* 75 (2019), pp. 961 –981. ISSN: 0307-904X. DOI: <https://doi.org/10.1016/j.apm.2019.04.036>.



F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. “A structure-preserving Partitioned Finite Element Method for the 2D wave equation”. In: *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*. Valparaíso, CL, 2018, pp. 1–6. URL: <https://doi.org/10.1016/j.ifacol.2018.06.033>.



V. Duindam et al. *Modeling and Control of Complex Physical Systems*. Springer Verlag, 2009. URL: <https://www.springer.com/us/book/9783642031953>.



G. Golo et al. “Hamiltonian Discretization of Boundary Control Systems”. In: *Automatica* 40.5 (May 2004), pp. 757–771. DOI: <http://dx.doi.org/10.1016/j.automatica.2003.12.017>.

References II



B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Operator Theory: Advances and Applications 223. Germany: Springer Verlag, 2012. DOI: [10.1007/978-3-0348-0399-1](https://doi.org/10.1007/978-3-0348-0399-1). URL: <https://link.springer.com/book/10.1007/978-3-0348-0399-1>.



P. Kotyczka, B. Maschke, and L. Lefèvre. “Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems”. In: *Journal of Computational Physics* 361 (2018), pp. 442–476. ISSN: 0021-9991. DOI: <https://doi.org/10.1016/j.jcp.2018.02.006>.



A. Macchelli, C. Melchiorri, and L. Bassi. “Port-based Modelling and Control of the Mindlin Plate”. In: *Proceedings of the 44th IEEE Conference on Decision and Control*. 2005, pp. 5989–5994. DOI: [10.1109/CDC.2005.1583120](https://doi.org/10.1109/CDC.2005.1583120).



R. Moulla, L. Lefevre, and B. Maschke. “Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws”. In: *Journal of Computational Physics* 231.4 (2012), pp. 1272–1292. URL: <https://doi.org/10.1016/j.jcp.2011.10.008>.



Dirk Pauly and Walter Zulehner. “On Closed and Exact Grad-grad-and div-Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D”. In: *arXiv preprint arXiv:1609.05873* (2016).



Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).



F. Rathgeber et al. “Firedrake: automating the finite element method by composing abstractions”. In: *ACM Transactions on Mathematical Software (TOMS)* 43.3 (2017), p. 24.



Jesus Toledo et al. *Observer-based boundary control of distributed port-Hamiltonian systems*. 2019. arXiv: 1910.04193.



J. A. Villegas et al. “Exponential Stability of a Class of Boundary Control Systems”. In: *IEEE Transactions on Automatic Control* 54.1 (2009), pp. 142–147. ISSN: 2334-3303. DOI: 10.1109/TAC.2008.2007176.

Institut Supérieur de l'Aéronautique et de l'Espace

10 avenue Édouard Belin – BP 54032

31055 Toulouse Cedex 4 – France

Phone: +33 5 61 33 80 80

www.isae-superaero.fr