Integration by parts for tensors

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1 Differential operators

The space of all, symmetric and skew-symmetric $d \times d$ matrices are denoted by $\mathbb{M}$, $\mathbb{S}$, $\mathbb{K}$ respectively. The space of $\mathbb{R}^d$ vectors is denoted by $\mathbb{V}$. $\Omega \subset \mathbb{R}^d$ is an open connected set. For a scalar field $u : \Omega \to \mathbb{R}$ the gradient is defined as

$$\mathrm{grad}(u) = \nabla u := \left(\partial_{x_1} u \dots \partial_{x_d} u\right)^\top.$$

For a vector field $\boldsymbol{u} : \Omega \to \mathbb{V}$, with components u_i, the gradient (Jacobian) is defined as

$$\mathrm{grad}(\boldsymbol{u})_{ij} := (\nabla \boldsymbol{u})_{ij} = \partial_{x_i} u_j.$$

The symmetric part of the gradient operator Grad (i. e. the deformation gradient in continuum mechanics) is thus given by

$$\mathrm{Grad}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\right) \in \mathbb{S}.$$

The Hessian operator of u is then computed as follows

$$\mathrm{Hess}(u) = \nabla^2 u = \mathrm{Grad}(\mathrm{grad}(u)),$$

For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{M}$, with components u_{ij}, the divergence is a vector, defined column-wise as

$$\mathrm{Div}(\boldsymbol{U}) = \nabla \cdot \boldsymbol{U} := \left(\sum_{i=1}^{d} \partial_{x_i} u_{ij}\right)_{j=1,\dots,d}.$$

The double divergence of a tensor field $\boldsymbol{U}$ is then a scalar field defined as

$$\mathrm{div}(\mathrm{Div}(\boldsymbol{U})) := \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_{x_i} \partial_{x_j} u_{ij}.$$

2 Integration by parts

Consider a smooth tensor-valued function $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ and vector-valued function $\boldsymbol{b} \in \mathbb{V} = \mathbb{R}^d$. The following integration by parts formula holds

$$\int_\Omega \left\{\mathrm{Div}(\boldsymbol{A}) \cdot \boldsymbol{b} + \boldsymbol{A} : \mathrm{grad}(\boldsymbol{b})\right\} \,\mathrm{d}\Omega = \int_\Omega \mathrm{div}(\boldsymbol{A}\boldsymbol{b}) \,\mathrm{d}\Omega = \int_{\partial\Omega} (\boldsymbol{A}^\top \boldsymbol{n}) \cdot \boldsymbol{b} \,\mathrm{d}S, \tag{1}$$

where $\boldsymbol{n}$ is the outward normal at the boundary and $\mathrm{d}S$ the infinitesimal surface.

Proof 1 *Consider the components expression of Eq.* (1)

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{A})\cdot\boldsymbol{b} + \boldsymbol{A} : \mathrm{grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega = \int_\Omega \sum_{i=1}^{d}\sum_{j=1}^{d}\{(\partial_{x_i}A_{ij})b_j + A_{ij}(\partial_{x_i}b_j)\}\ \mathrm{d}\Omega,$$
$$= \int_\Omega \sum_{i=1}^{d}\sum_{j=1}^{d}\partial_{x_i}(A_{ij}b_j)\ \mathrm{d}\Omega = \int_\Omega \mathrm{div}(\boldsymbol{A}\boldsymbol{b})\ \mathrm{d}\Omega, \tag{2}$$
$$= \int_{\partial\Omega}\sum_{i=1}^{d}\sum_{j=1}^{d}(n_iA_{ij})b_j\ \mathrm{d}S = \int_{\partial\Omega}(\boldsymbol{A}^\top\boldsymbol{n})\cdot\boldsymbol{b}\ \mathrm{d}S.$$

The previous result can be specialized for symmetric tensor field (see Chapter 1 of book mixed element by Boffi etc 2013). Consider a smooth tensor-valued function $\boldsymbol{M}\in\mathbb{S} = \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and vector-valued function $\boldsymbol{b}\in\mathbb{V}=\mathbb{R}^d$. Then, it holds

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{S})\cdot\boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega = \int_\Omega \mathrm{div}(\boldsymbol{M}\boldsymbol{b})\ \mathrm{d}\Omega = \int_{\partial\Omega}(\boldsymbol{M}\,\boldsymbol{n})\cdot\boldsymbol{b}\ \mathrm{d}S. \tag{3}$$

Proof 2 *Consider the components expression of Eq.* (3)

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{M})\cdot\boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega = \int_\Omega \sum_{i=1}^{d}\sum_{j=1}^{d}\left\{(\partial_{x_i}M_{ij})b_j + M_{ij}\frac{1}{2}(\partial_{x_i}b_j + \partial_{x_j}b_i)\right\}\ \mathrm{d}\Omega, \tag{4}$$

The term $M_{ij}\frac{1}{2}(\partial_{x_i}b_j+\partial_{x_j}b_i)$ *can be manipulated exploiting the symmetry of the tensor* $\boldsymbol{M}$

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}(M_{ij}\partial_{x_i}b_j + M_{ij}\partial_{x_j}b_i) = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}(M_{ij}\partial_{x_i}b_j + M_{ji}\partial_{x_i}b_j),$$
$$= \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}(M_{ij}+M_{ji})\partial_{x_i}b_j \qquad \textit{Since } \boldsymbol{M} \textit{ is symmetric},$$
$$= \sum_{i=1}^{d}\sum_{j=1}^{d}M_{ij}\partial_{x_i}b_j = \boldsymbol{M} : \mathrm{grad}(\boldsymbol{b}) \tag{5}$$

Then it holds

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{M})\cdot\boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega = \int_\Omega \{\mathrm{Div}(\boldsymbol{M})\cdot\boldsymbol{b} + \boldsymbol{M} : \mathrm{grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega \tag{6}$$

Using Eq (1) *then*

$$\int_\Omega \{\mathrm{Div}(\boldsymbol{M})\cdot\boldsymbol{b} + \boldsymbol{M} : \mathrm{Grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega = \int_\Omega \{\mathrm{Div}(\boldsymbol{M})\cdot\boldsymbol{b} + \boldsymbol{M} : \mathrm{grad}(\boldsymbol{b})\}\ \mathrm{d}\Omega,$$
$$= \int_{\partial\Omega}(\boldsymbol{M}^\top\boldsymbol{n})\cdot\boldsymbol{b}\ \mathrm{d}S, \qquad \textit{Since } \boldsymbol{M} \textit{ is symmetric},$$
$$= \int_{\partial\Omega}(\boldsymbol{M}\,\boldsymbol{n})\cdot\boldsymbol{b}\ \mathrm{d}S. \tag{7}$$

This concludes the proof.