A port-Hamiltonian formulation for the full Von-Karman plate model

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Summary. In this contribution, a port-Hamiltonian reformulation of the full von-Karman dynamical model for geometrically non-linear plates is detailed. Starting from the canonical equations, a set of variables is chosen so that that make the total energy quadratic. The model, reformulated in these variables, highlights a port-Hamiltonian structure ruled by a state-modulated interconnection operator.

Classical model

The classical full von-Karman dynamical model is detailed in Bilbao et al. [2015]. The problem, defined an open connected set $\Omega \subset \mathbb{R}^2$, takes the dimensionless form

$$
\begin{aligned}
&\ddot{\boldsymbol{u}} = \operatorname{Div} \boldsymbol{N}, && \boldsymbol{N} = \boldsymbol{\Phi}(\boldsymbol{\varepsilon}), && \boldsymbol{\varepsilon} = \operatorname{Grad} \boldsymbol{u} + 1/2 \operatorname{grad} w \otimes \operatorname{grad} w, \\
&\ddot{w} = -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}\left(\boldsymbol{N} \operatorname{grad} w\right), && \boldsymbol{M} = \boldsymbol{\Phi}(\boldsymbol{\kappa}), && \boldsymbol{\kappa} = \operatorname{Grad} \operatorname{grad} w,
\end{aligned} \tag{1}
$$

where $\boldsymbol{u} \in \mathbb{R}^2$ is the in-plane displacement, w is the vertical displacement, $\boldsymbol{\varepsilon}$ is the in-plane strain tensor, $\boldsymbol{\kappa}$ is the curvature tensor, $\boldsymbol{N}$ is the in-plane stress resultant and $\boldsymbol{M}$ is the bending stress resultant. The notation $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top$ denotes the dyadic product of two vectors. The operator div is the divergence of a vector field and grad the gradient of a scalar field. The operator $\operatorname{Grad} = \frac{1}{2}\left(\nabla + \nabla^\top\right)$ designates the symmetric part of the gradient (i. e. the deformation gradient in continuum mechanics). For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{R}^{2\times 2}$, with components U_{ij}, the divergence Div is a vector, defined column-wise as

$$
\operatorname{Div}(\boldsymbol{U}) := \sum_{i=1}^{2} \partial_{x_i} U_{ij}, \qquad \forall j = \{1, 2\}.
$$

The tensor mapping $\boldsymbol{\Phi}$ is positive and preserves the symmetry

$$
\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\mathbf{1} + (1-\nu)\boldsymbol{A}, \qquad \boldsymbol{A} = \boldsymbol{A}^\top \implies \boldsymbol{\Phi}(\boldsymbol{A}) = \boldsymbol{\Phi}(\boldsymbol{A})^\top, \qquad \text{where} \qquad \mathbf{1} = \operatorname{Diag}(1,1).
$$

Its inverse is given by

$$
\boldsymbol{\Phi}^{-1}(\boldsymbol{A}) = \frac{1}{1-\nu}\boldsymbol{A} - \frac{\nu}{1-\nu^2}\operatorname{Tr}(\boldsymbol{A})\mathbf{1}. \tag{2}
$$

The total energy of the model (Hamiltonian functional)

$$
H = \frac{1}{2}\int_\Omega \left\{ \|\dot{\boldsymbol{u}}\|^2 + \dot{w}^2 + \boldsymbol{N} : \boldsymbol{\varepsilon} + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \, \mathrm{d}\Omega, \qquad \text{where} \qquad \boldsymbol{A} : \boldsymbol{B} = \operatorname{Tr}(\boldsymbol{A}^\top \boldsymbol{B}) \tag{3}
$$

consists of the kinetic energy and membrane and bending deformation energies. This model is conservative Bilbao et al. [2015]. Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

The equivalent port-Hamiltonian system

To find a suitable port-Hamiltonian (pH) system, we first select a set of new variables to make the Hamiltonian functional quadratic. The selection is the same as for a linear plate problem Brugnoli [2020]

$$
\boldsymbol{\alpha}_u = \dot{\boldsymbol{u}}, \qquad \alpha_w = \dot{w}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}. \tag{4}
$$

The energy is quadratic in these variables

$$
H = \frac{1}{2}\int_\Omega \left\{ \|\boldsymbol{\alpha}_u\|^2 + \alpha_w^2 + \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon + \boldsymbol{\Phi}(\boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa \right\}. \tag{5}
$$

By computing the variational derivative of the Hamiltonian, one obtains the co-energy variables

$$
\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad \boldsymbol{e}_w := \delta_{\boldsymbol{\alpha}_w} H = \dot{w}, \qquad \boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon), \qquad \boldsymbol{E}_\kappa := \delta_{\boldsymbol{A}_\kappa} H = \boldsymbol{\Phi}(\boldsymbol{A}_\kappa). \tag{6}
$$

Before stating the final formulation, consider the operator $\mathcal{C}(w)(\cdot) : L^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}}) \to L^2(\Omega)$ acting on symmetric tensors

$$
\mathcal{C}(w)(\cdot) = \operatorname{div}(\cdot \operatorname{grad} w). \tag{7}
$$

Proposition 1 *The formal anti-adjoint of the $\mathcal{C}(w)(\cdot)$ is given by*

$$-\mathcal{C}(w)^*(\cdot) = \frac{1}{2}\left[\mathrm{grad}(\cdot) \otimes \mathrm{grad}(w) + \mathrm{grad}(w) \otimes \mathrm{grad}(\cdot)\right]. \tag{8}$$

Proof 1 *Consider a smooth scalar $v \in C_0^\infty(\Omega)$ and a smooth symmetric tensor field $\boldsymbol{U} \in C_0^\infty(\Omega, \mathbb{R}^{2\times 2}_{sym})$ with compact support. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation*

$$\langle v,\, \mathcal{C}(w)(\boldsymbol{U})\rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)(v)^*,\, \boldsymbol{U}\rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})}. \tag{9}$$

The proof follows the a simple computation

$$\begin{aligned}
\langle v,\, \mathcal{C}(w)(\boldsymbol{U})\rangle_{L^2(\Omega)} &= \langle v,\, \mathrm{div}(\boldsymbol{U}\,\mathrm{grad}\, w)\rangle_{L^2(\Omega)}, \qquad \textit{Integration by parts} \\
&= \langle -\,\mathrm{grad}\, v,\, \boldsymbol{U}\,\mathrm{grad}\, w\rangle_{L^2(\Omega, \mathbb{R}^2)}, \qquad \textit{Dyadic product properties} \\
&= \langle -\,\mathrm{grad}\, v \otimes \mathrm{grad}\, w,\, \boldsymbol{U}\rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})}, \qquad \textit{Since } \boldsymbol{U} \textit{ is symmetric} \\
&= \langle -1/2(\mathrm{grad}\, v \otimes \mathrm{grad}\, w + \mathrm{grad}\, w \otimes \mathrm{grad}\, v),\, \boldsymbol{U}\rangle_{L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})}.
\end{aligned} \tag{10}$$

This means

$$\mathcal{C}(w)^*(\cdot) = -\frac{1}{2}\left[\mathrm{grad}(\cdot) \otimes \mathrm{grad}(w) + \mathrm{grad}(w) \otimes \mathrm{grad}(\cdot)\right], \tag{11}$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\alpha}_u \\ \boldsymbol{A}_\varepsilon \\ \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \begin{bmatrix} \boldsymbol{0} & \mathrm{Div} & \boldsymbol{0} & \boldsymbol{0} \\ \mathrm{Grad} & \boldsymbol{0} & -\mathcal{C}(w)^* & \boldsymbol{0} \\ 0 & \mathcal{C}(w) & 0 & -\,\mathrm{div}\,\mathrm{Div} \\ \boldsymbol{0} & \boldsymbol{0} & \mathrm{Grad}\,\mathrm{grad} & \boldsymbol{0} \end{bmatrix}\begin{pmatrix} \boldsymbol{e}_u \\ \boldsymbol{E}_\varepsilon \\ e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}, \tag{12}$$

The second line of system (12) represents the time derivative of the membrane strain tensor.

Conclusions

References

Stefan Bilbao, Olivier Thomas, Cyril Touzé, and Michele Ducceschi. Conservative numerical methods for the full von kármán plate equations. *Numerical Methods for Partial Differential Equations*, 31(6):1948–1970, 2015. doi: 10.1002/num.21974. URL `https://onlinelibrary.wiley.com/doi/abs/10.1002/num.21974`.

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