

# Interconnection of the Kirchhoff plate within the port-Hamiltonian framework

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- 1 Introduction: discretization of pH Systems
- 2 The Kirchhoff plate as a port-Hamiltonian system
  - Classical formulation
  - Port-Hamiltonian formulation
- 3 Structure preserving discretization
  - The partitioned finite element method
  - Application to the Kirchhoff plate
- 4 Interconnection with rigid elements
- 5 Stabilization by boundary injection

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## General infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z, t) &= \mathcal{J} \frac{\delta H}{\delta x} + B u(z, t), \\ y(z, t) &= B^* \frac{\delta H}{\delta x}. \end{cases}$$

With boundary conditions

$$u_{\partial} = \mathcal{B} \frac{\delta H}{\delta x}, \quad y_{\partial} = \mathcal{C} \frac{\delta H}{\delta x}$$

$$\text{Energy rate: } \dot{H} = \int_{\partial\Omega} u_{\partial} y_{\partial} \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$$

- $x$  energy variables,  $e = \delta_x H =:$  co-energy variables ;
- $\mathcal{J}$ : skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$ : boundary operator;
- $u, y, B$ : distributed input, output and control operator.

## Linear infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z, t) &= \mathcal{J} \mathcal{Q} x + B u(z, t), \\ y(z, t) &= B^* \mathcal{Q} x. \end{cases}$$

With boundary conditions

$$u_{\partial} = \mathcal{B} \mathcal{Q} x, \quad y_{\partial} = \mathcal{C} \mathcal{Q} x$$

$$\text{Energy rate: } \dot{H} = \int_{\partial\Omega} u_{\partial} y_{\partial} \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$$

- $x$  energy variables,  $e = \delta_x H = \mathcal{Q} x$ : co-energy variables ( $\mathcal{Q}$  symmetric positive operator);
- $\mathcal{J}$ : skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$ : boundary operator;
- $u, y, B$ : distributed input, output and control operator.

# Structure preserving discretization

The main properties of the dpH (conservative, passive system) are preserved at a discrete level.

## Infinite dimensional pHs

PDE:

$$\partial_t x(z, t) = \mathcal{J} \frac{\delta H}{\delta x} + B u(z, t),$$

$$y(z, t) = B^* \frac{\delta H}{\delta x}.$$

Boundary conditions:

$$u_\partial = \mathcal{B} \frac{\delta H}{\delta x}, \quad y_\partial = \mathcal{C} \frac{\delta H}{\delta x}$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} u_\partial y_\partial \, ds + \int_{\Omega} u(z, t) y(z, t) \, d\Omega$$

## Finite dimensional pHs

ODE:

$$\dot{x} = J \partial_x H + B_d u_d + B_\partial u_\partial,$$

$$y_d = B_d^T \partial_x H,$$

$$y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

The main properties of the dpH (conservative, passive system) are preserved at a discrete level.

## Infinite dimensional pHs (linear case)

PDE:

$$\partial_t x(z, t) = \mathcal{J} Q x + B u(z, t),$$

$$y(z, t) = B^* Q x.$$

Boundary conditions:

$$u_{\partial} = \mathcal{B} Q x, \quad y_{\partial} = \mathcal{C} Q x$$

Power balance (Stokes Theorem):

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## Finite dimensional pHs (linear case)

ODE:

$$\dot{x} = J Q x + B_d u_d + B_{\partial} u_{\partial},$$

$$y_d = B_d^T Q x,$$

$$y_{\partial} = B_{\partial}^T Q x$$

Power balance:

$$\dot{H} = u_{\partial}^T y_{\partial} + u_d^T y_d$$

## Available methods

- Spectral methods (Moulla 2012):
  - ✓ Rapid spectral convergence;
  - ✗ Only for 1D problem;
- Finite differences (Trenchant 2018);
  - ✓ Valid up to 2D geometries;
  - ✗ Requires *ad hoc* implementation (staggered grids);
- Finite elements based
  - Golo 2004, Kotyczka 2018: the implementation requires exterior calculus knowledge and depends on the some parameters that ensure the preservation of power flow;
  - [Cardoso-Ribeiro 2018](#):
    - ✓ Natural extension of the mixed finite element method to pH systems;
    - ✓ Implementable using well-established libraries (Fenics, Firedrake);



# Why structure preserving discretization?

A distributed pH system interconnected to a strictly positive real boundary controller is asymptotically stable<sup>1</sup>.

This allows the construction of observed based controller based on a finite dimensional discretization of the plant<sup>2</sup> that guarantees stability of the infinite dimensional system (no spillover effect).

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<sup>1</sup>J. A. Villegas et al. “Exponential Stability of a Class of Boundary Control Systems”. In: *IEEE Transactions on Automatic Control* 54.1 (2009), pp. 142–147. ISSN: 2334-3303. DOI: 10.1109/TAC.2008.2007176.

<sup>2</sup>Jesus Toledo et al. *Observer-based boundary control of distributed port-Hamiltonian systems*. 2019. arXiv: 1910.04193.

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## Classical bilaplacian formulation

For an homogeneous isotropic material

$$\rho h \frac{\partial^2 w}{\partial t^2} + D \Delta^2 w = p, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2.$$

$\Delta^2 = \partial_{xxxx} + 2\partial_{xyyy} + \partial_{yyyy}$  is the bilaplacian operator

- $\rho$  [kg/m<sup>3</sup>] is the mass density;
- $h$  [m] is the plate thickness;
- $p$  [N/m<sup>2</sup>] is an external distributed force;
- $D$  [Pa m] is the bending stiffness;

## Bending moment formulation

$$\rho h \frac{\partial^2 w}{\partial t^2} + \operatorname{div} \operatorname{Div}(\mathbf{M}) = p, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2.$$

Where  $\mathbf{M} = \mathbb{D} \nabla^2 w \in \mathbb{R}_{\text{sym}}^{2 \times 2}$  is the bending moment tensor and  $\nabla^2 = \operatorname{Grad} \circ \operatorname{grad}$  the Hessian.

$$\operatorname{div} \operatorname{Div}(\mathbf{M}) = \partial_{xx} M_{11} + 2\partial_{xy} M_{12} + \partial_{yy} M_{22}$$

- $\rho$  [kg/m<sup>3</sup>] is the mass density;
- $h$  [m] is the plate thickness;
- $p$  [N/m<sup>2</sup>] is an external distributed force;
- $\mathbb{D}$  is the bending rigidity tensor (symmetric, positive). For an homogeneous isotropic material

$$\mathbb{D}\mathbf{A} = D \{ (1 - \nu)\mathbf{A} + \nu \operatorname{Tr}(\mathbf{A})\mathbf{I} \};$$

# Boundary conditions

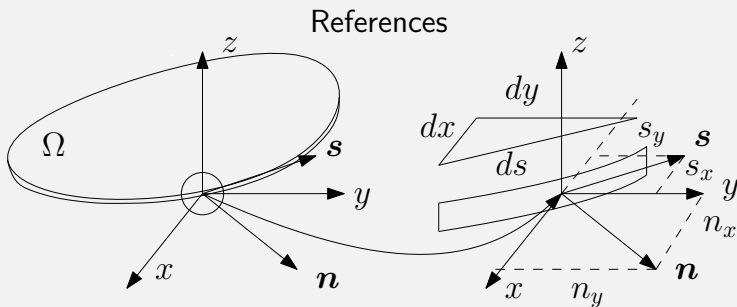
For the boundary variables consider the definitions

Flexural moment  $M_{nn} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{n})$ ,

Shear stress  $\mathbf{q} = \text{Div } \mathbf{M}$ ,

Torsional moment  $M_{ns} = \mathbf{M} : (\mathbf{n} \otimes \mathbf{s})$ , Kirchhoff shear force  $\tilde{q}_n = -\mathbf{q} \cdot \mathbf{n} - \partial_s M_{ns}$ .

$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$  is the tensor contraction and  $\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^\top \in \mathbb{R}^{2 \times 2}$  is the dyadic product.



# Boundary conditions

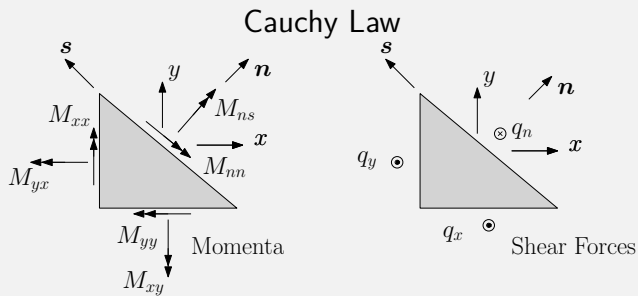
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Boundary conditions

$$\Gamma_f = \{\tilde{q}_n, M_{nn} \text{ known}\}$$

$$\Gamma_c = \{w, \frac{\partial w}{\partial n} \text{ known}\}$$

$$\Gamma_s = \{w, M_{nn} \text{ known}\}$$

$\Omega$

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The total energy of the system is given by the sum of kinetic and deformation energy

$$H = \frac{1}{2} \int_{\Omega} \left\{ \rho h (\partial_t w)^2 + \mathbb{D} \nabla^2 w : \nabla^2 w \right\} d\Omega$$

Consider the following choice for the energy variables

$$\alpha_1 := \rho h \partial_t w, \quad \text{Linear momentum}$$

$$\mathbf{A}_2 := \nabla^2 w, \quad \text{Curvature}$$

This leads to the following co-energy variables

$$e_1 := \frac{\delta H}{\delta \alpha_1} = \partial_t w, \quad \text{Vertical velocity}$$

$$\mathbf{E}_2 := \frac{\delta H}{\delta \mathbf{A}_2} = \mathbf{M}, \quad \text{Bending moment}$$

The system

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_1 \\ \mathbf{A}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_1 \\ \mathbf{E}_2 \end{pmatrix}, \quad \begin{pmatrix} e_1 \\ \mathbf{E}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} (\rho h)^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix}}_{\mathcal{Q}} \begin{pmatrix} \alpha_1 \\ \mathbf{A}_2 \end{pmatrix}$$

defines a Stokes-Dirac structure. The proof is readily obtained considering that the following holds

$$(\operatorname{div} \operatorname{Div})^* = \nabla^2$$

This means that the operator  $\mathcal{J}$  is formally skew-adjoint.

It is worth noticing that the boundary variables are defined by the power balance

$$\dot{H} = \int_{\partial\Omega} \{ \partial_t w \, \tilde{q}_n + \partial_n(\partial_t w) \, M_{nn} \} \, ds.$$

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# The partitioned finite element method (PFEM)

General form of a linear pH system in co-energy variables

$$\mathcal{M} \frac{\partial e}{\partial t} = \mathcal{J}e, \quad \mathcal{M} = \mathcal{Q}^{-1}$$

## General procedure for PFEM

- 1 Put the system into weak form:

$$\left( v, \mathcal{M} \frac{\partial e}{\partial t} \right)_{\Omega} = (v, \mathcal{J}e)_{\Omega}.$$

- 2 Apply integration by part on a partition of  $\mathcal{J}$ :

$$(v, \mathcal{J}e)_{\Omega} \stackrel{i.b.p.}{=} j(v, e)_{\Omega} + b(v, u_{\partial})_{\partial\Omega},$$

so that  $j(v, e)_{\Omega}$  is a skew-symmetric bilinear form.

- 3 Discretization by Galerkin method (same basis function for test and co-energy variables)

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Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the **first line of the operator  $\mathcal{J}$**  is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega \\ &= \underbrace{\int_\Omega \left\{ -\nabla^2 v_1 : \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega}_{j_{\text{Hess}}(v, e)} + \underbrace{\int_{\partial\Omega} \{ v_1 q_n + \partial_n v_1 M_{nn} \} ds}_{b_N(v, u_\partial)_{\partial\Omega}} \end{aligned}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the second line of the operator  $\mathcal{J}$  is integrated by parts

$$\begin{aligned} (v, \mathcal{J}e)_\Omega &= \int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \mathbf{V}_2 : \nabla^2 e_1 \right\} d\Omega \\ &= \underbrace{\int_\Omega \left\{ -v_1 \text{div Div } \mathbf{E}_2 + \text{div Div } \mathbf{V}_2 e_1 \right\} d\Omega}_{j_{\text{div Div}}(\mathbf{v}, \mathbf{e})} + \underbrace{\int_{\partial\Omega} \{ v_{q_n} \partial_t w + v_{m_n} \partial_n \partial_t w \} ds}_{b_D(\mathbf{v}, \mathbf{u}_\partial)_{\partial\Omega}}, \end{aligned}$$

where  $v_{q_n} = -(\text{Div } \mathbf{V}_2) \cdot \mathbf{n} - \partial_s v_{m_s}$ ,  $v_{m_s} = \mathbf{V}_2 : (\mathbf{n} \otimes \mathbf{s})$ ,  $v_{m_n} = \mathbf{V}_2 : (\mathbf{n} \otimes \mathbf{n})$



Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h, \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\text{div Div} \\ \nabla^2 & 0 \end{pmatrix}$$

The selection depends on the control variables. For **Neumann** control the first line is integrated by parts. For **Dirichlet control** the second.

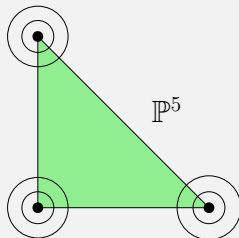
## Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

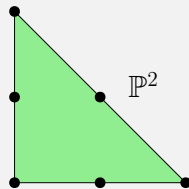
$$m(\boldsymbol{v}, \partial_t \boldsymbol{e}) = j_{\text{Hess}}(\boldsymbol{v}, \boldsymbol{e}) + b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}.$$

For both  $\boldsymbol{e}_1, \boldsymbol{E}_2$  the  $H^2$  conforming Bell elements are selected. Dirichlet boundary conditions are enforced by Lagrange multipliers, i.e.  $\boldsymbol{u}_\partial|_{\Gamma_D} = \boldsymbol{\lambda}$ . Those are discretized using quadratic Lagrange polynomials.

Bell element



Lagrange polynomials



## Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

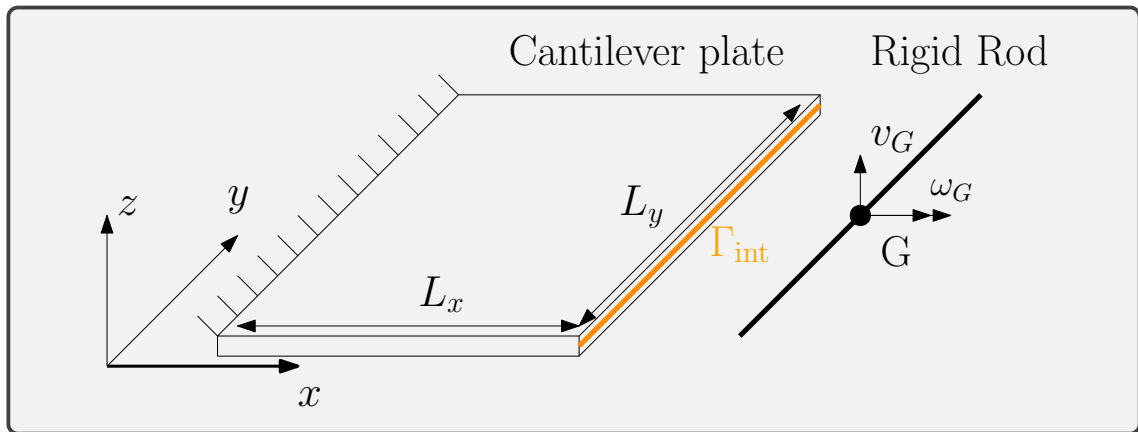
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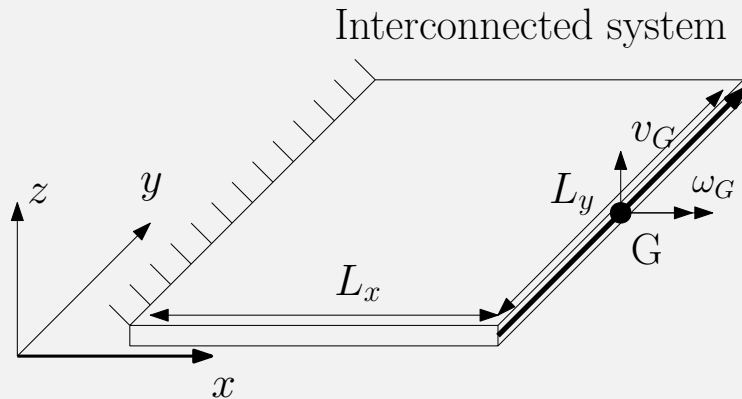
$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \boldsymbol{u},$$
$$\boldsymbol{y} = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

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# Rigid rod welded to a cantilever plate



# Rigid rod welded to a cantilever plate



# Boundary interconnection of the Kirchhoff plate

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod

$$\text{dpH} \begin{cases} \frac{\partial x_1}{\partial t} = \mathcal{J} \frac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \frac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \frac{\delta H_1}{\delta x_1} \end{cases} \quad \text{pH} \begin{cases} \frac{dx_2}{dt} = J \frac{\partial H_2}{\partial x_2} + B u_2 \\ y_2 = B^T \frac{\partial H_2}{\partial x_2} + D u_2 \end{cases},$$

where  $u_{\partial,1} \in \mathcal{U}$ ,  $y_{\partial,1} \in \mathcal{Y} = \mathcal{U}'$  belong to some Hilbert spaces and  $x_2 \in \mathbb{R}^n$ ,  $u, y \in \mathbb{R}^m$ .  
The interconnection is power-preserving if

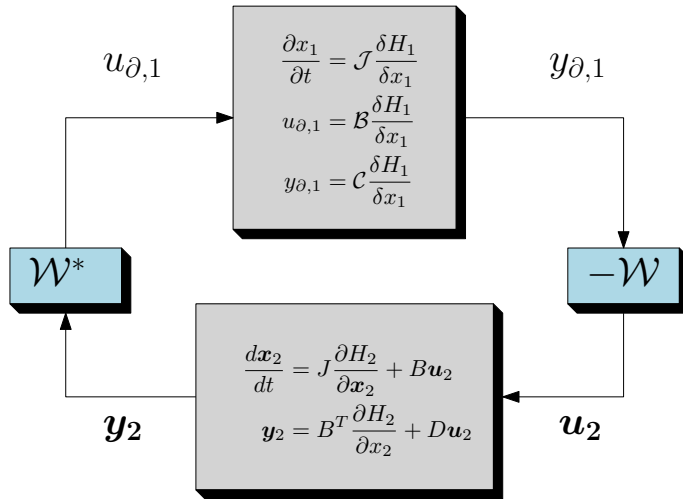
$$\langle u_{\partial,1}, y_{\partial,1} \rangle_{\mathcal{U} \times \mathcal{Y}} + \langle u_2, y_2 \rangle_{\mathbb{R}^m} = 0.$$

This is achieved by introducing a compact operator  $\mathcal{W} : \mathcal{Y} \rightarrow \mathbb{R}^m$

$$u_2 = -\mathcal{W} y_{\partial,1}, \quad u_{\partial,1} = \mathcal{W}^* y_2,$$

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The system is composed by a cantilever plate (distributed pH) connected to a rigid rod





# Boundary interconnection of the Kirchhoff plate

Plate ( $\Omega = [0, L_x] \times [0, L_y]$ )

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix}$$

$$u_{\partial, \text{pl}} = \partial_t w(x = L_x, y),$$

$$y_{\partial, \text{pl}} = \tilde{q}_n(x = L_x, y).$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \mathbf{u}_{\text{rod}},$$

$$\mathbf{y}_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space  $\mathcal{Y}$  is the space of square-integrable functions with support on  $\Gamma_{\text{int}} = \{(x, y) \mid x = L_x, 0 \leq y \leq L_y\}$ . The interconnection operator then provides the total force and torque acting on the rigid rod

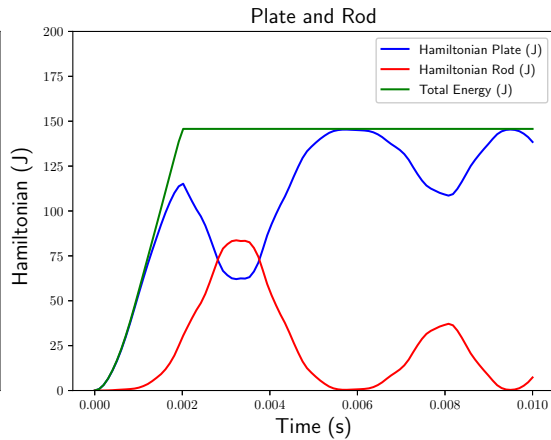
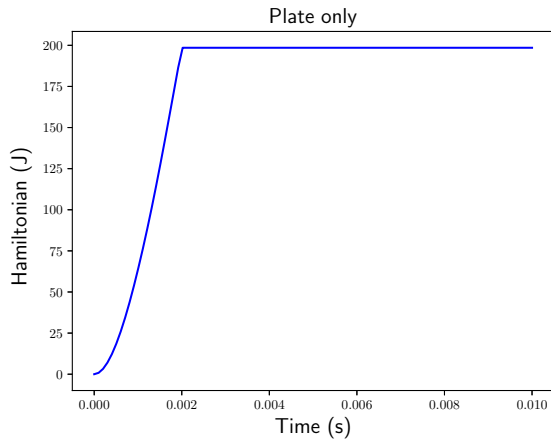
$$\mathcal{W}y_{\partial, \text{pl}} = - \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial, \text{pl}} \, ds \\ \int_{\Gamma_{\text{int}}} (y - L_y/2) y_{\partial, \text{pl}} \, ds \end{pmatrix}.$$

The adjoint operator provides a rigid movement as the plate input at  $\Gamma_{\text{int}}$

$$\langle \mathcal{W}y_{\partial, \text{pl}}, \mathbf{y}_{\text{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial, \text{pl}}, \mathcal{W}^* \mathbf{y}_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})},$$

$$\mathcal{W}^* \mathbf{y}_{\text{rod}} = v_G + \omega_G (y - L_y/2).$$

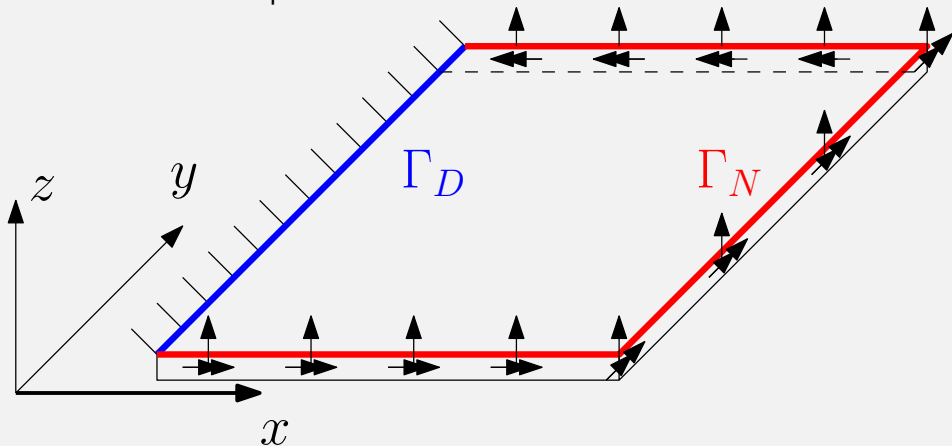
$$\begin{array}{c} \text{Distributed load } (t_{\text{end}} = 10 [\text{ms}]) \\ p = \begin{cases} 10^5 \left[ y + 10 (y - L_y/2)^2 \right] [Pa], & \forall t < 2 [\text{ms}], \\ 0, & \forall t \geq 2 [\text{ms}]. \end{cases} \end{array}$$



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# Boundary control of a cantilever plate

Cantilever plate at the left side and controlled elsewhere



# Boundary stabilization of the Kirchhoff plate

Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \mathbf{M} \end{bmatrix} \quad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

subjected to the following boundary conditions

$$\begin{aligned} \partial_t w|_{\Gamma_D} &= 0, \\ \partial_x \partial_t w|_{\Gamma_D} &= 0, \\ M_{nn}|_{\Gamma_N} &= u_M, \\ \tilde{q}|_{\Gamma_N} &= u_F, \end{aligned} \quad \begin{aligned} \Gamma_D &= \{x = 0\} \\ \Gamma_N &= \{y = 0 \cup x = 1 \cup y = 1\} \end{aligned}$$

with initial conditions (compatible with the constraints):

$$\partial_t w(x, y, 0) = x^2; \quad \mathbf{M}(x, y, 0) = 0.$$

Obtain a finite-dimensional uncontrolled system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

Apply the control law  $u = -Ky$ ,  $K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

with  $R = BKB^T \succeq 0$ .

The Hamiltonian  $\dot{H} = -e^T R e \leq 0$  is a non increasing function and by La Salle principle the equilibrium point  $e = 0$  is asymptotically stable.

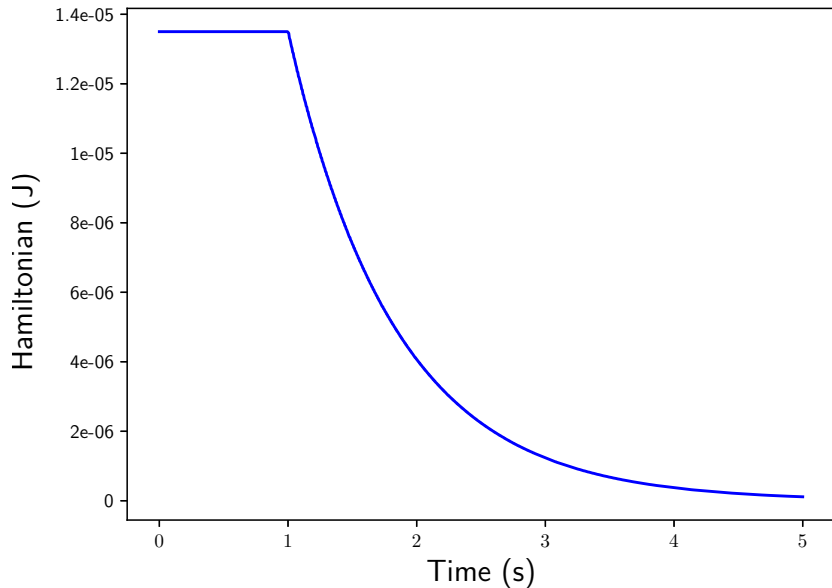
# Stabilization by boundary injection

Control parameter ( $t_{\text{end}} = 5[\text{s}]$ )

$$K = \begin{cases} 0, & \forall t < 1 [\text{s}], \\ 100, & \forall t \geq 1 [\text{s}]. \end{cases}$$



# Stabilization by boundary injection



The following has been presented:

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<sup>3</sup>H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: [10.1007/BF00350239](https://doi.org/10.1007/BF00350239).

<sup>4</sup>Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: [10.1137/17M1118427](https://doi.org/10.1137/17M1118427).

The following has been presented:

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Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

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Thanks for your attention  
Questions?

- [1] C. Beattie et al. “Linear port-Hamiltonian descriptor systems”. In: *Mathematics of Control, Signals, and Systems* 30.4 (2018), p. 17. URL: <https://link.springer.com/article/10.1007%2Fs00498-018-0223-3>.
- [2] K. Bell. “A refined triangular plate bending finite element”. In: *International Journal for Numerical Methods in Engineering* 1.1 (1969), pp. 101–122. DOI: 10.1002/nme.1620010108.
- [3] H. Blum and R. Rannacher. “On mixed finite element methods in plate bending analysis”. In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: 10.1007/BF00350239.
- [4] A. Brugnoli et al. “Port-Hamiltonian formulation and symplectic discretization of plate models. Part II: Kirchhoff model for thin plates”. In: *Applied Mathematical Modelling* 75 (2019), pp. 961 –981. ISSN: 0307-904X. DOI: <https://doi.org/10.1016/j.apm.2019.04.036>.
- [5] F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. “A structure-preserving Partitioned Finite Element Method for the 2D wave equation”. In: *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*. Valparaíso, CL, 2018, pp. 1–6. URL: <https://doi.org/10.1016/j.ifacol.2018.06.033>.
- [6] V. Duindam et al. *Modeling and Control of Complex Physical Systems*. Springer Verlag, 2009. URL: <https://www.springer.com/us/book/9783642031953>.
- [7] G. Golo et al. “Hamiltonian Discretization of Boundary Control Systems”. In: *Automatica* 40.5 (May 2004), pp. 757–771. DOI: <http://dx.doi.org/10.1016/j.automatica.2003.12.017>.
- [8] B. Jacob and H. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Operator Theory: Advances and Applications 223. Germany: Springer Verlag, 2012. DOI: 10.1007/978-3-0348-0399-1. URL: <https://link.springer.com/book/10.1007/978-3-0348-0399-1>.

- [9] P. Kotyczka, B. Maschke, and L. Lefèvre. “Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems”. In: *Journal of Computational Physics* 361 (2018), pp. 442–476. ISSN: 0021-9991. DOI: <https://doi.org/10.1016/j.jcp.2018.02.006>.
- [10] A. Macchelli, C. Melchiorri, and L. Bassi. “Port-based Modelling and Control of the Mindlin Plate”. In: *Proceedings of the 44th IEEE Conference on Decision and Control*. 2005, pp. 5989–5994. DOI: 10.1109/CDC.2005.1583120.
- [11] R. Moulla, L. Lefevre, and B. Maschke. “Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws”. In: *Journal of Computational Physics* 231.4 (2012), pp. 1272–1292. URL: <https://doi.org/10.1016/j.jcp.2011.10.008>.
- [12] Dirk Pauly and Walter Zulehner. “On Closed and Exact Grad-grad-and div-Div-Complexes, Corresponding Compact Embeddings for Tensor Rotations, and a Related Decomposition Result for Biharmonic Problems in 3D”. In: *arXiv preprint arXiv:1609.05873* (2016).
- [13] Katharina. Rafetseder and Walter. Zulehner. “A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach”. In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: 10.1137/17M1118427.
- [14] F. Rathgeber et al. “Firedrake: automating the finite element method by composing abstractions”. In: *ACM Transactions on Mathematical Software (TOMS)* 43.3 (2017), p. 24.
- [15] Jesus Toledo et al. *Observer-based boundary control of distributed port-Hamiltonian systems*. 2019. arXiv: 1910.04193.

- [16] J. A. Villegas et al. "Exponential Stability of a Class of Boundary Control Systems". In: *IEEE Transactions on Automatic Control* 54.1 (2009), pp. 142–147. ISSN: 2334-3303. DOI: 10.1109/TAC.2008.2007176.

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