The Euler-Bernoulli beam in differential forms

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October 20, 2020

1 Classical formulation

The Euler-Bernoulli beam model consists of one PDE, describing the vertical displacement along the beam length:

$$\rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) = 0, \quad x \in \Omega = \{0, L\}, \tag{1}$$

where $w(x,t)$ is the transverse displacement of the beam. The coefficients $\rho(x), A(x)E(x)$ and $I(x)$ are the mass density, cross section, Young's modulus of elasticity and the moment of inertia of a cross section. The energy variables are then chosen as follows:

$$\alpha_w = \rho A(x)\frac{\partial w}{\partial t}(x,t), \quad \text{Linear Momentum}, \quad \alpha_\kappa = \frac{\partial^2 w}{\partial x^2}(x,t), \quad \text{Curvature}. \tag{2}$$

Those variables are collected in the vector $\boldsymbol{\alpha} = (\alpha_w,\, \alpha_\kappa)^T$, so that the Hamiltonian can be written as a quadratic functional in the energy variables:

$$H = \frac{1}{2}\int_\Omega \left\{\frac{1}{\rho A}\alpha_w^2 + EI\alpha_\kappa^2\right\} \, \mathrm{d}\Omega \tag{3}$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}(x,t), && \text{Vertical velocity}, \\ e_\kappa &:= \frac{\delta H}{\delta \alpha_\kappa} = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t), && \text{Flexural momentum}. \end{aligned} \tag{4}$$

The underlying interconnection structure is then found to be:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w \\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix}\begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}. \tag{5}$$

The power flow gives access to the boundary variables:

$$
\begin{aligned}
\dot{H} &= \int_{\Omega} \left\{ e_w \partial_t \alpha_w + e_\kappa \partial_t \alpha_\kappa \right\} \, \mathrm{d}\Omega, \\
&= \int_{\Omega} \left\{ -e_w \partial_{xx} e_\kappa + e_\kappa \partial_{xx} e_w \right\} \, \mathrm{d}\Omega, \qquad \text{Integration by parts}, \\
&= \int_{\partial\Omega} \left\{ -e_w \partial_x e_\kappa + e_\kappa \partial_x e_w \right\} \mathrm{d}s = \langle -e_w, \, \partial_x e_\kappa \rangle_{\partial\Omega} + \langle e_\kappa, \, \partial_x e_w \rangle_{\partial\Omega}
\end{aligned}
\tag{6}
$$

Since the system is of differential order two, two pairing appears, giving rise to four combination of uniform boundary causality

- First case $u_{\partial,1} = e_w, \quad u_{\partial,2} = \partial_x e_w, \quad y_{\partial,1} = -\partial_x e_\kappa, \quad y_{\partial,2} = e_\kappa.$
  This imposes the vertical $e_w := \partial_t w$ and angular velocity $\partial_x e_w := \partial_{xt} w$ as boundary inputs. If the inputs are null a clamped boundary condition is obtained.

- Second case $u_{\partial,1} = e_w, \quad u_{\partial,2} = e_\kappa, \quad y_{\partial,1} = -\partial_x e_\kappa, \quad y_{\partial,2} = \partial_x e_w.$
  This imposes the vertical velocity and flexural momentum $e_\kappa := EI\partial_{xx} w$ as boundary inputs. Zero inputs lead to a simply supported condition is found.

- Third case $u_{\partial,1} = -\partial_x e_\kappa, \quad u_{\partial,2} = e_\kappa, \quad y_{\partial,1} = e_w, \quad y_{\partial,2} = \partial_x e_w.$
  This imposes the shear force $\partial_x e_\kappa := \partial_x (EI\partial_{xx} w)$ and flexural momentum as boundary inputs. Null inputs correspond to a free condition.

- Forth case $u_{\partial,1} = -\partial_x e_\kappa, \quad u_{\partial,2} = \partial_x e_w, \quad y_{\partial,1} = e_w, \quad y_{\partial,2} = e_\kappa.$
  This imposes the shear force and angular velocity as boundary inputs.

2 Differential forms formulation

The co-energy now are 0-forms $e_w, e_\kappa \in \Lambda^0(\Omega)$ whereas the flows are 1-forms $f_w = \partial_t \alpha_w, f_\kappa = \partial_t \alpha_\kappa \in \Lambda^1(\Omega)$. To recast (5) using the exterior derivative, the Hodge star operator is needed

$$
* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega), \qquad \Omega \subset \mathbb{R}^n. \tag{7}
$$

For one dimensional domains $\Omega \subset \mathbb{R}$ and using Euclidian coordinates, this operator can be either used on 1-forms or 0-forms

$$
\begin{aligned}
* : \Lambda^1(\Omega) &\longrightarrow \Lambda^0(\Omega), \\
f(x) \, \mathrm{d}x &\longrightarrow f(x)
\end{aligned}
\tag{8}
$$

or

$$
\begin{aligned}
* : \Lambda^0(\Omega) &\longrightarrow \Lambda^1(\Omega), \\
f(x) &\longrightarrow f(x) \, \mathrm{d}x
\end{aligned}
\tag{9}
$$

Then the equivalent system reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_w\\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix}0 & -\mathrm{d}{*}\mathrm{d}\\ \mathrm{d}{*}\mathrm{d} & 0\end{bmatrix}\begin{pmatrix}e_w\\ e_\kappa\end{pmatrix}. \tag{10}$$

Proof 1 *The operator* $\mathrm{d}*\mathrm{d} : \Lambda^0(\Omega) \to \Lambda^1(\Omega)$ *is a composition of operators that reads in Euclidean coordinates*

$$\begin{aligned}\mathrm{d}{*}\mathrm{d}e &= \mathrm{d}{*}(\frac{\partial e}{\partial x}\,\mathrm{d}x), \qquad e \in \Lambda^0(\Omega)\\ &= \ \mathrm{d}(\frac{\partial e}{\partial x}),\\ &= \frac{\partial^2 e}{\partial x^2}\,\mathrm{d}x \in \Lambda^1(\Omega)\end{aligned} \tag{11}$$

The Hamiltonian energy is then

$$H = \frac{1}{2}\int_\Omega e_w \wedge \alpha_w + e_\kappa \wedge \alpha_\kappa \tag{12}$$

To find the appropriate power balance, consider the integration by parts formula for smooth differential forms $\lambda \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^{n-k-1}(\Omega)$,

$$\langle \mathrm{d}\lambda,\, \mu\rangle = \langle \operatorname{Tr}\lambda,\, \operatorname{Tr}\mu\rangle - (-1)^k\,\langle \lambda,\, \mathrm{d}\mu\rangle \tag{13}$$

For uni-dimensional domains $n = 1$, and we take $\lambda \in \Lambda^0(\Omega)$ and $\mu \in \Lambda^0(\Omega)$, implying

$$\begin{aligned}\langle \mathrm{d}\lambda,\, \mu\rangle &= \langle \operatorname{Tr}\lambda,\, \operatorname{Tr}\mu\rangle - \langle \lambda,\, \mathrm{d}\mu\rangle\,,\\ \langle \lambda,\, \mathrm{d}\mu\rangle &= \langle \operatorname{Tr}\lambda,\, \operatorname{Tr}\mu\rangle - \langle \mathrm{d}\lambda,\, \mu\rangle\,.\end{aligned} \tag{14}$$

Then we can state

$$\langle -e_w,\, \mathrm{d}(*\mathrm{d}e_\kappa)\rangle = \langle -\operatorname{Tr}e_w,\, \operatorname{Tr}(*\mathrm{d}e_\kappa)\rangle + \langle \mathrm{d}e_w,\, *\mathrm{d}e_\kappa\rangle\,. \tag{15}$$

and

$$\langle e_\kappa,\, \mathrm{d}(*\mathrm{d}e_w)\rangle = \langle \operatorname{Tr}e_\kappa,\, \operatorname{Tr}(*\mathrm{d}e_w)\rangle - \langle \mathrm{d}e_\kappa,\, *\mathrm{d}e_w\rangle\,, \tag{16}$$

The power rate then reads

$$\begin{aligned}\dot{H} &= \int_\Omega e_w \wedge \partial_t\alpha_w + e_\kappa \wedge \partial_t\alpha_\kappa,\\ &= \int_\Omega e_w \wedge \partial_t\alpha_w + e_\kappa \wedge \partial_t\alpha_\kappa,\\ &= \int_\Omega -e_w \wedge (\mathrm{d}{*}\mathrm{d}e_\kappa) + e_\kappa \wedge (\mathrm{d}{*}\mathrm{d}e_w), \qquad \text{From (15), (16)}\\ &= \langle -\operatorname{Tr}e_w,\, \operatorname{Tr}(*\mathrm{d}e_\kappa)\rangle + \langle \mathrm{d}e_w,\, *\mathrm{d}e_\kappa\rangle + \langle \operatorname{Tr}e_\kappa,\, \operatorname{Tr}(*\mathrm{d}e_w)\rangle - \langle \mathrm{d}e_\kappa,\, *\mathrm{d}e_w\rangle\end{aligned} \tag{17}$$

The wedge product is such that for $\lambda \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^l(\Omega)$ it holds

$$\lambda \wedge \mu = (-1)^{kl} \mu \wedge \lambda \tag{18}$$

Furthermore, the Hodge star is such that

$$(\alpha, \beta) := \langle \alpha,\, *\beta \rangle = \langle *\alpha,\, \beta \rangle = (\beta, \alpha), \qquad \alpha, \beta \in \Lambda^k(\Omega) \tag{19}$$

Then for $e_w, e_\kappa \in \Lambda^0$ it holds

$$\begin{aligned} \langle \mathrm{d}e_w,\, *\mathrm{d}e_\kappa \rangle &= \langle *\mathrm{d}e_w,\, \mathrm{d}e_\kappa \rangle\,, \\ &= \int_\Omega (*\mathrm{d}e_w) \wedge (\mathrm{d}e_\kappa), \\ &= \int_\Omega (\mathrm{d}e_\kappa) \wedge (*\mathrm{d}e_w), \\ &= \langle \mathrm{d}e_\kappa,\, *\mathrm{d}e_w \rangle \end{aligned} \tag{20}$$

It can be then stated

$$\langle \mathrm{d}e_w,\, *\mathrm{d}e_\kappa \rangle - \langle \mathrm{d}e_\kappa,\, *\mathrm{d}e_w \rangle = 0 \tag{21}$$

The power balance is then

$$\dot{H} = \langle -\operatorname{Tr} e_w,\, \operatorname{Tr}(*\mathrm{d}e_\kappa) \rangle + \langle \operatorname{Tr} e_\kappa,\, \operatorname{Tr}(*\mathrm{d}e_w) \rangle \tag{22}$$

In vector calculus notation it reads

$$\dot{H} = \left\langle -\operatorname{Tr} e_w,\, \operatorname{Tr} \frac{\partial e_\kappa}{\partial x} \right\rangle + \left\langle \operatorname{Tr} e_\kappa,\, \operatorname{Tr} \frac{\partial e_w}{\partial x} \right\rangle = \langle -e_w,\, \partial_x e_\kappa \rangle_{\partial\Omega} + \langle e_\kappa,\, \partial_x e_w \rangle_{\partial\Omega} \tag{23}$$

Completely equivalent to Eq. (6).

References

[1] P. Kotyczka. *Numerical Methods for Distributed Parameter Port-Hamiltonian Systems.* TUM University Press, 2019.