Multibody System Dynamics manuscript No.
(will be inserted by the editor)

Port-Hamiltonian Flexible Multibody Dynamics

First Author · Second Author

Received: date / Accepted: date

Abstract Insert your abstract here. Include keywords, PACS and mathematical subject classification numbers as needed.

Keywords First keyword · Second keyword · More

1 Introduction

Your text comes here. Separate text sections with

2 Flexible dynamics

The coupled ODE-PDE system representing the motion of a single flexible body are here recalled. The model for the classical equations derived using the principle of virtual work can be found in [1]. The small difference with respect to the derivation therein is that the equation for the translation is now written in the body frame. Consider an open connected set $\Omega \in \mathbb{R}^d$, representing a floating flexible body. The dynamics is computed at a generic point P, that is not necessarily the center of mass. The velocity of a generic point is the expressed by considering a small flexible displacement superimposed to the rigid motion

$$\boldsymbol{v} = \boldsymbol{v}_P + [\boldsymbol{\omega}_P]_\times(\boldsymbol{x} + \boldsymbol{u}_f) + \boldsymbol{v}_f,$$

where $\boldsymbol{v}_P, \boldsymbol{\omega}_P$ is the linear and angular velocities of point P and $\boldsymbol{v}_f := \dot{\boldsymbol{u}}_f$ is the derivative of the deformation displacement $\boldsymbol{u}_f$. These quantities are evaluated

F. Author
first address
Tel.: +123-45-678910
Fax: +123-45-678910
E-mail: fauthor@example.com

S. Author
second address

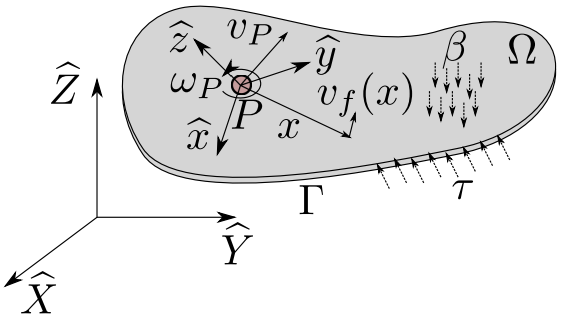

Fig. 1 Thin floating body undergoing a surface traction τ and body force density β

in the body reference frame (see Fig. 1). The notation $[\boldsymbol{a}]_\times$ denotes the skew-symmetric matrix associated to vector $\boldsymbol{a}$. The Eqs. of motion are then expressed as follows

$$\begin{aligned} m(\dot{\boldsymbol{v}}_P + [\boldsymbol{\omega}_P]_\times \boldsymbol{v}_P) + [\boldsymbol{s}_u]_\times^\top \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho \ddot{\boldsymbol{u}}_f \, \mathrm{d}\Omega = \\ - [\boldsymbol{\omega}_P]_\times [\boldsymbol{\omega}_P]_\times \boldsymbol{s}_u - \int_\Omega 2\rho [\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f \, \mathrm{d}\Omega + \int_\Omega \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, \mathrm{d}\Gamma \end{aligned} \tag{1}$$

$$\begin{aligned} [\boldsymbol{s}_u]_\times (\dot{\boldsymbol{v}}_P + [\boldsymbol{\omega}_P]_\times \boldsymbol{v}_P) + \boldsymbol{J}_u \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \ddot{\boldsymbol{u}}_f \, \mathrm{d}\Omega - [\boldsymbol{\omega}_P]_\times \boldsymbol{J}_u \boldsymbol{\omega}_P = \\ - \int_\Omega 2\rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times [\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f \, \mathrm{d}\Omega + \int_\Omega [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\tau} \, \mathrm{d}\Gamma \end{aligned} \tag{2}$$

$$\rho(\dot{\boldsymbol{v}}_P + [\boldsymbol{\omega}_P]_\times \boldsymbol{v}_P) + \rho([\dot{\boldsymbol{\omega}}_P]_\times + [\boldsymbol{\omega}_P]_\times [\boldsymbol{\omega}_P]_\times)(\boldsymbol{x} + \boldsymbol{u}_f) + \rho(2[\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f + \ddot{\boldsymbol{u}}_f) = \mathrm{Div}\, \boldsymbol{\Sigma} + \boldsymbol{\beta}, \tag{3}$$

where ρ is the mass density, $m = \int_\Omega \rho \, \mathrm{d}\Omega$ is the total mass, $\boldsymbol{s}_u = \int_\Omega \rho(\boldsymbol{x} + \boldsymbol{u}_f) \, \mathrm{d}\Omega$ is the static moment, $\boldsymbol{J}_u = -\int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times [\boldsymbol{x} + \boldsymbol{u}_f]_\times \, \mathrm{d}\Omega$ is the inertia matrix. Considering that $\dot{\boldsymbol{v}}_f = \ddot{\boldsymbol{u}}_f + [\boldsymbol{\omega}_P]_\times \dot{\boldsymbol{u}}_f$ and using the Jacobi identity Eqs. (1), (2), (3) can be rewritten equivalently as

$$\begin{aligned} m\dot{\boldsymbol{v}}_P + [\boldsymbol{s}_u]_\times^\top \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho \dot{\boldsymbol{v}}_f \, \mathrm{d}\Omega = \\ \left[m\boldsymbol{v}_P + [\boldsymbol{s}_u]_\times^\top \boldsymbol{\omega}_P + \int_\Omega \rho \boldsymbol{v}_f \, \mathrm{d}\Omega \right]_\times \boldsymbol{\omega}_P + \int_\Omega \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} \boldsymbol{\tau} \, \mathrm{d}\Gamma, \end{aligned} \tag{4}$$

$$\begin{aligned} [\boldsymbol{s}_u]_\times \dot{\boldsymbol{v}}_P + \boldsymbol{J}_u \dot{\boldsymbol{\omega}}_P + \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \dot{\boldsymbol{v}}_f \, \mathrm{d}\Omega = \\ \left[[\boldsymbol{s}_u]_\times^\top \boldsymbol{\omega}_P + \int_\Omega \rho \boldsymbol{v}_f \, \mathrm{d}\Omega \right]_\times \boldsymbol{v}_P + \left[[\boldsymbol{s}_u]_\times \boldsymbol{v}_P + \boldsymbol{J}_u \boldsymbol{\omega}_P + \int_\Omega \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times \dot{\boldsymbol{v}}_f \, \mathrm{d}\Omega \right]_\times \boldsymbol{\omega}_P + \\ \int_\Omega \left[\rho \boldsymbol{v}_P + \rho [\boldsymbol{x} + \boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P + \rho \boldsymbol{v}_f \right]_\times \boldsymbol{v}_f \, \mathrm{d}\Omega + \int_\Omega [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\beta} \, \mathrm{d}\Omega + \int_{\partial\Omega} [\boldsymbol{x} + \boldsymbol{u}_f]_\times \boldsymbol{\tau} \, \mathrm{d}\Gamma \end{aligned} \tag{5}$$

$$\begin{aligned} \rho\dot{\boldsymbol{v}}_P + \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top \dot{\boldsymbol{\omega}}_P + \rho\dot{\boldsymbol{v}}_f = \\ \left[\rho\boldsymbol{v}_P + [\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P + \rho\boldsymbol{v}_f\right]_\times \boldsymbol{\omega}_P + \operatorname{Div}\boldsymbol{\Sigma} + \boldsymbol{\beta}, \end{aligned} \tag{6}$$

together with the boundary condition

$$\boldsymbol{\Sigma}\cdot\boldsymbol{n}|_\Gamma = \boldsymbol{\tau}|_\Gamma, \quad \boldsymbol{n} \text{ is the outward normal.} \tag{7}$$

3 PH flexible dynamics

In this section the flexible dynamics of a floating body is written as a coupled system of ODEs and PDEs in pH form. First of only the kinetic and deformation energy are considered for sake of simplicity. Then a generic potential is introduced, so that the generalized coordinates are included in the formulation.

3.1 Formulation without generalized coordinates

Consider the total energy (Hamiltonian), given by the sum of kinetic and deformation energy (linear elasticity):

$$H = H_{\text{kin}} + H_{\text{def}} = \frac{1}{2}\int_\Omega \rho||\boldsymbol{v}_P + [\boldsymbol{\omega}_P]_\times(\boldsymbol{x}+\boldsymbol{u}_f) + \boldsymbol{v}_f||^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon}\ \mathrm{d}\Omega, \tag{8}$$

where $\boldsymbol{\Sigma}$ is the Cauchy stress tensor and $\boldsymbol{\varepsilon}$ is the infinitesimal stress tensor. From linear elasticity theory it is well known that $\boldsymbol{\varepsilon} = \operatorname{Grad}(\boldsymbol{u}_f)$, where $\operatorname{Grad} = \frac{1}{2}[\nabla + \nabla^\top]$ and $\boldsymbol{\Sigma} = \boldsymbol{\mathcal{D}}\boldsymbol{\varepsilon}$, where $\boldsymbol{\mathcal{D}}$ is the stiffness tensor. The inner product $\boldsymbol{A} : \boldsymbol{B} = \operatorname{Tr}(\boldsymbol{A}\boldsymbol{B}^T)$ is the tensor contraction.
The momenta (usually called energy variables in the pH framework) are then computed by derivation of the Hamiltonian. As the variables belong to finite- and infinite-dimensional spaces the derivative is either a classical gradient either a variational derivative:

$$\begin{aligned} \boldsymbol{p}_t &:= \frac{\partial H}{\partial \boldsymbol{v}_P} = m\boldsymbol{v}_P + [\boldsymbol{s}_u]_\times^\top \boldsymbol{\omega}_P + \int_\Omega \rho\boldsymbol{v}_f\ \mathrm{d}\Omega, \\ \boldsymbol{p}_r &:= \frac{\partial H}{\partial \boldsymbol{\omega}_P} = [\boldsymbol{s}_u]_\times \boldsymbol{v}_P + \boldsymbol{J}_u\boldsymbol{\omega}_P + \int_\Omega \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times \boldsymbol{v}_f\ \mathrm{d}\Omega, \\ \boldsymbol{p}_f &:= \frac{\delta H}{\delta \boldsymbol{v}_f} = \rho\boldsymbol{v}_P + \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top \boldsymbol{\omega}_P + \rho\boldsymbol{v}_f, \\ \boldsymbol{\varepsilon} &:= \frac{\delta H}{\delta \boldsymbol{\Sigma}} \end{aligned} \tag{9}$$

The relation between energy and co-energy variable is then given by

$$\begin{bmatrix} \boldsymbol{p}_t \\ \boldsymbol{p}_r \\ \boldsymbol{p}_f \\ \boldsymbol{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} m\boldsymbol{I} & [\boldsymbol{s}_u]_\times^\top & \mathcal{I}_\rho^\Omega & 0 \\ [\boldsymbol{s}_u]_\times & \boldsymbol{J}_u & \boldsymbol{\mathcal{I}}_{\rho x}^\Omega & 0 \\ (\mathcal{I}_\rho^\Omega)^* & (\boldsymbol{\mathcal{I}}_{\rho x}^\Omega)^* & \rho & 0 \\ 0 & 0 & 0 & \boldsymbol{\mathcal{D}}^{-1} \end{bmatrix}}_{\boldsymbol{\mathcal{M}}\text{: Mass operator}} \begin{bmatrix} \boldsymbol{v}_P \\ \boldsymbol{\omega}_P \\ \boldsymbol{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}, \tag{10}$$

where $\boldsymbol{I}$ is the identity matrix in $\mathbb{R}^3$ and the operators are defined as

$$\mathcal{I}_\rho^\Omega := \int_\Omega \rho(\cdot)\,\mathrm{d}\Omega, \qquad \boldsymbol{\mathcal{I}}_{\rho x}^\Omega := \int_\Omega \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times(\cdot)\,\mathrm{d}\Omega,$$
$$(\mathcal{I}_\rho^\Omega)^* := \rho, \qquad (\boldsymbol{\mathcal{I}}_{\rho x}^\Omega)^* := \rho[\boldsymbol{x}+\boldsymbol{u}_f]_\times^\top.$$

The superscript $*$ denotes the adjoint operator. The mass operator $\boldsymbol{\mathcal{M}}$ is a self-adjoint, positive operator.
If the dependency of $\boldsymbol{\mathcal{M}}$ on the flexible displacement $\boldsymbol{u}_f$ is neglected then Eqs. (5), (4), (6) can be recast into port Hamiltonian form in co-energy variables

$$\boldsymbol{\mathcal{M}}\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_P\\ \boldsymbol{\omega}_P\\ \boldsymbol{v}_f\\ \boldsymbol{\Sigma}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & [\boldsymbol{p}_t]_\times & 0 & 0\\ [\boldsymbol{p}_t]_\times & [\boldsymbol{p}_r]_\times & \boldsymbol{\mathcal{I}}_{p_f}^\Omega & 0\\ 0 & -(\boldsymbol{\mathcal{I}}_{p_f}^\Omega)^* & 0 & \mathrm{Div}\\ 0 & 0 & \mathrm{Grad} & 0\end{bmatrix}}_{\mathcal{J}\text{: Interconnection operator}}\begin{bmatrix}\boldsymbol{v}_P\\ \boldsymbol{\omega}_P\\ \boldsymbol{v}_f\\ \boldsymbol{\Sigma}\end{bmatrix} + \underbrace{\begin{bmatrix}\mathcal{I}^\Omega\\ \boldsymbol{\mathcal{I}}_x^\Omega\\ \boldsymbol{I}\\ 0\end{bmatrix}}_{\mathcal{B}_d}\boldsymbol{\beta} + \underbrace{\begin{bmatrix}\mathcal{I}^\Gamma\\ \boldsymbol{\mathcal{I}}_x^\Gamma\\ 0\\ 0\end{bmatrix}}_{\mathcal{B}_r}\boldsymbol{\tau}, \quad (11)$$

The operator $\boldsymbol{\mathcal{I}}_{p_f}^\Omega = \int_\Omega [\boldsymbol{p}_f]_\times(\cdot)\,\mathrm{d}\Omega$ has formal adjoint given by $(\boldsymbol{\mathcal{I}}_{p_f}^\Omega)^* = -[\boldsymbol{p}_f]_\times(\cdot)$.

The control operators read

$$\mathcal{I}^\Omega := \int_\Omega (\cdot)\,\mathrm{d}\Omega, \qquad \boldsymbol{\mathcal{I}}_x^\Omega := \int_\Omega [\boldsymbol{x}+\boldsymbol{u}_f]_\times(\cdot)\,\mathrm{d}\Omega,$$
$$\mathcal{I}^\Gamma := \int_{\partial\Omega} (\cdot)\,\mathrm{d}\Gamma, \qquad \boldsymbol{\mathcal{I}}_x^\Gamma := \int_{\partial\Omega} [\boldsymbol{x}+\boldsymbol{u}_f]_\times(\cdot)\,\mathrm{d}\Gamma.$$

It is important to highlight that Div and Grad are formally skew-adjoint operators, i.e. for homogeneous boundary conditions $\boldsymbol{\tau} = 0$

$$\int_\Omega \boldsymbol{\Sigma} : \mathrm{Grad}(\boldsymbol{v}_f)\,\mathrm{d}\Omega \underbrace{=}_{\text{I.B.P.}} -\int_\Omega \mathrm{Div}(\boldsymbol{\Sigma})\cdot\boldsymbol{v}_f\,\mathrm{d}\Omega,$$
$$\langle \boldsymbol{\Sigma},\, \mathrm{Grad}(\boldsymbol{v}_f)\rangle_{\mathscr{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})} \underbrace{=}_{\text{I.B.P.}} -\langle \mathrm{Div}(\boldsymbol{\Sigma}),\, \boldsymbol{v}_f\rangle_{\mathscr{L}^2(\Omega,\mathbb{R}^d)},$$

where $\langle , \rangle_{\mathscr{H}}$ denote an inner product over the Hilbert space $\mathscr{H}$.
$\mathscr{L}^2(\Omega,\mathbb{R}^d)$, $\mathscr{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ are the spaces of square integrable vector-valued or symmetric tensors-valued functions in a geometric domain of dimension d.
For this reason operator $\boldsymbol{\mathcal{J}}$ is skew-symmetric $\boldsymbol{\mathcal{J}}^* = -\boldsymbol{\mathcal{J}}$.

Remark 1 If case of vanishing deformation $\boldsymbol{u}_f \equiv 0$ the Newton-Euler equations on $SE(3)$ are retrieved

$$\begin{bmatrix}\dot{\boldsymbol{p}}_t\\ \dot{\boldsymbol{p}}_r\end{bmatrix} = \begin{bmatrix}0 & [\boldsymbol{p}_t]_\times\\ [\boldsymbol{p}_t]_\times & [\boldsymbol{p}_r]_\times\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_P\\ \boldsymbol{\omega}_P\end{bmatrix}. \quad (12)$$

This system is again in port Hamiltonian form with total energy equal to the kinetic energy.

3.2 Including the generalized coordinates

For the time being only kinetic and deformation energy were considered. If a generic potential energy contribution has to be considered one must account for the generalized coordinates

- ${}^i\boldsymbol{r}_P$: the position of point P in the inertial frame of reference;
- $\boldsymbol{R}$: the orientation matrix that transforms vectors from the body frame to the inertial frame (other attitude parametrization are possible, here the orientation matrix is considered for ease of presentation);
- $\boldsymbol{u}_f$ the flexible displacement;

In particular following [2] the orientation matrix is converted in a vector by concatenating its rows

$$\boldsymbol{R}_{\mathrm{v}} = \mathrm{vec}(\boldsymbol{R}^\top) = [\boldsymbol{R}_x\ \boldsymbol{R}_y\ \boldsymbol{R}_z]^\top,$$

where $\boldsymbol{R}_x, \boldsymbol{R}_y, \boldsymbol{R}_z$ are the first, second and third row of matrix $\boldsymbol{R}$. Furthermore the corresponding cross map will be given by

$$[\boldsymbol{R}_{\mathrm{v}}]_\times = \begin{bmatrix} [\boldsymbol{R}_x]_\times \\ [\boldsymbol{R}_y]_\times \\ [\boldsymbol{R}_z]_\times \end{bmatrix} \qquad [\boldsymbol{R}_{\mathrm{v}}]_\times : \mathbb{R}^9 \to \mathbb{R}^{9\times 3}$$

The total energy now includes a potential energy that depends on the generalized coordinates

$$H = H_{\mathrm{kin}} + H_{\mathrm{def}} + H_{\mathrm{pot}}$$

The overall port Hamitonian formulation is then an augmented version of system (11)

$$\underbrace{\left[\begin{array}{c|c} \boldsymbol{I} & 0 \\ \hline 0 & \boldsymbol{\mathcal{M}} \end{array}\right]}_{\boldsymbol{\mathcal{E}}} \frac{d}{dt} \underbrace{\begin{bmatrix} {}^i\boldsymbol{r}_P \\ \boldsymbol{R}_{\mathrm{v}} \\ \boldsymbol{u}_f \\ \hline \boldsymbol{v}_P \\ \boldsymbol{\omega}_P \\ \boldsymbol{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}}_{\boldsymbol{e}} = \underbrace{\left[\begin{array}{ccc:ccc:c} 0 & 0 & 0 & \boldsymbol{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\boldsymbol{R}_{\mathrm{v}}]_\times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} & 0 \\ \hline -\boldsymbol{R}^\top & 0 & 0 & 0 & [\boldsymbol{p}_t]_\times & 0 & 0 \\ 0 & -[\boldsymbol{R}_{\mathrm{v}}]_\times^\top & 0 & [\boldsymbol{p}_t]_\times & [\boldsymbol{p}_r]_\times & \boldsymbol{\mathcal{I}}_{p_f}^{\Omega} & 0 \\ 0 & 0 & -\boldsymbol{I} & 0 & -(\boldsymbol{\mathcal{I}}_{p_f}^{\Omega})^* & 0 & \mathrm{Div} \\ 0 & 0 & 0 & 0 & 0 & \mathrm{Grad} & 0 \end{array}\right]}_{\boldsymbol{\mathcal{J}}} \underbrace{\begin{bmatrix} \partial_{\boldsymbol{r}_P} H \\ \partial_{\boldsymbol{R}_{\mathrm{v}}} H \\ \delta_{\boldsymbol{u}_f} H \\ \hline \boldsymbol{v}_P \\ \boldsymbol{\omega}_P \\ \boldsymbol{v}_f \\ \boldsymbol{\Sigma} \end{bmatrix}}_{\boldsymbol{z}}, \tag{13}$$

where the contribution of external forces and torques has been omitted. For example if the gravity potential energy is considered

$$H_{\mathrm{pot}} = \int_\Omega \rho g\, {}^i r^z \,\mathrm{d}\Omega = \int_\Omega \rho g \left[{}^i r_{P,z} + \boldsymbol{R}_z(\boldsymbol{x} + \boldsymbol{u}_f)\right] \,\mathrm{d}\Omega,$$

the co-energy variables are easily obtained

$$\begin{aligned} &\partial_{\boldsymbol{r}_P} H = mg\, {}^i\widehat{\boldsymbol{Z}}, \quad {}^i\widehat{\boldsymbol{Z}} \text{ is the inertial frame vertical direction} \\ &\partial_{\boldsymbol{R}_{\mathrm{v}}} H = [\boldsymbol{0}_{(3,1)},\ \boldsymbol{0}_{(3,1)},\ \int_\Omega \rho g(\boldsymbol{x} + \boldsymbol{u}_f)^\top \,\mathrm{d}\Omega]^\top \\ &\delta_{\boldsymbol{u}_f} H = \rho g\, \boldsymbol{R}_z^\top \end{aligned}$$

These correspond to the forcing terms due to gravity. System (13) fits into the framework detailed in [3] and extends it as a coupled system of ODEs and PDEs is considered. It can be rewritten compactly as follows

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\boldsymbol{e})\dot{\boldsymbol{e}} &= \boldsymbol{\mathcal{J}}(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}) + \boldsymbol{\mathcal{B}}_d(e)\boldsymbol{u}_d + \boldsymbol{\mathcal{B}}_r(\boldsymbol{e})\boldsymbol{u}_\partial, \\ \boldsymbol{y}_d &= \boldsymbol{\mathcal{B}}_d^*(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}), \\ \boldsymbol{u}_\partial &= \boldsymbol{\mathcal{B}}_\partial \boldsymbol{z}(\boldsymbol{e}) = \boldsymbol{\Sigma}\cdot\boldsymbol{n}|_\Gamma, \\ \boldsymbol{u}_\partial &= \boldsymbol{\mathcal{C}}_\partial \boldsymbol{z}(\boldsymbol{e}) = \partial_t \boldsymbol{u}_f|_\Gamma \end{aligned} \tag{14}$$

where $\boldsymbol{u}_d = \boldsymbol{\beta}$, $\boldsymbol{u}_\partial = \boldsymbol{\tau}$. The gradient of the Hamiltonian gives $\partial_{\boldsymbol{e}} H = \boldsymbol{\mathcal{E}}^* \boldsymbol{z}$. Adopting the same nomenclature as in [3], e contains the state and z contains the effort functions. In this case the operators verify $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^*$, $\boldsymbol{\mathcal{J}} = -\boldsymbol{\mathcal{J}}^*$. The distributed control operator $\boldsymbol{\mathcal{B}}_d$ is bounded.

Remark 2 For sake of simplicity the linear elastic case $H_{\text{def}} = \int_\Omega \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}\Omega$ has been presented. If more complex behaviors have to be accounted for (e.g. geometric stiffening) more complex expression for the deformation energy can be considered. Furthermore, 3D linear elasticity can be simplified and the geometrical dimension reduced. Beam and plate models are represent by appropriate differential operators that replace the Div, Grad appearing in (14) (see §4.3).

4 Discretization procedure

A finite-element based technique to obtain a finite dimensional pH system is illustrated. This methodology relies on the results explained in [4] and boils down to three simple steps

1. The system is put into weak form;
2. An integration by parts is applied to highlight the proper boundary control;
3. A Galerkin method is employed to obtain a finite-dimensional system.

4.1 Illustration for the Elastodynamics PDE

To explain the methodology consider the elastodynamics PDE

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{Div}\left(\boldsymbol{\mathcal{D}}\,\mathrm{Grad}(\boldsymbol{u})\right) = \boldsymbol{u}_d,$$

where a distributed control u_d (a volumetric force) is considered. The total energy is simply given by

$$H = \frac{1}{2}\int_\Omega \left\{ \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} \right)^2 + \boldsymbol{\Sigma} : \boldsymbol{\varepsilon} \right\} \mathrm{d}\Omega.$$

To get a pH representation the energy variables have to selected. The corresponding co-energies are then computed taking the variational derivative of the Hamiltonian

$$\begin{aligned} \text{Energies} \quad & \boldsymbol{x}_1 = \rho\, \partial_t \boldsymbol{u}, && \boldsymbol{X}_2 = \boldsymbol{\varepsilon} = \mathrm{Grad}(\boldsymbol{u}). \\ \text{Co-energies} \quad & \boldsymbol{e}_1 := \frac{\delta H}{\delta \boldsymbol{x}_1} = \partial_t \boldsymbol{u}, && \boldsymbol{E}_2 := \frac{\delta H}{\delta \boldsymbol{X}_2} = \boldsymbol{\Sigma}. \end{aligned} \tag{15}$$

The port-Hamiltonian representation in co-energy variables becomes

$$\underbrace{\begin{bmatrix}\rho & 0\\ 0 & \boldsymbol{\mathcal{D}}^{-1}\end{bmatrix}}_{\boldsymbol{\mathcal{M}}}\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{e}_1\\ \boldsymbol{E}_2\end{bmatrix}=\underbrace{\begin{bmatrix}0 & \mathrm{Div}\\ \mathrm{Grad} & 0\end{bmatrix}}_{\boldsymbol{\mathcal{J}}}\begin{bmatrix}\boldsymbol{e}_1\\ \boldsymbol{E}_2\end{bmatrix}+\underbrace{\begin{bmatrix}\boldsymbol{I}\\ 0\end{bmatrix}}_{\boldsymbol{\mathcal{B}}_d}\boldsymbol{u}_d$$

The interconnection operator may be decomposed as $\boldsymbol{\mathcal{J}}=\boldsymbol{\mathcal{J}}_{\mathrm{Div}}+\boldsymbol{\mathcal{J}}_{\mathrm{Grad}}$

$$\underbrace{\begin{bmatrix}0 & \mathrm{Div}\\ \mathrm{Grad} & 0\end{bmatrix}}_{\boldsymbol{\mathcal{J}}}=\underbrace{\begin{bmatrix}0 & \mathrm{Div}\\ 0 & 0\end{bmatrix}}_{\boldsymbol{\mathcal{J}}_{\mathrm{Div}}}+\underbrace{\begin{bmatrix}0 & 0\\ \mathrm{Grad} & 0\end{bmatrix}}_{\boldsymbol{\mathcal{J}}_{\mathrm{Grad}}} \tag{16}$$

Assuming a Neumann boundary conditions (the normal traction $\boldsymbol{\tau}$ is known at the boundary), this system can be written compactly as a boundary control system

$$\begin{aligned}\boldsymbol{\mathcal{M}}\frac{\partial \boldsymbol{e}}{\partial t}&=\boldsymbol{\mathcal{J}}\boldsymbol{e}+\boldsymbol{\mathcal{B}}_d\boldsymbol{u}_d,\\ \boldsymbol{y}_d&=\boldsymbol{\mathcal{B}}_d^*\boldsymbol{e},\\ \boldsymbol{u}_\partial&=\boldsymbol{E}_2\cdot\boldsymbol{n}=\boldsymbol{\Sigma}\cdot\boldsymbol{n}|_\Gamma,\\ \boldsymbol{y}_\partial&=\boldsymbol{e}_1=\partial_t\boldsymbol{u}|_\Gamma.\end{aligned} \tag{17}$$

Now the system is put into weak form considering the inner product on space $\mathscr{X}=\mathscr{L}^2(\Omega,\mathbb{R}^d)\times\mathscr{L}^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Taking two elements $[\boldsymbol{a},\boldsymbol{A}],\ [\boldsymbol{b},\boldsymbol{B}]\in\mathscr{X}$ the inner product is computed as

$$\langle[\boldsymbol{a},\boldsymbol{A}],\ [\boldsymbol{b},\boldsymbol{B}]\rangle_{\mathscr{X}}=\int_\Omega\boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}\Omega+\int_\Omega\boldsymbol{A}:\boldsymbol{B}\,\mathrm{d}\Omega.$$

Considering a test function $w\in\mathscr{X}$ the weak form reads

$$\langle\boldsymbol{w},\ \boldsymbol{\mathcal{M}}\ \partial_t\boldsymbol{e}\rangle_{\mathscr{X}}=\langle\boldsymbol{w},\ \boldsymbol{\mathcal{J}}\boldsymbol{e}\rangle_{\mathscr{X}}+\langle\boldsymbol{w},\ \boldsymbol{\mathcal{B}}_d\boldsymbol{u}_d\rangle_{\mathscr{X}}$$

The bilinear form $m(\boldsymbol{w},\partial_t\boldsymbol{e})=\langle\boldsymbol{w},\ \boldsymbol{\mathcal{M}}\ \partial_t\boldsymbol{e}\rangle_{\mathscr{X}}$ is symmetric and coercive. The bilinear form $b_d(\boldsymbol{w},\boldsymbol{u}_d):=\langle\boldsymbol{w},\ \boldsymbol{\mathcal{B}}_d\boldsymbol{u}_d\rangle_{\mathscr{X}}$ takes into account distributed control. Now an integration by parts is applied on $\boldsymbol{\mathcal{J}}_{\mathrm{Div}}$

$$\langle\boldsymbol{w},\ \boldsymbol{\mathcal{J}}\boldsymbol{e}\rangle_{\mathscr{X}}=\langle\boldsymbol{w},\ \boldsymbol{\mathcal{J}}\boldsymbol{e}\rangle_{\mathscr{X}}-\langle\boldsymbol{\mathcal{J}}_{\mathrm{Grad}}\boldsymbol{w},\ \boldsymbol{e}\rangle_{\mathscr{X}}+\langle\boldsymbol{w},\ \boldsymbol{u}_\partial\rangle_{\mathscr{L}^2(\Gamma)} \tag{18}$$

The expression $j_{\mathrm{Grad}}(\boldsymbol{w},\boldsymbol{e}):=\langle\boldsymbol{w},\ \boldsymbol{\mathcal{J}}_{\mathrm{Grad}}\boldsymbol{e}\rangle_{\mathscr{X}}-\langle\boldsymbol{\mathcal{J}}_{\mathrm{Grad}}\boldsymbol{w},\ \boldsymbol{e}\rangle_{\mathscr{X}}$ is a skew symmetric bilinear form as it holds $j_{\mathrm{Grad}}(\boldsymbol{w},\boldsymbol{e})=-j_{\mathrm{Grad}}(\boldsymbol{w},\boldsymbol{e})$. The bilinear form $b_\partial(\boldsymbol{w},\boldsymbol{u}_\partial):=\langle\boldsymbol{w},\ \boldsymbol{u}_\partial\rangle_{\mathscr{L}^2(\Gamma)}$ imposes weakly the Neumann condition. System (17) is now rewritten in weak form

$$m(\boldsymbol{w},\partial_t\boldsymbol{e})=j_{\mathrm{Grad}}(\boldsymbol{w},\boldsymbol{e})+b_d(\boldsymbol{w},\boldsymbol{u}_d)+b_\partial(\boldsymbol{w},\boldsymbol{u}_\partial). \tag{19}$$

The output equation is discretized considering test function w_∂ defined over the boundary

$$\langle\boldsymbol{w}_\partial,\ \boldsymbol{y}_\partial\rangle_{\mathscr{X}}=\langle\boldsymbol{w}_\partial,\ \boldsymbol{e}_1\rangle_{\mathscr{X}}\,. \tag{20}$$

If a Galerkin method is applied then corresponding test and trial functions are discretized using the same basis

$$\begin{aligned}\boldsymbol{w}_1&=\boldsymbol{\phi}_1^\top\boldsymbol{v}_1, &\qquad \boldsymbol{W}_2&=\boldsymbol{\phi}_2^\top\boldsymbol{w}_2,\\ \boldsymbol{e}_1&=\boldsymbol{\phi}_1^\top\boldsymbol{e}_1, &\qquad \boldsymbol{E}_2&=\boldsymbol{\phi}_2^\top\boldsymbol{e}_2,\end{aligned}$$

where, with some abuse of notation, a physical vector and the numerical vector representing its approximation are denoted in the same manner. A finite dimensional pH system is readily obtained

$$\begin{aligned} \boldsymbol{M}\dot{\boldsymbol{e}} &= \boldsymbol{J}\boldsymbol{e} + \boldsymbol{B}_d\boldsymbol{u}_d + \boldsymbol{B}_\partial\boldsymbol{u}_\partial, \\ \boldsymbol{y}_d &:= \boldsymbol{M}_d\widetilde{\boldsymbol{y}}_d = \boldsymbol{B}_d^\top\boldsymbol{e}, \\ \boldsymbol{y}_\partial &:= \boldsymbol{M}_\partial\widetilde{\boldsymbol{y}}_\partial = \boldsymbol{B}_\partial^\top\boldsymbol{e}. \end{aligned} \tag{21}$$

Vectors $\widetilde{\boldsymbol{y}}_d, \widetilde{\boldsymbol{y}}_\partial$ correspond to the output degrees of freedom. The outputs $\boldsymbol{y}_d, \boldsymbol{y}_\partial$ have been defined incorporating the mass matrix in order get the discrete power balance $\dot{H}_d = \boldsymbol{u}_\partial^\top\boldsymbol{y}_\partial + \boldsymbol{u}_d^\top\boldsymbol{y}_d$.

Remark 3 Stable mixed finite elements for the elastodynamics problem are detailed in [5]. However the formulation therein is based on a weak form obtained by integration by parts of the $\boldsymbol{\mathcal{J}}_{\mathrm{Grad}}$ operator. The mixed finite element method for such a problem are then stable in the sense of Brezzi thanks to the properties of L^2/H^{Div} finite element spaces.

4.2 Discretized rigid-flexible Hamiltonian dynamics

The same methodology is applied to system (14). If corresponding test functions w, state e and the effort functions z are discretized using the same bases

$$\boldsymbol{w} = \boldsymbol{\phi}^\top\boldsymbol{w}, \quad \boldsymbol{e} = \boldsymbol{\phi}^\top\boldsymbol{e}, \quad \boldsymbol{z} = \boldsymbol{\phi}^\top\boldsymbol{z},$$

then a finite-dimensional pHDAE system is obtained (after integration by parts of the $\mathcal{J}_{\mathrm{Div}}$ operator as)

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{e})\dot{\boldsymbol{e}} &= \boldsymbol{J}(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}) + \boldsymbol{B}_d(\boldsymbol{e})\boldsymbol{u}_d + \boldsymbol{B}_\partial(\boldsymbol{e})\boldsymbol{u}_\partial, \\ \boldsymbol{y}_d &:= \boldsymbol{M}_d\widetilde{\boldsymbol{y}}_d = \boldsymbol{B}_d^\top\boldsymbol{z}(\boldsymbol{e}), \\ \boldsymbol{y}_\partial &:= \boldsymbol{M}_\partial\widetilde{\boldsymbol{y}}_\partial = \boldsymbol{B}_\partial^\top\boldsymbol{z}(\boldsymbol{e}). \end{aligned} \tag{22}$$

The computation of vector $\boldsymbol{z}$ is based on the gradient of the discrete Hamiltonian:

$$\frac{\partial H_d}{\partial \boldsymbol{e}} = \boldsymbol{E}^\top\boldsymbol{z}, \qquad H_d = H_{d,\mathrm{kin}} + H_{d,\mathrm{def}} + H_{d,\mathrm{pot}},$$

For the deformation and kinetic energy it is straightforward to find the between the state and effort functions as those energy are quadratic in the state variable:

$$H_{d,\mathrm{kin}} + H_{d,\mathrm{def}} = \frac{1}{2}\boldsymbol{e}_{\mathrm{kd}}^\top\boldsymbol{M}_{\mathrm{kd}}\,\boldsymbol{e}_{\mathrm{kd}} \longrightarrow \boldsymbol{z}_{\mathrm{kd}} = \boldsymbol{e}_{\mathrm{kd}}, \tag{23}$$

where $\boldsymbol{e}_{\mathrm{kd}} = [\boldsymbol{v}_P;\, \boldsymbol{\omega}_P;\, \boldsymbol{v}_f;\, \boldsymbol{\Sigma}]$ and M_{kd} is the discretization of the mass operator $\mathcal{M}$ given in Eq (10). The only term that requires additional care is the potential energy and particularly the variational derivative of the Hamiltonian with respect to the deformation displacement $\boldsymbol{z}_u = \delta_{\boldsymbol{u}_f}H$. Consider the continuous power balance associate to

$$\dot{H} = \int_\Omega \frac{\partial \boldsymbol{u}_f}{\partial t}\cdot\boldsymbol{z}_u\,\mathrm{d}\Omega = \int_\Omega \frac{\partial \boldsymbol{u}_f}{\partial t}\cdot\frac{\delta H}{\delta \boldsymbol{u}_f}\,\mathrm{d}\Omega$$

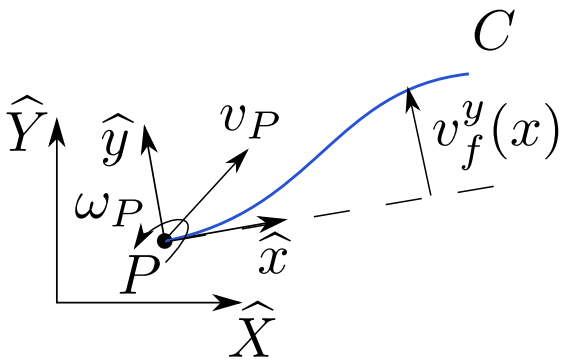

Fig. 2 Floating beam. The rigid motion is located at the initial point P

The deformation velocity and its corresponding effort variable are discretized using the same basis, i.e. $\boldsymbol{u}_f = \boldsymbol{\phi}_u^\top \boldsymbol{u}_f$, $\boldsymbol{z}_u = \boldsymbol{\phi}_u^\top \boldsymbol{z}_u$. The discrete Hamiltonian rate assumes two equivalent expressions

$$\dot{H}_d(\boldsymbol{u}_f) = \begin{cases} \dot{\boldsymbol{u}}_f^\top \boldsymbol{M}_u \, \boldsymbol{z}_u, \\ \dot{\boldsymbol{u}}_f^\top \dfrac{\partial H_d}{\partial \boldsymbol{u}_f}, \end{cases}$$

where $\boldsymbol{M}_u = \int_\Omega \boldsymbol{\phi}_u \, \boldsymbol{\phi}_u^\top \, \mathrm{d}\Omega$. To preserve the power balance at a discrete level it must hold $\boldsymbol{z}_u = \boldsymbol{M}_u^{-1} \frac{\partial H_d}{\partial \boldsymbol{u}_f}$.

Remark 4 So for a uniform Neumann boundary condition has been considered. Indeed, as an unconstrained flexible displacement also produces rigid body motion, it is necessary to constrain the flexible deformation in order to have a non singular mass matrix. If P belongs to the boundary then a natural choice is to assume zero deformation at point P. This is equivalent, in the case of a 1D structure, to a cantilever condition.

4.3 Application to thin planar beams

If a thin planar flexible beams is considered as mechanical model. P is placed at the origin of the local frame, while C is the ending point of the beam (see Fig. 2). The beam has length L, Young modulus E, density ρ, cross section A and second moment of area I. The model in strong form for a flexible beam (omitting the partition associated to the generalized coordinate and neglecting the dependence of the mass operator with respect to the deformation) is then written compactly as

$$\begin{aligned} \boldsymbol{\mathcal{M}} \dot{\boldsymbol{e}} &= \boldsymbol{\mathcal{J}}(\boldsymbol{e}) \boldsymbol{e} + \boldsymbol{\mathcal{B}}_r \boldsymbol{u}_\partial, \\ \boldsymbol{u}_\partial &= \boldsymbol{\mathcal{B}}_\partial \boldsymbol{e}, \\ \boldsymbol{y}_\partial &= \boldsymbol{\mathcal{C}}_\partial \boldsymbol{e}. \end{aligned} \tag{24}$$

Furthermore, the deformation at point P is assumed to be null (see Remark 4), i.e. $\boldsymbol{v}_f(P) = 0$. The state and boundary vectors are expressed as

$$\begin{aligned}\boldsymbol{e} &= [v_P^x,\ v_P^y,\ \omega_P^z,\ v_f^x,\ v_f^y,\ n_x,\ m_{xx}]^\top,\\ \boldsymbol{u}_\partial &= [F_P^x,\ F_P^y,\ T_P^z,\ F_C^x,\ F_C^y,\ T_C^z]^\top,\\ \boldsymbol{y}_\partial &= [v_P^x,\ v_P^y,\ \omega_P^z,\ v_C^x,\ v_C^y,\ \omega_C^z]^\top.\end{aligned}$$

The state contains the linear and angular velocity $v_P^x,\ v_P^y,\ \omega_P^z$ at point P, the deformation velocity $v_f^x,\ v_f^y$ and the traction and bending stress $n_x,\ m_{xx}$. The boundary input contains the forces and torques acting at the extremities of the beam, while the boundary output contains the velocities at the extremities of the beam. The cross product are then adapted to the planar case. The mass operator then reads

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} m & 0 & 0 & \mathcal{I}_\rho^L & 0 & 0 & 0 \\ 0 & m & s^x & 0 & \mathcal{I}_\rho^L & 0 & 0 \\ 0 & s^x & J^{zz} & 0 & \mathcal{I}_{\rho x}^L & 0 & 0 \\ (\mathcal{I}_\rho^L)^* & 0 & 0 & \rho A & 0 & 0 & 0 \\ 0 & (\mathcal{I}_\rho^L)^* & (\mathcal{I}_{\rho x}^L)^* & 0 & \rho A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & EA^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & EI^{-1} \end{bmatrix}, \tag{25}$$

where $s^x = \int_0^L \rho A x \,\mathrm{d}x$ is the static moment, $J^{zz} = \int_0^L \rho A x^2 \,\mathrm{d}x$ is the moment of inertia, $\mathcal{I}_\rho^L := \int_0^L \rho A(\cdot)\,\mathrm{d}x$, $\mathcal{I}_{\rho x}^L := \int_0^L \rho A x(\cdot)\,\mathrm{d}x$. The interconnection operator

$$\boldsymbol{\mathcal{J}}(\boldsymbol{e}) = \begin{bmatrix} 0 & 0 & +p_t^y & 0 & 0 & 0 & 0 \\ 0 & 0 & -p_t^x & 0 & 0 & 0 & 0 \\ -p_t^y & +p_t^x & 0 & -\mathcal{I}_{p_f^y}^L & +\mathcal{I}_{p_f^x}^L & 0 & 0 \\ 0 & 0 & +(\mathcal{I}_{p_f^y}^L)^* & 0 & 0 & \partial_x & 0 \\ 0 & 0 & -(\mathcal{I}_{p_f^x}^L)^* & 0 & 0 & 0 & -\partial_{xx} \\ 0 & 0 & 0 & \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{xx} & 0 & 0 \end{bmatrix} \tag{26}$$

where p_t^x, p_t^y are the canonical momenta components, $\mathcal{I}_{p_f^x}^L := \int_0^L p_f^x(\cdot)\,\mathrm{d}x$ and $\mathcal{I}_{p_f^y}^L := \int_0^L p_f^y(\cdot)\,\mathrm{d}x$. And finally the control operator

$$\boldsymbol{\mathcal{B}}_r = \begin{bmatrix} I_{3\times 3} & \boldsymbol{\tau}_{CP}^\top \\ 0_{4\times 3} & 0_{4\times 3} \end{bmatrix} \qquad \text{with} \quad \boldsymbol{\tau}_{CP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & L \\ 0 & 0 & 1 \end{bmatrix}, \tag{27}$$

The discretization procedure detailed in §4 is easily extended to this case considering that the operator to be integrated by parts here is

$$\boldsymbol{\mathcal{J}}_{\mathrm{Div}} = \begin{bmatrix} \partial_x & 0 \\ 0 & -\partial_{xx} \end{bmatrix}$$

The first line and second line are integrated by parts once and twice respectively, so that the boundary forces and momenta are naturally included in the discretized system as inputs. The finite-dimensional system then reads

$$\begin{aligned}\boldsymbol{M}\dot{\boldsymbol{e}} &= \boldsymbol{J}(\boldsymbol{e})\boldsymbol{e} + \boldsymbol{B}_\partial \boldsymbol{u}_\partial,\\ \boldsymbol{y}_\partial &= \boldsymbol{B}_\partial^\top \boldsymbol{e}.\end{aligned} \tag{28}$$

This model describes the motion of a flexible floating beam that undergoes small deformations.

5 Multibody systems in pH form

In Sections §3, §4 the pH formulation of a single flexible floating body in infinite- and finite-dimensional form were presented. The construction of a multibody system is accomplished by exploiting the modularity of the port-Hamiltonian framework. Each element of the system is interconnected to the others by means of classical pH interconnections.

5.1 Interconnections of pHDAE systems

Consider two geniric pHDAE systems of the form

$$\begin{cases} \boldsymbol{E}_i \dot{\boldsymbol{e}}_i = \boldsymbol{J}_i \boldsymbol{z}_i(\boldsymbol{e}_i) + \boldsymbol{B}_i^{\text{int}} \boldsymbol{u}_i^{\text{int}} + \boldsymbol{B}_i^{\text{ext}} \boldsymbol{u}_i^{\text{ext}} \\ \boldsymbol{y}_i = \boldsymbol{B}_i^T \boldsymbol{z}_i \end{cases} \qquad \forall i = 1, 2. \tag{29}$$

where $\frac{\partial H_i}{\partial \boldsymbol{e}_i} = \boldsymbol{E}_i^T \boldsymbol{z}_i$. The interconnection uses the internal control $\boldsymbol{u}_i^{\text{int}}$. An interconnection is said to be power preserving if and only if the following holds:

$$\langle \boldsymbol{u}_1^{\text{int}},\ \boldsymbol{y}_1^{\text{int}} \rangle + \langle \boldsymbol{u}_2^{\text{int}},\ \boldsymbol{y}_2^{\text{int}} \rangle = 0. \tag{30}$$

The power going out from one system flows in the other in a lossless manner. Two main interconnections of this kind are of interest when coupling system: the gyrator and transformer interconnection.

Gyrator interconnection The gyrator interconnection reads

$$\boldsymbol{u}_1^{\text{int}} = -\boldsymbol{C} \boldsymbol{y}_2^{\text{int}}, \qquad \boldsymbol{u}_2^{\text{int}} = \boldsymbol{C}^\top \boldsymbol{y}_1^{\text{int}}.$$

This interconnection verifies (30). It is easy to verify that from this interconnection, it is obtained

$$\begin{bmatrix} \boldsymbol{E}_1 & 0 \\ 0 & \boldsymbol{E}_2 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{e}}_1 \\ \dot{\boldsymbol{e}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1 & -\boldsymbol{B}_1^{\text{int}} \boldsymbol{C} \boldsymbol{B}_2^{\text{int}\top} \\ \boldsymbol{B}_2^{\text{int}} \boldsymbol{C} \boldsymbol{B}_1^{\text{int}\top} & \boldsymbol{J}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1^{\text{ext}} & 0 \\ 0 & \boldsymbol{B}_2^{\text{ext}} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{\text{ext}} \\ \boldsymbol{u}_2^{\text{ext}} \end{bmatrix}$$
$$\begin{bmatrix} \boldsymbol{y}_1^{\text{ext}} \\ \boldsymbol{y}_2^{\text{ext}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_1^{\text{ext}\top} & 0 \\ 0 & \boldsymbol{B}_2^{\text{ext}\top} \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{bmatrix},$$

with $H(\boldsymbol{e}_1, \boldsymbol{e}_2) = H_1(\boldsymbol{e}_1) + H_2(\boldsymbol{e}_2)$.

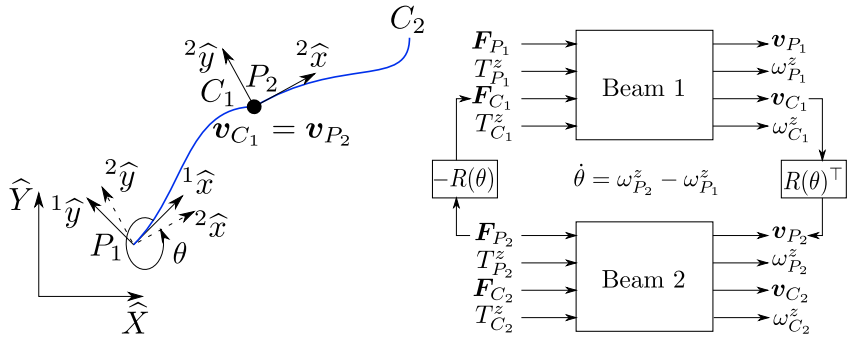

Fig. 3 Two beams interconnected by an hinge

Transformer interconnection The transformer interconnection reads

$$\boldsymbol{u}_1^{\text{int}} = -\boldsymbol{C}\boldsymbol{u}_2^{\text{int}}, \qquad \boldsymbol{y}_2^{\text{int}} = \boldsymbol{C}^\top \boldsymbol{y}_1^{\text{int}}.$$

Again, this interconnection verifies (30). After the interconnection the final system is differential algebraic:

$$\begin{bmatrix} \boldsymbol{E}_1 & 0 & 0 \\ 0 & \boldsymbol{E}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{e}}_1 \\ \dot{\boldsymbol{e}}_2 \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1 & 0 & -\boldsymbol{B}_1^{\text{int}}\boldsymbol{C} \\ 0 & \boldsymbol{J}_2 & \boldsymbol{B}_2^{\text{int}} \\ \boldsymbol{C}^\top \boldsymbol{B}_1^{\text{int}\top} & -\boldsymbol{B}_2^{\text{int}\top} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1^{\text{ext}} & 0 \\ 0 & \boldsymbol{B}_2^{\text{ext}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{\text{ext}} \\ \boldsymbol{u}_2^{\text{ext}} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{y}_1^{\text{ext}} \\ \boldsymbol{y}_2^{\text{ext}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_1^{\text{ext}\top} & 0 & 0 \\ 0 & \boldsymbol{B}_2^{\text{ext}\top} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{\lambda} \end{bmatrix}.$$

5.2 Application to multibody systems of beams

Once a discretized system is obtained by application of a transformer interconnection lossless joints can be introduced. A common example is an hinged link between two beams. In this case the internal variables are

$$\begin{aligned} \boldsymbol{u}_1^{\text{int}} &= [F^x_{C_1}, F^y_{C_1}]^\top := \boldsymbol{F}_{C_1}, & \boldsymbol{y}_1^{\text{int}} &= [v^x_{C_1}, v^y_{C_1}]^\top := \boldsymbol{v}_{C_1}, \\ \boldsymbol{u}_2^{\text{int}} &= [F^x_{P_2}, F^y_{P_2}]^\top := \boldsymbol{F}_{P_2}, & \boldsymbol{y}_2^{\text{int}} &= [v^x_{P_2}, v^y_{P_2}]^\top := \boldsymbol{v}_{P_2}. \end{aligned}$$

The interconnection matrix is the relative rotation matrix between the two local frames

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad \theta(t) = \theta(0) + \int_0^t (\omega^z_{P_2} - \omega^z_{P_1})\, \mathrm{d}\tau. \tag{31}$$

The transformer interconnection

$$\boldsymbol{F}_{C_1} = -\boldsymbol{R}(\theta)\boldsymbol{F}_{P_2}, \qquad \boldsymbol{v}_{P_2} = \boldsymbol{R}(\theta)^\top \boldsymbol{v}_{C_1}.$$

imposes the constraints on a velocity level and gives rise to a quasi-linear index 2 pHDAE:

$$\begin{bmatrix} \boldsymbol{E}_1 & 0 & 0 \\ 0 & \boldsymbol{E}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{e}}_1 \\ \dot{\boldsymbol{e}}_2 \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_1(\boldsymbol{e}_1) & 0 & -\boldsymbol{B}_1^{\text{int}}\boldsymbol{R}(\theta) \\ 0 & \boldsymbol{J}_2(\boldsymbol{e}_2) & \boldsymbol{B}_2^{\text{int}} \\ \boldsymbol{R}(\theta)^\top \boldsymbol{B}_1^{\text{int}\top} & -\boldsymbol{B}_2^{\text{int}\top} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{\partial 1}^{\text{ext}} & 0 \\ 0 & \boldsymbol{B}_{\partial 2}^{\text{ext}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{\text{ext}} \\ \boldsymbol{u}_2^{\text{ext}} \end{bmatrix}$$
$$\begin{bmatrix} \boldsymbol{y}_1^{\text{ext}} \\ \boldsymbol{y}_2^{\text{ext}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{\partial 1}^{\text{ext}\top} & 0 & 0 \\ 0 & \boldsymbol{B}_{\partial 2}^{\text{ext}\top} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{\lambda} \end{bmatrix}. \tag{32}$$

A transformer interconnection is indeed equivalent to a gyrator interconnection of pHDAE systems. It is sufficient to interchange the role of output and input of the second system $\boldsymbol{u}_2^{\text{int}} \leftrightarrow \boldsymbol{y}_2^{\text{int}}$. The output than plays the role of a Lagrange multiplier. Thus one may use SIMULINK® to interconnect pHDAE using an equivalent gyrator interconnection.

This approach allow to modularly construct multibody systems of arbitrary complexity. Other kind of lossless joints (prismatic, spherical) can be modeled by appropriate interconnections. The system can then be simulated by using specific DAE solvers [6] or appropriately reduced the system to get an ordinary differential equation. The port-Hamiltonian structure is preserved if a null space matrix is employed to eliminate the Lagrange multiplier. Consider a generic pHDAE where the differential and algebraic part are explicitly separated

$$\begin{aligned} \boldsymbol{E}\dot{\boldsymbol{e}} &= \boldsymbol{J}(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}) + \boldsymbol{G}^\top(\boldsymbol{e})\boldsymbol{\lambda} + \boldsymbol{B}\boldsymbol{u}, \\ 0 &= \boldsymbol{G}(\boldsymbol{e})\boldsymbol{z}(\boldsymbol{e}) \end{aligned} \tag{33}$$

Consider matrix $\boldsymbol{P}(\boldsymbol{e})$ that satisfies

$$\text{range}(\boldsymbol{P}(\boldsymbol{e})) = \text{null}(\boldsymbol{G}(\boldsymbol{e})),$$

then the admissible variable belongs to the range of $\boldsymbol{P}$, considering the transformations $\widehat{\boldsymbol{z}} = \boldsymbol{P}\boldsymbol{z}$, $\widehat{\boldsymbol{e}} = \boldsymbol{P}\boldsymbol{e}$ and pre-multiplying the system by $\boldsymbol{P}^\top$ an equivalent ODE is obtained

$$\widehat{\boldsymbol{E}}\dot{\widehat{\boldsymbol{e}}} = \widehat{\boldsymbol{J}}(\widehat{\boldsymbol{e}})\widehat{\boldsymbol{z}}(\widehat{\boldsymbol{e}}) + \widehat{\boldsymbol{B}}\boldsymbol{u},$$

with $\widehat{\boldsymbol{E}} = \boldsymbol{P}^\top\boldsymbol{E}\boldsymbol{P}$, $\widehat{\boldsymbol{J}} = \boldsymbol{P}^\top\boldsymbol{J}\boldsymbol{P}$, $\widehat{\boldsymbol{B}} = \boldsymbol{P}^\top\boldsymbol{B}$. The actual computation of $\boldsymbol{P}$ can be obtained by QR decomposition of matrix $\boldsymbol{G}$ [7].

5.3 The linear case

If the angular velocities and the relative orientations are small then the system may be linearized about a particular geometrical configuration. Considering that the Hamiltonian becomes a quadratic function of the state variables $H = \frac{1}{2}\boldsymbol{e}^\top\boldsymbol{E}\boldsymbol{e}$, the resulting equations are then expressed as

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{e}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{G}^\top \\ -\boldsymbol{G} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ 0 \end{bmatrix} \boldsymbol{u} \tag{34}$$

The modular construction of complex multi-body systems than is analogous to a sub-structuring technique [8]. As the mass matrix can be partition into rigid and flexible coordinate

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{rr} & \boldsymbol{M}_{rf} \\ \boldsymbol{M}_{rf}^\top & \boldsymbol{M}_{ff} \end{bmatrix}$$

References

1. B. Simeon. DAEs and PDEs in elastic multibody systems. *Numerical Algorithms*, 19(1):235–246, Dec 1998.
2. Paolo Forni, Dimitri Jeltsema, and Gabriel A.D. Lopes. Port-Hamiltonian formulation of rigid-body attitude control. *IFAC-PapersOnLine*, 48(13):164 – 169, 2015. 5th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2015.
3. Volker Mehrmann and Riccardo Morandin. Structure-preserving discretization for port-Hamiltonian descriptor systems. *arXiv preprint arXiv:1903.10451*, 2019.
4. F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A partitioned finite element method for power-preserving discretization of open systems of conservation laws. *arXiv preprint arXiv:1906.05965*, 2019.
5. D. Arnold and J. Lee. Mixed methods for elastodynamics with weak symmetry. *SIAM Journal on Numerical Analysis*, 52(6):2743–2769, 2014.
6. K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. Society for Industrial and Applied Mathematics, 1995.
7. Sigrid Leyendecker, Peter Betsch, and Paul Steinmann. The discrete null space method for the energy-consistent integration of constrained mechanical systems. part III: Flexible multibody dynamics. *Multibody System Dynamics*, 19(1):45–72, Feb 2008.
8. D. De Klerk, D. J. Rixen, and S. N. Voormeeren. General framework for dynamic substructuring: History, review and classification of techniques. *AIAA Journal*, 46(5):1169–1181, 2008.