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Port-Hamiltonian modeling and control of flexible structures.

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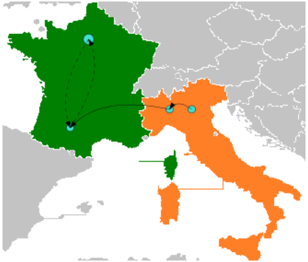
Outline

Outline

Formation:

- High school diploma in Humanities, **Vérone**;
- Bachelor in Mechanical Engineering, Politecnico di Milano, **Milan**;
- Master of science in Space Engineering, Politecnico di Milano, **Milan**;
- Double Degree Politecnico/Isae-SUPAERO, **Toulouse**;
- Research Master in Automatic Control, Supélec, **Paris/Toulouse**;



PHD title and purposes

PHD title

Modeling and control by the Port-Hamiltonian formalism of 2D flexible structures with varying boundary conditions.

Supervisors

Daniel Alazard Valerie Budinger Denis Matignon

Fundings

This work is funded by ISAE-SUPAERO.

Objectives

The knowledge of boundary conditions is mandatory for building models in Patran/Nastran (an a priori knowledge of overall system is required). The pH framework on the contrary is highly modular and allow building a complex system form its subcomponents. Moreover, we aim to examine performance specifications in the PH formalism.

Impressions and ideas?

What about the PHD reality

My personal vision of PHD after two years:

- Less remunerate (but not always) than an engineer position but greater freedom (and intellectual gratification);
- Freedom also implies working on your own for the majority of time. However collaborations, meetings, presentations, social networking are also of crucial importance;
- Motivation, autonomy, perseverance are necessary conditions. Curiosity and willingness to learn by trials and errors are the driving force;
- A worldwide experience: researchers exchange towards frontiers and continents;
- Very enriching under a personal and intellectual point of view;

A purely intellectual static activity?

Trips: conferences, meetings, schools

- Winter school in Paris, Supélec;
- Spring school and project meeting in Wuppertal;
- International mobility to Brasil, at ITA;
- Conference at Oaxaca, CPDE-CDPS;

Upcoming trips:

- Project meeting in Besançon;
- CDC conference in Nice;
- IFAC World Congress in Berlin;

Outline

What does that mean?

The port-Hamiltonian formalism (Maschke, van der Schaft, 1992) brings together:

- Hamiltonian mechanics;
- Port-based modeling approach (bond-graph)
  - Different domains (mechanical, electrical, hydraulic, thermal);
  - Energy is the lingua franca;
  - Complex systems are written as a composition of ideal components: energy-storage, energy-dissipation, energy-routing, etc;
- Passive systems and control theory.

Lagrangian function: $L(q, \dot{q}, t) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$.
Define the generalize momenta as

$$p := \frac{\partial L}{\partial \dot{q}}.$$

The Hamiltonian is obtained by application of the Legendre transform

$$H = p^T \dot{q} - L(q, \dot{q}, t).$$

This new quantity depends only on (q, p). Notice that

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \qquad \frac{\partial H}{\partial p} = \dot{q}$$

The Lagrange equation of motion for conservative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

can now be recast using the Hamiltonian

$$\frac{d}{dt}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix}$$

If the energy rate is computed

$$\dot{H} = \dot{p}^T \partial_p H + \dot{q}^T \partial_q H = 0$$

Finite dimensional pH systems

The Lagrange equation of motion for dissipative systems $D = \frac{1}{2} r ||\dot{q}||^2$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0$$

can now be recast using the Hamiltonian

$$\frac{d}{dt}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -r & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix}$$

If the energy rate is computed

$$\dot{H} = \dot{p}^T \partial_p H + \dot{q}^T \partial_q H = -r ||\partial_p(H)||^2$$

Finite dimensional pH systems

The Lagrange equation of motion for open dissipative systems $D = \frac{1}{2} r ||\dot{q}||^2$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = u$$

can now be recast using the Hamiltonian

$$\frac{d}{dt}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -r & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

If the energy rate is computed

$$\dot{H} = \dot{p}^T \partial_p H + \dot{q}^T \partial_q H = -r||\partial_p(H)||^2 + u^T y$$

The output is chosen to be the energy conjugated variable to the input $y = \partial_p H$.

General structure of an open Hamiltonian system with dissipation

$$\frac{d}{dt}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -r & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix}$$

This system are also called port-Hamiltonian systems as the communicate to the outer world through ports.

Finite-dimensional port-Hamiltonian systems

Non linear finite dimensional pH system

The Hamiltonian is a generic function of the state $H(x)$

$$
\begin{aligned}
\dot{x} &= (J - R)\partial_x H + Bu \\
y &= B^T \partial_x H
\end{aligned}
$$

Linear finite dimensional pH system

The Hamiltonian is a quadratic function of the energy variables $H = \frac{1}{2} x^T Q x$

$$
\begin{aligned}
\dot{x} &= (J - R)Qx + Bu \\
y &= B^T Q x
\end{aligned}
$$

Nomenclature:

- x energy variables;
- $e := \partial_x H$ co-energy variables;
- $J = -J^T$ interconnection matrix (skew symmetric);
- $R = R^T$, $R \succeq 0$ dissipation matrix (symmetric positive semi-definite);

Interconnection of pH systems

If the interconnection is power-preserving then the resulting system is again pH.

Individual systems

System 1:

$$\begin{aligned} \dot{x}_1 &= J_1 \; \partial_{x_1} H_1 + B_1 u_1 \\ y_1 &= B_1^T \; \partial_{x_1} H_1 \end{aligned}$$

System 2:

$$\begin{aligned} \dot{x}_2 &= J_2 \; \partial_{x_2} H_2 + B_2 u_2 \\ y_2 &= B_2^T \; \partial_{x_2} H_2 \end{aligned}$$

Interconnection of pH systems

If the interconnection is power-preserving then the resulting system is again pH.

Individual systems

System 1:

$$\dot{x}_1 = J_1 \ \partial_{x_1} H_1 + B_1 u_1$$
$$y_1 = B_1^T \ \partial_{x_1} H_1$$

System 2:

$$\dot{x}_2 = J_2 \ \partial_{x_2} H_2 + B_2 u_2$$
$$y_2 = B_2^T \ \partial_{x_2} H_2$$

Gyrator interconnection

$$u_1 = y_2 + u_e$$
$$u_2 = -y_1$$

Coupled system

$$H(x_1, x_2) = H_1(x_1) + H_2(x_2)$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} J_1 & B_1 B_2^T \\ -B_2 B_1^T & J_2 \end{bmatrix} \begin{bmatrix} \partial_{x_1} H \\ \partial_{x_2} H \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_e$$

$$y_e = \begin{bmatrix} B_1^T & 0 \end{bmatrix} \begin{bmatrix} \partial_{x_1} H \\ \partial_{x_2} H \end{bmatrix}$$

Interconnection of pH systems

If the interconnection is power-preserving then the resulting system is again pH.

Individual systems

System 1:

$$\dot{x}_1 = J_1 \; \partial_{x_1} H_1 + B_1 u_1$$
$$y_1 = B_1^T \; \partial_{x_1} H_1$$

System 2:

$$\dot{x}_2 = J_2 \; \partial_{x_2} H_2 + B_2 u_2$$
$$y_2 = B_2^T \; \partial_{x_2} H_2$$

Transformer interconnection

$$u_1 = -u_2 + u_e$$
$$y_2 = y_1$$

Coupled system

$$H(x_1, x_2) = H_1(x_1) + H_2(x_2)$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} J_1 & 0 & -B_1 \\ 0 & J_2 & B_2 \\ B_1^T & -B_2^T & 0 \end{bmatrix} \begin{bmatrix} \partial_{x_1} H \\ \partial_{x_2} H \\ \lambda \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} u_e$$

$$y_e = \begin{bmatrix} B_1^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_{x_1} H \\ \partial_{x_2} H \\ \lambda \end{bmatrix}$$

This is a differential-algebraic system (pHDAE)

Outline

Infinite dimensional systems

Even PDE models can be put in Hamiltonian form[1]. Consider the Euler-Bernoulli beam model

$$\rho A(z)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial z^2}\left(EI(z)\frac{\partial^2 w}{\partial z^2}\right) = 0 \quad z \in [0, L] \quad + \text{Boundary conditions}$$

Total energy of the system

$$H = \frac{1}{2}\int_0^L \rho A(z)\left(\frac{\partial w}{\partial t}\right)^2 + EI(z)\left(\frac{\partial^2 w}{\partial z^2}\right)^2 \, \mathrm{d}z$$

Select as energy variables $x_1 = \rho A(x)\frac{\partial w}{\partial t}$, $x_2 = \frac{\partial^2 w}{\partial z^2}$.

$$H = \frac{1}{2}\int_0^L \frac{1}{\rho A(z)} x_1^2 + EI(x) x_2^2 \, \mathrm{d}z$$

[1] P. J. Olver. *Applications of Lie groups to differential equations*. 2nd ed. Vol. 107. Graduate texts in mathematics. Springer-Verlag New York, 1993. ISBN: 978-0-387-95000-6.

The energy is no more a function but a functional. The co-energy variables are given by the variational derivative with respect to the state

$$e_1 := \frac{\delta H}{\delta x_1} = \frac{\partial w}{\partial t}, \qquad e_2 := \frac{\delta H}{\delta x_2} = EI(z)\frac{\partial^2 w}{\partial x^2}$$

The Euler Bernoulli beam model is then rewritten as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial z^2} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

The operator $\mathcal{J}$ is formally skew-adjoint (infinite dimensional extension of the skew-symmetric property of a matrix). That means that for homogeneous boundary conditions

$$< x, \mathcal{J} y >_{\mathcal{H}} \underbrace{=}_{\text{i.b.p.}} < -\mathcal{J} x, y >_{\mathcal{H}}$$

where $\mathcal{H}$ is an Hilbert space.

Energy rate

From the energy balance the boundary variables are readily found

$$\dot{H} = \int_0^L \frac{\partial x}{\partial t}^T \frac{\delta H}{\delta x} \,\mathrm{d}x = \int_0^L e^T \mathcal{J} e \,\mathrm{d}x \underbrace{=}_{\text{i.b.p.}} \left(e_2 \frac{\partial e_1}{\partial x} - e_1 \frac{\partial e_2}{\partial x} \right) \Big|_0^L.$$

The energy rate equals a scalar product of the standard boundary condition:

- $e_1|_{\partial\Omega} = \partial_t w|_{\partial\Omega}$: vertical velocities;
- $\partial_x e_1|_{\partial\Omega} = \partial_{xt} w|_{\partial\Omega}$: vertical angular velocities;
- $e_2|_{\partial\Omega} = EI\partial_{xx} w|_{\partial\Omega}$: bending momenta;
- $\partial_x e_2|_{\partial\Omega} = \partial_x(EI\partial_{xx} w)|_{\partial\Omega}$: shear forces;

Thanks to this structure important properties of pH boundary control system can be easily checked[2].

[2] Y. Le Gorrec, H. Zwart, and B. Maschke. "Dirac structures and Boundary Control Systems associated with Skew-Symmetric Differential Operators". In: *SIAM Journal on Control and Optimization* 44.5 (2005), pp. 1864–1892. URL: https://doi.org/10.1137/040611677.

General infinite dimensional pH system

$$\begin{cases} \dfrac{\partial x}{\partial t}(z,t) &= \mathcal{J}\frac{\delta H}{\delta x} + B u(z,t), \\ y(z,t) &= B^* \frac{\delta H}{\delta x}. \end{cases}$$

With boundary conditions

$$u_\partial = \mathcal{B}\frac{\delta H}{\delta x}, \quad y_\partial = \mathcal{C}\frac{\delta H}{\delta x}$$

Energy rate: $\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t) y(z,t)\, \mathrm{d}\Omega$

- x energy variables, $e = \delta_x H =:$ co-energy variables;
- $\mathcal{J}$: skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$: boundary operator;
- u, y, B: distributed input, output and control operator;

Linear infinite dimensional pH system

$$\begin{cases} \dfrac{\partial x}{\partial t}(z,t) & = \mathcal{J}\mathcal{Q}x + Bu(z,t), \\ y(z,t) & = B^* \frac{\delta H}{\delta x}. \end{cases}$$

With boundary conditions

$$u_\partial = \mathcal{B}\frac{\delta H}{\delta x}, \quad y_\partial = \mathcal{C}\frac{\delta H}{\delta x}$$

Energy rate: $\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t) y(z,t) \, \mathrm{d}\Omega$

- x energy variables, $e = \delta_x H = \mathcal{Q}x$: co-energy variables;
- $\mathcal{J}$: skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$: boundary operator;
- u, y, B: distributed input, output and control operator;

Linear elasticity in PH form

The general PDE for the elastodynamics (linear elasticity) reads

$$\rho \frac{\partial^2 u}{\partial t^2} - \mathrm{Div}\left(\mathbb{D}\, \mathrm{Grad}(u)\right) = f.$$

- ρ mass density, $\mathbb{D}$ stiffness tensor;
- u displacement vector;
- Div divergence of a tensor, $\mathrm{Grad} = \frac{1}{2}\left[\nabla + \nabla^T\right]$ symmetric gradient of a vector;
- f Body force;

Total energy: $H = \dfrac{1}{2}\displaystyle\int_\Omega \left\{\rho \left(\frac{\partial u}{\partial t}\right)^2 + \Sigma : \epsilon\right\} \mathrm{d}\Omega,$

- $\epsilon = \mathrm{Grad}(u)$, infinitesimal strain tensor,
- $\Sigma = \mathbb{D}\, \varepsilon$, Cauchy stress tensor.

Linear elasticity in PH form

The general PDE for the elastodynamics (linear elasticity) reads

$$\rho \frac{\partial^2 u}{\partial t^2} - \mathrm{Div}\left(\mathbb{D}\, \mathrm{Grad}(u)\right) = f.$$

To get a pH representation the energy variable have to selected:

$$\begin{aligned} x_1 &= \rho \frac{\partial u}{\partial t}, \quad \text{Linear Momentum}, \\ x_2 &= \mathrm{Grad}(u) = \varepsilon, \quad \text{Strain tensor}. \end{aligned}$$

The corresponding co-energy are then retrieved

$$\begin{aligned} e_1 &= \frac{\delta H}{\delta x_1} = \frac{\partial u}{\partial t}, \quad \text{Linear velocity}, \\ e_2 &= \frac{\delta H}{\delta x_2} = \Sigma, \quad \text{Stress tensor}. \end{aligned}$$

The general PDE for the elastodynamics (linear elasticity) reads

$$\rho \frac{\partial^2 u}{\partial t^2} - \mathrm{Div}\left(\mathbb{D}\, \mathrm{Grad}(u)\right) = f\,.$$

The pH system representing the elastodynamics equation becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix}\begin{bmatrix} \rho^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f$$

Together with appropriate boundary condition.

The pH system representing the elastodynamics equation becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix} \begin{bmatrix} \rho^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f$$

Together with appropriate boundary condition.

Neumann control

$$\begin{aligned} {\color{blue}u_\partial} &= \Sigma \cdot n \quad \text{on } \partial\Omega, \\ {\color{red}y_\partial} &= \frac{\partial u}{\partial t} \quad \text{on } \partial\Omega \end{aligned}$$

The pH system representing the elastodynamics equation becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix} \begin{bmatrix} \rho^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f$$

Together with appropriate boundary condition.

Dirichlet control

$$\begin{aligned} u_\partial &= \frac{\partial u}{\partial t} \quad \text{on } \partial\Omega \\ y_\partial &= \Sigma \cdot n \quad \text{on } \partial\Omega \end{aligned}$$

The pH system representing the elastodynamics equation becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix} \begin{bmatrix} \rho^{-1} & 0 \\ 0 & \mathbb{D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f$$

Together with appropriate boundary condition.

Mixed control ($\partial\Omega = \Gamma_D \cup \Gamma_N$)

$$\begin{aligned} u_\partial &= \partial_t u \quad \text{on } \Gamma_D, & u_\partial &= \Sigma \cdot n \quad \text{on } \Gamma_N \\ y_\partial &= \Sigma \cdot n \quad \text{on } \Gamma_D & y_\partial &= \partial_t u \quad \text{on } \Gamma_N \end{aligned}$$

Some mechanical models

Euler Bernoulli beam

$$\mathcal{J}_{\mathrm{EB}} = \begin{bmatrix} 0 & -\frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial z^2} & 0 \end{bmatrix}$$

Kirchhoff plate

$$\mathcal{J}_{\mathrm{K}} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \operatorname{Grad}\operatorname{grad} & 0 \end{bmatrix}$$

Timoshenko beam

$$\mathcal{J}_{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 1 \\ 0 & \frac{\partial}{\partial z} & 0 & 0 \\ \frac{\partial}{\partial z} & -1 & 0 & 0 \end{bmatrix}$$

Mindlin-Reissner plate

$$\mathcal{J}_{\mathrm{M}} = \begin{bmatrix} 0 & 0 & 0 & \operatorname{div} \\ 0 & 0 & \operatorname{Div} & \mathbb{1} \\ 0 & \operatorname{Grad} & 0 & 0 \\ \operatorname{grad} & -\mathbb{1} & 0 & 0 \end{bmatrix}$$

This model are obtained by imposing some assumptions on the displacement field from general 3D elasticity.

Outline

How to discretize pH systems?

Infinite dimensional pHs

PDE:

$$\dot{x}(z,t) = \mathcal{J}\delta_x H + B u(z,t),$$
$$y(z,t) = B^* \delta_x H.$$

Boundary conditions:

$$u_\partial = \mathcal{B}\ \delta_x H, \quad y_\partial = \mathcal{C}\ \delta_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t) y(z,t)\ \mathrm{d}\Omega$$

Finite dimensional pHs

ODE:

$$\dot{x} = J\partial_x H + B_d u_d + B_\partial u_\partial,$$
$$y_d = B_d^T \partial_x H,$$
$$y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

Available methods

- Spectral methods (Moulla 2012);
- Finite differences (Trenchant 2018);
- Finite elements based (Golo 2004, Kotyczka 2018, Cardoso-Riberio 2019);

FEM for pHS

A partitioned Finite Element method (PFEM) is our method of choice for its versatility:

- Any geometrical dimension can be treated;
- Existent finite element libraries may be used;
- Easy to be implemented;

General idea of PFEM

General form of a pH system

$$\frac{\partial x}{\partial t} = \mathcal{J} e, \qquad e = \frac{\delta H}{\delta x}$$

In general it holds $\mathcal{J} = \sum_i \mathcal{J}_i$ (sum of skew-symmetric differential and algebraic operators).

General procedure for PFEM

1. Put the system into weak form:

$$\left(v, \frac{\partial x}{\partial t}\right)_\Omega = \left(v, \sum_i \mathcal{J}_i e\right)_\Omega .$$

2. Apply integration by part on each differential $\mathcal{J}_i$:

$$\left(v, \sum_i \mathcal{J}_i e\right)_\Omega \overbrace{=}^{i.b.p.} j(v, e)_\Omega + b(v, u_\partial)_{\partial\Omega},$$

so that $j(v, e)_\Omega$ is a skew-symmetric bilinear form.

3. Discretization using a Galerkin method (same basis function for test, energy and co-energy variables)

Finite dimensional system

Once all the steps are carried out, the following system is obtained

$$M\dot{x}_d = Je_d + B_d u_d.$$

The Hamiltonian still need to be discretized

$$\dot{H}(x) = \int_\Omega \dot{x}^T \delta_x H \, \mathrm{d}\Omega = \int_\Omega \dot{x}^T e \, \mathrm{d}\Omega$$

If the approximated variable are introduced:

$$\dot{H}_d(x_d) = \dot{x}_d^T \partial_{x_d} H_d = \dot{x}_d^T M e_d$$

Leading to: $e_d = M^{-1}\partial_{x_d} H_d$. Now taking $J_d = M^{-1} J M^{-1}$ a standard pH system is obtained

$$\begin{aligned} \dot{x}_d &= J_d \partial_{x_d} H_d + B_d u_d, \\ y_d &= B_d \partial_{x_d} H_d \end{aligned}$$

Finite dimensional system (Linear case)

If the system is linear the relation between energy and co-energy is trivial $e = \mathcal{Q}\alpha$. Assuming Q coercive and symmetric (always verified for standard models), then the weak form may be rewritten as

$$\left(v, \mathcal{Q}^{-1}\frac{\partial e}{\partial t}\right)_{\Omega} = (v, \mathcal{J}e)_{\Omega}.$$

Applying the PFEM procedure it is obtained

$$\begin{aligned} M_d \dot{e}_d &= J_d e_d + B_d u_d, \\ y_d &= B_d e_d. \end{aligned}$$

This representation is also possible. When the problem is large inverting the mass matrix is unfeasible. The co-energy variables are even more meaningful than the energy ones.

The boundary term

Different boundary terms arise from the integration by parts.
Consider the Euler Bernoulli beam in weak form:

$$\int_{\Omega} v^T \dot{x} \, \mathrm{d}\Omega = \int_0^L -v_1 \frac{\partial^2 e_2}{\partial z^2} + v_2 \frac{\partial^2 e_1}{\partial z^2} \, \mathrm{d}x$$

The boundary term

Different boundary terms arise from the integration by parts.
Consider the Euler Bernoulli beam in weak form:

$$\int_\Omega v^T \dot{x} \, \mathrm{d}\Omega = \int_0^L -v_1 \frac{\partial^2 e_2}{\partial z^2} + v_2 \frac{\partial^2 e_1}{\partial z^2} \, \mathrm{d}x$$

The boundary term

Different boundary terms arise from the integration by parts.
Consider the Euler Bernoulli beam in weak form:

$$
\begin{aligned}
\int_\Omega v^T \dot{x} \, \mathrm{d}\Omega &= \int_0^L -v_1 \frac{\partial^2 e_2}{\partial z^2} + v_2 \frac{\partial^2 e_1}{\partial z^2} \, \mathrm{d}x \\
&= \int_0^L -\frac{\partial^2 v_1}{\partial z^2} e_2 + v_2 \frac{\partial^2 e_1}{\partial z^2} \, \mathrm{d}x - v_1 \frac{\partial e_2}{\partial z}\Big|_0^L + \frac{\partial v_1}{\partial z} e_2 \Big|_0^L
\end{aligned}
$$

Control given by forces $\frac{\partial e_2}{\partial z}|_{\partial\Omega} = F|_{\partial\Omega}$ and momenta $e_2|_{\partial\Omega} = M|_{\partial\Omega}$.
The uncontrolled system corresponds to the Free-Free case

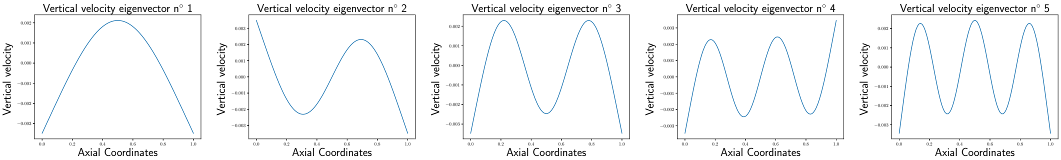


Figure: Eigenvectors for the uncontrolled system

The boundary term

Different boundary terms arise from the integration by parts.
Consider the Euler Bernoulli beam in weak form:

$$\int_{\Omega} v^T \dot{x} \, \mathrm{d}\Omega = \int_0^L -v_1 \frac{\partial^2 e_2}{\partial z^2} + v_2 \frac{\partial^2 e_1}{\partial z^2} \, \mathrm{d}x$$

The boundary term

Different boundary terms arise from the integration by parts.
Consider the Euler Bernoulli beam in weak form:

$$
\begin{aligned}
\int_\Omega v^T \dot{x} \, \mathrm{d}\Omega &= \int_0^L -v_1 \frac{\partial^2 e_2}{\partial z^2} + v_2 \frac{\partial^2 e_1}{\partial z^2} \, \mathrm{d}x \\
&= \int_0^L -v_1 \frac{\partial^2 e_2}{\partial z^2} + \frac{\partial^2 v_2}{\partial z^2} e_1 \, \mathrm{d}x + v_2 \frac{\partial e_1}{\partial z}\Big|_0^L + \frac{\partial v_2}{\partial z} e_1 \Big|_0^L
\end{aligned}
$$

Control given by angular velocities $\frac{\partial e_1}{\partial z}|_{\partial\Omega} = \partial_{xt} w|_{\partial\Omega}$ and linear velocities $e_1|_{\partial\Omega} = \partial_t w|_{\partial\Omega}$.
The uncontrolled system corresponds to the Clamped-Clamped case.

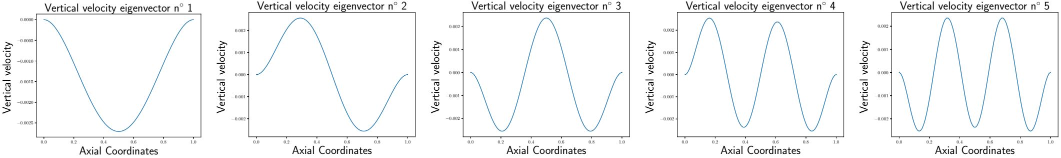


Figure: Eigenvectors for the uncontrolled system

Mixed boundary control

If a system is controlled with two different kind of boundary input, the PFEM has to be adjusted.

Two possible methodology can be adopted:

- Domain decomposition and interconnection of two models with different inputs (not discussed here);
- Lagrange multiplier;

The control input u arising from the integration by parts is not known everywhere. Lagrange multiplier must be introduced.

Port Hamiltonian descriptor system

Generalization of the standard formulation. More general formulations exist.

$$\begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{bmatrix} \partial_x H \\ \lambda \end{bmatrix} + \begin{bmatrix} B_x & 0 \\ 0 & B_\lambda \end{bmatrix} \begin{bmatrix} u_x \\ u_\lambda \end{bmatrix},$$

$$\begin{bmatrix} y_x \\ y_\lambda \end{bmatrix} = \begin{bmatrix} B_x^T & 0 \\ 0 & B_\lambda^T \end{bmatrix} \begin{bmatrix} \partial_x H \\ \lambda \end{bmatrix}$$

This formulation allow to consider mixed homogeneous and inhomogeneous boundary conditions and generic boundary control law.

Given a descriptor model the eigenvectors for arbitrary boundary condition may be computed.

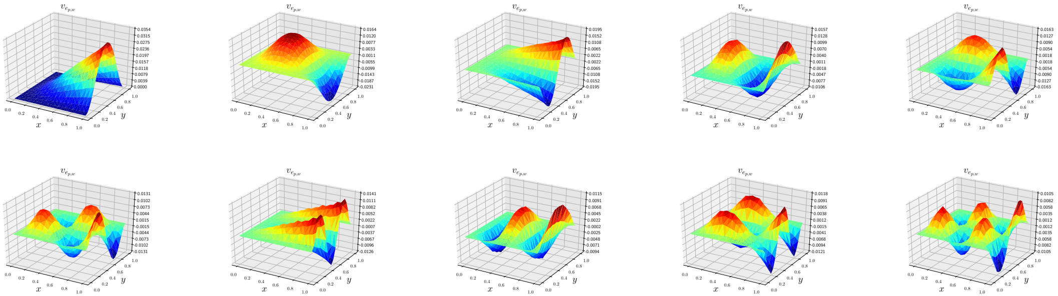


Figure: Eigenvectors for the CCCF condition of the Mindlin plate by solving $Ev = j\omega Jv$

Eliminating the Lagrange multipliers

In structural dynamics applications components are clamped on part of the boundary and free elsewhere (solar panels, robotic arms). However, the control is normally given either by distributed actuation, either by boundary forces and torques.

Consider a linear pHDAE system in co-energy variables

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} e \\ \lambda \end{bmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{bmatrix} e \\ \lambda \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{bmatrix} e \\ \lambda \end{bmatrix}$$

Multiplying by the left annihilator of the constraint $G^\perp G = 0$ and applying the variable change $(G^\perp)^T \widehat{e} = e$, the constraints disappear

$$\widehat{M} \, \dot{\widehat{e}} = \widehat{J} \widehat{e} + \widehat{B} u$$

Main research points

The numerical discretization of pHs is a very recent field.

- Ongoing work on the optimal convergence rate for the wave equation;
- Electrodynamics and structural mechanics require further investigation[3] [4];

[3] D. Arnold and J. Lee. "Mixed Methods for Elastodynamics with Weak Symmetry". In: *SIAM Journal on Numerical Analysis* 52.6 (2014), pp. 2743–2769. DOI: 10.1137/13095032X.

[4] Eliane Becache, Patrick Joly, and Chrysoula Tsogka. "A New Family of Mixed Finite Elements for the Linear Elastodynamic Problem". In: *SIAM Journal on Numerical Analysis* 39 (June 2001), pp. 2109–2132. DOI: 10.1137/S0036142999359189.

Outline

Damping assignment and control by interconnection[5]

Energy shaping

The interconnection with a suitable controller allows to shape the energy, so that its minimum is located at a desired configuration $u = u_{\mathrm{ES}} + v$:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \partial_p H \\ \partial_q H \end{bmatrix} + \begin{bmatrix} G(q) \\ 0 \end{bmatrix} u \quad \longrightarrow \quad \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -J_1^T(q) \\ J_1(q) & 0 \end{bmatrix} \begin{bmatrix} \partial_p H_d \\ \partial_q H_d \end{bmatrix} + \begin{bmatrix} G(q) \\ 0 \end{bmatrix} v$$

Damping Injection

PH system are collocated: input and output are energy conjugated. The simple control law

$$v = -K_{\mathrm{DI}} G^T(q) \partial_p H_d(q, p)$$

always inject damping into the system.

[5] Romeo Ortega and Eloísa García-Canseco. "Interconnection and Damping Assignment Passivity-Based Control: A Survey". In: *European Journal of Control* 10.5 (2004), pp. 432 –450.

Time domain methodologies

For non-linear pH system the Proper Orthogonal Decomposition is available. Given the optimization problem:

$$P_* = \underset{\mathrm{rank}(P)=r}{\arg\min} \int_0^\infty ||(I-P)x(t)||^2 \, \mathrm{d}t, \qquad Q_* = \underset{\mathrm{rank}(P)=r}{\arg\min} \int_0^\infty ||(I-P)\partial_x H(x(t))||^2 \, \mathrm{d}t.$$

The reduction system is given by the Petrov-Galerkin projection using as subspaces $\mathcal{V}_r = \mathrm{Ran}(P_*)$, $\mathcal{W}_r = \mathrm{Ran}(Q_*)$.

Frequency domain techniques

One may use $\mathcal{H}_2/\mathcal{H}_\infty$ optimal subspaces. They try to match the full and reduced transfer functions

$$G(s) = B^T(sQ^{-1} - (J-R))^{-1}B \qquad G_r(s) = B_r^T(sQ_r^{-1} - (J_r - R_r))^{-1}B_r$$

Krilov subspaces method (moment matching) can also be used and work fine for pHDAE.

Model Reduction

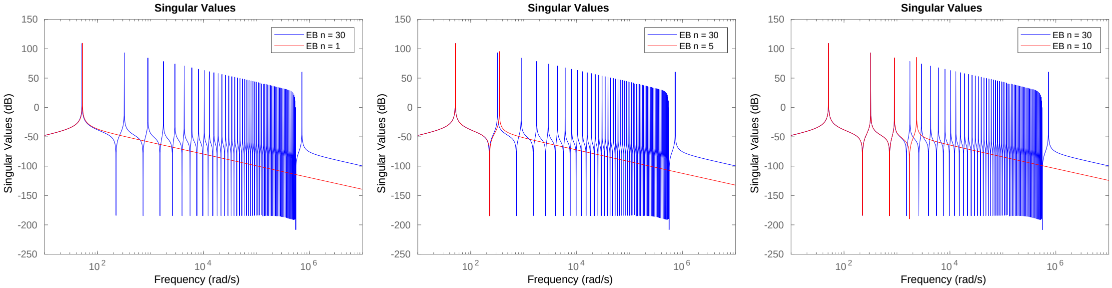


Figure: Krilov method[6] applied to the Euler Bernoulli beam

[6] H. Egger et al. "On Structure-Preserving Model Reduction for Damped Wave Propagation in Transport Networks". In: *SIAM Journal on Scientific Computing* 40.1 (2018), A331–A365. DOI: 10.1137/17M1125303.

Outline

Boundary stabilization of the Kirchhoff plate

Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} w_t \\ \Sigma \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \operatorname{Grad}\operatorname{grad} & 0 \end{bmatrix} \begin{bmatrix} w_t \\ \Sigma \end{bmatrix} \qquad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

Boundary conditions

$$\begin{aligned} w_t|\Gamma_D &= 0, \\ \partial_x w_t|\Gamma_D &= 0, \end{aligned} \qquad \Gamma_D = \{x = 0\}$$

$$\begin{aligned} \Sigma : (n \otimes n)|\Gamma_N &= u_M, \\ \mathcal{D}\Sigma|\Gamma_N := \widetilde{q}|\Gamma_N &= u_F, \end{aligned} \qquad \Gamma_N = \{x = 0,\ x = 1,\ y = 1\}$$

with initial conditions (compatible with the constraints):

$$w_t(x, y, 0) = x^2; \qquad \Sigma(x, y, 0) = 0.$$

Obtain a finite-dimensional uncontrolled system

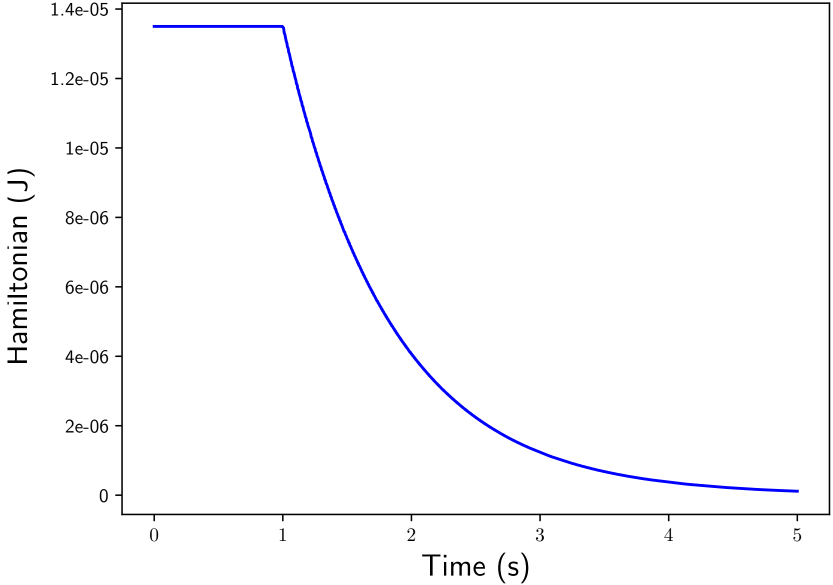
$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

Apply the control law $u = -Ky, \; K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix},$$

with $R = BKB^T \succeq 0$.
The Hamiltonian $\dot{H} = -e^T Re \leq 0$ is a non increasing function and by La Salle principle the equilibrium point $e = 0$ is asymptotically stable.

Damping Kirchhoff Plate


1.4e-05
1.2e-05
1e-05
8e-06
6e-06
4e-06
2e-06
0
Hamiltonian (J)
0
1
2
3
4
5
Time (s)

The system is composed by a cantilever plate connected to a rigid rod. The interconnection is given by a compact operator.

$$\text{dpH} \begin{cases} \dfrac{\partial x_1}{\partial t} = \mathcal{J} \dfrac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \dfrac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \dfrac{\delta H_1}{\delta x_1} \end{cases} \qquad \text{pH} \begin{cases} \dfrac{dx_2}{dt} = J \dfrac{\partial H_2}{\partial x_2} + Bu_2 \\ y_2 = B^T \dfrac{\partial H_2}{\partial x_2} + Du_2 \end{cases},$$

where $x_1 \in \mathscr{X}$, $u_{\partial,1} \in \mathscr{U}$, $y_{\partial,1} \in \mathscr{Y} = \mathscr{U}'$ belong to some Hilbert spaces (the prime denotes the topological dual of a space) and $x_2 \in \mathbb{R}^n, u, y \in \mathbb{R}^m$. The duality pairings for the boundary ports are denoted by

$$\langle u_{\partial,1},\ y_{\partial,1} \rangle_{\mathscr{U} \times \mathscr{Y}}, \qquad \langle u_2,\ y_2 \rangle_{\mathbb{R}^m}.$$

For the interconnection, consider the compact operator $\mathcal{W} : \mathscr{Y} \to \mathbb{R}^m$ and the following power preserving interconnection

$$u_2 = -\mathcal{W}\, y_{\partial,1}, \qquad u_{\partial,1} = \mathcal{W}^*\, y_2,$$

Kirchhoff plate

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} w_t \\ \Sigma \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \operatorname{Grad}\operatorname{grad} & 0 \end{bmatrix} \begin{bmatrix} w_t \\ \Sigma \end{bmatrix}$$

$$\begin{aligned} u_{\partial,\text{pl}} &= w_t(x = L_x, y), \\ y_{\partial,\text{pl}} &= \mathcal{D}\Sigma = \widetilde{q}_n(x = L_x, y). \end{aligned}$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = u_{\text{rod}},$$

$$y_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space $\mathscr{Y}$ is the space of square-integrable functions on with support $\Gamma_{\text{int}} = \{(x, y) |\ x = L_x, 0 \leq y \leq L_y\}$. The compact interconnection operator then reads

$$\mathcal{W} y_{\partial,\text{pl}} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial,\text{pl}} \, \mathrm{d}s \\ \int_{\Gamma_{\text{int}}} (y - L_y/2) \, y_{\partial,\text{pl}} \, \mathrm{d}s \end{pmatrix}.$$

The adjoint operator is then obtained considering that $u_{\text{rod}} = \mathcal{W} y_{\partial,\text{pl}}$ and that the inner product of $\mathbb{R}^m$ is easily converted to an inner product on the space $L^2(\Gamma_{\text{int}})$

$$\langle \mathcal{W} y_{\partial,\text{pl}}, \ y_{\text{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial,\text{pl}}, \ \mathcal{W}^* y_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})},$$

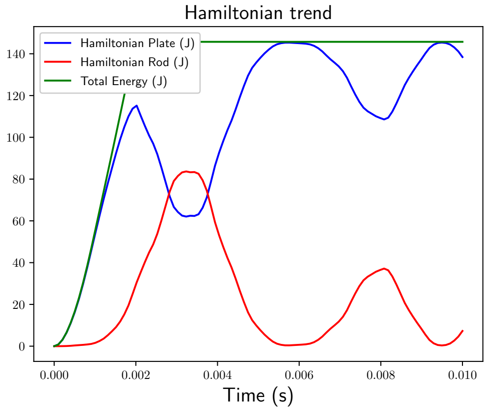
$$\mathcal{W}^* y_{\text{rod}} = v_G + \omega_G \, (y - L_y/2).$$

Results

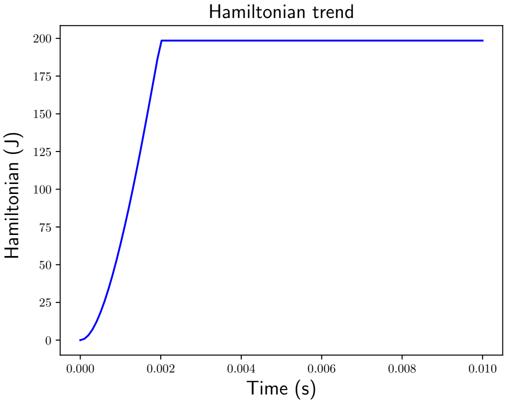
Plate and rod

Only plate

Results



Hamiltonian trend
Hamiltonian Plate (J)
Hamiltonian Rod (J)
Total Energy (J)
Time (s)



Hamiltonian trend
Hamiltonian (J)
Time (s)

Outline

Consider the non linear system PDE

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ \rho \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{bmatrix} \rho g \left(h + \frac{1}{2}||\boldsymbol{v}||^2\right) \\ h\boldsymbol{v} \end{bmatrix} \qquad (x, y) \in \Omega = \left\{x^2 + y^2 \leq 1\right\}$$

With boundary condition

$$h\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = u_\partial$$

This model represents the oscillations of waves in a reservoir with h the fluid height and $\boldsymbol{v}$ the velocity field, with mass flux controlled on the boundary.

The Hamiltonian

$$H = \frac{1}{2}\int_\Omega \rho g h^2 + \rho h ||\boldsymbol{v}||^2 \ \mathrm{d}\Omega$$

is non separable $H \neq E(\boldsymbol{v}) + V(h)$.
Selecting as energy variables $x_1 = h$, $\boldsymbol{x}_2 = \rho \boldsymbol{v}$, we get the pH system

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix}\begin{bmatrix} \delta_{x_1} H \\ \delta_{x_2} H \end{bmatrix}$$

With boundary condition

$$\delta_{x_1} H \cdot n|_{\partial\Omega} := \rho h \boldsymbol{v} \cdot n|_{\partial\Omega} = u_\partial$$

Considering as conjugated output the pressure $y_\partial = p = \rho g \left(h + \frac{1}{2}||\boldsymbol{v}||^2\right)$ and using the control law

$$u_\partial = -k(y_\partial - y_\partial^0)$$

with $y_\partial^0 = \rho g h_{\text{des}}$ the desired pressure at equilibrium.
It can be shown that the Lyapunov function

$$V = \frac{1}{2}\int_\Omega \rho g (h - h_{\text{des}})^2 + \rho h ||\boldsymbol{v}||^2 \, \mathrm{d}\Omega,$$

is non increasing

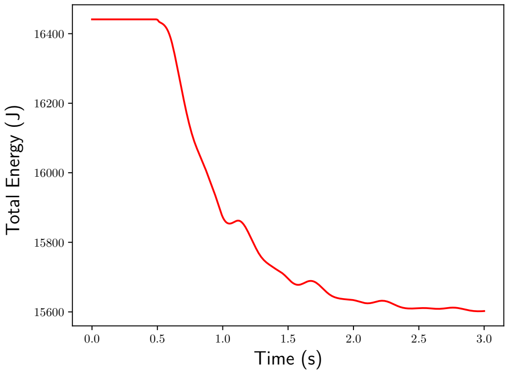
$$\dot{V} = -k \int_{\partial\Omega} (y_\partial - y_\partial^0)^2 \, \mathrm{d}\Gamma \leq 0,$$

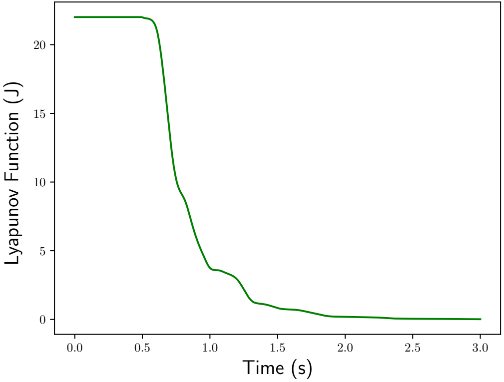
and the equilibrium point $(h, \boldsymbol{v}) = (h_{\text{des}}, 0)$ is asymptotically stable.

Initial condition

$$h(r, \theta, 0) = h_{\text{des}}(1 + \cos(\pi r)\cos(2\theta)/10), \quad \rho \boldsymbol{v} = 0$$

Saint Venant in circular domain


Total Energy (J)
16400
16200
16000
15800
15600
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Time (s)

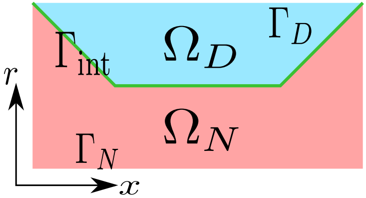

Lyapunov Function (J)
20
15
10
5
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Time (s)

Outline

Propagation of acoustic waves in a cylindrical axis-symmetrical channel

$$\frac{\partial}{\partial t}\begin{bmatrix}\chi_s p \\ \mu_0 \boldsymbol{v}\end{bmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix}\begin{bmatrix} p \\ \boldsymbol{v}\end{bmatrix}, \qquad \Omega = \{x \in [0, L],\ r \in [0, R],\ \theta \in [0, 2\pi)\}$$

with a constant axial flow and an impedance conditions on the lateral surface.
Two discretization methods are compared: DAE with Lagrange multiplier and ODE with domain decomposition.



Results

Vibroacoustic DAE

Vibroacoustic ODE

Future developments

- Flexible multibody dynamics;
- Thermoelasticity?

Thank you

Questions

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