A port-Hamiltonian formulation for the full Von-Karman plate model

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Summary. In this contribution, a port-Hamiltonian reformulation of the full von-Karman dynamical model for geometrically non-linear plates is detailed. Starting from the canonical equations, a set of variables is chosen so that that make the total energy quadratic. The model, reformulated in these variables, highlights a port-Hamiltonian structure ruled by a state-modulated interconnection operator.

Classical model

The classical full von-Karman dynamical model is detailed in Bilbao et al. [2015]. The problem, defined an open connected set $\Omega \subset \mathbb{R}^2$, takes the dimensionless form

$$\begin{aligned} \ddot{\boldsymbol{u}} &= \operatorname{Div} \boldsymbol{N}, & \boldsymbol{N} &= \boldsymbol{\Phi}(\boldsymbol{\varepsilon}), & \boldsymbol{\varepsilon} &= \operatorname{Grad} \boldsymbol{u} + 1/2 \operatorname{grad} w \otimes \operatorname{grad} w, \\ \ddot{w} &= -\operatorname{div} \operatorname{Div} \boldsymbol{M} + \operatorname{div}(\boldsymbol{N} \operatorname{grad} w), & \boldsymbol{M} &= \boldsymbol{\Phi}(\boldsymbol{\kappa}), & \boldsymbol{\kappa} &= \operatorname{Grad} \operatorname{grad} w, \end{aligned} \tag{1}$$

where $\boldsymbol{u} \in \mathbb{R}^2$ is the in-plane displacement, w is the vertical displacement, $\boldsymbol{\varepsilon}$ is the in-plane strain tensor, $\boldsymbol{\kappa}$ is the curvature tensor, $\boldsymbol{N}$ is the in-plane stress resultant and $\boldsymbol{M}$ is the bending stress resultant. The notation $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top$ denotes the outer product of two tensors. The operator div is the divergence of a vector field and grad the gradient of a scalar field. The operator $\operatorname{Grad} = \frac{1}{2}\left(\nabla + \nabla^\top\right)$ designates the symmetric part of the gradient (i. e. the deformation gradient in continuum mechanics). For a tensor field $\boldsymbol{U} : \Omega \to \mathbb{M}$, with components U_{ij}, the divergence Div is a vector, defined column-wise as

$$\operatorname{Div}(\boldsymbol{U}) := \sum_{i=1}^{2} \partial_{x_i} U_{ij}, \qquad \forall j = \{1, 2\}.$$

The tensor mapping $\boldsymbol{\Phi}$ is positive and preserves the symmetry

$$\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\mathbf{1} + (1-\nu)\boldsymbol{A}, \qquad \boldsymbol{A} = \boldsymbol{A}^\top \implies \boldsymbol{\Phi}(\boldsymbol{A}) = \boldsymbol{\Phi}(\boldsymbol{A})^\top, \qquad \text{where} \qquad \mathbf{1} = \operatorname{Diag}(1,1).$$

Its inverse is given by

$$\boldsymbol{\Phi}^{-1}(\boldsymbol{A}) = \frac{1}{1-\nu}\boldsymbol{A} - \frac{\nu}{1-\nu^2}\operatorname{Tr}(\boldsymbol{A})\mathbf{1}. \tag{2}$$

The total energy of the model (Hamiltonian functional)

$$H = \frac{1}{2}\int_\Omega \left\{ \|\dot{\boldsymbol{u}}\|^2 + \dot{w}^2 + \boldsymbol{N} : \boldsymbol{\varepsilon} + \boldsymbol{M} : \boldsymbol{\kappa} \right\} \mathrm{d}\Omega, \qquad \text{where} \qquad \boldsymbol{A} : \boldsymbol{B} = \operatorname{Tr}(\boldsymbol{A}^\top \boldsymbol{B}) \tag{3}$$

consists of the kinetic energy and membrane and bending deformation energies. This model is conservative Bilbao et al. [2015]. Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

The equivalent port-Hamiltonian system

To find a suitable port-Hamiltonian (pH) system, we first select a set of new variables to make the Hamiltonian functional quadratic. The selection is the same as for a linear plate problem Brugnoli [2020]

$$\boldsymbol{\alpha}_u = \dot{\boldsymbol{u}}, \qquad \alpha_w = \dot{w}, \qquad \boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}, \qquad \boldsymbol{A}_\kappa = \boldsymbol{\kappa}. \tag{4}$$

The energy is quadratic in these variables

$$H = \frac{1}{2}\int_\Omega \left\{ \|\boldsymbol{\alpha}_u\|^2 + \alpha_w^2 + \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon) : \boldsymbol{A}_\varepsilon + \boldsymbol{\Phi}(\boldsymbol{A}_\kappa) : \boldsymbol{A}_\kappa \right\}. \tag{5}$$

By computing the variational derivative of the Hamiltonian, one obtains the co-energy variables

$$\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad e_w := \delta_{\alpha_w} H = \dot{w}, \qquad \boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon), \qquad \boldsymbol{E}_\kappa := \delta_{\boldsymbol{A}_\kappa} H = \boldsymbol{\Phi}(\boldsymbol{A}_\kappa) \tag{6}$$

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Conclusions

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References

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A. Brugnoli. *A port-Hamiltonian formulation of flexible structures. Modelling and structure-preserving finite element discretization*. PhD thesis, Université de Toulouse, ISAE-SUPAERO, France, 2020.