

Tensorial Formulations for thin and thick plates

Weak Formulation and Discretization Procedure

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December 9, 2019



Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

The corresponding 1D models

Timoshenko beam

- Valid for thick beams
- Dimension of the PH model: 4
- Differential operator J of order 1

$$\alpha = [\rho v, I_\rho \omega_x, \frac{\partial \phi_x}{\partial x}, \frac{\partial w}{\partial x} - \phi_x]^T$$
$$e = [v, \omega_x, M_{xx}, T_x]^T$$
$$J = \begin{pmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 \end{pmatrix}$$

Euler-Bernoulli beam

- Valid for thin beams
- Dimension of the PH model: 2
- Differential operator J of order 2

$$\alpha = [\rho v, \frac{\partial^2 w}{\partial x^2}]^T$$
$$e = [v, M_{xx}]^T$$
$$J = \begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Energy and co-energy variables

Linear momenta and curvature are taken as energy variables. Additional the shear strain are considered, leading to

$$\boldsymbol{\alpha} = \left(\rho h v, \rho \frac{h^3}{12} \omega_x, \rho \frac{h^3}{12} \omega_y, \kappa_{xx}, \kappa_{yy}, \kappa_{xy}, \gamma_{xz}, \gamma_{yz} \right)^T$$

where $v = \frac{\partial w}{\partial t}$, $\omega_x = \frac{\partial \psi_x}{\partial t}$, $\omega_y = \frac{\partial \psi_y}{\partial t}$. The Hamiltonian density is quadratic in the energy variables

$$\mathcal{H} = \frac{1}{2} \boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{D}_b & 0 \\ 0 & 0 & 0 & 0 & \mathbf{D}_s \end{bmatrix} \boldsymbol{\alpha}$$

The variational derivative provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v, \omega_x, \omega_y, M_{xx}, M_{yy}, M_{xy}, Q_x, Q_y)^T$$

Definition of J and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 & 0 & 0 & 0 & \\ \frac{\partial}{\partial y} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \alpha}{\partial t} = J \mathbf{e}.$$

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
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 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Main References

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon.

Port-hamiltonian formulation and symplectic discretization of plate models. Part I : Mindlin model for thick plates.

arXiv preprint arXiv:1809.11131, 2018.

Under Review

A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon.

Port-hamiltonian formulation and symplectic discretization of plate models. Part II : Kirchhoff model for thin plates.

arXiv preprint arXiv:1809.11136, 2018.

Under Review

A scalar-vector-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now κ_{xy} now is half the value of the one in the vectorial case. The curvatures tensor is the linear deformation tensor applied to the rotation vector $\boldsymbol{\theta} = (\psi_x, \psi_y)^T$

$$\mathbb{K} = \text{Grad}(\boldsymbol{\theta}) = \frac{1}{2} \left(\nabla \otimes \boldsymbol{\theta} + \left(\nabla \otimes \boldsymbol{\theta} \right)^T \right),$$

where Grad is the symmetric gradient operator applied to a vector, which gives rise to a symmetric tensor.

The momenta are found by introducing a fourth order tensor \mathbb{D} , such that $\mathbb{M}_{ij} = \sum_{k,l} \mathbb{D}_{ijkl} \mathbb{K}_{kl}$, a linear relation between \mathbb{M} and \mathbb{K} .

Energy Variables

The energy variables are now distinguished with respect to their different nature

$$\begin{aligned}\alpha_w &= \rho h \frac{\partial w}{\partial t}, & \alpha_\theta &= \frac{\rho h^3}{12} \frac{\partial \theta}{\partial t}, \\ \mathbb{A}_\kappa &= \mathbb{K}, & \alpha_{\epsilon_s} &= \epsilon_s.\end{aligned}$$

The Hamiltonian is expressed as follows

$$H = \int_{\Omega} \frac{1}{2} \left\{ \rho h \left(\frac{\partial w}{\partial t} \right)^2 + \frac{\rho h^3}{12} \frac{\partial \theta}{\partial t} \cdot \frac{\partial \theta}{\partial t} + \mathbb{M} : \mathbb{K} + \mathbf{Q} \cdot \epsilon_s \right\} d\Omega,$$

where $\mathbb{M} : \mathbb{K}$ denotes the tensor contraction operation.

The momenta \mathbb{M} depends linearly on \mathbb{K} , hence $\frac{1}{2}\mathbb{M} : \mathbb{K}$ is a quadratic form in \mathbb{K} .

Co-energy variables

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$\begin{aligned}e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, & e_\theta &:= \frac{\delta H}{\delta \alpha_\theta} = \frac{\partial \theta}{\partial t}, \\ \mathbb{E}_\kappa &:= \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}, & e_{\epsilon_s} &:= \frac{\delta H}{\delta \epsilon_s} = \mathbf{Q}.\end{aligned}$$

where now the ϵ_s and \mathbf{Q} are the shear strain and stress respectively.

Proposition (see [1] for the proof)

The variational derivative of the Hamiltonian with respect to the curvatures tensor is the momenta tensor $\frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}$.

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- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
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- 5 Tensorial PH formulation of the Kirchhoff plate
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 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Strong form and Interconnection structure

From div , the scalar divergence of a vector, we construct Div , the vector-valued divergence of a symmetric tensor, defined by

$$\boldsymbol{\varepsilon} = \text{Div}(\mathbb{E}) \quad \text{with } \varepsilon_i = \text{div}(\mathbb{E}_{ji}) = \sum_{j=1}^n \frac{\partial \mathbb{E}_{ji}}{\partial x_j}.$$

The port-Hamiltonian system is expressed as follows

$$\begin{cases} \frac{\partial \alpha_w}{\partial t} &= \text{div}(\mathbf{e}_{\epsilon_s}), \\ \frac{\partial \boldsymbol{\alpha}_\theta}{\partial t} &= \text{Div}(\mathbb{E}_\kappa) + \mathbf{e}_{\epsilon_s}, \\ \frac{\partial \mathbb{A}_\kappa}{\partial t} &= \text{Grad}(\mathbf{e}_\theta), \\ \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} &= \text{grad}(e_w) - \mathbf{e}_\theta \end{cases}.$$

If the variables are concatenated together, the formally skew-symmetric operator J can be highlighted

Strong form for the Mindlin plate

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_\theta \\ \mathbb{A}_\kappa \\ \alpha_{\epsilon_s} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \text{div} \\ 0 & 0 & \text{Div} & \mathbf{I}_{2 \times 2} \\ 0 & \text{Grad} & 0 & 0 \\ \text{grad} & -\mathbf{I}_{2 \times 2} & 0 & 0 \end{bmatrix}}_J \begin{pmatrix} e_w \\ e_\theta \\ \mathbb{E}_\kappa \\ e_{\epsilon_s} \end{pmatrix},$$

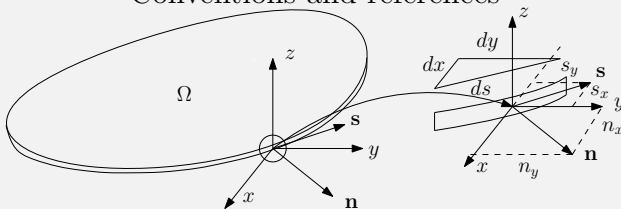
where all zeros are intended as nullifying operators from the space of input variables to the space of output variables.

Theorem (See [1] for additional details)

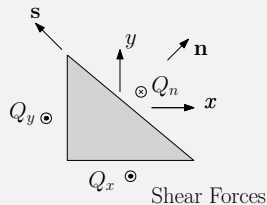
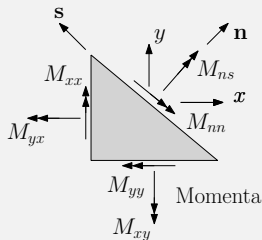
The adjoint of the tensor divergence Div is $-\text{Grad}$, the opposite of the symmetric gradient.

Boundary Variables

Conventions and references



Boundary variables



Energy flow

Again the boundary values can be found by evaluating the time derivative of the Hamiltonian

$$\begin{aligned}
 \dot{H} &= \int_{\Omega} \left\{ \frac{\partial \alpha_w}{\partial t} e_w + \frac{\partial \alpha_{\theta}}{\partial t} \cdot \mathbf{e}_{\theta} + \frac{\partial \mathbb{A}_{\kappa}}{\partial t} : \mathbb{E}_{\kappa} + \frac{\partial \alpha_{\epsilon_s}}{\partial t} \cdot \mathbf{e}_{\epsilon_s} \right\} d\Omega \\
 &= \int_{\Omega} \{ \operatorname{div}(\mathbf{e}_{\epsilon_s}) e_w + [\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}] \cdot \mathbf{e}_{\theta} + \operatorname{Grad}(\mathbf{e}_{\theta}) : \mathbb{E}_{\kappa} + (\operatorname{grad}(e_w) - \mathbf{e}_{\theta}) \cdot \mathbf{e}_{\epsilon_s} \} d\Omega \\
 &= \int_{\partial\Omega} \left\{ \underbrace{(\mathbf{n} \cdot \mathbf{e}_{\epsilon_s})}_{Q_n} e_w + (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \cdot \mathbf{e}_{\theta} \right\} d\Omega, \\
 &= \int_{\partial\Omega} \{ Q_n e_w + (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \cdot (\omega_n \mathbf{n} + \omega_s \mathbf{s}) \} d\Omega, \\
 &= \int_{\partial\Omega} \left\{ Q_n e_w + \omega_n \underbrace{\mathbf{n}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{nn}} + \omega_s \underbrace{\mathbf{s}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{ns}} \right\} d\Omega, \\
 &= \int_{\partial\Omega} \{ \color{blue}{Q_n} \color{red}{e_w} + \color{blue}{\omega_n} \color{red}{M_{nn}} + \color{blue}{\omega_s} \color{red}{M_{ns}} \} d\Omega.
 \end{aligned}$$

Same result as in the vectorial case but with intrinsic operators.

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 - Strong form
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- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
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 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Weak Formulation

The first line is multiplied by v_w (multiplication by a scalar)

$$\int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega = \int_{\Omega} v_w \operatorname{div}(\mathbf{e}_{\epsilon_s}) \, d\Omega,$$

the second and the fourth lines by \mathbf{v}_{θ} , \mathbf{v}_{ϵ_s} (scalar product of \mathbb{R}^2)

$$\begin{aligned} \int_{\Omega} \mathbf{v}_{\theta} \cdot \frac{\partial \boldsymbol{\alpha}_{\theta}}{\partial t} \, d\Omega &= \int_{\Omega} \mathbf{v}_{\theta} \cdot (\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}) \, d\Omega, \\ \int_{\Omega} \mathbf{v}_{\epsilon_s} \cdot \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} \, d\Omega &= \int_{\Omega} \mathbf{v}_{\epsilon_s} \cdot (\operatorname{grad}(e_w) - \mathbf{e}_{\theta}) \, d\Omega, \end{aligned}$$

the third one by \mathbb{V}_{κ} (tensor contraction)

$$\int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} \, d\Omega = \int_{\Omega} \mathbb{V}_{\kappa} : \operatorname{Grad}(\mathbf{e}_{\theta}) \, d\Omega.$$

Choice I: Boundary control through forces and momenta

The first two line of the system in weak form are integrated by parts

$$\int_{\Omega} v_w \operatorname{div}(\mathbf{e}_{\epsilon_s}) \, d\Omega = \int_{\partial\Omega} v_w \underbrace{\mathbf{n} \cdot \mathbf{e}_{\epsilon_s}}_{Q_n} \, ds - \int_{\Omega} \operatorname{grad}(v_w) \cdot \mathbf{e}_{\epsilon_s} \, d\Omega,$$

$$\int_{\Omega} \mathbf{v}_{\theta} \cdot (\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}) \, d\Omega = \int_{\partial\Omega} \mathbf{v}_{\theta} \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds - \int_{\Omega} \{ \operatorname{Grad}(\mathbf{v}_{\theta}) : \mathbb{E}_{\kappa} - \mathbf{v}_{\theta} \cdot \mathbf{e}_{\epsilon_s} \} \, d\Omega.$$

The usual additional manipulation is performed on the boundary term containing the momenta, so that the proper boundary values arise

$$\begin{aligned} \int_{\partial\Omega} \mathbf{v}_{\theta} \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds &= \int_{\partial\Omega} \left\{ \underbrace{(\mathbf{v}_{\theta} \cdot \mathbf{n})}_{v_{\omega_n}} \mathbf{n} + \underbrace{(\mathbf{v}_{\theta} \cdot \mathbf{s})}_{v_{\omega_s}} \mathbf{s} \right\} \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds \\ &= \int_{\partial\Omega} \{ v_{\omega_n} M_{nn} + v_{\omega_s} M_{ns} \} \, ds. \end{aligned}$$

The final system exhibits as **control inputs** the boundary forces and momenta

Weak Form with forces and momenta as inputs

$$\left\{ \begin{array}{l} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} d\Omega = - \int_{\Omega} \text{grad}(v_w) \cdot e_{\epsilon_s} d\Omega + \int_{\partial\Omega} v_w Q_n ds \\ \int_{\Omega} v_{\theta} \cdot \frac{\partial \alpha_{\theta}}{\partial t} d\Omega = - \int_{\Omega} \{ \text{Grad}(v_{\theta}) : \mathbb{E}_{\kappa} - v_{\theta} \cdot e_{\epsilon_s} \} d\Omega + \int_{\partial\Omega} \{ v_{\omega_n} M_{nn} + v_{\omega_s} M_{ns} \} ds \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} d\Omega = \int_{\Omega} \mathbb{V}_{\kappa} : \text{Grad}(e_{\theta}) d\Omega \\ \int_{\Omega} v_{\epsilon_s} \cdot \frac{\partial \alpha_{\epsilon_s}}{\partial t} d\Omega = \int_{\Omega} v_{\epsilon_s} \cdot (\text{grad}(e_w) - e_{\theta}) d\Omega \end{array} \right. .$$

In this first case, the boundary controls \mathbf{u}_{∂} and the corresponding output \mathbf{y}_{∂} are

$$\mathbf{u}_{\partial} = \begin{pmatrix} Q_n \\ M_{nn} \\ M_{ns} \end{pmatrix}_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \begin{pmatrix} e_w \\ \omega_n \\ \omega_s \end{pmatrix}_{\partial\Omega} .$$

Choice II: Boundary control through kinematic variables

If instead the last two lines are integrated by parts

Weak Form with linear and rotational velocities as inputs

$$\left\{ \begin{array}{lcl} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} d\Omega & = & \int_{\Omega} v_w \operatorname{div}(\mathbf{e}_{\epsilon_s}) d\Omega \\ \int_{\Omega} \mathbf{v}_{\theta} \cdot \frac{\partial \boldsymbol{\alpha}_{\theta}}{\partial t} d\Omega & = & \int_{\Omega} \mathbf{v}_{\theta} \cdot (\operatorname{Div}(\mathbb{E}_{\kappa}) + \mathbf{e}_{\epsilon_s}) d\Omega \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} d\Omega & = & - \int_{\Omega} \operatorname{Div}(\mathbb{V}_{\kappa}) \cdot \mathbf{e}_{\theta} d\Omega + \int_{\partial\Omega} \{v_{M_{nn}} \boldsymbol{\omega}_n + v_{M_{ns}} \boldsymbol{\omega}_s\} ds \\ \int_{\Omega} \mathbf{v}_{\epsilon_s} \cdot \frac{\partial \boldsymbol{\alpha}_{\epsilon_s}}{\partial t} d\Omega & = & - \int_{\Omega} \{\operatorname{div}(\mathbf{v}_{\epsilon_s}) \mathbf{e}_w + \mathbf{v}_{\epsilon_s} \cdot \mathbf{e}_{\theta}\} d\Omega + \int_{\partial\Omega} v_{Q_n} \mathbf{e}_w ds \end{array} \right.$$

where $v_{M_{nn}} = \mathbf{n}^T \mathbb{V}_{\kappa} \mathbf{n}$, $v_{M_{ns}} = \mathbf{s}^T \mathbb{V}_{\kappa} \mathbf{n}$ and $v_{Q_n} = \mathbf{v}_{\epsilon_s} \cdot \mathbf{n}$. In this second case, the boundary controls \mathbf{u}_{∂} and corresponding output \mathbf{y}_{∂} are

$$\mathbf{u}_{\partial} = \begin{pmatrix} \mathbf{e}_w \\ \boldsymbol{\omega}_n \\ \boldsymbol{\omega}_s \end{pmatrix}_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \begin{pmatrix} Q_n \\ M_{nn} \\ M_{ns} \end{pmatrix}_{\partial\Omega}.$$

Summary for the Mindlin Plate

Main points:

- variational derivative w.r.t. a tensor quantity;
- strong tensorial form;
- structure preserving weak formulation with different choices of the boundary inputs;
- only two choices for the boundary inputs were shown but others are possible.

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate**
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Energy and co-energy variables

This model is the 2D extension of the Bernoulli beam. It is logical to select as energy variable the linear momentum, together with the curvatures

$$\boldsymbol{\alpha} = (\mu v, \kappa_{xx}, \kappa_{yy}, \kappa_{xy})^T$$

where $v = \frac{\partial w}{\partial t}$. The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2} \boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & \mathbf{D} \end{bmatrix} \boldsymbol{\alpha}$$

So the variational derivative of the total Hamiltonian $H = \int_{\Omega} \mathcal{H} d\Omega$ provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v, M_{xx}, M_{yy}, M_{xy})^T$$

Definition of J and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -\left(\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial x \partial y}\right) \\ \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial x \partial y} & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \alpha}{\partial t} = J \mathbf{e}.$$

From the Schwarz theorem for C^2 functions the mixed derivative could be expressed as $2\frac{\partial^2}{\partial x \partial y}$, instead of $\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial x \partial y}$. However, in this way the symmetry intrinsically present in $\gamma_{xy} = -z \left(\frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 w}{\partial x \partial y} \right)$ would be lost. The mixed derivative is here split to reestablish the symmetric nature of curvatures and momenta (that are of tensorial nature).

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

A scalar-tensor formulation

The momenta and curvatures are of tensorial nature. In Cartesian coordinates

$$\mathbb{K} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix},$$

where now κ_{xy} now is half the value of the one in the vectorial case. The curvatures tensor is the linear deformation tensor applied to the rotation vector $\boldsymbol{\theta} = \text{grad}(w)$

$$\mathbb{K} = \text{Grad}(\boldsymbol{\theta}) = \text{Grad}(\text{grad}(w)).$$

The momenta are found by introducing a fourth order tensor \mathbb{D} , such that $\mathbb{M}_{ij} = \mathbb{D}_{ijkl} \mathbb{K}_{kl}$

For what concerns the choice of the energy variables a scalar and a tensor variable are grouped together

$$\alpha_w = \mu \frac{\partial w}{\partial t} \quad \mathbb{A}_\kappa = \mathbb{K}$$

The Hamiltonian energy is written as

$$H = \int_{\Omega} \left\{ \frac{1}{2} \mu \left(\frac{\partial w}{\partial t} \right)^2 + \frac{1}{2} \mathbb{M} : \mathbb{K} \right\} d\Omega,$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$e_w := \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, \quad \mathbb{E}_\kappa := \frac{\delta H}{\delta \mathbb{A}_\kappa} = \mathbb{M}.$$

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
- 2 Vectorial PH formulation of the Mindlin plate (Macchelli et al. 2005)
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 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
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 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Interconnection structure

The formally skew-symmetric operator J can be highlighted

Strong form for the Kirchhoff plate

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \mathbb{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\text{div} \circ \text{Div} \\ \text{Grad} \circ \text{grad} & 0 \end{bmatrix}}_J \begin{pmatrix} e_w \\ \mathbb{E}_\kappa \end{pmatrix}.$$

where all zeros are intended as nullifying operator from the space of input variables to the space of output variables.

Theorem (See [2] for additional details)

The adjoint of $\text{div} \circ \text{Div}$ is $\text{Grad} \circ \text{grad}$ (i.e. the Hessian operator)

Remark

The interconnection structure J now resembles that of the Bernoulli beam. The double divergence and the double gradient coincide, in dimension one, with the second derivative.

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
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- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
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- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Weak Formulation

The first line is multiplied by v_w (scalar multiplication)

$$\int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega = \int_{\Omega} -v_w \operatorname{div}(\operatorname{Div}(\mathbb{E}_{\kappa})) \, d\Omega,$$

the second line by \mathbb{V}_{κ} (tensor contraction)

$$\int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} \, d\Omega = \int_{\Omega} \mathbb{V}_{\kappa} : \operatorname{Grad}(\operatorname{grad}(e_w)) \, d\Omega.$$

Now depending on which line is integrated by parts different boundary control term can be selected.

Choice I: Boundary control through forces and momenta

The first line has to be integrated by parts **twice**

$$\int_{\Omega} -v_w \operatorname{div}(\operatorname{Div}(\mathbb{E}_{\kappa})) \, d\Omega = \int_{\partial\Omega} \underbrace{-\mathbf{n} \cdot \operatorname{Div}(\mathbb{E}_{\kappa})}_{Q_n} v_w \, ds + \int_{\Omega} \operatorname{grad}(v_w) \cdot \operatorname{Div}(\mathbb{E}_{\kappa}) \, d\Omega$$

Applying again the integration by parts

$$\int_{\Omega} \operatorname{grad}(v_w) \cdot \operatorname{Div}(\mathbb{E}_{\kappa}) \, d\Omega = \int_{\partial\Omega} \operatorname{grad}(v_w) \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds - \int_{\Omega} \operatorname{Grad}(\operatorname{grad}(v_w)) : \mathbb{E}_{\kappa} \, d\Omega$$

The usual additional manipulation is performed on the boundary terms, so that the proper boundary values arise

$$\begin{aligned} \int_{\partial\Omega} \operatorname{grad}(v_w) \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds &= \int_{\partial\Omega} \left(\frac{\partial v_w}{\partial n} \mathbf{n} + \frac{\partial v_w}{\partial s} \mathbf{s} \right) \cdot (\mathbf{n} \cdot \mathbb{E}_{\kappa}) \, ds \\ &= \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} \underbrace{\mathbf{n}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{nn}} + \frac{\partial v_w}{\partial s} \underbrace{\mathbf{s}^T \mathbb{E}_{\kappa} \mathbf{n}}_{M_{ns}} \right\} \, ds \\ &= \sum_{\Gamma_i \subset \partial\Omega} [M_{ns} v_w]_{\partial\Gamma_i} + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} - v_w \frac{\partial M_{ns}}{\partial s} \right\} \, ds \end{aligned}$$

Defining the effective shear stress as $\tilde{Q}_n = Q_n - \frac{\partial M_{ns}}{\partial s}$ If the boundary is regular the final expression becomes

$$\int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega = - \int_{\Omega} \text{Grad}(\text{grad}(v_w)) : \mathbb{E}_{\kappa} \, d\Omega + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} + v_w \tilde{Q}_n \right\} \, ds.$$

Weak form with forces and momenta as inputs

$$\begin{cases} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} \, d\Omega &= - \int_{\Omega} \text{Grad}(\text{grad}(v_w)) : \mathbb{E}_{\kappa} \, d\Omega + \int_{\partial\Omega} \left\{ \frac{\partial v_w}{\partial n} M_{nn} + v_w \tilde{Q}_n \right\} \, ds, \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} \, d\Omega &= \int_{\Omega} \mathbb{V}_{\kappa} : \text{Grad}(\text{grad}(e_w)) \, d\Omega. \end{cases}$$

The control input \mathbf{u}_{∂} and the corresponding conjugate outputs \mathbf{y}_{∂} are

$$\mathbf{u}_{\partial} = \begin{pmatrix} \tilde{Q}_n \\ M_{nn} \end{pmatrix}_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \begin{pmatrix} e_w \\ \frac{\partial e_w}{\partial n} \end{pmatrix}_{\partial\Omega}.$$

Choice II: Boundary control through kinematic variables

The same procedure can be performed on the second line of the system

Weak form with linear and angular velocities as inputs

$$\begin{cases} \int_{\Omega} v_w \frac{\partial \alpha_w}{\partial t} d\Omega &= \int_{\Omega} -v_w \operatorname{div}(\operatorname{Div}(\mathbb{E}_{\kappa})) d\Omega, \\ \int_{\Omega} \mathbb{V}_{\kappa} : \frac{\partial \mathbb{A}_{\kappa}}{\partial t} d\Omega &= \int_{\Omega} \operatorname{div}(\operatorname{Div}(\mathbb{V}_{\kappa})) e_w d\Omega + \int_{\partial\Omega} \left\{ v_{M_{nn}} \frac{\partial e_w}{\partial n} + v_{\tilde{Q}_n} e_w \right\} ds. \end{cases}$$

where $v_{M_{nn}} = \mathbf{n}^T \mathbb{V}_{\kappa} \mathbf{n}$ and $v_{\tilde{Q}_n} = -\operatorname{Div}(\mathbb{V}_{\kappa}) \cdot \mathbf{n} - \frac{\partial(\mathbf{s}^T \mathbb{V}_{\kappa} \mathbf{n})}{\partial s}$.

The control input \mathbf{u}_{∂} are now the kinematic boundary conditions, the corresponding conjugate outputs \mathbf{y}_{∂} are the dynamic boundary condition

$$\mathbf{u}_{\partial} = \left(\frac{e_w}{\frac{\partial e_w}{\partial n}} \right)_{\partial\Omega}, \quad \mathbf{y}_{\partial} = \left(\frac{\tilde{Q}_n}{M_{nn}} \right)_{\partial\Omega}.$$

Summary for the Kirchhoff Plate

Main points:

- new distributed port-Hamiltonian system involving second order differential operator in space dimension 2 for J ;
- strong tensorial form where $\text{div} \circ \text{Div} = (\text{Grad} \circ \text{grad})^*$;
- structure preserving weak formulation with different choices of the boundary inputs;
- only two choices for the boundary variables were shown but others are possible;

Other contribution (not presented here):

- definition of the underlying Stokes-Dirac structure for the vectorial formulation of the Kirchhoff plate [2].

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
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 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 **Partition Finite Element method (PFEM) for Numerics**
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. [A structure-preserving partitioned finite element method for the 2d wave equation.](#)

In *6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 1–6, Valparaíso, CL, 2018.

<http://oatao.univ-toulouse.fr/19965>

General principles of the structure preserving discretization

The energy, co-energy and test functions of the *same* index are discretized by using the *same* bases:

$$\begin{aligned}\alpha_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x, y) \alpha_i^k(t), & e_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x, y) e_i^k(t), & v_i^{ap} &:= \sum_{k=1}^{N_i} \phi_i^k(x, y) v_i^k, \\ \alpha_i^{ap} &:= \phi_i(x, y)^T \boldsymbol{\alpha}_i(t), & e_i^{ap} &:= \phi_i(x, y)^T \mathbf{e}_i(t), & v_i^{ap} &:= \phi_i(x, y)^T \mathbf{v}_i.\end{aligned}$$

The same procedure is applied for the boundary terms with a specific basis $\boldsymbol{\psi}$

$$u_{\partial,i} \approx u_{\partial,i}^{ap} := \sum_{k=1}^{n_{\partial,i}} \psi_i^k(s) u_{\partial,i}^k(t) = \boldsymbol{\psi}_i(s)^T \mathbf{u}_{\partial,i}(t).$$

Remark

The functions $\boldsymbol{\psi}_i(s)$ can be selected as the restriction of functions $\boldsymbol{\phi}$ over the boundary $\boldsymbol{\psi}(s) = \boldsymbol{\phi}(x(s), y(s))$ or in other ways.

Plan

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 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
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 - Mindlin Plate
 - Kirchhoff Plate
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Mindlin Plate: discretized operators

The formally skew-symmetric operator J is replaced with the following skew-symmetric matrix

$$J_d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -D_{x,71}^T & -D_{y,81}^T \\ 0 & 0 & 0 & -D_{x,42}^T & 0 & -D_{y,62}^T & D_{0,27} & 0 \\ 0 & 0 & 0 & 0 & -D_{y,53}^T & -D_{x,63}^T & 0 & D_{0,38} \\ 0 & D_{x,42} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{y,53} & 0 & 0 & 0 & 0 & 0 \\ 0 & D_{y,62} & D_{x,63} & 0 & 0 & 0 & 0 & 0 \\ D_{x,71} & -D_{0,27}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ D_{y,81} & 0 & -D_{0,38}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As a consequence of the integration by parts, a control input is included in the finite-dimensional system. Matrix B is defined by

$$B = \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \\ 0_{\nu_4 \times n_{\partial,1}} & 0_{\nu_4 \times n_{\partial,2}} & 0_{\nu_4 \times n_{\partial,3}} \end{bmatrix}.$$

Final discretized system

The final system is written as

$$\begin{aligned}M\dot{\boldsymbol{\alpha}} &= J_d \boldsymbol{e} + B \boldsymbol{u}_{\partial}, \\ \boldsymbol{y}_{\partial} &= B^T \boldsymbol{e}\end{aligned}$$

where

- $M = \text{Diag}[M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8]$ is a block diagonal matrix,
- $\boldsymbol{\alpha}$ is simply the concatenation of the $\boldsymbol{\alpha}_i$ (just like \boldsymbol{e})
- $\boldsymbol{u}_{\partial}$ is the concatenation of the $\boldsymbol{u}_{\partial,i}$ ($\boldsymbol{u}_{\partial,1} = \boldsymbol{Q}_n$, $\boldsymbol{u}_{\partial,2} = \boldsymbol{M}_{nn}$ and $\boldsymbol{u}_{\partial,3} = \boldsymbol{M}_{ns}$).

Plan

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- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
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 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Kirchhoff Plate: discretized operators

The discretized system is written as

$$\begin{pmatrix} M_1 \dot{\alpha}_1 \\ M_2 \dot{\alpha}_2 \\ M_3 \dot{\alpha}_3 \\ M_4 \dot{\alpha}_4 \end{pmatrix} = \begin{bmatrix} 0 & -D_{xx}^T & -D_{yy}^T & -2D_{xy}^T \\ D_{xx} & 0 & 0 & 0 \\ D_{yy} & 0 & 0 & 0 \\ 2D_{xy} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_{\partial,1} \\ u_{\partial,2} \end{pmatrix},$$

where M_i are square matrices (of size $N_i \times N_i$), D_{xx} is an $N_2 \times N_1$ matrix, D_{yy} is an $N_3 \times N_1$ matrix, D_{xy} is an $N_4 \times N_1$ matrix, B_1 is an $N_1 \times N_{\partial,1}$ matrix and finally B_2 is an $N_2 \times N_{\partial,2}$ matrix. The collocated output are defined as

$$y_{\partial} = \begin{bmatrix} B_1^T & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}.$$

Plan

- 1 The 1D case: Euler-Bernoulli and Timoshenko beams
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- 3 Tensorial PH formulation of the Mindlin plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 4 Vectorial PH formulation of the Kirchhoff plate
- 5 Tensorial PH formulation of the Kirchhoff plate
 - Strong form
 - Weak Formulation with different choices of boundary inputs
- 6 Partition Finite Element method (PFEM) for Numerics
 - Mindlin Plate
 - Kirchhoff Plate
- 7 General PFEM Variational Formulation and Numerical Results

Variational formulation using PFEM

The general variational form can be now stated:

- Energy variables belong to the mixed space $\alpha \in V_\alpha = V_{\alpha_1} \times \cdots \times V_{\alpha_n}$, where $\alpha_i \in V_{\alpha_i}$;
- Coenergy variables $e = \mathcal{Q}\alpha$ where \mathcal{Q} is a coercive symmetrical operator $\mathcal{Q} : V_\alpha \rightarrow V_\alpha$;
- Test functions belong to the same space as α $v \in V_\alpha$

General structure preserving weak form with known inputs

If the boundary input u_∂ is known (i.e for the Mindlin plate free or clamped conditions) then

$$m \left(v, \frac{\partial \alpha}{\partial t} \right) = j(v, e) + b_{u_\partial}(v_{y_\partial}) \quad \forall v \in V$$

where m is the mass bilinear, symmetric and coercive form, j is the interconnection bilinear and antisymmetric form and b is a linear functional.

Dealing with algebraic constraints

If the boundary input vector is partly unknown then Lagrange multipliers and corresponding test function have to be introduced. Variables $\lambda_{u_\partial}, v_{u_\partial} \in V_\lambda$ defined over Γ_λ , the boundary subset where the input are unknown and the conjugated output are set to zero (e.g. clamped plate with a free side).

General structure preserving weak form with unknown inputs

$$\begin{cases} m\left(v, \frac{\partial \alpha}{\partial t}\right) = j(v, e) + b(v_{y_\partial}, \lambda_{u_\partial}) \\ 0 = c(v_{u_\partial}, e_{y_\partial}) \end{cases} \quad \forall v \in V, \forall v_{u_\partial} \in V_\lambda$$

where $v_{y_\partial} = \mathcal{B}_{y_\partial} v$ and $e_{y_\partial} = \mathcal{B}_{y_\partial} e$ (\mathcal{B} is the boundary variables operator). Now b is a bilinear form on spaces $V_\alpha \times V_\lambda$ and c is defined over $V_\lambda \times V_\alpha$ and is obtained from b by swapping variables and applying \mathcal{B}_{y_∂} on e instead of v .

Numerical Study of the Mindlin Plate

This PH system can be discretized using a FE software. These numerical results were obtained using Fenics. To compare the eigenvalues publications [4, 5, 6] were used :

R. Durán, L. Hervella-Nieto, E. Liberman, and J. Solomin. [Approximation of the vibration modes of a plate by Reissner-Mindlin equations.](#)

Mathematics of Computation of the American Mathematical Society, 68(228):1447–1463, 1999

D.J. Dawe and O.L. Roufaeil. [Rayleigh-Ritz vibration analysis of Mindlin plates.](#)

Journal of Sound and Vibration, 69(3):345–359, 1980

H.C. Huang and E. Hinton. [A nine node Lagrangian Mindlin plate element with enhanced shear interpolation.](#)

Engineering Computations, 1(4):369–379, 1984

In the next tables these references will be denoted respectively by D-H, D-R, H-H. Our results are denoted by B-A.

Study case

Properties of the plate

- Square plate with $l_x = l_y = 1$;
- Young modulus $E = 70 \text{ GPa}$;
- Density $\rho = 2000 \text{ kg/m}^3$;
- Poisson modulus $\nu = 0.3$.

Boundary conditions considered

- All clamped CCCC ($w = 0, \omega_s = 0, \omega_n = 0$);
- Simply supported hard SSSS ($w = 0, \omega_s = 0$);
- Half clamped half simply supported SCSC;
- All clamped but one side free CCCF (F stands for free, i.e. $M_{nn} = 0, M_{ns} = 0, Q_n = 0$).

A scalar viable discretized using n P^1 elements for each side gives rise to $(n + 1) \times (n + 1)$ dofs and one discretized using n P^2 elements contains $(2n + 1) \times (2n + 1)$ dofs. Hence since the system is of dimension 8:

- if 10 P^1 elements are used for each side the final system contains 968 states;
- if 10 P^2 elements are used for each side the final system contains 3528 states.






Benchmark variables

The variables are discretized by using Lagrange polynomials of order 1 (same order for each variable) or 2. Normalized eigenfrequency found in [5] are used as benchmark

$$\hat{\omega}_{mn} = \omega_{mn}^h L \left(\frac{2(1+\nu)\rho}{E} \right)^{\frac{1}{2}} \quad \text{Final result independent on } E, \rho,$$

m and n being the numbers of half-waves occurring in the modes shapes in the x and y directions, respectively.

Colors used for comparing of the first 4 eigenvalues

-  benchmark results;
-  1% error or less;
-  up to 5% error;
-  up to 15% error;
-  spurious eigenvalue;

Eigenvalues for $t/L = 0.1$ using P^1

BCs	Mode	$N : 10(\text{B-A})$	$N : 10(\text{D-H})$	$N : 20(\text{B-A})$	$N : 20(\text{D-H})$	H-H	D-R
CCCC	$\hat{\omega}_{11}$	1.5999	1.5947	1.5917	1.5921	1.591	1.594
	$\hat{\omega}_{21}$	3.0615	3.1181	3.0410	3.0595	3.039	3.046
	$\hat{\omega}_{12}$	3.0615	3.1181	3.0410	3.0595	3.039	3.046
	$\hat{\omega}_{22}$	4.3161	4.4477	4.2682	4.3106	4.263	4.285
SSSS	$\hat{\omega}_{11}$	0.9324	0.9384	0.9324	0.9323	0.930	0.930
	$\hat{\omega}_{21}$	2.2227	2.2893	2.2226	2.2366	2.219	2.219
	$\hat{\omega}_{12}$	2.2227	2.2893	2.2226	2.2366	2.219	2.219
	$\hat{\omega}_{22}$	3.4142	3.5657	3.3608	3.4450	3.405	3.406
SCSC	$\hat{\omega}_{11}$	1.3111	1.3060	1.3013	1.3016	1.300	1.302
	$\hat{\omega}_{21}$	2.4155	2.4664	2.3966	2.4120	2.394	2.398
	$\hat{\omega}_{12}$	2.9082	2.9617	2.8871	2.9043	2.885	2.888
	$\hat{\omega}_{22}$	3.8906	4.0126	3.8458	3.8830	3.839	3.852
CCCF	$\hat{\omega}_{\frac{1}{2}1}$	1.0855	1.0812	1.0982	1.0848	1.081	1.089
	$\hat{\omega}_{\frac{3}{2}1}$	1.7636	1.7759	1.7461	1.7525	1.744	1.758
	$\hat{\omega}_{\frac{1}{2}2}$	2.6696	2.7413	2.6575	2.6787	2.657	2.673
	$\hat{\omega}_{\frac{5}{2}1}$	3.2248	3.3186	3.1997	3.2282	3.197	3.216

Eigenvalues for $t/L = 0.01$ using P^1

BCs	Mode	$N : 10(\text{B-A})$	$N : 10(\text{D-H})$	$N : 20(\text{B-A})$	$N : 20(\text{D-H})$	H-H	D-R
CCCC	$\hat{\omega}_{11}$	0.1967	.1754	.1765	.1754	.1754	.1754
	$\hat{\omega}_{21}$	0.4030	.3668	.3604	.3599	.3574	.3576
	$\hat{\omega}_{12}$	0.4030	.3668	.3604	.3599	.3574	.3576
	$\hat{\omega}_{22}$	0.6431	.5487	.5358	.5323	.5264	.5274
SSSS	$\hat{\omega}_{11}$	0.1706	.0972	.1128	.0965	.0963	.0963
	$\hat{\omega}_{21}$	0.3576	.2486	.2660	.2426	.2406	.2406
	$\hat{\omega}_{12}$	0.3576	.2486	.2660	.2426	.2406	.2406
	$\hat{\omega}_{22}$	0.5803	.4035	.3865	.3893	.3847	.3848
SCSC	$\hat{\omega}_{11}$	0.1864	.1417	.1487	.1413	.1411	.1411
	$\hat{\omega}_{21}$	0.3649	.2748	.2829	.2688	.2668	.2668
	$\hat{\omega}_{12}$	0.3987	.3474	.3485	.3402	.3377	.3377
	$\hat{\omega}_{22}$	0.6075	.4814	.4933	.4657	.4604	.4608
CCCF	$\hat{\omega}_{\frac{1}{2}1}$	0.1238	.1182	.1166	.1170	.1166	.1171
	$\hat{\omega}_{\frac{3}{2}1}$	0.2207	.1977	.1954	.1956	.1949	.1951
	$\hat{\omega}_{\frac{1}{2}2}$	0.3204	.3193	.3078	.3109	.3080	.3093
	$\hat{\omega}_{\frac{5}{2}1}$	0.4144	.3874	.3751	.3771	.3736	.3740

Eigenvalues for $t/L = 0.1$ using P^2

BCs	Mode	$N = 5$ (B-A)	$N = 10$ (B-A)	H-H	D-R
CCCC	$\hat{\omega}_{11}$	1.5976	1.5914	1.591	1.594
	$\hat{\omega}_{21}$	3.0584	3.0405	3.039	3.046
	$\hat{\omega}_{12}$	3.0677	3.0405	3.039	3.046
	$\hat{\omega}_{22}$	4.3109	4.2662	4.263	4.285
SSSS	$\hat{\omega}_{11}$	0.9304	0.9302	0.930	0.930
	$\hat{\omega}_{21}$	2.2223	2.2194	2.219	2.219
	$\hat{\omega}_{12}$	2.2224	2.2194	2.219	2.219
	$\hat{\omega}_{22}$	3.4128	3.4061	3.405	3.406
SCSC	$\hat{\omega}_{11}$	1.3053	1.3004	1.300	1.302
	$\hat{\omega}_{21}$	2.4040	2.3946	2.394	2.398
	$\hat{\omega}_{12}$	2.9060	2.8858	2.885	2.888
	$\hat{\omega}_{22}$	3.8721	3.8415	3.839	3.852
CCCF	$\hat{\omega}_{\frac{1}{2}1}$	1.0845	1.0797	1.081	1.089
	$\hat{\omega}_{\frac{3}{2}1}$	1.7559	1.7425	1.744	1.758
	$\hat{\omega}_{\frac{1}{2}2}$	2.6762	2.6547	2.657	2.673
	$\hat{\omega}_{\frac{5}{2}1}$	3.2186	3.1954	3.197	3.216

Eigenvalues for $t/L = 0.01$ using P^2

BCs	Mode	$N = 5$ (B-A)	$N = 10$ (B-A)	H-H	D-R
CCCC	$\hat{\omega}_{11}$	0.1872	0.1762	0.1754	0.1754
	$\hat{\omega}_{21}$	0.3725	0.3598	0.3574	0.3576
	$\hat{\omega}_{12}$	0.4055	0.3598	0.3574	0.3576
	$\hat{\omega}_{22}$	0.6043	0.5335	0.5264	0.5274
SSSS	$\hat{\omega}_{11}$	0.0963	0.0963	0.0963	0.0963
	$\hat{\omega}_{21}$	0.2422	0.2406	0.2406	0.2406
	$\hat{\omega}_{12}$	0.2430	0.2406	0.2406	0.2406
	$\hat{\omega}_{22}$	0.3874	0.3848	0.3847	0.3848
SCSC	$\hat{\omega}_{11}$	0.1492	0.1418	0.1411	0.1411
	$\hat{\omega}_{21}$	0.2827	0.2683	0.2668	0.2668
	$\hat{\omega}_{12}$	0.3608	0.3394	0.3377	0.3377
	$\hat{\omega}_{22}$	0.4940	0.4654	0.4604	0.4608
CCCF	$\hat{\omega}_{\frac{1}{2}1}$	0.1197	0.1169	0.1166	0.1171
	$\hat{\omega}_{\frac{3}{2}1}$	0.2092	0.1960	0.1949	0.1951
	$\hat{\omega}_{\frac{1}{2}2}$	0.3188	0.3089	0.3080	0.3093
	$\hat{\omega}_{\frac{5}{2}1}$	0.3938	0.3757	0.3736	0.3740

Temporal Simulation

Consider a uniform specific load acting on a clamped plate at rest

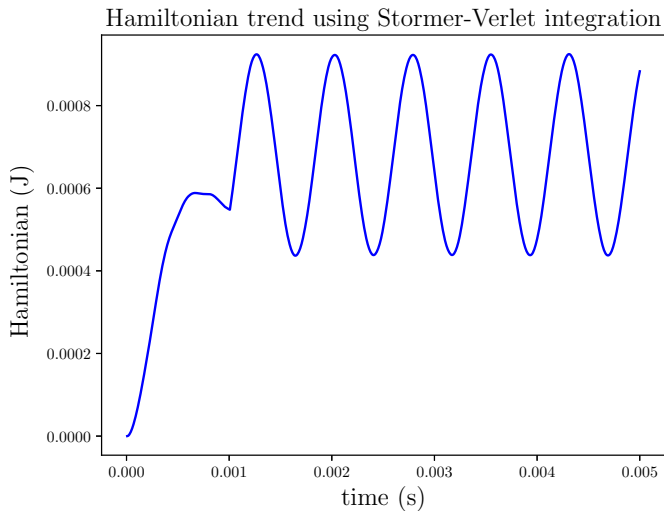
$$\mathbf{f}(t) = \begin{cases} -1 \text{ m/s}^2 \hat{\mathbf{z}} & t < 0.01 \text{ s} \\ 0 & t > 0.01 \text{ s} \end{cases}$$

Corresponding linear form: $l(v) = \int \rho h v_w f \, d\Omega$.

Parameters:

- Total simulation time: $T = 0.05$
- Time step: $dt = 6.85 \cdot 10^{-6} \text{ s}$
- Integration in time: Stormer-Verlet

Simulation output $e_w = \frac{\partial w}{\partial t}$
(scale factor $\times 500$, 100 frame per second)



- [1] A. Brugnoli, D. Alazard, V. Budinger, and D. Matignon. Port-hamiltonian formulation and symplectic discretization of plate models. Part I : Mindlin model for thick plates. *arXiv preprint arXiv:1809.11131*, 2018. Under Review.
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Thank you for your attention. Questions?