Explicit structure-preserving discretization of port-Hamiltonian systems with mixed boundary control

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Introduction

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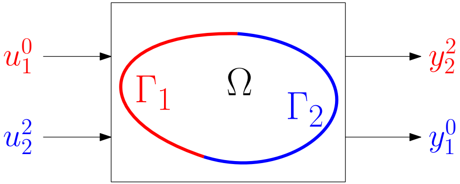
Outline

Introduction

Given a port-Hamiltonian PDE with mixed boundary conditions (BCs), can these be imposed in an explicit manner (i.e. without algebraic conditions)?



This can be achieved by weak imposition of the boundary conditions via the Hellinger-Reissner variational principle[1].

[1] Tobias Thoma and Paul Kotyczka. *Explicit Port-Hamiltonian FEM-Models for Linear Mechanical Systems with Non-Uniform Boundary Conditions*. arXiv:2110.15608. 2021.

Weak and strong imposition

There is no general consensus upon which approach performs better:

- "For boundary layer solutions of the incompressible Navier-Stokes eqs, it is found that **weakly enforced conditions are superior to strongly enforced ones**."[2]
- "Numerical experiments also show that **subtle effects close to solid walls are more efficiently captured with strong boundary condition imposition** methods rather than weak (less dofs. required)."[3]

Strong impositon of BCs. is cumbersome for H^2 conforming elements and penalty methods are preferred[4].

[2] Yuri Bazilevs and Thomas JR Hughes. "Weak imposition of Dirichlet boundary conditions in fluid mechanics". In: *Computers & Fluids* 36.1 (2007), pp. 12–26.

[3] Gustav Eriksson and Ken Mattsson. "Weak Versus Strong Wall Boundary Conditions for the Incompressible Navier-Stokes Equations". In: *Journal of Scientific Computing* 92.3 (2022).

[4] Robert C. Kirby and Lawrence Mitchell. "Code Generation for Generally Mapped Finite Elements". In: *ACM Trans. Math. Softw.* 45.4 (2019).

Outline

Notation

$\Omega \subset \mathbb{R}^d$ (bounded connected) with $d = \{1, 2, 3\}$ and $\partial\Omega = \overline{\Sigma}_1 \cup \overline{\Sigma}_2$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$.
Each Σ_i associated to a specific BC, captured by the trace operator γ_i.
Let $\mathbb{A}$, $\mathbb{B}$ be vector spaces and L an unbounded operator

$$L : L^2(\Omega, \mathbb{A}) \to L^2(\Omega, \mathbb{B}), \qquad D(L) = \{u \in L^2(\Omega, \mathbb{A}) |\ Lu \in L^2(\Omega, \mathbb{B})\}.$$

We denote by $L^\dagger$, with domain

$$D(L^\dagger) = \{u \in L^2(\Omega, \mathbb{B}) |\ L^\dagger u \in L^2(\Omega, \mathbb{A})\},$$

a formal adjoint operator of L with respect to the γ_i operators, *i.e.*

$$(Le_1,\, e_2)_\Omega = (e_1,\, L^\dagger e_2)_\Omega, \qquad \forall e_i \in \ker\left(\gamma_i^{\Sigma_i}\right), \qquad i \in \{1, 2\}$$

where $\gamma_i^{\Sigma_i}$ is the restriction of the operator γ_i to Σ_i and $(\cdot,\, \cdot)_\Omega$ is the L^2 inner product for a suitable vector space ($\mathbb{A}$ or $\mathbb{B}$).

Main assumption

Assumption (Abstract integration by parts)

The operators L and $L^\dagger$ are assumed to satisfy the integration by parts formula

$$(Le_1,\, e_2)_{L^2(\Omega,\mathbb{B})} - (e_1,\, L^\dagger e_2)_{L^2(\Omega,\mathbb{A})} = \langle \gamma_1 e_1 \,|\gamma_2 e_2 \rangle_{V_\partial, V_\partial'},$$

where $\langle \cdot \,|\cdot \rangle_{V_\partial, V_\partial'}$ is the duality product between the boundary space V_∂ and its dual V_∂'.

Example: $L = \mathrm{grad} : L^2(\Omega) \to L^2(\Omega, \mathbb{R}^d),\ D(\mathrm{grad}) = H^1(\Omega)$

Operator γ_1 is the Dirichlet trace, γ_2 is the normal trace

$$\gamma_1 u = u|_{\partial\Omega}, \qquad \gamma_2 \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega}.$$

$L^\dagger = -\,\mathrm{div}$ with domain $H^{\mathrm{div}}(\Omega)$. The assumption is simply the Green's formula

$$\int_\Omega \mathrm{grad}\, u \cdot \boldsymbol{v} = -\int_\Omega u \,\mathrm{div}\, \boldsymbol{v} + \langle u \,|\boldsymbol{v} \cdot \boldsymbol{n} \rangle_{H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega)}.$$

Boundary control operator and compatibility conditions

Let $\mathcal{L}(X,Y)$ be the set of bounded linear operators from X to Y. The boundary control operator is given by

$$G_u = \begin{bmatrix} \gamma_1^{\Sigma_1} & 0 \\ 0 & \gamma_2^{\Sigma_2} \end{bmatrix} \in \mathcal{L}(D(L) \times D(L^\dagger), \mathcal{U}_1 \times \mathcal{U}_2), \qquad \mathcal{U}_i \text{ control spaces.}$$

Compatibility relations at the interface(s) $\overline{\Sigma_1} \cap \overline{\Sigma_2}$ must be fulfilled by $u_i \in \mathcal{U}_i$.
If the boundary control system is well-posed, then $\mathcal{U}_i \subset \mathrm{Range}(\gamma_i^{\Sigma_i})$ and it holds

$$\begin{aligned}
\langle \gamma_1 e_1 \,|\gamma_2 e_2 \rangle_{V_\partial, V_\partial'} =& \langle \gamma_1^{\Sigma_1} e_1 \,|\gamma_2^{\Sigma_1} e_2 \rangle_{V_{\partial,1}, V_{\partial,1}'} + \langle \gamma_1^{\Sigma_2} e_1 \,|\gamma_2^{\Sigma_2} e_2 \rangle_{V_{\partial,2}, V_{\partial,2}'}, \\
=& \langle u_1 \,| y_1 \rangle_{V_{\partial,1}, V_{\partial,1}'} + \langle y_2 \,| u_2 \rangle_{V_{\partial,2}, V_{\partial,2}'}, \\
=& \langle u_1 \,| y_1 \rangle_{\mathcal{U}_1, \mathcal{Y}_1} + \langle y_2 \,| u_2 \rangle_{\mathcal{Y}_2, \mathcal{U}_2}.
\end{aligned}$$

The input and output functional spaces are defined accordingly to the splitting

$$\begin{aligned}
\mathcal{U}_1 \subset V_{\partial,1} &:= \mathrm{Range}(\gamma_1^{\Sigma_1}), \qquad & \mathcal{Y}_1 \supset V_{\partial,1}', \\
\mathcal{U}_2 \subset V_{\partial,2}' &:= \mathrm{Range}(\gamma_2^{\Sigma_2}), \qquad & \mathcal{Y}_2 \supset V_{\partial,2}.
\end{aligned}$$

Second assumption

Assumption (Compatibility conditions)

The compatibility conditions at $\overline{\Sigma_1} \cap \overline{\Sigma_2}$ are fulfilled.
Consequently we do not discriminate between boundary spaces

$$\langle \cdot \,|\, \cdot \rangle_{\Sigma_1} := \langle \cdot \,|\, \cdot \rangle_{\mathcal{U}_1, \mathcal{Y}_1} = \langle \cdot \,|\, \cdot \rangle_{V_{\partial,1}, V'_{\partial,1}},$$

$$\langle \cdot \,|\, \cdot \rangle_{\Sigma_2} := \langle \cdot \,|\, \cdot \rangle_{\mathcal{Y}_2, \mathcal{U}_2} = \langle \cdot \,|\, \cdot \rangle_{V_{\partial,2}, V'_{\partial,2}},$$

$$\langle \cdot \,|\, \cdot \rangle_{\partial\Omega} := \langle \cdot \,|\, \cdot \rangle_{V_{\partial}, V'_{\partial}}$$

Hence, the abstract integration by parts formula of Assumption 1 can be rewritten as

$$(L e_1,\, e_2)_\Omega - \langle \gamma_1 e_1 \,|\, \gamma_2 e_2 \rangle_{\Sigma_1} = (e_1,\, L^\dagger e_2)_\Omega + \langle \gamma_1 e_1 \,|\, \gamma_2 e_2 \rangle_{\Sigma_2}. \tag{1}$$

Abstract linear hyperbolic port-Hamiltonian systems

We focus on linear pH systems of the form (co-energy formulation)

$$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{bmatrix} 0 & -L^\dagger \\ L & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

$$H = \frac{1}{2}(e_1,\, Q_1 e_1)_\Omega + \frac{1}{2}(e_2,\, Q_2 e_2)_\Omega.$$

The operators $Q_1,\ Q_2$ are bounded, symmetric and uniformly positive.
The boundary inputs and outputs are

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} \gamma_1^{\Sigma_1} & 0 \\ 0 & \gamma_2^{\Sigma_2} \end{bmatrix} = G_u \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 0 & \gamma_2^{\Sigma_1} \\ \gamma_1^{\Sigma_2} & 0 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = G_y \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

where $G_y \in \mathcal{L}(D(L) \times D(L^\dagger), \mathcal{Y}_1 \times \mathcal{Y}_2)$. Thanks to Eq. (1)

$$\dot{H} = \langle u_1 \,| y_1 \rangle_{\Sigma_1} + \langle y_2 \,| u_2 \rangle_{\Sigma_2}.$$

The wave equation

$$\begin{bmatrix} \kappa^{-1} & 0 \\ 0 & \rho \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} p \\ \boldsymbol{u} \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{pmatrix} p \\ \boldsymbol{u} \end{pmatrix},, \qquad \Omega \subset \mathbb{R}^d$$

Unknowns:

- the pressure scalar field $p : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}$;
- the velocity vector field $\boldsymbol{u} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}^d$.

Parameters:

- the bulk modulus $\kappa : \Omega \to \mathbb{R}_+$
- the mass density $\rho : \Omega \to \mathbb{R}_+$.

Linear Elastodynamics

$$\begin{bmatrix} \rho & 0 \\ 0 & \boldsymbol{\mathcal{C}} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Sigma} \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Sigma} \end{pmatrix}, \qquad \Omega \subset \mathbb{R}^d$$

where $\mathrm{Grad}\, \boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^\top \boldsymbol{u})$ and $\mathrm{Div}\, \boldsymbol{\Sigma} = \sum_{i=1}^d \partial_{x_i} [\boldsymbol{\Sigma}]_{ij}$.
Unknowns

- the velocity field $\boldsymbol{u} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}^d$;
- the symmetric stress tensor $\boldsymbol{\Sigma} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}^{d\times d}_{\mathsf{sym}}$.

$\boldsymbol{\mathcal{C}} : \Omega \to \mathcal{L}(\mathbb{R}^{d\times d}_{\mathsf{sym}})$ is the compliance fourth order tensor (positive and symmetric).

$$(\mathrm{Grad}\, \boldsymbol{u},\, \Sigma)_\Omega + (\boldsymbol{u},\, \mathrm{Div}\, \Sigma)_\Omega = \langle \boldsymbol{u} \,|\, \boldsymbol{\Sigma} \cdot \boldsymbol{n} \rangle_{\partial\Omega},$$

The boundary duality product involve the spaces

$$V_\partial = H^{1/2}(\partial\Omega, \mathbb{R}^d), \qquad V_\partial' = H^{-1/2}(\partial\Omega, \mathbb{R}^d).$$

The Maxwell equations

$$\begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix},, \qquad \Omega \subset \mathbb{R}^3$$

Unknowns:

- the electric field $\boldsymbol{E} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}^3$;
- the magnetic field $\boldsymbol{H} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}^3$.

Parameters:

- the electric permittivity $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}_{\mathsf{sym}} > 0$;
- the magnetic permeability $\mu : \Omega \to \mathbb{R}^{3\times 3}_{\mathsf{sym}} > 0$.

$$(-\mathrm{curl}\, \boldsymbol{E}, \boldsymbol{H})_\Omega + (\boldsymbol{E}, \mathrm{curl}\, \boldsymbol{H})_\Omega = \langle \boldsymbol{E} \times \boldsymbol{n} \,|\, \boldsymbol{n} \times (\boldsymbol{H} \times \boldsymbol{n}) \rangle_{\partial\Omega}.$$

The trace space V_∂ then corresponds to

$$V_\partial = \{ \boldsymbol{f} \in (H^{-1/2}(\partial\Omega))^3 |\ \exists \boldsymbol{\xi} \in H^{\mathrm{curl}} \text{ s.t. } \boldsymbol{\xi} \times \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{f} \}.$$

V_∂' corresponds to the topological dual (whose characterization is really involved).

Outline

Weak formulation

Consider a weak formulation of the dynamics

$$
\begin{aligned}
(v_1,\, Q_1 \partial_t e_1)_\Omega &= -(v_1,\, L^\dagger e_2)_\Omega, \\
(v_2,\, Q_2 \partial_t e_2)_\Omega &= +(v_2,\, L e_1)_\Omega.
\end{aligned}
$$

A completely analogous formulation is obtained by summing a zero contribution

$$
u_1 - \gamma_1^{\Sigma_1} e_1 = 0, \qquad u_2 - \gamma_2^{\Sigma_2} e_2 = 0.
$$

Taking the duality product of these expressions with test functions $v_1,\ v_2$ leads to a modified weak formulation

$$
\begin{aligned}
(v_1,\, Q_1 \partial_t e_1)_\Omega &= -(v_1,\, L^\dagger e_2)_\Omega + \langle \gamma_1 v_1 \,|u_2 - \gamma_2 e_2 \rangle_{\Sigma_2}, \\
(v_2,\, Q_2 \partial_t e_2)_\Omega &= +(v_2,\, L e_1)_\Omega + \langle u_1 - \gamma_1 e_1 \,|\gamma_2 v_2 \rangle_{\Sigma_1}.
\end{aligned}
$$

Integration by parts of the $L^\dagger$ operator

Using (1), if $L^\dagger$ is integrated by parts the first weak formulation is obtained:
find $e_1 \in D(L),\ e_2 \in D(L^\dagger)$ such that

$$\begin{aligned}
(v_1,\, Q_1 \partial_t e_1)_\Omega &= -\,(L v_1,\, e_2)_\Omega + \langle \gamma_1 v_1 \,|\gamma_2 e_2\rangle_{\Sigma_1} + \langle \gamma_1 v_1 \,| u_2\rangle_{\Sigma_2}, \qquad && \forall v_1 \in D(L),\\
(v_2,\, Q_2 \partial_t e_2)_\Omega &= +\,(v_2,\, L e_1)_\Omega - \langle \gamma_1 e_1 \,|\gamma_2 v_2\rangle_{\Sigma_1} + \langle u_1 \,|\gamma_2 v_2\rangle_{\Sigma_1}. \qquad && \forall v_2 \in D(L^\dagger).
\end{aligned}$$

The test functions do include boundary conditions.
Notice that, the bilinear form

$$\begin{aligned}
j_{L,\Sigma_1}((v_1, v_2), (e_1, e_2)) = &-\,(L v_1,\, e_2)_\Omega + \langle \gamma_1 v_1 \,|\gamma_2 e_2\rangle_{\Sigma_1} \\
&+\,(v_2,\, L e_1)_\Omega - \langle \gamma_1 e_1 \,|\gamma_2 v_2\rangle_{\Sigma_1},
\end{aligned}$$

is skew symmetric.

Integration by parts of the L operator

Using (1), if L is integrated by parts the second weak formulation is obtained: find $e_1 \in D(L),\ e_2 \in D(L^\dagger)$ such that

$$
\begin{aligned}
(v_1,\, Q_1 \partial_t e_1)_\Omega &= -\,(v_1,\, L^\dagger e_2)_\Omega - \langle \gamma_1 v_1 \,|\gamma_2 e_2 \rangle_{\Sigma_2} + \langle \gamma_1 v_1 \,| u_2 \rangle_{\Sigma_2}, \qquad && \forall v_1 \in D(L), \\
(v_2,\, Q_2 \partial_t e_2)_\Omega &= +\,(L^\dagger v_2,\, e_1)_\Omega + \langle \gamma_1 e_1 \,|\gamma_2 v_2 \rangle_{\Sigma_2} + \langle u_1 \,| \gamma_2 v_2 \rangle_{\Sigma_1}, \qquad && \forall v_2 \in D(L^\dagger).
\end{aligned}
$$

The bilinear form

$$
\begin{aligned}
j_{L^\dagger, \Sigma_2}((v_1, v_2), (e_1, e_2)) = &-\,(v_1,\, L^\dagger e_2)_\Omega - \langle \gamma_1 v_1 \,|\gamma_2 e_2 \rangle_{\Sigma_2} \\
&+ (L^\dagger v_2,\, e_1)_\Omega + \langle \gamma_1 e_1 \,|\gamma_2 v_2 \rangle_{\Sigma_2},
\end{aligned}
$$

is skew symmetric.

Properties of the weak formulations

Proposition (Equivalence of the formulations)

Since $v_1,\ e_1 \in D(L)$ *and* $v_2,\ e_2 \in D(L^\dagger)$, *by using the integration by parts* (1) *on the appropriate line of the bilinear forms* j_{L,Σ_1} *or* $j_{L^\dagger,\Sigma_2}$, *it holds* $j_{L,\Sigma_1} = j_{L^\dagger,\Sigma_2}$.

Proposition (Connection with Lagrange multiplier method)

If the BC on Σ_1 *is imposed via a Lagrange multiplier. then* λ *is the multiplier associated to the constraint* $u_1 - \gamma_1^{\Sigma_1} e_1 = 0$. *Using Eq.* (1) *an the extended system in weak form is obtained: find* $e_1 \in D(L)$, $e_2 \in D(L^\dagger)$, $\lambda \in \gamma_1^{\Sigma_1}(D(L))$ *such that*

$$
\begin{aligned}
(v_1,\, Q_1 \partial_t e_1)_\Omega &= -(L v_1,\, e_2)_\Omega + \langle \gamma_1 v_1 \,|\lambda \rangle_{\Sigma_1} + \langle \gamma_1 v_1 \,| u_2 \rangle_{\Sigma_2}, \\
(v_2,\, Q_2 \partial_t e_2)_\Omega &= +(v_2,\, L e_1)_\Omega, \\
0 &= \langle u_1 - \gamma_1 e_1 \,| v_\lambda \rangle_{\Sigma_1},
\end{aligned}
$$

Since $\lambda = y_1 := \gamma_2^{\Sigma_1} e_2$, *if* $v_\lambda = \gamma_2 v_2$ *and substituting the third line in the second one obtains to previous* L *weak formulation (and analogously for* $L^\dagger$*).*

First finite dimensional systems

Galerkin expansion for test functions, states and control inputs ($\boldsymbol{x} \in \Omega,\ \boldsymbol{s}_i \in \Sigma_i$)

$$v_i = \sum_{m=1}^{N_i} \phi_i^m(\boldsymbol{x}) v_i^m, \qquad e_i = \sum_{m=1}^{N_i} \phi_i^m(\boldsymbol{x}) e_i^m(t), \qquad u_i = \sum_{m=1}^{N_{i,\partial}} \psi_i^m(\boldsymbol{s}_i) u_i^m(t),$$

The following finite-dimensional system is obtained from the first weak formulation

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{K}_L^\top \\ \mathbf{K}_L & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix},$$

$$[\mathbf{M}_i]_{mn} = (\phi_i^m,\, Q_i \phi_i^n)_\Omega, \qquad [\mathbf{K}_L]_{mn} = (\phi_2^m,\, L\phi_1^n)_\Omega - \langle \gamma_1 \phi_1^n \,|\gamma_2 \phi_2^m \rangle_{\Sigma_1}.$$

The control matrices are computed via

$$[\mathbf{B}_1]_{mn} = \langle \psi_1^n \,|\gamma_2 \phi_2^m \rangle_{\Sigma_1}, \qquad [\mathbf{B}_2]_{mn} = \langle \gamma_1 \phi_1^m \,|\psi_2^n \rangle_{\Sigma_2}.$$

Second finite dimensional systems

Symmetrically, starting from the second formulation, the following system is readily obtained

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{K}_{L^\dagger} \\ \mathbf{K}_{L^\dagger}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix},$$

where the differentiation matrix $\mathbf{D}_{L^\dagger}$ now reads

$$[\mathbf{K}_{L^\dagger}]_{mn} = (\phi_1^m, \, L^\dagger \phi_2^n)_\Omega + \langle \gamma_1 \phi_1^m \, | \gamma_2 \phi_2^n \rangle_{\Sigma_2}.$$

Proposition (Algebraic Stokes theorem)

From the equivalence of the formulation, for conforming discrete spaces

$$\begin{aligned} \mathrm{span}(\phi_1^1, \dots, \phi_1^{N_1}) &= V_L \subset D(L), \\ \mathrm{span}(\phi_2^1, \dots, \phi_2^{N_2}) &= V_{L^\dagger} \subset D(L^\dagger), \end{aligned}$$

the Stokes theorem leads to the algebraic relation $\mathbf{K}_L^\top = \mathbf{K}_{L^\dagger}$.

Outline

Eigenproblem for the 2D wave equation with unitary parameters

Domain data:

$$\Omega = \{(x, y) \in [0, \pi] \times [0, \pi]\}, \qquad \Sigma_1 = \{x = 0 \cup x = \pi\}, \qquad \Sigma_2 = \{y = 0 \cup y = \pi\}.$$

Analytical eigenvalues take the form

$$\lambda_{\mathrm{ex}} = \pm j\omega_{\mathrm{ex}}, \qquad \omega_{\mathrm{ex}} = \sqrt{n^2 + m^2}, \qquad \forall n \in \mathbb{N}_0,\ \forall m \in \mathbb{N}_{>0}, \qquad j = \sqrt{-1}$$

The grad formulation is employed and

$$p_h \in \mathrm{CG}_r, \quad \boldsymbol{u}_h \in \mathrm{RT}_r.$$

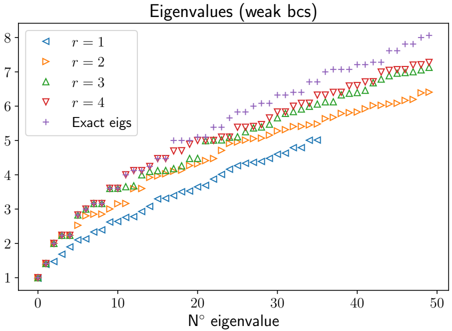
For the discretization 5 triangular elements per side are used.

Remark

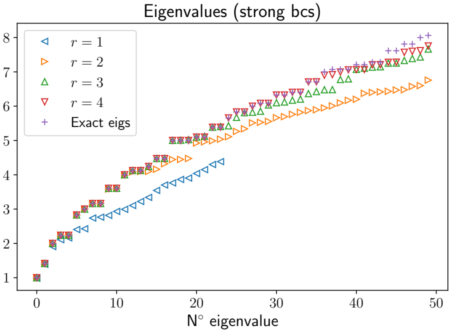
The weak formulations are more restrictive for the choice of finite elements, since both spaces need to be conforming. In particular FE spaces will not form a de Rham subcomplex in general.

Results

Eigensolver: Krylov-Schur solver from SLEPc with shift and invert spectral transform.



(a) Weak imposition



(b) Strong imposition

Eigenvalues for different polynomial orders

Conclusion

This approach allows avoiding the need to deal with differential algebraic systems but:

- the choice of the finite elements is restricted to more regular elements. These may not satisfy de Rham subcomplex property.
- The results for the considered test case show that the approach performs rather poorly compared to the standard strong imposition.

Bibliography

Bazilevs, Yuri and Thomas JR Hughes. "Weak imposition of Dirichlet boundary conditions in fluid mechanics". In: *Computers & Fluids* 36.1 (2007), pp. 12–26.

Eriksson, Gustav and Ken Mattsson. "Weak Versus Strong Wall Boundary Conditions for the Incompressible Navier-Stokes Equations". In: *Journal of Scientific Computing* 92.3 (2022).

Kirby, Robert C. and Lawrence Mitchell. "Code Generation for Generally Mapped Finite Elements". In: *ACM Trans. Math. Softw.* 45.4 (2019).

Thoma, Tobias and Paul Kotyczka. *Explicit Port-Hamiltonian FEM-Models for Linear Mechanical Systems with Non-Uniform Boundary Conditions*. arXiv:2110.15608. 2021.