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Interconnection of the Kirchhoff plate within the port-Hamiltonian framework

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Outline

1 The Kirchhoff plate as a port-Hamiltonian system
- Classical formulation
- Port-Hamiltonian formulation

2 Structure preserving discretization
- The partitioned finite element method
- Application to the Kirchhoff plate

3 Interconnection with rigid elements

4 Stabilization by boundary injection

Outline

1. The Kirchhoff plate as a port-Hamiltonian system
   - Classical formulation
   - Port-Hamiltonian formulation
2. Structure preserving discretization
3. Interconnection with rigid elements
4. Stabilization by boundary injection

Infinite dimensional pH systems

General infinite dimensional pH system

$$\begin{cases} \frac{\partial x}{\partial t}(z,t) &= \mathcal{J}\frac{\delta H}{\delta x} + Bu(z,t), \\ y(z,t) &= B^* \frac{\delta H}{\delta x}. \end{cases}$$

With boundary conditions

$$u_\partial = \mathcal{B}\frac{\delta H}{\delta x}, \quad y_\partial = \mathcal{C}\frac{\delta H}{\delta x}$$

Energy rate: $\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t)y(z,t)\, \mathrm{d}\Omega$

- x energy variables, $e = \delta_x H =:$ co-energy variables ;
- $\mathcal{J}$: skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$: boundary operator;
- u, y, B: distributed input, output and control operator;

Linear infinite dimensional pH system

$$\begin{cases} \dfrac{\partial x}{\partial t}(z,t) & = \mathcal{J}\mathcal{Q}x + Bu(z,t), \\ y(z,t) & = B^*\mathcal{Q}x. \end{cases}$$

With boundary conditions

$$u_\partial = \mathcal{B}\frac{\delta H}{\delta x}, \quad y_\partial = \mathcal{C}\frac{\delta H}{\delta x}$$

Energy rate: $\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t)y(z,t) \, \mathrm{d}\Omega$

- x energy variables, $e = \delta_x H = \mathcal{Q}x$: co-energy variables ($\mathcal{Q}$ symmetric positive operator);
- $\mathcal{J}$: skew-symmetric differential operator;
- $\mathcal{B}, \mathcal{C}$: boundary operator;
- u, y, B: distributed input, output and control operator;

The classical Kirchhoff model

Classical bilaplacian formulation

For an homogeneous isotropic material

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\Delta^2 w = p, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2.$$

$\Delta^2 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$ is the bilaplacian operator

- ρ $[\mathrm{kg/m^3}]$ is the mass density;
- h $[\mathrm{m}]$ is the plate thickness;
- p $[\mathrm{N/m^2}]$ is an external distributed force;
- D $[\mathrm{Pa\,m}]$ is the bending stiffness;

The classical Kirchhoff model

Bending moment formulation

$$\rho h \frac{\partial^2 w}{\partial t^2} + \operatorname{div}\operatorname{Div}(\boldsymbol{M}) = p, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2.$$

Where $\boldsymbol{M} = \mathbb{D}\nabla^2 w \in \mathbb{R}^{2\times 2}_{\text{sym}}$ is the bending moment tensor and $\nabla^2 = \operatorname{Grad} \circ \operatorname{grad}$ the Hessian.

$$\operatorname{div}\operatorname{Div}(\boldsymbol{M}) = \partial_{xx} M_{11} + 2\partial_{xy} M_{12} + \partial_{yy} M_{22}$$

- ρ $[\mathrm{kg/m^3}]$ is the mass density;
- h $[\mathrm{m}]$ is the plate thickness;
- p $[\mathrm{N/m^2}]$ is an external distributed force;
- $\mathbb{D}$ is the bending rigidity tensor (symmetric, positive). For an homogeneous isotropic material

$$\mathbb{D}\boldsymbol{A} = D\left\{(1-\nu)\boldsymbol{A} + \nu \operatorname{Tr}(\boldsymbol{A})\boldsymbol{I}\right\};$$

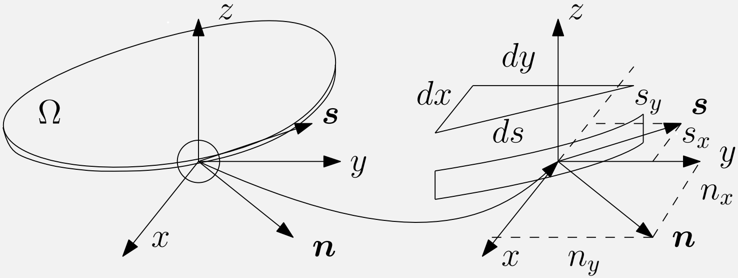
Boundary conditions

For the boundary variables consider the definitions

Flexural moment	$M_{nn} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n})$,	Shear stress	$\boldsymbol{q} = \mathrm{Div}\, \boldsymbol{M}$,
Torsional moment	$M_{ns} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{s})$,	Kirchhoff shear force	$\widetilde{q}_n = -\boldsymbol{q} \cdot \boldsymbol{n} - \partial_{\boldsymbol{s}} M_{ns}$.

$\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j} A_{ij} B_{ij}$ is the tensor contraction and $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}^\top \in \mathbb{R}^{2\times 2}$ is the dyadic product.

References



Boundary conditions

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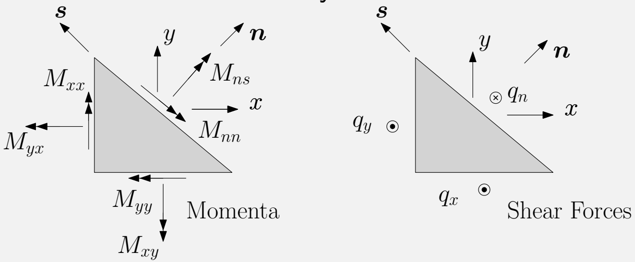
Cauchy Law



For the boundary variables consider the definitions

$$\text{Flexural moment} \quad M_{nn} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{n}), \qquad \text{Shear stress} \quad \boldsymbol{q} = \mathrm{Div}\, \boldsymbol{M},$$

$$\text{Torsional moment} \quad M_{ns} = \boldsymbol{M} : (\boldsymbol{n} \otimes \boldsymbol{s}), \qquad \text{Kirchhoff shear force} \quad \widetilde{q}_n = -\boldsymbol{q} \cdot \boldsymbol{n} - \partial_{\boldsymbol{s}} M_{ns}.$$

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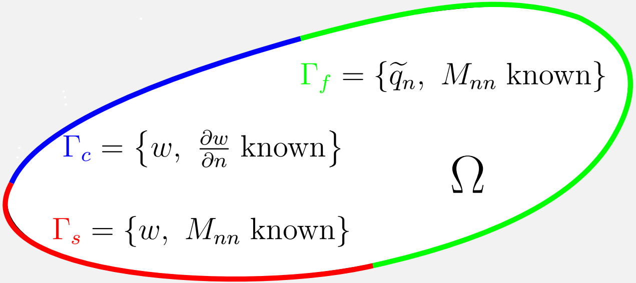
Boundary conditions



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Hamiltonian and energy variables

The total energy of the system is given by the sum of kinetic and deformation energy

$$H = \frac{1}{2}\int_{\Omega} \left\{\rho h \left(\partial_t w\right)^2 + \mathbb{D}\nabla^2 w : \nabla^2 w\right\} \, \mathrm{d}\Omega$$

Consider the following choice for the energy variables

$$\begin{aligned} \alpha_1 &:= \rho h \partial_t w, \quad \text{Linear momentum} \\ \boldsymbol{A}_2 &:= \nabla^2 w, \quad \text{Curvature} \end{aligned}$$

This leads to the following co-energy variables

$$\begin{aligned} e_1 &:= \frac{\delta H}{\delta \alpha_1} = \partial_t w, \quad \text{Vertical velocity} \\ \boldsymbol{E}_2 &:= \frac{\delta H}{\delta \boldsymbol{A}_2} = \boldsymbol{M}, \quad \text{Bending moment} \end{aligned}$$

The system

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_1\\ \boldsymbol{A}_2\end{pmatrix} = \underbrace{\begin{bmatrix}0 & -\operatorname{div}\operatorname{Div}\\ \nabla^2 & 0\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}e_1\\ \boldsymbol{E}_2\end{pmatrix}, \quad \begin{pmatrix}e_1\\ \boldsymbol{E}_2\end{pmatrix} = \underbrace{\begin{bmatrix}(\rho h)^{-1} & 0\\ 0 & \mathbb{D}\end{bmatrix}}_{\mathcal{Q}}\begin{pmatrix}\alpha_1\\ \boldsymbol{A}_2\end{pmatrix}$$

defines a Stokes-Dirac structure. The proof is readily obtained considering that the following holds

$$(\operatorname{div}\operatorname{Div})^* = \nabla^2$$

This means that the operator $\mathcal{J}$ is formally skew-adjoint.

It is worth noticing that the boundary variables are defined by the power balance

$$\dot{H} = \int_{\partial\Omega}\{\partial_t w\,\widetilde{q}_n + \partial_{\boldsymbol{n}}(\partial_t w)\,M_{nn}\}\,\mathrm{d}s.$$

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How to discretize pH systems?

Infinite dimensional pHs

PDE:

$$\partial_t x(z,t) = \mathcal{J}\delta_x H + B u(z,t),$$
$$y(z,t) = B^* \delta_x H.$$

Boundary conditions:

$$u_\partial = \mathcal{B}\ \delta_x H, \quad y_\partial = \mathcal{C}\ \delta_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + \int_\Omega u(z,t) y(z,t)\ \mathrm{d}\Omega$$

Finite dimensional pHs

ODE:

$$\dot{x} = J \partial_x H + B_d u_d + B_\partial u_\partial,$$
$$y_d = B_d^T \partial_x H,$$
$$y_\partial = B_\partial^T \partial_x H$$

Power balance:

$$\dot{H} = u_\partial^T y_\partial + u_d^T y_d$$

Available methods

- Spectral methods (Moulla 2012);
- Finite differences (Trenchant 2018);
- Finite elements based (Golo 2004, Kotyczka 2018, Cardoso-Ribeiro 2018);

The partitioned finite element method (PFEM)

General form of a linear pH system in co-energy variables

$$\mathcal{M}\frac{\partial e}{\partial t} = \mathcal{J} e, \qquad \mathcal{M} = \mathcal{Q}^{-1}$$

General procedure for PFEM

1. Put the system into weak form:
$$\left(v, \mathcal{M}\frac{\partial e}{\partial t}\right)_\Omega = (v, \mathcal{J} e)_\Omega \,.$$
2. Apply integration by part on a partition of $\mathcal{J}$:
$$(v, \mathcal{J} e)_\Omega \overbrace{=}^{i.b.p.} j(v, e)_\Omega + b(v, u_\partial)_{\partial\Omega},$$
so that $j(v, e)_\Omega$ is a skew-symmetric bilinear form.
3. Discretization by Galerkin method (same basis function for test and co-energy variables)

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Application to the Kirchhoff plate

Consider the Kirchhoff plate operators

$$\mathcal{M} = \mathsf{Diag}(\rho h, \ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Application to the Kirchhoff plate

Consider the Kirchhoff plate operators

$$\mathcal{M} = \mathsf{Diag}(\rho h,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the first line of the operator $\mathcal{J}$ is integrated by parts

$$\begin{aligned}(v, \mathcal{J} e)_\Omega &= \int_\Omega \left\{ -v_1 \operatorname{div}\operatorname{Div} \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1 \right\} \, \mathrm{d}\Omega \\ &= \underbrace{\int_\Omega \left\{ -\nabla^2 v_1 : \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1 \right\} \, \mathrm{d}\Omega}_{j_{\mathsf{Hess}}(\boldsymbol{v}, \boldsymbol{e})} + \underbrace{\int_{\partial\Omega} \left\{ v_1 q_n + \partial_{\boldsymbol{n}} v_1 M_{nn} \right\} \, \mathrm{d}s}_{b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}}\end{aligned}$$

Consider the Kirchhoff plate operators

$$\mathcal{M} = \mathrm{Diag}(\rho h,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\mathrm{div}\,\mathrm{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

Either the second line of the operator $\mathcal{J}$ is integrated by parts

$$\begin{aligned} (v, \mathcal{J} e)_\Omega &= \int_\Omega \left\{ -v_1 \,\mathrm{div}\,\mathrm{Div}\, \boldsymbol{E}_2 + \boldsymbol{V}_2 : \nabla^2 e_1 \right\} \,\mathrm{d}\Omega \\ &= \underbrace{\int_\Omega \left\{ -v_1 \,\mathrm{div}\,\mathrm{Div}\, \boldsymbol{E}_2 + \mathrm{div}\,\mathrm{Div}\, \boldsymbol{V}_2 \; e_1 \right\} \,\mathrm{d}\Omega}_{j_{\mathrm{div}\,\mathrm{Div}}(\boldsymbol{v},\boldsymbol{e})} + \underbrace{\int_{\partial\Omega} \left\{ v_{q_n} \partial_t w + v_{m_n} \partial_{\boldsymbol{n}} \partial_t w \right\} \,\mathrm{d}s}_{b_D(\boldsymbol{v},\boldsymbol{u}_\partial)_{\partial\Omega}}, \end{aligned}$$

where $v_{q_n} = -(\mathrm{Div}\, \boldsymbol{V}_2) \cdot \boldsymbol{n} - \partial_{\boldsymbol{s}} v_{m_s}$, $v_{m_s} = \boldsymbol{V}_2 : (\boldsymbol{n} \otimes \boldsymbol{s})$, $v_{m_n} = \boldsymbol{V}_2 : (\boldsymbol{n} \otimes \boldsymbol{n})$

Application to the Kirchhoff plate

Consider the Kirchhoff plate operators

$$\mathcal{M} = \text{Diag}(\rho h,\ \mathbb{D}^{-1}), \quad \mathcal{J} = \begin{pmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{pmatrix}$$

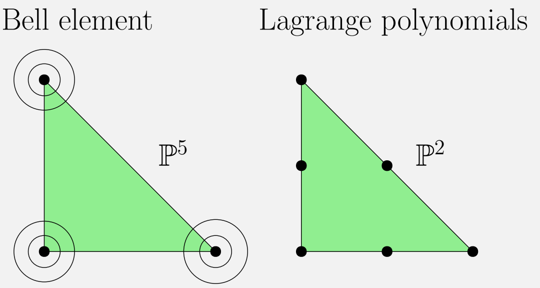
The selection depends on the control variables. For Neumann control the first line is integrated by parts. For Dirichlet control the second.

Finite element choice

Selecting as control variables the forces and torques (Neumann boundary conditions), the following weak form is obtained:

$$m(\boldsymbol{v}, \partial_t \boldsymbol{e}) = j_{\mathsf{Hess}}(\boldsymbol{v}, \boldsymbol{e}) + b_N(\boldsymbol{v}, \boldsymbol{u}_\partial)_{\partial\Omega}.$$

For both $\boldsymbol{e}_1, \boldsymbol{E}_2$ the H^2 conforming Bell elements are selected. Dirichlet boundary conditions are enforced by Lagrange multipliers, i.e. $\boldsymbol{u}_\partial|_{\Gamma_D} = \boldsymbol{\lambda}$. Those are discretized using quadratic Lagrange polynomials.



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$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \boldsymbol{u},$$
$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

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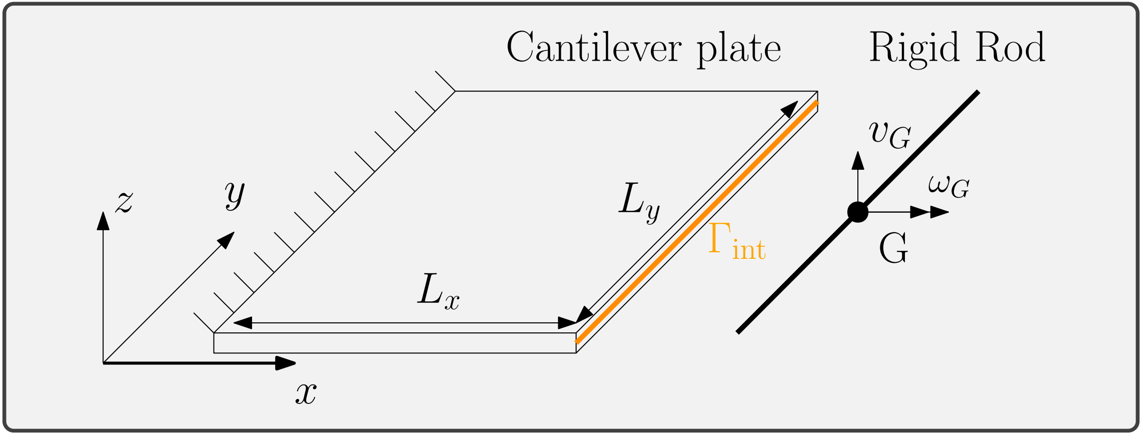
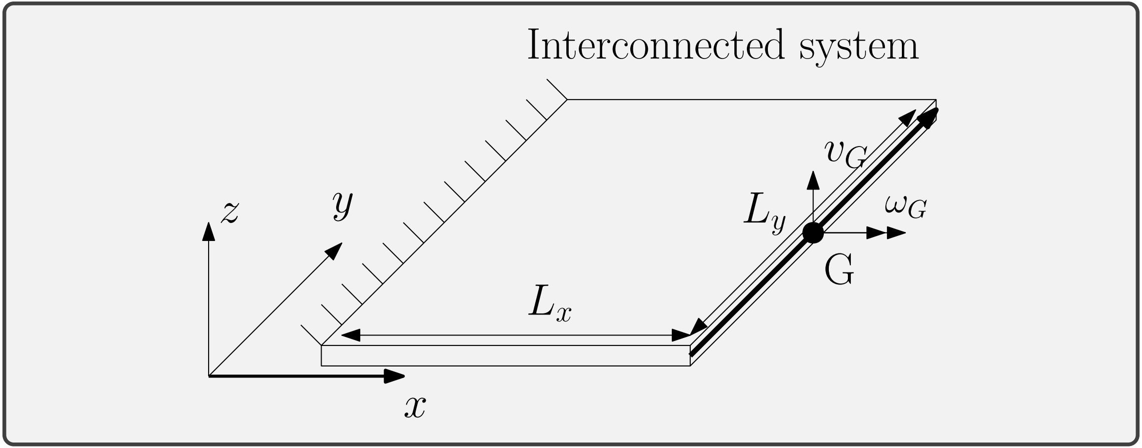

Cantilever plate
Rigid Rod
v_G
ω_G
z
y
L_y
Γ_{int}
G
L_x
x



Interconnected system
z
y
x
L_x
L_y
v_G
ω_G
G

The system is composed by a cantilever plate (distributed pH) connected to a rigid rod

$$\text{dpH} \begin{cases} \dfrac{\partial x_1}{\partial t} = \mathcal{J} \dfrac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \dfrac{\delta H_1}{\delta x_1} \\ y_{\partial,1} = \mathcal{C} \dfrac{\delta H_1}{\delta x_1} \end{cases} \qquad \text{pH} \begin{cases} \dfrac{d\boldsymbol{x}_2}{dt} = J \dfrac{\partial H_2}{\partial \boldsymbol{x}_2} + B\boldsymbol{u}_2 \\ \boldsymbol{y}_2 = B^T \dfrac{\partial H_2}{\partial x_2} + D\boldsymbol{u}_2 \end{cases},$$

where $u_{\partial,1} \in \mathscr{U}$, $y_{\partial,1} \in \mathscr{Y} = \mathscr{U}'$ belong to some Hilbert spaces and $x_2 \in \mathbb{R}^n, u, y \in \mathbb{R}^m$.
The interconnection is power-preserving if

$$\langle u_{\partial,1},\ y_{\partial,1} \rangle_{\mathscr{U} \times \mathscr{Y}} + \langle u_2,\ y_2 \rangle_{\mathbb{R}^m} = 0.$$

This is achieved by introducing a compact operator $\mathcal{W} : \mathscr{Y} \to \mathbb{R}^m$

$$u_2 = -\mathcal{W}\, y_{\partial,1}, \qquad u_{\partial,1} = \mathcal{W}^*\, y_2,$$

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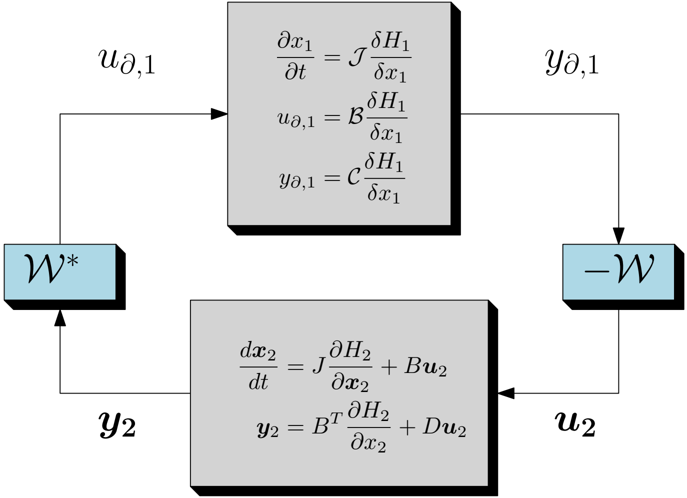

Boundary interconnection of the Kirchhoff plate

Plate $(\Omega = [0, L_x] \times [0, L_y])$

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix}$$

$$u_{\partial,\text{pl}} = \partial_t w(x = L_x, y),$$
$$y_{\partial,\text{pl}} = \widetilde{q}_n(x = L_x, y).$$

Rigid rod

$$\begin{bmatrix} M & 0 \\ 0 & J_G \end{bmatrix} \frac{d}{dt} \begin{pmatrix} v_G \\ \omega_G \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \boldsymbol{u}_{\text{rod}},$$

$$\boldsymbol{y}_{\text{rod}} = \begin{pmatrix} v_G \\ \omega_G \end{pmatrix},$$

Space $\mathscr{Y}$ is the space of square-integrable functions with support on $\Gamma_{\text{int}} = \{(x, y) |\, x = L_x, 0 \leq y \leq L_y\}$. The interconnection operator then provides the total force and torque acting on the rigid rod

$$\mathcal{W} y_{\partial,\text{pl}} = -\begin{pmatrix} F_z \\ T_x \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_{\text{int}}} y_{\partial,\text{pl}} \, \mathrm{d}s \\ \int_{\Gamma_{\text{int}}} (y - L_y/2)\, y_{\partial,\text{pl}} \, \mathrm{d}s \end{pmatrix}.$$

The adjoint operator provides a rigid movement as the plate input at Γ_{int}

$$\langle \mathcal{W} y_{\partial,\text{pl}},\, \boldsymbol{y}_{\text{rod}} \rangle_{\mathbb{R}^m} = \langle y_{\partial,\text{pl}},\, \mathcal{W}^* \boldsymbol{y}_{\text{rod}} \rangle_{L^2(\Gamma_{\text{int}})},$$
$$\mathcal{W}^* y_{\text{rod}} = v_G + \omega_G\, (y - L_y/2)\,.$$

Results

Distributed load ($t_{\text{end}} = 10\,[\text{ms}]$)

$$p = \begin{cases} 10^5 \left[y + 10\,(y - L_y/2)^2\right] [Pa], & \forall\, t < 2\,[\text{ms}], \\ 0, & \forall\, t \geq 2\,[\text{ms}]. \end{cases}$$

Play/Pause

Results

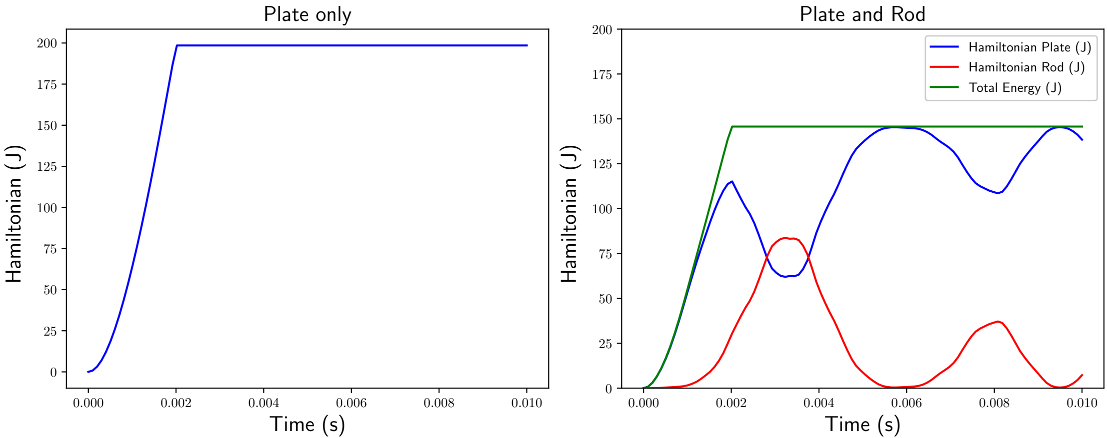

Plate only
Hamiltonian (J)
200
175
150
125
100
75
50
25
0
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)
Plate and Rod
Hamiltonian Plate (J)
Hamiltonian Rod (J)
Total Energy (J)
Hamiltonian (J)
200
175
150
125
100
75
50
25
0
0.000
0.002
0.004
0.006
0.008
0.010
Time (s)

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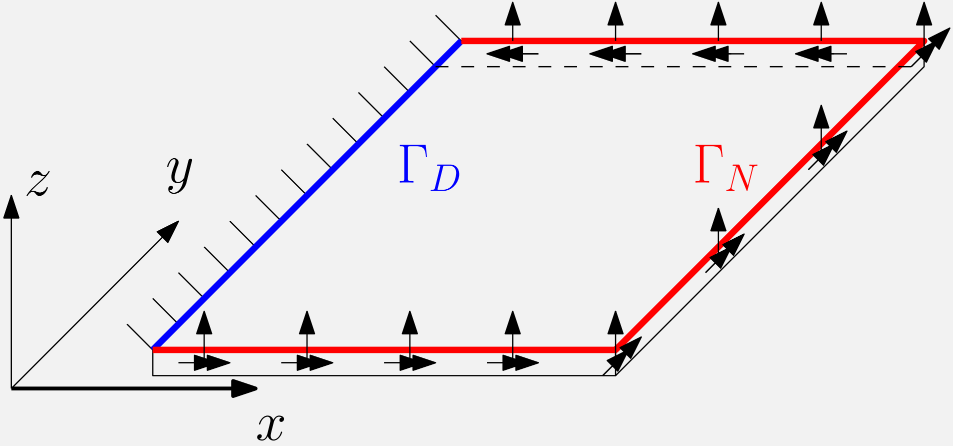
Boundary control of a cantilever plate

Cantilever plate at the left side and controlled elsewhere



Consider the problem

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathbb{D}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} \partial_t w \\ \boldsymbol{M} \end{bmatrix} \qquad (x, y) \in \Omega = [0, 1] \times [0, 1]$$

subjected to the following boundary conditions

$$\begin{aligned} \partial_t w|\Gamma_D &= 0, \\ \partial_x \partial_t w|\Gamma_D &= 0, \\ M_{nn}|\Gamma_N &= u_M, \\ \widetilde{q}|\Gamma_N &= u_F, \end{aligned} \qquad \begin{aligned} \Gamma_D &= \{x = 0\} \\ \Gamma_N &= \{y = 0 \cup x = 1 \cup y = 1\} \end{aligned}$$

with initial conditions (compatible with the constraints):

$$\partial_t w(x, y, 0) = x^2; \qquad \boldsymbol{M}(x, y, 0) = 0.$$

Boundary stabilization of the Kirchhoff plate

Obtain a finite-dimensional uncontrolled system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \boldsymbol{u},$$

$$y = \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

Apply the control law $\boldsymbol{u} = -K\boldsymbol{y}, \;\; K > 0$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{bmatrix} J - R & G \\ -G^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda} \end{pmatrix},$$

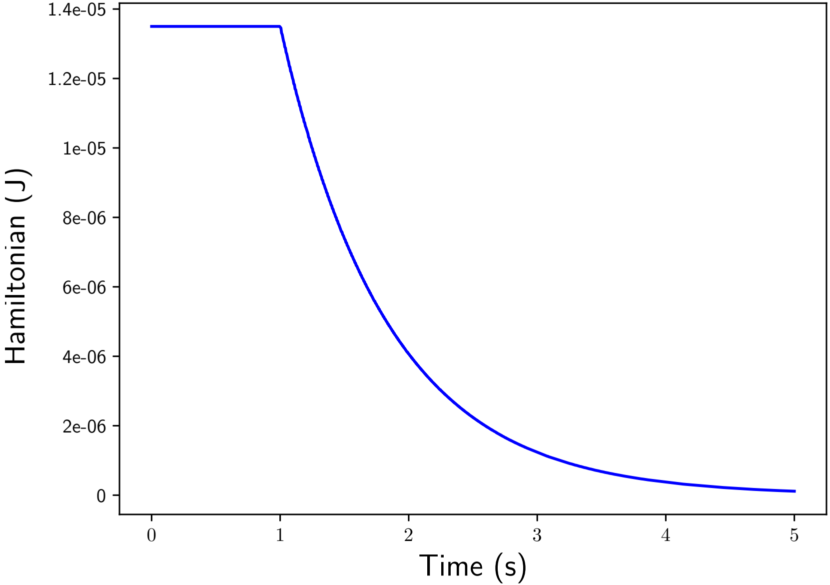
with $R = BKB^T \succeq 0$.
The Hamiltonian $\dot{H} = -\boldsymbol{e}^T R \boldsymbol{e} \leq 0$ is a non increasing function and by La Salle principle the equilibrium point $\boldsymbol{e} = 0$ is asymptotically stable.

Control parameter ($t_{\text{end}} = 5[\mathrm{s}]$)

$$K = \begin{cases} 0, & \forall t < 1\,[\mathrm{s}], \\ 100, & \forall t \geq 1\,[\mathrm{s}]. \end{cases}$$

Play/Pause

Stabilization by boundary injection



Conclusion

The following has been presented:

- the Kirchhoff plate model as a port Hamiltonian system;
- a structure preserving discretization method capable of dealing with generic interconnections;
- interconnection with rigid elements (multibody framework);
- a simple control application by damping injection;

Still no rigorous proof of convergence for the finite elements. Existing solutions (only for static problems):

- The Hellan-Herrmann-Johnson method[1], but difficulties when dealing with inhomogeneous bcs;
- New discretization method capable that handles inhomogeneous bcs[2]

[1] H. Blum and R. Rannacher. "On mixed finite element methods in plate bending analysis". In: *Computational Mechanics* 6.3 (1990), pp. 221–236. ISSN: 1432-0924. DOI: 10.1007/BF00350239.

[2] Katharina. Rafetseder and Walter. Zulehner. "A Decomposition Result for Kirchhoff Plate Bending Problems and a New Discretization Approach". In: *SIAM Journal on Numerical Analysis* 56.3 (2018), pp. 1961–1986. DOI: 10.1137/17M1118427.

Thanks for your attention
Questions?

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