Control by interconnection of the Kirchhoff plate within the Port-Hamiltonian framework*

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***Abstract*— The Kirchhoff plate model is here detailed by using a tensorial port-Hamiltonian (pH) formulation. A structure-preserving discretization of this model is then achieved by using the partitioned finite element (PFEM). This methodology easily accounts for the boundary variables and the finite dimensional system can be interconnected to the surrounding environments in an simple and structured manner. The algebraic constraints to be considered are deduced by the boundary conditions, that may be homogeneous or defined by an interconnection with another dynamical system.**

The versatility of the proposed approach is presented by means of numerical simulations. A first illustration considers a rectangular plate clamped on one side and interconnected to a rod rigidly attached to the opposite side. A second example exploits the collocated output feature of pH systems to perform damping injection in a plate undergoing an external forcing. A stability proof is obtained effortlessly as the Hamiltonian is a Lyapunov function.

I. Introduction

The port-Hamiltonian (pH) framework has proved to be a powerful framework for modeling and control multi-physics system [1]. During the last years distributed systems, i.e. systems ruled by partial differential equations (PDEs) have attracted a lot of interest [2]. The modularity of the pH paradigm is particularly appealing as it provides a structured and coherent way to build complex system. Infinite [3] and finite [4] dimensional pH systems can be connected together giving rise to another pH system. In order to simulate and control such systems, a finite dimensional representation of the distributed system has to be found and it is advantageous to use a discretization procedure that preserves the port-Hamiltonian nature.

The first attempt to perform a structure-preserving discretization dates back to [5], where the authors proposed a mixed finite element spatial discretization for 1D hyperbolic system. Pseudo-spectral methods were studied in [6] . A 2D finite difference method with staggered grids was used in [7]. In [8] the prototypical example of hyperbolic systems of two conservation laws was discretized by a weak formulation, leading to a Galerkin numerical approximations. All these methods require a specific implementation and cannot be related to standard numerical methods. An extension of the Mixed finite element method to pH system was proposed in [9]. The main point of this methodology is that the integration by parts to be performed so that the symplectic structure is preserved. Several choice of the boundary control are possible. For this reason this method is referred to as partitioned finite element method (PFEM). If mixed boundary conditions have to be considered the discretized system is an algebraic differential one (pHDAEs), which can be analyzed by referring to [10], [11].

In this paper the Kirchhoff plate model is presented in a pH fashion. The tensorial calculus is employed in order to clearly identify the skew-symmetric nature of the differential operator. The PFEM methodology is then used to obtain a finite dimensional system. Depending on the application the weak form may contain forces and momenta or linear and angular velocities as control inputs. Numerical applications are then carried out using the Firedrake platform [12]. First an interconnection along the boundary is then presented to model an cantilever plate welded to a rigid bar. A control application by damping injection follows. The plate is interconnected along part of the boundary with a dissipative system. The Hamiltonian for the interconnected system is a Lyapunov function, therefore the system will tend to the equilibrium point, i.e. the undeformed configuration.

In section II the Kirchhoff Plate model in strong form as a port-Hamiltonian system is described. In section **??** BLABLA

II. pH formulation of the Kirchoff plate

In this section the classical formulation of the Kirchhoff plate is recalled. Then the tensorial pH formulation is illustrated. The boundary variables are highlighted thanks to the energy balance.

A. Notations

First, the differential operators needed for the following are recalled. For a scalar field $u : \mathbb{R}^d \rightarrow \mathbb{R}$ the gradient is defined as

$$\mathrm{grad}(u) = \nabla u := \left(\partial_{x_1} u \dots \partial_{x_d} u\right)^T.$$

For a vector field $\boldsymbol{v} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ the symmetric part of the gradient operator Grad (i. e. the deformation gradient

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in continuum mechanics) is given by

$$\mathrm{Grad}(\boldsymbol{v}) := \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\right).$$

The Hessian operator of u is then computed as follows

$$\mathrm{Hess}(\boldsymbol{u}) = \mathrm{Grad}(\mathrm{grad}(u)),$$

For a tensor field $\boldsymbol{U} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, with elements u_{ij}, the divergence is a vector defined column-wise as

$$\mathrm{Div}(\boldsymbol{U}) = \nabla \cdot \boldsymbol{U} := \left(\sum_{i=1}^{d} \partial_{x_i} u_{ij}\right)_{j=1,\dots,d}.$$

The double divergence of a tensor field $\boldsymbol{U}$ is then a scalar field defined as

$$\mathrm{div}(\mathrm{Div}(\boldsymbol{U})) := \sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j} u_{ij}.$$

Furthermore, $\mathbb{S}$ denotes the space of symmetric $d \times d$ matrices. The geometrical dimension of interest in this paper is $d = 2$. In the following vectors (tensor) fields and numerical vectors (matrices) will be denote by a lower (upper) case bold letter. It will be clear by the context which mathematical objects is considered.

B. *Kirchhoff Model for Thin Plates*

The Kirchhoff Model is a generalization to the 2D case of the Euler-Bernoulli beam model and accounts for the shear deformation. Given an open and connected set $\Omega \in \mathbb{R}^2$, the classical equations for this model ([13]) are

$$\rho h \frac{\partial^2 w}{\partial t^2} = -\mathrm{div}(\mathrm{Div}(\boldsymbol{M})), \tag{1}$$

where ρ is the material density, h is the plate thickness, the scalar w is the vertical displacement and $\boldsymbol{M}$ is the symmetric momenta tensor. This tensor is related to the symmetric curvatures tensor $\boldsymbol{K}$ by the bending rigidity tensor, so that $\boldsymbol{M}_{ij} = \boldsymbol{D}_{ijkl}\boldsymbol{K}_{kl}$. The curvature tensor is defined as

$$\boldsymbol{K} := \mathrm{Grad}(\mathrm{grad}(w)).$$

For an homogeneous isotropic material the components of $\boldsymbol{M}, \boldsymbol{K} \in \mathbb{S}$ are relatedby the relations (x denotes index 1, y index 2)

$$\begin{aligned} m_{xx} &= D\left(\kappa_{xx} + \nu \kappa_{yy}\right), \\ m_{yy} &= D\left(\kappa_{yy} + \nu \kappa_{xx}\right), \\ m_{xy} &= D(1-\nu)\kappa_{xy}, \end{aligned}$$

with ν the Poisson ratio, D the bending module. The kinetic and potential energy density $\mathcal{K}$ and $\mathcal{U}$ read

$$\mathcal{K} = \frac{1}{2}\rho h \left(\frac{\partial w}{\partial t}\right)^2, \quad \mathcal{U} = \frac{1}{2}\boldsymbol{M} : \boldsymbol{K}, \tag{2}$$

where $\boldsymbol{M} : \boldsymbol{K} := \sum_{i,j} m_{ij}\kappa_{ij}$ is the tensor contraction. The Hamiltonian is easily written as

$$H = \int_\Omega (\mathcal{K} + \mathcal{U}) \ \mathrm{d}\Omega. \tag{3}$$

C. *Tensorial Port-Hamiltonian formulation*

In order to rewrite the system as a port-Hamiltonian one, the energy variables have to be selected first. This choice is analogous to that of the pH Euler-Bernoulli beam model ([14]), but with the additional complication that the variable mome:

$$\begin{aligned} \alpha_w &= \rho h \frac{\partial w}{\partial t}, && \text{Linear momentum}, \\ \boldsymbol{A}_\kappa &= \boldsymbol{K}, && \text{Curvature tensor}. \end{aligned} \tag{4}$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}, && \text{Vertical Velocity}, \\ \boldsymbol{E}_\kappa &:= \frac{\delta H}{\delta \boldsymbol{A}_\kappa} = \boldsymbol{M}, && \text{Momenta tensor}. \end{aligned} \tag{5}$$

The port-Hamiltonian system is expressed as follows

$$\begin{cases} \dfrac{\partial \alpha_w}{\partial t} & = -\mathrm{div}(\mathrm{Div}(\boldsymbol{E}_\kappa)), \\ \dfrac{\partial \boldsymbol{A}_\kappa}{\partial t} & = \mathrm{Grad}(\mathrm{Grad}(e_w)),, \end{cases} \tag{6}$$

If the variables are concatenated together, the formally skew-symmetric operator J can be highlighted

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -\mathrm{div} \circ \mathrm{Div} \\ \mathrm{Grad} \circ \mathrm{grad} & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix}, \tag{7}$$

Remark 1: It can be observed that the interconnection structure given by $\mathcal{J}$ mimics that of the Euler-Bernoulli beam (in one spatial dimension the double divergence and the Hessian reduce to the second derivative).

Theorem 1 ([14]): The adjoint of the double divergence of a tensor $\mathrm{div} \circ \mathrm{Div}$ is $-\mathrm{Grad} \circ \mathrm{grad} = -\mathrm{Hess}$, the opposite of the Hessian operator.

The boundary values can be found by evaluating the time derivative of the Hamiltonian

$$\dot{H} = \int_{\partial\Omega} \left\{ w_t \widetilde{q}_n + \omega_n m_{nn} \right\} \ \mathrm{d}s. \tag{8}$$

where s is the curvilinear abscissa and the result integral is obtained by applying the Green-Gauss theorem [14]. The boundary variable are defined as follows

$$\begin{aligned} &\text{Effetive Shear Force} && \widetilde{q}_n := -\mathrm{Div}(\boldsymbol{E}_\kappa) \cdot \boldsymbol{n} - \frac{\partial m_{ns}}{\partial s}, \\ &\text{Flexural momentum} && m_{nn} := \boldsymbol{E}_\kappa : (\boldsymbol{n} \otimes \boldsymbol{n}), \end{aligned} \tag{9}$$

where $m_{ns} := \boldsymbol{E}_\kappa : (\boldsymbol{s} \otimes \boldsymbol{n})$ is the torsional momentum and $\boldsymbol{u} \otimes \boldsymbol{v}$ denotes the outer product of vectors equivalent to a matrix given by $\boldsymbol{u}\boldsymbol{v}^T$. Vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ designate the normal and tangential unit vector to the boundary. The corresponding power conjugated variables are

$$\begin{aligned} &\text{Vertical velocity} && w_t := e_w, \\ &\text{Flexural rotation} && \omega_n := \frac{\partial e_w}{\partial n}. \end{aligned} \tag{10}$$

III. Structure preserving discretization

In this section the structure preserving discretization procedure is detailed. Three steps are needed:

1) put the system in weak form;
2) second perform integrations by parts to get the boundary control of choice;
3) select the finite element spaces to achieve a finite dimensional system.

A. Weak Form

In order to put the system into weak form the first line of (7) is multiplied by v_w (multiplication by a scalar), the second line one by $\boldsymbol{V}_\kappa$ (tensor contraction).

$$\int_\Omega v_w \frac{\partial \alpha_w}{\partial t} \, \mathrm{d}\Omega = -\int_\Omega v_w \mathrm{div}(\mathrm{Div}(\boldsymbol{E}_\kappa)) \, \mathrm{d}\Omega, \tag{11}$$

$$\int_\Omega \boldsymbol{V}_\kappa : \frac{\partial \boldsymbol{A}_\kappa}{\partial t} \, \mathrm{d}\Omega = +\int_\Omega \boldsymbol{V}_\kappa : \mathrm{Grad}(\mathrm{grad}(e_w)) \, \mathrm{d}\Omega, \tag{12}$$

For sake of simplicity, all test and unknown functions can be collected in one variable

$$v := (v_w, \boldsymbol{V}_\kappa), \qquad \alpha := (\alpha_w, \boldsymbol{A}_\kappa), \qquad e := (e_w, \boldsymbol{E}_\kappa), \tag{13}$$

so that the previous system is rewritten compactly as

$$\left(v, \frac{\partial \alpha}{\partial t}\right) = (v, \mathcal{J} e), \tag{14}$$

where the bilinear form $(v, u) = \int v \cdot u \, \mathrm{d}\Omega$, is the inner product on space $\mathscr{L}^2(\Omega) := L^2(\Omega) \times L^2(\Omega; \mathbb{S})$. The operator $\mathcal{J}$ was defined in equation (7). It can be decomposed into the sum of three operators

$$\mathcal{J} = \mathcal{J}_{\mathrm{divDiv}} + \mathcal{J}_{\mathrm{Hess}}, \tag{15}$$

where $\mathcal{J}_{\mathrm{divDiv}}, \mathcal{J}_{\mathrm{Hess}}$ contain only the double divergence (divDiv) and Hessian operator respectively.

The integration by part has to be performed so that the final bilinear form on the right-hand side remains skew-symmetric. Obviously, since $\mathcal{J}$ is skew-symmetric $\mathcal{J}_{\mathrm{divDiv}} = -\mathcal{J}_{\mathrm{Hess}}^*$, where A^* is the formal adjoint of operator A. Depending on which of the two differential operators is chosen for the integration by parts, two different boundary controls can arise [14] (other choices are possible but less meaningful under a physical point of view).

B. Boundary control through forces and momenta

Applying the integration by parts twice on $\mathcal{J}_{\mathrm{divDiv}}$ is integrated by parts (meaning that the right-hand side of equation (11) is integrated by parts twice) then

$$(v, Je) = j_{\mathrm{Hess}}(v, e) + f_N(v), \tag{16}$$

where now the bilinear form

$$j_{\mathrm{Hess}}(v, e) = (\mathcal{J}_{\mathrm{divDiv}}^* v, e) + (v, \mathcal{J}_{\mathrm{Hess}} e)$$

is skew symmetric and can be expressed as follows

$$\begin{aligned} j_{\mathrm{Hess}}(v, e) := &-\int_\Omega \mathrm{Grad}(\mathrm{grad}(v_w)) : \boldsymbol{E}_\kappa \, \mathrm{d}\Omega \, \mathrm{d}s \\ &+ \int_\Omega \boldsymbol{V}_\kappa : \mathrm{Grad}(\mathrm{grad}(e_w)) \, \mathrm{d}\Omega. \end{aligned} \tag{17}$$

The linear functional $f_N(v)$ represents the boundary term associated with forces and momenta. The subscript N denotes the fact that classical Neumann conditions appear as boundary input. It reads

$$f_N(v) = \int_{\partial\Omega} \{v_w \widetilde{q}_n + v_{\omega_n} m_{nn}\} \, \mathrm{d}s, \tag{18}$$

where $v_{\omega_n} = \frac{\partial v_w}{\partial n}$. In this first case, the boundary controls $\boldsymbol{u}_\partial$ and the corresponding output $\boldsymbol{y}_\partial$ are

$$\boldsymbol{u}_\partial = \begin{pmatrix} q_n \\ m_{nn} \end{pmatrix}_{\partial\Omega}, \qquad \boldsymbol{y}_\partial = \begin{pmatrix} w_t \\ \omega_n \end{pmatrix}_{\partial\Omega}.$$

C. Finite Dimensional System

In this subsection the formulation (16) is used is order to explain the discretization procedure and the associated finite elements.

a) Discretization Procedure: Test and co-energy variables are discretized using the same basis function (Galerkin Method)

$$\begin{aligned} v_w &= \sum_{i=1}^{N_w} \phi_w^i(x, y) \, v_w^i, & e_w &= \sum_{i=1}^{N_w} \phi_w^i(x, y) \, e_w^i(t), \\ \boldsymbol{V}_\kappa &= \sum_{i=1}^{N_\kappa} \boldsymbol{\Phi}_\kappa^i(x, y) \, v_\kappa^i, & \boldsymbol{E}_\kappa &= \sum_{i=1}^{N_\kappa} \boldsymbol{\Phi}_\kappa^i(x, y) \, e_\kappa^i(t), \end{aligned} \tag{19}$$

The basis function ϕ_w^i, $\boldsymbol{\Phi}_\kappa^i$, have to be chosen in a suitable function space $\mathcal{V}^h$ in the domain of operator $\mathcal{J}$, i.e. $\mathcal{V}^h \subset \mathcal{V} \in \mathcal{D}(\mathcal{J})$. The discretized skew-symmetric bilinear form given in (17) then reads

$$\boldsymbol{J}_d = \begin{bmatrix} 0 & -\boldsymbol{D}_{\mathrm{H}}^T \\ \boldsymbol{D}_{\mathrm{H}} & 0 \end{bmatrix}, \tag{20}$$

where $\boldsymbol{A}^T$ is the transpose of the $\boldsymbol{A}$ matrix. The matrix $\boldsymbol{D}_{\mathrm{H}}$ is computed in the following way

$$\boldsymbol{D}_{\mathrm{H}}(i, j) = \int_\Omega \boldsymbol{\Phi}_\kappa^i : \mathrm{Grad}(\mathrm{grad}(\phi_w^j)) \, \mathrm{d}\Omega, \quad \in \mathbb{R}^{N_\kappa \times N_w}, \tag{21}$$

where $A(i, j)$ indicates the entry in the matrix corresponding to the i th row and j th column. The energy variables are deduced from the co-energy variables

$$\alpha_w = \rho h e_w, \qquad \boldsymbol{A}_\kappa = \boldsymbol{D}^{-1} \boldsymbol{E}_\kappa. \tag{22}$$

The symmetric bilinear form on the left side of (16) is discretized as $\boldsymbol{M} = \mathrm{diag}[\boldsymbol{M}_w, \, \boldsymbol{M}_\kappa]$ with

$$\begin{aligned} \boldsymbol{M}_w(i, j) &= \int_\Omega \rho h \, \boldsymbol{\phi}_w^i \, \boldsymbol{\phi}_w^j \, \mathrm{d}\Omega \; \in \mathbb{R}^{N_w \times N_w}, \\ \boldsymbol{M}_\kappa(i, j) &= \int_\Omega \left(\boldsymbol{D}^{-1} \boldsymbol{\Phi}_\kappa^i\right) : \boldsymbol{\Phi}_\kappa^j \, \mathrm{d}\Omega \; \in \mathbb{R}^{N_\kappa \times N_\kappa}, \end{aligned} \tag{23}$$

To deal with generic boundary conditions the Lagrange multipliers have to be introduced in (18)

$$\lambda_{\widetilde{q}_n} = \sum_{i=1}^{N_{\widetilde{q}_n}} \boldsymbol{\phi}_{\widetilde{q}_n}^i(s) \lambda_{\widetilde{q}_n}^i, \quad \lambda_{m_{nn}} = \sum_{i=1}^{N_{m_{nn}}} \boldsymbol{\phi}_{m_{nn}}^i(s) \lambda_{m_{nn}}^i, \tag{24}$$

If inhomogeneous Neumann boundary conditions are then discretized as

$$\widetilde{q}_n = \sum_{i=1}^{N_{\widetilde{q}_n}} \phi^i_{\widetilde{q}_n}(s)\,\widetilde{q}^i_n, \qquad m_{nn} = \sum_{i=1}^{N_{m_{nn}}} \phi^i_{m_{nn}}(s)\, m^i_{nn}. \tag{25}$$

It is now possible to construct the following matrices

$$\begin{aligned}
\boldsymbol{G}_w(i,j) &= \int_{\Gamma_C \cup \Gamma_S} \phi^i_w\, \phi^j_{\widetilde{q}_n}\, \mathrm{d}s, \quad \in \mathbb{R}^{N_w \times N_{q_n}}, \\
\boldsymbol{B}_{\widetilde{q}_n}(i,j) &= \int_{\Gamma_{\widetilde{q}_n}} \phi^i_w\, \phi^j_{\widetilde{q}_n}\, \mathrm{d}s, \quad \in \mathbb{R}^{N_w \times N_{q_n}}, \\
\boldsymbol{G}_{\omega_n}(i,j) &= \int_{\Gamma_C} \frac{\partial \phi^i_w}{\partial n}\, \phi^j_{m_{nn}}\, \mathrm{d}s, \quad \in \mathbb{R}^{N_w \times N_{m_{nn}}}, \\
\boldsymbol{B}_{m_{nn}}(i,j) &= \int_{\Gamma_{m_{nn}}} \frac{\partial \phi^i_w}{\partial n}\, \phi^j_{m_{nn}}\, \mathrm{d}s, \quad \in \mathbb{R}^{N_w \times N_{m_{nn}}},
\end{aligned} \tag{26}$$

where Γ_C, Γ_S are subsets of the boundary where clamped and simply supported boundary conditions apply and $\Gamma_{\widetilde{q}_n}, \Gamma_{m_{nn}}$ are subsets where inhomogeneous Neumann conditions hold. Consequently, the input matrix reads The final port-Hamiltonian descriptor system (pHDAEs), as defined in [10], is written as

$$\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda}_D \end{pmatrix} = \begin{bmatrix} \boldsymbol{J}_d & \boldsymbol{G}_D \\ -\boldsymbol{G}_D^T & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{\lambda}_D \end{pmatrix} + \begin{bmatrix} \boldsymbol{B}_N \\ 0 \end{bmatrix} \boldsymbol{f}_N, \tag{27}$$

where $\boldsymbol{e} = (\boldsymbol{e}_w, \boldsymbol{e}_\kappa)$, $\boldsymbol{\lambda}_D = (\boldsymbol{\lambda}_{\widetilde{q}_n}, \boldsymbol{\lambda}_{m_{nn}})$, $\boldsymbol{f}_N = (\widetilde{\boldsymbol{q}}_n, \boldsymbol{m}_{nn})$ are the concatenation of the co-energy variables, Lagrange multipliers and boundary forces and momenta at the borders and

$$\boldsymbol{G}_D = \begin{bmatrix} \boldsymbol{G}_w & \boldsymbol{G}_{\omega_n} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B}_N = \begin{bmatrix} \boldsymbol{B}_{\widetilde{q}_n} & \boldsymbol{B}_{m_{nn}} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

. The subscript N, D refer to Neumann and Dirichlet boundary conditions.

b) Finite Element Choice: The domain of the operator $\mathcal{J}$ in (7) is $\mathcal{D}(J) = H^2(\Omega) \times H^{\text{div Div}}(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}}) +$ boundary conditions. For this reason a suitable choice for the functional space is

$$(v_w, \boldsymbol{V}_\kappa) \in H^2(\Omega) \times H^2(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}}) \equiv \mathscr{H}, \tag{28}$$

since $\mathscr{H} \subset \mathcal{D}(J)$.

Remark 2: It has to be appointed that the space $H^{\text{div Div}}(\Omega, \mathbb{R}^{2\times 2}_{\text{sym}})$ was never addressed in the mathematical literature. For this reason the only way to deal with this problem numerically is to use $H^2(\Omega)$ conforming finite elements.

The Firedrake library [12] was used to implement the numerical analysis as it provides functionalities to automate the generalized mappings for H^2 conforming finite elements (like the Hermite, Bell or Argyris finite elements). All the variables, i.e. the velocity e_w and the momenta tensor $\boldsymbol{E}_\kappa$ as well as the corresponding test functions, are discretized by the same finite element space, the Bell finite element [15], denoted by $H^2_r(\mathbb{P}_5, \Omega)$. The multipliers are therefore discretized by using second degree Lagrange polynomials defined over the boundary and denoted with $H^1_r(\mathbb{P}_2, \partial\Omega)$.

IV. Interconnection with a finite dimensional pH system

In this section the interconnection of an infinite and finite port system is explained in both the infinite and finite dimensional setting.

A. General Ideas for the infinite dimensional setting

Consider an infinite dimensional pH system (or distributed pH system, dpH) and a finite dimensional pH system denoted by equations

$$\text{pH} \begin{cases} \dot{\boldsymbol{x}}_2 = \boldsymbol{J} \dfrac{\partial H_2}{\partial \boldsymbol{x}_2} + \boldsymbol{B} \boldsymbol{u}_2 \\ \boldsymbol{y}_2 = \boldsymbol{C} \dfrac{\partial H_2}{\partial \boldsymbol{x}_2} + \boldsymbol{D} \boldsymbol{u}_2 \end{cases}, \tag{29}$$

$$\text{dpH} \begin{cases} \dfrac{\partial x_1}{\partial t} = \mathcal{J} \dfrac{\delta H_1}{\delta x_1} \\ u_{\partial,1} = \mathcal{B} \dfrac{\delta H_1}{\delta x_1}, \\ y_{\partial,1} = \mathcal{C} \dfrac{\delta H_1}{\delta x_1} \end{cases} \tag{30}$$

where $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u}, \boldsymbol{y} \in \mathbb{R}^m$ $x_1 \in \mathscr{X}$ and $u_{\partial,1} \in \mathscr{U}$, $y_{\partial,1} \in \mathscr{Y} = \mathscr{U}'$ belong to some Hilbert spaces (the prime denotes the dual of a space) and $\mathcal{B} : \mathscr{X} \to \mathscr{U}$, $\mathcal{C} : \mathscr{X} \to \mathscr{Y}$ are boundary operators. The duality pairings for the boundary ports are then denoted by

$$\langle u_{\partial,1},\, y_{\partial,1} \rangle_{\mathscr{U} \times \mathscr{Y}} \qquad \langle \boldsymbol{u}_2,\, \boldsymbol{y}_2 \rangle_{\mathbb{R}^m}$$

Given a compact interconnection operator $\mathcal{I} : \mathscr{Y} \to \mathbb{R}^m$ consider the following power preserving interconnection

$$\boldsymbol{u}_2 = \mathcal{I}\, y_{\partial,1} \qquad u_{\partial,1} = -\mathcal{I}^*\, \boldsymbol{y}_2 \tag{31}$$

a) Example 1: Rigid rod welded to the plate: A rigid rod can be written undergoing small displacements about the z axis and small rotation about the x axis is a port-Hamiltonian system with structure

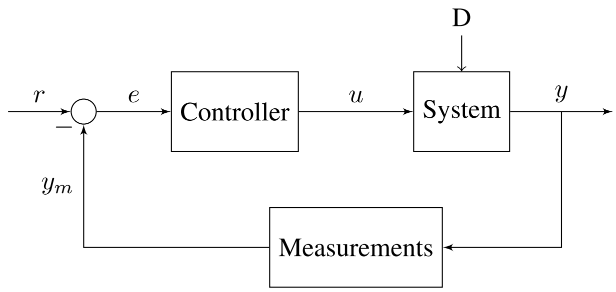
$$\begin{aligned} \dot{\boldsymbol{x}}_{\text{rod}} &= \frac{d}{dt} \begin{pmatrix} p_w \\ p_\theta \end{pmatrix} = \begin{pmatrix} F_z \\ T_x \end{pmatrix} = \boldsymbol{u}_{\text{rod}} \\ \boldsymbol{y}_{\text{rod}} &= \begin{pmatrix} v_G \\ \theta_G \end{pmatrix} = \frac{\partial H_{\text{rod}}}{\partial \boldsymbol{x}_{\text{rod}}} \end{aligned}, \tag{32}$$

with p_w, p_θ the linear and angular momentum about the center of mass, v_G, θ_G the linear and angular velocity about the center of mass G and F_z, T_x the force along z and the torque along x. The Hamiltonian reads $H_{\text{rod}} = \frac{1}{2}\left(\frac{p_w^2}{M_G} + \frac{p_\theta^2}{J_G}\right)$, with M, J_G the mass and rotary inertia about the x axis. The rod is welded to a rectangular thin plate of sides L_x, L_y on side $x = L_x$. The boundary variables for the plate involved in the interconnection are

$$u_{\partial,\text{pl}} = w_t|_{x=L_x} \qquad y_{\partial,\text{pl}} = \widetilde{q}_n|_{x=L_x}$$

The space $\mathscr{Y}$ is the space of square integrable functions on with support $\partial\Omega_{\text{int}} = \{(x,y) |\ x = L_x, 0 \leq y \leq L_y\}$ compact interconnection operator then reads

$$\mathcal{I}\widetilde{q}_n|_{x=L_x} \begin{pmatrix} \int_{y=0}^{y=L_y} \widetilde{q}\, \mathrm{d}y \\ \int_{y=0}^{y=L_y} \widetilde{q}\,(y - L_y/2)\ \mathrm{d}y \end{pmatrix} \tag{33}$$



V. Conclusion

The conclusion goes here.

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