A port-Hamiltonian formulation
for the full von-Kármán plate model

Andrea Brugnoli and Denis Matignon

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UNIVERSITY OF TWENTE.

isae SUPAERO
Institut Supérieur de l'Aéronautique et de l'Espace

Outline

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A unified language for multiphysics in engineering

The port-Hamiltonian (pH) paradigm provides a language to understand multiphysics:

- **Physics** is at the core: pH systems are **passive** with respect to the **energy storage function**.
- The **topological** and **metrical** structure of the equations is clearly separated (mimetic discretization).
- PH systems are **closed under interconnection**.

Finite dimensional pH systems

A theory still under developement

There is **not a unique definition** of pH systems, even in finite dimension.

Definition (Finite dimensional pH system)

The following time-invariant dynamical system is a pH system

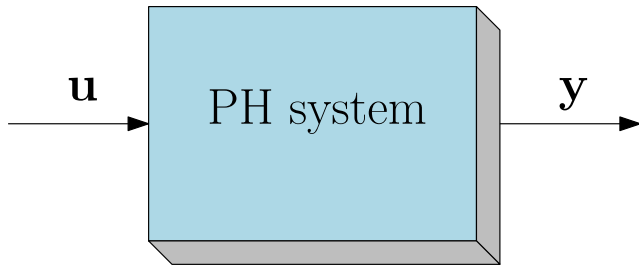$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{B}^\top \mathbf{x}.$$

$\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u}(t), \mathbf{y}(t) \in \mathbb{R}^m$ the input and output and

▶ $\mathbf{J}(\mathbf{x}) = -\mathbf{J}(\mathbf{x})^\top \in \mathbb{R}^{n \times n}$ the interconnection operator

▶ $\mathbf{B} \in \mathbb{R}^{n \times m}$ the control operator.

▶ $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{M} > 0$, the Hamiltonian.
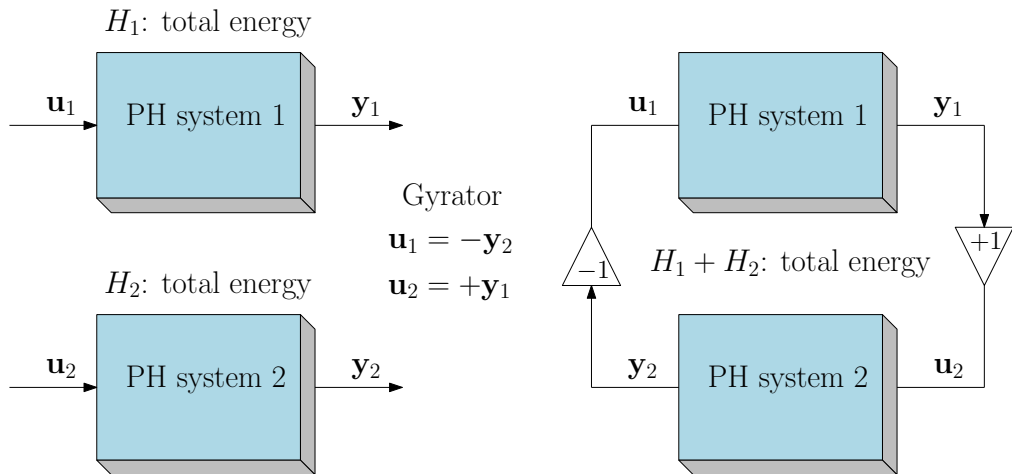
Finite dimensional pH systems

H: total energy

Lossless: $\dot{H} = \mathbf{u}^\top \mathbf{y}$

Passive: $\dot{H} \leq \mathbf{u}^\top \mathbf{y}$

Interconnection of pH systems
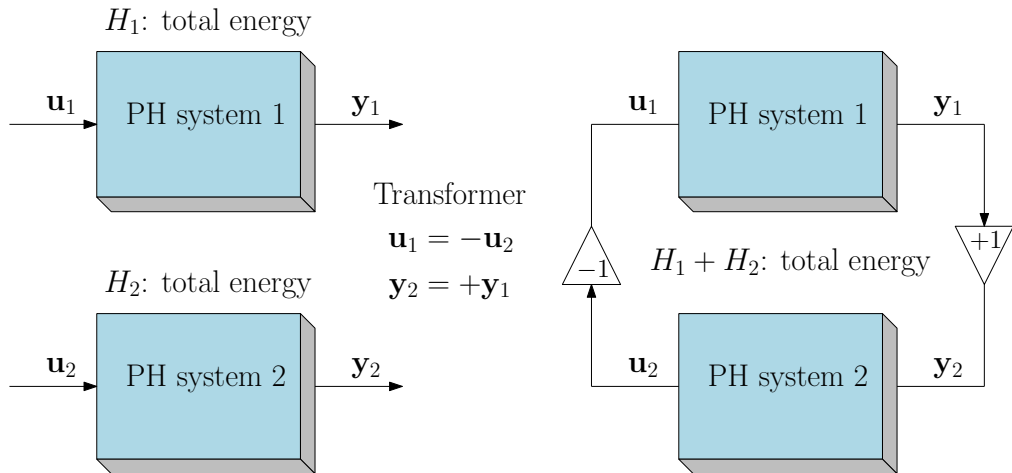
H_1: total energy

PH system 1

$\mathbf{u}_1$ $\mathbf{y}_1$

H_2: total energy

PH system 2

$\mathbf{u}_2$ $\mathbf{y}_2$

Gyrator

$\mathbf{u}_1 = -\mathbf{y}_2$

$\mathbf{u}_2 = +\mathbf{y}_1$

$\mathbf{u}_1$ PH system 1 $\mathbf{y}_1$

-1

$H_1 + H_2$: total energy

$+1$

$\mathbf{y}_2$ PH system 2 $\mathbf{u}_2$

Interconnection of pH systems

H_1: total energy

$\mathbf{u}_1 \rightarrow$ PH system 1 $\rightarrow \mathbf{y}_1$

H_2: total energy

$\mathbf{u}_2 \rightarrow$ PH system 2 $\rightarrow \mathbf{y}_2$

Transformer
$$\mathbf{u}_1 = -\mathbf{u}_2$$
$$\mathbf{y}_2 = +\mathbf{y}_1$$

$\mathbf{u}_1 \rightarrow$ PH system 1 $\rightarrow \mathbf{y}_1$

$H_1 + H_2$: total energy

$\mathbf{u}_2 \rightarrow$ PH system 2 $\rightarrow \mathbf{y}_2$

$\boxed{-1}$ $\boxed{+1}$

Outline

The von-Kármán assumption

Second-order approximation of geometrically exact beam/plate theory **capturing the axial bending coupling.**
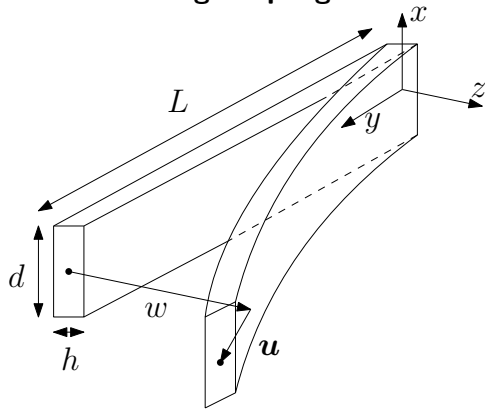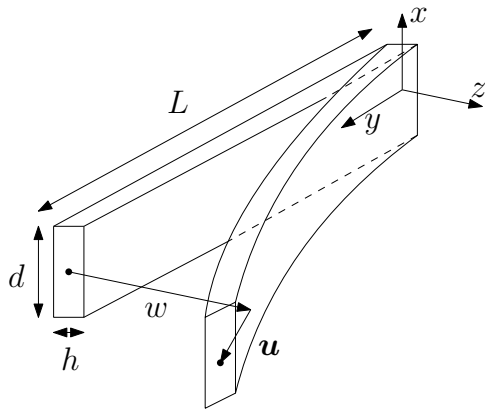

Basic geometric assumptions

▶ Out of plane deflection comparable to the thickness: $w/h = \mathcal{O}(1)$.

▶ The squares of the in-plane stretching terms are negligible compared to the square of the rotations.

The von-Kármán assumption

Second-order approximation of geometrically exact beam/plate theory **capturing the axial bending coupling.**

Geometrical non-linearities allow describing bifurcations (i.e. buckling).

Linear isotropic plates ($\Omega \subset \mathbb{R}^2$)

The axial and bending behavior are uncoupled if $w/h \ll 1$:

Membrane displacement (2D elastodynamics)

$$\rho h \partial_{tt} \boldsymbol{u} = \operatorname{Div} \boldsymbol{N},$$
$$\boldsymbol{N} = D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m),$$
$$\boldsymbol{\varepsilon}_m = \operatorname{Sym}(\nabla \boldsymbol{u}) = \operatorname{Grad} \boldsymbol{u}$$

Total membrane energy

$$H_m = \frac{1}{2} \int_\Omega \rho h ||\partial_t \boldsymbol{u}||^2 + D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m) : \boldsymbol{N} \; \mathrm{d}\Omega.$$

Vertical displacement (Kirchhoff plate)

$$\rho h \partial_{tt} w = - \operatorname{div} \operatorname{Div} \boldsymbol{M},$$
$$\boldsymbol{M} = D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}),$$
$$\boldsymbol{\kappa} = \operatorname{Hess} w = \operatorname{Grad} \operatorname{grad} w.$$

Total bending energy

$$H_b = \frac{1}{2} \int_\Omega \rho h (\partial_t w)^2 + D_b \boldsymbol{\Phi}(\boldsymbol{\kappa}) : \boldsymbol{M} \; \mathrm{d}\Omega.$$

The linear mapping $\boldsymbol{\Phi}(\boldsymbol{A}) = \nu \operatorname{Tr}(\boldsymbol{A})\boldsymbol{1} + (1 - \nu)\boldsymbol{A}$ is positive and preserves symmetry.

Von-Kármán plates

Decomposition strain field

$$\boldsymbol{\varepsilon} = \boxed{\text{Grad}\,\boldsymbol{u}} + \boxed{1/2\,\text{grad}\,w \otimes \text{grad}\,w} - z\,\boxed{\text{Hess}\,w} = \boldsymbol{\varepsilon}_m - z\boldsymbol{\kappa}.$$

Linear membrane def.

Quadratic membrane def.

Linear bending def.

Von-Kármán plate Dynamics

$$\rho h \partial_{tt} u = \text{Div}\ \boldsymbol{N},$$
$$\rho h \partial_{tt} w = -\text{div}\,\text{Div}\,\boldsymbol{M} + \text{div}(\boldsymbol{N}\,\text{grad}\,w),$$

Total energy

$$H = \frac{1}{2}\int_\Omega \rho h\{||\partial_t \boldsymbol{u}||^2 + ||\partial_t w||^2\} + D_m \boldsymbol{\Phi}(\boldsymbol{\varepsilon}_m) : \boldsymbol{N} + D_b \boldsymbol{\Phi}(\kappa) : \boldsymbol{M}\ \mathrm{d}\Omega$$

Port-Hamiltonian Von-Kármán plates

Energy variables

The Hamiltonian functional is quadratic in the following variables

$\boldsymbol{\alpha}_u = \rho h \partial_t \boldsymbol{u}$,   Axial momentum, $\alpha_w = \rho h \partial_t w$,   Bending momentum,

$\boldsymbol{A}_\varepsilon = \boldsymbol{\varepsilon}_m$,   Membrane strain, $\boldsymbol{A}_\kappa = \boldsymbol{\kappa}$,   Curvature

Co-energy variables

The variational derivative of the Hamiltonian gives the co-energy variables

$$\boldsymbol{e}_u := \delta_{\boldsymbol{\alpha}_u} H = \dot{\boldsymbol{u}}, \qquad\qquad e_w := \delta_{\alpha_w} H = \dot{w},$$

$$\boldsymbol{E}_\varepsilon := \delta_{\boldsymbol{A}_\varepsilon} H = D_m \boldsymbol{\Phi}(\boldsymbol{A}_\varepsilon), \qquad \boldsymbol{E}_\kappa := \delta_{\boldsymbol{A}_\kappa} H = D_b \boldsymbol{\Phi}(\boldsymbol{A}_\kappa)$$

or more compactly $\boldsymbol{e} := \delta_\alpha H = \mathcal{Q} \boldsymbol{\alpha}$ with

$$\mathcal{Q} = \mathrm{Diag} \left[(\rho h)^{-1}, \ D_m \boldsymbol{\Phi}, \ (\rho h)^{-1}, \ D_b \boldsymbol{\Phi} \right].$$

The port-Hamiltonian realization

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\alpha}_u \\ \boldsymbol{A}_\varepsilon \\ \alpha_w \\ \boldsymbol{A}_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ \mathbf{0} & \mathcal{C}(w) & \mathbf{0} & -\mathrm{div\,Div} \\ \mathbf{0} & \mathbf{0} & \mathrm{Grad\,grad} & \mathbf{0} \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \delta_{\boldsymbol{\alpha}_u} H \\ \delta_{\boldsymbol{A}_\varepsilon} H \\ \delta_{\alpha_w} H \\ \delta_{\boldsymbol{A}_\kappa} H \end{pmatrix}.$$

The operator $\mathcal{J}$ is formally skew-adjoint.

The coupling term reads

$$\mathcal{C}(w)(\cdot) : L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathsf{sym}}) \to L^2(\Omega),$$
$$\boldsymbol{X} \to \mathrm{div}(\boldsymbol{X}\,\mathrm{grad}\,w),$$

and its formal adjoint

$$\mathcal{C}(w)^*(\cdot) : L^2(\Omega) \to L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathsf{sym}}),$$
$$y \to -\mathrm{Sym}\left[\mathrm{grad}(y) \otimes \mathrm{grad}(w)\right].$$

Pure coenergy formulation

Incorporation of the constitutive equations

Once the $\mathcal{Q}$ operator (matrix) is inverted, the dynamics is expressed :

$$\begin{pmatrix} \rho h \partial_t \boldsymbol{e}_u \\ (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon \\ \rho h \partial_t e_w \\ (D_b \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & \mathcal{C}(w) & 0 & -\mathrm{div}\,\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathrm{Grad}\,\mathrm{grad} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_u \\ \boldsymbol{E}_\varepsilon \\ e_w \\ \boldsymbol{E}_\kappa \end{pmatrix},$$

where the vertical position is given by

$$w(t) = w_0 + \int_0^t e_w(\tau)\,\mathrm{d}\tau.$$

Energy rate and boundary conditions

Proposition

The energy rate reads

$$\dot{H} = \langle \gamma_0 \boldsymbol{e}_u \,|\, \gamma_\perp \boldsymbol{E}_\varepsilon \rangle_{\partial\Omega} + \langle \gamma_0 \boldsymbol{e}_w \,|\, \gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0 (\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w) \rangle_{\partial\Omega} + \langle \gamma_1 \boldsymbol{e}_w \,|\, \gamma_{\perp\perp} \boldsymbol{E}_\kappa \rangle_{\partial\Omega},$$

▶ $\gamma_0 \boldsymbol{e}_u = \boldsymbol{e}_u|_{\partial\Omega}$ *is the Dirichlet trace;*

▶ $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{E}_\varepsilon \boldsymbol{n}|_{\partial\Omega}$ *is the normal trace;*

▶ $\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa = -\boldsymbol{n} \cdot \operatorname{Div} \boldsymbol{E}_\kappa - \partial_{\boldsymbol{s}}(\boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{s})|_{\partial\Omega}$ *is the effective shear force;*

▶ $\gamma_1 \boldsymbol{e}_w = \partial_{\boldsymbol{n}} \boldsymbol{e}_w|_{\partial\Omega}$ *is the normal derivative trace;*

▶ $\gamma_{\perp\perp} \boldsymbol{E}_\kappa = \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n}$ *is the normal to normal trace.*

Boundary conditions classification

BCs	Traction	Bending	
Kinematical/Dirichlet	$\gamma_0 \boldsymbol{e}_u$	$\gamma_0 \boldsymbol{e}_w$	$\gamma_1 \boldsymbol{e}_w$
Dynamical/Neumann	$\gamma_\perp \boldsymbol{E}_\varepsilon$	$\gamma_{\perp\perp,1} \boldsymbol{E}_\kappa + \gamma_0 (\boldsymbol{E}_\varepsilon \boldsymbol{n} \cdot \operatorname{grad} w)$	$\gamma_{\perp\perp} \boldsymbol{E}_\kappa$

Outline
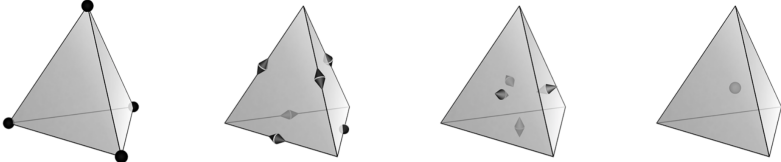
Hilbert complexes and stable discretization of PDEs

Crucial concept to derive stable convergent approximations: **Hilbert complexes**.

$$H^1(\Omega) \xrightarrow{\text{grad}} H^{\text{curl}}(\Omega) \xrightarrow{\text{curl}} H^{\text{div}}(\Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

$$\Big\downarrow \Pi_{s,h}^{-,0} \qquad \Big\downarrow \Pi_{s,h}^{-,1} \qquad \Big\downarrow \Pi_{s,h}^{-,2} \qquad \Big\downarrow \Pi_{s,h}^{-,1}$$

$$\text{CG}_s(\Omega_h) \xrightarrow{\text{grad}} \text{NED}_s^1(\Omega_h) \xrightarrow{\text{curl}} \text{RT}_s(\Omega_h) \xrightarrow{\text{div}} \text{DG}_{s-1}(\Omega_h)$$

The Whitney forms $s = 0$ form a subcomplex of the de Rham complex

This framework is well developed for linear problems[1] and open source finite elements libraries implement them. Definitely less for non linear elasticity problems.

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[1] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther (2006). "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numerica* 15, pp. 1–155.

Weak formulation for 2D Elastodynamics

Grad primal formulation

Suppose the traction (Neumann) boundary condition $\gamma_\perp \boldsymbol{E}_\varepsilon = \boldsymbol{g}_N$.
Find $\boldsymbol{e}_u \in H^1(\Omega; \mathbb{R}^2)$, $\boldsymbol{E}_\varepsilon \in H(\operatorname{rot}\operatorname{rot}, \Omega; \mathbb{R}^{2\times2}_{\mathsf{sym}})$
such that $\forall \boldsymbol{\psi}_u \in H^1(\Omega; \mathbb{R}^2)$, $\forall \boldsymbol{\Psi}_\varepsilon \in H(\operatorname{rot}\operatorname{rot}, \Omega; \mathbb{R}^{2\times2}_{\mathsf{sym}})$ the following holds

$$(\boldsymbol{\psi}_u, \, \rho h \partial_t \boldsymbol{e}_u)_\Omega = -(\operatorname{Grad}\boldsymbol{\psi}_u, \, \boldsymbol{e}_u)_\Omega + \langle \gamma_0 \boldsymbol{\psi}_u \, | \boldsymbol{g}_N \rangle_{\partial\Omega},$$

$$(\boldsymbol{\Psi}_\varepsilon, \, (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon)_\Omega = (\boldsymbol{\Psi}_\varepsilon, \, \operatorname{Grad}\boldsymbol{e}_u)_\Omega.$$

This formulation is related to the strain elasticity complex in 2D

$$H^1(\Omega; \mathbb{R}^2) \xrightarrow{\operatorname{Grad}} H(\operatorname{rot}\operatorname{rot}, \Omega; \mathbb{R}^{2\times2}_{\mathsf{sym}}) \xrightarrow{\operatorname{rot}\operatorname{rot}} L^2(\Omega).$$

Weak formulation for 2D Elastodynamics

> ### Div dual formulation
>
> Suppose the kinematic (Dirichlet) boundary condition $\gamma_0 e_u = g_D$.
> Find $e_u \in L^2(\Omega; \mathbb{R}^2)$, $\boldsymbol{E}_\varepsilon \in H(\text{Div}, \Omega; \mathbb{R}^{2 \times 2}_{\text{sym}})$
> such that $\forall \boldsymbol{\psi}_u \in L^2(\Omega; \mathbb{R}^2)$, $\forall \boldsymbol{\Psi}_\varepsilon \in H(\text{Div}, \Omega; \mathbb{R}^{2 \times 2}_{\text{sym}})$ the following holds
>
> $$(\boldsymbol{\psi}_u, \rho h \partial_t e_u)_\Omega = (\boldsymbol{\psi}_u, \text{Div}\,\boldsymbol{E}_\varepsilon)_\Omega,$$
>
> $$(\boldsymbol{\Psi}_\varepsilon, (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon)_\Omega = -(\text{Div}\,\boldsymbol{\Psi}_\varepsilon, e_u)_\Omega + \langle \gamma_\perp \boldsymbol{E}_\varepsilon \,|\, g_D \rangle_{\partial\Omega}.$$
>
> This formulation is related to the Elasticity complex in 2D
>
> $$H^2(\Omega) \xrightarrow{\text{curl curl}} H(\text{Div}, \Omega; \mathbb{S}) \xrightarrow{\text{Div}} L^2(\Omega; \mathbb{R}^2).$$

The theory is not directly applicable to this problem

Semi discretization in space is obtained using finite elements forming a **subcomplex of original Hilbert complex**. But the theory is still under development[2] and no open source software.

So heuristic choice of the finite elements.

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[2]Long Chen and Xuehai Huang (2022). "Finite Element Complexes in Two Dimensions". In: *arXiv preprint arXiv:2206.00851*.

Heuristic Finite element choice and final system

For the proposed weak formulation, the following FE spaces are selected

$$e_u^h \in \mathrm{CG}_{2s-1}, \qquad \boldsymbol{E}_\varepsilon^h \in \mathrm{DG}_{2s-2}, \qquad e_w^h \in \mathrm{CG}_s, \qquad \boldsymbol{E}_\kappa^h \in \mathrm{HHJ}_{s-1}, \quad s \geq 1.$$

Combination of spectral FE for the membrane part and the non conforming Hellan-Hermann-Johnson method for bending.

$$(\boldsymbol{\psi}_u^h, \, \rho h \partial_t \boldsymbol{e}_u^h)_\Omega = -(\mathrm{Grad}\, \boldsymbol{\psi}_u^h, \, \boldsymbol{e}_u^h)_\Omega,$$

$$(\boldsymbol{\Psi}_\varepsilon^h, \, (D_m \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\varepsilon^h)_\Omega = (\boldsymbol{\Psi}_\varepsilon^h, \, \mathrm{Grad}\, \boldsymbol{e}_u^h)_\Omega + (\boldsymbol{\Psi}_\varepsilon^h, \, \mathrm{Sym}\{\mathrm{grad}\, e_w^h \otimes \mathrm{grad}\, w^h\})_\Omega,$$

$$(\psi_w^h, \, \rho b \partial_t e_w^h)_\Omega = +d_h(\psi_w^h, \, \boldsymbol{E}_\kappa^h) - (\mathrm{Sym}\{\mathrm{grad}\, \psi_w^h \otimes \mathrm{grad}\, w^h\}, \, \boldsymbol{E}_\varepsilon^h)_\Omega,$$

$$(\boldsymbol{\Psi}_\kappa^h, \, (D_b \boldsymbol{\Phi})^{-1} \partial_t \boldsymbol{E}_\kappa^h)_\Omega = -d_h(e_w^h, \, \boldsymbol{\Psi}_\kappa^h).$$

> ### Finite dimensional system (Galerkin projection)
>
> $$\mathbf{M}\dot{\mathbf{e}} = \mathbf{J}(\mathbf{w})\mathbf{e} + \mathbf{B}\mathbf{u},$$
> $$\dot{\mathbf{w}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{e},$$
> $$\mathbf{y} = \mathbf{B}^\top \mathbf{e}.$$

Outline

Manufactured solution

Consider $\Omega = [0,1] \times [0,1]$. The following manufactured solution is considered

$$\boldsymbol{u}^{\text{ex}} = \begin{pmatrix} x^4(1-x^4)\sin^2(\pi y) \\ \sin^2(\pi x)y^4(1-y^4) \end{pmatrix} \sin(2\pi t), \qquad w^{\text{ex}} = \sin(\pi x)\sin(\pi y)\sin(2\pi t),$$

together with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad w|_{\partial\Omega} = 0, \quad \boldsymbol{n}^\top \boldsymbol{M} \boldsymbol{n}|_{\partial\Omega} = 0.$$

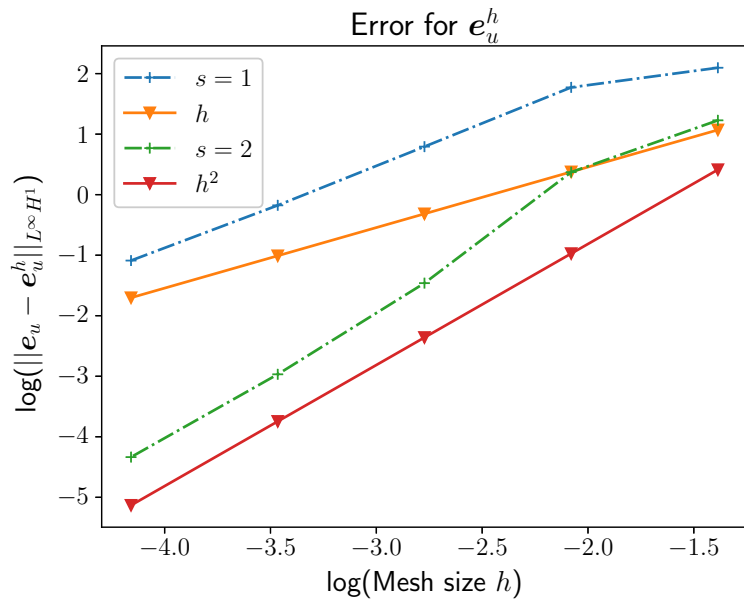A Crank-Nicholson scheme is used for time integration.

Convergence measure

The discrete time-space norm $L^\infty_{\Delta t}(\mathcal{X})(\mathcal{X} = H^1 \text{or } L^2)$ is used to measure convergence
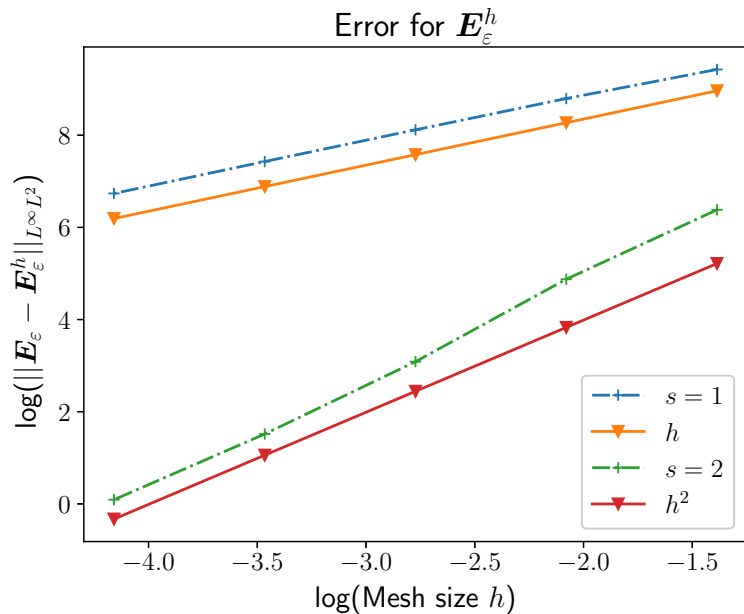
$$||\cdot||_{L^\infty(\mathcal{X})} \approx ||\cdot||_{L^\infty_{\Delta t}(\mathcal{X})} = \max_{t\in t_i}||\cdot||_{\mathcal{X}},$$
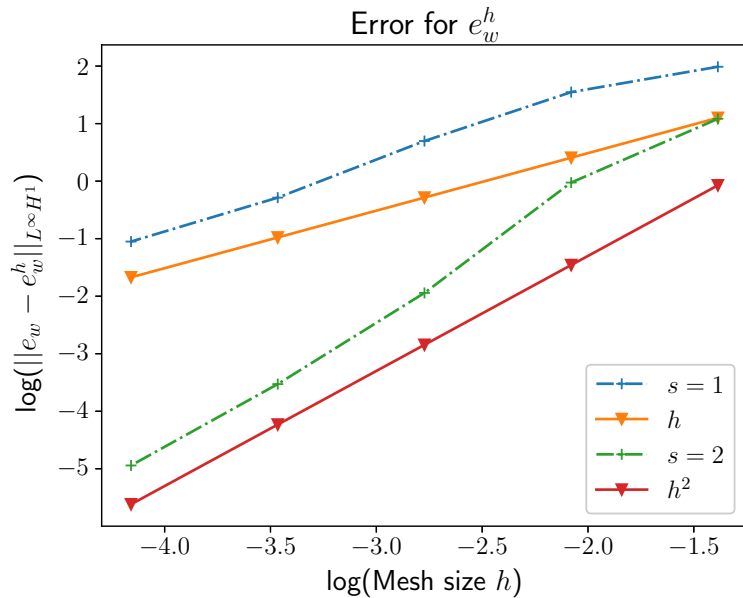
where t_i are the discrete simulation instants.

Results

Results

Error for $\boldsymbol{E}_\varepsilon^h$

Results

Error for e_w^h

Results

Error for $\boldsymbol{E}_\kappa^h$

x-axis: $\log(\text{Mesh size } h)$

y-axis: $\log(\|\boldsymbol{E}_\kappa - \boldsymbol{E}_\kappa^h\|_{L^\infty L^2})$

Legend: $s = 1$, h, $s = 2$, h^2
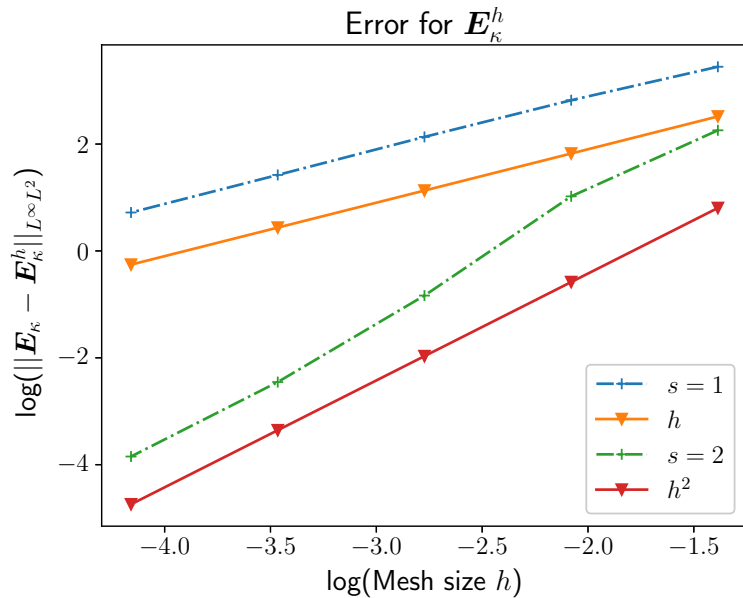
Conclusion and Outlook

▶ First step into pH non linear mechanics. The **geometrical non linearities** belong to the **interconnection operator**.

▶ Can be used to study more complex phenomena. The discretization method guarantees **exact discrete energy conservation** when **symplectic time integration** is used.

▶ **System theory perspective**: incorporates **interactions** with the environment by means of the **boundary conditions**.

References I

Arnold, Douglas N., Richard S. Falk, and Ragnar Winther (2006). "Finite element exterior calculus, homological techniques, and applications". In: *Acta Numerica* 15, pp. 1–155.

Chen, Long and Xuehai Huang (2022). "Finite Element Complexes in Two Dimensions". In: *arXiv preprint arXiv:2206.00851*.

Mixed finite element construction: Kirchhoff plate

Find $(e_w, \boldsymbol{E}_\kappa) \in \mathrm{CG}_s \times \mathrm{HHJ}_{s-1}$ such that

$$(\psi_w,\ \rho b \partial_t e_w)_\Omega = +d_h(\psi_w,\ \boldsymbol{E}_\kappa), \qquad \forall\ \psi_w \in \mathrm{CG}_s,$$

$$(\boldsymbol{\Psi}_\kappa,\ (D_b\boldsymbol{\Phi})^{-1}\partial_t\boldsymbol{E}_\kappa)_\Omega = -d_h(e_w,\ \boldsymbol{\Psi}_\kappa), \qquad \forall\ \boldsymbol{\Psi}_\kappa \in \mathrm{HHJ}_{s-1}.$$

This method is non-conforming and includes inter-cells terms

$$d_h(v_w,\ \boldsymbol{E}_\kappa) := -\sum_{T \in \mathcal{T}_h} (\mathrm{Hess}\, v_w,\ \boldsymbol{E}_\kappa)_\Omega + \sum_{E \in \mathcal{E}_h} ([\![\partial_n v_w]\!],\ \boldsymbol{n}^\top \boldsymbol{E}_\kappa \boldsymbol{n})_\Omega.$$

The $\mathrm{div}\,\mathrm{Div}$ distributional complex in 2D and the HHJ finite element complex:

$$
\begin{array}{ccccc}
H^1(\Omega; \mathbb{R}^2) & \xrightarrow{\mathrm{Sym\,curl}} & H^{-1}(\mathrm{div}\,\mathrm{Div}, \Omega; \mathbb{R}^{2\times 2}_{\mathsf{sym}}) & \xrightarrow{\mathrm{div}\,\mathrm{Div}} & L^2(\Omega) \\
\downarrow{\scriptstyle \boldsymbol{I}_h} & & \downarrow{\scriptstyle \boldsymbol{\Pi}_h} & & \downarrow{\scriptstyle Q_h} \\
\mathrm{CG}_s & \xrightarrow{\mathrm{Sym\,curl}} & \mathrm{HHJ}_{s-1} & \xrightarrow{\mathrm{div}\,\mathrm{Div}_h} & \mathrm{CG}_s,
\end{array}
$$