Mixed finite elements for port-Hamiltonian von Kármán beams

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Abstract:

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1. INTRODUCTION

2. VON KÁRMÁN BEAMS

The classical von-Kármán beam model is presented in (Reddy, 2010, Chapter 4). Under the hypothesis of isotropic material, the extensional-bending stiffness is zero when the x-axis is taken along the geometric centroidal axis. With this assumption, the problem, defined on an open interval $\Omega = (0, L)$, takes the following form

$$\begin{aligned} \rho A \ddot{u} &= \partial_x n_{xx}, \\ \rho A \ddot{w} &= -\partial_{xx}^2 m_{xx} + \partial_x (n_{xx} \partial_x w), \end{aligned} \tag{1}$$

together with the stresses and strains expressions

$$\begin{aligned} n_{xx} &= EA\varepsilon_{xx}, \quad \varepsilon_{xx} = \partial_x u + 1/2(\partial_x w)^2, \\ m_{xx} &= EI\kappa_{xx}, \quad \kappa_{xx} = \partial_{xx}^2 w. \end{aligned} \tag{2}$$

Variable u is the horizontal displacement, w is the vertical displacement, n_{xx} is the axial stress resultant and m_{xx} is the bending stress resultant. The coefficients ρ, A, E, I are the mass density, the cross section, the Young module and the second moment of area.

The total energy of the model (Hamiltonian functional)

$$H = \frac{1}{2}\int_\Omega \left\{\rho A \dot{u}^2 + \rho A \dot{w}^2 + n_{xx}\varepsilon_{xx} + m_{xx}\kappa_{xx}\right\} \, \mathrm{d}\Omega, \tag{3}$$

consists of the kinetic energy and both membrane and bending deformation energies. This model proves conservative, see Bilbao et al. (2015). Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

3. THE EQUIVALENT PORT-HAMILTONIAN REALIZATION

To find a suitable port-Hamiltonian system, we first select a set of new energy variables to make the Hamiltonian functional quadratic

$$\alpha_u = \rho A \dot{u}, \quad \alpha_\varepsilon = \varepsilon_{xx}, \quad \alpha_w = \rho A \dot{w}, \quad \alpha_\kappa = \kappa_{xx}. \tag{4}$$

The energy is quadratic in these variables

$$H = \frac{1}{2}\int_\Omega \left\{\frac{\alpha_u^2}{\rho A} + \frac{\alpha_w^2}{\rho A} + EA\varepsilon_{xx}^2 + EI\kappa_{xx}^2\right\} \, \mathrm{d}\Omega. \tag{5}$$

By computing the variational derivative of the Hamiltonian, one obtains the so-called co-energy variables:

$$\begin{aligned} e_u &:= \delta_{\alpha_u} H = \dot{u}, \qquad & e_\varepsilon &:= \delta_{\alpha_\varepsilon} H = n_{xx}, \\ e_w &:= \delta_{\alpha_w} H = \dot{w}, \qquad & e_\kappa &:= \delta_{\alpha_\kappa} H = m_{xx}. \end{aligned} \tag{6}$$

Before stating the final formulation, consider the unbounded operator operator $\mathcal{C}(w)(\cdot) : L^2(\Omega) \to L^2(\Omega)$, that acts as follows

$$\mathcal{C}(w)(n_{xx}) = \partial_x(n_{xx}\partial_x w). \tag{7}$$

Proposition 1. The formal adjoint of the $\mathcal{C}(w)(\cdot)$ is given by

$$\mathcal{C}(w)^*(\cdot) = -\partial_x(\cdot)\partial_x(w). \tag{8}$$

Proof 1. Consider a smooth scalar field $\psi \in C_0^\infty(\Omega)$ and a smooth scalar field $\xi \in C_0^\infty(\Omega)$ with compact support. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation

$$\langle \psi, \mathcal{C}(w)(\xi)\rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)(\psi)^*, \xi\rangle_{L^2(\Omega)}. \tag{9}$$

The proof follows from the computation

$$\begin{aligned} \langle \psi, \mathcal{C}(w)(\xi)\rangle_{L^2(\Omega)} &= \langle \psi, \partial_x(\xi\, \partial_x w)\rangle_{L^2(\Omega)}, \\ &= \langle -\partial_x \psi, \xi \partial_x w\rangle_{L^2(\Omega, \mathbb{R}^2)}, \\ &= \langle -\partial_x \psi\, \partial_x w, \xi\rangle_{L^2(\Omega)}. \end{aligned} \tag{10}$$

This means that

$$\mathcal{C}(w)^*(\cdot) = -\partial_x(\cdot)\partial_x w, \tag{11}$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_u \\ \alpha_\varepsilon \\ \alpha_w \\ \alpha_\kappa \end{pmatrix} = \begin{bmatrix} 0 & \partial_x & 0 & 0 \\ \partial_x & 0 & \partial_x w\, \partial_x & 0 \\ 0 & \partial_x(\cdot\, \partial_x w) & 0 & -\partial_{xx}^2 \\ 0 & 0 & \partial_{xx}^2 & 0 \end{bmatrix} \begin{pmatrix} \delta_{\alpha_u} H \\ \delta_{\alpha_\varepsilon} H \\ \delta_{\alpha_w} H \\ \delta_{\alpha_\kappa} H \end{pmatrix}, \tag{12}$$

The second line of system (12) represents the time derivative of the membrane strain tensor. To close the system, variable w has to be accessible. For this reason, its dynamics has to be included. The augmented system reads

$$\frac{\partial}{\partial t}\begin{pmatrix} \alpha_u \\ \alpha_\varepsilon \\ w \\ \alpha_w \\ \alpha_\kappa \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \partial_x & 0 & 0 & 0 \\ \partial_x & 0 & 0 & \partial_x w\, \partial_x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \partial_x(\cdot\, \partial_x w) & -1 & 0 & -\partial_{xx}^2 \\ 0 & 0 & 0 & \partial_{xx}^2 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \delta_{\alpha_u} H \\ \delta_{\alpha_\varepsilon} H \\ \delta_w H \\ \delta_{\alpha_w} H \\ \delta_{\alpha_\kappa} H \end{pmatrix}. \tag{13}$$

The operator $\mathcal{J}$ is formally skew-adjoint. If only the kinetic and deformation energies are considered, it holds $\delta_w H =$

0. In general this terms allows accommodating other potentials, for example the gravitational one. Suitable boundary variables are then obtained considering the power balance

$$\dot{H} = \langle e_u,\, e_\varepsilon \rangle_{\partial\Omega} + \langle e_w,\, -\partial_x e_\kappa + e_\varepsilon \partial_x w \rangle_{\partial\Omega} + \langle \partial_x e_w,\, e_\kappa \rangle_{\partial\Omega}\,, \tag{14}$$

The boundary conditions are consistent with the ones assumed in Puel and Tucsnak (1996) for deriving a global existence result for this model.

4. CONCLUSION

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