PH formulation for thin and thick plates

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Plan

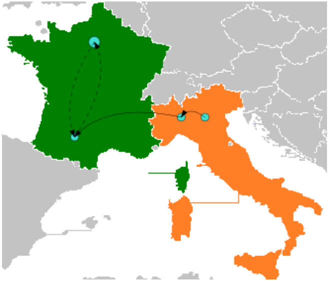
1. Me and my thesis
2. The 1D case: Euler-Bernoulli and Timoshenko beams
3. Kirchhoff-Love theory
4. PH formulation of the Kirchhoff plate
    - Underlying Stokes-Dirac structure
5. Mindlin theory for thick plates
6. PH formulation of the Mindlin plate (Macchelli et al. 2005)
    - Underlying Stokes-Dirac structure

Plan

Me

Formation:

- High school diploma in Humanities, **Vérone**;
- Bachelor in Mechanical Engineering, Politecnico di Milano, **Milan**;
- Master of science in Space Engineering, Politecnico di Milano, **Milan**;
- Double Degree Politecnico/Isae-SUPAERO, **Toulouse**;
- Research Master in Automatic Control, Université Paris Saclay/ Supélec, **Paris/Toulouse**;



PHD title and purposes

PHD title

Modeling and control by the Port-Hamiltonian formalism of 2D flexible structures with varying boundary conditions.

Supervisors

Daniel Alazard Valerie Budinger Denis Matignon

Fundings

This work is funded by ISAE-SUPAERO.

Objectives

Patran/Nastran cannot model flexible structures with a priori unknown boundary conditions. The PH framework can overcome this problem, making possible the analysis of system even when boundary conditions are unknown. Moreover, we aim to examine performance specifications in the PH formalism.

Plan

The corresponding 1D models

Euler-Bernoulli beam

- Valid for thin beams
- Dimension of the PH model: 2
- Differential operator J of order 2

Timoshenko beam

- Valid for thick beams
- Dimension of the PH model: 4
- Differential operator J of order 1

$$
\boldsymbol{\alpha} = [\rho v, \ \frac{\partial^2 w}{\partial x^2}]^T
$$

$$
\boldsymbol{e} = [v, \ M_{xx}]^T
$$

$$
J = \begin{pmatrix} 0 & -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}
$$

$$
\boldsymbol{\alpha} = [\rho v, \ I_\rho \omega_x, \ \frac{\partial w}{\partial x} - \phi_x, \ \frac{\partial \phi_x}{\partial x}]^T
$$

$$
\boldsymbol{e} = [v, \ \omega_x, \ T_x, \ M_{xx}]^T
$$

$$
J = \begin{pmatrix} 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 1 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -1 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 \end{pmatrix}
$$

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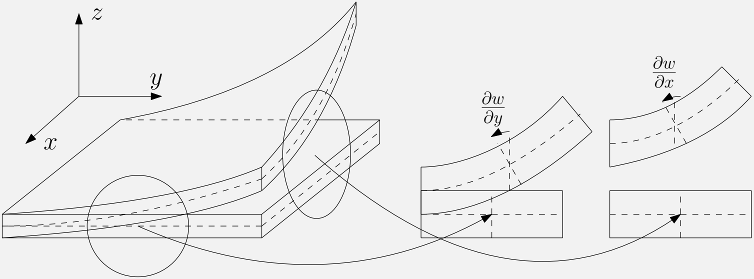
Model Hypothesis

Plane cross sections remains plane and normal to the middle-plane during deformations.



Consequently the displacement field assumes the following expression

$$u(x, y, z) = -z\frac{\partial w}{\partial x} \qquad v(x, y, z) = -z\frac{\partial w}{\partial y} \qquad w(x, y, z) = w(x, y)$$

Constitutive relations

Plane strain condition

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} -z\,\frac{\partial w}{\partial x} \\ -z\,\frac{\partial w}{\partial y} \end{pmatrix} = -z \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\,\frac{\partial^2 w}{\partial x\,\partial y} \end{pmatrix} = -z \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{pmatrix} = -z\boldsymbol{\kappa}$$

Generalized momenta

$$\boldsymbol{M} = \begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} -z\boldsymbol{\sigma}\,dz\boldsymbol{\kappa} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Ez^2\,dz\boldsymbol{\kappa} = \boldsymbol{D}\boldsymbol{\kappa}$$

$$\boldsymbol{D} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Ez^2\,dz = D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

where $D = \frac{Eh^3}{12(1-\nu)}$ is the flexural stiffness and ν is the Poisson ratio.

Variational description

Constitutive relations (physical parameters, namely D, ν do not depend on z but may depend on x, y)

$$
\begin{aligned}
\kappa_{xx} &= \frac{\partial^2 w}{\partial x^2} \qquad & M_{xx} &= D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \\
\kappa_{yy} &= \frac{\partial^2 w}{\partial y^2} \qquad & M_{yy} &= D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \\
\kappa_{xy} &= 2\frac{\partial^2 w}{\partial x \, \partial y} \qquad & M_{xy} &= D(1-\nu)\frac{\partial^2 w}{\partial x \, \partial y}
\end{aligned}
$$

The Euler-Lagrange equations arise form the following extrema problem (where $\mathcal{V}$ is a suitable functional space):

$$
w_0 = \{w \in \mathcal{V};\; \delta F(w)|_{w_0} = 0\}
$$

$$
F(w) = \int_0^T \int_\Omega \{\mathcal{K} - \mathcal{U} + \mathcal{W}\}\, d\Omega \, dt
$$

Euler Lagrange equations

Work and energy densities defined by:

$$\mathcal{W} = pw \qquad \text{Work of external forces}$$

$$\mathcal{K} = \frac{1}{2}\mu\left(\frac{\partial w}{\partial t}\right)^2 \qquad \text{Kinetic energy}$$

$$\mathcal{U} = \frac{1}{2}\boldsymbol{\kappa}^T \boldsymbol{D}\boldsymbol{\kappa} \qquad \text{Deformation energy}$$

Euler-Lagrange equation for the Kirchhoff plate

$$\mu\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = p$$

In the homogeneous case $(D, \nu$ constant) the ruling equation is

$$\mu\frac{\partial^2 w}{\partial t^2} + D\Delta^2 w = p$$

In the sequel it is assumed $p = 0$.

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Energy and co-energy variables

This model is the 2D extension of the Bernoulli beam. It is logical to select as energy variable the linear momentum, together with the curvatures

$$\boldsymbol{\alpha} = (\mu v,\ \kappa_{xx},\ \kappa_{yy},\ \kappa_{xy})^T$$

where $v = \frac{\partial w}{\partial t}$. The Hamiltonian density is given by

$$\mathcal{H} = \mathcal{K} + \mathcal{U} = \frac{1}{2}\boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\mu} & 0 \\ 0 & \boldsymbol{D} \end{bmatrix} \boldsymbol{\alpha}$$

So the variational derivative of the total Hamiltonian $H = \int_\Omega \mathcal{H}\ d\Omega$ provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v,\ M_{xx},\ M_{yy},\ M_{xy})^T$$

Definition of J and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ 2\frac{\partial^2}{\partial x \partial y} & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \boldsymbol{\alpha}}{\partial t} = J\mathbf{e}$$

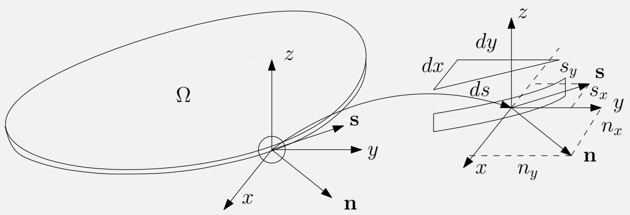
The boundary variables are obtained by evaluating the time derivative of the Hamiltonian

$$\dot{H} = \int_\Omega \frac{\delta H}{\delta \boldsymbol{\alpha}} \cdot \frac{\partial \boldsymbol{\alpha}}{\partial t} \, d\Omega = \int_\Omega \left\{ e_1 \left(-\frac{\partial^2 e_2}{\partial x^2} - \frac{\partial^2 e_3}{\partial y^2} - 2\frac{\partial^2 e_4}{\partial x \partial y} \right) \right.$$
$$\left. + e_2 \frac{\partial^2 e_1}{\partial x^2} + e_3 \frac{\partial^2 e_1}{\partial y^2} + 2 e_4 \frac{\partial^2 e_1}{\partial x \partial y} \right\} d\Omega$$

By Green thm, equivalent results are found due to the mixed derivative:

$$\dot{H} = \int_{\partial\Omega} \left\{ n_x \left(e_2 \frac{\partial e_1}{\partial x} - e_1 \frac{\partial e_2}{\partial x} + 2 e_4 \frac{\partial e_1}{\partial y} \right) + n_y \left(e_3 \frac{\partial e_1}{\partial y} - e_1 \frac{\partial e_3}{\partial y} - 2 e_1 \frac{\partial e_4}{\partial x} \right) \right\} ds$$

$$\dot{H} = \int_{\partial\Omega} \left\{ n_x \left(e_2 \frac{\partial e_1}{\partial x} - e_1 \frac{\partial e_2}{\partial x} - 2 e_1 \frac{\partial e_4}{\partial y} \right) + n_y \left(e_3 \frac{\partial e_1}{\partial y} - e_1 \frac{\partial e_3}{\partial y} + 2 e_4 \frac{\partial e_1}{\partial x} \right) \right\} ds$$



The half-sum of both terms does restore symmetry

$$\dot{H} = \int_{\partial\Omega} \left\{ n_x \left(e_2 \frac{\partial e_1}{\partial x} + e_4 \frac{\partial e_1}{\partial y} - e_1 \frac{\partial e_2}{\partial x} - e_1 \frac{\partial e_4}{\partial y} \right) \right.$$
$$\left. + n_y \left(e_3 \frac{\partial e_1}{\partial y} + e_4 \frac{\partial e_1}{\partial x} - e_1 \frac{\partial e_3}{\partial y} - e_1 \frac{\partial e_4}{\partial x} \right) \right\} ds$$

If the physical variables are introduced

$$\dot{H} = \int_{\partial\Omega} \left\{ n_x \left(M_{xx} \frac{\partial v}{\partial x} + M_{xy} \frac{\partial v}{\partial y} - v \, \frac{\partial M_{xx}}{\partial x} - v \, \frac{\partial M_{xy}}{\partial y} \right) \right.$$
$$\left. + n_y \left(M_{yy} \frac{\partial v}{\partial y} + M_{xy} \frac{\partial v}{\partial x} - v \, \frac{\partial M_{yy}}{\partial y} - v \, \frac{\partial M_{xy}}{\partial x} \right) \right\} ds$$

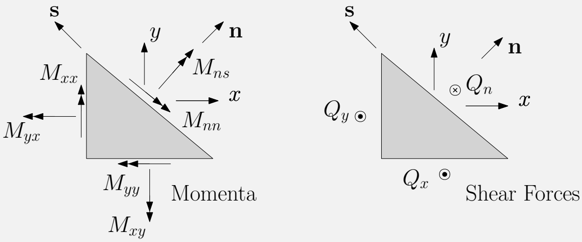
The boundary forces and momenta need to be defined:

$$\text{Shear Force} \qquad Q_n := n_x Q_x + n_y Q_y$$

$$\text{Flexural momentum} \qquad M_{nn} := \boldsymbol{n}^T \begin{pmatrix} M_{xx} n_x + M_{xy} n_y \\ M_{xy} n_x + M_{yy} n_y \end{pmatrix} \qquad \boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$$

$$\text{Torsional momentum} \qquad M_{ns} := \boldsymbol{s}^T \begin{pmatrix} M_{xx} n_x + M_{xy} n_y \\ M_{xy} n_x + M_{yy} n_y \end{pmatrix} \qquad \boldsymbol{s} = \begin{pmatrix} -n_y \\ n_x \end{pmatrix}$$

where $Q_x = -\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y}$ and $Q_y = -\frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x}$.



Using the previous definition of Q_n into the energy balance

$$\dot{H} = \int_{\partial\Omega} \left\{ \begin{pmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} M_{xx} n_x + M_{xy} n_y \\ M_{xy} n_x + M_{yy} n_y \end{pmatrix} + v Q_n \right\} ds$$

To get the proper boundary conditions the gradient of the vertical velocity has to be projected along the normal and tangential direction

$$\nabla v = \left(\nabla v^T \boldsymbol{n} \right) \, \boldsymbol{n} + \left(\nabla v^T \boldsymbol{s} \right) \, \boldsymbol{s} = \frac{\partial v}{\partial n} \, \boldsymbol{n} + \frac{\partial v}{\partial s} \, \boldsymbol{s}$$

So the time derivative of the Hamiltonian can be finally written as

$$\dot{H} = \int_{\partial\Omega} \left\{ v \, Q_n + \frac{\partial v}{\partial s} \, M_{ns} + \frac{\partial v}{\partial n} \, M_{nn} \right\} ds$$

Variables v, $\frac{\partial v}{\partial \boldsymbol{s}}$ are kinematically related. Another integration by part is needed to highlight the power conjugated variables. Given a closed and regular boundary the integration by parts leads to

$$\int_{\partial\Omega} \frac{\partial v}{\partial s} \, M_{ns} = - \int_{\partial\Omega} \frac{\partial M_{ns}}{\partial s} \, v$$

Final energy balance and boundary variables

The energy balance can be finally written as

$$\dot{H} = \int_{\partial\Omega} \left\{ v \, \tilde{Q}_n + \frac{\partial v}{\partial n} \, M_{nn} \right\} ds$$

where $\tilde{Q}_n = Q_n - \frac{\partial M_{ns}}{\partial s}$ is the effective shear force.

This energy balance highlights the power conjugated variables between input and output ports. No a priori causality is imposed.

Indeed this formulation enables us to consider different boundary conditions

Functional spaces

A possible choice for the flow space $\mathcal{F}$ is the space of the squared integrable function on the open bounded set Ω, i.e $L^2(\Omega, \mathbb{R}^4)$.

For the effort space $\mathcal{E}$ a subset of the Sobolev space $H^2(\Omega, \mathbb{R}^4)$ is a possible choice.

The space of boundary conditions is found by considering the previous energy balance

$$\mathcal{Z} = \{z | \, z = B_\partial(e), \forall \, e \in \mathcal{E}\} \qquad z = \left(V_n, v, M_{nn}, \frac{\partial v}{\partial n} \right)^T$$

The operator B_∂ is a differential operator which contains derivatives and normal to retrieve the conjugated variable at the boundary. It is not simply a trace operator of the effort variables. It also involves the derivatives, both normal and tangential.

Stokes-Dirac structure

Theorem

The set

$$\mathbb{D} := \left\{ (f, e, z) \in \mathcal{F} \times \mathcal{E} \times \mathcal{Z} \mid f = -\frac{\partial \boldsymbol{\alpha}}{\partial t} = -Je, z = B_\partial(e) \right\}$$

is a Stokes-Dirac structure with respect to the pairing

$$\ll (f_1, e_1, z_1), (f_2, e_2, z_2) \gg = \int_\Omega \left[e_1^T f_2 + e_2^T f_1 \right] d\Omega + \int_{\partial\Omega} B_J(z_1, z_2) ds$$

where B_J is a symmetric operator, arising from the application of the Green theorem. It reads

$$B_J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{aligned} B_J(z_1, z_2) = {} & \tilde{Q}_{n,2} v_1 + M_{nn,2} \frac{\partial v_1}{\partial n} \\ & + \tilde{Q}_{n,1} v_2 + M_{nn,1} \frac{\partial v_2}{\partial n} \end{aligned}$$

Plan

Constitutive Relations

Deflection of the cross section about x : ψ_y
Deflection of the cross section about y : $-\psi_x$

$$u(x,y,z) = -z\psi_x \qquad v(x,y,z) = -z\psi_y \qquad w(x,y,z) = w(x,y)$$

The strain field can be separated into the bending and a shear part $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_b,\ \boldsymbol{\epsilon}_s)^T$

$$\boldsymbol{\epsilon}_b = \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = -z \begin{pmatrix} \frac{\partial \psi_x}{\partial x} \\ \frac{\partial \psi_y}{\partial y} \\ \frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y} \end{pmatrix} = -z \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{pmatrix} = -z\boldsymbol{\kappa}$$

$$\boldsymbol{\epsilon}_s = \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial x} - \psi_x \\ \frac{\partial w}{\partial y} - \psi_y \end{pmatrix}$$

The stresses are split in the same manner

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_b \\ \boldsymbol{\sigma}_s \end{pmatrix} = \begin{bmatrix} E_b & 0 \\ 0 & E_s \end{bmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_b \\ \boldsymbol{\epsilon}_s \end{pmatrix} \qquad \boldsymbol{\sigma}_b = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \qquad \boldsymbol{\sigma}_s = \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix}$$

The stresses integrated along the thickness generate momenta and forces

$$\begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{Q} \end{pmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} -z\boldsymbol{\sigma}_b \\ \boldsymbol{\sigma}_s \end{pmatrix} dz = \begin{bmatrix} \boldsymbol{D}_b & 0 \\ 0 & \boldsymbol{D}_s \end{bmatrix} \begin{pmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\epsilon}_s \end{pmatrix}$$

Matrices $\boldsymbol{D}_b, \boldsymbol{D}_s$ come from the integration along the fiber of the constitutive relation

$$\boldsymbol{D}_b = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 E_b \ dz \qquad \boldsymbol{D}_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 E_s \ dz$$

Since the kinematic model of the plate is too stiff, a correction factor is introduced inside matrix $\overline{\boldsymbol{D}}_s = k\boldsymbol{D}_s$.

If the mechanical properties are constant along the z axis (but not necessarily in the x, y plane), momenta and shear forces are given by

$$
\begin{aligned}
M_{xx} &= D\left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y}\right) \\
M_{yy} &= D\left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x}\right) \\
M_{xy} &= D\frac{(1-\nu)}{2}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right)
\end{aligned}
$$

$$
\begin{aligned}
Q_x &= kGh\left(\frac{\partial w}{\partial x} - \psi_x\right) \\
Q_y &= kGh\left(\frac{\partial w}{\partial y} - \psi_y\right)
\end{aligned}
$$

Variational formulation

These preliminary definitions allow defining the following extrema problem

$$\{w_0, \psi_{x,0}, \psi_{y,0}\} = \{w, \psi_x, \psi_y \in \mathcal{V};\ \delta F(w, \psi_x, \psi_y)|_{w_0, \psi_{x,0}, \psi_{y,0}} = 0\}$$

$$F(w, \psi_x, \psi_y) = \int_0^T \int_\Omega \{\mathcal{K} - \mathcal{U} + \mathcal{W}\}\, d\Omega\, dt = \int_0^T \int_\Omega \left\{ \frac{1}{2}\rho \left[h \left(\frac{\partial w}{\partial t}\right)^2 + \frac{h^3}{12}\left(\frac{\partial \psi_x}{\partial t}\right)^2 + \frac{h^3}{12}\left(\frac{\partial \psi_y}{\partial t}\right)^2 \right] - \frac{1}{2}\boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa} - \frac{1}{2}\boldsymbol{\epsilon}_s^T \boldsymbol{D}_s \boldsymbol{\epsilon}_s + pw \right\} d\Omega\, dt$$

Euler Lagrange Equations

Work and energy densities defined by

$$
\begin{aligned}
\mathcal{W} =& pw && \text{Work} \\
\mathcal{K} =& \frac{1}{2}\rho\left[h\left(\frac{\partial w}{\partial t}\right)^2 + \frac{h^3}{12}\left(\frac{\partial \psi_x}{\partial t}\right)^2 + \frac{h^3}{12}\left(\frac{\partial \psi_y}{\partial t}\right)^2\right] && \text{Kinetic energy} \\
\mathcal{U} =& \frac{1}{2}\boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\epsilon}_s^T \boldsymbol{D}_s \boldsymbol{\epsilon}_s && \text{Deformation energy}
\end{aligned}
$$

Euler Lagrange equations for the Mindlin plate

$$
\begin{aligned}
\rho h \frac{\partial^2 w}{\partial t^2} &= p + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \\
\rho \frac{h^3}{12}\frac{\partial^2 \psi_x}{\partial t^2} &= Q_x + \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \\
\rho \frac{h^3}{12}\frac{\partial^2 \psi_y}{\partial t^2} &= Q_y + \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{y}}{\partial y}
\end{aligned}
$$

Plan

Energies and co-energies variables

Linear momenta and curvature are taken as energy variables. Additional the shear strain are considered, leading to

$$\boldsymbol{\alpha} = \left(\rho h v,\ \rho\frac{h^3}{12}\omega_x,\ \rho\frac{h^3}{12}\omega_y,\ \kappa_{xx},\ \kappa_{yy},\ \kappa_{xy},\ \gamma_{xz},\ \gamma_{yz}\right)^T$$

where $v = \frac{\partial w}{\partial t},\ \omega_x = \frac{\partial \psi_x}{\partial t},\ \omega_y = \frac{\partial \psi_y}{\partial t}$. The Hamiltonian density is quadratic in the energy variables

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\alpha}^T \begin{bmatrix} \frac{1}{\rho h} & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{D}_b & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{D}_s \end{bmatrix} \boldsymbol{\alpha}$$

The variational derivative provides as co-energy variables

$$\mathbf{e} := \frac{\delta H}{\delta \boldsymbol{\alpha}} = (v,\ \omega_x,\ \omega_y,\ M_{xx},\ M_{yy},\ M_{xy},\ Q_x,\ Q_y)^T$$

Definition of J and boundary variables

The skew-adjoint operator relating energies and co-energies is found to be

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 1 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & -1 & 0 & 0 & 0 & 0 & 0 & \\ \frac{\partial}{\partial y} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \boldsymbol{\alpha}}{\partial t} = J\mathbf{e}$$

The boundary variables are found by evaluating the time derivative of the Hamiltonian

$$
\begin{aligned}
\dot{H} = \int_\Omega \Bigg\{ & v \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) + \omega_x \left(Q_x + \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) \\
& + \omega_y \left(Q_y + \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \right) + M_{xx} \frac{\partial \omega_x}{\partial x} + M_{yy} \frac{\partial \omega_y}{\partial y} \\
& + M_{xy} \left(\frac{\partial \omega_y}{\partial x} + \frac{\partial \omega_x}{\partial y} \right) + Q_x \left(\frac{\partial v}{\partial x} - \omega_x \right) + Q_y \left(\frac{\partial v}{\partial y} - \omega_y \right) \Bigg\} d\Omega \\
= \int_\Omega \Bigg\{ & \frac{\partial}{\partial x} \left(v Q_x + \omega_x M_{xx} + \omega_y M_{xy} \right) + \frac{\partial}{\partial y} \left(v Q_y + \omega_x M_{xy} + \omega_y M_{yy} \right) \Bigg\} d\Omega
\end{aligned}
$$

Boundary Variables

By applying Green theorem and using definitions of Q_n, M_{nn}, M_{ns}, together with the decomposition of vector $(\omega_x,\ \omega_y)^T$ along the normal and tangential direction

$$\begin{pmatrix}\omega_x \\ \omega_y\end{pmatrix} = \left(\begin{pmatrix}\omega_x \\ \omega_y\end{pmatrix}^T \boldsymbol{n}\right)\boldsymbol{n} + \left(\begin{pmatrix}\omega_x \\ \omega_y\end{pmatrix}^T \boldsymbol{s}\right)\boldsymbol{s} = \omega_n \boldsymbol{n} + \omega_s \boldsymbol{s}$$

the energy balance can be expressed through the boundary values

$$\dot{H} = \int_{\partial\Omega} \left\{ v\, Q_n + \omega_n M_{nn} + \omega_s M_{ns} \right\} ds$$

Functional spaces

A possible choice for the flow space $\mathcal{F}$ is the space of the squared integrable function on the open bounded set Ω, i.e $L^2(\Omega, \mathbb{R}^4)$.

For the effort space $\mathcal{E}$ a subset of the Sobolev space $H(\Omega, \mathbb{R}^4)$ is a possible choice.

The space of the boundary conditions is found by considering the previous energy balance

$$\mathcal{Z} = \{z |\, z = B_\partial(e), \forall\, e \in \mathcal{E}\} \qquad z = \left(Q_n,\ v,\ M_{nn},\ \omega_n,\ M_{ns},\ \omega_s\right)^T$$

The operator B_∂ is a linear operator which enables to retrieve the conjugated variables at the boundary. It is expressed by a simple matrix on the trace of the efforts at the boundary.

Stokes-Dirac Structure

This is a recall of what was published by Macchelli et al. (2005)

Theorem

The set

$$\mathbb{D} := \left\{ (f, e, z) \in \mathcal{F} \times \mathcal{E} \times \mathcal{Z} \mid f = -\frac{\partial \boldsymbol{\alpha}}{\partial t} = -Je, z = B_\partial(e) \right\}$$

is a Stokes-Dirac structure with respect to the pairing

$$\ll (f_1, e_1, z_1), (f_2, e_2, z_2) \gg = \int_\Omega \left[e_1^T f_2 + e_2^T f_1 \right] d\Omega + \int_{\partial\Omega} B_J(z_1, z_2) ds$$

where B_J is a symmetric operator arising from the application of the Green theorem.

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Thank you for your attention. Questions?