Numerics for PH-PDEs:

the Partitioned Finite Element Method,

a structure-preserving method for physics-based PDEs with boundary control.

Denis Matignon[1]

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ANR DFG Deutsche Forschungsgemeinschaft INFIDHEM

Outline

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1 Introduction
- Goal of the presentation
- A few references
- Main objective of PFEM

2 Linear Wave equations: towards PH-DAEs and PH-ODEs

3 Nonlinear wave equation: the 2D Shallow Water Equation

Goal of the presentation

- Make links with other courses given at this Spring School:
  1. course on infinite-dimensional PH-PDEs by Birgit Jacob and Hans Zwart,
  2. course on PH-DAEs by Volker Mehrmann and Arjan van der Schaft,
  3. course on Thermodynamics by Hans-Christian Öttinger.
- **Main Goal**: the underlying structure of physical systems must be preserved by numerical methods at the discrete level, i.e. from infinite dimension to finite dimension (but still in continuous time).
- The question of specific time discretization addressed in the course by Paul Kotyczka and Laurent Lefèvre, just before, will not be tackled in the sequel.

Outline

Twenty years of distributed port-Hamiltonian systems: a literature review
Rashad, R., Califano, F., van der Schaft, A.J. and Stramigioli, S.
IMA J. Mathematics of Control and Information,
vol.37 (4), pp. 1400–1422 (2020)

$\Longrightarrow$ More than 170 up-to-date references on:

- Theoretical Framework
- Modeling
- Analysis and Control
- Discretization

and more specifically

- Numerical Methods for Distributed Parameter Port-Hamiltonian Systems **Kotyczka P.** *TUM University Press, Munich (2019)*,
- Structure preserving approximation of dissipative evolution problems **Egger H.** *Numerische Mathematik vol.143(1), pp. 85–106 (2019)*
- A Partitioned Finite-Element Method for power-preserving discretization of open systems of conservation laws, **Cardoso-Ribeiro F.L., Matignon D., Lefèvre L.** *IMA J. Mathematics of Control and Information, vol.38(2), pp. 493–533 (2021)*
- Numerical Approximation of Port-Hamiltonian Systems for Hyperbolic or Parabolic PDEs with Boundary Control, **Brugnoli A., Haine G., Serhani A., Vasseur X.** *Journal of Applied Mathematics and Physics, vol.9, pp. 1278–1321 (2021)*.

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Simulate complex open physical systems by ensuring the *conservation of the* ***power balance*** for a chosen functional: the **Hamiltonian**.

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  $\rightarrow$ The power balance is *encoded* in a **Stokes-Dirac structure**.
- *Partitioned Finite Element Method (PFEM):*
  $\rightarrow$ It approximates the Stokes-Dirac structure into a **Dirac structure**.
  $\rightarrow$ The **discrete Hamiltonian** satisfies a "discrete" power balance.

A Partitioned Finite-Element Method for power-preserving discretization of open systems of conservation laws
Cardoso-Ribeiro F.L., Matignon D., Lefèvre L.
IMA J. Mathematics of Control and Information, vol.38(2) , pp. 493–533 (2021)

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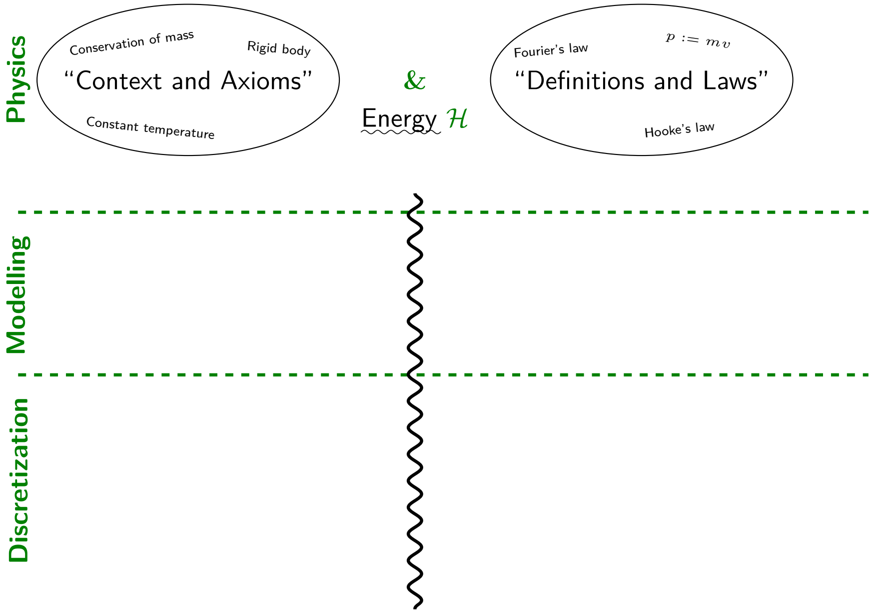

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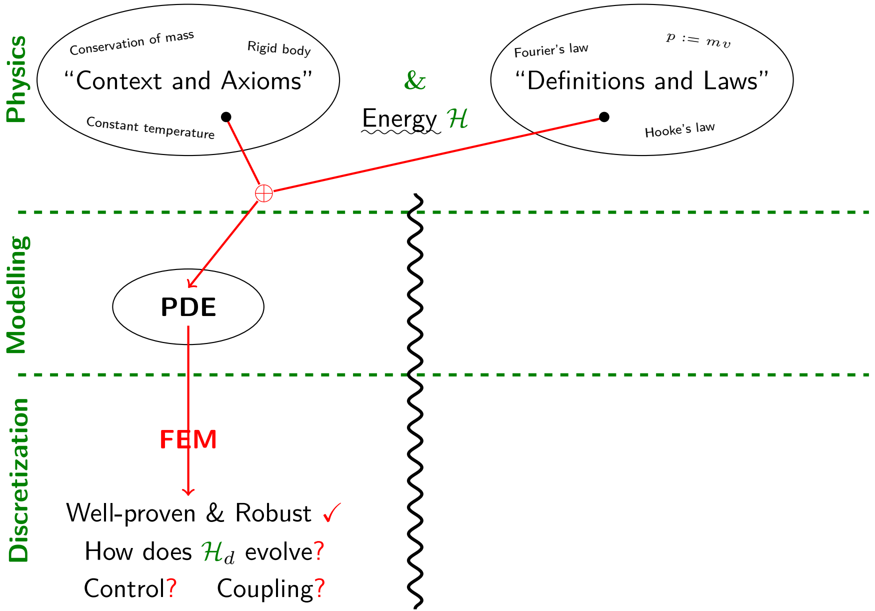

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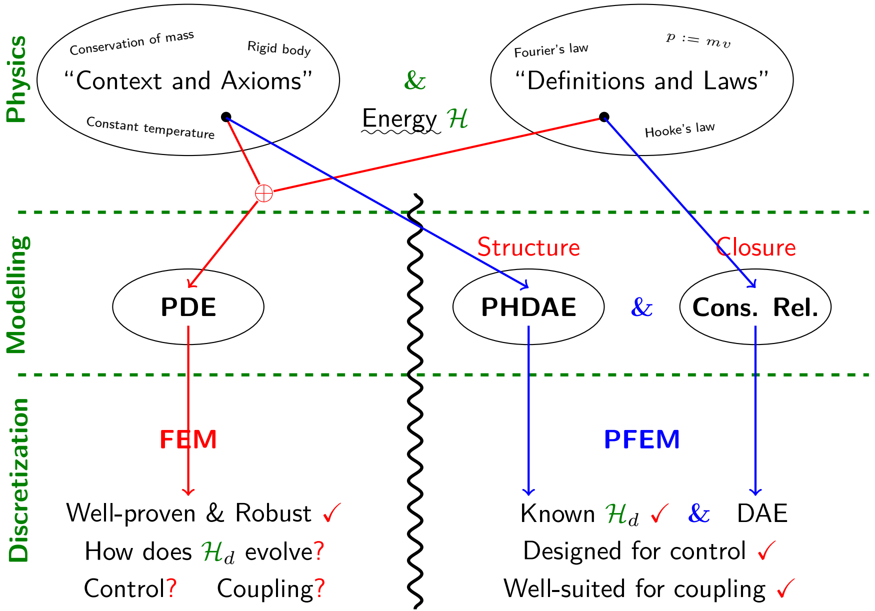

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\alpha}(t)) = -\left\langle R\overrightarrow{e}_{\overrightarrow{\alpha}}(t), \overrightarrow{e}_{\overrightarrow{\alpha}}(t)\right\rangle_J + \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B \;\leq\; \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B \,.$$

Although **the underlying geometry** is well-determined with the above equality, **constitutive relations** between $\overrightarrow{\alpha}$ and $\overrightarrow{e}_{\overrightarrow{\alpha}}$ are also needed to solve the system!

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Conservative System: Wave as PH-DAE

Deflection w **of a 2D-membrane,** ***boundary deflection velocity*** **as control.**

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$\Longrightarrow$ for infinite-dimensional pHs in general, see the course by Birgit Jacob and Hans Zwart on Wednesday morning.

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For all test functions $\overrightarrow{\boldsymbol{v}}_q$, v_p and v_∂ (smooth enough):

$$\begin{cases} \left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = \left\langle \overrightarrow{\mathbf{grad}}\left(e_p\right), \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2}, \\ \left\langle \partial_t \alpha_p, v_p \right\rangle_{L^2} = \left\langle \operatorname{div}\left(\overrightarrow{\boldsymbol{e}}_q\right), v_p \right\rangle_{L^2}, \\ \left\langle \boldsymbol{y}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = \left\langle \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \end{cases}$$

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Applying Green's formula on the 1st line and using the definition of $\boldsymbol{u}$:

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Green's formula applied on the 2nd line would lead to normal stress control $\boldsymbol{u} = \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}$. The energy variables are **partitioned** accordingly.

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t),$$

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$$\boldsymbol{u}^{ap}(t,\overrightarrow{\boldsymbol{s}}) := \sum_{m=1}^{N_\partial} \psi^{m}(\overrightarrow{\boldsymbol{s}})\boldsymbol{u}^{m}(t) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{u}}(t), \quad \boldsymbol{y}^{ap}(t,\overrightarrow{\boldsymbol{s}}) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{y}}(t),$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix, $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix and $\boldsymbol{\Psi}$ an $N_\partial \times 1$ matrix.

Conservative System: FEM Application

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$$
\begin{aligned}
&\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \textstyle\sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad && \overrightarrow{\boldsymbol{e}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),\\
&\alpha_p^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \textstyle\sum_{k=1}^{N_p} \varphi_p^{k}(\overrightarrow{\boldsymbol{x}})\alpha_p^{k}(t) = \boldsymbol{\phi}_p^{\top} \cdot \underline{\alpha}_p(t), && e_p^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \boldsymbol{\phi}_p^{\top} \cdot \underline{e}_p(t),\\
&\boldsymbol{u}^{ap}(t,\overrightarrow{\boldsymbol{s}}) := \textstyle\sum_{m=1}^{N_\partial} \psi^{m}(\overrightarrow{\boldsymbol{s}})\boldsymbol{u}^{m}(t) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{u}}(t), && \boldsymbol{y}^{ap}(t,\overrightarrow{\boldsymbol{s}}) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{y}}(t),
\end{aligned}
$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix, $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix and $\boldsymbol{\Psi}$ an $N_\partial \times 1$ matrix.

The discretized system (giving the structure) then reads:

$$
\begin{cases}
\overrightarrow{\boldsymbol{M}}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) = D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t),\\
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\end{cases}
$$

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$
\begin{aligned}
&\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \textstyle\sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad && \overrightarrow{\boldsymbol{e}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),\\
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where:

$$
\overrightarrow{\boldsymbol{M}}_q := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_q \cdot \overrightarrow{\boldsymbol{\Phi}}_q^{\top}, \qquad M_p := \int_\Omega \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^{\top}, \qquad M_\partial := \int_{\partial\Omega} \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^{\top},
$$

$$
D := -\int_\Omega \mathrm{div}\left(\overrightarrow{\boldsymbol{\Phi}}_q\right) \cdot \boldsymbol{\phi}_p^{\top}, \qquad B := \int_{\partial\Omega} \left(\overrightarrow{\boldsymbol{\Phi}}_q \cdot \overrightarrow{\boldsymbol{n}}\right) \cdot \boldsymbol{\Psi}^{\top}.
$$

Finite-Dimensional extended structure operator

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix}.$$

Conservative System: Power Balance

Finite-Dimensional extended structure operator

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The inner product on $\mathbb{R}^{N_q}$, $\mathbb{R}^{N_p}$ and $\mathbb{R}^{N_\partial}$ has to be taken w.r.t. the mass matrices $\overrightarrow{\boldsymbol{M}}_q$, M_p and M_∂: *e.g.* $\langle \overrightarrow{\boldsymbol{v}}_1, \overrightarrow{\boldsymbol{v}}_2 \rangle_{N_q} := \overrightarrow{\boldsymbol{v}}_2^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \overrightarrow{\boldsymbol{v}}_1$.

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Discrete Hamiltonian

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

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Conservative System: Power Balance

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Denote $\underline{f} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q, \quad \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p, \quad -\underline{\boldsymbol{y}}\right)^\top$ and $\underline{e} := \left(\underline{e}_q, \quad \underline{e}_p, \quad \underline{\boldsymbol{u}}\right)^\top$, then:

Discrete Lossless **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = \underline{\boldsymbol{u}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

A Proof of the Discrete Power Balance

Since by definition the discrete Hamiltonian reads:

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

we can compute its time derivative along the trajectories:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) &= (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q)^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p)^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p, \\
&= (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q)^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \underline{e}_q + (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p)^\top \cdot M_p \cdot \underline{e}_p, \\
&= (\overrightarrow{\boldsymbol{M}}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q)^\top \cdot \underline{e}_q + (M_p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p)^\top \cdot \underline{e}_p, \\
&= (D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t))^\top \cdot \underline{e}_q + (-D^\top \cdot \underline{e}_q(t))^\top \cdot \underline{e}_p, \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot B^\top \cdot \underline{e}_q \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}(t). \quad \square
\end{aligned}$$

PFEM for pHs gives rise to PH-DAEs

Summarizing the main steps: discretization of the structure and of the constitutive relations are made separately.

The discretized system is a PH-DAE:

$$\begin{cases} \overrightarrow{\boldsymbol{M}}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) = D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t), \\ M_p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) = -D^\top \cdot \underline{e}_q(t), \\ M_\partial \cdot \underline{\boldsymbol{y}}(t) = B^\top \cdot \underline{e}_q(t), \end{cases}$$

together with

$$\begin{cases} \overrightarrow{\boldsymbol{M}}_q \cdot \underline{e}_q(t) = \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q(t), \\ M_p \cdot \underline{e}_p(t) = M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p(t) \end{cases}$$

$\Longrightarrow$ in general, PFEM for pHs gives rise to finite-dimensional PH-DAEs, for which efficient numerical methods can be used (see the course by Volker Mehrmann on Tuesday morning)

Outline

Impedance Boundary Condition (IBC)

The Impedance Boundary Condition, with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}} \left(w \right) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

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It can be seen as an *output feedback law* $\boldsymbol{u} = -Z\boldsymbol{y} + \boldsymbol{\nu}$ in the previous case.

Lossy Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) = -\left\langle \boldsymbol{y}, Z\boldsymbol{y} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \left\langle \boldsymbol{y}, \boldsymbol{\nu} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

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Add **impedance ports** (f_i, e_i) and **dissipative constitutive relation** $e_i = Z f_i$, and approximate f_i and e_i in the boundary FEM basis $\boldsymbol{\Psi}$:

$$\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 & 0 \\ 0 & 0 & M_p & 0 & 0 \\ 0 & 0 & 0 & M_\partial & 0 \\ 0 & 0 & 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_i(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix} = \begin{pmatrix} 0 & D & -B & B \\ -D^\top & 0 & 0 & 0 \\ B^\top & 0 & 0 & 0 \\ -B^\top & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_i(t) \\ \underline{\boldsymbol{\nu}}(t) \end{pmatrix}$$

and $M_\partial \cdot \underline{e}_i = \langle Z \rangle \cdot \underline{f}_i$, with $\langle Z \rangle := \int_{\partial\Omega} Z \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^\top \geq 0.$

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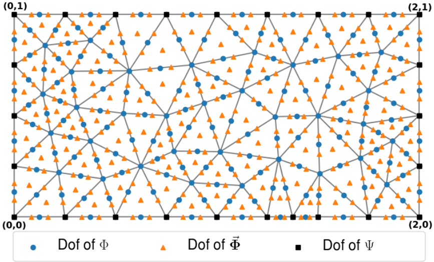
and $M_\partial \cdot \underline{e}_i = \langle Z \rangle \cdot \underline{f}_i$, with $\langle Z \rangle := \int_{\partial\Omega} Z \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^\top \geq 0$.

Discrete Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = -\underline{\boldsymbol{y}}^\top \cdot \langle Z \rangle \cdot \underline{\boldsymbol{y}} + \underline{\boldsymbol{\nu}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

Boundary Dissipation: Simulations

- Heteregenous ($\rho \not\equiv constant$);
- Anisotropic (tensor $\overline{\overline{T}} \not\equiv constant$);
- $\epsilon \equiv 0$;
- $Z \neq 0$ for $t \geq 2$;
- Raviart-Thomas FEM for q-variables;
- Lagrange FEM for p-variables;
- Lagrange FEM for ∂-variables;



Outline

Discretization in terms of co-energy

In order to transform the PH-DAEs into PH-ODEs, in the linear case, the constitutive relations can be first inverted, second discretized.

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \overline{\overline{\boldsymbol{T}}}^{-1} \cdot \overrightarrow{\boldsymbol{e}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) \quad \text{and} \quad \alpha_p^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \rho \, e_p^{ap}(t, \overrightarrow{\boldsymbol{x}}).$$

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The discretization in the same bases as previously gives:

$$\overrightarrow{\boldsymbol{M}}_q \cdot \underline{\alpha}_q = \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q \quad \text{and} \quad M_p \cdot \underline{\alpha}_p = M_\rho \cdot \underline{e}_p,$$

where new mass matrices, or spatial averages, have been defined:
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Discretization in terms of co-energy variables: PH-ODEs

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The discretized system now is a PH-ODE:

$$\begin{cases} \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \underline{e}_q(t) = D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t), \\ M_\rho \cdot \frac{\mathrm{d}}{\mathrm{d}t} \underline{e}_p(t) = -D^\top \cdot \underline{e}_q(t), \\ M_\partial \cdot \underline{\boldsymbol{y}}(t) = B^\top \cdot \underline{e}_q(t), \end{cases}$$

and enjoys the same conservative power balance at the discrete level.

With the same definition the discrete Hamiltonian:

$$\tilde{\mathcal{H}}_d\left(\underline{e}_q, \underline{e}_p\right) := \mathcal{H}\left(\vec{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{e}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\underline{\underline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q + \underline{e}_p^\top \cdot M_\rho \cdot \underline{e}_p\right),$$

we can easily compute its time derivative along the trajectories:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{H}}_d\left(\underline{e}_q, \underline{e}_p\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_q\right)^\top \cdot \overrightarrow{\boldsymbol{M}}_{\underline{\underline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q + \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_p\right)^\top \cdot M_\rho \cdot \underline{e}_p, \\
&= \left(\overrightarrow{\boldsymbol{M}}_{\underline{\underline{\boldsymbol{T}}}^{-1}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_q\right)^\top \cdot \underline{e}_q + \left(M_\rho \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_p\right)^\top \cdot \underline{e}_p, \\
&= \left(D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t)\right)^\top \cdot \underline{e}_q + \left(-D^\top \cdot \underline{e}_q(t)\right)^\top \cdot \underline{e}_p, \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot B^\top \cdot \underline{e}_q \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}(t). \quad \square
\end{aligned}$$

Remark: both definitions do coincide, i.e. $\tilde{\mathcal{H}}_d\left(\underline{e}_q, \underline{e}_p\right) = \mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right)$, since the discretization of the constitutive relations now provides: $\overrightarrow{\boldsymbol{M}}_q \cdot \underline{\alpha}_q = \overrightarrow{\boldsymbol{M}}_{\underline{\underline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q$ and $M_p \cdot \underline{\alpha}_p = M_\rho \cdot \underline{e}_p$ (exercise).

Outline

Mixed boundary control: principle

1. The basic idea is: $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\int_{\partial\Omega} = \int_{\Gamma_D} + \int_{\Gamma_N}$.
2. Where the control is not known, a Lagrange multiplier λ is introduced instead $+$ a constraint is added to the system, an extended skew-symmetric J_e matrix is obtained.
3. $\Longrightarrow$ a PH-DAE is readily obtained, with a Lagrange multiplier of very small dimension.
4. This method is detailed in one early reference, but several other possibilities have been explored since then.

Outline

Convergence rate: theory

Theorem (Haine, Matignon & Serhani, 2020)

Let $\kappa \geq \delta$ be an integer, where $\delta = 0$ if Ω is convex and 1 otherwise, and $T > 0$.

Let $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_{q0} \\ \alpha_{p0} \end{pmatrix} \in \mathcal{Z}_\kappa := \begin{pmatrix} \overline{\overline{\boldsymbol{T}}} & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{H}^{\kappa+1}_{\mathrm{div}}(\Omega) \\ H^{\kappa+1}(\Omega) \end{bmatrix}$, $\boldsymbol{u} \in C^2([0,\infty); H^{\kappa+1}(\partial\Omega))$.

Let $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_{q}^{ap}(0) \\ \alpha_{p}^{ap}(0) \end{pmatrix}$, $\boldsymbol{u}^{ap}$ be their interpolations with $\left(\mathbb{P}^k\right)^N \times \mathbb{P}^\ell \times \mathbb{P}^m$.

Let $\mathbf{E}(t) := \left\| \left((\overrightarrow{\boldsymbol{\alpha}}_q - \overrightarrow{\boldsymbol{\alpha}}_q^{ap})(t), \quad (\alpha_p - \alpha_p^{ap})(t) \right)^\top \right\|_{\mathbf{L}^2 \times L^2}$,

$\exists C_T > 0$, independent of $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_{q0} \\ \alpha_{p0} \end{pmatrix}$, and $\boldsymbol{u}$: for all h and all $t \in [0,T]$

$$\mathbf{E}(t) \leq C_T h^{\min\{\ell-\delta\,;\,k\,;\,m\}} \left(\left\| \begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_q \\ \alpha_p \end{pmatrix} \right\|_{L^\infty([0,T];\mathcal{Z}_\kappa)} + \|\boldsymbol{u}\|_{L^\infty([0,T];H^{\kappa+1}(\partial\Omega))} \right).$$

The **optimal** order is $\kappa - \delta$, when $k = \kappa - \delta$, $\ell = \max\{\kappa; 1\}$ and $m = \kappa - \delta$.

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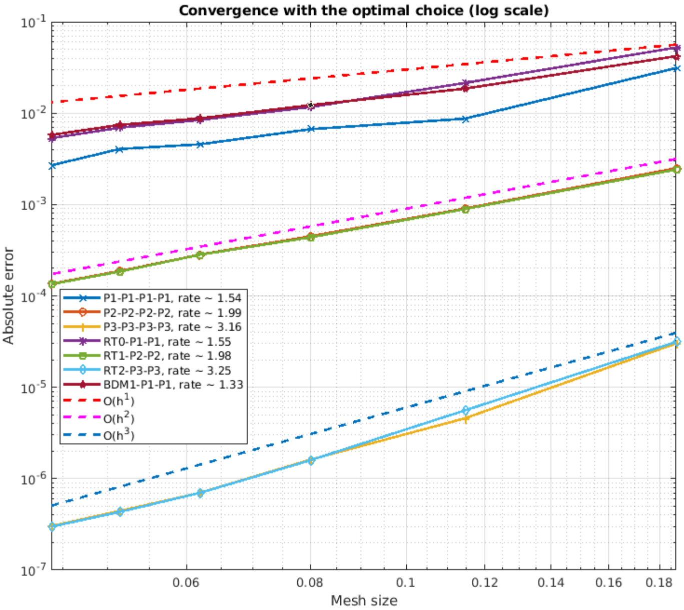
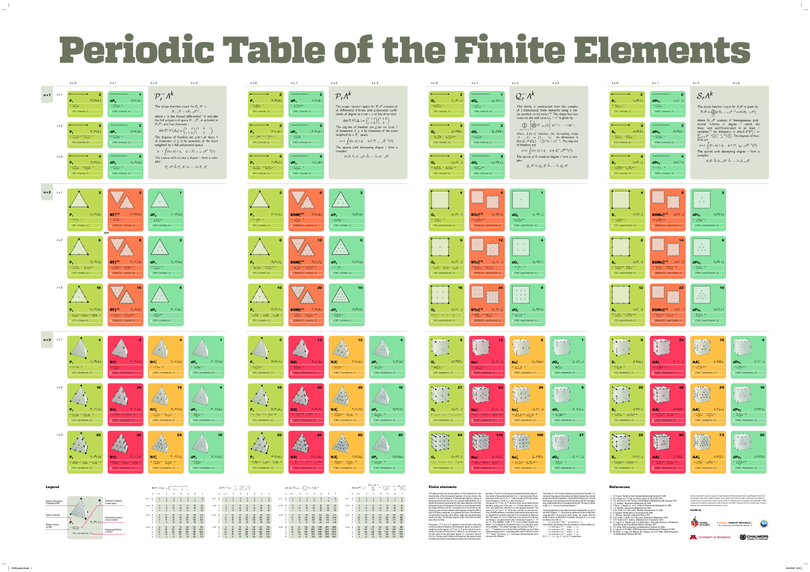
$RT_{\kappa-1-\delta} \times \mathbb{P}^\kappa \times \mathbb{P}^{\kappa-\delta}$ $\quad BDM_{\kappa-\delta} \times \mathbb{P}^\kappa \times \mathbb{P}^{\kappa-\delta}$ $\quad BDFM_{\kappa-\delta} \times \mathbb{P}^\kappa \times \mathbb{P}^{\kappa-\delta}$

Convergence rate: numerics



A map for choosing the appropriate conforming finite elements



Outline

The irrotational shallow water equations

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$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2} \int_\Omega \left\{ \frac{1}{\rho} \alpha_h \left\| \boldsymbol{\alpha}_v \right\|^2 + \rho g \alpha_h^2 \right\} \, \mathrm{d}\Omega.$$

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* Dynamical system:

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div} \\ -\overrightarrow{\mathbf{grad}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_h \\ \mathbf{e}_v \end{pmatrix}, \qquad (x, y) \in \Omega = \{x^2 + y^2 \leq R\},$$

$$\begin{pmatrix} e_h \\ \mathbf{e}_v \end{pmatrix} = \begin{pmatrix} \delta_{\alpha_h} H \\ \delta_{\boldsymbol{\alpha}_v} H \end{pmatrix} = \begin{pmatrix} \frac{1}{2\rho} \left\| \boldsymbol{\alpha}_v \right\|^2 + \rho g \alpha_h \\ \frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v \end{pmatrix},$$

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* Consider a uniform Neumann boundary control

$$u_\partial = -\mathbf{e}_v \cdot \mathbf{n}|_{\partial\Omega} = -\frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v \cdot \mathbf{n}|_{\partial\Omega}, \qquad \text{Volumetric inflow rate.}$$

The corresponding output reads

$$y_\partial = e_h|_{\partial\Omega} = (\rho g \alpha_h + \frac{1}{2\rho} \left\| \boldsymbol{\alpha}_v \right\|^2)|_{\partial\Omega}.$$

Outline

Numerics: PFEM in the polynomial case

The difficulty lies in the non-linear nature of both constitutive relations. However, since they remain polynomial, off-line Finite Element computations can be performed, and makes possible the online computation of the discrete constitutive relations at each time step.

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A general result

$$M_h \cdot \underline{e_h} := \nabla_{\underline{\alpha_h}} \mathcal{H}_d(\underline{\alpha_h}, \underline{\alpha_v}), \text{ and } M_v \cdot \underline{e_v} := \nabla_{\underline{\alpha_v}} \mathcal{H}_d(\underline{\alpha_h}, \underline{\alpha_v}).$$

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Here quadratic quantities have to be computed in the integrals, namely $\underline{q_h} := \int_\Omega \phi_h \frac{1}{2\,\rho} \underline{\alpha_v}^\top \cdot \overrightarrow{\boldsymbol{\Phi}}_v \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \cdot \underline{\alpha_v}$ and $\underline{q_v} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_v \frac{1}{\rho} \underline{\alpha_h}^\top \cdot \phi_h \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \cdot \underline{\alpha_v}$.

$$\Rightarrow 1 \leq i \leq N_h, \quad q_h^i(t) = \underline{\alpha_v}(t)^\top \cdot \left(\int_\Omega \phi_h^i \frac{1}{2\,\rho} \overrightarrow{\boldsymbol{\Phi}}_v \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \right) \cdot \underline{\alpha_v}(t),$$

$$\Rightarrow 1 \leq k \leq N_v, \quad q_v^k(t) = \underline{\alpha_h}(t)^\top \cdot \left(\int_\Omega \phi_h \frac{1}{\rho} \overrightarrow{\boldsymbol{\phi}}_v^{k\,\top} \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \right) \cdot \underline{\alpha_v}(t).$$

Remark: the sizes of the vectors and matrices do match as well (exercise).
$\Longrightarrow$ Off-line computation proves possible!

Outline

A simple proportional control stabilizes the system around the desired point h^{des}

$$u_\partial = -k(y_\partial - y_\partial^{\mathsf{des}}), \qquad y_\partial^{\mathsf{des}} = \rho g h^{\mathsf{des}}, \quad k > 0.$$

This control law ensures that the Lyapunov functional

$$V = \frac{1}{2}\int_\Omega \left\{ \frac{1}{2}\rho g(\alpha_h - \alpha_h^{\mathsf{des}})^2 + \frac{1}{2\rho}\alpha_h \left\|\boldsymbol{\alpha}_v\right\|^2 \right\} \, \mathrm{d}\Omega \geq 0,$$

where $\alpha_h^{\mathsf{des}} = h^{\mathsf{des}}$, has negative semi-definite time derivative

$$\dot{V} = -k \int_{\partial\Omega} \left(y_\partial - y_\partial^{\mathsf{des}}\right)^2 \, \mathrm{d}\Gamma \leq 0.$$

Simulation Results for the 2D SWE

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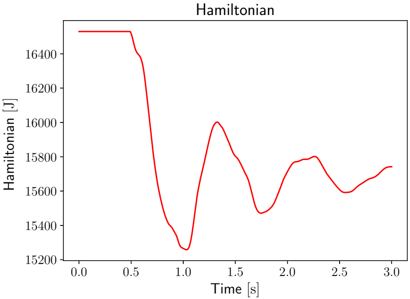

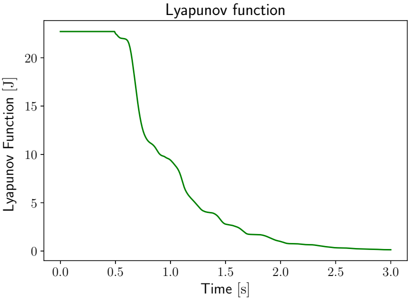

PFEM applies to many more models

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- **Timoshenko beam:** $J := \begin{pmatrix} 0 & 0 & 0 & \partial_x \\ 0 & 0 & \partial_x & 1 \\ 0 & \partial_x & 0 & 0 \\ \partial_x & -1 & 0 & 0 \end{pmatrix}$;

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- **3D Maxwell's equation:** $J := \begin{pmatrix} 0 & \overrightarrow{\mathbf{curl}} \\ -\overrightarrow{\mathbf{curl}} & 0 \end{pmatrix}$.

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Constitutive relations are <u>postponed</u>!
Dissipation **is not** a drawback!

- SCRIMP project at Isae-Supaero: Python as programming language, modules `FEniCS` (Finite elements), `PETSc` (Time integration), and SCRIMP (a wrapper to speed up the coding process), as well as usual modules such as `NumPy` and `Matplotlib`.
- Supplementary material to the Open Access paper by A. Brugnoli, G. Haine, A. Serhani, and X. Vasseur, Numerical approximation of port-Hamiltonian systems for hyperbolic or parabolic PDEs with boundary control, (2021) :
  supplementary material `https://doi.org/10.5281/zenodo.3938600`.
- Github and Supplementary material in MatLab associated to the IEEE CDC paper by F. L. Cardoso-Ribeiro, A. Brugnoli, D. Matignon, and L. Lefèvre, Port-Hamiltonian modeling, discretization and feedback control of a circular water tank,
  supplementary material `https://doi.org/10.5281/zenodo.4029598`

`https://github.com/flavioluiz/Circular-Tank-PFEM-CDC-supplementary-materia`

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