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Portwings Internal Meeting
Challenges and outlook for the numerics

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April 29, 2021

What numerics for the portwings project?

Methods should preserve the continuous structure at the discrete level. Which structure?

1. Cohomology: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$;
2. Variational structure $\delta \int l = 0$, (l Lagrangian density);
3. Hamiltonian structure $\dot{\mathcal{F}} = \{F, H\}$, $\{\cdot, \cdot\}$ Poisson brackets.
4. ...

Recent developments:

- splitting of topological and metric operators (Bauer and Behrens 2018);
- Lie group structure and underlying variational formulation (Gawlik and Gay-Balmaz 2020);
- connection with algebraic topology, i.e. de Rham complex and more general Hilbert complexes, e.g. elasticity (Bochev and Hyman 2006; Arnold, Falk, and Winther 2006; Palha et al. 2014);

Principle behind split discretization

- Fluid equation written in covariant form (exterior calculus);
- Split Hamiltonian form $\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\}$.
  - Topological braket depending on d (exterior derivative) or ι_v (interior product).
  - Metric dependent $\mathcal{H}$, since it depends on $*$

Linear shallow water waves in Hamiltonian form

- $\mathcal{H} = \frac{1}{2}\int_\Omega \left\{\bar{h}\,\|\mathbf{u}\|^2 + gh^2\right\}\, \mathrm{d}\Omega$, with $\frac{\delta\mathcal{H}}{\delta\mathbf{u}} = \bar{h}\mathbf{u}$, $\frac{\delta\mathcal{H}}{\delta h} = gh$, where g gravity acc. and $\bar{h}$ equilibrium fluid height.
- $\{\mathcal{F}, \mathcal{G}\} = -(\frac{\delta\mathcal{F}}{\delta\mathbf{u}}, \nabla\frac{\delta\mathcal{G}}{\delta h}) - (\frac{\delta\mathcal{F}}{\delta h}, \nabla\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{u}})$.

$$\mathcal{F} = \int \mathbf{u}\, \mathrm{d}\Omega: \ \dot{\mathcal{F}} = -(\frac{\delta\mathcal{F}}{\delta\mathbf{u}}, \nabla\frac{\delta\mathcal{H}}{\delta h}) \rightarrow \partial_t\mathbf{u} = -g\nabla h.$$

$$\mathcal{F} = \int h\, \mathrm{d}\Omega: \ \dot{\mathcal{F}} = -(\frac{\delta\mathcal{F}}{\delta h}, \nabla\cdot\frac{\delta\mathcal{H}}{\delta\mathbf{u}}) \rightarrow \partial_t h = -\bar{h}\nabla\cdot\mathbf{u}.$$

Split and weak (or mixed) form

De Rham complex: $V^0(\mathbb{R}) \xrightarrow{\nabla} V^1(\mathbb{R}^3) \xrightarrow{\nabla\times} V^2(\mathbb{R}^3) \xrightarrow{\nabla\cdot} V^3(\mathbb{R})$.

Split form

$$\begin{array}{ccc} h \in V^0 & \xrightarrow{\nabla} & V^1 \ni \mathbf{u} \\ {\scriptstyle \tilde{h}=\tilde{*}h}\downarrow & & \downarrow{\scriptstyle \mathbf{u}=\tilde{*}\mathbf{u}} \\ \tilde{h} \in V^3 & \xleftarrow{\nabla\cdot} & V^2 \ni \tilde{\mathbf{u}} \end{array}$$

- Assume full $\tilde{*}$ in metric eqs.
- Both $\nabla, \nabla\cdot$ are imposed strongly.
- Both diff. eqs exact

Weak form (Mixed FE)

$$\begin{array}{ccc} h \in V^3 & \xrightarrow{\hat{\nabla}} & V^2 \ni \mathbf{u} \\ {\scriptstyle \tilde{h}=h}\downarrow & & \downarrow{\scriptstyle \tilde{\mathbf{u}}=\mathbf{u}} \\ \tilde{h} \in V^3 & \xleftarrow{\nabla\cdot} & V^2 \ni \tilde{\mathbf{u}} \end{array}$$

- Assume $\tilde{*} = \mathrm{Id}$, i.e. $\tilde{\mathbf{u}} = \mathbf{u}$, $\tilde{h} = h$.
- Weak gradient $\hat{\nabla}$.
- Moment Eq. weak

Application of mixed finite elements

Infinite-dimensional pH system

PDE with boundary control:

$$\frac{\partial \boldsymbol{\alpha}}{\partial t}(\boldsymbol{x}, t) = \mathcal{J}\delta_{\boldsymbol{\alpha}} H.$$

Boundary conditions:

$$\boldsymbol{u}_\partial = \mathcal{B}_\partial \delta_{\boldsymbol{\alpha}} H, \quad \boldsymbol{y}_\partial = \mathcal{C}_\partial \delta_{\boldsymbol{\alpha}} H.$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial \, \mathrm{d}S.$$

Structure-preserving discretization

Resulting ODE:

Tools available

FEniCS: https://fenicsproject.org/.

Fluid Structure Interaction in Fenics: Bergersen et al. 2020.

Mesh morphing in FEniCS: https://bitbucket.org/Epoxid/femorph/src/c7317791c8f00d70fe16d593344cb164a53cad9b/?at=dokken%2Frestructuring

PyDec: https://github.com/hirani/pydec

Learning Python for scientific computing https://faculty.math.illinois.edu/~hirani/cbmg/index.html