Mixed finite elements for port-Hamiltonian von Kármán beams

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Abstract: A port-Hamiltonian formulation of von Kármán beams is presented. The variables selection lead to a non linear interconnection operator, while the constitutive laws are linear. The model can be readily discretized by exploiting a coenergy formulation and a mixed finite element method. The mixed formulation does not demand the H^2 regularity requirement typical of standard Galerkin discretization of thin structures. A numerical test is performed to assess the convergence rate of the solution. The error behavior exhibit the same convergence rate of the associated linear problem.

Keywords: Port-Hamiltonian systems (pHs), von Kármán beams, Mixed Finite Elements

1. INTRODUCTION

Linear elastic structures have been largely investigated into the port-Hamiltonian (pH) framework as well as the heat equation (consult for instance Macchelli and Melchiorri (2004) for the Timoshenko beam, Aoues et al. (2017) for the Euler-Bernoulli beam, Brugnoli et al. (2019b,a) for thick and thin plates). Recently, more complicated models arising from fluid dynamics have also been considered Cardoso-Ribeiro et al. (2020a); Rashad et al. (2020a,b); Altmann and Schulze (2017).

The Hamiltonian foundation of non-linear elasticity dates back to the late 80' Simo et al. (1988). However, the field of non-linear elasticity remains an unexplored topic in the pH framework

The development of new models within the pH framework has been accompanied with an increased interest in numerical discretization methods, capable of retaining the main features of the distributed system in its finite-dimensional counterpart. Recently, it has become evident that there is a strict link between discretization of port-Hamiltonian systems and mixed finite elements Cardoso-Ribeiro et al. (2020b). An example of this connection is given in Kirby and Kieu (2015), where a velocity-stress formulation for the wave dynamics is shown to be Hamiltonian and its mixed discretization preserves such a structure.

2. VON KÁRMÁN BEAMS

The classical von-Kármán beam model is presented in (Reddy, 2010, Chapter 4). Under the hypothesis of isotropic material, the extensional-bending stiffness is zero when the x-axis is taken along the geometric centroidal axis. With this assumption, the problem, defined on an open interval $\Omega = (0, L)$, takes the following form

$$\begin{aligned} \rho A \ddot{u} &= \partial_x n_{xx}, \\ \rho A \ddot{w} &= -\partial^2_{xx} m_{xx} + \partial_x (n_{xx} \partial_x w), \end{aligned} \tag{1}$$

together with the stresses and strains expressions

$$\begin{aligned} n_{xx} &= EA\varepsilon_{xx}, & m_{xx} &= EI\kappa_{xx}, \\ \varepsilon_{xx} &= \partial_x u + 1/2(\partial_x w)^2, & \kappa_{xx} &= \partial^2_{xx} w. \end{aligned} \tag{2}$$

Variable u is the horizontal displacement, w is the vertical displacement, n_{xx} is the axial stress resultant and m_{xx} is the bending stress resultant. The coefficients ρ, A, E, I are the mass density, the cross section, the Young module and the second moment of area.

The total energy of the model (Hamiltonian functional)

$$H = \frac{1}{2} \int_\Omega \left\{ \rho A(\dot{u}^2 + \dot{w}^2) + n_{xx}\varepsilon_{xx} + m_{xx}\kappa_{xx} \right\} \, \mathrm{d}\Omega, \tag{3}$$

consists of the kinetic energy and both membrane and bending deformation energies. This model proves conservative, see Bilbao et al. (2015). Indeed, this implies that a port-Hamiltonian realization of the system exists. We shall demonstrate how to construct a port-Hamiltonian realization, equivalent to (1).

3. THE EQUIVALENT PORT-HAMILTONIAN REALIZATION

To find a suitable port-Hamiltonian system, we first select a set of new energy variables to make the Hamiltonian functional quadratic

$$\alpha_u = \rho A \dot{u}, \quad \alpha_\varepsilon = \varepsilon_{xx}, \quad \alpha_w = \rho A \dot{w}, \quad \alpha_\kappa = \kappa_{xx}. \tag{4}$$

The energy is quadratic in these variables

$$H = \frac{1}{2} \int_\Omega \left\{ \frac{\alpha_u^2 + \alpha_w^2}{\rho A} + EA\varepsilon^2_{xx} + EI\kappa^2_{xx} \right\} \, \mathrm{d}\Omega. \tag{5}$$

By computing the variational derivative of the Hamiltonian, one obtains the so-called co-energy variables:

$$\begin{aligned} e_u &:= \delta_{\alpha_u} H = \dot{u}, & e_\varepsilon &:= \delta_{\alpha_\varepsilon} H = n_{xx}, \\ e_w &:= \delta_{\alpha_w} H = \dot{w}, & e_\kappa &:= \delta_{\alpha_\kappa} H = m_{xx}. \end{aligned} \tag{6}$$

Before stating the final formulation, consider the unbounded operator operator $\mathcal{C}(w)(\cdot) : L^2(\Omega) \to L^2(\Omega)$, that acts as follows

$$\mathcal{C}(w)(\cdot) = \partial_x(\cdot\, \partial_x w). \tag{7}$$

Proposition 1. The formal adjoint of the $\mathcal{C}(w)(\cdot)$ is given by

$$\mathcal{C}(w)^*(\cdot) = -\partial_x(\cdot)\partial_x(w). \tag{8}$$

Proof 1. Consider a smooth scalar fields with compact support $\psi \in C_0^\infty(\Omega)$ and $\xi \in C_0^\infty(\Omega)$. The formal adjoint of $\mathcal{C}(w)(\cdot)$ satisfies the relation

$$\langle \psi,\, \mathcal{C}(w)(\xi)\rangle_{L^2(\Omega)} = \langle \mathcal{C}(w)^*(\psi),\, \xi\rangle_{L^2(\Omega)}\,. \tag{9}$$

The proof follows from the computation

$$\begin{aligned}\langle \psi,\, \mathcal{C}(w)(\xi)\rangle_{L^2(\Omega)} &= \langle \psi,\, \partial_x(\xi\, \partial_x w)\rangle_{L^2(\Omega)}\,,\\ &= \langle -\partial_x\psi,\, \xi\partial_x w\rangle_{L^2(\Omega,\mathbb{R}^2)}\,,\\ &= \langle -\partial_x\psi\, \partial_x w,\, \xi\rangle_{L^2(\Omega)}\,.\end{aligned} \tag{10}$$

This means that

$$\mathcal{C}(w)^*(\cdot) = -\partial_x(\cdot)\partial_x w, \tag{11}$$

leading to the final result.

The pH realization is then given by the following system

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_u\\ \alpha_\varepsilon\\ \alpha_w\\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix}0 & \partial_x & 0 & 0\\ \partial_x & 0 & \partial_x w\, \partial_x & 0\\ 0 & \partial_x(\cdot\, \partial_x w) & 0 & -\partial^2_{xx}\\ 0 & 0 & \partial^2_{xx} & 0\end{bmatrix}\begin{pmatrix}e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\end{pmatrix}, \tag{12}$$

The second line of system (12) represents the time derivative of the membrane strain tensor. To close the system, variable w has to be accessible. For this reason, its dynamics has to be included. The augmented system reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_u\\ \alpha_\varepsilon\\ \alpha_w\\ \alpha_\kappa\\ w\end{pmatrix} = \underbrace{\begin{bmatrix}0 & \partial_x & 0 & 0 & 0\\ \partial_x & 0 & \partial_x w\, \partial_x & 0 & 0\\ 0 & \partial_x(\cdot\, \partial_x w) & 0 & -\partial^2_{xx} & -1\\ 0 & 0 & \partial^2_{xx} & 0 & 0\\ 0 & 0 & 1 & & 0\end{bmatrix}}_{\mathcal{J}}\begin{pmatrix}e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\\ \delta_w H\end{pmatrix}. \tag{13}$$

The operator $\mathcal{J}$ is formally skew-adjoint. If only the kinetic and deformation energies are considered, it holds $\delta_w H = 0$. In general this terms allows accommodating other potentials, for example the gravitational one. Suitable boundary variables are then obtained considering the power balance

$$\dot H = \langle e_u,\, e_\varepsilon\rangle_{\partial\Omega} + \langle e_w,\, e_\varepsilon\partial_x w - \partial_x e_\kappa\rangle_{\partial\Omega} + \langle \partial_x e_w,\, e_\kappa\rangle_{\partial\Omega}\,. \tag{14}$$

The boundary conditions are consistent with the ones assumed in Puel and Tucsnak (1996) for deriving a global existence result for this model.

4. MIXED FINITE ELEMENT DISCRETIZATION

To perform the numerical discretization, the constitutive relations are first incorporated in the dynamics. The link between the energy variables (4) and the conergy variables (6) is given by the linear transformation

$$\begin{pmatrix}\alpha_u\\ \alpha_\varepsilon\\ \alpha_w\\ \alpha_\kappa\end{pmatrix} = \begin{bmatrix}\rho A & 0 & 0 & 0\\ 0 & C_a & 0 & 0\\ 0 & 0 & \rho A & 0\\ 0 & 0 & 0 & C_b\end{bmatrix}\begin{pmatrix}e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\end{pmatrix}, \tag{15}$$

where $C_a = (EA)^{-1}$ and $C_b = (EI)^{-1}$ are the axial and bending compliance respectively. A pure coenergy formulation can then be employed once (15) is plugged into (13)

$$\begin{pmatrix}\rho A\dot e_u\\ C_a\dot e_\varepsilon\\ \rho A\dot e_w\\ C_b\dot e_\kappa\\ \dot w\end{pmatrix} = \begin{bmatrix}0 & \partial_x & 0 & 0 & 0\\ \partial_x & 0 & \partial_x w\, \partial_x & 0 & 0\\ 0 & \partial_x(\cdot\, \partial_x w) & 0 & -\partial^2_{xx} & -1\\ 0 & 0 & \partial^2_{xx} & 0 & 0\\ 0 & 0 & 1 & & 0\end{bmatrix}\begin{pmatrix}e_u\\ e_\varepsilon\\ e_w\\ e_\kappa\\ \delta_w H\end{pmatrix}. \tag{16}$$

To derive the discrete system, first (16) is put into weak form. To this aim the test functions $(\psi_u,\, \psi_\varepsilon,\, \psi_w,\, \psi_\kappa,\, \psi)$ are introduced. For sake of simplicity, no dependency between the displacements and the energy is considered, i.e. $\delta_w H = 0$:

$$\begin{aligned}\langle \psi_u,\, \rho A\, \dot e_u\rangle_\Omega &= \langle \psi_u,\, \partial_x e_\varepsilon\rangle_\Omega\,.\\ \langle \psi_\varepsilon,\, C_a\, \dot e_\varepsilon\rangle_\Omega &= \langle \psi_\varepsilon,\, \partial_x e_u\rangle_\Omega + \langle \psi_\varepsilon,\, \partial_x w\, \partial_x e_w\rangle_\Omega\,,\\ \langle \psi_w,\, \rho A\dot e_w\rangle_\Omega &= \langle \psi_w,\, \partial_x(e_\varepsilon\partial_x w)\rangle_\Omega - \left\langle \psi_w,\, \partial^2_{xx} e_\kappa\right\rangle_\Omega\,,\\ \langle \psi_\kappa,\, C_b\, \dot e_\kappa\rangle_\Omega &= \left\langle \psi_\kappa,\, \partial^2_{xx} e_w\right\rangle_\Omega\,,\\ \langle \psi,\, \dot w\rangle_\Omega &= \langle \psi,\, e_w\rangle_\Omega\,.\end{aligned} \tag{17}$$

Then the integration by parts is performed on the first line, the third and fourth line. This choice is such to retain the skew-symmetric structure at the discrete level and to lower the regularity requirement for the finite elements (Brugnoli, 2020, Chap. 8). The weak formulation then looks for $(e_u, e_w, e_\kappa, w) \in H^1(\Omega)$, $e_\varepsilon \in L^2(\Omega)$ such that the following system

$$\begin{aligned}\langle \psi_u,\, \rho A\, \dot e_u\rangle_\Omega &= -\langle \partial_x\psi_u,\, e_\varepsilon\rangle_\Omega + \langle \psi_u,\, e_\varepsilon\rangle_{\partial\Omega}\,.\\ \langle \psi_\varepsilon,\, C_a\, \dot e_\varepsilon\rangle_\Omega &= \langle \psi_\varepsilon,\, \partial_x e_u\rangle_\Omega + \langle \psi_\varepsilon,\, \partial_x w\, \partial_x e_w\rangle_\Omega\,,\\ \langle \psi_w,\, \rho A\dot e_w\rangle_\Omega &= -\langle \partial_x\psi_w\partial_x w,\, e_\varepsilon\rangle_\Omega + \langle \partial_x\psi_w,\, \partial_x e_\kappa\rangle_\Omega\\ &\quad + \langle \psi_w,\, e_\varepsilon\partial_x w - \partial_x e_\kappa\rangle_{\partial\Omega}\,,\\ \langle \psi_\kappa,\, C_b\, \dot e_\kappa\rangle_\Omega &= -\langle \partial_x\psi_\kappa,\, \partial_x e_w\rangle_\Omega + \langle \psi_\kappa,\, \partial_x e_w\rangle_{\partial\Omega}\,,\\ \langle \psi,\, \dot w\rangle_\Omega &= \langle \psi,\, e_w\rangle_\Omega\,.\end{aligned} \tag{18}$$

holds $\forall(\psi_u, \psi_w, \psi_\kappa, \psi) \in H^1(\Omega)$, $\forall\psi_\varepsilon \in L^2(\Omega)$. In this formulation, the boundary axial forces $e_\varepsilon|_0^L$, vertical forces $e_\varepsilon\partial_x w - \partial_x e_\kappa|_0^L$ and rotations $\partial_x e_w|_0^L$ are enforced weakly. To obtain the associated finite-dimensional system, the following Galerkin approximation is considered

$$\begin{aligned} e_u^h &= \sum_{i=1}^{n_u}\xi_u^i(x)e_u^i(t), & \psi_u^h &= \sum_{i=1}^{n_u}\xi_u^i(x)\psi_u^i,\\ e_\varepsilon^h &= \sum_{i=1}^{n_\varepsilon}\xi_\varepsilon^i(x)e_\varepsilon^i(t), & \psi_\varepsilon^h &= \sum_{i=1}^{n_\varepsilon}\xi_\varepsilon^i(x)\psi_\varepsilon^i,\\ e_w^h &= \sum_{i=1}^{n_w}\xi_w^i(x)e_w^i(t), & \psi_w^h &= \sum_{i=1}^{n_w}\xi_w^i(x)\psi_w^i,\\ e_\kappa^h &= \sum_{i=1}^{n_\kappa}\xi_\kappa^i(x)e_\kappa^i(t), & \psi_\kappa^h &= \sum_{i=1}^{n_\kappa}\xi_\kappa^i(x)\psi_\kappa^i,\\ w^h &= \sum_{i=1}^{n_w}\xi_w^i(x)w^i(t), & \psi^h &= \sum_{i=1}^{n_w}\xi_w^i(x)\psi^i. \end{aligned} \tag{19}$$

Notice that w, e_w are discretized using the same test functions. Plugging (19) into (18), the following finite dimensional system is obtained

$$\begin{pmatrix} \mathbf{M}_u\dot{\mathbf{e}}_u \\ \mathbf{M}_\varepsilon\dot{\mathbf{e}}_\varepsilon \\ \mathbf{M}_w\dot{\mathbf{e}}_w \\ \mathbf{M}_\kappa\dot{\mathbf{e}}_\kappa \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\varepsilon u}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{D}_{\varepsilon u} & \mathbf{0} & \mathbf{D}_{\varepsilon w}(\mathbf{w}) & \mathbf{0} \\ \mathbf{0} & -\mathbf{D}_{\varepsilon w}^\top(\mathbf{w}) & \mathbf{0} & \mathbf{D}_{w\kappa} \\ \mathbf{0} & \mathbf{0} & -\mathbf{D}_{w\kappa}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_u \\ \mathbf{e}_\varepsilon \\ \mathbf{e}_w \\ \mathbf{e}_\kappa \end{pmatrix}, \tag{20}$$

where the boundary terms have been omitted for simplicity. The mass matrices are defined as follows

$$\begin{aligned} M_u^{ij} &= \left\langle \xi_u^i,\, \rho A \xi_u^j \right\rangle_\Omega, \qquad & M_w^{ij} &= \left\langle \xi_w^i,\, \rho A \xi_w^j \right\rangle_\Omega, \\ M_\varepsilon^{ij} &= \left\langle \xi_\varepsilon^i,\, C_a \xi_\varepsilon^j \right\rangle_\Omega, \qquad & M_\kappa^{ij} &= \left\langle \xi_\kappa^i,\, C_b \xi_\kappa^j \right\rangle_\Omega. \end{aligned} \tag{21}$$

The interconnection matrices are given by

$$\begin{aligned} D_{\varepsilon u}^{ij} &= \left\langle \xi_\varepsilon^i,\, \partial_x \xi_u^j \right\rangle_\Omega, \\ D_{\varepsilon w}^{ij}(\mathbf{w}) &= \left\langle \xi_\varepsilon^i,\, \sum_{k=1}^{n_w} \partial_x \xi_w^k(x) w^k(t) \partial_x \xi_w^j \right\rangle_\Omega, \\ D_{w\kappa}^{ij} &= \left\langle \partial_x \xi_w^i,\, \partial_x \xi_\kappa^j \right\rangle_\Omega. \end{aligned} \tag{22}$$

For what concerns the choice of the underlying finite elements, a simple selection conforming to the weak form (18) is given by continuous Galerkin to discretize the space $H^1(\Omega)$ and discontinuous Galerkin for the space $L^2(\Omega)$. Consider an interval mesh $\mathcal{I}_h$ with elements E. The space of polynomials of order k on a mesh cell is denoted by P_k. The following conforming family of finite elements is selected for this problem:

$$\begin{aligned} H_h^1(\Omega) &= \left\{ v_h \in H^1(\Omega) | \forall E \in \mathcal{I}_h,\ v_h|_E \in P_k \right\}, \\ L_h^2(\Omega) &= \left\{ u_h \in L^2(\Omega) | \forall E \in \mathcal{I}_h,\ u_h|_E \in P_{k-1} \right\}. \end{aligned} \tag{23}$$

The space H_h^1 corresponds to the space of continous Galerkin elements of order k, whereas L_h^2 corresponds to the space of discontinuous Galerkin elements of order $k-1$. This choice guarantees that $\partial_x H_h^1(\Omega) \subset L_h^2(\Omega)$. So the discrete weak formulation becomes: find $(e_u^h, e_w^h, e_\kappa^h, w^h) \in H_h^1(\Omega)$, $e_\varepsilon^h \in L_h^2(\Omega)$ such that system (18) holds $\forall (\psi_u^h, \psi_w^h, \psi_\kappa^h, \psi^h) \in H_h^1(\Omega)$, $\forall \psi_\varepsilon^h \in L_h^2(\Omega)$.

5. NUMERICAL TESTS

Consider the following analytical solution for the axial and vertical displacement

$$\begin{aligned} u^{\text{ex}} &= x(1 - x/L)\sin(2\pi t), \\ w^{\text{ex}} &= \sin(\pi x/L)\sin(2\pi t), \end{aligned} \tag{24}$$

together with the boundary conditions

$$u|_0^L = 0, \quad w|_0^L = 0, \quad m_{xx}|_0^L = 0. \tag{25}$$

For u and w given in (24) to be the solution of (1), appropriate forcing term have to be introduced. These are given by

$$\begin{aligned} f_u &= \rho A \partial_{tt}^2 u - \partial_x n_{xx}, \\ f_w &= \rho A \partial_{tt}^2 w + \partial_{xx}^2 m_{xx} - \partial_x (n_{xx} \partial_x w), \end{aligned} \tag{26}$$

The numerical values of the parameters for the simulation are reported in Table 1. The Firedrake library (Rathgeber et al. (2017)) is used to generate the matrices. To integrate the equations in time a Crank-Nicholson scheme has been used, for all simulations. The time step is set to $\Delta t = h/(2\pi)$ to have the same impact of the time discretization error with respect to the spatial error. The final time is set to one $t_f = 1$[s]. The non-linear system is solved using Newton-Krylov iterations with a line search and direct factorisation of the linear system.

To measure the convergence suitable norms have to be introduced. Let $\mathcal{X}$ be a Hilbert space, and t_f a positive real number. We denote by $L^\infty([0,t_f];\mathcal{X})$ or $L^\infty(\mathcal{X})$ the space of functions $f : [0,t_f] \to X$ for which the time-space norm $||\cdot||_{L^\infty([0,t_f];\mathcal{X})}$ satisfies

$$||f||_{L^\infty([0,t_f];\mathcal{X})} = \operatorname*{ess\,sup}_{t\in[0,t_f]} ||f||_{\mathcal{X}} < \infty.$$

To compute the $L^\infty(\mathcal{X})$ space-time dependent norm from the numerical results, the discrete norm $L_{\Delta t}^\infty(\mathcal{X})$ is used

$$||\cdot||_{L^\infty(\mathcal{X})} \approx ||\cdot||_{L_{\Delta t}^\infty(\mathcal{X})} = \max_{t\in t_i} ||\cdot||_{\mathcal{X}},$$

where t_i are the discrete simulation instants. The Hilbert space $\mathcal{X}$ depends on the regularity of the considered variable. For variables (e_u, e_w, e_κ, w) the error is measured in the $L_{\Delta t}^\infty(H^1)$, whereas for e_ε is measured in the $L_{\Delta t}^\infty(L^2)$. The simulation results are reported in Fig. 1. It can be noticed that all variable converge at rate given by the polynomial degree k. This is in accordance with the order of convergence of the the linear problem (cf. Brugnoli (2020) for the Euler-Bernoulli beam).

Beam parameters				
E	ρ	L	A	I
70 [GPa]	2700 [kg/m^3]	1 [m]	0.01 [m^2]	8.3 10^{-6} [m^4]

Table 1. Physical parameters for the beam.

6. CONCLUSION

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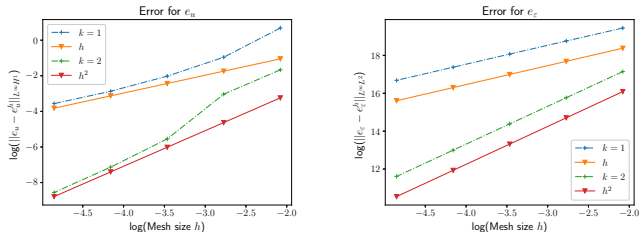
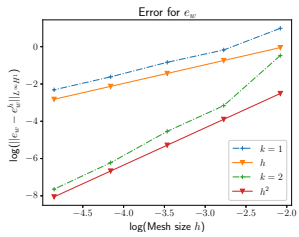
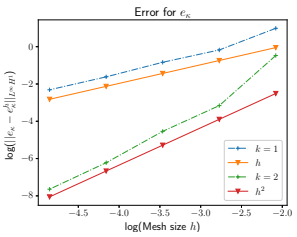


(a) $L^\infty_{\Delta t}(H^1)$ error for e_u

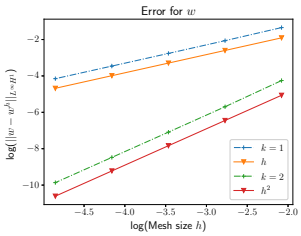
(b) $L^\infty_{\Delta t}(L^2)$ error for e_ε



(c) $L^\infty_{\Delta t}(H^1)$ error for e_w



(d) $L^\infty_{\Delta t}(H^1)$ error for e_κ



(e) $L^\infty_{\Delta t}(H^1)$ error for w

Fig. 1. Error trend for the different variables

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