

The Euler-Bernoulli beam in differential form

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1 Classical formulation

The Euler-Bernoulli beam model consists of one PDE, describing the vertical displacement along the beam length:

$$\rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t) + \frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) = 0, \quad x \in \Omega = \{0, L\}, \quad (1)$$

where $w(x, t)$ is the transverse displacement of the beam. The coefficients $\rho(x)$, $A(x)E(x)$ and $I(x)$ are the mass density, cross section, Young's modulus of elasticity and the moment of inertia of a cross section. The energy variables are then chosen as follows:

$$\alpha_w = \rho A(x) \frac{\partial w}{\partial t}(x, t), \quad \text{Linear Momentum}, \quad \alpha_\kappa = \frac{\partial^2 w}{\partial x^2}(x, t), \quad \text{Curvature}. \quad (2)$$

Those variables are collected in the vector $\boldsymbol{\alpha} = (\alpha_w, \alpha_\kappa)^T$, so that the Hamiltonian can be written as a quadratic functional in the energy variables:

$$H = \frac{1}{2} \int_{\Omega} \left\{ \frac{1}{\rho A} \alpha_w^2 + EI \alpha_\kappa^2 \right\} d\Omega \quad (3)$$

The co-energy variables are found by computing the variational derivative of the Hamiltonian:

$$\begin{aligned} e_w &:= \frac{\delta H}{\delta \alpha_w} = \frac{\partial w}{\partial t}(x, t), & \text{Vertical velocity,} \\ e_\kappa &:= \frac{\delta H}{\delta \alpha_\kappa} = EI(x) \frac{\partial^2 w}{\partial x^2}(x, t), & \text{Flexural momentum.} \end{aligned} \quad (4)$$

The underlying interconnection structure is then found to be:

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\partial_{xx} \\ \partial_{xx} & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}. \quad (5)$$

The power flow gives access to the boundary variables:

$$\begin{aligned}
\dot{H} &= \int_{\Omega} \{e_w \partial_t \alpha_w + e_{\kappa} \partial_t \alpha_{\kappa}\} \, d\Omega, \\
&= \int_{\Omega} \{-e_w \partial_{xx} e_{\kappa} + e_{\kappa} \partial_{xx} e_w\} \, d\Omega, \quad \text{Integration by parts,} \\
&= \int_{\partial\Omega} \{-e_w \partial_x e_{\kappa} + e_{\kappa} \partial_x e_w\} \, ds = \langle -e_w, \partial_x e_{\kappa} \rangle_{\partial\Omega} + \langle e_{\kappa}, \partial_x e_w \rangle_{\partial\Omega}
\end{aligned} \tag{6}$$

Since the system is of differential order two, two pairing appears, giving rise to four combination of uniform boundary causality

- First case $u_{\partial,1} = e_w$, $u_{\partial,2} = \partial_x e_w$, $y_{\partial,1} = -\partial_x e_{\kappa}$, $y_{\partial,2} = e_{\kappa}$.
This imposes the vertical $e_w := \partial_t w$ and angular velocity $\partial_x e_w := \partial_{xt} w$ as boundary inputs. If the inputs are null a clamped boundary condition is obtained.
- Second case $u_{\partial,1} = e_w$, $u_{\partial,2} = e_{\kappa}$, $y_{\partial,1} = -\partial_x e_{\kappa}$, $y_{\partial,2} = \partial_x e_w$.
This imposes the vertical velocity and flexural momentum $e_{\kappa} := EI \partial_{xx} w$ as boundary inputs. Zero inputs lead to a simply supported condition is found.
- Third case $u_{\partial,1} = -\partial_x e_{\kappa}$, $u_{\partial,2} = e_{\kappa}$, $y_{\partial,1} = e_w$, $y_{\partial,2} = \partial_x e_w$.
This imposes the shear force $\partial_x e_{\kappa} := \partial_x (EI \partial_{xx} w)$ and flexural momentum as boundary inputs. Null inputs correspond to a free condition.
- Forth case $u_{\partial,1} = -\partial_x e_{\kappa}$, $u_{\partial,2} = \partial_x e_w$, $y_{\partial,1} = e_w$, $y_{\partial,2} = e_{\kappa}$.
This imposes the shear force and angular velocity as boundary inputs.

2 Differential forms formulation

The co-energy now are 0-forms $e_w, e_{\kappa} \in \Lambda^0(\Omega)$ whereas the flows are 1-forms $f_w = \partial_t \alpha_w, f_{\kappa} = \partial_t \alpha_{\kappa} \in \Lambda^1(\Omega)$. To recast (5) using the exterior derivative, the Hodge star operator is needed.

$$* : \Lambda^k(\Omega) \rightarrow \Lambda^{n-k}(\Omega), \quad \Omega \subset \mathbb{R}^n. \tag{7}$$

For one dimensional domains $\Omega \subset \mathbb{R}$ and using Euclidian coordinates, this operator can be either used on 1-forms or 0-forms

$$\begin{aligned}
* : \Lambda^1(\Omega) &\longrightarrow \Lambda^0(\Omega), \\
f(x) \, dx &\longrightarrow f(x)
\end{aligned} \tag{8}$$

or

$$\begin{aligned}
* : \Lambda^0(\Omega) &\longrightarrow \Lambda^1(\Omega), \\
f(x) &\longrightarrow f(x) \, dx
\end{aligned} \tag{9}$$

Then the equivalent system reads

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_w \\ \alpha_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -d*d \\ d*d & 0 \end{bmatrix} \begin{pmatrix} e_w \\ e_\kappa \end{pmatrix}. \quad (10)$$

Proof 1 *The operator $d*d : \Lambda^0(\Omega) \rightarrow \Lambda^1(\Omega)$ is a composition of operators that reads in Euclidean coordinates*

$$\begin{aligned} d*de &= d*\left(\frac{\partial e}{\partial x} dx\right), \quad e \in \Lambda^0(\Omega) \\ &= d\left(\frac{\partial e}{\partial x}\right), \\ &= \frac{\partial^2 e}{\partial x^2} dx \in \Lambda^1(\Omega) \end{aligned} \quad (11)$$

The Hamiltonian energy is then

$$H = \frac{1}{2} \int_{\Omega} e_w \wedge \alpha_w + e_\kappa \wedge \alpha_\kappa \quad (12)$$

To find the appropriate power balance, consider the integration by parts formula for smooth differential forms $\lambda \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^{n-k-1}(\Omega)$,

$$\langle d\lambda, \mu \rangle = \langle \text{Tr } \lambda, \text{Tr } \mu \rangle - (-1)^k \langle \lambda, d\mu \rangle \quad (13)$$

For uni-dimensional domains $n = 1$, and we take $\lambda \in \Lambda^0(\Omega)$ and $\mu \in \Lambda^0(\Omega)$, implying

$$\begin{aligned} \langle d\lambda, \mu \rangle &= \langle \text{Tr } \lambda, \text{Tr } \mu \rangle - \langle \lambda, d\mu \rangle, \\ \langle \lambda, d\mu \rangle &= \langle \text{Tr } \lambda, \text{Tr } \mu \rangle - \langle d\lambda, \mu \rangle. \end{aligned} \quad (14)$$

Then we can state

$$\langle -e_w, d(*de_\kappa) \rangle = \langle -\text{Tr } e_w, \text{Tr}(*de_\kappa) \rangle + \langle de_w, *de_\kappa \rangle. \quad (15)$$

and

$$\langle e_\kappa, d(*de_w) \rangle = \langle \text{Tr } e_\kappa, \text{Tr}(*de_w) \rangle - \langle de_\kappa, *de_w \rangle, \quad (16)$$

The power rate then reads

$$\begin{aligned} \dot{H} &= \int_{\Omega} e_w \wedge \partial_t \alpha_w + e_\kappa \wedge \partial_t \alpha_\kappa, \\ &= \int_{\Omega} e_w \wedge \partial_t \alpha_w + e_\kappa \wedge \partial_t \alpha_\kappa, \\ &= \int_{\Omega} -e_w \wedge (d*de_\kappa) + e_\kappa \wedge (d*de_w), \quad \text{From (15), (16)} \\ &= \langle -\text{Tr } e_w, \text{Tr}(*de_\kappa) \rangle + \langle de_w, *de_\kappa \rangle + \langle \text{Tr } e_\kappa, \text{Tr}(*de_w) \rangle - \langle de_\kappa, *de_w \rangle \end{aligned} \quad (17)$$

The wedge product is such that for $\lambda \in \Lambda^k(\Omega)$ and $\mu \in \Lambda^l(\Omega)$ it holds

$$\lambda \wedge \mu = (-1)^{kl} \mu \wedge \lambda \quad (18)$$

Furthermore, the Hodge star is such that

$$(\alpha, \beta) := \langle \alpha, *\beta \rangle = \langle *\alpha, \beta \rangle = (\beta, \alpha), \quad \alpha, \beta \in \Lambda^k(\Omega) \quad (19)$$

Then for $e_w, e_\kappa \in \Lambda^0$ it holds

$$\begin{aligned} \langle de_w, *de_\kappa \rangle &= \langle *de_w, de_\kappa \rangle, \\ &= \int_{\Omega} (*de_w) \wedge (de_\kappa), \\ &= \int_{\Omega} (de_\kappa) \wedge (*de_w), \\ &= \langle de_\kappa, *de_w \rangle \end{aligned} \quad (20)$$

It can be then stated

$$\langle de_w, *de_\kappa \rangle - \langle de_\kappa, *de_w \rangle = 0 \quad (21)$$

The power balance is then

$$\dot{H} = \langle -\text{Tr } e_w, \text{Tr}(*de_\kappa) \rangle + \langle \text{Tr } e_\kappa, \text{Tr}(*de_w) \rangle \quad (22)$$

That in vector calculus notation reads

$$\dot{H} = \left\langle -\text{Tr } e_w, \text{Tr} \frac{\partial e_\kappa}{\partial x} \right\rangle + \left\langle \text{Tr } e_\kappa, \text{Tr} \frac{\partial e_w}{\partial x} \right\rangle = \langle -e_w, \partial_x e_\kappa \rangle_{\partial\Omega} + \langle e_\kappa, \partial_x e_w \rangle_{\partial\Omega} \quad (23)$$

Completely equivalent to Eq. (6).

References

- [1] P. Kotyczka. *Numerical Methods for Distributed Parameter Port-Hamiltonian Systems*. TUM University Press, 2019.