1 GS Notes

GS dispersion relation:

$$q^3 + A(\vec{k})q^2 + B(\vec{k})q + C(\vec{k}) = 0 \tag{1}$$

$$A(\vec{k}) = \frac{k^2}{\Omega}\Big(2\nu + \frac{1}{\gamma}\xi\Big) \tag{2}$$

$$B(\vec{k}) = \Big(\frac{k_z}{k}\Big)^2\Big[\frac{1}{\gamma\Omega^2\rho}(-\tilde{D}P + \rho\Omega^2 R)\tilde{D}\sigma - \frac{2}{\Omega R}\tilde{D}l\Big] + \frac{2}{\gamma}\Big(\frac{\xi k^2}{\Omega}\Big)\Big(\frac{\nu k^2}{\Omega}\Big) \tag{3}$$

$$C(\vec{k}) = \Big(\frac{k_z}{k}\Big)^2\Big[\frac{1}{\gamma\Omega^2\rho}\Big(\frac{\nu k^2}{\Omega}\Big)(-\tilde{D}P + \rho\Omega^2 R)\tilde{D}\sigma - \frac{2}{\gamma\Omega R}\Big(\frac{\xi k^2}{\Omega}\Big)\tilde{D}l\Big] + \frac{1}{\gamma}\Big(\frac{\xi k^2}{\Omega}\Big)\Big(\frac{\nu k^2}{\Omega}\Big)^2 \tag{4}$$

$$\tilde{D} = \frac{k_R}{k_z}\frac{\partial}{\partial z} - \frac{\partial}{\partial R} \tag{5}$$

There are unstable modes if $C(\vec{k}) < 0$. We discard the last term in C, which is always positive, and neglect $\rho\Omega^2 R$ compared to $\tilde{D}P$. The condition $C(\vec{k}) < 0$ can be rewritten as ($\mathrm{Pr} \equiv \nu/\xi$):

$$\mathrm{Pr} < -\Big(\frac{\Omega r}{c_s}\Big)^2 \frac{2\gamma}{(-\frac{d\log P}{d\log r})(\frac{d\sigma}{d\log r})} \times \frac{2 + \frac{d\log\Omega}{d\log R} - x\tan\theta\frac{d\log\Omega}{d\log z}}{(x\cos\theta - \sin\theta)^2}, \tag{6}$$

where $c_s^2 = \gamma P/\rho$, $\sigma = \log(P\rho^{-\gamma})$. Typical values at $r = 0.6R_\odot$:

$$\Big(\frac{\Omega r}{c_s}\Big)^2 \sim 2\cdot 10^{-5}, \qquad \frac{d\log P}{d\log r} \sim -6.9, \qquad \frac{d\log\sigma}{d\log r} \sim 2.6, \tag{7}$$

$$\frac{d\log\Omega}{d\log R} \sim -0.001, \qquad \frac{d\log\Omega}{d\log z} \sim -0.02. \tag{8}$$

At $r = 0.7R_\odot$, the logarithmic derivatives of P and σ are only moderately different, while those of Ω are larger by an order of magnitude.

We note that the numerical factor $2\gamma/\Big[(-\frac{d\log P}{d\log r})(\frac{d\sigma}{d\log r})\Big]$ is therefore positive and of order unity or less. For mid-latitudes, $\tan\theta\frac{d\log\Omega}{d\log z}$ is less than 0.1 and the numerator of the last term is positive. The condition reduces to $\mathrm{Pr} < Q$ with $Q < 0$ which is clearly satisfied for any viscosity. The only cases in which the instability may occur are those for which the numerator of the last term is negative. Unless $\tan\theta$ is large, i.e. θ is close to $\pi/2$, this implies that $|x| >> 1$. In this case, however, the denominator of is also large, of order x^2, and the term results small overall. We explored the maximum of the function for values of θ in the $5-85°$ and $r = 0.5R_\odot - 0.7R_\odot$ and found that it is always well below $\sim 10^{-2}$. We show the maximum of the function:

$$N(x;r,\theta) = \frac{2\gamma}{(-\frac{d\log P}{d\log r})(\frac{d\sigma}{d\log r})} \times \frac{2 + \frac{d\log\Omega}{d\log R} - x\tan\theta\frac{d\log\Omega}{d\log z}}{(x\cos\theta - \sin\theta)^2} \tag{9}$$

for $-10^6 < x < 10^6$ as a function of r for 3 values of the co-latitude θ in figure 1.

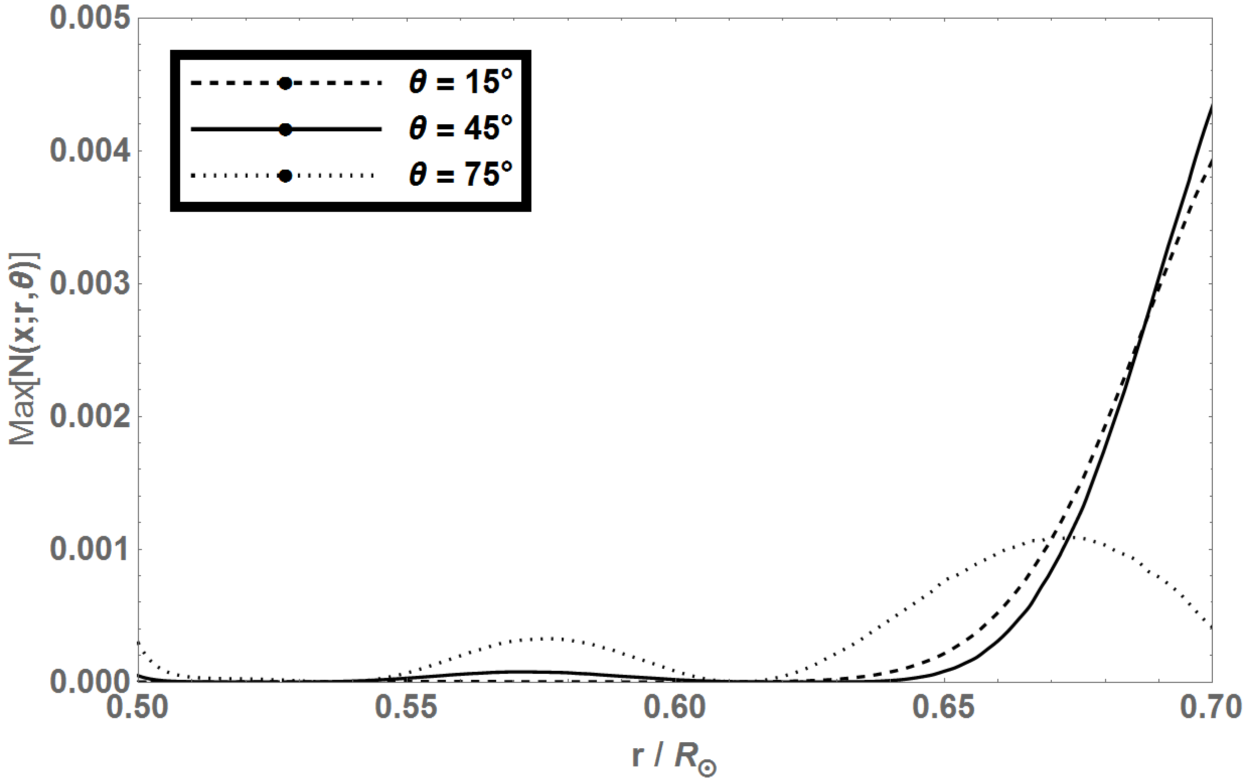

Figure 1: Maximum of $N(x;r,\theta)$ for $-10^6 < x < 10^6$.

Finally, we explored the maximum of such term for $\theta \to \pi/2$. We have not found any higher value of it. The reason is as follows: while $\tan\theta \to +\infty$ for $\theta \to \pi/2$, it's also true that:

$$\frac{d\log\Omega}{d\log z} = \frac{z}{\Omega}\frac{d\Omega}{dz} \to 0 \text{ for } \theta \to \pi/2 \tag{10}$$

The condition for instability is therefore:

$$\mathrm{Pr} \lesssim 10^{-2}\left(\frac{\Omega r}{c_s}\right)^2 \tag{11}$$

This condition (except for the numerical factor 10^{-2}) holds as long as $|\frac{d\log\Omega}{d\log R}|$ and $|\frac{d\log\Omega}{d\log z}|$ are smaller than 2. At this point, values of $x \sim 1$ may be sufficient to make the numerator of the last term of (6) positive where the denominator vanishes, and the system may be unstable even for large values of Pr.

So in general we can say:

$$\mathrm{Pr} = \frac{\nu}{\xi} \lesssim \left(\frac{\Omega r}{c_s}\right)^2 \tag{12}$$

The viscosity $\nu = \nu_{\mathrm{dyn}} + \nu_{\mathrm{rad}}$ where ν_{dyn} and ν_{rad} are the dynamical and radiative viscosity respectively. In most cases, ν_{dyn} dominates in a star. However, let's consider the case in which ν_{rad} is significant. It is known that:

$$\mathrm{Pr}_{\mathrm{rad}} = \frac{\nu_{\mathrm{rad}}}{\xi} \cong \left(\frac{c_s}{c}\right)^2, \tag{13}$$

where c_s is the sound speed; see e.g. Mihalas (1983). The condition (12) therefore prescribes that the onset of the instability is only possible if:

$$\Omega r \gtrsim c_s \cdot \frac{c_s}{c} \tag{14}$$

Typically, in a stellar core, $c_s/c \sim 10^{-3}$; hence the condition is $v_{rot} \gtrsim 10^{-3} c_s$. For example, for the angular velocity with the discontinuity for model D of Deheuvels et al. (2014), this would require $\Omega \gtrsim 9 \times 10^4$ nHz, which is not observed (their fit gives $\Omega \sim 5 \times 10^3$ nHz).