AFEMT - Lecture 1

0. Motivation for nonconforming FE

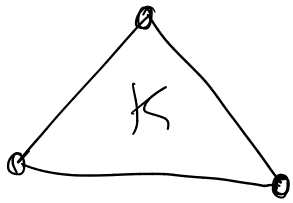
nonconforming : - V ambient solution space

- V_h FE space

$V_h \not\subset V \Rightarrow$ variational crime

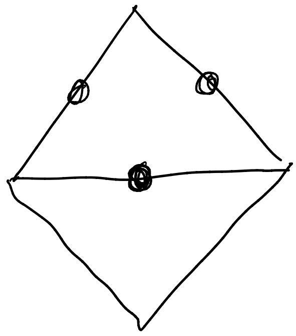
example: $V = H^1$ conf. FE $V_h^c \subset C^0$

linear FEM



$\mathbb{P}^1(K)$

linear CR

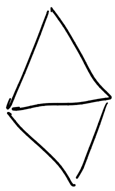


$\mathbb{P}^1(K)$

$V_h^{CR} \not\subset C^0$

but with some level of continuity

$V_h^{DG} \not\subset C^0$ $\quad V_h^{DG} \subset L^2$ only



$\mathbb{P}^1(K)$

hystory

- Crouzeix-Raviart 73
  - low order element for Stokes which is div-free

$P_1^{CR} - P_0$ element

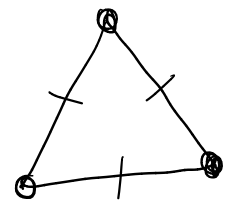
vel. pres

- Morley, Aeronautical Quarterly, 1968

  - low order locking-free for elasticity problems modeling membranes plates

    ↳ 4th order problems

    $\mathbb{P}_2$ element "minimal"



$V_h^M \not\subset C^0$

as opposed to C^1 conforming elements, e.g. Argyris element contains some $\mathbb{P}_5$ functions !

Note: generalisations of CR element not obvious:

- $\mathbb{P}_2^{CR}$ for quadrilateral meshes Fortin-Soulie 1983
- P_1^{CR} " " Park-Sheen 2003
- Morley for tetrahedra: Ming-Xu 2006
- ⋮
- Virtual element Method 2016 nonconforming Ayuso, Lipnikov, Manzini

extension of CR to
- any order
- any polygonal/polyhedral elements

<u>DG</u>

- Reed-Hill (Los Alamos) 1973
  neutron transport problems developing discontinuous solutions

- analysed by Lesaint + Raviart 1974
  Johnson-Pitkaranta 1986

- DG for time-stepping : Jamet (1978)
  parabolic problems by space/time FEM
  C^0 / DG
  ↳ eg: $u_t = \Delta u$
  $(u_t, v) + (\nabla u, \nabla v) = 0 \quad \forall v \in H^1 \qquad u \in L^2(0,T;H^1)$

  DG in time allows to decouple time-steps ⇒ solve by time-marching

- Mitsche (1971) : method to impose essential boundary cond. weakly

- classical DG :

  Baker 1977, Wheeler 1978, Arnold 1982

4th order problems
eg. bilaplacian

2nd order
elliptic
problems

IPDG
(Interior-Penalty DG)

precursor: Babuska 1973 - penalty method

see review Arnold, Brezzi, Cockburn, Marini 2002

↳ motivation:
- flexible in problems
|| mesh / order adaptivity

1 Classical nonconforming FE

- $\Omega \subset \mathbb{R}^d$ open polyhedron
  $\partial\Omega$ the boundary, $\underline{n}$ = unit outward normal
- mesh < classical simplicial / hexahedral $\mathcal{T}_h$
  polytopic

classical $\mathcal{T}_h$ collection of disjoint "elements"
$K \in \mathcal{T}_h$ s.t. $\bigcup_{K\in\mathcal{T}_h} \overline{K} = \overline{\Omega}$

$h_K = \operatorname{diam}(K)$

$h = \max_{K\in\mathcal{T}_h} h_K$

$F \subset \overline{\Omega}$ a mesh (inter)face

- $(d-1)$ - dimensional
- $\exists K_1, K_2 \in \mathcal{T}_h : F = \partial K_1 \cap \partial K_2$ → $\mathcal{F}_h^i$

or $\exists K_1 \in \mathcal{T}_h : F \subset \partial K \cap \partial \Omega$ → $\mathcal{F}_h^b$

$\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$

- Averages + jumps

v scalar smooth enough function

$\forall F \in \mathcal{F}_h^i \qquad \{\{v\}\} = \frac{1}{2}\left(v|_{K_1} + v|_{K_2}\right)$ a.e. on F

$$[\![v]\!] = v|_{K_1} \underline{n}_{K_1} + v|_{K_2} \underline{n}_{K_2} = \left(v|_{K_1} - v|_{K_2}\right) \underline{n}_{K_1}$$

- $\underline{v}$ vector smooth function

$$\{\{\underline{v}\}\} = \frac{1}{2}\left(\underline{v}|_{K_1} + \underline{v}|_{K_2}\right)$$

$$[\![\underline{v}]\!] = \underline{v}|_{K_1} \cdot \underline{n}_{K_1} + \underline{v}|_{K_2} \cdot \underline{n}_{K_2}$$

1.2 Broken spaces

Given $\mathcal{T}_h$ mesh,

$$W^{m,p}(\mathcal{T}_h) = \{ v \in L^p(\Omega) : \forall K \in \mathcal{T}_h,\ v|_K \in W^{m,p}(K) \}$$

in particular $p=2$

$$H^m(\mathcal{T}_h) = \{ v \in L^2(\Omega) : \forall K \in \mathcal{T}_h,\ v|_K \in H^m(K) \}$$

with norm / seminorm

$$|||v|||_m = \left(\sum_{K \in \mathcal{T}_h} \|v\|^2_{H^m(K)} \right)^{1/2}, \quad |v|_{m,\mathcal{T}_h} = \left(\sum_{K \in \mathcal{T}_h} |v|^2_{H^m(\mathcal{T}_h)} \right)^{1/2}$$

Broken Gradient:

$$\nabla_h : W^{1,p}(\mathcal{T}_h) \to [L^p(\Omega)]^d$$

$$\forall K \in \mathcal{T}_h \quad (\nabla_h v)|_K = \nabla(v|_K)$$

Lemma 1.1 (Di Pietro – Ern). Let $m \geq 0$ $1 \leq p \leq +\infty$. Then $W^{m,p}(\Omega) \subset W^{m,p}(\mathcal{T}_h)$ and $\forall v \in W^{1,p}(\Omega)$, we have $\nabla_h v = \nabla v$

Lemma 1.2 (characterisation $W^{1,p}(\Omega)$):
For $1 \leq p \leq \infty$, $v \in W^{1,p}(\mathcal{T}_h)$, then

$$v \in W^{1,p}(\Omega) \iff [\![v]\!] = 0 \quad \forall F \in \mathcal{F}_h^i.$$