AFEMT 2024

VIRTUAL ELEMENT METHOD

(Ern - Guermond Chap. 2)

FEM analysis setting

for general (abstract) discrete problem

find $u_h \in W_h$:

(GM) $$a_h(u_h, v_h) = f_h(v_h) \quad \forall v_h \in V_h$$

(Remark: in VEM, a_h, f_h are always "approximations" of "true a, f". In fact, same is true for FEM due to quadrature
[Ciarlet "The FEM for elliptic problems"] chap 4)

Introduce

$$W(h) = W + W_h \quad \text{with norm } \|\cdot\|_{W(h)}$$

such that

- $\|\cdot\|_{W(h)}$ defines a norm $\|\cdot\|_{W_h}$

- $\|w\|_{W(h)} \leq C \|w\|_{W} \quad \forall w \in W$

Approximability property

$$\lim_{h \to 0} \left(\inf_{w_h \in W_h} \|w - w_h\|_{W(h)} \right) = 0 \quad \forall w \in W$$

(in FEM analysis based on interp. error bounds)

Consistency: (GM) is consistent (fully consistent) if we can extend

$$a_h : W(h) \times W(h) \to \mathbb{R} \quad \text{and}$$
$$f_h : W(h) \longrightarrow \mathbb{R}$$

$$a_h(u, v_h) = f_h(v_h) \quad \forall v_h \in V_h$$

Otherwise (GM) is said inconsistent.

If we introduce truncation error

$$T_h = \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{|T_h(v_h)|}{\|v_h\|_{V_h}}$$

with $T_h(v_h) = a_h(u, v_h) - f_h(v_h)$

GM consistent $\Leftrightarrow$ $T_h \equiv 0$

$\Leftrightarrow$ $\boxed{a_h(u-u_h, v_h)=0}$

Galerkin orthogonality

If GM consistent, $a_h=a$, $f_h=f$

$W_h \subset W$, $V_h \subset V$

- BNB applies to GM, so $\exists!\, u_h$ sol. of (GM) and

$$\|u_h\|_W \leq \frac{1}{\alpha}\|f\|_{V'}$$

- Cèa Lemma: $\exists\, c>0$:

$$\|u-u_h\|_W \leq c \inf_{w_h \in W_h} \|u-w_h\|_W$$

Asymptotic consistency:

assume

w.r.t. h $\quad \exists\, \gamma>0$:

$a_h(w_h, v_h) \leq \gamma \|w_h\|_{W_h} \|v_h\|_{V_h}$

- a_h is uniformly bounded (continuous)
- $\exists \Pi_h : W \to W_h : \|\Pi_h w - w\|_{W(h)} \le c \inf_{w_h \in W_h} \|w - w_h\|_{W(h)}$

$\forall w \in W$

then GM is asimptotically consistent if.

$$\lim_{h\to 0} \underbrace{\left(\sup_{v_h \in V_h} \frac{|f_h(v_h) - a_h(\Pi_h u, v_h)|}{\|v_h\|_{V_h}} \right)}_{R_h(u) \text{ consistency error}} = 0$$

<u>Theorem</u>: If $\dim(W_h) = \dim(V_h)$

and $BNB1_h$ holds, namely

$\exists \alpha_h > 0 : \forall w_h \in W_h$

$$\sup_{v_h \in V_h} \frac{a_h(w_h, v_h)}{\|v_h\|_{V_h}} \ge \alpha_h \|w_h\|_{W_h}$$

then (GM) is well posed and

$$\|u_n\|_{W_n} \le \frac{1}{\alpha_n} \|f_n\|_{V_n'}$$

(follows from BNB theorem applied to GM + fact that when $\dim V_n = \dim W_n$, BNB1 $\Leftrightarrow$ BNB2)

Lemma (strong 2): If

(1) BNB1_h holds

(2) a_n is unif. bounded in $W(h) \times V_n$

then sol. u_n of GM satisfies

$$\|u - u_n\|_{W(h)} \le \left(1 + \|a_n\|_{W(h), V_n}\right) \inf_{w_n \in W_n} \|u - w_n\|_{W(h)}$$

$$+ \underbrace{\frac{1}{\alpha_n} \sup_{v_n \in V_n} \frac{|f_n(v_n) - a_n(u, v_n)|}{\|v_n\|_{V_n}}}_{=0 \text{ if method is consistent}}$$

otherwise this measures the inconsistency

Proof : $\forall\, w_n \in W_n$

$$\|u - u_n\|_{W(h)} \leq \underbrace{\|u - w_n\|_{W(h)}} + \|w_n - u_n\|_{W(h)}$$

Use BNB1$_h$:

$$\alpha_h \|u_h - w_h\|_{W(h)} \leq \sup_{v_h \in V_h} \frac{a_h(u_h - w_h, v_h)}{\|v_h\|_{V_h}}$$

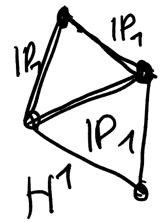
$$|a_h(u_h - w_h, v_h)| \leq |a_h(u_h - u, v_h)| + |a_h(u - w_h, v_h)|$$

use GM : $\leq |f_h(v_h) - a_h(u, v_h)| + \|a_h\| \, \|u - w_h\|_{W(h)} \, \|v_h\|_{V_h}$

Plug this

Classical nonconforming (low order) Crouzeix-Raviart

Courant

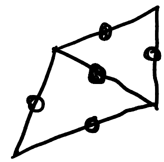


Proposed by CR in 1973

(Poisson, Stokes)

P_1^{nc} element , $(P_1^{nc})^2 - P_0$

for triangular meshes with dof



Question: is this generalisable to

- higher orders
- quadrilateral meshes
- 3D

examples:

- P_2^{NC} on triangles (Fortin+Soulie, 1983)
- P_1^{NC} quadrilateral (Park+Sheen, 2003)

⋮

- NC-VEM of any order, any shape, any dim (Ayuso+Lipnikov+Manzini, 2016)

<u>POISSON</u> (Ayuso+Lipnikov+Manzini, 2016) (?)

<u>STOKES</u> (Cangiani+Manzini+Sutton 2019)

Def: (Broken spaces)

$$H^1(\mathcal{T}_h) = \{ v \in L^2(\Omega) : v|_T \in H^1(T) \ , \forall T \in \mathcal{T}_h \}$$

Def: Jump of

v scalar $[\![v]\!] = v|_{T_1} \underline{n}_{T_1} + v|_{T_2} \underline{n}_{T_2}$, $\overline{T}_1 \cap \overline{T}_2$
$T_1, T_2 \in \mathcal{T}_h$

$\underline{v}$ vector $[\![\underline{v}]\!] = \underline{v}|_{T_1} \cdot \underline{n}_{T_1} + \underline{v}|_{T_2} \cdot \underline{n}_{T_2}$

Def: $\forall v \in H^1(\mathcal{T}_h)$

$$\|v\|_{1,\mathcal{T}_h} = \left(\sum_{T \in \mathcal{T}_h} \|v\|^2_{H^1(T)} \right)^{1/2}$$

$$|v|_{1,\mathcal{T}_h} = \left(\sum_{T \in \mathcal{T}_h} |v|^2_{H^1(T)} \right)^{1/2}$$

Def (Broken gradient): $\forall v \in H^1(\mathcal{T}_h)$

$$\nabla_h v : \quad (\nabla_h v)|_T = \nabla (v|_T).$$

- if $v \in H^1(\Omega)$, then $\nabla_h v = \nabla v$

- $v \in H^1(\Omega) \iff \begin{cases} v \in H^1(\mathcal{T}_h) \\ [\![v]\!]\big|_F = 0 \quad \forall F \in \mathcal{F}_h^i \end{cases}$

  where $\mathcal{F}_h^i = \{ \text{internal edges of } \mathcal{T}_h \}$

Def (Broken H(div)):

$$H(\mathrm{div};\mathcal{T}_h) = \left\{ \underline{v} \in [L^2(\Omega)]^d : \underline{v}|_T \in H(\mathrm{div};T) \ \forall T \in \mathcal{T}_h \right\}$$

- $\underline{v} \in H(\mathrm{div};\Omega) \iff \begin{cases} \underline{v} \in H(\mathrm{div};\mathcal{T}_h) \\ + \text{some extra regularity} \\ [\![\underline{v}]\!] = 0 \quad \forall F \in \mathcal{F}_h^i \end{cases}$

  $\left(\underline{v} \in [W^{1,1}(\mathcal{T}_h)]^d \right)$

back CR

Given $\Omega \in \mathbb{R}^2$, $\mathcal{T}_h$ tria

$$\overset{\circ}{V}{}_h^{nc} = \left\{ v : \Omega \to \mathbb{R} : \begin{cases} v_{|T} \in \mathbb{P}^1(T) \\ v_{|T_1}(x_F) = v_{|T_2}(x_F) & \forall F \in \mathcal{F}_h^i \\ v(x_F) = 0 & \forall F \in \mathcal{F}_h^b \end{cases} \right.$$

$\mathbb{P}^1(T) = \{$ polynomials of degree ≤ 1 over $T \}$

$\forall F \in \mathcal{F}_h$ $\quad x_F =$ midpoint of F

$$\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$$

$\mathcal{F}_h^i =$ internal tria edges

$\mathcal{F}_h^b =$ boundary " "

$\forall F \in \mathcal{F}_h^i$ call T_1, T_2 elements sharing F

$\overset{\circ}{V}{}_h^{nc} \not\subset H^1(\Omega)$ but

$\mathring{V}_h^{nc} \subset H^{1,nc}(\mathcal{T}_h)$ where

$$H^{1,nc}(\mathcal{T}_h) = \left\{ v \in H^1(\mathcal{T}_h) : \int_e [\![v]\!]\, ds = 0 \quad \forall e \in \mathcal{F}_h \right\}$$

$$\left([\![v]\!]|_F = v\, \underline{n}_F \qquad F \in \mathcal{F}_h^b \right)$$

$|\cdot|_{1,\mathcal{T}_h}$ is a norm over $H^{1,nc}(\mathcal{T}_h)$

($\Rightarrow$ also for $\mathring{V}_h^{nc}$)

proof: suppose $|v|_{1,\mathcal{T}_h} = 0 \Rightarrow \|\nabla v\|_T = 0 \quad \forall T$

$\Rightarrow v|_T \equiv \text{const}$

Also $\int [\![v]\!] = 0 \Leftrightarrow [\![v]\!] = 0$ over e

$\forall e \in \mathcal{F}_h^b \quad v|_e = 0 \quad \Rightarrow v|_T = 0$

$\forall T$ □

CR method

① Poisson problem $\begin{cases} a(u,v) = \int_\Omega \nabla u \cdot \nabla v \\ f(v) = \int_\Omega f v \end{cases}$

$\rightarrow \quad u \in H^1_0(\Omega) : \; a(u,v) = f(v) \quad \forall v \in V$

find $u_h \in \overset{\circ}{V}{}^{nc}_h$:

$$a_h(u_h, v_h) = f(v_h) \quad \forall v_h \in \overset{\circ}{V}{}^{nc}_h$$

where

$$a_h(u_h, v_h) = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h$$

note tho $\forall u, v \in H^1(\Omega)$

$$a_h(u,v) = a(u,v)$$

a_n is unif bounded $H^1 + H^{1,nc}(\mathcal{T}_n)$

$W(n) =$

hence, theorem above applies (strong 2)

$$\|u-u_h\|_{1,\mathcal{T}_h} \le c \inf_{v_h \in \overset{\circ}{V}{}_h^{nc}} \|u-v_h\|_{1,\mathcal{T}_h} + \sup_{v_h \in \overset{\circ}{V}{}_h^{nc}} \frac{|(f,v_h) - a_h(u,v_h)|}{\|v_h\|_{1,\mathcal{T}_h}}$$

TASK: estimate two terms above!

CR interpolant

- fix or degrees of freedom for $\overset{\circ}{V}{}_h^{nc}$
  - value at midpoint x_F $\forall F \in \mathcal{F}_h$

    or equivalently

  - $\frac{1}{|F|} \int_F v_h|_F$
- local interpolant $\forall T \in \mathcal{T}_h$

$\Pi_T^{nc} : H^1(T) \to \mathbb{P}_1(T)$ by

$= \Pi_T^{nc} v(x_F) = \frac{1}{|F|}\int_F v$

by midpoint rule

$= \frac{1}{|F|}\int_F \Pi_T^{nc} v$

- Global intepolant

$\Pi^{nc} : H_0^1(\Omega) \to \overset{\circ}{V}_h^{nc}$

$\Pi^{nc} v|_T = \Pi_T^{nc} v \qquad \forall T \in \mathcal{G}_h$

Properties

① Prop: $\forall z \in H_0^1(\Omega)$

$a_h(z - \Pi^{nc} z, v_h) = 0 \quad \forall v_h \in \overset{\circ}{V}_h^{nc}$

that Π^{nc} coincides with a_h-orthog. projection

proof (exercise; see Brenner paper)

② note that Π^{nc} is define for H^1 functions (not true for Courant H^1-conforming element)

③ Assume $\mathcal{T}_h$ is shape-regular

that is $\exists\, \sigma \geq 1 : \forall T \in \mathcal{T}_h \quad \dfrac{h_T}{\rho_T} \leq \sigma$

where $\begin{cases} h_T = \text{diameter of } T \\ \rho_T = \text{radius of largest inscribed ball} \end{cases}$

then $\quad \| z - \Pi^{nc} z \|_{1,\mathcal{T}_h} \leq c\, h \, | z |_{H^2(\Omega)}$

$\forall\, z \in H^2(\Omega)$

So, using this

$$\inf_{v_h \in \mathring{V}_h^{nc}} \| u - u_h \|_{1,\mathcal{T}_h} = \| u - \Pi^{nc} u \|_{1,\mathcal{T}_h} \leq c h |u|_{H^2}$$

hence it remains to bound the inconsistency term :

$$a_h(u, v_h) - (f, v_h)$$

$$= \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v_h - \int_\Omega f\, v_h$$

$$= -\sum_{T\in\mathcal{T}_h} \int_T \Delta u \, v_h + \int_{\partial T} (\underline{n}\cdot\nabla u) v_h - \int_\Omega f v_h$$

$$= \sum_{F\in\mathcal{F}_h} \int_F \nabla u \cdot [\![v_h]\!]$$

or

$$= \sum_{F\in\mathcal{F}_h} \int_F (\nabla u - \underline{c}(\nabla u)) \cdot [\![v_h]\!] \qquad \int_F [\![v_h]\!] = 0$$

Schwarz

$$\leq \left(\sum_{F\in\mathcal{T}_h} h_F \| \nabla u - \underline{c}(\nabla u) \|^2_{L^2(F)} \right)^{1/2} \left(\sum_{F\in\mathcal{T}_h} \frac{1}{h_F} \| [\![v_h]\!] \|^2_{L^2(F)} \right)^{1/2}$$

use :

- continuous trace ineq. with scaling:

$$\forall z \in H^1(T)$$

$$\| z \|^2_{L^2(F)} \leq C \left(h_T^{-1} \| z \|^2_{L^2(T)} + h_T | z |^2_{H^1(T)} \right)$$

$$F \subset \partial T$$

- for $v_h \in V_h^{onc}$, we have (Brenner)

$$\sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \| [\![v_h]\!] \|^2_{L^2(F)} \leq C \| v_h \|^2_{1,\mathcal{T}_h}$$

$$\leq C\left(\sum_{F\in\mathcal{F}_h} h_e\left(h_T^{-1}\|\nabla u-\underline{c}(\nabla u)\|_{L^2(T_F)}^2 + h_T|u|_{H^2(T)}^2\right)\right)^{1/2}\|v_h\|_{1,\mathcal{T}_h}$$

- use $h_e \sim h_T$
- fix $\underline{c}$: $\|\nabla u-c\| \lesssim h_T |u|_{H^2}$

implies

$$\underbrace{\frac{|a_h(u,v_h)-(f,v_h)|}{\|v_h\|_{1,\mathcal{T}_h}}} \leq C h \, |u|_{H^2}$$

$\Rightarrow$ inconsistency is $O(h)$

The pair $(\mathring{V}_h^{nc})^2 \times \mathbb{P}^0$

$(\underline{u}_h , p_h)$ discrete sol. for Stokes

- yields discretisation of Stokes
  - well posed (inf-sup stable)
  - solution $u_h \in \{ \underline{v}_h \in (V_h^{nc})^2 : \nabla_h \cdot \underline{v}_h = 0 \}$

(see Brenner paper)

Before next lecture (in 2 weeks) read Basic VEM paper !