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VIRTUAL ELEMENT METHOD

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BASIC PRINCIPLES OF VIRTUAL ELEMENT METHOD
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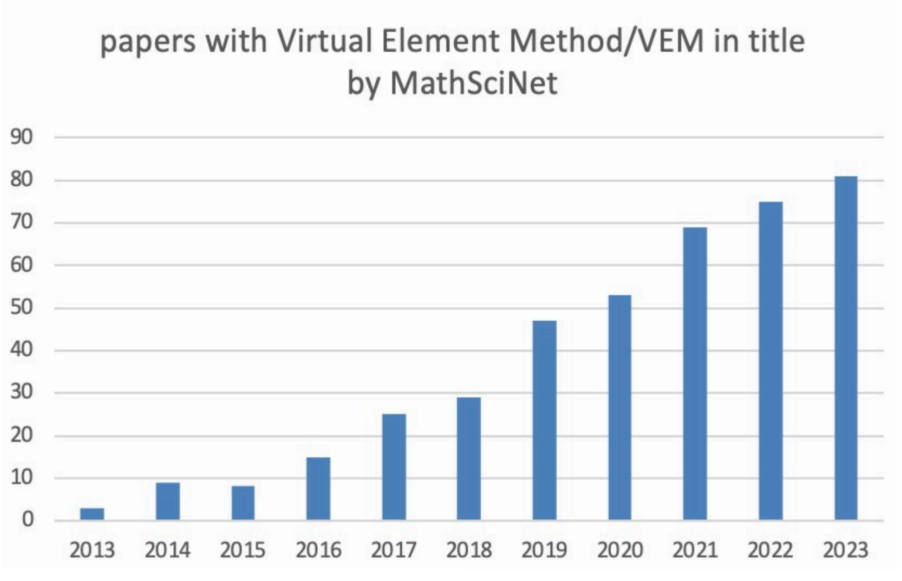

General setting for FEM

(Ern, Guermond "Theory and Practice of FEMs", Springer) 2004

chap. 2

- V, W Banach normed vector spaces with V reflexive ← could also be $\mathbb{C}$
- $a \in \mathcal{L}(W \times V; \mathbb{R})$ continuous bilinear forms

$$\exists \gamma > 0 \quad |a(u,v)| \le \gamma \, \|u\|_W \, \|v\|_V \quad \begin{cases} \forall u \in W \\ \forall v \in V \end{cases}$$

- $f \in \mathcal{L}(V; \mathbb{R}) = V'$ cont. linear form

$$\underset{\langle f, v \rangle_{V',V}}{\underset{\parallel}{|f(v)|}} \le \|f\|_{V'} \, \|v\|_V$$

General (linear) abstract problem:

Find $u \in W$:

(AP) $\qquad a(u,v) = f(v) \qquad \forall v \in V$

(AP) is well-posed iff

- $\exists!$ solution
- $\exists c > 0 : \forall f \in V' \quad \|u\| \le c \, \|f\|_{V'}$.

For us (AP) stems from a given weak formulation (WP) of a PDE problem

In turns, the WF typically is founded on a variational formulation

- a minimization principle
- an equilibrium condition (saddle point)

Analysis of (AP)

CASE $V = W$ Hilbert

Lax-Milgram Lemma: Let V Hilbert,

- $a \in \mathcal{L}(V \times V, \mathbb{R})$ coercive

$$\exists \alpha > 0 : \boxed{a(v,v) \geq \alpha \|v\|_V^2} \quad \forall v \in V$$

- $f \in \mathcal{L}(V; \mathbb{R})$

Then (AP) is well-posed, i.e. $\exists!$ u sol of

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}$$ (a priori estimate)

Remarks:

1) Generalization of core symmetric a in which core (AP) is equivalent to minimizing

energy

$$J(v) = \frac{1}{2} a(v,v) - f(v)$$

and coercivity of a $\equiv$ strong convexity of J

2) LM can be proven directly or just as a consequence of:

Theorem (Banach-Nečas-Babuška) [1962] [1972]:

↳ or consequence of
Closed Range Theorem (CRT)
Open Mapping Theorem (OMT)

Let W, V Banach, V reflexive

Then (AP) is well-posed iff:

(BNB 1) $\exists \alpha > 0 : \inf_{w \in W} \sup_{v \in V} \frac{a(w,v)}{\|w\|_W \|v\|_V} \geq \alpha$

(BNB 2) if $v \in V : a(w,v) = 0 \;\; \forall w \in W$

$$\Downarrow$$

$$v = 0$$

Moreover the sol. $u \in W$ satisfies the a priori estimate

$$\|u\|_W \leq \frac{1}{\alpha} \|f\|_{V'} .$$

Remarks.

1) (BNB1) termed inf-sup condition, can be written

$$\forall w \in W, \quad \sup_{v \in V} \frac{a(w,v)}{\|v\|_V} \geq \alpha \|w\|_W$$

2) $\exists!$ consequence of CRT OMT and a priori bound of BNB1:

$$\alpha \|u\|_W \leq \sup_{v \in V} \frac{a(u,v)}{\|v\|_V} = \sup_{v \in V} \frac{f(v)}{\|v\|_V} = \|f\|_{V'}$$

3) $A \in \mathcal{L}(W; V')$ by

$$\begin{cases} \langle Aw, v \rangle_{V',V} = a(w,v) \\ = \langle w, A^t v \rangle_{W,W'} \end{cases}$$

Then (AP) is equivalent to

$u \in W: \quad Au = f \quad$ in V'

and can show that

BNB 1 $\Leftrightarrow \begin{cases} \operatorname{Ker} A = \{0\} \\ \operatorname{Im} A \text{ is closed} \end{cases} \Leftrightarrow A^T$ is surjective

BNB 2 $\Leftrightarrow \operatorname{Ker}(A^T) = \{0\} \Leftrightarrow A^T$ is injective

BNB $\Leftrightarrow$ A invertible

(LM $\Leftrightarrow$ A is positive)

4) lemma. Let $W = V$, a coercive. Then BNB 1, BNB 2 holds

proof: <u>BNB 1</u>: $\alpha \|w\|_V^2 \leq a(w, w)$

$$\Rightarrow \quad \alpha \, \|w\|_V \le \frac{a(w,w)}{\|w\|_V} \le \sup_{v \in V} \frac{a(w,v)}{\|v\|_V}$$

<u>BNB2:</u> $\forall v \in V$,

$$\sup_{w \in W} a(w,v) \ge a(v,v) \ge \alpha \|v\|_V^2$$

$$\overset{\|}{0} \qquad \Rightarrow \qquad v = 0$$

$\square$

Examples:

① Poisson (LM)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$W = V = H_0^1(\Omega)$

$f(v) = \langle f, v \rangle_{H^{-1}, H_0^1}$ $\qquad \left(f \in L^2 \Rightarrow \underset{L^2\text{-prod.}}{=} (f,v) \right)$

$a(u,v) = (\nabla u, \nabla v)$

minimization problem

$$J(v) = \frac{1}{2} (\nabla v, \nabla v) - \langle f, v \rangle$$

② Stokes (BHB)

dynamic viscosity

$$\begin{cases} -\mu \Delta \underline{u} + \nabla p = \underline{s} & \text{in } \Omega \\ \nabla \cdot \underline{u} = 0 & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

$$V = W = [H_0^1(\Omega)]^d \times L_0^2(\Omega)$$

where $L_0^2(\Omega) = L^2(\Omega) / P_0(\Omega)$

$$f(\underline{v}) = \langle \underline{s}, \underline{v} \rangle_{H^{-1}, H_0^1}$$

trial, test, by int. by parts

$$a((\underline{u}, p), (\underline{v}, q)) = \int_\Omega \nabla u : \nabla v - \int_\Omega p \nabla \cdot v + \int_\Omega q \nabla \cdot u$$

$$\nabla u : \nabla v = \sum_i \nabla u_i \cdot \nabla v_i$$

comes from $= \tilde{a}$, b

$$\begin{cases} \underbrace{\mu \int_\Omega \nabla \underline{u} : \nabla \underline{v}}_{=\tilde{a}} - \underbrace{\int_\Omega p \nabla \cdot \underline{v}}_{b} = f(\underline{v}) \\ \int_\Omega q \nabla \cdot \underline{u} = 0 \end{cases} \iff \begin{cases} \tilde{a}(\underline{u}, \underline{v}) - b(\underline{v}, p) = f(\underline{v}) \\ b(\underline{u}, q) = 0 \end{cases}$$

Saddle point problem

$$\inf_{\underline{v} \in [H_0^1(\Omega)]^d} \sup_{q \in L^2(\Omega)} \left(\mu \int_\Omega |\nabla \underline{v}|^2 - \int_\Omega q \, \nabla \cdot \underline{v} - \langle \underline{S}, \underline{v} \rangle \right)$$

Note: can also define

$$V = \{ \underline{v} \in [H_0^1(\Omega)]^d : \nabla \cdot \underline{v} = 0 \}$$

find $\underline{u} \in V$:

$$\tilde{a}(\underline{u}, \underline{v}) = \langle \underline{S}, \underline{v} \rangle$$

then $p \in L_0^2(\Omega)$ given solving

$$(p, \nabla \cdot v) = 0 \qquad \forall v \in (H_0^1(\Omega))^d$$

③ Advection eq. $(V \neq W)$

outward normal

$$\begin{cases} \underline{b} \cdot \nabla u = f & \text{in} \\ u = 0 & \text{on } \partial\Omega^- = \{ x \in \partial\Omega : (\underline{b} \cdot \underline{n})(x) < 0 \} \end{cases}$$

$$a(u, v) = \int_\Omega (\underline{b} \cdot \nabla u) \, v$$

$$W = \{ w \in L^2(\Omega) : \underline{b} \cdot \nabla w \in L^2(\Omega),\ w|_{\partial\Omega} = 0 \}$$

$$V := L^2(\Omega)$$

"Graph space"

④ Biharmonic problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \partial_{\underline{n}} u = 0 & \text{on } \partial\Omega \end{cases}$$

$$W = V = H^2_0(\Omega) = \{ v \in H^2(\Omega) : v|_{\partial\Omega} = 0 ;\ \partial_{\underline{n}} v|_{\partial\Omega} = 0 \}$$

$f \in L^2(\Omega)$ $\quad f(v) = (f, v)$

$$J(v) = \frac{1}{2} (\Delta v, \Delta v) - (f, v)$$

models:
- Plates (limit elasticity model of plates of thickness $\delta \ll 1$
- Stream function formulation of Stokes

stream f.

2D: $\nabla \cdot \underline{u} = 0 \Rightarrow \exists\ \psi :$

$$\underline{u} = (\partial_2 \varphi, -\partial_1 \varphi)$$

$$\Rightarrow \mu \, \Delta^2 \varphi = \partial_1 f_2 - \partial_2 f_1$$

$$\left(\underline{f} = (f_1, f_2) \right)$$

(5) Time harmonic electric wave equation

$$\begin{cases} \underbrace{\nabla \times}_{\text{curl}} \mu^{-1} \nabla \times E + k^2 E = f \\ E \times \underline{n} = 0 \end{cases}$$

μ = magnetic permeability

[Green's: $(\nabla \times \underline{u}, \underline{v}) = (\underline{u}, \nabla \times \underline{v}) - (\underline{u} \times \underline{n}, \underline{v})_{\partial\Omega}$

$$V = W = H_0(\text{curl}; \Omega) = \{ \underline{v} \in H(\text{curl}; \Omega) : \underline{u} \times \underline{n}|_{\partial\Omega} = 0 \}$$

$$\boxed{H(\text{curl}; \Omega)} = \{ \underline{v} \in [L^2(\Omega)]^d : \nabla \times \underline{v} \in [L^2(\Omega)]^d \}$$

$$a(\underline{u}, \underline{v}) = (\mu^{-1} \nabla \times \underline{u}, \nabla \times \underline{v}) + k^2 (\underline{u}, \underline{v})$$

⑥ Poisson in mixed form

$$\begin{cases} -\Delta p = f & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega \end{cases}$$

setting $\underline{u} = -\nabla p$

$$\Rightarrow \begin{cases} \underline{u} + \nabla p = 0 \\ \nabla \cdot \underline{u} = f \\ p = 0 \quad \text{on } \partial\Omega \end{cases}$$

$$H(\mathrm{div};\Omega) = \left\{ \underline{v} \in [L^2(\Omega)]^d : \nabla\cdot\underline{v} \in L^2(\Omega) \right\}$$

Hilbert with $(\underline{u},\underline{v})_{H(\mathrm{div})} = (\underline{u},\underline{v}) + (\nabla\cdot\underline{u}, \nabla\cdot\underline{v})$

find $(\underline{u}, p) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$:

$$\begin{cases} (\underline{u},\underline{v}) - (\nabla\cdot\underline{v}, p) = 0 & \forall \underline{v} \in H(\mathrm{div};\Omega) \\ (\nabla\cdot\underline{u}, q) = (f,q) & \forall q \in L^2(\Omega) \end{cases}$$

$$\begin{cases} \tilde{a} - b \\ b \end{cases}$$

div-free

H^1, L^2, $H(\mathrm{div})$, $H(\mathrm{curl})$, H^2, Graph, $\nabla\cdot u = 0$

FEM or Galerkin methods

FEM: (Petrov-) $\begin{cases} \text{Galerkin} \\ \text{generalized Galerkin} \end{cases}$ methods

based on piecewise polynomial

Petrov-Galerkin method: $(W_h, V_h) \leq (W, V)$
(h indicates a parametrisation of the disc.)
and restrict (AP): find $u_h \in W_h$:

$$a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$$

Galerkin method: special case $W_h = V_h$

Generalised (Petrov) Galerkin:

$V_h \not\leq V$, $W_h \not\leq W$ and/or $a, f \leftarrow a_h\, f_h$ approximate forms

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V_h, W_h — subspaces $\rightarrow$ conforming
V_h, W_h — not subspaces $\rightarrow$ nonconforming

$\Rightarrow$ design conforming discretizations requires set up of appropriate subspaces V_h, W_h (eg $H^1, H(\mathrm{div}), \ldots$)

or could "relax" conformity

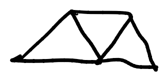
Example: Stokes

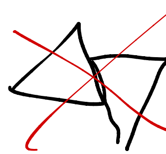
Recall $\mathring{V} = V = W = [H^1_0(\Omega)]^d \times L^2_0(\Omega)$
$(\underline{u} \qquad , \qquad p \ )$

<u>Taylor-Hood (1973)</u>

A famous example of conforming pair:

- $\mathcal{T}_h$ conforming triangular mesh
  ↳ triangles share an edge or a vertex





- $k \geq 2$
- $\mathring{V}^k_h = \{(\underline{u}_h, p) \in \mathring{V} : \underline{u}_h|_T \in \mathbb{P}^d_k(T),\ p_h|_T \in \mathbb{P}_{k-1}(T)$ $\quad \forall T \in \mathcal{T}_h\}$
- $a_h = a$

note: why not $k = 1$?
because $\mathring{V}^1_h$ does not satisfy BNB

Taylor-Hood: find $(\underline{u}_h, p_h) \in \overset{\circ}{V}_h^k$:

$$\begin{cases} \overbrace{\mu \int_\Omega \nabla \underline{u}_h : \nabla \underline{v}_h}^{\tilde{a}(u_h, v_h)} - \int_\Omega p \, \nabla \cdot v_h = f(\underline{v}_h) \\ \underbrace{\int_\Omega q_h \, \nabla \cdot \underline{u}_h}_{b(\underline{u}_h, q_h)} = 0 \end{cases}$$

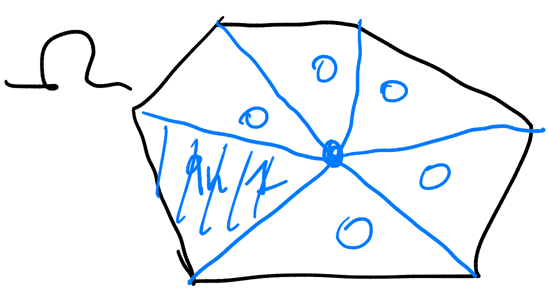
$$V_h = \{ \underline{v}_h : b(\underline{v}_h, q_h) = 0 \quad \forall q_h \}$$

$\rightarrow$ equiv. form: find $\underline{u}_h \in V_h$:

$$\tilde{a}(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Problem: it can happen $(k=1)$ that

$$V_h = \{0\}$$



V_h has only 2 degrees of freedom

def of V_h five conditions

(velocity space not reach enough !)

for $k \geq 2$ Taylor-Hood satisfies (BNB)

$\longrightarrow \exists ! \, (\underline{u}_h, p_h)$

however $\underline{u}_h$ is not exactly divergence free

TH is conforming but not compatible with cond. $\nabla \cdot \underline{u} = 0$

A second example

Classical nonconforming FEM of Crouzeix-Raviart (1973)

- nonconforming $\underline{v}_h \notin [H^1_0(\Omega)]^d$
- $\exists ! \, (\underline{u}_h, p_h)$ and $\forall T \; \nabla \cdot \underline{u}_h|_T = 0$

$(k = 1)$