ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 4

Galerkin method

Ingredients:

WP: Find $u \in \overset{\text{Hilbert}}{V}$: $a(u,v) = f(v) \quad \forall v \in V$

Lemma (Lax–Milgram): Let V Hilbert, $a \in \mathcal{L}(V \times V, \mathbb{R})$, $f \in V'$ $\left(f(v) = \langle f, v \rangle_{V' \times V} \quad \forall v \in V \right)$

and assume a is coercive

$$\exists\, \alpha > 0 : \quad a(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$$

Then $\exists!$ solution to (WP) and it holds

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}$$

a priori bound

Galerkin method: given $V_H < V$ finite

dimensional subspace, look for $u_H \in V_H$:

(GM) $a(u_H, v_H) = f(v_H) \quad \forall v_H \in V_H$.

Proposition: The Galerkin method (GM) has a unique solution $u_H \in V_H$ and

$$\|u_H\|_V \leq \frac{1}{\alpha} \|f\|_{V'}$$

Proposition: It holds Galerkin orthogonality

$$a(u - u_H, v_H) = 0 \qquad \forall u_H \in V_H$$

or $a(u, v_H) = f(v_H) \quad \forall v_H \in V_H \subset V$

subtracting (GM): $a(u - u_H, v_H) = f(v_H) - f(v_H) = 0$

Lemma (Céa): $\|u - u_H\|_V \leq C \underbrace{\inf_{v_H \in H_H} \|u - v_H\|_V}_{\text{quasi optimality}}$

$(C = \gamma/\alpha$, where γ continuity of a)

Example: if (WP) comes from Poisson and $V_H = V_h^k$ C^0- conf. FEM, then $V = H_0^1(\Omega)$

$$\inf_{v_h \in V_h^k} |u - v_h|_{H^1(\Omega)} \leq |u - I_h^k u|$$

$$\underbrace{|u - u_h|_{H^1(\Omega)} \leq c\, h^k |u|_{H^{k+1}(\Omega)} \qquad O(h^k)}_{\text{a priori bound for FEM}}$$

$c = c(k)$

A posteriori error estimation

[see Ainsworth + Oden, Wiley, 2000]

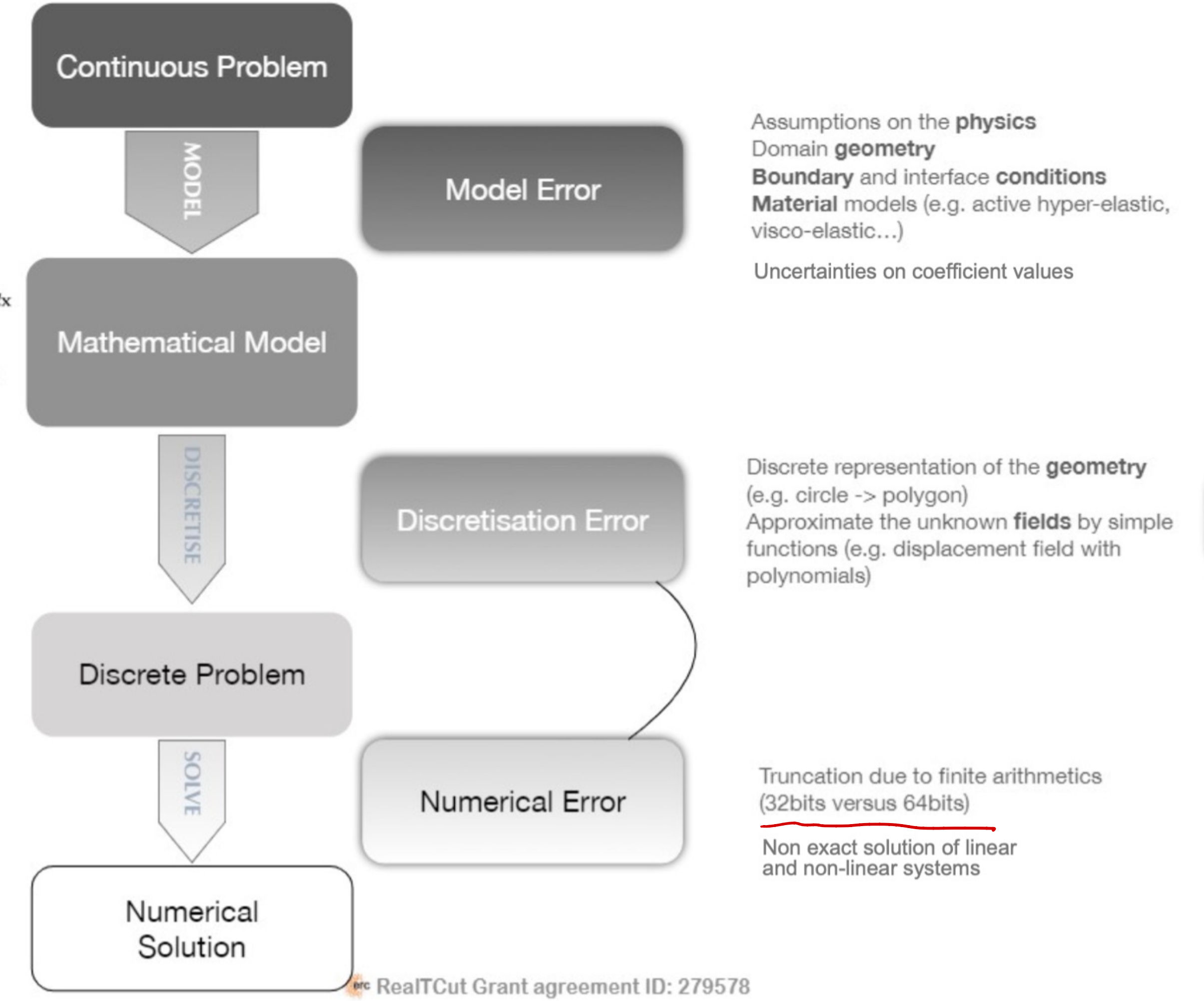
[E.G. book, chapter 10]

- General Intro

A priori bounds do not provide an estimate on the error attained by computing u_h because they depend on exact sol. u_0

A posteriori error estimator is a quantity η_h that depends on u_h (and data) but not on u which (ideally) is equivalent to the discretisation error, i.e.

$$\eta_h(u_h) \simeq \underbrace{\mathcal{E}_h(u, u_h)}_{\text{error measured in some norm}}$$


Continuous Problem
MODEL
Mathematical Model
DISCRETISE
Discrete Problem
SOLVE
Numerical Solution
Model Error
Assumptions on the physics
Domain geometry
Boundary and interface conditions
Material models (e.g. active hyper-elastic, visco-elastic…)
Uncertainties on coefficient values
Discretisation Error
Discrete representation of the geometry (e.g. circle -> polygon)
Approximate the unknown fields by simple functions (e.g. displacement field with polynomials)
Numerical Error
Truncation due to finite arithmetics (32bits versus 64bits)
Non exact solution of linear and non-linear systems
erc RealTCut Grant agreement ID: 279578

Is it possible to derive a post. error est.?

Consider Poisson problem again,

$V = H_0^1(\Omega) \qquad V' = H^{-1}(\Omega)$

From $\begin{cases} \text{a priori est.} & \|\nabla u\|_{L^2(\Omega)} \le \|f\|_{H^{-1}} \\ \text{coercivity:} & a(v,v) \ge \alpha \|\nabla v\|_{L^2}^2 \quad \forall v \in V \end{cases}$

$\forall v \neq 0$

$$\alpha \|\nabla v\|_{L^2} \le \frac{a(v,v)}{\|\nabla v\|_{L^2}} \le \sup_{w \in V} \frac{a(v,w)}{\|\nabla w\|_{L^2}}$$

$$\Rightarrow \alpha \le \inf_{v} \sup_{w \in V} \frac{a(v,w)}{\|\nabla v\|_{L^2} \|\nabla w\|_{L^2}}$$

"INF-SUP" CONDITION

$v = u - u_h$

$$\alpha \underbrace{\|\nabla u - \nabla u_h\|_{L^2}}_{= |u-u_h|_{H^1}} \le \sup_{w \in V} \frac{\overbrace{a(u-u_h, w)}^{\alpha_{u-u_h}(w)}}{\|\nabla w\|_{L^2}}$$

$$\le \sup_{w \in V} \frac{\langle \overbrace{\Delta u}^{f} - \Delta u_h, w \rangle_{V', V}}{\|\nabla w\|_{L^2}}$$

dual norm $\qquad = \|f + \Delta u\|_{V'}$

$$\Rightarrow \quad |u-u_h|^2_{H^1} \le \frac{1}{2} \| \underbrace{f + \Delta u_h}_{\text{RESIDUAL}} \|_{H^{-1}}$$

a posteriori error bound !

... but not a very useful one

- computing H^{-1} norm $\Rightarrow$ problem
- not localizable

A post. est. desiderata

(1) Reliability : $\quad \varepsilon_h(u, u_h) \le C_2 \, \eta_h(u_h)$

- $C_2 = 1$ exact guaranteed error bound
- otherwise important estimate C_2

(2) Efficiency : $\quad c_1 \, \eta_h(u_h) \le \varepsilon_h(u, u_h)$

(1)+(2) difficult, typically efficiency of form

see later

$$c_1 \, \eta_K(u_h) \le \varepsilon_h(u, u_h) + \Pi_K(\Delta_K)$$

Patch around K

with Π_K perturbation

at most of order of ε_h.

measure of "effectivity"

$$\mathrm{eff}_h = \frac{\eta_h(u_h)}{\varepsilon_h(u, u_h)} \geq 1$$

③ Localization:

local error

$$\eta_h = \left(\sum_K \eta_K^2 \right)^{1/2}$$

adaptivity

④ cheap to compute

Types of estimators

- residual type (1978)
- recovery type (1983)
- goal oriented (1996)

Recovery est.

- often quantity of interest is not u but its ∇u $\Rightarrow$ codes have routines to post process ∇u (eg C^0-FEM(k) grad is p.w. $\mathbb{P}^{k-1}$ discontinuous!) to a C^0 gradient say using some space as for u

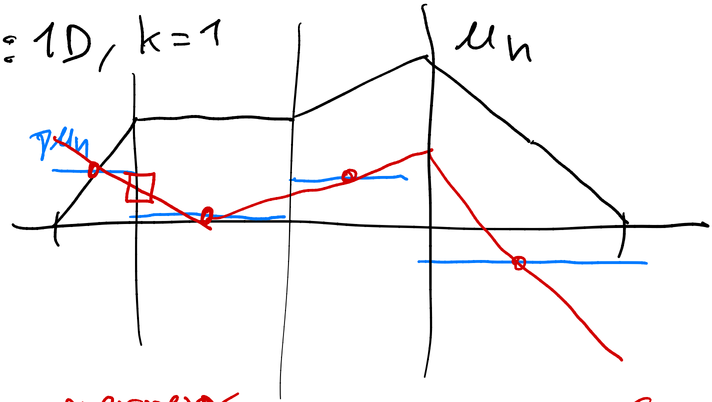
From $(u, \nabla u) \xrightarrow{\text{postprocess}} Gu$

approx of ∇u of higher order (!)

Estimator : (heuristic!)

$$\eta_h^2 = \sum_K \underbrace{\int_K |G u_h - \nabla u_h|^2}_{\eta_K^2}$$

Example: 1D, $k=1$



averaging neigh. grad $\Rightarrow$ p.w. linear G

Kelly: $$\eta_K^2 = \frac{h}{\sigma_K} \boxed{\int_{\partial K} \left[\frac{\partial u_h}{\partial n}\right]^2 dS}$$

$$\left[\frac{\partial u_h}{\partial n}\right] = \frac{\partial u_h}{\partial n_K}\Big|_K + \frac{\partial u_h}{\partial n_{K'}}\Big|_{K'}$$

normal jump

can be recast as a gradient recovery est.

is a form of Gradient