ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 5

A posteriori error estimators

Kelly estimator [Kelly, Gago, Zienkiewicz, Babuska, IJNME, 1983]

$$\eta(u_h) = \Big(\sum_{K \in \mathcal{T}_h} \eta_K (u_h)^2 \Big)^{1/2}$$

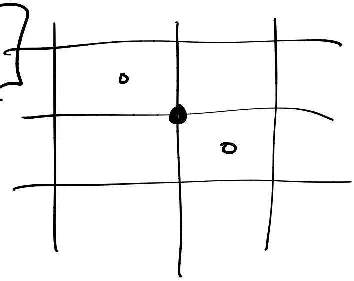
$$\eta_K^2 = \frac{h}{24} \int_{\partial K} \left[\frac{\partial u_h}{\partial n} \right]^2 ds$$

$$\frac{\partial u_h}{\partial n} = \frac{\partial u_h}{\partial n_K}\Big|_K + \frac{\partial u_h}{\partial n_{K'}}\Big|_{K'}$$ for K' the neighbouring element

for square meshes, linear FEM is equivalent to

$$\eta_K(u_h) = \int_K \left[G u_h - \nabla u_h \right]^2$$

$$Gv(x,y) = \frac{1}{2}\left[\nabla v\big|_{x-\frac{1}{2}h,\, y+\frac{1}{2}h} + \nabla v\big|_{x+\frac{1}{2}h,\, y-\frac{1}{2}}\right]$$



Idea behind this is the fact

Proposition ([Zlamal, 1975] superconvergence)

$$|u_h - I_h^1 u|_{H^1(\Omega)} \le c\, h^2 |u|_{H^3(\Omega)} \quad \text{if } u \in H^3$$

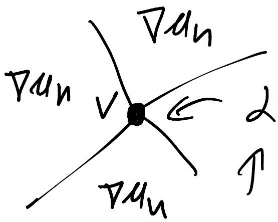
$k=1$

(while $|u-u_h|_{H^1(\Omega)} \le c\, h\, |u|_{H^2(\Omega)}$)

$$G(I_h^1 v) = I_h^1(v')$$

[Z+Z, IJNME, 1992]

Zienkiewicz – Zhu recovery very famous and often used recovery



$\leftarrow \alpha + \beta x + \gamma y$

to fix the value of the reconstructed linear gradient at v by least squares fitting of the neighbouring gradients values

Residual estimator (Ainsworth + Oden, Ern+Guermond)

last week: $\underset{\substack{\| \\ e}}{|u-u_h|^2_{H^1(\Omega)}} \leq \frac{1}{\alpha} \|\underbrace{f+\Delta u_h}_{\text{Residual}}\|^2_{H^{-1}(\Omega)}$

We used the coercivity of a:

Let V Hilbert, $V_h \subseteq V$ finite dimensional

suppose

$$u \in V \qquad a(u,v) = f(v) \qquad \forall v \in V$$

$$u_h \in V_h \qquad a(u_h, v_h) = f(v_h) \qquad \forall v_h \in V$$

with a coercive:

$$\alpha \|e\|_V \overset{\downarrow}{\leq} \frac{a(e,e)}{\|e\|_V} \leq \sup_{v \in V} \frac{a(e,v)}{\|v\|_V}$$

$$a(e,v) = a(u,v) - a(u_h,v) = f(v) - a(u_h,v)$$

$$\overset{\downarrow}{=:} R(v)$$

$\forall v_h \in V_h, R(v_h) = 0$

$$= \sup_{v \in V} \frac{R(v)}{\|v\|_V} = \|R\|_{V'}$$

$$\Rightarrow \|e\|_V \leq \frac{1}{\alpha} \|R\|_{V'}$$

$$\|R\|_{V'} = \sup_v \frac{a(e,v)}{\|v\|_V} \underset{\text{continuity } (\gamma)}{\le} \gamma \sup_v \frac{\|e\|_V \|v\|}{\|v\|} = \gamma \|e\|_V$$

$$\Rightarrow \quad \boxed{\alpha \|e\|_V \le \|R\|_{V'} \le \gamma \|e\|_V}$$

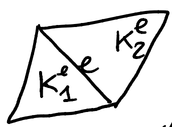
__Poisson case__ : $V = H^1_0(\Omega)$, $V_h = V_h^k$ H^1-conforming FE space

Remains to provide estimate of the dual norm of the residual

$$R(v) = R(v - v_h) \qquad \forall v_h \in V_h$$

$$= f(v - v_h) - a(u_h, v - v_h)$$

$$= f(v - v_h) + \sum_{K \in \mathcal{T}_h} \int_K \Delta u_h (v - v_h) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u_h}{\partial n} (v - v_h)$$



$$= \sum_K \int_K \underbrace{(f + \Delta u_h)}_{\text{elemental residual}} (v - v_h) - \sum_{\substack{e \text{ face} \\ e \notin \partial\Omega}} \int_e \left[\frac{\partial u_h}{\partial n}\right] v - v_h$$

$$\left[\frac{\partial u_h}{\partial n}\right] = \frac{\partial u_h}{\partial n_{K_1^e}}\Big|_{K_1^e} + \frac{\partial u_h}{\partial n_{K_2^e}}\Big|_{K_2^e}$$

$$R(v) \le \sum_K \left[\|f+\Delta u_h\|_{L^2(K)} \, \|v-v_h\|_{L^2(K)} + \frac{1}{2}\left\|\left[\frac{\partial u}{\partial n}\right]\right\|_{L^2(\partial K)} \|v-v_h\|_{L^2(\partial K)}\right]$$

Goal: bound of

$$\frac{|R(v)|}{\|v\|_V} \le \frac{\text{"estimate }(u_h)\text{"} \; \cancel{\|v\|_V}}{\cancel{\|v\|_V}}$$

$v \in C(K)$

interp. $v \in H^l$ valid for $l > 1$

$\Rightarrow$ need to work in V-norm (for us, H^1-norm)

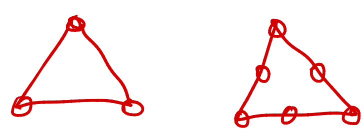
QUASI-INTERPOLANT (e.g. Clément, Scott-Zhang)

Clément: $R_h : H^1(\Omega) \longrightarrow V_h$ $\swarrow$ $v(V_j)$ Lagrange

$$v \longrightarrow \sum_j (P_j v)(V_j)\, \varphi_j(\cdot)$$

where:

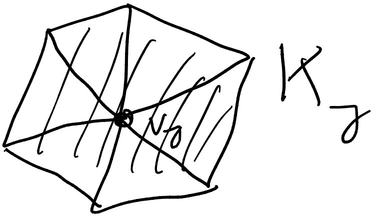
V_j are the nodes used in def of Lagrangian FEM space V_h^k

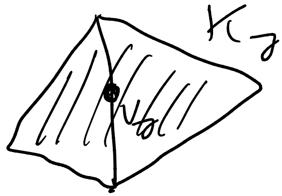


φ_j corresponding Lagrangian basis

P_j local L^2-projector onto $\mathbb{P}^1$ functions over K_γ = patch of elements around v_γ

ex:





$P_\gamma v \in \mathbb{P}^1(K_\gamma)$:

$$\int_{K_\gamma} P_\gamma v \, \varphi = \int_{K_\gamma} v \, \varphi \qquad \forall \varphi \in \mathbb{P}^1(K_\gamma)$$

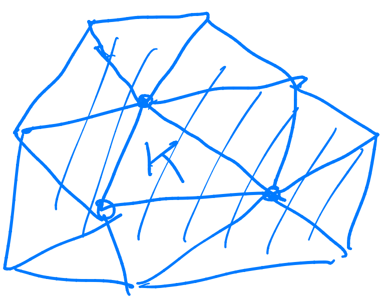
(that is $\int_{K_\gamma} (v - P_\gamma v)\varphi = 0$)

$\mathcal{T}_h$ shape regular triangulation

Prop: $\|v - R_h v\|_{L^2(K)} \le C_1 h_K |v|_{H^1(\tilde{K})}$

$\forall v \in H^1(\Omega)$ $\forall K \in \mathcal{T}_h$

where



$$\tilde{K} := \bigcup_{K \cap K' \neq \emptyset} K'$$

$$\|v - R_h v\|_{L^2(\partial K)} \le C_2\, h_K^{1/2}\, \|v\|_{H^1(\tilde{K})}$$

(Trace estimate: $\forall\, w \in H^1(K)$,

$$\|w\|_{L^2(\partial K)} \le C\left(h_K^{-1/2}\|w\|_{L^2(K)} + h_K^{1/2}|w|_{H^1(K)}\right)\Big)$$

Back to

$$R(v) \le \sum_K \left[\|f+\Delta u_h\|_{L^2(K)}\|v-v_h\|_{L^2(K)} + \frac{1}{2}\left\|\left[\frac{\partial u}{\partial n}\right]\right\|_{L^2(\partial K)}\|v-v_h\|_{L^2(\partial K)}\right]$$

choose $v_h = R_h v$

$$\le \underset{\max\{C_1, C_2\}}{\underset{\parallel}{C}} \sum_K \left\{\|f+\Delta u_h\|_{L^2(K)}\, h_K |v|_{H^1(\tilde{K})} + \frac{1}{2}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(\partial K)} h_K^{1/2}\|v\|_{H^1(\tilde{K})}\right]$$

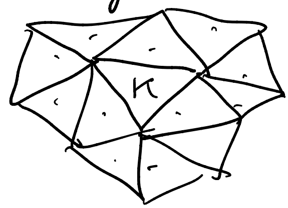
internal residual

Kelly

$$\underset{\text{Schwarz}}{\le} C\left[\sum_K \underbrace{\left(h_K\|f-\Delta u_h\|_{L^2(K)} + \frac{1}{2}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(\partial K)}\right)^2}_{\eta_K(u_h)}\right]^{1/2} \times$$

$$\times \left(\sum_K \|v\|^2_{H^1(\tilde{K})}\right)^{1/2} \underset{?}{\le} \|v\|_{H^1(\Omega)}$$

Given $\mathcal{T}_h$ is shape regular triangulation



the number of neighbours of each K is bounded

Fix n = largest number of neighbours

$$\le \left(n \sum_K |v|^2_{H^1(K)} \right)^{1/2}$$

$$\Rightarrow \langle R, v \rangle \le \sqrt{n}\, c \sqrt{1 + C_\Omega^2}\, \|v\|_{H^1(\Omega)}$$

$$H^1(\Omega) = V$$

[C_Ω = Poincaré Constant

$\forall v \in H^1_0(\Omega) \quad \|v\|_{L^2(\Omega)} \le C_\Omega |v|_{H^1(\Omega)}$]

$$\Rightarrow \quad \alpha\|e\|_V \le \|R\|_{H^1(\Omega)} \le \gamma\|e\|_V \quad \le c\sqrt{n} \underbrace{\left(\sum_K \eta_K (u_h)^2 \right)^{1/2}}_{\eta(u_h)}$$

Reliability