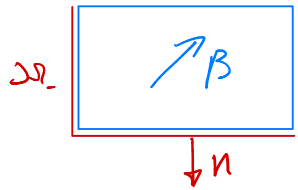
ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 7



Streamline diffusion

transport

$$\mathcal{L}u = \begin{cases} \beta\cdot\nabla u + \mu u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega_- \end{cases}$$

$(\beta\cdot n < 0)$

ass: $\mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0$

Test with $v_h + h\,\beta\cdot\nabla v_h \rightarrow$

Find $u_h \in V_h^k$:

$$\underbrace{(\mathcal{L}u_h, v_h + h\,\beta\cdot\nabla v_h) - (1+h)(u_h, \beta\cdot n\, v_h)_{\partial\Omega_-}}_{a(u_h, v_h)}$$

$$= (f, v_h + h\,\beta\cdot\nabla v_h) - (1+h)(g, \beta\cdot n\, v_h)_{\partial\Omega_-}$$

$\forall v_h \in V_h^k$

weak imposition of B. C.

[Nitsche, 1971

streamline diffusion norm

$$\|v\|_\beta = \left(h \|\beta\cdot\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \frac{1+h}{2} |v|^2_{\beta,\partial\Omega} \right)^{1/2}$$

$$|v|^2_{\beta,\partial\Omega} = \int_{\partial\Omega} |\beta\cdot n|\, v^2$$

Lemma: $a(v,v) \geq \|v\|^2_\beta \quad \forall v \in H^1(\Omega)$

Theorem: If $u \in H^{k+1}(\Omega)$, $\exists\, C > 0$:

$$\|u-u_h\|_\beta \leq C h^{k+1/2} \|u\|_{H^{k+1}(\Omega)}$$

(why full norm ?)

Convection-diffusion

$V = H^1_0(\Omega) \qquad u \in V$:

$$(-\varepsilon \Delta u) + \beta\cdot\nabla u + \mu u = f$$

$\varepsilon \ll 1$

(perturbation of transport problem)

Streamline diffusion: test with

$v_h + \delta\, \beta \cdot \nabla v_h \qquad \forall v_h \in V_h^k$

$\delta =$ streamline diff. parameter

For exact sol, $\forall v \in V$

$$\underline{\varepsilon(\nabla u, \nabla v)} - \varepsilon(\delta \Delta u, \beta \cdot \nabla v)$$
$$+ (\beta \cdot \nabla u + \mu u, v + \delta \beta \cdot \nabla v)$$
$$= (f, v + \delta\, \beta \cdot \nabla v) \qquad \forall v \in V$$

For $u \in V_h^k$ $\quad$ $\underline{\Delta u_h \text{ not def. in } \Omega}$

but it is define elemenwise

Δ_h = broken Laplacian with respect to $\mathcal{T}_h$

$$\int_\Omega \Delta_h u\, v := \sum_K \int_K \Delta u\, v$$

$\Rightarrow$ define

$$a_h^{SD}(u,v) = \varepsilon(\nabla u, \nabla v) +$$

$$-\varepsilon(\delta \Delta_h u, \beta\cdot\nabla v) + (\beta\cdot\nabla u + \mu u, v + \delta \beta\cdot\nabla v)$$

$$\ell_h^{SD}(v) = (f, v + \delta \beta\cdot\nabla v$$

SD-FEM: find $u_h \in V_h^k (\subset V)$:

$$a_h^{SD}(u_h, v_h) = \ell_h^{SD}(v_h) \quad \forall v_h \in V_h^k$$

pick p-w constant δ

$\Rightarrow \delta = \delta_K$

SD norm:

$$\|v\|_{SD} = \left(\varepsilon|v|_{H^1}^2 + \sum_K \delta_K \|\beta\cdot\nabla v\|_{L^2(K)}^2 + \mu_0 \|v\|_{L^2(K)}^2\right)^{1/2}$$

Mesh Peclét number:

$$Pe_K = \frac{\|\beta\|_{\infty(K)} \, h_K}{2 \, \varepsilon} \qquad \forall K \in \mathcal{T}_h$$

Peclét Number

$$Pe_K \begin{cases} > 1 & \text{convection dominated} \\ \leq 1 & \text{diffusion} \quad '' \end{cases}$$

see e.g.
[Quarteroni: Numerical Methods for Differential Problems, Springer, 2017.]

Theorem: Assuming $\mathcal{T}_h$ shape reg, if

$$\delta_K = \begin{cases} \delta_0 \, h_K & \text{if } Pe_K > 1 \\ \delta_1 \, \frac{h_K^2}{\varepsilon} & \text{if } Pe_K \leq 1 \end{cases}$$

with δ_0, δ_1 constants, then $\exists C > 0$:

$$\|u - u_h\|_{SD} \leq C \left(\varepsilon^{1/2} + h^{1/2}\right) h^k \, |u|_{H^{k+1}(\Omega)}.$$

preasimptotic $O(h^{k+1/2})$

ε, h, H

$$\left\langle \begin{array}{ll} \varepsilon^{1/2} h^{k} & \text{dominant when } h \ll \varepsilon \\ \\ h^{k+1/2} \quad \text{''} & \text{when } \varepsilon \ll h \end{array} \right.$$

DG for transport

Books on DG:

- Di Pietro - Ern. Mathematical aspects of Discontinuous Galerkin Methods, Springer, 2012.
- Kanschat. Discontinuous Galerkin for viscous incompressible flows, Teuber, 2007.

Recall:

$$\mathcal{L}u = \begin{cases} \beta \cdot \nabla u + \mu u = f & \text{in } \Omega \\ u_{|\partial\Omega^-} = g \end{cases}$$

$$V = \{ v \in L^2(\Omega) : \beta \cdot \nabla v \in L^2(\Omega) \} \subset L^2(\Omega)$$

Hilbert: $(u,v)_V = (u,v)_{L^2} + (\beta\cdot\nabla u, \beta\nabla v)_{L^2}$

$\rightarrow \|v\|_V^2 = \|v\|_{L^2}^2 + \|\beta\cdot\nabla v\|_{L^2}$

W.P.: $u \in V$:

$(\mathcal{L}u, v) - (u, \beta\cdot n v)_{\partial\Omega^-} = (f,v) - (g, \beta n v)_{\partial\Omega^-}$

(WP) $\forall v \in V$

coercive w.r.t.

$\mu_0 \|v\|_{L^2(\Omega)}^2 + \frac{1}{2} |v|_{\beta,\partial\Omega}^2$

(not in $\|\cdot\|_V$!)

$\Rightarrow$ Lax-Milgram does not apply

well-posedness through inf-sup framework

Theorem : (WP) is well-posed in V

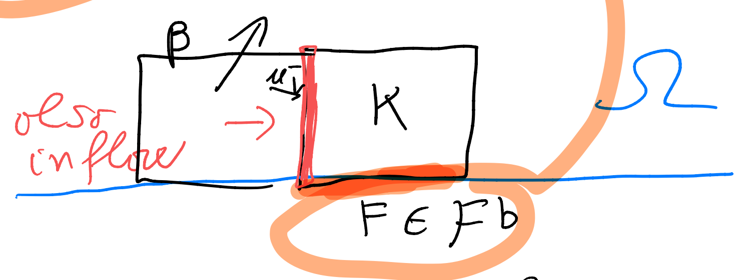
DG formulation

- Ω polygone
- $\{\mathcal{T}_h\}_h$ family of meshes of Ω

(1) restrict equation to each $K \in \mathcal{T}_h$:

for u exact sol. / $\forall K$

$$\int_K \mathcal{L} u\, v - \int_{\partial K \cap \partial_- \Omega} (\beta \cdot n)\, u\, v = \int f v - \int_{\partial K \cap \partial_- \Omega} \beta \cdot n\, g\, v$$



$$\mathcal{F}^b = \{ F \subset \partial K \cap \partial \Omega \ ,\ \exists K \in \mathcal{T}_h \}$$

$$\mathcal{F}^i = \{ F = \partial K^+ \cap \partial K^- \quad \exists K^-, K^+ \in \mathcal{T}_h \}$$

Define

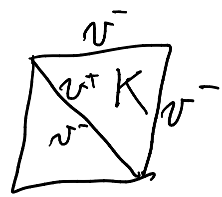
$$\partial_- K = \{ x \in \partial K : \beta(x) \cdot n(x) < 0 \}$$

$$\partial_+ K = \partial K \setminus \partial_- K$$

If above we test with v potentially discontinuous across K, name

v^+ = interior of K trace

v^- = exterior of K trace



Over $\partial_- K \setminus \partial\Omega$ add

$$\rightarrow \int_{\partial_- K \setminus \partial\Omega} b \cdot n \, u^+ v^+ = \int_{\partial_- K \setminus \partial\Omega} b \cdot n \, u^- v^+$$

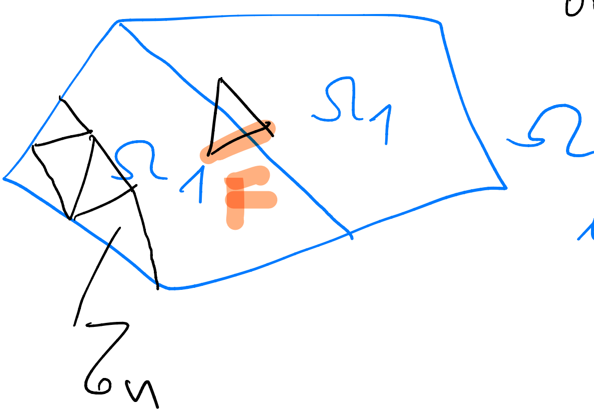
or, with $\lfloor u \rfloor := u^+ - u^-$ "upwind jump"

$$\boxed{\int_{\partial_- K \setminus \partial\Omega} \beta\cdot n \,[u]\, v^+ = 0}$$

indeed on $\partial_- K$ $\beta\cdot n < 0$ so u is continuous across $\partial_- K$:

lemma: (2.14 Di Pietro + Ern): Assume that exists a finite partition $P_\Omega = \{\Omega_i\}$ into disjoint polyhedra.

$$u \in V_* = V \cap \underbrace{H^1(P_\Omega)}_{\text{broken } H^1 \text{ w.r.t. } P_\Omega}$$



$u \in H^1(\Omega_i)\ \forall i$

Then $\forall F \in \mathcal{F}^i \quad \beta\cdot n\,[u] = 0$ a.e in F

$\beta\cdot n = 0 \qquad [u] = 0$

<u>DG method</u> : recall

$$V_{DG} = V_h^{DG,k} = \{ v_h \in L^2(\Omega) : v_h \circ T_K \in P_{\hat{K}} \,,\ \forall K \in \mathcal{T}_h \}$$

where $P_{\hat{K}}$ $<$ $\mathbb{P}_k(\hat{K})$, $\mathbb{Q}_k(\hat{K})$

$$B(u,v) = \sum_{K\in\mathcal{T}_h} \left[\int_K \mathcal{L}u\, v - \int_{\partial_- K \setminus \partial\Omega} \beta\cdot n\,[u]\,v^+ - \int_{\partial_- K \cap \partial\Omega} \beta\cdot n\, u^+ v^+ \right]$$

$$\ell(v) = \int_\Omega f v - \sum_{K\in\mathcal{T}_h} \int_{\partial_- K\cap\partial\Omega} (\beta\cdot n)\, g\, v^+$$

UPWIND DG method : Find $u_h \in V_h^{DG,k}$:

$$(DG)\quad B(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V_h$$

Remarks:

(1) Can be implemented from inflow elements and proceeding along β

(2) full consistency:

$$B(u-u_h, v_h) = 0 \quad \forall v_h \in V_h^{DG,k}$$

(3) suppose $\beta \equiv \text{const}$. Then

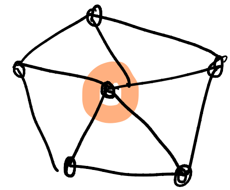
$$v_h + \delta_K \, \beta \cdot \nabla v_h \in V_{DG} \quad \forall v_h \in V_{DG}$$

hence DG has naturally stronger stability. Due to additional richness of DG space + upwind scheme

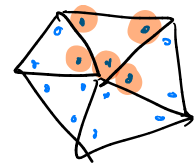
(4) Indeed, we have

$$\|u-u_h\|^2_{L^2(\Omega)} + \sum_{F \in F_h} \left\| \frac{|b \cdot n|}{2} \underbrace{[u-u_h]}_{(0)} \right\|^2_{L^2(F)} \le C h^{2p+1} \|u\|^2_{H^{k+1}(\Omega)}$$

$\Rightarrow \mathcal{O}(p+1/2)$ in L^2



(5) e.g. on triangulations, DG spaces is larger. But communication is only through edges (faces) Assembly requires also face integrals !



(6) Starting with [Arnold, SINUM, 1982] DG for elliptic problems also available giving method for all regimes.

DG "take 2" – dual-II version

for transport in divergence form:

(also $f=0$, $\mu=0$)

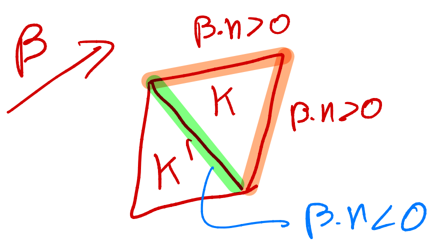
$$\begin{cases} \nabla\cdot(\beta u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega_- \end{cases}$$

Test with v elementwise + integrate by parts:

$$-\int_K u\,\beta\cdot\nabla v + \int_{\partial K} \beta\cdot n\; u\, v = 0$$

choice of numerical flux $\partial K = \bigcup_{F\in\partial\Omega} F$

$$u \to u^{up} : \begin{cases} F \subset \partial_-\Omega \to u^{up} = g \\ F \subset \partial_+\Omega \to u^{up} = u_{|K} \quad \text{where } K\in\mathcal{T}_h : F\subset\partial K \\ F \in \mathcal{F}^i \to u^{up} = u^{+} \text{ if } \beta\cdot n \geq 0 \;:= (u_{|K})_{|F} \\ \qquad\qquad\qquad = u^{-} \text{ if } \beta\cdot n < 0 \;:= (u_{|K'})_{|F} \end{cases}$$



K' = neighbour

$$\Downarrow$$

$$\sum_K \text{ fix } \int_{\partial K} \beta\cdot n \, u^{up} v$$

as a sum over faces

$$\sum_{F\in\mathcal{F}^i} \int_F |\beta\cdot n| \, u^{up} [v]$$

$$[v] = v^+ - v^-$$

$$u^{up} \quad \beta\cdot n^+ v^+ + \beta\cdot n^- v^-$$

DG upwind method: $u_h \in V_{DG}$:

$$-\sum_K \int_K u_h \beta\cdot\nabla v + \sum_{F\in\mathcal{F}^i} \int_F |\beta\cdot n| u_h^{up} [v]$$

$$+ \sum_{F\in\mathcal{F}^b\cap\partial_+\Omega} \int_F \beta\cdot n \, u^{up} v = -\sum_{F\in\mathcal{F}^b\cap\partial_-\Omega} \int_F \beta n g v$$