ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 11

Boffi, Brezzi, Fortin *Mixed Finite Element Methods and Applications*. Springer, 2013.

Given spaces V, Q , $(u, q) \in V \times Q$:

$$(SP) \begin{cases} 1 & a(u,v) + b(v,p) = g(v) \quad \forall v \in V \\ 2 & b(u,q) \qquad\qquad = f(q) \quad \forall q \in Q \end{cases}$$

If introduce

$$Z = \{ v \in V : \ b(v,q) = 0 \quad \forall q \in Q \}$$

$(SP)_1$ equivalent to : $u \in Z$:

$$a(u,v) = g(v) \qquad \forall v \in Z \qquad (*)$$

well-posed if a is coercive in Z !

Given $u \in Z \subset V$ solution of $(*)$, it remains to: find $p \in Q$:

$$b(v,p) = \underbrace{a(u,v) - g(v)}_{=(r,v) \text{ weak residual}} \qquad \forall v \in V$$

or $(r,z) = 0 \quad \forall z \in Z$

$$r \in Z^{\perp} = \{ v \in V : (v,z) = 0 \quad \forall z \in Z \}$$

solvability depends on compatibility between V, Q.

Example: $b(v,q) = \int_\Omega \nabla \cdot v \, q$

Solvability $\Leftrightarrow$ find $p \in Q : B^T = \nabla p = r \quad \forall r \in Z^{\perp}$

$\Leftrightarrow$ range $B^T = Z^{\perp}$

Let's look at it from the point of view of the inf-sup condition ...

Theorem (Brezzi): V, Q Hilbert

$a : V\times V \to \mathbb{R}$; $b : V\times Q \to \mathbb{R}$ bilinear, continuous,

Define $Z = \{ v \in V : b(v,q) = 0 \quad \forall q \in Q \}$

If • $a(\cdot,\cdot)$ is coercive in Z

• $b(\cdot,\cdot)$ satisfies : $\exists \beta > 0$:

$$\beta \|q\|_Q \leq \sup_{\substack{v\in V \\ v\neq 0}} \frac{b(v,q)}{\|v\|_V} \qquad \forall q \in Q$$

$$\left(\iff \quad \beta \leq \inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \qquad \begin{matrix}\text{INF} \\ \text{SUP}\end{matrix} \right)$$

then (SP) is well-posed

$V = H(\mathrm{div};\Omega)$, $Q = L^2(\Omega)$, Darcy

a is coercive in Z

$$a(v,v) \geq \alpha \|v\|^2_{L^2(\Omega)} = \alpha \|v\|^2_{H(\mathrm{div};\Omega)}$$

$\boxed{\text{if } \nabla\cdot v = 0}$

$\Updownarrow$

$\boxed{v \in Z}$

$\forall q \in Q$, consider problem

Find $\phi \in H^1$: $\begin{cases} -\Delta\phi = q & \Omega \\ \phi = 0 & \partial\Omega \end{cases}$

is well posed and so can fix

$u \in H(\mathrm{div};\Omega)$ or $u = \nabla\phi$

$\Rightarrow \nabla\cdot u = q$ (surjectivity of the divergence operator (B))

implies inf-sup :

given q, pick such u : $\nabla \cdot u = q$

then for (u, q) inf-sup relation holds

Take home message: well-posedness of saddle point problems:

- surjectivity op. B related to $b(\cdot,\cdot)$
- coercivity in Z of a

STOKES (homog. Dirichlet conditions)

$V = (H_0^1(\Omega))^d$; $Q = L_0^2(\Omega)$

$$\begin{cases} -2\nabla\cdot\varepsilon(u) + \nabla p = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

Weak form

$$a(u,v) = 2\mu \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx$$

$$b(v,q) = - \int_\Omega q \; \nabla \cdot v \; dx$$

$u \in V,\ p \in Q:$

$$(WS) \quad \begin{cases} a(u,v) + b(v,p) = (f,v) & \forall v \in V \\ b(u,q) \qquad\qquad = 0 & \forall q \in Q \end{cases}$$

Coercivity of a: we have (Korn): $\exists\, C > 0$:

$$\|\varepsilon(v)\|^2_{L^2(\Omega)} \geq C \, \|\nabla v\|^2_{L^2(\Omega)}$$

$\Rightarrow$
- a is coercive V $\boxed{?}$
- $\nabla\cdot$ is surjective

$\longrightarrow$ Brezzi theory holds

Discretisations of Stokes

A pair of discrete spaces may not satisfy "discrete" inf-sup (see below) issue: surjectivity of the B (divergence operator) discrete version

$$B_h : -\nabla_h : V_h \longrightarrow Q_h$$

$$(\nabla_h \cdot u_h, q_h) = \int_\Omega q_h \, \nabla \cdot u_h \quad \forall q_h \in Q$$

i.e. L^2-proj of divergence of u_h

If $\nabla \cdot V_h \subset Q_h \Rightarrow \nabla_h \cdot V_h = \nabla \cdot V$

(exactly RT case $\Rightarrow$ OK ?)

In general this may not be the case and u_h may not be div-free

Theorem (discrete inf-sup): If $V_h \times Q_h$ satisfy

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|q_h\|_Q \|v_h\|_V} \geq \beta_h \geq \beta_0 > 0$$

$\exists \, \beta_h, \ \beta_0,$

Then $\exists!\ (u_h, p_h) \in V_h \times Q_h$ sol. of (WS) and

$$\|u-u_h\|_V + \|p-p_h\|_Q \le C \left[\inf_{v_h\in V_h} \|u-v_h\|_V + \inf_{q_h\in Q_h} \|p-q_h\|_Q\right]$$

Examples $\quad \mathcal{T}_h$ triangulation

1. $\mathbb{P}^1-\mathbb{P}^0$

$$V_h = \left\{ v \in [H_0^1(\Omega)]^d : v_{|K} \in [\mathbb{P}^1(K)]^d, \forall K \in \mathcal{T}_h \right\}$$

$$Q_h = \left\{ q \in L_0^2(\Omega) : q_{|K} \in \mathbb{P}^0(K), \forall K \in \mathcal{T}_h \right\}$$

DOES NOT WORK: (LOCKING)

Define $Z_h = \{ v_h \in V_h : b(v_h, q_h) = 0, \forall q_h \in Q_h \}$

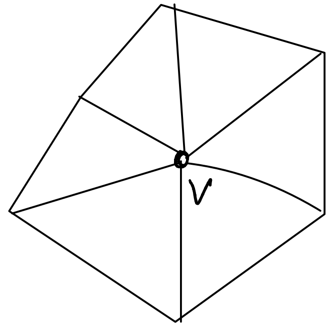
as before, 1^{st} eq. equivalent to: $u_h \in Z_h$:

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Issue: Z_h may not be rich enough.

Indeed, may even have $Z_h = \{0\}$

Consider :



V_h has dim 2 (two components values at v)

but def. of Z_h involves 6 conditions ($\equiv \dim Q_h$)

② $\mathbb{P}_1 - \mathbb{P}_1$ take $k=1=\ell$ in

$$V_h = \{ v_h \in [H_0^1(\Omega)]^d : v_{h|K} \in [\mathbb{P}^k(K)]^d, \forall K \in \mathcal{T}_h \}$$

$$Q_h = \{ q_h \in H^1(\Omega) : q_h|_K \in \mathbb{P}^\ell(K), \forall K \in \mathcal{T}_h \}$$
$$\left(+ \int q_h = 0 \right)$$

$k=\ell=1$ does not work (see example in the BBF book). Again inf-sup not satisfied or kernel of discrete grad is non-trivial $\to$ spurious pressure modes may appear.

- $\mathbb{P}_2 - \mathbb{P}_1$ ($\mathbb{P}_k - \mathbb{P}_{k-1}$, $\mathbb{Q}_k - \mathbb{Q}_{k-1}$ in quad case)

Taylor - Hood

take $\ell = k-1$

- discrete inf-sup Th. holds
- $\|u-u_h\|_{L^2(\Omega)} + h\left(|u-u_h|_{H^1(\Omega)} + \|p-p_h\|_{L^2(\Omega)}\right)$

$$\leq C h^{k+1} \left(\|u\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right)$$