ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 10

See e.g. Larson-Bengzon
Boffi-Brezzi-Fortin, Springer, 2013
Boffi et al. C.I.M.E. summer school
Girault-Raviart, Springer, 1986.

Fluid Mechanics

Equations derived from

1) conservation of mass
2) " of momentum

1) Consider generic volume ω,

ρ = density, u = velocity

$$\int_\omega \dot{\rho} \, dx + \int_{\partial\omega} \rho \, u \cdot n \, ds = 0 \qquad \left(\dot{\rho} = \frac{\partial \rho}{\partial t}\right)$$

$$\Rightarrow \int_\omega \left(\dot{\rho} + \nabla \cdot (\rho u)\right) dx = 0$$

$$\Rightarrow \boxed{\dot{\rho} + \nabla \cdot (\rho u) = 0}$$

If $\rho \equiv \text{const} \Rightarrow \boxed{\nabla \cdot u = 0}$ incompressibility condition

2) Momentum balance

$\hookrightarrow \; q = m u \;, \quad \dot{q} \overset{\text{Newton}}{=} F$

look at dx, $\quad dq = \rho u \, dx$

$$\int_\omega \overset{\circ}{(\rho u)} \, dx + \int_{\partial\omega} \rho u \; u \cdot n \, ds = \int_\omega F = \int (\nabla \cdot \underbrace{\sigma}_{\text{stress}} + f) \, dx$$

$$\Rightarrow \boxed{\underbrace{\overset{\circ}{(\rho u)} + \nabla \cdot (\rho u \otimes u)}_{\text{LHS}} = \nabla \cdot \sigma + f}$$

For incompressible case $\left(\overset{\circ}{\rho} = 0 \quad \nabla \cdot (\rho u) = 0\right)$

$$\Rightarrow \text{LHS} = \rho \overset{\circ}{u} + \rho (u \cdot \nabla) u$$

Hence,

$$\boxed{\begin{aligned} \rho \overset{\circ}{u} + \rho (u \cdot \nabla) u &= \nabla \cdot \sigma + f \\ \nabla \cdot u &= 0 \end{aligned}}$$

Constitutive equations for Newtonian fluids

only two types of stresses:

- internal stress do to pressure: $-pI$

- " viscous stress

$$\underset{\substack{\uparrow \\ \text{viscosity}}}{\mu}\left(\nabla u + \nabla u^T\right)$$

$$\Rightarrow \quad \sigma = -pI + 2\mu\,\varepsilon(u)$$

$$\boxed{\varepsilon(u) = \frac{1}{2}\left[\nabla u + \nabla u^T\right]} \quad \text{strain}$$

Inserting in incompressible fluids eq.:

$$\rho\dot{u} + \rho(u\cdot\nabla)u = -\nabla p + 2\nabla\cdot\mu\,\varepsilon(u) + f$$

If $\mu \equiv \text{const}$; $\nu = \mu/\rho$

$$\Rightarrow \begin{cases} \dot{u} + (u\cdot\nabla)u = -\nabla p + 2\nu\, \underbrace{\nabla\cdot\varepsilon(u)}_{\nu\Delta u} + f \\ \nabla\cdot u = 0 \end{cases}$$

if $u \in C^2(\Omega)$

NAVIER-STOKES

$\left(+ \frac{1}{|\Omega|}\int_\Omega p \, dx = 0 \right.$ to fix the constant $\left. \right)$

boundary conditions

- wall conditions
  - $u\cdot n = 0$ (slip)
  - $u = g_D$ (no slip) (fixed wall $g_D = 0$)
- outflow boundary
  - $\sigma\cdot n = 0$ (stress free)
  - $n\cdot\nabla u - p n = 0$ (do-nothing)
- inflow boundary
  - $u = u_{in}$

Stokes system

Omitting convective term $(u\cdot\nabla)u$

+ assuming steady-state

$$\begin{cases} -2\mu\nabla\cdot\varepsilon(u)+\nabla p=f & \text{in } \Omega \\ \nabla\cdot u=0 & \text{in } \Omega \\ u=g_D & \text{on } \partial\Omega \end{cases}$$

Darcy's law

describes flow in porous media

can be obtained by homogenisation of NS.

(At constant elevation) Darcy's law expresses relationship between

- discharge Q
- pressure drop P_1-P_a
- viscosity μ

$$Q = -\frac{k\, \overset{\text{area}}{A}\,(P_b - P_a)}{\mu \quad L - \text{length}}$$

permeability

$$\boxed{q = -\frac{k}{\mu}\nabla p}$$

volumetric flux

Darcy system

$$\boxed{\begin{array}{lll} \text{D1} & K^{-1} u + \nabla p = 0 & \text{in } \Omega \\ \text{D2} & -\nabla \cdot u = -f & \text{in } \Omega \\ & p = g & \text{on } \partial\Omega \end{array}}$$

Mixed formula of Poisson problem

$$\begin{cases} -\nabla \cdot (K \overset{u}{\overset{\parallel}{\nabla}} p) = f & \text{in } \Omega \\ p = g & \text{on } \partial\Omega \end{cases}$$

Weak form for Darcy $\Omega \subset \mathbb{R}^d$

$$H(\mathrm{div};\Omega) = \{ v \in [L^2(\Omega)]^d : \nabla \cdot v \in L^2(\Omega) \}$$

Hilbert

$$\| v \|_{H(\mathrm{div})} = \left(\| v \|^2_{L^2(\Omega)} + \| \nabla \cdot v \|^2_{L^2(\Omega)} \right)^{1/2}$$

$$\rightarrow \quad H(\mathrm{div};\Omega) \times L^2(\Omega)$$

Test in D1 with $v \in H(\mathrm{div};\Omega)$

$$\int_\Omega K^{-1} u \cdot v \, dx - \int_\Omega p \, \nabla \cdot v + \int_{\partial\Omega} \underbrace{p}_{= g} \, v \cdot n = 0$$

natural condition

Test in D2 with $q \in L^2(\Omega)$

$$- \int_\Omega q \, \nabla \cdot u = - \int_\Omega f q$$

Darcy:

Find $(u, p) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$:

(MF) $$\begin{cases} \int_\Omega K^{-1} u\cdot v - \int_\Omega p\, \nabla\cdot v = -\int_{\partial\Omega} g\, v\cdot n, & \forall v \in H(\mathrm{div}) \\ -\int_\Omega q\, \nabla\cdot u = -\int_\Omega f q & \forall q \in L^2(\Omega) \end{cases}$$

Define: • $a(u,v) = \int_\Omega K^{-1} u\cdot v \, dx$

• $b(u,q) = \int_\Omega (\nabla\cdot u)\, q \, dx$

→ (MF) appears in general saddle point form:

$$\boxed{\begin{aligned} a(u,v) - b(v,p) &= (g,v)_{\partial\Omega} \\ -b(u,q) &= -(f,q)_\Omega \end{aligned}}$$

Note:

• The Poisson problem is equivalent to minimisation problem (case $g=0$):

Find $p \in H^1_0(\Omega)$: $p = \underset{q \in H^1_0(\Omega)}{\arg\min} \left[\frac{1}{2} \int_\Omega K \nabla q \cdot \nabla q - \int_\Omega f q \right]$

- The mixed formulation (Darcy) is equivalent to constraint minimisation problem

Find $u \in H(\mathrm{div}; \Omega)$:

$$u = \underset{\substack{v \in H(\mathrm{div};\Omega) \\ \boxed{-\nabla \cdot v + f = 0}}}{\arg\min} \int_\Omega K^{-1} v \cdot v + \int_{\partial\Omega} g \, v \cdot n$$

deemed to be more difficult ?

For the moment assume (MP) well posed and look for conforming discretisation

- $p \in L^2(\Omega) \rightarrow p_h \in V^k_{DG}$ piecewise disc. polynomials
- $u \in H(\mathrm{div}; \Omega)$ $u_h \in$? $H(\mathrm{div})$-conforming space

Raviart-Thomas $H(\mathrm{div})$ - conforming space

$\mathcal{T}_h$ mesh made of simplices

$K \in \mathcal{T}_h, \quad k \geq 0$

$$RT_k(K) = \underbrace{\mathbb{P}_k(K)^d + x\,\mathbb{P}_k(K)}_{\text{polynomial of degree } k+1}$$

$v \in RT_k(K)$ then $v = \overset{\mathbb{V}}{w} + x \underbrace{\sum_{|\alpha|=k} a_\alpha x^\alpha}_{p}$

α = multi-index

Lemma :

(a) $\dim RT_k = d\binom{k+d}{k} + \binom{k+d-1}{k}$

[as $\dim \mathbb{P}_k(K) = \binom{k+d}{k}$; $\#(|\alpha|=k) = \binom{k+d-1}{k}$

$\forall v \in RT_k(K)$

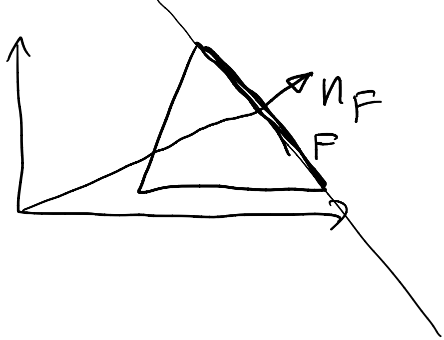
(b) $\boxed{v \cdot n_F \in \mathbb{P}_k(F)} \quad F \in \partial K$

(c) $\nabla \cdot v \in \mathbb{P}_k(K)$ and $\nabla \cdot RT_k(K) = \mathbb{P}_k(K)$

(d) If $\nabla \cdot v = 0 \Rightarrow v \in \mathbb{P}_k(K)$

proof of (b): $v = w + x\, p$, $w \in \mathbb{P}_k^d$, $p \in \mathbb{P}_k$

over $F \subset \partial K$: $v \cdot n_F = \underbrace{w \cdot n_F}_{\in \mathbb{P}_k(F)} + \underbrace{(x \cdot n_F)}_{= s \text{ constant}}\, p$



DoF for $RT_k(K)$

lemma: (RT local interpolant):

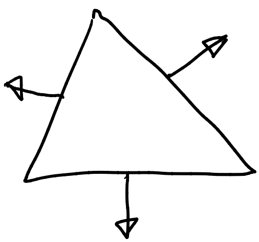
Given $v \in H^1(K)^d$ $\exists!\ I_K v \in RT_k(K)$:

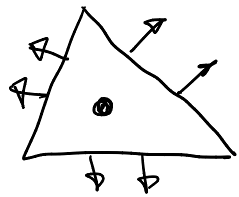
- $\int_F I_K v \cdot n_F\, p_k \, dx = \int_F v \cdot n_F\, p_k \, dx \quad \forall p_k \in \mathbb{P}_k(F), \forall F \subset \partial K$
- (if $k \geq 1$) $\int_K I_K v\, p_{k-1} \, dx = \int_K v\, p_{k-1} \, dx \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(K)$

The RHSs above can be used DoF $RT_k(K)$

Examples:

RT_0 $\qquad\qquad$ RT_1





Theorem: $\forall v \in H^m(K) \quad 1 \leq m \leq k+1$

$$\| v - I_K v \|_{L^2(K)} \leq C h_K^m |v|_{H^m(K)}$$

C depends on d, k, shape reg.

Global RT space

$$RT_k = \{ v \in H(\mathrm{div}; \Omega) : v_K \in RT_k(K), \forall K \in \mathcal{T}_h \}$$

with global interpolation operator

$$I_h : H(\mathrm{div};\Omega) \cap \prod_{K\in\mathcal{T}_h} H^1(K)^d \longrightarrow RT_k$$

$$v \longrightarrow I_h v$$

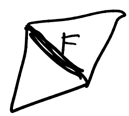
by $\quad I_h v|_K = I_K v \quad \forall K \in \mathcal{T}_h$

proof that $I_h v \in RT_k$

(1) by def. $\quad I_K v \in RT_k(K)$

(2) $\quad I_h v \in H(\mathrm{div};\Omega)^d$

true $\Leftrightarrow$ $\quad v\cdot n_F$ continuous on all F mesh faces



as $I_h v|_K$ is polynomial and

Lemma : $v \in H(\mathrm{div};\mathcal{T}_h) \cap [W^{1,1}(\mathcal{T}_h)]^d$
(= piecewise)

belongs to $H(\mathrm{div};\Omega)$ $\Leftrightarrow$ $[\![v]\!]|_F = 0$
$\forall F \in \mathcal{F}_h^i$

$v \in H(\mathrm{div};\Omega) \Rightarrow v\cdot n$ is continuous on F

by (6) $(I_K v)\cdot n_F \in \mathbb{P}_k(F)$

and, by the definition of the interpolant,

$$\int_F I_K v\cdot n_F\, p_k = \int_F v\cdot n_F\, p_k$$

such polynomial is fully determined by

$v_{|F}\cdot n_F$.

Lemma: $\forall v \in H(\mathrm{div};\Omega) \cap \prod_{K\in\mathcal{T}_h} H^1(K)^d$

$$(\bullet) \int_\Omega \nabla\cdot(v - I_h v)\, q\, dx = 0 \quad \forall q \in V^k_{DG}$$

and $\boxed{\nabla\cdot RT_k = V^k_{DG}}$

RT FEM : Find $(u_h, p_h) \in RT_h \times V_{DG}^k$:

$$
(RT) \quad \begin{cases} a(u, v) - b(v, p) = -\int_{\partial\Omega} g\, v \cdot n & \forall v \in RT_k \\ -b(u, q) = -\int_{\Omega} f q & \forall q \in V_{DG}^k \end{cases}
$$

Galerkin orthogonality

$$
\begin{cases} a(u-u_h, v) - b(v, p-p_h) = 0 & \forall v \in RT_k \quad (GO1) \\ b(u-u_h, q) = 0 & \forall q \in V_{DG}^k \quad (GO2) \end{cases}
$$

Lemma :

$$
\|u-u_h\|_{L^2(\Omega)} \leq \left(1 + \|K\|_{L^\infty(\Omega)} \|K^{-1}\|_{L^\infty(\Omega)}\right) \cdot \|u - I_h u\|_{L^2(\Omega)}
$$

Note: the velocity error does not depend on the pressure error ?

For instance, if $f=0$ $\nabla\cdot u=0$
also the RT solution is div-free

Proof: $\|u-u_h\|_{L^2} \leq \|u-I_h u\|_{L^2} + \|I_h u-u_h\|_{L^2}$

$$\|I_h u-u_h\|^2_{L^2(\Omega)} = \int_\Omega (I_h u-u_h)(I_h u-u_h)$$

$$(A) \quad \leq \|K\|_{L^\infty(K)} \int_\Omega K^{-1}(I_h u-u_h)(I_h u-u_h)$$

$$= \|K\|_{L^\infty(K)} \int_\Omega K^{-1}(I_h u-u)(I_h u-u_h)$$

Proof: use Galerkin orthogonality

by (GO2) $\int_\Omega \nabla\cdot(u-u_h)q = 0$

by $(\bullet)$ $\quad \stackrel{!}{=} \int_\Omega \nabla\cdot(I_h u - u_h)\, q$

by fact that $\nabla\cdot RT_h = V_{DG}^k$ can pick

$$q = \nabla\cdot(I_h u - u_h)$$

$$0 = \int_\Omega \left(\nabla\cdot(I_h u - u_h)\right)^2 \Rightarrow \boxed{\nabla\cdot(I_h u - u_h) = 0}$$

(601) test with $v = I_h u - u_h$

$$\underbrace{\int_\Omega K^{-1}(u - u_h)\cdot(I_h u - u_h)}_{\text{Hence we can add this term to A and get the required equality.}} = 0$$

In conclusion,

$$\| u_h - I_h u\|^{\cancel{2}}_{L^2(\Omega)} \leq \|K\|_{L^\infty(K)} \int_\Omega K^{-1}(I_h u - u)(I_h u - u_h)$$

$$\leq \|K\|_{L^\infty(\Omega)} \|K^{-1}\|_{L^\infty(\Omega)} \|u - I_h u\|_{L^2(\Omega)} \cancel{\|u_h - I_h u\|_{L^2(\Omega)}}$$

$\square$