

ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 2

Videos of lectures available here:

<https://www.youtube.com/playlist?list=PL2754MI3oCY8732lmyQlvKU2zEykN3oK4>

Def (FEM triplet [ciarlet 2002])

- K an admissible mesh cell (element)
- P vector space of functions $p: K \rightarrow \mathbb{R}^m$ of dim n_{sh}
- Σ set of n_{sh} linear forms $\{\sigma_i\}_{i=1}^{n_{sh}}$ acting on P , such that the linear map

$$\Phi(P) = (\sigma_1(P), \dots, \sigma_{n_{sh}}(P))$$

is bijective

Obviously, \exists basis $\{\phi_i\}_{i=1}^{n_{sh}}$ determined by property

$$\sigma_i(\phi_j) = \delta_{ij}$$

ϕ_i turned local shape functions

σ_i || Degrees of Freedom
node FE

Lagrange FE: σ_i are point evaluations

not the only choice, e.g. nodal FE.

[See Ern-Guermond, Ciarlet for many examples]

Def (FE interpolant): let (K, P, Σ) FE triple be given.

Let $V(K)$ Banach space over K s.t.

$$\begin{cases} V(K) \subset L^1(K) \\ P \subset V(K) \end{cases}$$

Assume that σ_i can be extended or
linear operator to $V(K)$

Then, define the interpolant

$I_K : V(K) \rightarrow P$ by

$$I_K v(x) = \sum_{i=1}^{N_K} \alpha_i(v) \phi_i(x)$$

I_K is linear operator with norm

$$\| I_K \|_{\mathcal{L}(V(K))} = \sup_{v \in V(K)} \frac{\| I_K(v) \|_{V(K)}}{\| v \|_{V(K)}} = L(I_K)$$

= Lebesgue constant

- $L(I_K) \geq 1$
 - If $P = P_k$ and ϕ_i Lagrange basis w.r.t. a set of points, then
- $$L(I_K) \geq \frac{2}{\pi} \ln(k) - c \quad \exists c > 0$$

[Erdős, 1961]

Theorem: in the setting of interpolant definition

$$\|v - I_K v\|_{V(K)} \leq (1 + L(I_K)) \underbrace{\inf_{P \in P} \|v - P\|}_{\|v - P\|_{V(K)}}$$

"quasi optimality"

<u>Lagrange poly:</u>	<u>$L(I_K)$</u>	
equidistant	exponential in k	\rightarrow Runge phenomenon
Gauss-Lobatto	$\leq \frac{2}{\pi} \ln(k) + C$	\rightarrow asymptotically optimal

0

Main FE in deal.II: fe-q

- \hat{K} = unit square $[0, 1]^d$
- $\hat{P} = Q_{k,d} = \text{span} \{x_1^{\beta_1} \cdots x_d^{\beta_d}, 0 \leq \beta_1, \dots, \beta_d \leq k\}$

↳ ϕ_i basis Gauss-Lobatto-Lagrange
(Tensor Prod)

- \sum : corresponding tensor product nodes evaluations

Implementation :

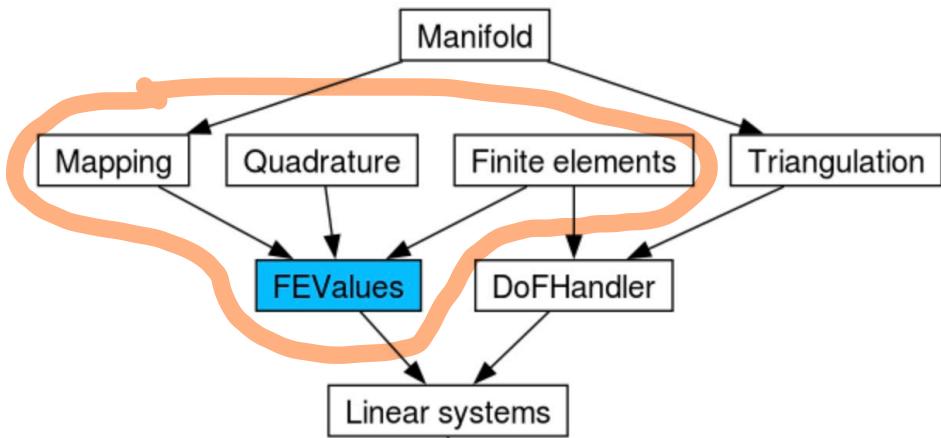
• FE class : { support points DoF
+ feq FE shape functions (and their
gradients) over \tilde{K}

• Quadratures : { locations of quadrature
Gauss-Legendre Points in \tilde{K}
- corresponding weights

+
• mappings : T_K

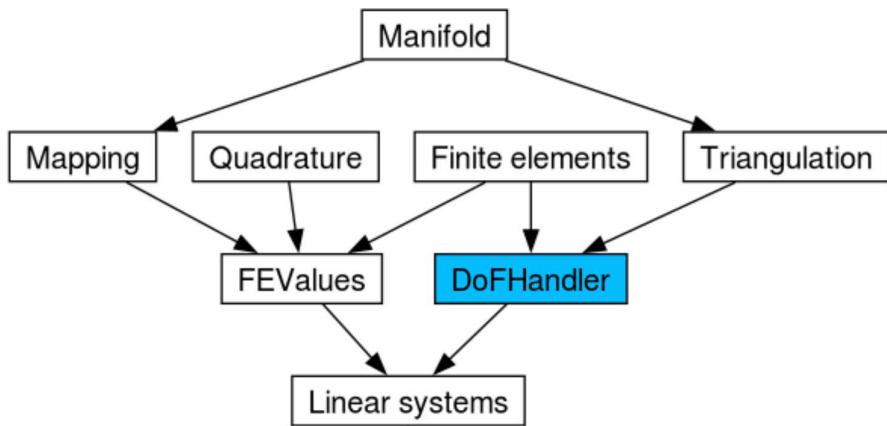
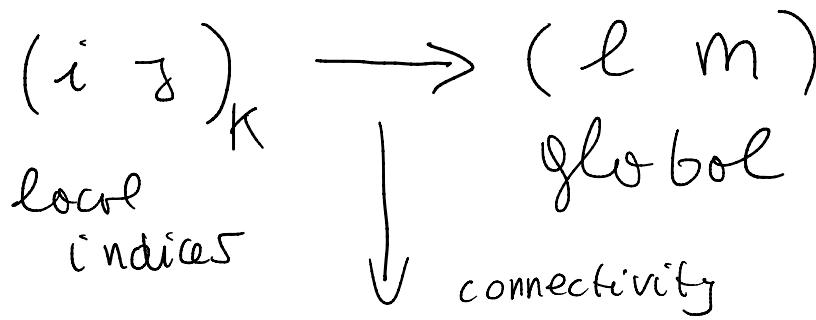
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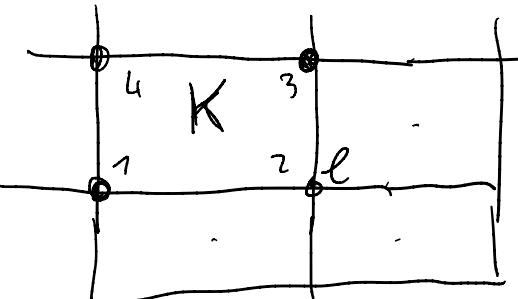
FE values



$$\begin{aligned}
 & \int_K c(x) \varphi_j(x) \varphi_i(x) dx \quad M_{ij}^K \\
 &= \int_R c(T_K(\tilde{x})) \hat{\varphi}_j(\tilde{x}) \hat{\varphi}_i(\tilde{x}) |\det J_x(\tilde{x})| d\tilde{x} \\
 &\approx \sum_{e=1}^{n_{qp}} c(T_K(\tilde{x}_e)) \hat{\varphi}_j(\tilde{x}_e) \hat{\varphi}_i(\tilde{x}_e) \cdot \\
 & \quad \circ |\det J_k(\tilde{x}_e)| \hat{w}_e \\
 & \text{quadrature}
 \end{aligned}$$

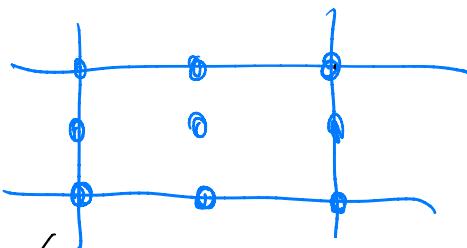
where $(\tilde{x}_e, \hat{w}_e)_{e=1}^{n_{qp}}$ quad. nodes and weights



$Q_{1,2}$ -FEM $\int_{S^2} C \varphi_e \varphi_e$

 $= \sum \int_K C \varphi_e \varphi_e$
 M_{ee}

- 1) Fix sparsity pattern
 - 2) Assemble system using
DoF Handler
- a second example

Q_2 - FEM

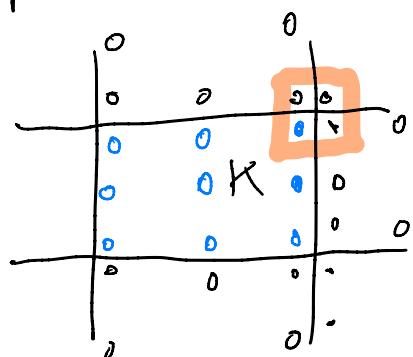


C^0 -conforming

space

shared by the
neighbouring cells

DG space



in deal.II : fe_q(2)

fe_dgq(2)