ADVANCED TOPICS IN SCIENTIFIC COMPUTING

see e.g. E.G. LECTURE 9

Model problem: $\begin{cases} \Omega \subset \mathbb{R}^{2,3} & \text{bounded, Lipchitz,} \\ I =]0, T[\end{cases}$

$$(HP) \begin{cases} u_t - \Delta u = f & (x,t) \in \Omega \times I \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = g(x,t) & (x,t) \in \partial\Omega \times I \end{cases}$$

with $f = f(x,t)$, u_0, g given.

weak formulation

Def (Bochner spaces): Let $(V, \|\cdot\|_V)$ Banach.
The Bochner space $L^p(I; V)$, $1 \le p \le +\infty$, is the space of V-valued functions with norm in V in $L^p(I)$.

$v \in L^p(I; V)$: $v(t) = v(\cdot, t) \in V$

Normed with:

$$\|v\|_{L^p(I;V)} := \begin{cases} \left(\int_I \|v(t)\|_V^p\right)^{1/p} & p<+\infty \\ \operatorname*{esssup}_{t\in I} \|u(t)\|_V & p=+\infty \end{cases}$$

Propoposition: $\left(L^p(I;V), \|\cdot\|_{L^p(I;V)}\right)$ Banach

For (HP), if $g\equiv 0$ then $V = H^1_0(\Omega)$.

Otherwise $H^1_0(\Omega) \subset V \subset H^1(\Omega)$.

In view of 6.c,

Def: $C^J(\bar{I};V)$, $J\geq 0$, is the space V-valued functions of class $C^J(\bar{I})$

$$\|v\|_{C^J(\bar{I};V)} = \sup_{t\in\bar{I}} \sum_{l=0}^{J} \|d_t^l v(t)\|_V$$

Prop: $C^J(\bar{I};V)$ is Banach

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

Test with $v = v(x)$; $g = 0$

$$\left\langle \frac{d\,u(t)}{dt}, v \right\rangle + a(u(t), v) = \langle f(t), v \rangle$$

In <u>general</u> $\langle \cdot , \cdot \rangle$ duality pairing V, V'

so that $\langle \phi, v \rangle$ def. $\forall \phi \in V'$.

Here $V = H^1_0(\Omega)$ $V' = H^{-1}(\Omega)$

$$H^{-1} \supset L^2 \supset H^1_0 \quad \begin{bmatrix} \text{GELFAND} \\ \text{TRIPLE} \end{bmatrix}$$

so that if $\phi \in L^2$ $\langle \phi, v \rangle = (\phi, v)_{L^2}$

$\Rightarrow$ works with $f \in L^2(I; H^{-1}(\Omega))$

$\rightsquigarrow$ weak formulation (HP):

Find $u \in L^2(I; V) \cap C(\bar{I}; L^2(\Omega))$

$\partial_t u \in L^2(I; V')$: for a.e. $t \in I$

(WP) $\left\{ \langle \partial_t u, v \rangle + a(u, v) = \langle f, v \rangle \quad \forall v \in V \right.$

$$\begin{cases} u(0) = u_0 \end{cases}$$

for $f \in L^2(I; V')$, $u_0 \in L^2(\Omega)$.

Well-posedness

In Theorem 6.6 in [E.G.] well posedness of more general parabolic weak problems is shown for any $a(\cdot,\cdot)$ weakly coercive

$$\exists\, \alpha, \lambda > 0 : \quad a(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geq \alpha \|v\|^2_V \quad \forall v \in V$$

(GÅRDING INEQ.)

In our case a is coercive

Corollary: Problem (WP) is well-posed

$$\|u\|^2_{C^0(I; L^2(\Omega))} + \alpha \|u\|^2_{L^2(I;V)} \leq \|u_0\|^2_{L^2(\Omega)} + \frac{1}{\alpha} \|f\|^2_{L^2(I;V')}$$

a priori bound

$(\alpha = 1)$

ϑ-method + Conforming FEM

Fix time steps $k_n = t_n - t_{n-1}$, $n = 1, \dots, N_t$, $t_0 = 0$; $t_{N_t} = T$ and

V_h^k standard H^1-conforming FE space of order k. $\vartheta \in [0,1]$

$\forall n = 0, \dots, N_t - 1$; find $u_h^{n+1} \in V_h^k$

$$\begin{cases} \frac{1}{k_{n+1}} (u_h^{n+1} - u_h^n, v_h) + a(\vartheta u_h^{n+1} + (1-\vartheta) u_h^n, v_h) \\ \qquad = (\vartheta f(t_{n+1}) + (1-\vartheta) f(t_n), v_h) \qquad \forall v_h \in V_h^k \\ u_h^0 = u_{0,h} \in V_h^k \end{cases}$$

$\uparrow$ some approximation from V_h^k of u_0.

In particular: $\vartheta = \begin{cases} 0 & \text{explicit Euler} \\ 1 & \text{implicit " } \\ 1/2 & \text{Crank-Nicolson} \end{cases}$ — Galerkin methods

generalises a sym.

Theorem: Assume $k_n = \Delta t$ $\forall n$ and that

if $0 \leq \vartheta < 1/2$ assume that $\Delta t \, \mu_M < \frac{2}{1-2\vartheta}$

with $\mu_M =$ max eigenvalue of a on V_h^k

Then $\exists !$ discrete sol. u_h^n :

$$\max_n \|u_h^n\| \le \|u_{0,h}\|_{L^2(\Omega)} + T \max_{t\in[0,T]} \|f(t)\|_{L^2(\Omega)}$$

For uniform mesh, condition on $\Delta t \sim h^2$ → superlinear dependence due to ∞-speed of propagation

A priori analysis by Wheeler (1973) based on idea of

- elliptic projection: $w_h \in V_h^k : a(w_h, v_h) = a(u(t^n), v_h) \quad \forall v_h \in V_h^k$
- split error : $u_h^n - u(t^n) = (u_h^n - w_h) + (w_h - u(t^n))$ As w_h is the FEM solution of an elliptic problem having $u(t^n)$ as exact solution, the error $w_h - u(t^n)$ can be estimated by elliptic a priori bounds.

Theorem (Wheeler 1973) :

Assume u_0, f, and u are sufficiently regular.

Then $\forall n \ge 1$,

$$\|u(t_n) - u_h^n\|^2_{L^2(\Omega)} + 2\alpha \, \Delta t \sum_{m=1}^{n} \|u(t^m) - u_h^m\|^2_{H^1(\Omega)}$$

$$\le \|u_0 - u_{0,h}\|^2_{L^2(\Omega)} + C(u_0, f, u) \cdot \left(h^{2k} + \begin{cases} \Delta t^2 & \vartheta \ne 1/2 \\ \Delta t^4 & \vartheta = 1/2 \end{cases} \right.$$

If $u_0 \in H^2(\Omega)$, then $u_{0,h} := I_h u_0 \in V_h$ gives $O(h^k + \Delta t^{1,2})$. Otherwise, can use e.g.

L^2 or H^1 projections.

Algebraic form:

1) Fixed mesh: $u_h^n = \sum_{j=1}^{M_x} U_j^n \varphi_j$ $\quad \varphi_j$ = Lagrange basis

M = mass matrix $\quad M_{ij} = \int_\Omega \varphi_i \varphi_j$

S = stiffness matrix $\quad S_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j$

F^n = load vector $\quad F_i^n = \int_\Omega f(t_n), \varphi_i$

$$\Rightarrow \quad (M + \vartheta k_n S)U^n = (M - (1-\vartheta) k_n S) U^{n-1}$$

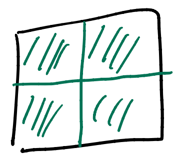
$$+ k_n \left(\vartheta F^n + (1-\vartheta) F^{n-1} \right)$$

2) $\mathcal{T}_h^{n-1} \neq \mathcal{T}_h^n$ old / new meshes

$$u_h^n = \sum_{j=1}^{M_n} U_j^n \varphi_j^n \qquad u_h^{n-1} = \sum_{j=1}^{M_{n-1}} U_j^{n-1} \varphi_j^{n-1}$$

need to evaluate

$$\left(\varphi_i^n, u_h^{n-1}\right)_\Omega = \sum_{j=1}^{M_{n-1}} U_j^{n-1} \underline{\left(\varphi_i^n, \varphi_j^{n-1}\right)_\Omega}$$



alternatively:

1) compute $\boxed{I_h^n u_h^{n-1}} \in V_h^n$

and resort on fixed mesh M.