ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 12

Some textbooks on numerical PDEs in general or on NS in particular:

Quarteroni. *Numerical Methods for Differential Problems*, Springer, 2027.
Quarteroni & Valli. *Numerical Approximation of Partial Differential Equations.* Springer, 2008.
Temam *Navier-Stokes Equations. Theory and Numerical Analysis.* AMS Chelsea, 1984.
Girault & Raviart. *Finite Element Methods for Navier-Stokes Equations.* Springer, 1986.
Kanschat. *Discontinuous Galerkin Methods for Viscous Incompressible Flow.* Teubner, 2007.

Circumventing inf-sup

Galerkin Least-square

From : Bilinear forms of Stokes with laplacian

$$B((u,p),(v,q)) = a(u,v) + \underbrace{b(v,p) - b(u,q)}_{\text{skew-symmetric form}}$$

$$F(v,q) = (f,v)$$

$$B_h((u,p),(v,q)) = B((u,p),(v,q)) + \delta \sum_{K\in\mathcal{T}_h} \left(-\Delta u + \nabla p,\ -\Delta v + \nabla q\right)_K$$

consistent stab

$$F_h((v,q)) = F(v,q) + \delta \sum_{K\in\mathcal{T}_h} \left(f,\ -\Delta v + \nabla q\right)_K$$

$\delta = \delta(h)$ Stabilisation parameter

example : $\mathbb{P}^1 - \mathbb{P}^1$ $\quad \Delta u|_K = \Delta v|_K = 0$

$\rightarrow$ yields $\quad \mathcal{B} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}_*$

$$B_h((u,p),(u,p)) = a(u,u) + \delta(\nabla p, \nabla p)$$

$$= \|\nabla u\|_{L^2(\Omega)} + \delta \|\nabla p\|_{L^2(\Omega)}$$

B_h is coercive depending on $\boxed{\delta} \sim ch^2$

Navier - Stokes

Some issues

- well-posedness in 3D not fully clear
- Turbulence : chaotic flows when small viscosity + $\boxed{\text{fast flow}}$

Reynolds number $Re = \frac{UL}{\nu}$

U, L = characteristic velocity and length-scale

$Re \gg 1 \rightarrow$ turbulent flow

$Re \sim 1 \rightarrow$ laminar flow $\sim$ Stokes

Numerically: as turbulence occurs at all scales down to $Re^{-3/4}$ = Kolmogorov scale $\rightarrow$ unaffordable fine meshes are necessary $\rightarrow$ DIFFICULT !

(adding to the issue of the need of stabilising (nonlinear) convection-dominated flows.

Possible remedy: turbulence modelling

= averaging of the small scales

For instance, resulting in a local change of viscosity $\nu \to \nu + \nu_K$ $\forall K \in \mathcal{T}_h$ where ν_K added viscosity ("subgrid") depends on velocity

- choice of time-stepping (exp/imp/semi-implicit)
- inf-sup elements vs stabilise
- nonlinear $\to$ initial guess

steady N-S

Recall N-S. system (with Δ)

$$\begin{cases} -\nu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } \Omega \\ -\nabla \cdot u = 0 & \text{in } \Omega \\ u = g_D & \text{in } \partial\Omega \end{cases}$$

with if $g_D \equiv 0$ $\begin{cases} u \in V = [H_0^1(\Omega)]^d \\ p \in Q = L_0^2(\Omega) \end{cases}$

otherwise $u \in [H^1(\Omega)]^d$ with $u_{|\partial\Omega} = g_D$

weak form: $(g_D \equiv 0)$ Find $(u,p) \in V \times Q$:

$$(NS) \begin{cases} \overbrace{a(u,v) + c(u;u,v)}^{\tilde{a}(u;u,v)} + b(v,p) = (f,v) & \forall v \in V \\ b(u,q) = 0 & \forall q \in Q \end{cases}$$

$$c(w;u,v) = \int_\Omega [(w \cdot \nabla)u] \cdot v \, dx = \sum_{i=1}^{d} \int_\Omega [(w \cdot \nabla)u]_i v_i \, dx$$

$$= \sum_{i,j=1}^{d} \int_\Omega \underbrace{w_j \partial_j u_i v_i \, dx}$$

Is the form $c(\cdot;\cdot,\cdot)$ continuous?

$$\left| \int_\Omega w_j (\partial_j u_i) v_i \, dx \right| \underset{\text{Hölder}}{\leq} \|w_j\|_{L^4(\Omega)} \|\partial_j u_i\|_{L^2(\Omega)} \|v_i\|_{L^4(\Omega)}$$

if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ $\quad \int_\Omega f g h \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}$

use with : $\boxed{2 \quad 4 \quad 4}$

$f \in L^p, g \in L^q, h \in L^r$

$\le \hat{C} \, |w|_{H^1(\Omega)} \, |u|_{H^1(\Omega)} \, |v|_{H^1(\Omega)}$

$H^1(\Omega) \subset L^4(\Omega)$

Poincare ineq.

that is, c is continuous in $[H^1_0(\Omega)]^d$.

well-posedness (Q-V) depending on f

if $\begin{cases} \nabla \cdot f = 0 \\ \|f\|_{L^2(\Omega)} < \frac{\nu^2}{\hat{C} \, C_\Omega^{1/2}} \end{cases}$ C_Ω = Poicare

then (NS) is well-posed

Need to tackle "new" difficulty given by nonlinear convection term

Linearise and then discretise (step-57)

Consider : $F(u,p)$:

$$F(u,p) = \begin{pmatrix} -\nu \Delta u + (u\cdot\nabla)u + \nabla p - f \\ -\nabla\cdot u \end{pmatrix}$$

issue : find (u,p) : $F(u,p) = 0$

By Newton iterations.

Denote $x = (u,p)$.

- fix x^0 initial guess
- $x^{k+1} = x^k - \nabla F(x^k)^{-1} F(x^k)$

i.e. solve for $\delta x^k = x^{k+1} - x^k$ the system

$$\boxed{\nabla F(x^k)\,\delta x^k = -F(x^k)}$$

Gateau derivative

$$\nabla F(u^k,p^k)(\delta u^k, \delta p^k) \stackrel{\text{Gateau derivative}}{=}$$

$$= \lim_{\varepsilon\to 0} \frac{F(u^k + \varepsilon\delta u^k, p^k + \varepsilon\delta p^k) - F(u^k,p^k)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \Big(\frac{-\nu\Delta(\cancel{u^k} + \varepsilon\delta u^k) + [(u^k + \varepsilon\delta u^k)\cdot\nabla](u^k + \varepsilon\delta u^k)}{\varepsilon}$$

$$
\begin{pmatrix}
\dfrac{+\nabla\cancel{(p^k}+\varepsilon\,\delta p^k)\cancel{-f}+\nu\cancel{\Delta u^k}-\cancel{(u^k\cdot\nabla)}u^k-\cancel{\nabla(p^k)}\cancel{+f}}{} \\[2ex]
\dfrac{-\nabla\cdot(\cancel{u^k}+\varepsilon\,\delta u^k)+\cancel{\nabla\cdot u^k}}{\varepsilon}
\end{pmatrix}
$$

$$
=\lim_{\varepsilon\to 0}\begin{pmatrix}
\dfrac{-\nu\varepsilon\,\Delta\,\delta u+\varepsilon^2(\delta u^k\cdot\nabla)\delta u^k+\varepsilon(\delta u^k\cdot\nabla)u^k+\varepsilon(u^k\cdot\nabla)\delta u^k+\varepsilon\,\nabla\delta p^k}{\varepsilon} \\[2ex]
\dfrac{\varepsilon\,\nabla\cdot\delta u^k}{\varepsilon}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\nu\,\Delta\,\delta u^k+(u^k\cdot\nabla)\delta u^k+(\delta u^k\cdot\nabla)u^k+\nabla\delta p^k \\
-\nabla\cdot\delta u^k
\end{pmatrix}
$$

$$
=-F(u^k,p^k)
$$

2nd eq.- $\left\{ -\nabla\cdot\delta u^k=\nabla\cdot u^k \right.$ incompressibility constraint

Linear system which can solve for δu^k

- In particular, if $u=g_D\big|_{\partial\Omega}=u^0\big|_{\partial\Omega}$

hence $\delta u^k\big|_{\partial\Omega}=0$

How to fix u^0

1) solve for Stokes problem

2) Continuation method:

If $\nu > \frac{1}{1000}$ Stokes good initial guess.

starting with Stokes solution, solve for ν^1 and then iterate (using obtained $NS(\nu^1)$ sol. as initial guess) up to required $NS(\nu)$ solution.