ADVANCED TOPICS IN SCIENTIFIC COMPUTING

[Ern-Guermond]

LECTURE 3

$\hat{K}$ = reference el.

$$
\begin{tikzcd}
\{\hat{K}, P_{\hat{K}}, \Sigma_{\hat{K}}\} \arrow[dr] & & \mathcal{T}_h = \{K\}_K \arrow[dl] \\
& \{K, P_K, \Sigma_K\} \arrow[d, "\text{rule on glueing (conformity)}"] & \\
& V_h &
\end{tikzcd}
$$

T_K

$P_K = T_K(\hat{P}_K)$

V_h FE space

$(m=1, d)$ $\Omega \subset \mathbb{R}^d$

$$V_h = \{ v_h \in [L^2(\Omega)]^m : v_h \circ T_K \in P_{\hat{K}} \ , \forall K \in \mathcal{T}_h \}$$

$$\|$$

$V_{DG} =$ discontinuous piecewise w.r.t. $\mathcal{T}_h$

$$V_h = \{ v_h \in [H^1(\Omega)]^m : v_h \circ T_K \in P_{\hat{K}} \ , \forall K \in \mathcal{T}_h \}$$

$$\|$$

$V_{CG} =$ H^1-conforming FEM

"crucial lemma" broken $H^1 \equiv$

Lemma: $v \in H^1(\mathcal{T}_h) = \{ v \in L^2(\Omega) : v|_K \in H^1(K) \ \forall K \in \mathcal{T}_h \}$

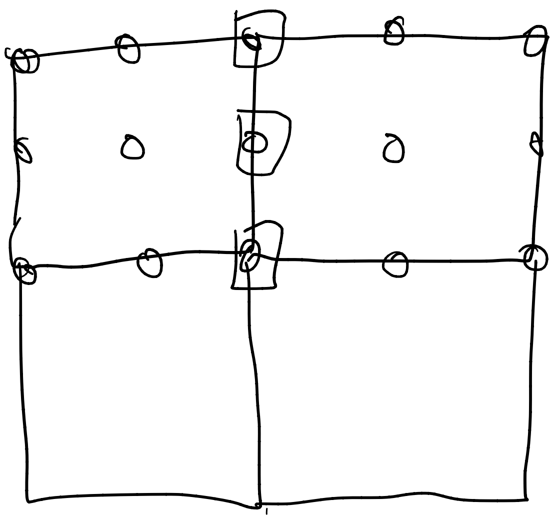
$$v \in H^1(\Omega) \Leftrightarrow [\![v]\!]_F = 0 \quad \forall F \text{ face of } \mathcal{T}_h$$

where $F = K^+ \cap K^- \quad \exists K^+, K^- \in \mathcal{T}_h$

$$[\![v]\!]|_F = v^+ \underline{n}^+ + v^- \underline{n}^-$$

$\underline{n}^{\pm}$ = outward normal of F from $K^{\pm}$

$$V_{CG} = \left\{ v \in V_{DG} : [\![v]\!]|_F = 0 \quad \forall F \text{ face} \right\}$$



$\Rightarrow$ continuity imposed acting on face-based DoF only.

Prototype: $P_{\hat{K}} = \mathbb{P}_k(\hat{K})$; $\underbrace{\mathbb{Q}_k(\hat{K})}$

deal.ii $\quad$ fe_q (CG) $\quad$ fe_dgq (DG)

Analysis of FE interpolants

Assume $V(K) = C^0(K) \quad \forall K$

and define **global interpolant**

$$I_h : C^0(\Omega) \to V_h \qquad I_h v\big|_K = \underbrace{I_h v|_K}_{\text{local interp}}$$

Also assume $k \in \mathbb{N}$

- $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset H^{k+1}(\hat{K}) \subset V(\hat{K})$

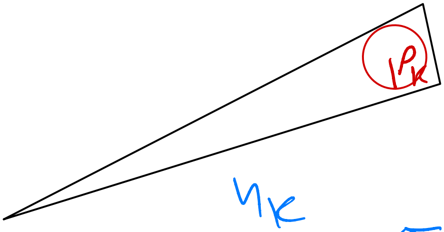
- $\begin{cases} T_K = J_K \hat{x} + b_K \\ V(K) = T_K(V(\hat{K})) \end{cases} \quad \forall K \in \mathcal{T}_h$

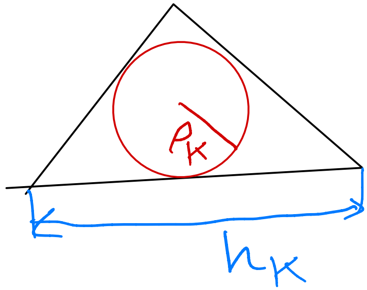
Call I_K^k = local interp $\to$ I_h^k = global interp

- h_K = diam K

- ρ_K = diam of largest ball within K

example:





$$\sigma_K = h_K/\rho_K$$

Theorem (local FE interp. error)

Let $0 \le \ell \le k$, assume $H^{\ell+1}(\hat{K}) \subset V(\hat{K})$

Then $\exists\ \boxed{C>0}$ (indep. of h):

$\forall\ 0 \le m \le \ell+1,\ v \in H^{\ell+1}(K)$

$$|v - I_K^k v|_{H^m(K)} \le C\, h_K^{\ell+1-m}\, \sigma_K^m\, |v|_{H^{\ell+1}(K)}$$

$$\left(\frac{h_K}{\rho_K}\right)^m$$

- quality depends on aspect ratio σ_K

• bounds depends on regularity of v

Special case: $\forall v \in H^{k+1}(K)$

$$|v - I_K^k v|_{H^m(K)} \le c\, h_K^{k+1-m} \sigma_K^m |v|_{H^{k+1}(K)}$$

Def (Shape regularity): A family $\{\mathcal{T}_h\}_{h>0}$ of meshes is shape regular if $\exists\, \sigma_0 > 0$: $\sigma_K \le \sigma_0 \quad \forall K \in \mathcal{T}_h \;\forall h$

Corollary (Global interpolation)

Assume $\{\mathcal{T}_h\}_h$ shape regular. then $\exists\, c > 0$: $\forall h, \forall v \in H^{\ell+1}(\Omega)$

$$\|v - I_h^k v\|_{L^2(\Omega)} + \sum_{m=1}^{\ell+1} h^m \left(\sum_{K \in \mathcal{T}_h} |v - I_K^k v|^2_{H^m(K)} \right)^{1/2}$$

$$\leq C h^{\ell+1} |v|_{H^{\ell+1}(\Omega)}$$

C depends on σ_0 i.e the shape regularity of the family of meshes

special case: suppose $V_h \subset H^1$

$$\boxed{|v - I_h^k v|_{H^1(\Omega)} \leq C h^k |v|_{H^{k+1}(\Omega)}}$$

Case of general quadrilaterals

. $T_K \in [Q_1(\hat{K})]^2$

Corollary: if every K is convex ,

then $|v - I_h^k v|_{H^1(\Omega)} \leq C h^k |v|_{H^{k+1}(\Omega)}$

More in general, the result cannot be guaranteed

[Arnold, Boffi, Falk, Math. Comp., 2002]

showing possible reductions in convergence rates are seen for special sequences of non-convex meshes

(generally non seen on reasonably refine meshes ?)

First example of FEM for PDE (Tutorial 3)

Poisson : $\begin{cases} -\Delta u = f & \text{on } \Omega \in \mathbb{R}^d, \ d=2,3 \\ u = 0 & \partial\Omega \end{cases}$

Discretised with $V_{CG} = V_h^k = \{ v_h \in C^0(\Omega) : v_h \circ T_K \in Q_k(\hat{K}) \ \forall K \in \mathcal{T}_h \}$

- Consider weak form: find $u \in V = H_0^1(\Omega)$:

(WP) $a(u, v) = f(v) \quad \forall v \in V$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\underline{x}$

$$f(v) = \int_\Omega f v \, d\underline{x}$$

- FEM $\equiv$ restriction of (WP) into V_h^k :

$u_h \in V_h$: $a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$

- Fix $\{\phi_i\}_{i=1}^{N}$ shape func. (basis of V_h^k)

  $N = \dim V_h^k$

$$u_h = \sum U_i \phi_i \quad , \quad \text{test with } \phi_j$$

$$\sum_i U_i \underbrace{a(\phi_j, \phi_i)}_{A_{ij}} = \underbrace{f(\phi_j)}_{= F_j} \quad \forall j$$

$$U = (U_1, \dots, U_N)$$

- FEM $\Rightarrow$ sol. of linear system

$$\boxed{AU = F}$$