ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 6

Theorem (Residual error estimator [E-G: Th 10.4])

Let $u \in V$ of Poisson problem and $u_h \in V_h^k$ corresponding H^1-conforming FEM sol. of order k over $\mathcal{T}_h$ shape regular triangulation.

Then $\exists\, c > 0$:

$$\|u - u_h\|_{H^1(\Omega)} \leq c \frac{\sqrt{n}}{\alpha} \underbrace{\left(\sum_K \eta_K(u_h)^2 \right)^{1/2}}_{\eta(u_h)},$$

where c depends on
- shape regularity constants from
  - trace inequality
  - Poincare constant
  - quasi-interpolation error bounds

where

$$\eta_K = h_K \|f + \Delta u_h\|_{L^2(K)} + \frac{h_K^{1/2}}{2} \left\| \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right] \right\|_{L^2(\partial K)}$$

Theorem (Local efficiency of residual estimator [E-G: Th 10.10])

Let $\ell \in \mathbb{N}$, $Z_h^\ell = V_{DG}^\ell = \{v_h \in L^2 : v_h|_K \in \mathbb{P}^\ell(K), \forall K \in \mathcal{T}_h\}$

Then, under assumption of previous theorem,

$\exists\, c = c(\ell)$:

$$\underline{\eta_K(u_h)} \le c \left(|u - u_h|_{H^1(\tilde{K})} + h_K \underbrace{\inf_{v_h \in Z_h^\ell} \|f - v_h\|_{L^2(\tilde{K})}}_{\text{oscillation}} \right)$$

where $\tilde{K}$ = patch of elements around K

General elliptic problems:

The differential operator

$$\mathcal{L}u = -\overbrace{\nabla \cdot (A \nabla u)}^{\text{diffusion}} + \overbrace{\beta \cdot \nabla u}^{\text{advection}} + \overbrace{\mu\, u}^{\text{reaction}}$$

where $A = A(\underline{x})$ tensor in $\mathbb{R}^d$, β vector

is called elliptic if

$A(\underline{x})$ is positive definite a.e. $\underline{x}$

Indeed, in this case the quadratic term

$$a(u,v) = \int_\Omega A(\underline{x}) \nabla u \cdot \nabla v$$

is coercive : $\exists \alpha_0 > 0 : a(u,u) \geq \alpha_0 |u|_{H^1(\Omega)}$

Issue: • well-posedness of

$$\begin{cases} \mathcal{L} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

• Discretization of lower-order terms

$V = H^1_0(\Omega)$, V_h^k H^1-conforming FEM space

weak form:

Find $u \in V$:

$$\text{(WP)} \int_\Omega A \nabla u \cdot \nabla v + \int_\Omega \widehat{\beta \cdot \nabla u \, v} + \int_\Omega \mu u v = \int_\Omega f v \quad \forall v \in V$$

$$\underbrace{\hphantom{aaaaaaaaaaaaaaaa}}_{a(u,v)}$$

coercivity depends also on the lower order terms.

For instance, form in (WP) is coercive if $\boxed{\mu - \frac{1}{2}\nabla\cdot\beta \geq 0}$

Take, for instance, $\mu = 0$; $\nabla\cdot\beta = 0$

then

$$a(u,u) = \int_\Omega A\nabla u\cdot\nabla u + \frac{1}{2}\int_\Omega \beta\cdot\nabla u^2$$

$$\geq \alpha_0 |u|^2_{H^1(\Omega)} - \frac{1}{2}\int_\Omega \overset{=0}{\cancel{\nabla\cdot\beta}}\, u^2 + \frac{1}{2}\int_{\partial\Omega} \cancel{\beta\cdot n\, u^2}$$

$\Rightarrow$ a is indeed coercive but

if $\begin{cases} \alpha_0 \ll 1 \\ \gamma \text{ large } (\gamma \sim \|A\|_\infty + \|\beta\|_\infty) \end{cases}$

continuity const

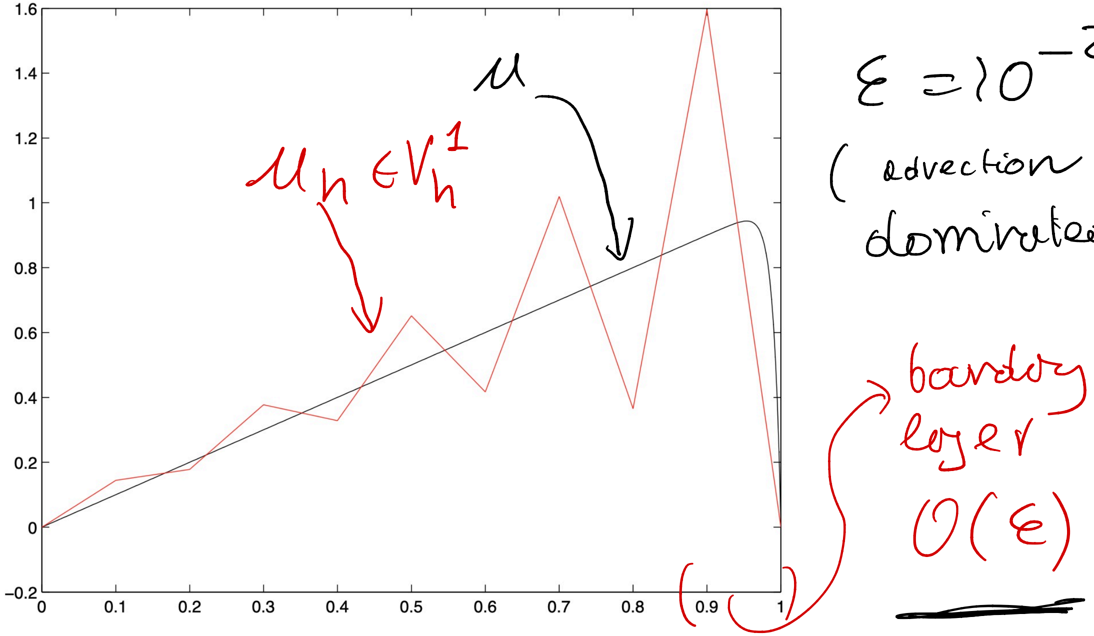
Corresponding bound for standard FEM discretisation

$$\|u-u_h\|_{H^1(\Omega)} \leq \frac{\gamma}{\alpha_0} \inf_{v_h \in V_h} \|u-v_h\|_{H^1(\Omega)}$$

$h \to 0$

is potentially very large ! 0

Example: $\begin{cases} -\varepsilon u'' + u' = 1 & (0,1) \ (\beta = 1) \\ u(0) = 0 = u(1) \end{cases}$



$\varepsilon = 10^{-2}$

(advection dominated)

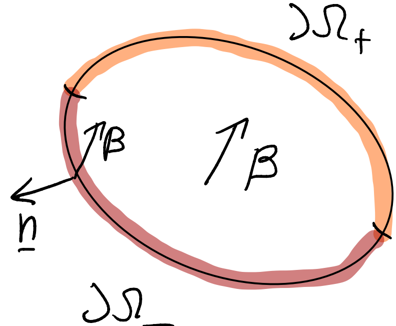
10 elements uniform mesh

Pure transport [Johnson, 2009]

Model problem: (E-G. sec 5.2)

$\Omega \subset \mathbb{R}^d$, $\beta \in [L^\infty(\Omega)]^d$, $\nabla\cdot\beta \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$, $f \in L^2(\Omega)$

$$\mathcal{L}u = \begin{cases} \beta\cdot\nabla u + \mu u = f \\ u|_{\partial\Omega^-} = g \end{cases}$$



$\partial\Omega_- = \{\underline{x} \in \partial\Omega : \beta\cdot\underline{n} < 0\}$

$\partial\Omega_+ = \partial\Omega \setminus \partial\Omega^-$

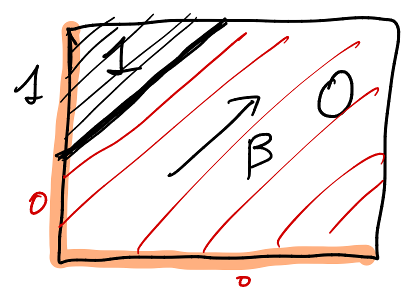
Right solution space: the graph space

$$W = \{w \in L^2(\Omega) : \beta\cdot\nabla w \in L^2(\Omega)\} \subset L^2(\Omega)$$

$\boxed{\text{"}\frac{\partial w}{\partial \beta}\text{"}}$

example $\beta \equiv$ const, $\mu = 0$, $f = 0$

$\Omega = [0,1]^2$, g discontinuous



Apply V_h^k H^1-conf FEM but, with a twist:

WEAK IMPOSITION OF BOUNDARY CONDITIONS $\quad u_h \in V_h^k \subset H^1(\Omega)$ (FREE AT BOUNDARY)

$$\underbrace{\int_\Omega (\beta\cdot\nabla u_h + \mu u_h) v - \int_{\partial\Omega^-} u_h v}_{a(u_h, v)} = \int_\Omega f v - \int_{\partial\Omega^-} g v \quad \forall v \in V_h^k$$

- $a(u - u_h, v) = 0 \quad \forall v \in V_h^k$ — Galerkin orthogonality (consistency)
- $a(v,v) \geq \|\mu_0^{1/2} v\|^2_{L^2(\Omega)} + \frac{1}{2} |v|^2_{\beta,\partial\Omega} \quad \forall v \in H^1(\Omega)$

weak!

$$\boxed{\mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0}$$

where $|v|^2_{\beta,\partial\Omega} = \int_{\partial\Omega} |\underline{n}\cdot\underline{\beta}|\, v^2$

(proof: exercise!)

- $\|u - u_h\|_{L^2} + |u - u_h|_{H^1} \leq C h^k \|u\|_{H^{k+1}(\Omega)}$

if $u \in H^{k+1}$

- (= asking a lot !)
- sub optimal

Streamline Diffusion method

Test instead $\boxed{v_h + h\beta \cdot \nabla v_h}$

test function $\notin V_h^k$

τ_{SD} parameter

Find $u_h \in V_h^k$:

$$(\beta \cdot \nabla u_h + \mu v_h, v_h + h\beta \cdot \nabla v_h) - (u_h, (1+h)\beta \cdot n\, v_h)_{\partial\Omega}$$

$$= (f, v_h + \delta h \beta \cdot \nabla v_h) - (g, (1+\delta h)\beta \cdot n\, v_h)_{\partial\Omega}$$

streamline diffusion term

also named **Petrov-Galerkin**

(test function from different space)