ADVANCED TOPICS IN SCIENTIFIC COMPUTING

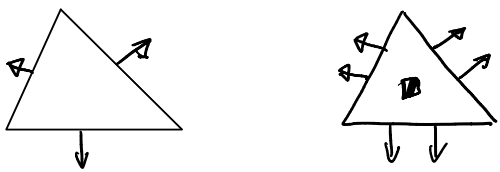
LECTURE 10a

Raviart-Thomas : $\mathcal{T}_h$ mesh, $K \in \mathcal{T}_h$, $k \in \mathbb{N}$

- On triangles:

$$RT_k(K) = \mathbb{P}_k(K)^d + x\,\mathbb{P}_k(K) \Rightarrow \boxed{\nabla \cdot RT_k(K) = \mathbb{P}_k(K)}$$

$k = 0$ $\qquad$ $k = 1$



- On ~~quadrilaterals~~ rectangles/cubes

$$RT_k(K) = \begin{cases} Q_{k+1,k} \times Q_{k,k+1} & d = 2 \\ Q_{k+1,k,k} \times Q_{k,k+1,k} \times Q_{k,k,k+1} \end{cases}$$

generalisations of Q_k = degree k in each variable

$Q_{\ell,m,p}$ = degree ℓ, m, p in x, y, z

$$\Rightarrow \boxed{\nabla RT_k(K) = Q_k(K)}$$

Applied To Darcy : Find $(u,p) \in RT_k \times V_{DG}^h$

$$\begin{cases} \int_\Omega K^{-1} u \cdot v - \int_\Omega \nabla \cdot v \, p = -\int_{\partial\Omega} g \, v \cdot n \, ds & \forall v \in RT_k \\ -\int_\Omega \nabla \cdot u \, q = -\int_\Omega f q & \forall q \in V_{DG}^k \end{cases}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq \left(1 + \|K\|_{L^\infty} \|K^{-1}\|_{L^\infty}\right) \|u - \underbrace{I_{RT}}_{\text{RT- interpolant}} u\|_{L^2}$$

$$\text{if } u \in [H^m(\Omega)]^d \leq C \, h^m |u|_{H^m(\Omega)}$$

Then, look at pressure error

Lemma : $\|p - p_h\|_{L^2(\Omega)} \leq C \left\{ \|p - \underbrace{\Pi_h^0}_{L^2\text{-proj}} p\|_{L^2(\Omega)} + \|u - I_{RT} u\|_{L^2(\Omega)} \right.$

$$\left[\begin{array}{ll} \forall p \in L^2(\Omega) : & \int_K \Pi_h^0 p \, q = \int_K p \, q \quad \forall K \in \mathcal{T}_h \\ \Pi_h^0 p \in V_{DG}^h & \forall q \in V_{DG}^k(K) \end{array} \right]$$

$$\leq C h^m \left[|p|_{H^m(\Omega)} + |u|_{H^m(\Omega)} \right.$$

if $p \in H^m(\Omega)$; $u \in [H^m(\Omega)]^d$

if, e.g. Ω convex $\leq C\, h^{m+1}$

Saddle point problems

V, Q Banach + reflexive

$$(SD)\quad \begin{cases} a(u,v) - b(v,p) = g(v) & \forall v \in V \\ -b(u,q) = -f(v) & \forall q \in Q \end{cases}$$

If a is symmetric (SD) relates to constrained minimisation:

Find $u = \underset{v \in V}{\operatorname{argmin}} \left[\frac{1}{2} a(v,v) - g(v) \right]$

subject to $f(q) - b(u,q) = 0 \quad \forall q \in Q$

Define the Lagrangian

$$\mathcal{L}(v,q) = \frac{1}{2} a(v,v) - g(v) - b(v,q) + f(q)$$

and consider saddle point problem

$$\inf_{v \in V} \sup_{q \in Q} \mathcal{L}(v,q) = \mathcal{L}(u,p)$$

Proposition: If a is sym. and pos. def then (u,p) solves (SP) $\iff$ (u,p) is a saddle point for $\mathcal{L}$.

Proof: see Th 2.39 in E.-G.

Rather, let's check the optimality conditions

$$\left(\partial_u \mathcal{L}(u,p), \partial_p \mathcal{L}(u,p) \right) = 0$$

$$\mathcal{L}(u+tv, p+sq) = \frac{1}{2} a(u+tv, u+tv) - g(u+tv) - b(u+tv, p+sq) + f(p+sq)$$

$$\begin{cases} \left.\dfrac{\partial \mathcal{L}}{\partial t}(u+tv, p+sq)\right|_0 = a(u,u) - g(u) - b(u,p) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial s} = -b(u,q) + f(q) = 0 \end{cases}$$

hence the name of saddle point problem!

From the linear algebra point of view

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{A} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

stabilisation

In general may have C from stabilisation if inf-sup well-posedness condition not satisfied or C may originate from regularisation (e.g. plates)

We have that if A is SPD and B has full rank then system $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ is invertible.