ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 1

Videos of lectures available here:
https://studio.youtube.com/playlist/PL2754Ml3oCY8732lmyQlvKU2zEykN3oK4/videos

Def (FEM triplet [Ciarlet 2002])

- K an admissible mesh cell (element)
- P vector space of functions $p : K \to \mathbb{R}^m$ of dim n_{sh}
- Σ set of n_{sh} linear forms $\{\sigma_i\}_{i=1}^{n_{sh}}$ acting on P, such that the linear map

$$\Phi(p) = (\sigma_1(p) \ldots \sigma_{n_{sh}}(p))$$

is bijective

Obviously, $\exists$ basis $\{\phi_i\}_{i=1}^{n_{sh}}$ determined by property

$$\sigma_i(\phi_j) = \delta_{ij}$$

ϕ_i termed local shape functions

σ_i " " Degrees of Freedom

nodal FE

Lagrange FE: σ_i are point evaluations

not the only choice, e.g. modal FE.

[See Ern-Guermond, Ciarlet for many examples]

Def (FE interpolant): let (K, P, Σ) FE triple be given.

Let $V(K)$ Banach space over K s.t.

$$\begin{cases} V(K) \subset L^1(K) \\ P \subset V(K) \end{cases}$$

assume that σ_i can be extended as a linear operator to $V(K)$

Then, define the interpolant

$$I_K : V(K) \longrightarrow P \quad \text{by}$$

$$I_K v(x) = \sum_{i=1}^{n_{sh}} \sigma_i(v)\, \phi_i(x)$$

I_K is linear operator with norm

$$\| I_K \|_{\mathcal{L}(V(K))} = \sup_{v \in V(K)} \frac{\| I_K(v) \|_{V(K)}}{\| v \|_{V(K)}} = L(I_K)$$

$$= \text{Lebesgue constant}$$

- $L(I_K) \geq 1$
- If $P = \mathbb{P}_k$ and ϕ_i Lagrange basis w.r.t. a set of points, then

$$L(I_K) \geq \frac{2}{\pi} \ln(k) - C \qquad \exists\, C > 0$$

[Erdös, 1961]

Theorem: in the setting of interpolant definition

$$\|v - I_K v\|_{V(K)} \le \underbrace{(1 + L(I_K))}_{\text{"quasi optimality"}} \underbrace{\inf_{p\in P} \|v - p\|_{V(K)}}$$

Lagrange poly:	$L(I_K)$	
equidistant	exponential in k	→ Runge phenomenon
Gauss-Lobatto	$\le \frac{2}{\pi} \ln(k) + C$	→ asymptotically optimal

——— o ———

Main FE in deal.II: fe-q

- $\hat{K}$ = unit square $[0,1]^d$
- $\hat{P} = Q_{k,d} = \text{span}\{x_1^{\beta_1} \dots x_d^{\beta_d},\ 0 \le \beta_1, \dots, \beta_d \le k\}$

↳ ϕ_i basis Gauss-Lobatto Lagrange
(Tensor prod)

- $\hat{\Sigma}$: corresponding tensor product ~~nodes~~ evaluations

Implementation :

- FE class : { support points DoF; FE shape functions (and their gradients) over $\hat{K}$ }

\+ fe_q

- Quadratures : { locations of quadrature points in $\hat{K}$; corresponding weights }

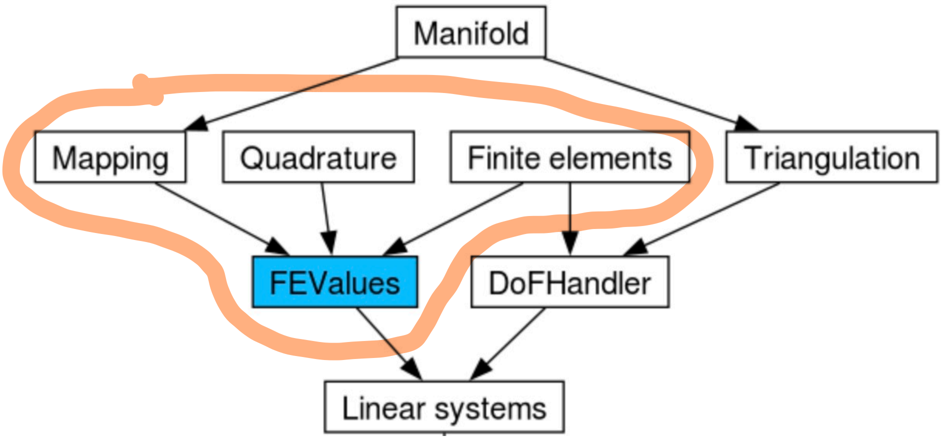
Gauss-Legendre

\+

- mappings : T_K

⇓

FE values



$$\int_K c(\underline{x})\,\varphi_j(\underline{x})\,\varphi_i(\underline{x})\,dx$$

$$= \overbrace{\int_{\hat K} c(T_K(\hat x))\,\hat\varphi_j(\hat x)\,\hat\varphi_i(\hat x)\,|\det J_K(\hat x)|\,d\hat x}^{M^K_{ij}}$$

$$\underset{\substack{|\\ \text{quadrature}}}{\approx} \sum_{\ell=1}^{n_{qp}} c(T_K(\hat\xi_\ell))\,\hat\varphi_j(\hat\xi_\ell)\,\hat\varphi_i(\hat\xi_\ell)\cdot$$

$$\cdot \underbrace{|\det J_K(\hat\xi_\ell)|\,\hat w_\ell}_{JxW}$$

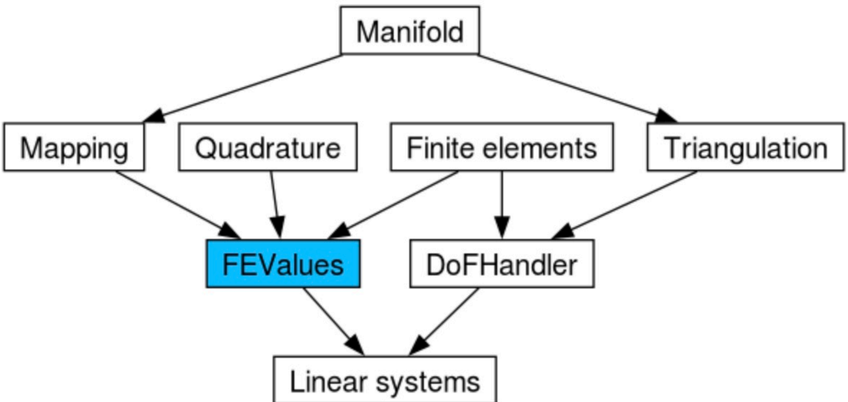
where $(\hat\xi_\ell, \hat w_\ell)_{\ell=1}^{n_{qp}}$ quad. nodes and weights

$$(i\ j)_K \longrightarrow (\ell\ m)$$

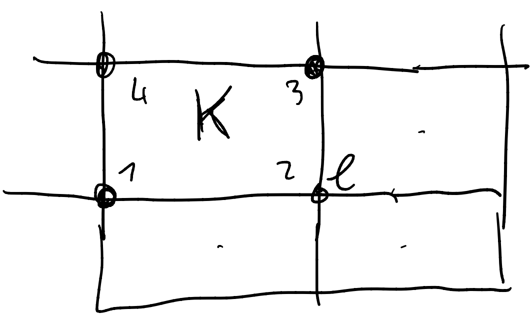
local indices

global

$\downarrow$ connectivity



$Q_{1,2}$-FEM



$\rightsquigarrow$ continuous FEM

$$\int_\Omega c\,\varphi_\ell \varphi_\ell = \sum \int_K c\,\varphi_\ell \varphi_\ell$$

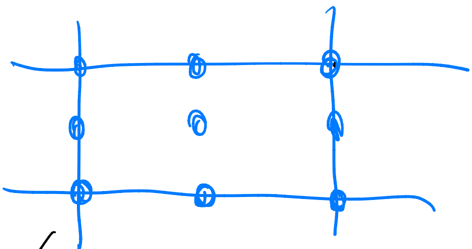
$$\rightsquigarrow M_{\ell\ell}$$

1) Fix sparsity pattern

2) Assemble system using DoF Handler

a second example

Q_2-FEM

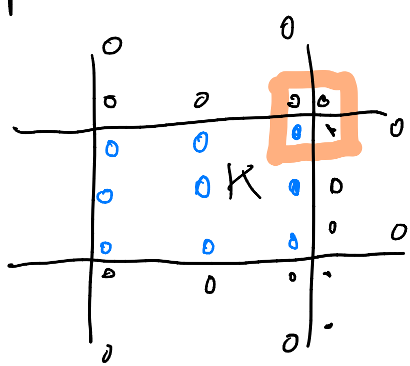


C^0-conforming space

shared by the neighbouring cells

in deal.II: fe_q(2)

DG space



fe_dgq(2)