ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 8

References:

- Ciarlet
- Larson Bengzon. The Finite Element Method. Springer, 2013.

Linear elasticity Ω domain in $\mathbb{R}^3$

Unknowns: displacements u (vectors)

∇u (gradient) tensor (each row is a gradient)

σ = stress tensor

A, B tensors, $\boxed{A : B = \sum_{ij} A_{ij} B_{ij}}$

$\nabla \cdot \sigma$ vector of components $\nabla \cdot \sigma_i$

On each portion $\omega \subset \Omega$ occupied by the elastic material, we have

- body forces (e.g. gravity): f
- boundary forces $\rightarrow$ $\sigma \cdot n$ $\quad \sigma$: stress tensor

σ_{ij} = force / unit area in x_i-direction over surface with normal along x_j-direction

$\sigma_{ij} = \sigma_{ji}$ (conservation angular momentum)

Net force $F = \int_{\omega} f \, dx + \int_{\partial\omega} \sigma \cdot n \, ds$

$$= \int_{\omega} (f + \nabla \cdot \sigma) \, dx$$

A equilibrium $F = 0$ + ω arbitrary

$$\boxed{- \nabla \cdot \sigma = f}$$ Cauchy equilibrium

- 6 stress components, 3 equations

→ constitutive equations

Hooke's Law

Introduce displacement vector

$$u = x - x_0$$

↑ initial position

Assumption: small displacements →

gradients measure deformation hence the strain

$$\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

$$\left(\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \right)$$

Linear elasticity: any rotation is small

Stress-to-deformation relationship:

$$\sigma_{ij} = \sum_{kl} C_{ijkl}\, \varepsilon_{kl}$$

(C = 4^{th} order tensor (36 entries))

Hooke's law

+ isotropic material

$$\boxed{\sigma = 2\mu\, \varepsilon(u) + \lambda\, (\nabla\cdot u)\, I^{3\times 3}}$$

λ, μ Lamé parameters

$$\mu = \frac{E}{2(1+\nu)}$$ Young's elastic modulus

$$\lambda = \frac{E\nu}{(+\nu)(1-2\nu)}$$ Poisson's ratio

In particular, if material homogeneous, then

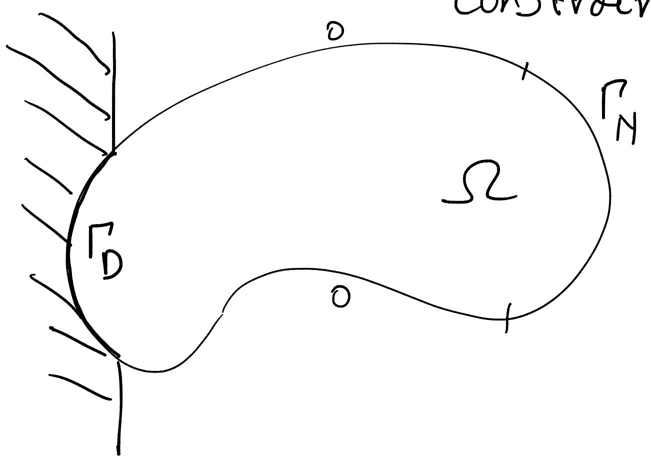
$$\lambda, \mu \equiv \text{const.}$$

Boundary Conditions

Dirichlet : $u = g_D$ displacement is constrained

Neumann : $\sigma \cdot n = g_N$ g_N = traction load (vector

$\longrightarrow$ normal stress is constrained



($\sigma \cdot n = 0$ free boundary)

Model problem

$$
\begin{cases}
-\nabla\cdot\sigma = f & \text{in } \Omega \quad (1)\\
\sigma = 2\mu\,\varepsilon(u) + \lambda(\nabla\cdot u)I & \text{in } \Omega \quad (2)\\
u = 0 & \text{in } \Gamma_D\\
\sigma\cdot n = g_N & \text{on } \Gamma_N
\end{cases}
$$

Weak formulation:

$$V = \left[H^1_{\Gamma_D}(\Omega)\right]^d$$

Test (1) with $v \in V$

$$
\begin{aligned}
(f,v)_\Omega &= (-\nabla\cdot\sigma, v)_\Omega \\
&= \sum_{i=1}^{3} \left(-\nabla\cdot\sigma_i, v_i\right)_\Omega \\
&= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} -\partial_j \sigma_{ij}, v_i\right)_\Omega \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \left[(\sigma_{ij}, \partial_j v_i)_\Omega - (\sigma_{ij} n_j, v_i)_{\partial\Omega}\right]
\end{aligned}
$$

$$= \int_\Omega \sigma : \nabla v \, dx - \int_{\partial\Omega} \underset{\substack{\| \\ g_H \text{ on } \Gamma_H}}{\sigma \circ n} \; \underset{\substack{\| \\ 0 \text{ on } \Gamma_D}}{v} \, dS$$

$$= \int_\Omega \sigma : \nabla v \, dx - \int_{\Gamma_H} g_H v \, dS$$

sym / anti-sym with zero diag

$$\sigma : \nabla v = \sigma : \tfrac{1}{2}(\nabla v + \nabla v^T) + \underbrace{\sigma : \tfrac{1}{2}(\nabla v - \nabla v^T)}_{=0}$$

$$= \sigma : \varepsilon(v)$$

$$= \boxed{\int_\Omega \sigma(u) : \varepsilon(v) \, dx - \int_{\Gamma_H} g_H v = \int_\Omega f v \, dx}$$

$u \in V$ $\qquad\qquad$ $\forall v \in V$

$$\underbrace{\qquad\qquad}_{a(u,v)}$$

$$\ell(v) = \int_\Omega f v \, dx + \int_{\Gamma_H} g_H v$$

Find $u \in V$: $a(u,v) = \ell(v) \quad \forall v \in V$

Using (2) $\sigma = 2\mu\, \varepsilon(u) + \lambda (\nabla\cdot u) I$

$$\boxed{\begin{aligned} a(u,v) &= \sum_{k,l} \left[(\lambda \partial_l u_l, \partial_k v_k)_\Omega + (\mu \partial_k u_l, \partial_k v_l)_\Omega + (\mu \partial_k u_l, \partial_l v_k)_\Omega \right] \\ &= \int_\Omega \left[\lambda (\nabla\cdot u)(\nabla\cdot v) + \mu \nabla u : \nabla v + \mu \nabla u : \nabla v^T \right] dx \end{aligned}}$$

Well-posedness

$$\boxed{\text{Korn's inequality:} \quad \exists c>0: \quad c\,\|\nabla v\|^2_{L^2(\Omega)} \le \|\varepsilon(v)\|^2_{L^2(\Omega)}}$$

Corollary (coercivity of $a(\cdot,\cdot)$):

$$a(u,u) \geq \alpha \|v\|_V^2 \qquad (\text{if } \mu>0)$$

$$a(u,u) = 2\mu \|\varepsilon(u)\|_0^2 + \lambda \|\nabla\cdot u\|_0^2$$

$$\geq 2\mu \|\varepsilon(u)\|_0^2 \geq 2\mu c \|\nabla u\|_0^2$$

Coercivity + continuity $\Rightarrow$ Lax-Milgram

$\Rightarrow$ well-posedness

FEM approximation

Pick again standard H^1 conf. FE p.w. polynomials w.r.t. a mesh $\mathcal{T}_h$ and apply it componentwise

V_h

FEM: Find $u_h \in V_h$: $a(u_h, v_h) = \ell(v_h)$ $\forall v_h \in V_h$

Theorem: $\Omega \in \mathbb{R}^3$ Lipschitz, $\Gamma_D = \partial\Omega$, $f \in [L^2(\Omega)]^3$,

$\lambda, \mu \in L^\infty(\Omega)$: $\exists \mu_{min}, K_{min} > 0$ s.t. $\begin{cases} \mu(x) \geq \mu_{min} \\ \lambda(x) + \frac{2}{3}\mu(x) \geq K_{min} \end{cases}$

Let $u \in V$ exact sol, u_h f.e. sol with $\mathcal{T}_h$ shape regular, then

Cèa: $$|u - u_h|_{H^1(\Omega)} \leq c \inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega)}$$

Error bound: $$|u - u_h|_{H^1(\Omega)} \leq c\, h^r \, |u|_{H^{r+1}(\Omega)}$$

if $u \in H^{r+1}(\Omega)$

$\forall r \leq k$.

$$\left(c\, h^k \, |u|_{H^{k+1}(\Omega)} \quad \text{if } u \in H^{k+1}(\Omega) \right)$$