ADVANCED TOPICS IN SCIENTIFIC COMPUTING

LECTURE 1

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This part:
Ern Guermond Theory and Practice of Finite Elements,
Springer, 2004

Course material -> github site

https://github.com/andreacangiani/ATSC

Idea: 1 intro to PDE problem rigorous + numerical analysis

- elliptic problems
- advection
- adaptivity
- Discontinuous Galerkin
- Mixed FEM
- elasticity
- Stokes, Navier-Stokes

② Implementation in deal.II

The mesh

Def: (Domain) $\Omega \subset \mathbb{R}^d$ nonempty, open, bounded, connected, Lipschitz.

Def: (Mesh) A mesh is a finite collection of closed, connected, Lipschitz sets with non-empty interior, denoted

$$\mathcal{T}_h = \{ K_m \}_{m=1}^{M_{el}}$$

forming a partition of Ω, i.e.

(1) $\overline{\Omega} = \bigcup_{m=1}^{M_{el}} K_m$

(2) $\mathring{K}_m \cap \mathring{K}_n = \emptyset \quad \forall\, m \neq n$

K_m called $\begin{cases} \text{mesh elements} \\ \text{mesh cells} \end{cases}$

$$h_K = \mathrm{diam}(K_m)$$

$$h = \max_{K \in \mathcal{T}_h} h_K$$

Mesh generation / Reference cell

Each mesh element image of a

Reference element $\hat{K}$

typically
- unit interval, unit square $[0,1]^d$
- unit simplex

$$\hat{K} = \{ \underline{x} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i \leq 1 ,\ x_i \geq 0 \}$$

if $\mathcal{T}_h$ is simplicial mesh, then each $K \in \mathcal{T}_h$ is obtained by affine transforming $\hat{K}$:

$$T_K : \hat{K} \longrightarrow K$$

{ geometric transf
{ reference-to-physical

$$T_K(\hat{x}) = b_K + J_K \hat{x}$$

if $\begin{cases} \hat{a}_i = \text{vertices of } \hat{K} \\ a_i = \text{'' of } K \end{cases}$ $\qquad T_K(\hat{a}_i) = a_i \quad \forall i = 1, \dots, d$

In deal.II, mainly hexahedral meshes are developed, with

$$\hat{K} = [0,1]^d$$

using d-linear T_K's instead

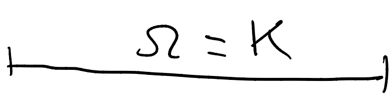
(Note: in 3D faces are not necessarily planar)

Higher order mappings also implemented.

Shape functions

Basic idea: approximate by polynomials

<u>1D</u>



$\mathbb{P}^k(K) = \{$ polynomials of degree $\le k \}$

$k = 0, 1, \dots$

require a basis ...

Lagrange polynomials:

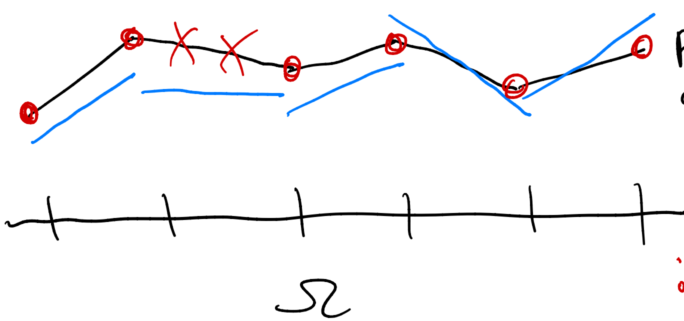
- fix a set of nodes $s_0, \dots, s_k$ $\quad s_j \neq s_i \;\forall i$
- fix $\{\mathcal{L}_0^k, \dots, \mathcal{L}_k^k\}$ or

$$\mathcal{L}_m^k(t) = \frac{\prod_{l \neq m} (t - s_l)}{\prod_{l \neq m} (s_m - s_l)} \quad \forall m$$

$$\Rightarrow \quad \mathcal{L}_m^k(s_l) = \delta_{ml} \qquad 0 \le l, m \le k$$

Global setting: spectral (collocation)

Local setting: piecewise polynomials



p.w. linear continuous

p.w. linear discontinuous

... higher order continuity ...

Choice of nodes

dof. II : equidistant $(k \leq 2)$

Gauss-Lobatto $(k > 2)$

Gauss-Legendre polynomials over

$\hat{K} = [0, 1]$

$$\hat{\xi}_k(t) = \frac{1}{k!} \frac{d^k}{dt^k} (t^2 - t)^k \quad k \geq 0$$

① $\hat{\xi}_k$ has k roots in $\hat{K}$

② $L^2(\hat{K})$-orthogonal

$$\underbrace{\int_{\hat{K}} \hat{\xi}_m(t)\, \hat{\xi}_n(t)}_{\text{mass matrix}} = \frac{1}{2m+1} \delta_{mn}$$

mass matrix $\Rightarrow$
- is diagonal
- conditioning $\frac{1}{2k+1}$

③ hyerarchical

Why useful :
- high order methods
- explicit time-stepping

  M^{-1} trivial

→ • ideal basis for discontinuous Galerkin

- nodes of the Gauss-Legendre poly + Gauss weights give quadrature rules of maximal order $\underline{2k-1}$

  (exact over $\mathbb{P}^{2k-1}$)

MAIN QUADRATURE USED IN DEAL.II

Gauss-Legendre not ideal for C^0-conforming FEM (on the end points are <u>not</u> nodes)

$\Rightarrow$ Gauss-Lobatto

Lagrange basis w.r.t. the Gauss-Lobatto points:

- $\{\xi_\ell\}_{\ell=1}^{k}$ roots $(1-t^2)\,\mathcal{L}'_{k-1}(t)$

$\downarrow$ Legendre polynomial over $[-1,1]$

corresp. nodes:

- $\omega_\ell = \dfrac{2}{(1-\xi_\ell^2)\,\mathcal{L}'_\ell(\xi_\ell)^2}$

$\rightarrow$ G-L quadrature, order $2k-2$

$\Rightarrow$ Gauss-Lobatto basis (def. II):

in $[0, 1]$

$$\ell_0 = 1 - x$$

$$\ell_1 = x$$

$$\ell_m = \frac{1}{\| \hat{\varepsilon}_{m-1} \|_2} \int_0^x \hat{\varepsilon}_{m-1}(t)\, dt \qquad m \geq 2$$