NMPDE/ATSC 2025
Lecture 19

Recall classification of 2^{nd} order PDEs

(2D case) : $\mathcal{L}u := a\, u_{xx} + 2b\, u_{xy} + c\, u_{yy}$ + L.O.T.

- $\mathcal{L}u = \Delta u = u_{xx} + u_{yy}$ $\qquad D = \underset{0}{\underset{\|}{b^2}} - \underset{1}{\underset{\|}{a}}\underset{1}{\underset{\|}{c}} = -1 < 0$ elliptic

- $\mathcal{L}u = u_t - u_{xx}$ $\qquad D = \underset{0}{\underset{\|}{b^2}} - \underset{0}{\underset{\|}{a}}\underset{-1}{\underset{\|}{c}} = 0$ parabolic

- $\mathcal{L}u = u_{tt} - c^2 u_{xx}$ $\qquad D = \underset{0}{\underset{\|}{b^2}} - \underset{1}{\underset{\|}{a}}\underset{-c^2}{\underset{\|}{c}} = c^2 > 0$ hyperbolic

<u>wave equation</u>

Hyperbolic model problems

- wave equation
- linear transport $\quad u_t + a u_x = 0$
- conservation laws: $\quad u_t + (f(u))_x = 0$

  example: Burger's eq.: $\quad u_t + \left(\frac{u^2}{2}\right)_x = 0$

See eg: Q, LT, MM (Finite differences)

Cauchy problem for linear transport:

$$\begin{cases} u_t + a(t,x)\, u_x = 0 & \text{in } (0,T] \times \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

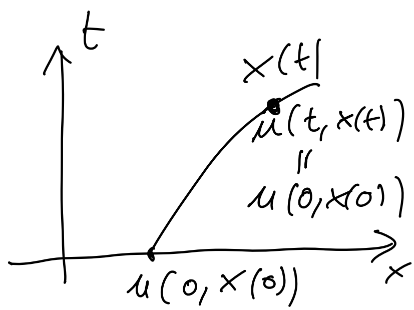
$$\underbrace{(1,a) \cdot \nabla_{t,x} u}_{\text{directional derivative along } (1,a)} = 0$$

(characteristic equations

$$\frac{dx}{dt} = a(t,x)$$

along the solution curves of this ODE u is constant or

$$\frac{d\,u(x(t))}{dt} = \underbrace{\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \overbrace{\frac{dx}{dt}}^{=a}}_{=0 \text{ by the PDE}}$$



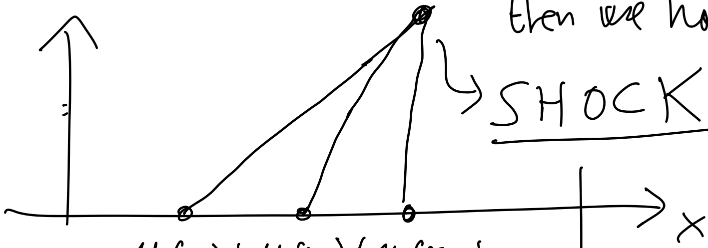
ex: • $a \equiv$ const $\rightarrow$ charact. curves ODE solution:

$x - at =$ const. $\Rightarrow$ straight lines !

$$\Rightarrow \boxed{u(t,x) = u_0(x - at)}$$

• $a = a(u)$ still $u \equiv$ const along characteristic

if these meet at one point:



then we have a

SHOCK

$u_0(x_0) \neq u_0(x_1) \neq u_0(x_2)$

solution afterword??

Systems

• linear case $\bar{u}_t + A u_x = 0$

assuming A has all real eigenvalues

can $T^{-1} A T = \Lambda = \mathrm{diag}(\lambda_i)$

$$\bar{w} = T^{-1} \bar{u}$$

so that $\bar{w}_t + \Lambda w_x = 0$

the component w_i propagates at speed λ_i

• nonlinear case : $\dfrac{\partial \bar{u}}{\partial t} + \dfrac{\partial \bar{f}(\bar{u})}{\partial x} = 0$

for esample Euler eq. :

$$\begin{cases} \frac{d u_i}{dt} = \frac{\partial u_i}{\partial t} + \underline{u} \cdot \nabla u_i & \forall i = 1, \dots, d \\ \nabla \cdot \overline{u} = 0 \end{cases}$$

in conservative form:

$$\frac{d u_i}{d t} = \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i} (u_i^2) + \sum_{j \neq i} \frac{\partial u_i u_j}{\partial x_j}$$

or ~~or~~ system

$$\overline{u}_t + A(\overline{u}) \, \overline{u}_x = 0$$

$$A(\overline{u}) = \frac{\partial \overline{f}}{\partial \overline{u}} (\overline{u}) = \text{Jacobian of } \overline{f}$$

i.e system of coupled transport eq. whose charact. curved in generale, most often impossible to treat by the method of characteristics.

Also the wave eq. can be conf or system of transport eq.

Consider $\underline{u_{tt} - u_{xx} = 0}$

$$\Updownarrow$$

$$\begin{cases} u_t + v_x = 0 & (1) \\ u_x + v_t = 0 & (2) \end{cases} \iff U_t + AU_x = 0$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(1)_t \quad 0 = u_{tt} + v_{tx} \overset{(2)_x}{=} u_{tt} - u_{xx}$$

$$(2)_t \quad 0 = u_{tx} + v_{tt} \overset{(1)_x}{=} -v_{xx} + v_{tt}$$

Conservation

$$\begin{cases} u(0,x) = u_0(x) \\ u_t + a u_x = 0 \end{cases}$$

test by u and integrate in x

$a \equiv$ const

$$\int_{\mathbb{R}} u_t u \, dx + \int_{\mathbb{R}} a u_x u \, dx = 0$$

$$\therefore \quad \frac{1}{2}\frac{d}{dt}\underbrace{\int_{\mathbb{R}} u^2 \, dx}_{= \|u\|_0^2} + \frac{1}{2} a \underbrace{\int_{\mathbb{R}} \frac{\partial}{\partial x} u^2 \, dx}_{\frac{1}{2} a u^2 \Big|_{-\infty}^{+\infty} = 0} = 0$$

assuming $u \in L^2(\Omega)$

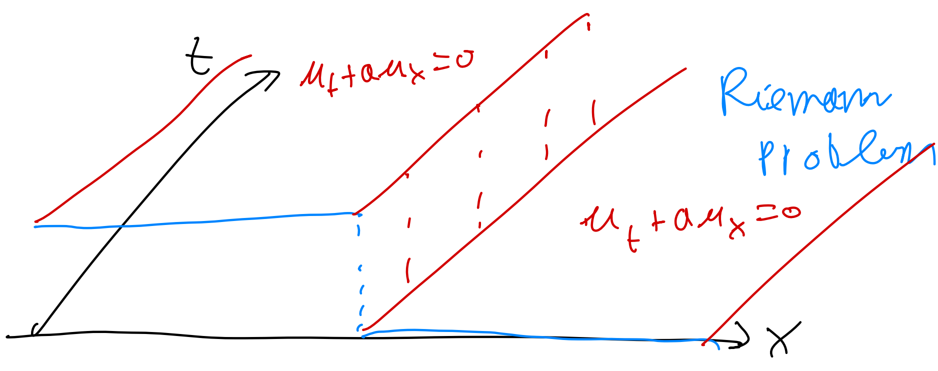
$$\boxed{\|u(t)\|_0^2 = \|u_0\|_0^2}$$ L^2 norm is conserved

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,x) = u_0(x) \\ u_t(0,x) = v_0(x) \end{cases} \qquad (0,T] \times \underbrace{\mathbb{R}}_{\Omega}$$

Total energy: $\mathcal{E}(t) = \frac{1}{2} \int_\Omega (u_t^2 + u_x^2)\, dx = \mathcal{E}(0)$

Proof: (ex) hint test with u_t

- For these models there is NO DISSIPATION
- Discontinuities of u_0 are propagated



solution satisfies PDE only piecewise

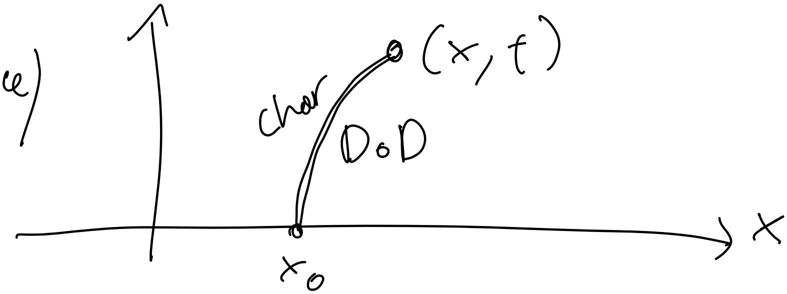
Bad news for numerics ! or jump requires dissipation.

CFL condition

PDE has finite propagation speed, and

DoD is the characteristic curve

(Domain of Dependence)

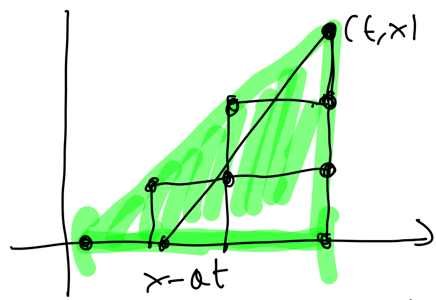


DoD of FD schemes:

$$\begin{cases} u_t + a u_x = 0 \\ u(0,x) = u_0(x) \end{cases} \qquad a \equiv \text{const} \qquad a > 0$$

$\Rightarrow u(t,x) = u_0(x - at)$

(forward int)



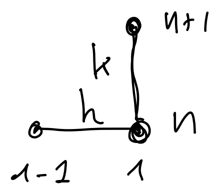
use backward FD in x:

$$\frac{U_i^{n+1} - U_i^n}{k} + a \frac{U_i^n - U_{i-1}^n}{h} = 0$$

$\Updownarrow$

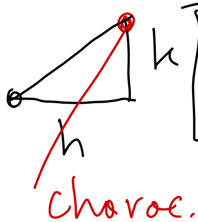
$$\delta_{k,+}^t U_i^n + a \, \delta_{h,-}^x U_i^n = 0$$

upwind method



$$\left(U_i^0 = u_0(x_i) \right)$$

CFL: DoD of scheme must include DoD of PDE



$$\boxed{\nu := \frac{k}{h}} \le \frac{1}{a} \quad \text{i.e.} \quad \boxed{a\nu \le 1}$$

Conservation:

$$\begin{cases} U_i^{n+1} = (1 - a\nu) U_i^n + a\nu U_{i-1}^n & \forall i \\ U_i^0 = u_0(x_i) \end{cases}$$

$$\boxed{\sum_i U_i^{n+1}} = (1-a\nu)\underline{\sum_i U_i^n} + a\nu \underline{\sum_i U_{i-1}^n}$$

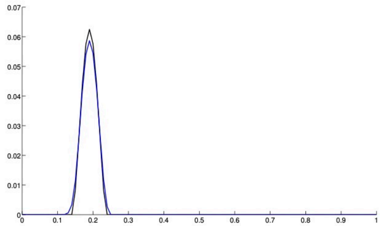
$$= \sum_i U_i^n$$

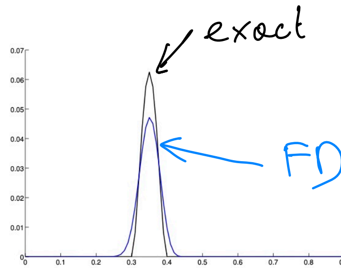
$$= \boxed{\sum_i U_i^0}$$

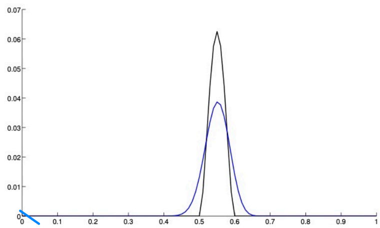
$\Omega = \mathbb{R}$

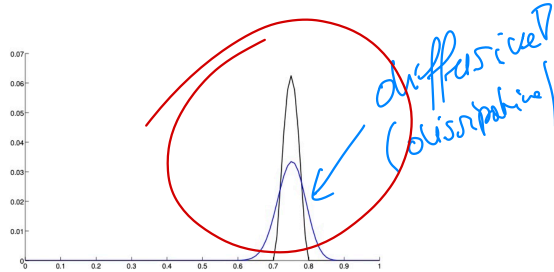
mass is conserved

example: $u_t + 2u_x = 0$ $\quad h = 1/100 \quad k = 1/250 \quad \nu < 0.5$

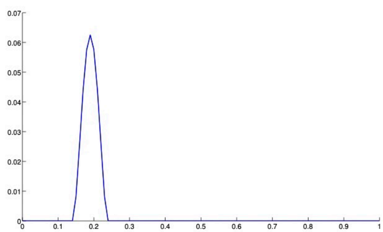


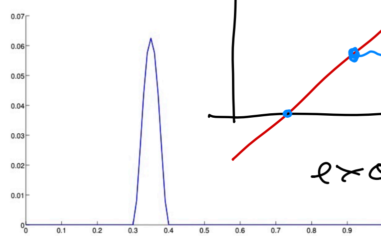


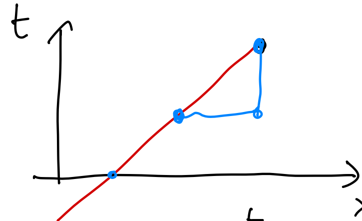




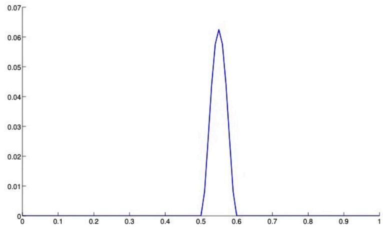
$k = 1/200 \qquad \underline{\nu = 0.5}$

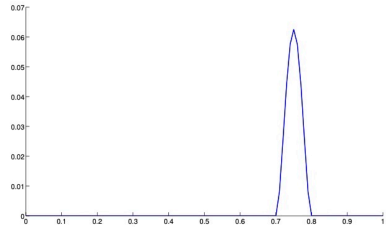






exact scheme?





von neumann analysis: $U_j^n = \lambda^n e^{i\xi j h}$

$$\lambda = (1-a\nu) + a\nu e^{-i\xi h}$$

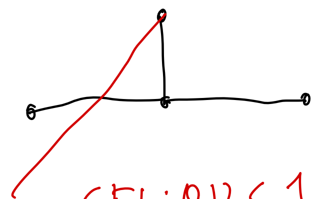
$$|\lambda|^2 = \dots = 1 - 4a\nu\underbrace{(1-a\nu)}_{\text{positive}}\sin^2\left(\frac{\xi h}{2}\right) \le 1$$

$$|\lambda| \le 1 \iff \boxed{a\nu \le 1} \equiv \text{CFL cond!}$$

Central FD Scheme:

$a>0$

$$\frac{U_j^{n+1}-U_j^n}{k} + a\,\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$



stability: $U_j^{n+1} = \frac{\nu}{2}U_{j-1}^n + U_j^n - \frac{\nu}{2}U_{j+1}^n$

$(a=1)$

$$\lambda = 1 - \nu i \sin kh$$

$$|\lambda|^2 = 1 + \nu^2 \sin^2 kh > 1 \quad \text{always unstable!!}$$

Explanation:

artificial diffusion!

$a>0$

$$a\,\underbrace{\delta^x_{h,-}U_j^n}_{\text{"upwind"}} = a\,\underbrace{\delta^x_{2h}U_j^n}_{\text{central}} - \boxed{\frac{ah}{2}\left(\delta^x_h\right)^2 U_j^n}$$

+ central for $-\frac{ah}{2}u_{xx}$

central scheme is without ↗

... but upwind "pays" in terms of accuracy

Truncation error of upwind scheme

$$|T_i^n| \le \frac{1}{2}\left(k M_{tt} + h M_{xx}\right) \qquad \begin{cases} M_{tt} = \max u_{tt} \\ M_{xx} = \max u_{xx} \end{cases}$$

i.e. $O(k, h)$

Convergence: If $0 < a\nu \le 1$ then

$$\max_i |u_i^n - U_i^n| \le \frac{T}{2}\left(k M_{tt} + h M_{xx}\right)$$

Proof: $e_i^{n+1} = u_i^{n+1} - U_i^{n+1} = (1 - a\nu) e_i^n + a\nu e_{i-1}^n + kT$

$a\nu \le 1 \Rightarrow$ coeff ≥ 0 and sum to 1

$E^n = \max_i |e_i^n|$ $\qquad \le E^n + kT^n \le \dots \le \underbrace{nk}_{\le T} \max_n T^n$

$T^n = \max_i |T_i^n|$

Upwind method (general form): for $u_t + a(t,x) u_x = 0$

$$U_i^{n+1} = \begin{cases} (1 - a_i^n \nu) U_i^n + a_i^n \nu U_{i-1}^n & \text{if } a > 0 \\ (1 + a_i^n \nu) U_i^n - a_i^n \nu U_{i+1}^n & \text{if } a < 0 \end{cases}$$