(LT, online notes)

NMPDE/ATSC 2025
Lecture 12

Analysis of 5-points FD scheme

Lemma (DMP): If $V = (V_{ij})_{ij}$: $\Delta_h V_{ij} \leq 0$

where $\Delta_h V_{ij} = -\left[(\delta_h^x)^2 V_{ij} + (\delta_h^y)^2 V_{ij}\right]$

then $\max_{ij} V_{ij} = \max_{(x_i, y_j)\in\partial\Omega} V_{ij}$

Proof: $0 \leq -\frac{h^2}{4}\Delta_h V_{ij} = -V_{ij} + \frac{V_{i+1,j} + V_{i-1,j} + V_{i,j-1} + V_{i,j+1}}{4}$

$$\Rightarrow V_{ij} \leq \underbrace{\frac{V_{i+1,j} + V_{i-1,j} + V_{i,j-1} + V_{i,j+1}}{4}}_{\text{average}} \leq V_{ij}$$

If V_{ij} is of maximum

$\Rightarrow$ all values are equal!

Repeat argument until reach the boundary. ◁

Lemma (stability): $\forall\ V = (V_{ij})_{ij}$ /

$$\overset{U}{\|V\|}_{\ell_\infty(\bar\Omega)} \le \overset{U}{\|V\|}_{\ell_\infty(\partial\Omega)} + \frac{1}{8}\,\|\overset{f_{ij}}{\mathcal{L}_h V}\|_{\ell_\infty(\Omega)}$$

$$\|V\|_{\ell_\infty(\bar\Omega)} = \max_{0\le ij\le N} |V_{ij}|$$

Proof: (LT) As in 1D-case based on comparison function $\omega(x) = x + y - x^2 - y^2$.

Lemma (consistency/truncation error bound)

If $u \in C^4(\Omega) \cap C^0(\bar\Omega)$, then

$$|T(x)| \le \frac{h^2}{12}\left(M_{xxxx} + M_{yyyy}\right)$$

where

$$T(x) = \underset{\substack{\| \\ f(x)}}{\mathcal{L}u(x)} - \mathcal{L}_h u(x)$$

$$M_{xxxx} = \left\|\frac{\partial^4 u}{\partial x^4}\right\|_C \;;\; M_{yyyy} = \left\|\frac{\partial^4 u}{\partial y^4}\right\|_C$$

$$\|\cdot\|_C = \max_{\bar\Omega}$$

Proof: exercise.

Theorem (convergence): Some assumptions on u, with $U = (U_{ij})$ 5-points FD scheme solution satisfies

$$|u_{ij} - U_{ij}| \le \frac{h^2}{96}\left(M_{xxxx} + M_{yyyy}\right) \quad \forall\, i,j$$

$$u_{ij} = u(x_i, y_j)$$

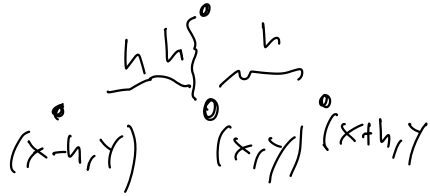
Proof: let $e_{ij} = u_{ij} - U_{ij}$ use consistency stability (exercise)

Generalisations

1) $\underbrace{a_{11} u_{xx}}_{\approx\, a_{11}(x,y)\left(\delta_h^x\right)^2 u(x,y)} + \boxed{a_{12} u_{xy}} + a_{22} u_{yy}$

$a_{11} = a_{11}(x,y), \ldots$

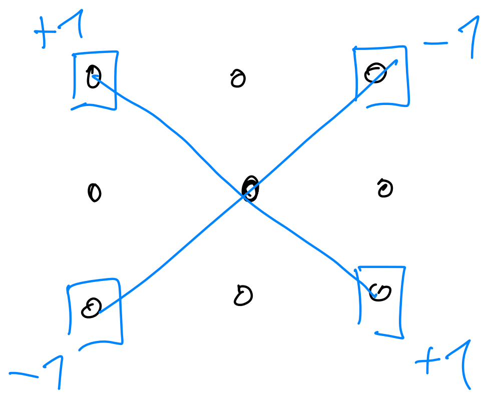
$\approx a_{11}(x,y)\left(\delta_h^x\right)^2 u(x,y)$



$$u_{xy}(x,y) \approx \left(\frac{u(x+h,y)-u(x-h,y)}{2h}\right)_y$$

$$\approx \frac{\frac{u(x+h,y+h)-u(x+h,y-h)}{2h} - \frac{u(x-h,y+h)-u(x-h,y-h)}{2h}}{2h}$$

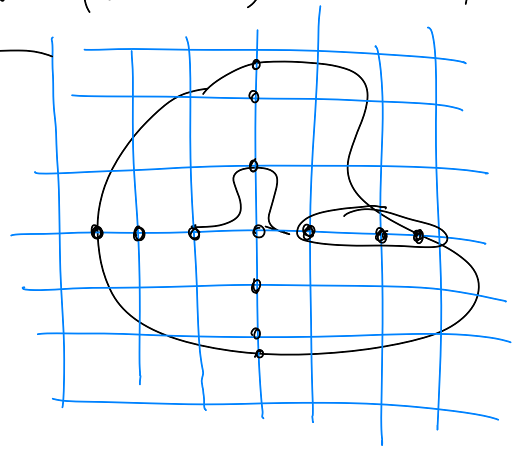
$$= \frac{u(x+h,y+h)-u(x+h,y-h)-u(x-h,y+h)+u(x-h,y-h)}{4h^2}$$



(2) Lower order terms (trivial), e.g.

$$b_x u_x + b_y u_y \approx b_x(x,y)\,\delta^x_{2h} u(x,y) + b_y(x,y)\,\delta^y_{2h} u(x,y)$$

③ Ω general (smooth) domain



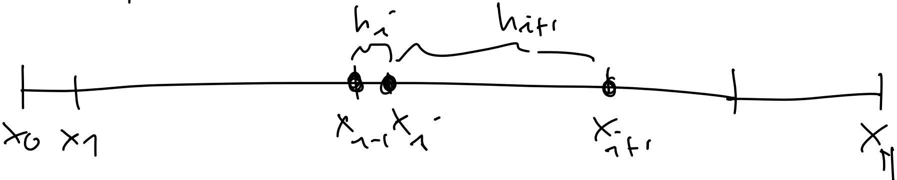
FD on non uniform grids

Find best possible 3-points formula for

$$\rightarrow \begin{cases} -u'' = f & \Omega = [a,b] \\ u(a) = 0 = u(b) \end{cases}$$

Let h_i, $i = 1, \dots, N$ $\qquad \sum_i h_i = (b-a)$

grid $x_0 = a$; $x_i = x_{i-1} + h_i \quad \forall i = 1, \dots N$



$$u''(x_i) \approx \alpha \underset{\substack{\shortparallel \\ u(x_{i-1})}}{u_{i-1}} + \beta u_i + \gamma u_{i+1} =$$

Taylor by x_i

$$= \alpha\left(u_i - h_i u_i' + \frac{h_i^2}{2} u_i'' - \frac{h_i^3}{6} u'''(\xi_i)\right) \quad \exists\, \xi_i \in [x_{i-1}, x_i]$$

$$+ \beta\, u_i$$

$$+ \gamma\left(u_i + h_{i+1} u_i' + \frac{h_{i+1}^2}{2} u_i'' + \frac{h_{i+1}^3}{6} u'''(\eta_i)\right) \quad \exists\, \eta_i \in [x_i, x_{i+1}]$$

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ -\alpha h_i + \gamma h_{i+1} = 0 \quad \Rightarrow \gamma h_{i+1} = \alpha h_i \\ \frac{\alpha}{2} h_i^2 + \frac{\gamma}{2} h_{i+1}^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = \frac{2}{h_i (h_i + h_{i+1})} \\ \beta = \frac{2}{h_i h_{i+1}} \\ \gamma = \frac{2}{h_{i+1}(h_i + h_{i+1})} \end{cases}$$

$\Rightarrow$ FD scheme:

$$\Rightarrow \frac{2}{h_i (h_i + h_{i+1})} U_{i-1} - \frac{2}{h_i h_{i+1}} U_i + \frac{2}{h_{i+1}(h_i + h_{i+1})} U_{i+1} = -f_i$$

Note: $h_1 = h_{i+1} = h \Rightarrow$ back to the second central FD

In general

$$|T_i| = \frac{2}{3} h_{max} |u'''(x_i)| + \frac{1}{12} \frac{h_{max}^3}{h_{min}} M_4$$

$O(h_{max})$ but requires

$$\boxed{\frac{h_{max}^3}{h_{min}} \longrightarrow 0}$$ to ensure convergence.

Back to the general domain problem.

- use nonuniform FD scheme only to comply with boundary adding grid points where grid hits the boundary.

- truncation error:

$O(h^2)$ internally

$O(h)$ boundary

It can be proved that this scheme (Shortley-Weller) is overall 2^{nd} order!

(LT)

Convection/reaction dominated diffusion problems (Q Ch. 13)

A 1D problem with known exact solution

$a, b > 0$ positive $\Omega = (0,1)$

$$\begin{cases} -\underbrace{a u''}_{\text{DIFF}} + \underbrace{b u'}_{\text{CONV}} = 0 & \text{in } \Omega \\ u(0) = 0 \;;\; u(1) = 1 \end{cases}$$

Recall DMP satisfied by FD scheme only if $a \pm \frac{1}{2} b \geq 0$

that is, require $\boxed{\frac{bh}{2a} \leq 1}$

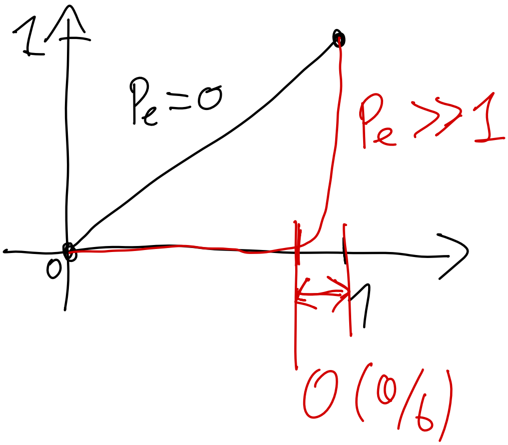
$= Pe_h$ Mesh Péclet Number

Exact solution

Let $P_e = \frac{bL}{2}$ $\quad L = |\Omega| = 1$

Pèclet Number

$$u(x) = \frac{e^{b/a\, x} - 1}{e^{b/a} - 1} \quad (\Rightarrow \text{monotone})$$



$\rightsquigarrow$ exponential boundary layer

$$-\underbrace{\frac{a}{b}}_{\varepsilon} u'' + u' = 0$$

$$-\varepsilon u'' + u' = 0$$

singular perturbation of pure convection problem:

$$\begin{cases} v' = 0 \\ v(0) = 0 \end{cases}$$

Applying (Centered) FD gives:

- grid M Intervals $h = 1/M$ $\quad x_i = ih$
- $U = (U_1, \dots, U_{M-1})$

• $AU = F$ where

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \\ 0 & -\frac{a}{h} - \frac{b}{2} & \frac{2a}{h} & \boxed{-\frac{a}{h} + \frac{b}{2}} & 0 \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

$$b u' \approx b \frac{U_{i+1} - U_{i-1}}{2h}$$

$$F = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \hline \frac{a}{h} - \frac{b}{2} \end{pmatrix}$$

or $Pe_h = \frac{bh}{2a}$ $\qquad -\frac{a}{h} - \frac{b}{2} = -(1 + Pe_h)$

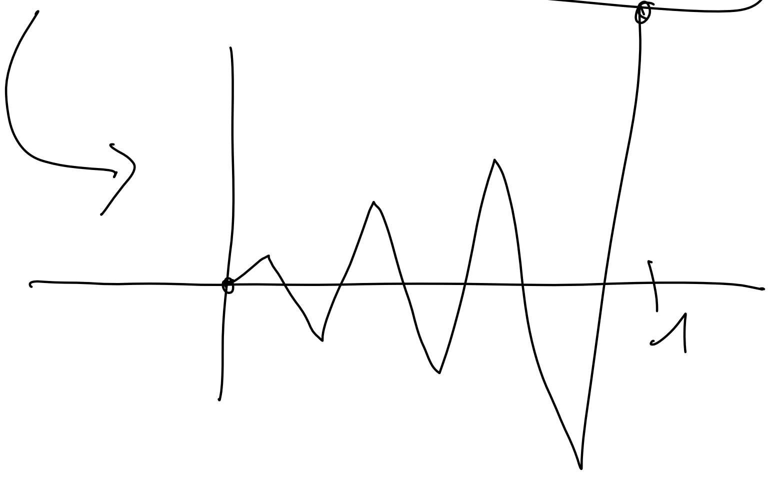
gives

$$U_j = \frac{1 - \left(\frac{1 + Pe_h}{1 - Pe_h}\right)^j}{1 - \left(\frac{1 + Pe_h}{1 - Pe_h}\right)^N} \qquad j = 1, \dots, N$$

Pe_h

$< 1 \quad \dfrac{1+Pe_h}{1-Pe_h} > 0$

$> 1 \quad \dfrac{1+Pe_h}{1-Pe_h} < 0 \Rightarrow U_j$ oscillates



Problem !!