# NUMERICAL METHODS FOR PDES
## Lecture 1

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HOUSE KEEPING:

- lectures: 45+45 minutes
- SISSA lecture room A-133 or zoom
- computer practicals on line

Course material:
https://github.com/andreacangiani/NMPDE2025/tree/main

ASSESSMENT:
By oral exam. Just before your oral, you will be asked to attempt a written theoretical exercise.
To be admitted to the oral, you need to have completed the computer classes.

BIBLIOGRAPHY:
- Lecture notes
- Morton & Mayers Numerical Solution of Partial Differential Equations. Cambridge, 1994. (MM)
- Larsson & Thomee Partial Differential Equations with Numerical Methods. Springer, 2009. (LT)
- Quarteroni Numerical Models for Differential Problems. Springer, 2017. (Q)

IDEA OF THE COURSE:
- present the fundamental ideas/methods and the interlink between different approaches.
- explore different problems/methods to see that fundamental ideas are ubiquitous
- choice of appropriate methods depend on the problem
- mainly theoretical but with python classes on basic implementation of the methods seen at the lectures

# Q 1.2

Why numerical methods?

- Theory: study well-posedness, etc. But most often close form solutions not available
- Numerical methods: approximate solutions computable in practice
- Numerical analysis: Study of numerical methods

Let $P(u, g) = 0$ a PDE problem

$u$ = solution ; $g$ = data

Well-Posed (Hadamard):

- $\exists!$ solution $u$
- solution depends cont. on data

$\delta g$ = admissible perturbation of data

$\delta u$ = associated change of solution

i.e. $P(u + \delta u, g + \delta g) = 0$

$\forall \eta > 0, \exists K(\eta, g) : \|\delta g\| < \eta \Rightarrow \|\delta u\| \leq K \|\delta g\|$

some norms

→ <u>condition number</u>

$$K(g) := \sup_{\delta g \in B} \frac{\|\delta u\| / \|u\|}{\|\delta g\| / \|g\|}$$

$B$: neighbourhood of admissible perturbations

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Numerical method : $P_N(u_N, g_N) = 0$

$g_N$ = some approximation of the data

$N$ = dimension of the discrete problem

<u>Numerical Analysis</u> :

$P_N(u_N, g_N)$ is

- STABLE : continuous dep. on data

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \|\delta g_N\| \leq \delta \Rightarrow \|\delta u_N\| \leq \varepsilon$$

$$\Big( P_N(u_N + \delta u_N, g_N + \delta g_N) \Big)$$

- CONSISTENT:

if $\lim_{N\to+\infty} P_N(u, g) = 0$

(equiv: $\underbrace{P_N(u,g) - P(u,g)}_{\text{truncation error}} \xrightarrow{N\to+\infty} 0$ )

STRONGLY (FULLY) CONSISTENT

if $P_N(u, g) = 0 \quad \forall N$

- CONVERGENT: if $\lim_{N\to+\infty} \|u - u_N\| = 0$

some norm

see also Richtmyer-Morton 1967

Theorem (Lax-Richtmayer): for linear PDEs:

consistency + stability $\Rightarrow$ convergence

Remarks:

1) analogue to Dahlquist equivalence theorem for (nonlinear!) ODEs

2) Consistency says that the exact solution satisfies the numerical scheme but is not sufficient to imply convergence

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Poisson b.v.p.: given $\Omega \subset \mathbb{R}^d$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

EVANS ("Partial Differential Equations", AMS, 2010)

Strong solutions in $L^2(\Omega)$

## Harmonic functions:

$u : \Omega \to \mathbb{R}$ : $\begin{cases} u \in C^2(\Omega) \\ \Delta u = 0 \text{ in } \Omega \end{cases}$

**Theorem** (mean-value formula): $u$ is harmonic in $\Omega$ if and only if $\forall x \in \Omega$

$$u(x) = \fint_{\partial B_r(x)} u \qquad \text{with } B_r(x) \subset \Omega$$

$$\left( \fint_\omega u := \frac{\int_\omega u}{|\omega|} \right)$$

we also have

$$u(x) = \fint_{B_r(x)} u$$

**Theorem** (Strong Max Principle - SMP):

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$, harmonic in $\Omega$, then

(i) $\max_{\bar{\Omega}} u = \max_{\Omega} u$

(ii) If $\Omega$ is connected and $x_0 \in \Omega$ s.t.

$$u(x_0) = \max_{\bar{\Omega}} u$$

then $u \equiv \text{const}$ on $\Omega$

Proof of (ii): suppose $x_0 \in \Omega$ of max $M$

take $B_{x_0}(r) \subset \Omega$, then

$$M \geq u(x_0) = \fint_{B_{x_0}(r)} u \leq M \Rightarrow u = M \text{ over } B_{x_0}(r)$$

$\Rightarrow \{x \in \Omega : u(x) = M\}$ is
- relatively closed
- open

$\Rightarrow \; = \Omega$

## Corollaries

(1) positiveness: if $u \in C^2(\Omega) \cap C(\bar{\Omega})$: $\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$

- $g \geq 0 \Rightarrow u \geq 0$ in $\Omega$
- $\begin{cases} g \geq 0 \\ \exists x : g(x) > 0 \end{cases} \Rightarrow u > 0$ in $\Omega$

(2) uniqueness: let $g \in C(\partial\Omega)$, $f \in C(\Omega)$

Then $\exists$ at most 1 solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$

of $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

Proof:

- by SMP, $w_{\pm} := \pm(\underset{\text{solutions}}{u - \tilde{u}}) \longrightarrow u \equiv \tilde{u}$

- by energy method: let $u, \tilde{u}$ solutions, $w = u - \tilde{u}$

since $\Delta w = 0$,

$$0 = \int_{\Omega} w \, \Delta w \overset{\text{parts}}{=} -\int_{\Omega} |\underset{\text{gradient}}{\nabla} w|^2 + \underset{=0}{\cancel{\int_{\partial\Omega} w^2}}$$

$$\Rightarrow \begin{cases} \nabla w \equiv 0 \\ w|_{\partial\Omega} = 0 \end{cases} \Rightarrow w \equiv 0$$

<u>Existence</u>: typically by construction via Green's functions, which however can be constructed only for $\Omega$ with "simple" geometries

Theorem (a priori bound)

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solve (BVP) $\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$

then $\underbrace{\|u\|_{C(\overline{\Omega})}}_{\max} \leq \|g\|_{C(\partial\Omega)} + C \underbrace{\|\Delta u\|_{C(\Omega)}}_{= f}$

Corollary (well-posedness): if $u, \tilde{u}$ solve (BVP) w.r.t $(f, g)$ and $(\tilde{f}, \tilde{g})$ respectively then

$$\|u - \tilde{u}\|_{C(\overline{\Omega})} \leq \|g - \tilde{g}\|_{C(\partial\Omega)} + c\|f - \tilde{f}\|_{C(\Omega)}$$

FD for Poisson in 1 space dim (ODE)

$$\begin{cases} \underbrace{-u''}_{=: \mathcal{L}u} = f & \text{in } \Omega = (0, 1) \\ u(0) = 0 = u(1) \end{cases}$$

second divided difference

$$\forall x \in \Omega, \quad \delta_h^2 u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$h$ : step size

## FD method

- Fix $N$, and $h = \frac{|\Omega|}{N} = \frac{1}{N}$
- $x_i = h\,i$, $i = 0, \dots, N$ grid points
- Discretize $U_i \simeq u(x_i)$ :

$U_0 = 0 = u(0)$

$$L_N U_i \begin{cases} -\delta_h^2 U_i = f_i := f(x_i) \\ U_0 = 0 = u(1) \end{cases} \longrightarrow U = \{U_i\}$$

grid function

## ANALYSIS

# STABILITY

**Theorem** (Discrete Max. Princ. – DMP)

Let $V = \{V_i\}_{i=0}^{M}$ a grid function

If $\quad L_M V_i \le 0 \quad \forall\, i = 1, \dots, M-1$

then $\quad \max\limits_i V_i = \max\{V_0, V_M\}$

**Proof**: suppose $\exists\, n$: $\quad V_n = \max\limits_{1 \le i \le M-1} V_i \quad : V_n > V_0$ and/or $V_n > V_M$

then $\quad 0 \le -h^2 L_M V_n = V_{n+1} - 2V_n + V_{n-1}$

$$\Rightarrow \quad V_n \le \frac{V_{n+1} + V_{n-1}}{2} \le V_n$$

$$\Rightarrow \quad V_n = \frac{V_{n+1} + V_{n-1}}{2} \longrightarrow V_n = V_{n+1} = V_{n-1}$$

repeat to the boundary

Lemma (stability) LT: For any $V = \{V_i\}_{i=0}^{N}$

$$|V|_{h,\infty} \leq \max\{|V_0|, |V_N|\} + \frac{1}{8}|\mathcal{L}_h V|_{h,\infty}$$

$$|V|_{h,\infty} = \max_i |V_i|$$

proof (tomorrow !)

Corollary (uniqueness): The discrete pbm

$$\begin{cases} \mathcal{L}_h U_i = f_i \\ U_0 = 0 = U_N \end{cases} \qquad \text{(FD method)}$$

has a unique solution

- by stability homog. prob. $\begin{cases} \mathcal{L}_h U_i^0 = 0 \\ U_0^0 = 0 = U_N^0 \end{cases}$

  the $U^0 \equiv 0 \Rightarrow$ uniqueness

  existence or finite dimensional