NMPDE/ATSC 2025
Lecture 5

Computer Practicals @11AM on:

- **22/10 in room 138**
- **29/10, 5/11 12/11, 19/11, 03/12 in room 139**
- **26/11 in room 005**

NOTE: no practical and no lecture on 10/12

NOTE: unfortunately, this lecture's registration came out without audio, making it pointless. Hence you will not find it in the You Tube channel

Lemma (Lax-Milgram): Let

- $(V, (\cdot,\cdot))$ Hilbert, $\|\cdot\|$ its assoc. norm
- $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ bilinear
  - ☆ continuous : $\exists \gamma : |A(w,v)| \le \gamma \|w\| \|v\|$
  - ☆ coercive : $\exists \alpha_0 : A(v,v) \ge \alpha_0 \|v\|^2$ $\forall v, w \in V$

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Lecture

- $F(\cdot) : V \longrightarrow \mathbb{R}$ in V', so linear,

$$|F(v)| \le C_0 \|v\|_V$$
$$\hookrightarrow \|F\|_{V'}$$

Then $\exists!\, u \in V : A(u,v) = F(v) \quad \forall v \in V$

and $\|u\| \le \dfrac{1}{\alpha_0} \|F\|_{V'}$

Proof: see e.g. LT (appendix)

Based on
- Closed Range Theorem
- Riesz Representation

Theorem (Riesz) Let $(H, (\cdot,\cdot))$ Hilbert

$\forall L \in H'$, $\exists! \, u \in H$:

$$\begin{cases} L(v) = (u, v) & \forall v \in H \\ \|L\|_{H'} = \|u\|_H \end{cases}$$

exercise: show that well-posedness in case of $a(\cdot,\cdot)$ symmetric

(example $a(u,v) = (c \nabla u, \nabla v)_0 + (c\, u, v)$)

follows directly from Riesz.

Hint: a defines a scalar product!

$$(u,v)_a := a(u,v)$$

Lax-Milgram: is non-trivial generalisation

Boundary Conditions

Consider:

$$\mathcal{L}u = -\sum_{ij} D_i(a_{ij} D_j u) + \sum_i D_i(b_i u) + cu \quad \text{in } \Omega \subset \mathbb{R}^d$$

From $\mathcal{L}u = f$

Test with v and integrate over Ω:

$$\underbrace{\int_\Omega A \nabla u \cdot \nabla v - \int_\Omega u \, \underline{b} \cdot \nabla v + \int_\Omega c u v}_{a(u,v)} + \int_{\partial\Omega} \overbrace{(-A\nabla u + \underline{b} u) \cdot \underline{n}}^{\frac{\partial u}{\partial n_L}} \, v = \int_\Omega f v =: \ell(v)$$

In general, we write weak problem as

Find $u \in V$: $\mathcal{A}(u,v) = F(v) \quad \forall v \in V$

where V, $\mathcal{A}$, F depend also on b.c.

Homog. Dirichlet: $u = 0$ on $\partial\Omega$

$\rightarrow V = H_0^1(\Omega)$; $\mathcal{A} = a$; $F = \ell$

ESSENTIAL B.C.

- Neumann : define conormal derivative

NATURAL B.C.

$$\frac{\partial u}{\partial n_L} := (A\nabla u - bu)\cdot n$$

and fix $\frac{\partial u}{\partial n_L} = g \in L^2(\partial\Omega)$

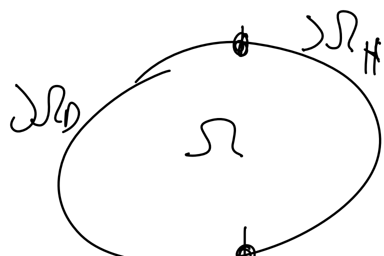
$$V = H^1(\Omega)\ ;\ \mathcal{A} = a\ ;\ F(v) = \ell(v) + \int_{\partial\Omega} g v$$

- ROBIN : $\frac{\partial u}{\partial n_L} + \mu u = g$ $\quad \begin{cases} g \in L^2(\partial\Omega) \\ \mu \in L^\infty(\partial\Omega) \end{cases}$

$$V = H^1(\Omega)\ ;\ \mathcal{A}(u,v) = a(u,v) + \int_{\partial\Omega} \mu u v$$

$$F(v) = \ell(v) + \int_{\partial\Omega} g v$$

- MIXED $\left(\begin{array}{l} N \text{ on } \partial\Omega_N \\ D \text{ on } \partial\Omega_D \end{array} \right.$ $\quad \overline{\partial\Omega_N} \cup \overline{\partial\Omega_D} = \partial\Omega$



$$\begin{cases} Lu = f & \text{in } \Omega \\ \frac{\partial u}{\partial n_L} = g_N & \text{on } \partial\Omega_N \\ u = g_D & \text{on } \partial\Omega_D \end{cases}$$

$g_N \in L^2(\Omega)\ ;\ g_D \in H^{1/2}(\Gamma_D)$ = space of all traces of H^1 functions

$$-\int_{\partial\Omega} \frac{\partial u}{\partial n_L} v = -\cancel{\int_{\partial\Omega_D} \frac{\partial u}{\partial n_L} v} - \int_{\partial\Omega_N} \underbrace{\frac{\partial u}{\partial n_L}}_{=g} v$$

$$\cancel{}\;\; = 0$$

- Fix test space $\boxed{H^1_{\partial\Omega_D}} = \{ v \in H^1(\Omega) : v|_{\partial\Omega_D} = 0 \}$ → TEST

$u \in \boxed{V_g} = \{ u \in H^1(\Omega) : u|_{\partial\Omega_D} = g \}$ → TRIAL

$$\mathcal{A} = a \; ; \; F(v) = \ell(v) + \int_{\partial\Omega_N} g v$$

BUT : TEST $\neq$ TRIAL

SOLUTION : lift the Dirichlet data !

Theorem (Traces) : Let $\Omega \subset \mathbb{R}^d$ bounded on open with $\left\{\begin{array}{l}\text{Lipschitz} \\ \text{Polygonal}\end{array}\right.$ boundary $\partial\Omega$.

Then $\exists\ \gamma_0 : H^1(\Omega) \longrightarrow \boxed{\overbrace{H^{1/2}(\partial\Omega)}^{\text{traces}}}$ trace operator

- linear
- $\forall v \in H^1(\Omega) \cap C^0(\bar{\Omega})$, then $\gamma_0 v = v|_{\partial\Omega}$

- $\exists\, c^* > 0 : \|\gamma_0 v\|_{0,\partial\Omega} \le c^* \|v\|_{H^1(\Omega)}$

  $\uparrow L^2(\partial\Omega)$ (= bounded)

Given $g_D \in H^{1/2}(\partial\Omega_D)$ $\exists\, r_g \in H^1(\Omega) : g_D = \gamma_0(r_g)$

Lifting

Set $\mathring{u} = u - r_g$

$\Rightarrow$ - $\mathring{u}|_{\partial\Omega_D} = u|_{\partial\Omega_D} - r_g|_{\partial\Omega_D} = g_D - g_D = 0$

- $\mathring{u} \in H^1_0(\Omega)$ solves:

$$\mathcal{A}(\mathring{u}, v) = F(v) \quad \forall v \in H^1_0(\Omega)$$

with $\mathcal{A} = a$;

$$F(v) = \ell(v) + \int_{\partial\Omega_N} g_N v - \mathcal{A}(r_g, v)$$

exercise: in each case,

- weak solution $\Rightarrow$ strong sol.
- weak problems are well-posed

Regularity of the solution

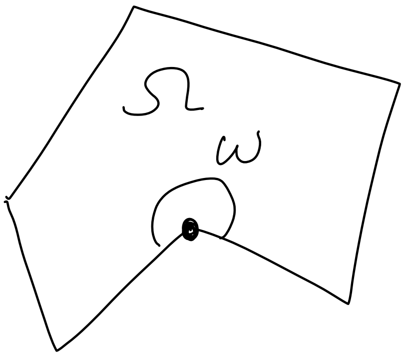
- $\Omega \subset \mathbb{R}^2$ and $\partial\Omega$ smooth + Homog. Dirichlet B.C. $\left(\cap H_0^1(\Omega)\right)$

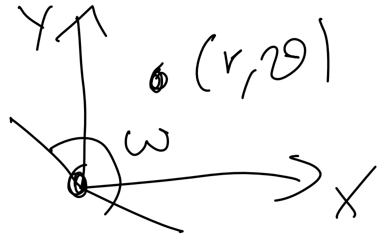
If $f \in H^s(\Omega)$, then $u \in H^{s+2}(\Omega)$
$\left(s \in \mathbb{R}^{\geq 0} ; \; s = 0 \Rightarrow L^2(\Omega) \right)$
and $\|u\|_{H^{s+2}(\Omega)} \leq C \|f\|$

Example: $f \in L^2(\Omega) \Rightarrow u \in H^2(\Omega)$

- If $\partial\Omega$ is <u>not</u> smooth
"2" - shift not guaranteed

Example: Ω = polygon





For the Poisson problem, near the corner in polar coordinates with the corner as origin,

$$u(r,\vartheta) = r^{\pi/\omega}\alpha(\vartheta) + \beta(r,\vartheta)$$

regular

If $\omega > \pi$ then $u \notin H^2(\Omega)$

indeed $u \in H^s(\Omega)$ $\forall\ \frac{3}{2} < s < 1+\frac{\pi}{\omega} < 2$

study $r^{\pi/\omega} \in H^s$ $s = ?$

if $D^s(r^{\pi/\omega}) = r^{\pi/\omega - s} \in L^2$

check: $\int r^{(\pi/\omega - s)^2} r^{d-1} \, dr$

$$= \int_0^R r^{2\pi/\omega - 2s + d - 1} \, dr$$

finite $\Leftrightarrow -2\frac{\pi}{\omega} + 2s - d + 1 < 1$

$\Leftrightarrow 2s < d + 2\pi/\omega = 2 + 2\pi/\omega$

$\Leftrightarrow s < 1 + \pi/\omega$

- same (shift) applies to Neumann / and Robin conditions
- mixed problem $u \notin H^2$ also for regular domains due to matching of Dirichlet and Neumann