NMPDE/ATSC 2025
Lecture 7

1D FEM

Reaction-Diffusion 2 points b.v.p:

$$\begin{cases} \mathcal{L}u = -(a u')' + c u = f & \text{in } \Omega = (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

$f \in L^2(\Omega)$; $a(x) > a_0 > 0$, $c \geq 0$.

→ weak formulation: find $u \in V := H_0^1(\Omega)$:

$$A(u,v) = F(v) \qquad \forall v \in V$$

with $A(u,v) = \int_0^1 a u' v' + \int_0^1 c u v$; $F(v) = \int_0^1 f v$

FEM:

- fix grid $\begin{cases} x_i = x_{i-1} + h_i \\ i = 1, \dots, N \\ x_0 = 0 \end{cases}$ with $\sum h_i = 1$ → $I_i = [x_{i-1}, x_i)$, $\bigcup \bar{I}_i = [0,1]$ mesh
- fix $k \in \mathbb{N}$ poly degree, $\mathcal{T}_h = \{I_i\}_i$, $h = \max_i h_i$

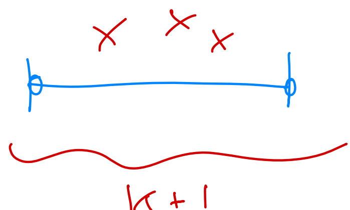
- $V_h^k = \{ v \in C^0(\bar{\Omega}) : v \in \mathbb{P}^k(I_i)\ \forall i,\ v(0) = 0 = v(1) \} \subset V$

- FEM : Find $u_h \in V_h^k : A(u_h, v) = F(v)\ \forall v \in V_h^k$

- Algebric formulation
  - fix a basis, a Lagrangian w.r.t. a set of FE nodes (points in each I_h) y_j, so that $\varphi_i(y_j) = \delta_{ij}$

    # mesh vertices

$$\dim V_h^k = \underbrace{N-1}_{\#\text{ internal nodes}} + N(k-1) = M$$



Test with φ_i and writing $u_h = \sum_j U_j \varphi_j$,

$$\sum_j U_j \underbrace{\left[\int_0^1 a \varphi_j' \varphi_i' + \int_0^1 c \varphi_j \varphi_i \right]}_{A_{ij}} = \overbrace{\int_0^1 f \varphi_i}^{F_i} \qquad i = 1, \dots, M$$

$$\boxed{AU = F}$$

A priori analysis

Lemma of Céa applies:

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha_0} \inf_{v_h \in V_h} \|u-v_h\|_V$$

where, in this case,

$$= |v|_1^2 = \|v'\|_0^2 \qquad = \|v\|_0^2$$

$$\|v\|_V^2 = \begin{cases} \int_0^1 (v')^2\,dx + \underbrace{\int_0^1 v^2\,dx} \\ \int_0^1 (v')^2\,dx \end{cases}$$

analysis of H^1-norm term

quantification of the inf w.r.t. the discretization parameters.

We have,

$$\inf_{v_h \in V_h} \|u-v_h\|_V \leq \|u - I_h u\|_V$$

for some $I_h u \in V_h^k$

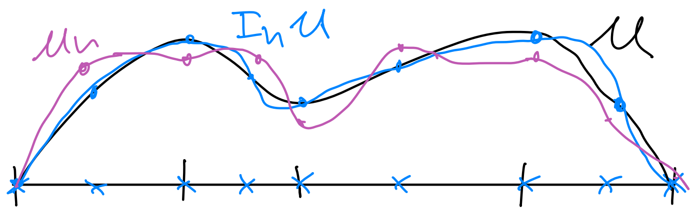
Also

$$\|u - I_h u\|_V^2 = \sum_{i=1}^{M} \|(u - I_h u)|_{I_i}\|_V$$

Def (FEM interpolant): Define $I_h u \in V_h$ or, on each I_i, $I_h u|_{I_h} \in \mathbb{P}_k(I_i)$ such that $I_h u(x_\ell^i) = u(x_\ell^i)$ where $\{x_\ell^i\}_\ell$ are the local to I_i nodes.

example

$k = 2$



Theorem: If $u \in H^{k+1}(I_i)$, then

(h-bound)

$$\|u - I_h u\|_{L^2(I_i)} \le C\, h_i^{k+1} |u|_{H^{k+1}(I_i)}$$

$\neq$

(hp-bounds)

$$|u - I_h u|_{H^1(I_i)} \le \overset{C(k)}{\overset{\|}{C}}\, h_i^{k} |u|_{H^{k+1}(I_i)}$$

where C do not depend on h (they depend on k.

Corollary. If $u \in H^{k+1}(\Omega)$, then

$\left(\|v\|_0 = \|v\|_{L^2(\Omega)}\right)$ $$\|u - I_h u\|_0 \le C\, h^{k+1} |u|_{k+1}$$

$$|u - I_h u|_1 \leq C h^k |u|_{k+1}$$

(note that $u \in H^2(\Omega)$)

<u>Theorem</u>: If $u \in H^{k+1}(\Omega)$, then

$$|u - u_h|_1 \leq C h^k |u|_{k+1} .$$

k-degree FEM is order k in the H^1-norm

proof: From Céa + interpolation error bound

<u>Remarks</u>:

① Assuming $u \in H^{k+1}(I_i)$ $\forall i$, then

$$|u - u_h|_1 \leq C \left(\sum_i h_i^{2k} |u|_{k+1}^2 \right)^{1/2}$$

② If u is only in H^{s+1} for $s \leq k$, then

$$|u - u_h|_1 \leq c h^s |u|_{H^{s+1}(\Omega)}$$

still optimal

k	H^2	H^3	H^4
1	$\boxed{1}$	1	1
2	1	$\boxed{2}$	2
3	1	2	$\boxed{3}$

order of the FEM

③ L^2-norm analysis

<u>Theorem</u>: If $u \in H^{s+1}(\Omega)$ for some $s \geq 1$,

$$\|u - u_h\|_0 \leq C h^{s+1} |u|_{s+1}$$

$$\left(u \in H^2(\Omega),\ u \in H^{s+1}(I_i)\ \forall i \quad \|u - u_h\|_0 \leq C h \left(\sum_i h_i^{2s} |u|^2_{H^{s+1}(I_i)} \right)^{1/2} \right)$$

proof: (Aubin-Nitsche duality trick):

Introduce a dual (adjoint) problem:

$$\begin{cases} \mathcal{L}^* \phi = u - u_h =: e \\ \phi(0) = 0 = \phi(1) \end{cases}$$

(see Quarteroni sec. 2.6 same argument in dim d, and general def. of dual problems)

Weak form ← NOTE! ϕ appears as 2nd argument

$$a(w, \phi) = (e, w) \quad \forall w \in V$$

note $\phi \in H^2(\Omega)$ at least and

$$\|\phi\|_2 \overset{(*)}{\le} c\|e\|_0$$

Test with $w = e$,

$$\|e\|_0^{\cancel{2}} = (e,e) \underset{\text{adj. problem}}{=} A(e,\phi) \underset{\text{Gal. orthog}}{=} A(\overset{u-u_h}{\overset{\|}{e}}, \phi - I_h\phi)$$

$$\underset{\text{continity}}{\le} \gamma \|e\|_V \|\phi - I_h\phi\|_V$$

$$\underset{\text{interpolation error bound}}{\le} C\gamma h \|e\|_V |\phi|_2 \underset{(*)}{\le} c\gamma h \|e\|_V \cancel{\|e\|_0}$$

$$\underset{\text{Poincaré}}{\le} c\gamma h |e|_1$$

$$\Rightarrow \|e\|_0 \le C h \underbrace{|e|_1}_{\text{apply known } H^1\text{-norm error bound}}$$

④ hp-error bounds (Babuska-Suri)

$$|u-u_h|_1 \le C \frac{h^{r-1}}{k^s} |u|_r$$

$r-1 = k$ if $s=k$

C indepen of both h, k

$r = \min(s+1, k+1)$

$u \in H^{s+1}(\Omega)$

valid $h \to 0$ and/or $k \to +\infty$