NMPDE/ATSC 2025 Lecture 4

About Strong and week solutions (Q, Ch.2,3)

Def: (durl space).

Given (V, 11.11) worned space, ils dual V'= [(V; IR) liver

ord bounded functionals $(|\Gamma(\mathcal{N})| \leq C \|\mathcal{N}\|)$

operator norm V' is Banach w.r.t. ren 11 TIN = 20b Tr(n)

Notation: duality $\langle \cdot, \cdot \rangle : \vee' \times \vee \longrightarrow \mathbb{R}$ Pairing

$$\langle L(N) := L(N)$$

representation
Theorem (Riesz-Frèclet): Let
 $(H,(\cdot,\cdot))$ Hilbert, LEH. Then
 $\exists !$ $u \in H$ & $L(v) = (u,v)$
 $\forall v \in H$
 $\exists l \in H$
 $\exists l \in H$

Example: ([7(51), (·/·)o)

 $f \in (L^{2}(\Omega))^{\prime} \Rightarrow M \in L^{2}(\Omega) := f(N) = (M/N)_{0}$ = 2f, N

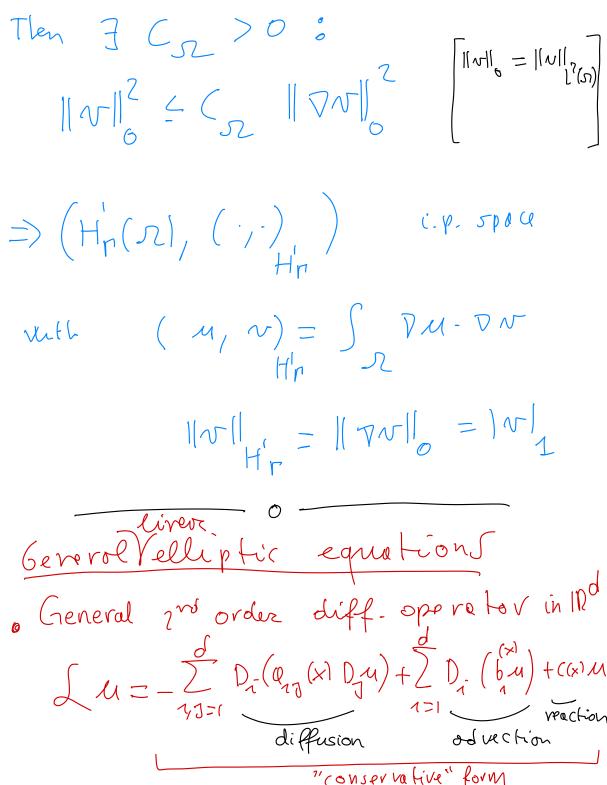
exercise: $\|f\| = \|u\|_{L^2}$

 $D(\Omega)$ as the space $Z_0(\Omega)$ veith Schwartz to pology (= uniform convergence of every combination of portol derivatores) D'(si): space of distribu $T: \mathbb{O}(\mathbb{N}) \longrightarrow \mathbb{R}$ $\varphi \longrightarrow \langle T, \varphi \rangle = T(\varphi)$ example: note L'(r) (D'(s) FRE [1, ta] Ivoleed, e.g. for P=2

Distributions: Let

giver f E L? (52), le con identify g E D'(s) by bounded by schwortz But, S= Dirac's delta is in D'(s) but not in L*(s) S(v) = V(o)Week derivetive. For TED(si DT ED'(si) "weak derivative": (DT, φ)=-(T, q)+φε06

ord, for on & multiinolex $(D^{2}T, y) = (-1)^{n}(T, D^{2}y) + q \in D(n)$ Soboler spaces W/K, P(S), KEH, PE[1,0] $W^{k,p}(n) := \{ v \in L^{p}(x) : D^{p} \in L^{p}(x), \forall |x| \leq k \}$ $\left[p = 2 \Rightarrow W^{k/p}(\Omega) := H^{k}(\Omega) \right]$ Grilbert w.v.t. $(N,W)_{N} = \sum_{|\mathcal{L}| \leq k} (D^{2}V, D^{2}W)$ e.g. H^{2} $(v_{1}v_{1})_{1} = (M_{1}v_{1})_{0} + (V_{1}V_{1}V_{1})_{0}$ $(||v_{1}||_{1}^{2} = ||v_{1}||_{0}^{2} + ||V_{1}V_{1}|_{0}^{2})$ Theorem: (Poincare ineq.): Let $SCIR^{d}$ open and boun ded, with Lipschitz boundary) Il let [(I) := {veH(s): Np=0} Snon-zero measure



· Associated bilinear form F(u,v)= Szaz DuDN-ZbruDN+ (MN N / AVU DN I M b. VN S valid and / bri C E LO(SZ) ond M, NEH (SZ) o E Phic if the water X $A(x) = \{ a_{1}(x) \}$ is positive definite o.e. in Ω .

That is, $(A^{(x)}_{\overline{3}}, \overline{3}) = \sum u_{ij}(x) \overline{3}, \overline{3} > 0$ $\overline{3} = 2 \text{ and}$ · Coercivity: Q: V×V -> IR is (seruve if]20>0 ; HNEV Q(N,N) > 20 ||N|

=) An Second o voler op. is elleptic if it is coercive in its 2nd order term. Weak solutions Consider 6.v.p. in N $\begin{cases} LM = f \\ M = 0 \end{cases}$ on) N = < f/v> $\rightarrow \bullet V = H'_{0}(\Omega)$ $0 + \epsilon l^{2}(x) \rightarrow F(w) = \int_{0}^{\infty} f x^{2}$ · Weak problem: First MEV: $\mathcal{A}(u,v) = F(v) \uparrow \forall v \in V$ wook sol. Strong solutions: $M \in O'(SZ)$: $LM = f \quad \text{in } O'(SZ)$

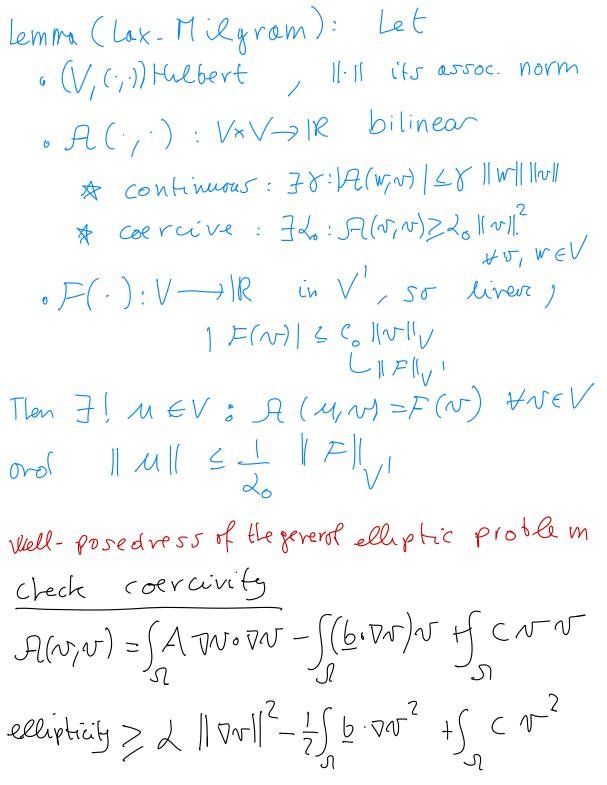
The rem: u & V is weak solution

refl u is strong solution in D(n) Proof: ">" (Lu = - DM Poisson) Assume MEV is Weak sol, / HNGZO(SZ) $= \mathcal{F}(\mathcal{M}/\mathcal{N}) = \mathcal{F}(\mathcal{M}/\mathcal{N}) = \mathcal{F}(\mathcal{M}/\mathcal{N})$ - S 74.70 1- Sam v = (-DM/v) $\Rightarrow -\Delta M = 1 \quad \text{in} \quad O'(\Omega)$ "E" some calculation + Lensity of Dan in V Q: When 11 is also classical solution? Again, consider Poisson problem ({-In=f} u=0) if $f \in L^{7}(\Sigma)$ then a weak solution SS is $Z^{7}(SL)$ $M \in H^{7}(SL)$.

Or is a convex polygon Regularity for elliptic plans GRISVARD

 $-\Delta u - f \in L^{2}(\Omega) \quad \therefore -\Delta u = f \quad L^{2}(\Omega) \quad (\alpha e - in \Omega)$ \$ classical sol, scm²

For that, need & EH?(52) Yord Dr is 24 then $M \in H^4(n) \subset Z^2(n)$ · if si cird smooth/polygonal then H" G Z (I) if h > 6/2 od FCV: IIVII E EVIIVII (Lopolor) -> weak notion of sol. more giverol > proving Well-posedvess for weak problem is losier (particulouly for nonlinear problem 5 ?)



$$\frac{1}{2} > 2 || \nabla v ||^{2}$$

$$\frac{1}{2} || \nabla v ||^{2}$$

$$\frac{1}{2} || \nabla v ||^{2} \leq || \nabla v ||^{2}$$

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=> well-posedNess (LM) if C+ > D.6 30

= 7 | Dalls + 2 (C+ 2 1.p) 2

 $-\frac{1}{2}\int_{0}^{2}\overline{p}\cdot\nabla v^{2}=\frac{1}{2}\int_{0}^{2}(\nabla \cdot b)v^{2}$

Assume C+= 7.6 20