# NMPDE/ATSC 2025
# Lecture 8

Once again, here is the zoom link in case you need to follow remotely:

https://sissa-it.zoom.us/j/7306190508?pwd=TXhCQkFHMlgrVERqNUw2aDNxSDVUQT09

## FEM in higher dimensions

Take $d=2$, $\Omega$ polygonal domain, model

pbm: $\begin{cases} \mathcal{L}u = -\nabla\cdot(a\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

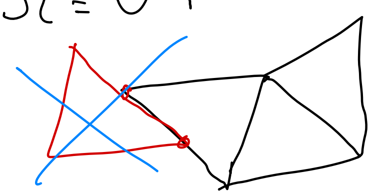
WP: Find $u \in V = H^1_0(\Omega)$ : $\underbrace{A(u,v)}_{\int_\Omega a\nabla u\cdot\nabla v} = \underbrace{F(v)}_{\int_\Omega f v}$ $\forall v \in V$

FEM space $V_h^k \subset V$ $\quad (\subset C^0(\bar\Omega))$

1. Triangulation $\mathcal{T}_h = \{T\}$ : $\bar\Omega = \bigcup T$

and such that

$T_1, T_2 \in \mathcal{T}_h$ $\quad T_1 \cap T_2 = \begin{cases} \emptyset \\ \text{a vertex} \\ \text{an entire edge} \end{cases}$

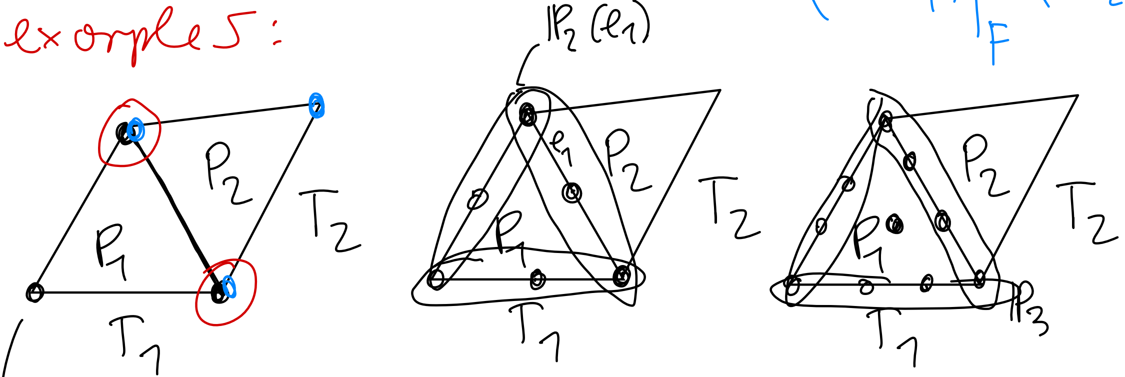


2.

$$V_h^k = \{ v \in \mathcal{C}^0(\Omega) \cap H_0^1(\Omega) : v|_T \in \mathbb{P}^k(T), \ \forall T \in \mathcal{T}_h \}$$

Proposition: $v \in H^1(\Omega) \iff \begin{cases} v|_T \in H^1(T) & \forall T \\ \text{if } F = T_1 \cap T_2 \\ \text{then } (v|_{T_1})|_F = (v|_{T_2})|_F \end{cases}$

examples:




$P_1, P_2 \in \mathbb{P}_1$ $\qquad$ $P_1, P_2 \in \mathbb{P}_2$ $\qquad$ $P_1, P_2 \in \mathbb{P}_3$

degrees of freedom

$$\dim \mathbb{P}_k(\mathbb{R}^d) = \binom{d+k}{k} = \frac{(d+k)!}{k!\, d!}$$

Remark: quadrilateral FE spaces can also be constructed using $\mathbb{Q}_k$ = polynomials of degree k separately in each variable

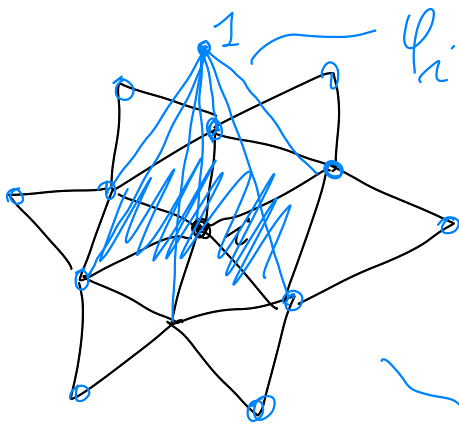
3. Basis: Lagrange Basis $\{\varphi_i\}$

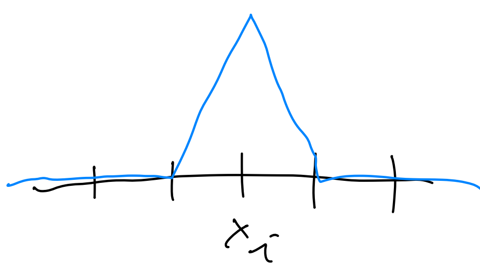
$$\varphi_i(x_j) = \delta_{ij} \quad \text{where } \{x_j\}$$

Lagrange nodes

( ⟶ local in $T$ finite element made of 3 ingredients ① $T$ ② $\mathbb{P}_k$ ③ degrees of freedom )

Example: $k = 1$ Lagrange nodes ≡ triangulation mesh vertices




⟶ sparse

4. The FEM : $u_h = \sum_j U_j \varphi_j$ :

$$\sum_j U_j \underbrace{\int_\Omega a \nabla\varphi_j \cdot \nabla\varphi_i}_{A_{ij}} = \underbrace{\int_\Omega f \varphi_i}_{F_i} \quad \forall i$$

ASSEMBLY

$$A_{ij} = \sum_T \int_T a \nabla\varphi_j \cdot \nabla\varphi_i$$

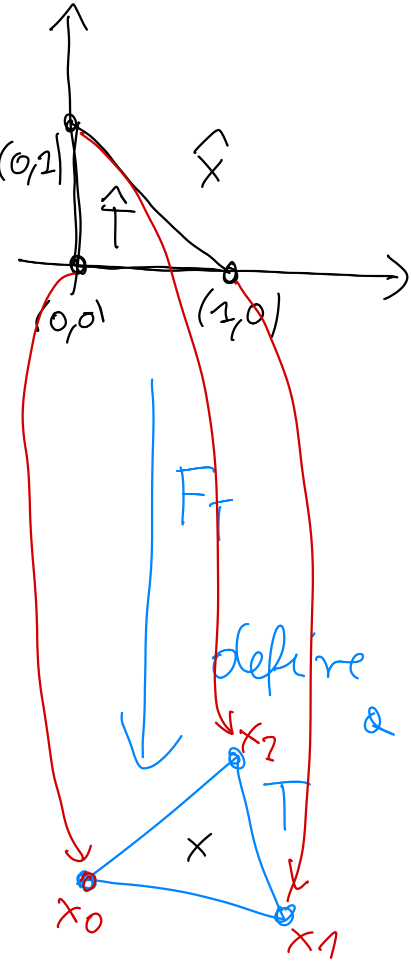
$\rightarrow$ local to T : $T \subset \operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j$

$\int_T$ computed by "appropriate quadrature

in this case we require quadrature exact on space $\mathbb{P}_{2k-2}$.

A practical way of performing assembly:

use REFERENCE ELEMENT




- $\hat{T}$
- $\hat{P}$ $\quad (\mathbb{P}_k(\hat{T}))$
- $\hat{\Sigma}$ $\quad$ degrees of freedom (DoF)

$\underbrace{\phantom{xxxxxxxxxx}}_{\text{linear functionals acting on } \hat{P}}$

$\underbrace{\phantom{xxxxxxxxxx}}_{\text{def. of F.E. by Ciarlet}}$

define the "physical" FE by a affine mapping

$$T_F(\hat{x}) = \underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{B_T} \hat{x} + \underset{\substack{\uparrow \\ \text{position } x_0}}{b_T}$$

- $\hat{T} \to T$
- $\hat{P} \to \{ p : T \to \mathbb{R} : p \circ \phi \in \hat{P} \}$
- $\Sigma = F_T(\hat{\Sigma})$

$$\int_T a(x) \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx = \int_{\hat{T}} a(F_T(\hat{x})) \, B_T^{-T} \hat{\nabla} \hat{\varphi}_j \circ B_T^{-1} \hat{\nabla} \hat{\varphi}_i \det(B_T) \, d\hat{x}$$

Advantage: define quadrature only on $\hat{T}$ and evaluate (and store) just once the values of the $\varphi_i$ at the quadrature points.

## A priori analysis

We have Cèa →

$$|u-u_h|_1 \le C \inf_{v_h \in V_h^k} \|u-v_h\|_1 \le$$

(1) define interpolant $I_h u$ $\le C \|u - I_h u\|_1$

(2) Evaluation of interpolation error

**Def**: Lagrange interpolant:

$$I_h : C^0(\bar{\Omega}) \longrightarrow V_h^k$$

Piecewise define $I_h v_{|T} = \sum_{\ell} \gamma_\ell^T \varphi_\ell(x)$

with $\gamma_\ell^T = v(x_\ell^T)$

where $\varphi_\ell$ Lagrange basis so that

$$I_h v = \sum_j v(a_j)\, \varphi_j(x)$$

with $\{a_j\}$ the global numbering of the nodes so that $\varphi_j(a_i) = \delta_{ij}$

To define the Lagrange interpolant requires point evaluation $\Leftarrow$ restrict to continuous functions

$\Rightarrow$ We fix $\underset{\substack{\uparrow \\ u}}{v} \in H^s(\Omega)$ $\quad s \geq 2$

so that point eval makes sense by

Theorem (Sobolev): If $\Omega \in \mathbb{R}^d$, $d = 2, 3$, then if $s \geq 2$ $\quad H^s(\Omega) \hookrightarrow C^0(\overline{\Omega})$

$$[v] \longrightarrow v$$

crucial idea for analysing interp error

$$|u - I_h u|_{1,T}$$

is map back $T$ to $\hat{T}$.

Lemma 1 (Q-4.2, seminorm scaling):

For each $m \geq 0$, and $v \in H^m(T)$

and let $\hat{v} : \hat{T} \to \mathbb{R}$ def. by

$\hat{v} = v \circ F_T$. Then:

$(F_T \hat{x} = B_T \hat{x} + b_T)$

- $\hat{v} \in H^m(\hat{T})$
- $\exists\, C = C(m) > 0$:

$$\begin{cases} |v|_{H^m(T)} \leq C \, \|B_T^{-1}\|^m \, |\det B_T|^{1/2} \, |\hat{v}|_{H^m(\hat{T})} \\ |\hat{v}|_{H^m(\hat{T})} \leq C \, \|B_T\|^m \, |\det B_T|^{-1/2} \, |v|_{H^m(T)} \end{cases}$$

with $\|B_T\| = \sup\limits_{\substack{\xi \in \mathbb{R}^2 \\ \xi \neq 0}} \frac{|B\xi|}{|\xi|}$ ← euclidean norm in $\mathbb{R}^2$

see e.g.

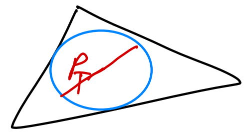
Ciarlet

Quarteroni–Valli

for proofs.

<u>**Def**</u> (diameter, sphericity, regularity):

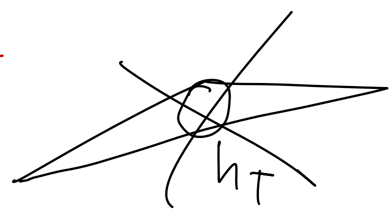
Let $\mathcal{T}_h = \{T\}$ a triangulation, and




- $h_T = \text{diam } T$, $\forall T$
- $\rho_T = \sup\left\{\text{diam}(S) : \begin{array}{l} \cdot\ S \text{ is sphere} \\ \cdot\ S \subset T \end{array}\right\}$ sphericity
- $\hat{h}, \hat{\rho}$ corresp. diam, sphericity of $\hat{T}$
- A family of meshes $\mathcal{T}_h$, $h \in \mathbb{R}^+$, is shape regular if $\exists\, \sigma > 1$:

$\forall h$, $\forall T \in \mathcal{T}_h$ we have

$$\underbrace{\frac{h_T}{\rho_T}}_{} \le \sigma$$




this ratio measures the regularity (fatness) of the triangles

Lemma 2 : We have

$\bullet\ \|B_T\| \le \frac{h_T}{\hat{\rho}}$ $\bullet\ \|B_T^{-1}\| \le \frac{\hat{h}}{\rho_T}$

Lemma 3 : (Bramble-Hilbert) :

Let $\hat{L} : H^{k+1}(\hat{T}) \longrightarrow H^m(\hat{T})$, with $m, r \geq 0$

linear, continuous $\Rightarrow \hat{L} \in \mathcal{L}^m_{k+1}(\hat{T})$ and such that

$$\hat{L}(\hat{p}) = 0 \qquad \forall\, \hat{p} \in \mathbb{P}_k(\hat{T})$$

Then $\forall\, \hat{v} \in H^{k+1}(\hat{T})$

$$|\hat{L}(\hat{v})|_{H^m(\hat{T})} \le \|\hat{L}\|_{\mathcal{L}^m_{k+1}(\hat{T})} \inf_{\hat{p} \in \mathbb{P}_k(\hat{T})} \|\hat{v} + \hat{p}\|_{H^{k+1}(\hat{T})}$$

Lemma 4 (Deny-Lions) : For $k \geq 0$ $\exists C = C(k, \hat{T})$

$$\inf_{\hat{p} \in \mathbb{P}_k(\hat{T})} \|\hat{v} + \hat{p}\|_{H^{k+1}(\hat{T})} \le C\, |\hat{v}|_{H^{k+1}(\hat{T})}$$

$\forall\, \hat{v} \in H^{k+1}(\hat{T})$

Corollary (Lemma 3 + Lemma 4):

$$|\hat{L}(\hat{v})|_{H^m(\hat{T})} \leq C \|\hat{L}\|_{\mathcal{L}^m_{k+1}(\hat{T})} |\hat{v}|_{H^{k+1}(\hat{T})}$$

use with $\hat{L} = \underset{\substack{\uparrow \\ \text{identity}}}{I_d} - \underset{\substack{\uparrow \\ \text{interpolant defined on } \hat{T}}}{\hat{I}_k}$

$$\hat{L}\hat{v} = \hat{v} - \hat{I}_k \hat{v}$$

$\Rightarrow$ in particular, $\hat{v} = \hat{p} \in \mathbb{P}_k(\hat{T})$

$$\underline{\hat{L}\hat{p}} = \hat{p} - \hat{I}_k \hat{p} = \hat{p} - \hat{p} = 0$$

<u>Theorem</u>: (Local interpolation error):

Let $k \geq 1$, $0 \leq m \leq k+1$. Then $\exists\, C = C(k, m, \hat{T}) > 0$

$$\boxed{|v - I_h^k v|_{H^m(T)} \leq C \frac{h_T^{k+1}}{\rho_T^m} |v|_{H^{k+1}(T)}} \quad \forall v \in H^{k+1}(T)$$