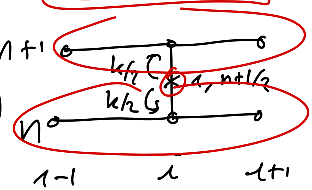
The ϑ - (family of) method, $\vartheta \in [0,1]$

(k time step, h space step)

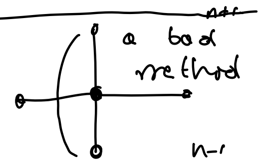
$$\boxed{u_t - u_{xx} = 0}$$

$$\left(\delta_k^t U_i^{n+1/2} = \right) \quad \frac{U_i^{n+1} - U_i^n}{k} - \vartheta \left(\delta_h^x\right)^2 U_i^{n+1} + (1-\vartheta)\left(\delta_h^x\right)^2 U_i^n = 0$$



$$\mu = \frac{k}{h^2}$$

$$\boxed{\begin{aligned} -\vartheta\mu\, U_{i+1}^{n+1} + (1+2\vartheta\mu)\, U_i^{n+1} - \vartheta\mu\, U_{i-1}^{n+1} \\ = \mu(1-\vartheta)\, U_{i+1}^n + \big(1-2\mu(1-\vartheta)\big) U_i^n + \mu(1-\vartheta)\, U_{i-1}^n \end{aligned}}$$



$$\delta_{2k}^t U_i^n - \left(\delta_h^x\right)^2 U_i^n = 0$$

Consistency: $\quad - \; O(k, h^2)$

$\vartheta \neq 1/2$ $\qquad \left|T_i^{n+1/2}\right| \leq \frac{1}{2}\left(k M_{tt} + \frac{h^2}{6} M_{xxxx}\right) + \text{H.O.T.}$

$\vartheta = 1/2$ (C.N.) $\qquad \left|T_i^{n+1/2}\right| \leq \frac{1}{12}\left(k^2 M_{ttt} + h^2 M_{xxxx}\right)$

Proof:

CH case

$u(t_{n+1/2}, x_i)$

20. We expand all the terms in the truncation error about the point $\boxed{(t_{n+\frac{1}{2}}, x_i)}$. We have

$=k$

$$\boxed{u(t_{n+1}, x_i)} = u(t_{n+1/2} + \frac{\tau}{2}, x_i) = u + \frac{\tau}{2}u_t + \frac{\tau^2}{8}u_{tt} + \frac{\tau^3}{48}u_{ttt}(\rho_n, x_i),$$

for $\rho_n \in (t_{n+1/2}, t_{n+1})$, where we have dropped the arguments from the function u and its derivatives, whenever they are evaluated at the point $(t_{n+1/2}, x_i)$. Similarly, we expand $u(t_n, x_i) = u(t_{n+1/2} - \frac{\tau}{2}, x_i)$ about $(t_{n+1/2}, x_i)$. This gives

$u_t(t_{n+1/2}, x_i)$

$$u_x \approx \boxed{\frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\tau}} = u_t + \frac{\tau^2}{48}(u_{ttt}(\rho_n, x_i) + u_{ttt}(\sigma_n, x_i))$$

for some $\sigma_n \in (t_n, t_{n+1/2})$. We also get ϑ

$$\frac{u(t_{n+1}, x_{i+1}) - 2u(t_{n+1}, x_i) + u(t_{n+1}, x_{i-1})}{h^2} = u_{xx}(t_{n+1}, x_i) + \frac{h^2}{24}(u_{xxxx}(t_{n+1}, \xi_i) + u_{xxxx}(t_{n+1}, \zeta_i));$$

similarly we get

$$\frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1})}{h^2} = u_{xx}(t_n, x_i) + \frac{h^2}{24}(u_{xxxx}(t_n, \xi_i) + u_{xxxx}(t_n, \zeta_i)),$$

for some $\xi_i \in (x_i, x_{i+1})$ and some $\zeta_i \in (x_{i-1}, x_i)$, which, in turn, should be expanded about the point $(t_{n+1/2}, x_i)$ in a similar manner to the above. Putting all these together into the truncation error, we deduce

$$\begin{aligned}
T_i^{n+\frac{1}{2}} &:= \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{\tau} - \frac{1}{2}\frac{u(t_{n+1}, x_{i+1}) - 2u(t_{n+1}, x_i) + u(t_{n+1}, x_{i-1})}{h^2} \\
&\quad - \frac{1}{2}\frac{u(t_n, x_{i+1}) - 2u(t_n, x_i) + u(t_n, x_{i-1})}{h^2} \\
&= u_t - u_{xx} - \frac{h^2}{24}((u_{xxxx}(t_{n+1/2}, \xi_i) + u_{xxxx}(t_{n+1/2}, \zeta_i)) \\
&\quad + \frac{\tau^2}{48}(u_{ttt}(\rho_n, x_i) + u_{ttt}(\sigma_n, x_i)) - \frac{\tau^2}{16}(u_{xxtt}(\tilde{\rho_n}, x_i) + u_{xxtt}(\tilde{\sigma_n}, x_i)),
\end{aligned}$$

for some for $\tilde{\rho}_n \in (t_{n+1/2}, t_{n+1})$ and some for $\tilde{\sigma}_n \in (t_n, t_{n+1/2})$. For the last term on the right-hand side we can use the PDE to arrive to $u_{xxtt} = u_{ttt}$, from which point the result follows by taking the maximum.

$(t_{n+1/2}, x_i)$

$$u_{xx}(t_{n+1}, x_i) = u_{xx} - \frac{\tau}{2}u_{xxt} + \frac{\tau^2}{8}u_{xxtt}(\tilde{\rho}_n, x_i) \quad (1)$$

$$u_{xx}(t_n, x_i) = u_{xx} + \frac{\tau}{2}u_{xxt} + \frac{\tau^2}{8}u_{xxtt}(\tilde{\sigma}_n, x_i) \quad (2)$$

$$\frac{1}{2}\,①+\frac{1}{2}\,② = u_{xx} + \frac{\tau^2}{16}\Big(\underbrace{u_{xxtt}(\tilde\rho_n, x_i) + u_{xxtt}(\tilde\sigma_n, x_i)}_{u_{ttt} \qquad + u_{ttt}}\Big)$$

$$u_t = u_{xx}$$

Instead, for the ϑ-method

$\vartheta\,① + (1-\vartheta)\,②$ gives $\frac{\tau}{2}\Big(u_{tt}(\tilde\rho_n, x_i) + u_{tt}(\tilde\sigma_n, x_i)\Big)$

stability - 1

$$\underline{(1+2\vartheta\mu)}\,U_i^{n+1} = \vartheta\mu\left(U_{i+1}^{n+1} + U_{i-1}^{n+1}\right) + \cancel{(1-\vartheta)}\mu\left(U_{i+1}^{n} + U_{i-1}^{n}\right)$$
$$+ \underline{\left(1 - 2\mu\cancel{(1-\vartheta)}\right)}\,U_i^n$$

if $1 - 2\mu\underline{(1-\vartheta)} \geq 0$

$\therefore\ \mu(1-\vartheta) \leq 1/2$

then all coefficients are nonnegative and their sum is $1+2\vartheta\mu$. Hence,

$$(1+2\vartheta\mu)\,|U_i^{n+1}| \le 2\vartheta\mu \|U^{n+1}\|_{\infty,h} + \|U^n\|_{\infty,h}$$

$$\overset{\forall i}{\Longrightarrow}$$

$$(1+2\vartheta\mu)\,\|U^{n+1}\|_{\infty,h} \le 2\vartheta\mu\|U^{n+1}\|_{\infty,h} + \|U^n\|_{\infty,h}$$

$$\Rightarrow \boxed{\|U^{n+1}\|_{\infty,h} \le \|U^n\|_{\infty,h}}$$

$\Rightarrow$ if $\mu(1-2\vartheta) \le 1/2$ (EE: $\mu \le 1/2$)

then, convergence follows

Remark

- EE ($\vartheta=0$) (same result): stable if $\mu \le 1/2$ $\overset{\text{(iif)}}{\Leftarrow}$ von Neumann analysis $\quad O(\tau, h^2)$
- IE ($\vartheta=1$) unconditionally stable (in ℓ_∞-norm !) ($\Rightarrow$ monotone) $\quad O(\tau, h^2)$

• CN ($\vartheta = 1/2$) stable in ℓ_∞ for $\mu \le 1$

$O(\tau^2, h^2)$

Von Neumann stability for ϑ-method

Plug mode $U_i^n = \lambda(\gamma)^n e^{i\gamma i h}$

$$-\vartheta\mu\, \lambda^{\cancel{n+1}} e^{i\gamma(\cancel{i}+1)h} + (1+2\vartheta\mu)\, \lambda^{\cancel{n+1}} \cancel{e^{i\gamma i h}} - \vartheta\mu\, \lambda^{\cancel{n+1}} e^{i\gamma(\cancel{i}-1)h}$$

$$= \mu(1-\vartheta)\, \cancel{\lambda^{n}} e^{i\gamma(\cancel{i}+1)h} + (1-2\mu(1-\vartheta))\, \cancel{\lambda^{n}}\, \cancel{e^{i\gamma i h}} + \mu(1-\vartheta)\, \cancel{\lambda^{n}} e^{i\gamma(\cancel{i}-1)h}$$

$$\lambda\left(-\vartheta\mu\, e^{i\gamma h} + 1 + 2\vartheta\mu - \vartheta\mu\, e^{-i\gamma h}\right)$$

$$= \mu(1-\vartheta)\, e^{i\gamma h} + 1 - 2\mu(1-\vartheta) + \mu(1-\vartheta)\, e^{-i\gamma h}$$

$$\left[-e^{i\gamma h} + 2 - e^{-i\gamma h} = 4\sin^2(\gamma h/2)\right]$$

$$\left(1 + 4\mu\vartheta \sin^2\left(\frac{\gamma h}{2}\right)\right)\lambda = 1 - 4\mu(1-\vartheta)\sin^2\left(\frac{\gamma h}{2}\right)$$

$$\Rightarrow \lambda = \lambda(\gamma) = \frac{1 - 4\mu(1-\vartheta)\sin^2(\gamma h/2)}{1 + 4\mu\vartheta\sin^2(\gamma h/2)}$$

$$|\lambda| \le 1 \iff \lambda \ge -1 \qquad \le 1$$

$$1 - 4\mu(1-\vartheta)\sin^2(\xi h/2) \ge -1 - 4\mu\vartheta\sin^2(\xi h/2)$$

$\therefore$

$$4\mu(1-\vartheta)\sin^2(\xi h/2) \le 2 + \underline{4\mu\vartheta\sin^2(\xi h/2)}$$

$$\overset{2}{\cancel{4}}\mu(1-2\vartheta)\underbrace{\sin^2(\xi h/2)}_{\le 1} \le \overset{1}{\cancel{2}}$$

$\therefore$

$$\boxed{\mu(1-2\vartheta) \le 1/2}$$

So we get a <u>stability condition</u> if $1-2\vartheta>0$

that is if $\vartheta < 1/2$

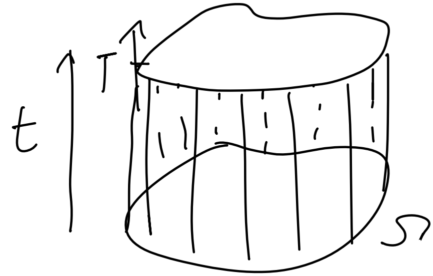
(again, $\vartheta = 0$ (EE) $\Rightarrow \mu \le 1/2$)

but indeed $\boxed{\vartheta \ge 1/2}$ ϑ-method is unconditionally stable ! (while max principle analysis we had conditional stability !)

Weak formulations

① heat equation $\Omega \subset \mathbb{R}^d$

$$\begin{cases} u_t - \boxed{\Delta u} = f(x,t) & \Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega \end{cases}$$



$$\underline{u} = \underline{u(t) = u(t,\cdot)} \in \underline{H_0^1(\Omega)}$$

$\forall t$, test eq. with $v \in H_0^1(\Omega) =: V$ and integrate w.r.t. Ω:

$$\int_\Omega u_t v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in V$$

$$\text{or} \quad \frac{d}{dt}\int_\Omega u v + \underbrace{a(u,v)}_{\text{spatial bilinear form}} = F(v) \quad \forall v \in V$$

$$\frac{d}{dt}\langle u,v\rangle + a(u,v) = \langle f,v\rangle$$

Q: $u \in$???

$$\boxed{u} : \left[\begin{array}{ccc} \boxed{I} & \longrightarrow & V \\ t & \longrightarrow & \boxed{u(t)} \end{array}\right]$$

$$u \in L^2(I; V) \cap \underbrace{C^0(\bar{I}; L^2(\Omega))}_{\text{to make sense of initial condition}}$$

$\underline{u \in L^2(I;V)}$ if $u \equiv u(t) \in V$ and $\int_I \|u\|_V^2 \, dt < +\infty$ $\quad \hat{=} \|u\|_{L^2-H^1}$

$u \in C^0(\bar{I}; L^2(\Omega))$ $\quad \forall \bar{t} \in \bar{I} \quad \lim_{t \to \bar{t}} \|u(t) - u(\bar{t})\|_{L^2(\Omega)} = 0$

norm: $\sup_{t \in \bar{I}} \|u(t)\|_{L^2(\Omega)}$

Right functional setting: $\left| L^2 - H^1 \right.$ and $\left| L^\infty - L^2 \right.$

$$f \in L^2(I; L^2(\Omega)) \quad ; \quad u_0 \in L^2(\Omega)$$

② General elliptic operator

$$Lu = -\sum_{ij} D_i(\underline{a_{ij}} D_j u) + D \cdot (\underline{b} u) + cu$$

elliptic , $a_{ij}, b_i, c \in L^\infty(\Omega)$

Weak form: Find $u \in L^2(I;V) \cap C^0(\bar{I};L^2(\Omega))$ with $V = H^1_0(\Omega)$, such that

$$(WP)\begin{cases} \frac{d}{dt}(u(t),v) + a(u(t),v) = F(t;v) \quad \forall v \in V \\ u(0) = u_0 \end{cases}$$

where $a(u,v) = \int_\Omega \left(a \nabla u \cdot \nabla v + \underline{b} u v + c u v \right)$

well-posedness

We shall require:

- a, F are continuous
- a is weakly coercive: $\exists\, \alpha_0 > 0$, $\lambda \geq 0$:

$$a(v,v) + \lambda \|v\|_0^2 \geq \alpha_0 \|v\|_V^2$$

Theorem (well-posedness): (Evans)

Assume $\mathcal{A}$ is (weakly) coercive in $V \times V$

$f \in L^2(I; L^2(\Omega))$, $u_0 \in L^2(\Omega)$

Then $\exists!\ u \in L^2(I; V) \cap C^0(\bar{I}; L^2(\Omega))$ of (WP)

and

$u_t \in L^2(I; V')$

$$\boxed{\max_{t \in \bar{I}} \|u(t)\|_0^2 + \alpha_0 \int_0^T \|u(t)\|_V^2 \leq \|u_0\|_0^2 + \frac{1}{\alpha_0} \int_0^T \|f(t)\|_0^2}$$

$\boxed{L^\infty - L^2}$ $\boxed{L^2 - H^1}$ $\boxed{L^2 - L^2}$

$\boxed{\ell^\infty - \ell^2}$ setting for FD analysis...

$\boxed{\text{No lecture on Wed. 10/12}}$