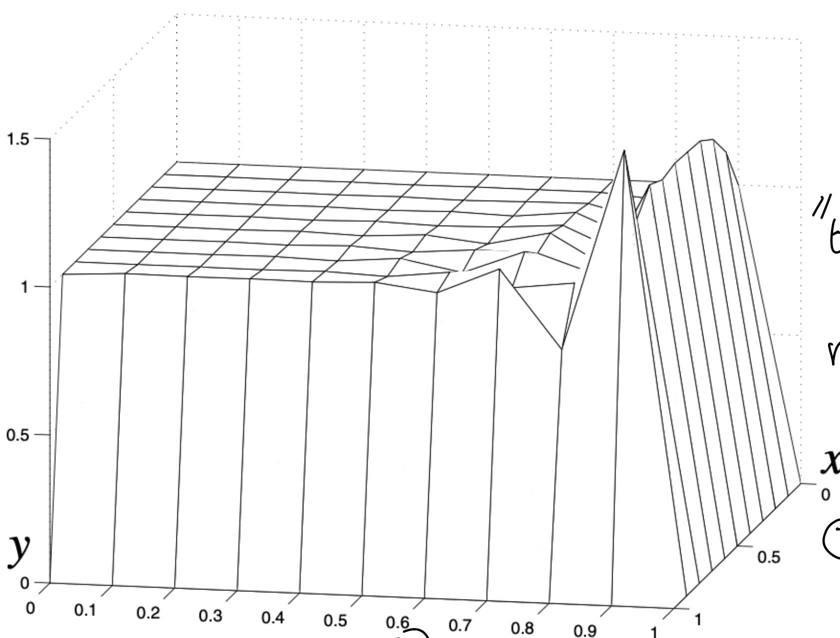
NMPDE/ATSC 2025
Lecture 14

$$\begin{cases} -a\,\Delta u + \underline{b}\cdot\nabla u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

$f \equiv 0 \qquad g(x,y) = \begin{cases} 1 & \text{at } x=0 \text{ or } y=0 \\ 0 & \text{at } x=1 \text{ or } y=1 \end{cases}$

$\underline{b} = (1,2) \qquad a = \varepsilon = 10^{-3}$

Solution by the SD-FEM $k=1$



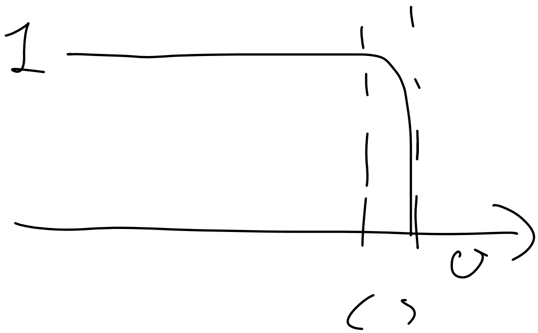
"boundary layer" region

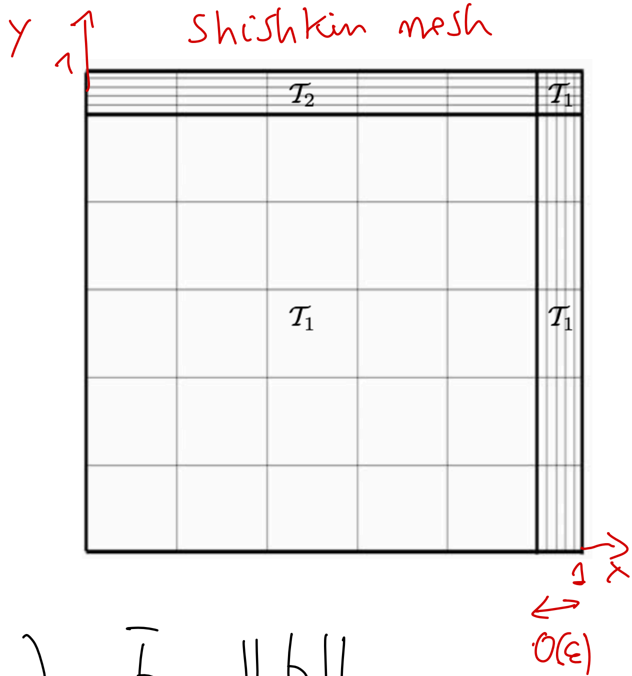
ε

$x = 1/2$

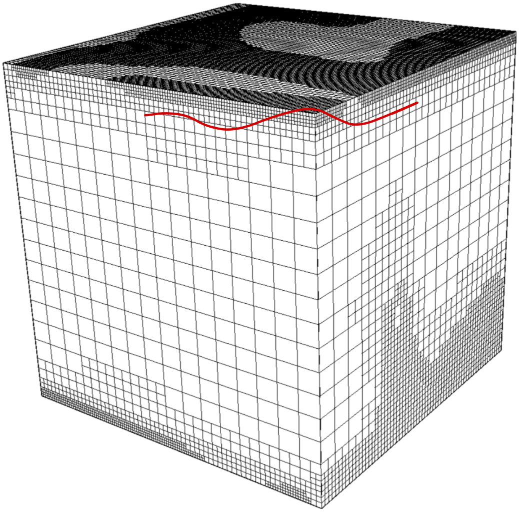
or

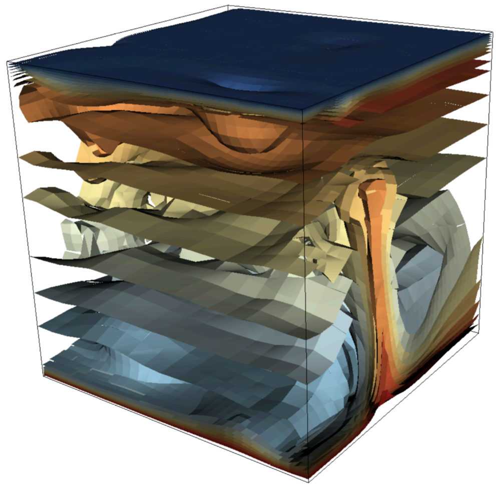
$y = 1/2$





$O(\bar{b}/a)$ $\quad \bar{b} = \|b\|_\infty$





Mesh adaptivity and a posteriori error estimators

$$\|u - u_h\| \le C h^k |u|_{k+1}$$

a priori error bound

exact

computed

$$\|u - u_h\| \le C \eta(u_h)$$

estimator

a posteriori error bound

$$\boxed{\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2}$$

η_T = local error estimator

used to drive

automatic adaptivity:

① SOLVE (Compute u_h on given mesh)

② ESTIMATE (" η_T $\forall T$)

③ MARK subset of elements T with larger η_T

④ REFINE the mesh based on ③

example (motivation):

Suppose want to solve the linear system

$$AU = F$$

and suppose U_0 is given approx. of U.

Can I estimate error $U - U_0$?

$$A(U-U_0) = AU - AU_0 = \underbrace{F - AU_0}_{\text{residual}}$$

$$U - U_0 = \underbrace{A^{-1}(F - AU_0)}_{\text{error representation}}$$

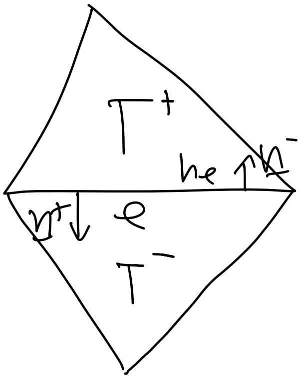
$$\|U - U_0\| \le \underbrace{\|A^{-1}\|}\,\|F - AU_0\|$$

estimating this norm
would provide a residual based
error estimator

example: on residual based a posteriori
error estimator for

$$\begin{cases} -\nabla\cdot(a\nabla u) = f & \text{in } \Omega \\ + \text{ b.c.} & \end{cases}$$

$$|u - u_h|_1 \le C\Big(\sum_{T\in\mathcal{T}_h} \underbrace{\Big(h_T^2\,\|\overbrace{f + \nabla\cdot a\nabla u_h}^{\text{elemental residual}}\|_{0,T}^2}_{\eta_T^2}$$



$$+\sum_{e\in\partial T} h_e \left\| a\, [\![\nabla u_h\cdot n]\!] \right\|_{0,e}^2 \Big)\Big)^{1/2}$$

edges of T

outward normal

$$[\![\nabla u_h\cdot n]\!]\Big|_e = \Big(\nabla u_h\cdot n^+\Big|_{T^+} + \nabla u_h\cdot n^-\Big|_{T^-}\Big)$$

$h_e = |e|$

Jump residual

Parabolic problems (LT, online notes, Q, MM)

$(\mathcal{L} u = f)$

Model problem:

- $\mathcal{L}$ a linear elliptic operator in $\mathbb{R}^d$.
- $t \in (0, T]$
- PDE: $u_t + \mathcal{L} u = f$

example: $\mathcal{L} u = -\Delta u \rightarrow u_t - \Delta u = f$

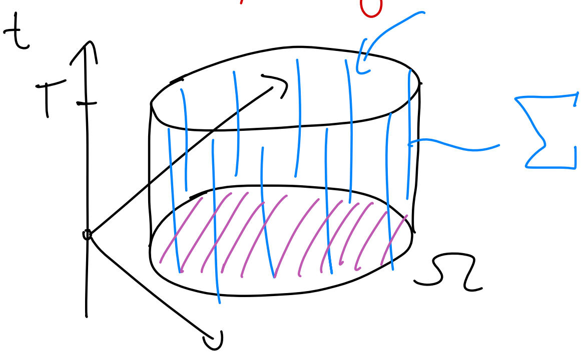
<u>heat equation</u>

Cauchy-Dirichlet (CD) problem: $\underset{\displaystyle I}{\overset{\shortparallel}{}}$

$$\begin{cases} u_t + \mathcal{L}u = f & \text{over } \Omega \times [0, T] \\ u(x,0) = u_0(x) & \text{on } \Omega \\ u(t,x) = g(t,x) & \text{on } \underbrace{\partial\Omega \times I}_{\Sigma} \end{cases}$$

initial cond.

boundary condition



Analysis of the heat equation

$$(*) \begin{cases} u_t - \Delta u = f & \Omega \times I \\ u(x,0) = u_0 & \text{in } \Omega \\ u(x,t) = 0 & \text{in } \Sigma \end{cases}$$

(LT)

Maximum principle: If u smooth function s.t. $u_t - \Delta u \leq 0$ in $\Omega \times I$ then u attains maximum on $\Omega \times \{0\} \cup \Sigma$.

Corollary: if u solves $(*)$, then
(stability)

$$\|u\|_{C(\bar{\Omega}\times\bar{I})} \leq \|u_0\|_{C(\bar{\Omega})} + \frac{r^2}{2d}\|f\|_{C(\bar{\Omega}\times\bar{I})}$$

$d = \dim$ of Ω

$r =$ radius of a ball containing $\bar{\Omega}\times\bar{I}$

Existence by providing solution:

$$u(\cdot, t) = u(t) = E(t)u_0 + \int_0^t \underline{E(t-s)}\, f(s)\, ds$$

where $E(t)$ solution of the homogeneous problem $(f=0)$

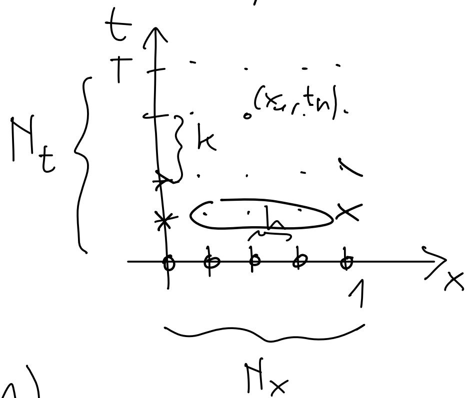
Duhamel principle

proof in LT.

FD Discretisation

$$\Omega = (0, 1)$$

$$\begin{cases} u_t - a\, u_{xx} = 0 & (0,1)\times(0,T] \\ u(x,0) = u_0(x) & \\ u(x,t) = 0 & x=0;\ x=1 \end{cases}$$



$h = 1/N_x$; $k = T/N_t$

$(x_i, t_n) = (hi, kn)$ $i = 0, \dots, N_x$

space-time grid $n = 0, \dots, N_t$

$u(x_i, t_n) = U_i^n$

Explicit Euler method

$$u_t(x,t) \approx \delta_{k,+}^{t}\, u(x,t) = \frac{u(x,t+k) - u(x,t)}{k}$$

$$u_{xx}(x,t) \approx \left(\delta_h^x\right)^2 u(x,t) \qquad a_i^n := a(x_i, t_n)$$

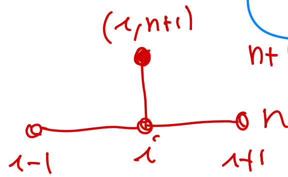
time stepping

$$\begin{cases} \dfrac{U_i^{n+1} - U_i^n}{k} - a_i^n \, \dfrac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0 & n = 1, \dots, N_t - 1 \\ U_0^{n+1} = 0 = U_{N_x}^{n+1} & \text{given } U_i^0 = u_0(x_i) \end{cases}$$

$$\begin{cases} U_i^{n+1} = -a_i^n \underbrace{\frac{k}{h^2}}_{\mu} U_{i+1}^n + \left(2a_i^n \underbrace{\frac{k}{h^2}}_{\mu} + 1\right) U_i^n - a_i^n \underbrace{\frac{k}{h^2}}_{\mu} U_{i-1}^n \\ U_0^{n+1} = 0 = U_{N_x}^{n+1} \end{cases}$$

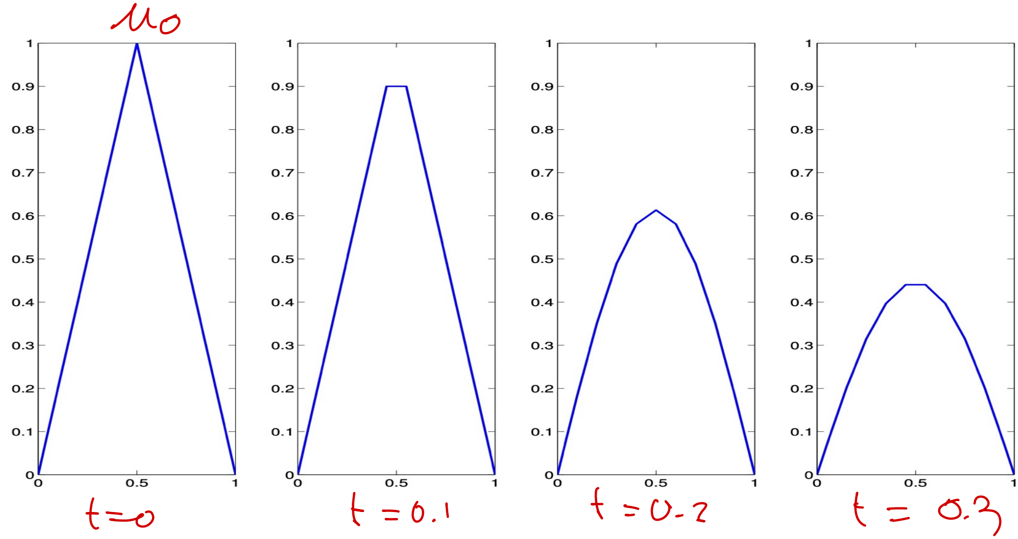
μ courant number

explicit, stencil



$a = 1$

h=0.05, k=0.00125 giving mu=0.5



h=0.05, k=0.0013 giving mu=0.52

instability