NMPDE/ATSC 2025
Lecture 18

# Fully discrete schemes

Model problem $\forall t \in (0,T] = I$, find $u(t) \in V$:

$$\begin{cases} \frac{d}{dt}(u(t), v) + A(u(t), v) = F(t, v) & \forall v \in V \\ u(0) = u_0 \end{cases}$$

$$\underset{\displaystyle k = T/N_t}{\underset{\downarrow}{\vartheta\text{-method}}} \quad + \quad \underset{\displaystyle V_h^k}{\underset{\downarrow}{\text{FEM}}}$$

$\forall n = 1, \dots, N_t$, find $u_h^n \in V_h^k$:

$$\begin{cases} \left(\frac{u_h^{n+1} - u_h^n}{k}, v_h\right) + A\left(\underbrace{\vartheta u_h^{n+1} + (1-\vartheta) u_h^n}, v_h\right) = F_\vartheta^{n+1}(v_h) & v_h \in V_h^k \\ u_h^0 = u_{0,h} \in V_h^k \end{cases}$$

$$\begin{cases} F_\vartheta^{n+1}(v_h) := \left(\vartheta f(t^{n+1}) + (1-\vartheta) f(t^n), v_h\right) \\ \partial_t u_h^{n+1} := \frac{u_h^{n+1} - u_h^n}{k} \\ u_\vartheta^{n+1} := \vartheta u_h^{n+1} + (1-\vartheta) u_h^n \end{cases}$$

so, equivalently

$$(\partial_t u_h^{n+1}, v_h) + A(u_\vartheta^{n+1}, v_h) = F_\vartheta^{n+1}(v_h)$$

e.g. $\begin{cases} \vartheta = 0 & \text{Exp. (Forward) Euler} \\ \vartheta = 1 & \text{Imp (Backward) Euler} \\ \vartheta = 1/2 & \text{CN} \end{cases}$

## Analysis

- Stability

example: Imp. Euler

- test the scheme with $v_h = u_h^{n+1}$ gives

$$\|u_h^{n+1}\|^2 - \underbrace{(u_h^n, u_h^{n+1})} + k\,\Omega(u_h^{n+1}, u_h^{n+1}) = k\left(f^{n+1}, u_h^{n+1}\right)$$

$$\Omega(u_h^{n+1}, u_h^{n+1}) > 0$$

$$(u_h^n, u_h^{n+1}) \le \|u_h^n\|_0 \|u_h^{n+1}\|_0$$

$$(f^{n+1}, u_h^{n+1}) \le \|f^{n+1}\|_0 \|u_h^{n+1}\|_0$$

$$\|u_h^{n+1}\|^{\cancel{2}} - \|u_h^n\|_0 \cancel{\|u_h^{n+1}\|} \le k\|f^{n+1}\|_0 \cancel{\|u_h^{n+1}\|_0}$$

$$\Rightarrow \|u_h^{n+1}\|_0 \le \underline{\|u_h^n\|_0} + k\|f^{n+1}\|_0$$

$$\Rightarrow \boxed{\|u_h^n\|_0 \le \|u_h^0\|_0 + k\sum_{j=1}^{n} \|f^j\|_0}$$

exercise: do same analysis for CN.
(Quarteroni - Valli, 1994)

**Theorem**: Assume $\Omega$ coercive, $\|f(t)\|_0$ bounded

Moreover, if $0 \le \vartheta < 1/2$ assume the $\mathcal{T}_h$ is quasi uniform

$$k\left(1 + C_I^2 h^{-2}\right) < \frac{2\,\alpha_0}{(1-2\vartheta)\gamma^2}$$

$C_I$: inverse est. const

Then $\|u_h^n\|_0 \le C_\vartheta \left(\|u_{0,h}\| + T \times \sup_{t\in[0,T]} \|f(t)\|_0\right)$

(Q, p. 135)

**Theorem** (convergence). Assume $u_0, f, u$ are sufficiently smooth, under the condition of stability theorem, $\forall n \geq 1$

$$\|u(t_n)-u_h^n\|_0^2 + 2\alpha_0 k \sum_{m=1}^{n} \|u(t^m)-u_h^m\|_V^2 \leq C(u_0, f, u) \cdot \left(h^{2k} + \begin{cases} k^2 & \theta \neq 1/2 \\ k^4 & \theta = 1/2 \end{cases}\right)$$

$\ell_\infty - L^2$ $\qquad$ $\ell_2 - H^1$

## higher-order time-steppings

① FD: use higher-order FD formulas

Example: $2^{nd}$ order Backward Differentiation Formula (BDF)

[ $\exists$ BDF formulas of any order ]

$$\bar{D}u^{n+1} := \bar{\partial} u^{n+1} + \frac{1}{2} k \bar{\partial}^2 u^{n+1}$$

$$\stackrel{!}{=} \frac{u^{n+1}-u^n}{k} + \frac{1}{2} k \bar{\partial}\left(\frac{u^{n+1}-u^n}{k}\right)$$

$$\stackrel{!}{=} \frac{u^{n+1}-u^n}{k} + \frac{1}{2} k \boxed{\frac{u^{n+1}-2u^n-u^{n-1}}{k}}$$

exercise:

$$\bar{D}u(t_{n+1}) = u_t(t_{n+1}) + O(k^2)$$

To use this starting from some $u_h^0$ requires also $u_h^1$ which can be provided by any 1-step method (e.g. $\vartheta$-method)

Theorem (Thomee): Under usual assumptions, using IE for 1st step and BDF of order 2 afterwords,

$$\|u(t_n)-u_h^n\| \le C h^{k+1}\left(\|u_0\|_{k+1} + \int_0^{t_n} |u_t|_{k+1}\right)$$
$$+ C k \int_0^k \|u_{tt}\|_0 + C k^2 \int_0^t \|u_{ttt}\|_0$$

## (2) Discontinuous Galerkin time-stepping

- use FE in space over the time steps

(• + continuous FE in space $(V \to V_h^k)$)

semidiscrete version

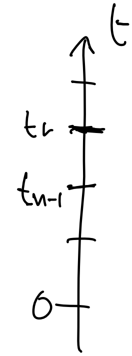
$$0 = t_0 < t_1 < \dots < t_n < \dots < T \qquad t_n = kn$$
$$N_t = T/k$$

$$J_n = (t_{n-1}, t_n) \qquad \text{Fix } q \ge 0$$

$$W_k^q = \{ v : [0,T] \to V : v(t)\big|_{J_n} \in \mathbb{P}_q(V), \ \forall n = 1, \dots, N_t$$

$\forall n,\quad v(t)\big|_{J_n} = \sum_{j=0}^{q} v_j \, t^j \quad ; \; v_j \in V$

* $v$ is <u>not</u> required to be cont. at $t_n$




$\forall n \quad v^n_{\pm} = \lim_{\tau \to 0^{\pm}} v(t_n \pm \tau)$

$[v^n] = v^n_+ - v^n_-$ "jump"

Derivation of DG method

$$\text{(1)} \quad \int_0^T (u_t, v) = -\int_0^T (u, v_t) + (u(T), \underset{0}{\underset{\parallel}{v(T)}}) - (u(0), v(0))$$

some $\boxed{\text{Smooth } v}$ such that $v(T) = 0$

$$\text{(2)} \Rightarrow -\int_0^T (u, v_t) + \int_0^T A(u, v) = (u_0, v(0)) + \int_0^T (f, v)\, dt$$

(3) Replace $u$ with $u_k \in W_k^q$ and integrate back:

$$-\int_0^T (u_k, v') = -\sum_{n=1}^{N_t} \int_{J_n} (u_k, v_t)$$

$$= \sum_{n=1}^{N_t} \int_{J_n} (u_k', v) - \sum_{n=1}^{N_t} (u_k, v)\Big|_{t=t_{n-1}}^{t=t^n}$$

$$= \int_0^T (u', v) + \sum_{n=1}^{N_t-1} ([u_k^n], v)\Big|_{t_n} + (u_{k,+}, v)\Big|_{t=0} - \cancel{\left(u^{N_t}_{\Delta t}, v^{N_t}\right)}\Big|_{t=T}$$

or $v^{N_t}=0$

(4) substitute back into (2)

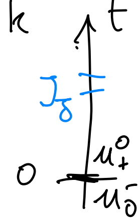
Find $u^k \in W_k^q$:

choice !

$$\begin{cases} \displaystyle\int_0^T \left[(u_k', v) + a(u_k, v)\right] dt + \sum_{n=1}^{N_t-1} \left([u_k]^n, v_-^n\right)\Big|_{t_n} \\ \left((u^k)^0_+, v^0_+\right) \\ = \left(u^0_{k,-}, v^0_+\right) + \displaystyle\int_0^T (f, v) \\ u^0_{k,-} = u_0 \end{cases}$$

PDE

$\forall v \in W_k^q$




DG-in-time method (monolithic version)

The test functions $v$ are also discontinuous in time $\Rightarrow$ we can test, for any $n$, with all and only test functions $v$ with support in $J_n$.

$\forall\, n = 1, \dots, N_t$ , find $u_{\Delta t}|_{J_n} \in \mathbb{P}^q(V)$ :

DG method in time (time-stepping version)

$$\begin{cases} \displaystyle\int_{J_n} \left[ \left(\frac{\partial}{\partial t} u_k, v\right) + a(u_k, v) \right] dt + \left(u_{k,+}^{n-1}, v_+^{n-1}\right) \\ \displaystyle\qquad = \left(u_{k,-}^{n-1}, v_+^{n-1}\right) + \int_{J_n} (f, v)\, dt \quad \forall v \in \mathbb{P}^q(V) \\ \text{with } u_{k,-}^{0} = u_0 \end{cases}$$

solve one time-step at a time,

starting from $(u^k)_-^0 = u_0$ !

Remark: the method is fully-consistent as it is satisfied by the exact solution.