NMPDE/ATSC 2025
Lecture 17

- Q
- LT $(k=1)$

Weak form: Find $u \in L^2(I;V) \cap C^0(\bar{I};L^2(\Omega))$ with $V = H^1_0(\Omega)$, such that

$$(WP)\begin{cases} \dfrac{d}{dt}\left(u(t),v\right) + a\left(u(t),v\right) = F(t;v) \quad \forall v \in V \\ \\ u(0) = u_0 \end{cases}$$

where $a(u,v) = \displaystyle\int_\Omega \left(\alpha \nabla u \cdot \nabla v + \underline{b}\, u\, v + c\, u\, v\right)$

Theorem (well-posedness): (Evans)

Assume a is (weakly) coercive in V ← yields appropriately modified bound

$f \in L^2(I;L^2(\Omega))$, $u_0 \in L^2(\Omega)$

Then $\exists!\, u \in L^2(I;V) \cap \underbrace{C^0(\bar{I};L^2(\Omega))}_{L^\infty - L^2}$ of (WP) and

$u_t \in L^2(I;V')$

$$(EE)\quad \boxed{\max_{t\in\bar{I}} \|u(t)\|_0^2 + \alpha_0 \int_0^T \|u(t)\|_V^2 \leq \|u_0\|_0^2 + \frac{1}{\alpha_0}\int_0^T \|f(t)\|_0^2}$$

energy estimate

Proof:

Existence: based on <u>Faedo-Galerkin method</u> (EVANS)

UNIQUENESS follows from (EE)

(EE) Test WP with $v = u(t)$ $\forall t$

$$(u_t, u) + A(u,u) = (f, u)$$

- $(u_t, u) = \int_\Omega u_t u = \frac{1}{2}\int_\Omega (u^2)_t = \frac{1}{2}\frac{d}{dt}\|u\|_0^2$
- $A(u,u) \geq \alpha_0 \|u\|_V^2$
- $(f, u) \leq \|f\|_0 \|u\|_0 \leq \|f\|_0 \|u\|_V$

$$\boxed{\text{Young's inequality } \quad ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall \varepsilon > 0}$$

Choosing $\varepsilon = \frac{1}{2\alpha_0}$ $\quad \leq \frac{1}{2\alpha_0}\|f\|_0^2 + \frac{\alpha_0}{2}\|u\|_V^2$

$$\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + \alpha_0\|u\|_V^2 \le \frac{1}{2\alpha_0}\|f\|_0^2 + \underbrace{\frac{\alpha_0}{2}\|u\|_V^2}$$

$$= \frac{\alpha_0}{2}\|u\|_V^2$$

$$\|u(t)\|_0^2 + \alpha_0\int_0^t \|u\|_V^2 \le \|u_0\|_0^2 + \frac{1}{2\alpha_0}\int_0^t \|f\|_0^2$$

$L^\infty - L^2$

$\square$

back to ϑ method, we had $(\mu = k/h^2)$

- ℓ_∞-norm : Stability if $\mu(1-\vartheta) \le 1/2$
  ($\Rightarrow$ only IE unconditional)

- von Neumann :
  stab. if $\mu(1-2\vartheta) \le 1/2$ for $\mu < 1/2$
  otherwise unconditionally stable (hence, including CN)

It is possible to derive alternative analysis of ϑ-method in $\ell_\infty - \ell_2$ (discrete ℓ_∞ in time, ℓ_2 in space) where

$$(V, W)_{0,h} = h \sum_{i=0}^{N_x} V_i W_i \qquad \forall V, W \in \mathbb{R}^{N_x+1}$$

$$\|V\|_{0,h} = (V, V)_{0,h}^{1/2}$$

Theorem: The Crank-Nicolson (CN) scheme is convergent $\forall \mu = k/h^2$ and

$$\max_n \|u^n - U^n\|_{0,h} = O(k^2, h^2) \qquad \square$$

Finite element semi-discretization in space

- Fix a mesh $\mathcal{T}_h$ of Ω
- Fix $V_h^k \subset V$ FE of order k
- FE space disc: $\forall t \in (0,T]$: $u_h(t) \in V_h^k$ s.t.

$$\frac{d}{dt}(u_h(t), v) + a(u_h(t), v) = (f, v) \quad \forall v \in V_h^k$$

with

$$u_h(0) = u_{0,h}$$

$\hookrightarrow$ approx. of u_0 (e.g. interpolant)

Well-posedness follows as for continuous pbm

$\exists ! \, u_h$ satisfying

$$\|u_h(t)\|_0^2 + \alpha_0 \int_0^t \|u(\tau)\|_V^2 \, d\tau \le \|u_{0,h}\|_0^2 + \int_0^t \|f(\tau)\|_0^2 \, d\tau \quad \forall t \le T$$

Remark (see later):

- Fix $\{\varphi_j\}$ basis for V_h^k

$$u_h(t) = \sum_j U_j(t) \varphi_j \; ; \; u_{0,h} = \sum_j U_j^0 \varphi_j$$

- Test FE scheme with $\varphi_i \; \forall i$:

$$\begin{cases} M \, U'(t) + A \, U(t) = F(t) \\ U(0) = U^0 \end{cases}$$

indeed $\displaystyle \frac{d}{dt} \int_\Omega \Big(\sum_j U_j(t) \varphi_j \Big) \varphi_i$

$$= \sum_j U_j' \underbrace{\int_\Omega \varphi_j \varphi_i}_{M_{ij}}$$

"mass" matrix

- Later: time disc. by e.g. FD: $M \dfrac{U^{n+1} - U^n}{k}$

not fully explicit even if explicit time stepping is used

EE in time

$$M \frac{U^{n+1} - U^n}{k} + A U^n = F^n$$

$\to$ still requires sol. w.r.t. M.

A priori analysis of FE semi discrete method

Key tool:

Def (Elliptic or Ritz projection) by Wheeler 1973

Given $v \in V$, the elliptic projection of v is the unique $R_h v \in V_h^k$ such that

$$A(R_h v, v_h) = \underbrace{A(v, v_h)}_{F(v_h)} \quad \forall v_h \in V_h^k$$

$R_h v$ is the FEM solution of an elliptic problem having v as the exact solution

$\Rightarrow$ by FEM a priori bounds we know

$$\left(|v - R_h v|_m \le C h^{(k+1)-m} |v|_{k+1} \right) \quad \begin{matrix} m = 0, 1 \\ \uparrow \quad \vert \\ L^2 \quad H^1 \end{matrix}$$

$m-1$

e.g. if Ω convex for $\underline{m=0}$.

Theorem (a priori analysis - semidiscrete scheme)

Let
- $\mathcal{T}_h$ family of shape-regular meshes of Ω
- a is cont. and coercive
- $u_0 \in H^{k+1}(\Omega)$, $\boxed{k} \ge 1$
- u : $\frac{\partial u}{\partial t} \in L^1(0, T; H^{k+1}(\Omega))$
- $V_h^k \subset V$ corresponding FE space

Then, $\forall t [0, T]$, for the semidiscrete solution u_h, we have

$$\|u(t) - u_h(t)\|_0 \le \|u_0 - u_{0,h}\|_0 + C h^{k+1} \left(\|u_0\|_{k+1} + \int_0^t \left\| \frac{\partial u}{\partial t}(z) \right\|_{k+1} dz \right)$$

<u>Proof:</u> (Q p. 130) split error

$$u_h(t)-u(t) = \underbrace{u_h(t)-R_h u(t)}_{\vartheta} + \underbrace{R_h u(t)-u(t)}_{\rho}$$

- <u>bound of ρ</u>

$$\|\rho\|_0 \underset{\substack{\text{elliptic projection}\\ \text{bound}}}{\le} C h^{k+1} \|u(t)\|_{k+1} \le$$

$$u(t) = u_0 + \int_0^t \frac{du}{dt}(\tau)\, d\tau$$

$$\le C h^{k+1} \left(\|u_0\|_{k+1} + \int_0^t \left\| \frac{du}{dt}(\tau) \right\|_{k+1} d\tau \right)$$

- <u>bound of ϑ</u> $\quad \forall v_h \in V_h^h$

error equation for ϑ

$$\boxed{(\vartheta_t, v_h) + a(\vartheta, v_h)}$$

$$= \underbrace{\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h)}_{\text{FE}} - \left(\frac{\partial R_h u}{\partial t}, v_h\right) - \boxed{\begin{matrix} a(u, v_h) \\ \shortparallel \\ a(R_h u, v_h) \end{matrix}}$$

ellip. proj.

$$= (f, v_h) - (R_h u_t, v_h) - a(u, v_h)$$

(IVP) $= \left(u_t - R_h u_t, v_h\right) = \boxed{-\left(\rho_t, v_h\right)}$

Test with $v_h = \vartheta = \underbrace{u_h(t) - R_h u(t)}_{\in V_h^k}$

$$\boxed{\frac{1}{2}\frac{d}{dt}\|\vartheta\|_0^2 + \alpha_0\|\vartheta\|_V^2 \leq \|\rho_t\|_0\|\vartheta\|_0}$$

$$\left(\frac{d}{dt}\|\vartheta\|_0\right)\|\vartheta\|_0$$

$$\Rightarrow \frac{d}{dt}\|\vartheta\|_0 \leq \|\rho_t\|_0$$

from which, integrating in time between 0 and t

$$\|\vartheta(t)\|_0 \leq \|\vartheta(0)\|_0 + \int_0^t \|\rho_t(\tau)\|_0 \, d\tau$$

Use again elliptic projection bound, this time on ρ_t

$$\|\vartheta(0)\|_0 = \|u_{0,h} - R_h u_0 \|$$
$$\le \|u_{0,h} - u_0\| + \|u_0 - R_h u_0\|$$
$$\le \|u_{0,h} - u_0\| + C h^{k+1} \|u_0\|_{k+1}$$

$$\int_0^t \|\rho_t(\tau)\|_0 \, d\tau \underset{\text{as before}}{\le} C h^{k+1} \left(\|u_0\|_{k+1} + \int_0^t \left\| \frac{du(\tau)}{dt} \right\|_{k+1} d\tau \right)$$

Theorem (L^2-H^1 error estimate). Under some assumptions

$$\|u - u_h\|_{L^2(I;V)} \le C h^k \left(\int_0^t |u(\tau)|^2_{k+1} + \int_0^t |\partial_t u(t)|^2_k \right) + \|u_0 - u_{0,h}\|_0$$

$(I = [0,t])$.

Proof: $u - u_h = u - \underbrace{I_h}_{\text{e.g. the interpolant}} u + I_h u - u_h = \xi(t) + \vartheta(t)$

$\forall v_h \in V_h^k$

$$\left(\frac{\partial}{\partial t}(u - u_h), v_h \right) + a(u - u_h, v_h) = 0$$

$$\left(\frac{\partial}{\partial t}\vartheta(t), v_h\right) + a(\vartheta(t), v_h) = \left(\frac{\partial}{\partial t}\eta(t), v_h\right) + a(\eta(t), v_h)$$

test with $v_h = \vartheta$

$$\frac{1}{2}\frac{d}{dt}\|\vartheta(t)\|_0^2 + \alpha_0 \|\vartheta(t)\|_V^2 \leq \frac{2}{\alpha_0}\left(\left\|\frac{\partial}{\partial t}\eta\right\|_0^2 + \|\eta\|_V^2\right) + \frac{\alpha}{2}\|\vartheta\|_1^2$$

For L^2-error analysis this term would give $\mathcal{O}(k) \Rightarrow$ sub-optimal !

$$\|\vartheta(t)\|_0^2 + \alpha_0 \int_0^t \|\vartheta(\tau)\|_V^2 \, d\tau \leq \|\vartheta(0)\|_0^2 + \frac{4}{\alpha}\int_0^t \underbrace{\left(\|\eta_t\|_0^2 + \|\eta\|_V^2\right)}_{\text{use optimal bounds for } u - I_h u}$$