NMPDE/ATSC 2025
Lecture 4

About strong and weak solutions

(Q, Ch. 2, 3)

Def: (dual space).

Given $(V, \|\cdot\|)$ normed space, its dual $V' = \mathcal{L}(V; \mathbb{R})$ linear and bounded functionals

$(|L(v)| \leq C \|v\|)$

V' is Banach w.r.t. operator norm

$L \in V'$ $\quad \|L\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|L(v)|}{\|v\|_V}$

Notation:

$\langle \cdot, \cdot \rangle : V' \times V \longrightarrow \mathbb{R}$ duality pairing

$$\langle L, v \rangle := L(v)$$

Theorem (Riesz-Fréchet representation): Let $(H, (\cdot,\cdot))$ Hilbert, $L \in H'$. Then

$$\exists! \; u \in H : L(v) = (u, v) \quad \forall v \in H$$

and $\|L\|_{H'} = \|u\|_H$

Example: $(L^2(\Omega), (\cdot,\cdot)_0)$

$$f \in (L^2(\Omega))' \Rightarrow u \in L^2(\Omega) : f(v) = (u, v)_0$$
$$\qquad = \langle f, v \rangle$$

exercise: $\|f\|_{(L^2)'} = \|u\|_{L^2}$

Distributions : Let

$\mathcal{D}(\Omega)$ as the space $C_0^\infty(\Omega)$ with Schwartz topology (= uniform convergence of every combination of partial derivatives)

$\mathcal{D}'(\Omega)$: space of distributions

$$T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$$

$$\varphi \longrightarrow \langle T, \varphi \rangle = T(\varphi)$$

example: note $L^p(\Omega) \subsetneq \mathcal{D}'(\Omega)$ $\forall p \in [1, +\infty]$

Indeed, e.g. for $p = 2$

given $f \in L^? (\Omega)$, we can identify $g \in \mathcal{D}'(\Omega)$ by

$$\varphi \longrightarrow g(\varphi) := \int_\Omega f \varphi$$

bounded by Schwartz

But, δ_0 = Dirac's delta is in $\mathcal{D}'(\Omega)$ but not in $L^p(\Omega)$

$$\delta(v) = v(0)$$

<u>weak derivative</u>: For $T \in \mathcal{D}'(\Omega)$

$DT \in \mathcal{D}'(\Omega)$ "<u>weak derivative</u>":

$$\langle DT, \varphi \rangle = - \langle T, \varphi' \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

and, for any α multiindex

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

<u>Sobolev spaces</u> $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1, \infty]$

$$W^{k,p}(\Omega) := \{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), \forall |\alpha| \leq k \}$$

$$\left[p = 2 \Rightarrow W^{k,p}(\Omega) := H^k(\Omega) \right]$$

↳ Hilbert w.r.t.

$$(v, w)_k = \sum_{|\alpha| \leq k} (D^\alpha v, D^\alpha w)$$

e.g. H^1 $\quad (u, v)_1 = (u, v)_0 + (\nabla u, \nabla v)_0$

$$\left(\|v\|_1^2 = \|v\|_0^2 + \|\nabla v\|_0^2 \right)$$

Theorem: (Poincaré ineq.): Let $\Omega \subset \mathbb{R}^d$ open and bounded, with Lipschitz boundary $\partial\Omega$, let $\Gamma \subset \partial\Omega$, and $H^1_\Gamma(\Omega) := \{ v \in H^1(\Omega) : v|_\Gamma = 0 \}$.

↳ non-zero measure

Then $\exists\, C_\Omega > 0$:

$$\|v\|_0^2 \leq C_\Omega \|\nabla v\|_0^2$$

$\left[\|v\|_0 = \|v\|_{L^2(\Omega)} \right]$

$$\Rightarrow \left(H^1_\Gamma(\Omega), (\cdot,\cdot)_{H^1_\Gamma} \right) \quad \text{i.p. space}$$

with $(u, v)_{H^1_\Gamma} = \int_\Omega \nabla u \cdot \nabla v$

$$\|v\|_{H^1_\Gamma} = \|\nabla v\|_0 = |v|_1$$

General linear elliptic equations

- General 2nd order diff. operator in $\mathbb{R}^d$

$$\underbrace{\mathcal{L}u = \underbrace{-\sum_{i,j=1}^{d} D_i(a_{ij}(x) D_j u)}_{\text{diffusion}} + \underbrace{\sum_{i=1}^{d} D_i(b_i(x) u)}_{\text{advection}} + \underbrace{c(x) u}_{\text{reaction}}}_{\text{"conservative" form}}$$

- Associated bilinear form

$$a(u,v) = \int_\Omega \sum_{ij} a_{ij} D_j u \cdot D_i v - \sum_i \int_\Omega b_i u D_i v + \int_\Omega c u v$$

$$A \nabla u \cdot \nabla v \qquad u\, \underline{b} \cdot \nabla v$$

valid a_{ij}, b_i, $c \in L^\infty(\Omega)$ and $u, v \in H^1(\Omega)$

- Elliptic if the matrix

$$A(x) = \{a_{ij}(x)\}_{ij}$$

is positive definite a.e. in Ω.

That is, $(A(x)\xi, \xi) = \sum_{ij} a_{ij}(x) \xi_i \xi_j > 0$.

$\forall \xi \in \mathbb{R}^d$

- Coercivity: $a : V \times V \to \mathbb{R}$ is coercive if $\exists\, \alpha_0 > 0$:

$$a(v,v) \geq \alpha_0 \|v\| \quad \forall v \in V$$

$\Rightarrow$ An second order op. is elliptic if it is coercive in its 2^{nd} order term.

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weak solutions

Consider b.v.p.

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\to$ • $V = H_0^1(\Omega)$

• $f \in L^2(\Omega) \to F(v) = \int_\Omega f v = \langle f, v \rangle$

• weak problem: Find $u \in V$:

$$a(u,v) = F(v) \quad \forall v \in V$$

weak sol.

strong solutions: $u \in \mathcal{D}'(\Omega)$:

$$\mathcal{L}u = f \quad \text{in } \mathcal{D}'(\Omega)$$

Therem: $u \in V$ is weak solution iff u is strong solution in $\mathcal{D}'(\Omega)$

Proof: "$\Rightarrow$" ($\mathcal{L}u = -\Delta u$ Poisson)

Assume $u \in V$ is weak sol., $\forall v \in C_0^\infty(\Omega)$

$$= a(u,v) = \int_\Omega f v = \langle f, v\rangle$$

$$= \int_\Omega \nabla u \cdot \nabla v$$

$$= -\int_\Omega \Delta u \, v = \langle -\Delta u, v\rangle$$

$$\Rightarrow \quad -\Delta u = f \quad \text{in } \mathcal{D}'(\Omega)$$

"$\Leftarrow$" same calculation + density of $\mathcal{D}(\Omega)$ in V.

(Q): When u is also classical solution?

Again, consider Poisson problem $\left(\begin{cases} -\Delta u = f \\ u = 0 \end{cases}\right)$

if $\begin{cases} f \in L^2(\Omega) \\ \partial\Omega \text{ is } C^2(\Omega) \text{ or is a convex Polygon} \end{cases}$ then a weak solution $u \in H^2(\Omega)$.

Regularity for elliptic pbms
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so $-\Delta u - f \in L^2(\Omega)$ $\therefore -\Delta u = f$ $L^2(\Omega)$
(a.e. in Ω)

$\not\Rightarrow$ classical sol.

For that, need $f \in H^2(\Omega)$ and $\partial\Omega$ is C^4 $\Omega \subset \mathbb{R}^2$

then $u \in H^4(\Omega) \hookrightarrow C^2(\Omega)$

- if $\Omega \subset \mathbb{R}^d$ smooth/polygonal then

  $H^k \hookrightarrow C(\bar{\Omega})$ if $k > d/2$ and

  $\exists C_\Omega : \|v\|_C \le C_\Omega \|v\|_k$ (Sobolev ineq.)

$\longrightarrow$ weak notion of sol. more general

$\longrightarrow$ proving well-posedness for weak problem is easier (particularly for nonlinear problems!)

Lemma (Lax-Milgram): Let

- $(V, (\cdot,\cdot))$ Hilbert, $\|\cdot\|$ its assoc. norm
- $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ bilinear
  - ☆ continuous: $\exists \gamma : |A(w,v)| \leq \gamma \|w\| \|v\|$
  - ☆ coercive: $\exists \alpha_0 : A(v,v) \geq \alpha_0 \|v\|^2$ $\forall v, w \in V$
- $F(\cdot) : V \longrightarrow \mathbb{R}$ in V', so linear;

$$|F(v)| \leq \underbrace{c_0}_{\|F\|_{V'}} \|v\|_V$$

Then $\exists!\, u \in V : A(u,v) = F(v) \quad \forall v \in V$

and $\|u\| \leq \frac{1}{\alpha_0} \|F\|_{V'}$

Well-posedness of the general elliptic problem

<u>check coercivity</u>

$$A(v,v) = \int_\Omega A \nabla v \cdot \nabla v - \int_\Omega (\underline{b} \cdot \nabla v) v + \int_\Omega c\, v\, v$$

$$\text{ellipticity} \geq \alpha \|\nabla v\|^2 - \frac{1}{2} \int_\Omega \underline{b} \cdot \nabla v^2 + \int_\Omega c\, v^2$$

$$-\frac{1}{2}\int_\Omega \underline{b}\cdot\nabla v^2 = \frac{1}{2}\int_\Omega (\nabla\cdot b)\, v^2$$

$$= \alpha\,\|\nabla v\|^2 + \int_\Omega \underbrace{\left(c+\tfrac{1}{2}\nabla\cdot b\right)}\, v^2$$

Assume $\underline{c+\frac{1}{2}\nabla\cdot b \geq 0}$

$$\Rightarrow \;\geq \alpha\,\|\nabla v\|^2 \qquad (*) \qquad \|v\|^2_{H^1_0}$$

Poincaré : $\frac{1}{C_\Omega}\|v\|^2 \leq \|\nabla v\|^2 \qquad (**)$

$$A(v,v) \geq \alpha\,\|\nabla v\|^2 \geq \frac{\alpha}{C_\Omega}\|v\|^2 \qquad (***)$$

$$(*) + (***) \Rightarrow \qquad \frac{1+C_\Omega}{\alpha}\,A(v,v) \geq \|v\|_1^2$$

$\Rightarrow$ coercivity with $\alpha_0 := \frac{\alpha}{1+C_\Omega}$ $\square$

$\Rightarrow$ well-posedness (LM) if $c+\frac{1}{2}\nabla\cdot b \geq 0$