LT
MM
online notes

NMPDE/ATSC 2025
Lecture 15

1 D heat equation:

$$\begin{cases} u_t - u_{xx} = 0 & (0,\pi) \times (0,T] \\ u(x,0) = u_0(x) & x \in (0,\pi) \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

By separation of variables:

$$u(x,t) = \sum_{j=1}^{\infty} \hat{u}_j \, \underbrace{e^{-j^2 t}}_{\text{amplitude of}} \, \underbrace{\sin(jx)}_{\text{sine wave}} \quad \text{mode}$$

frequency

$$\hat{u}_j = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u_0(x) \sin(jx)\, dx \quad , \; j = 1, \ldots$$

- $e^{-j^2 t}$ small for $j^2 t$ large $\Rightarrow$ mode $\sin jx$ dumped (relevant only at time scale $O(j^{-2})$)
- u is smooth away from $t = 0$ (while a $t \to 0^+$ u may be rough)

- – initial transient for small t
  – smooth u after

- $\|u(t)\|_0 \leq \|u_0\|_0$

Explicit Euler

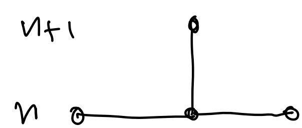
$$U_i^0 = u_0(x_i)$$

$$\begin{cases} U_0^{n+1} = 0 \\ \dfrac{U_i^{n+1} - U_i^n}{k} - \dfrac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} = 0 \\ U_H^{n+1} = 0 \end{cases}$$

$$\boxed{\mu = k/h^2}$$

Courant number of the EE scheme

$$U_i^{n+1} = \mu U_{i+1}^n + (1-2\mu)U_i^n + \mu U_{i-1}^n$$



Analysis

- $U^n = (U_0^n, \ldots, U_{M_x}^n)$
- $\|U^n\|_{\infty,h} = \max_{0 \le i \le M_x} |U_i^n|$
- $e_i^n = u_i^n - U_i^n \qquad u_i^n := u(x_i, t_n)$

  $E^n = \max_i \|e_i^n\|$
- solution operator $E_k U^n = U^{n+1}$

  $(\Rightarrow \underline{U^n = E_k^n U^0})$
- Truncation error:

  $$T(x,t) = \int_{k,t}^{t} u(x,t) - \left(\int_h^x\right)^? u(x,t)$$

  $$T_i^n = T(x_i, t_n) \qquad T^n = \max_i T_i^n$$

Stability: The EE method is

$\boxed{\text{stable}} \iff \mu \leq 1/2$.

$$\boxed{\|U^n\|_{\infty,h} \leq \|U^0\|_{\infty,h}}$$

(max norm stability)

Proof: "$\Leftarrow$" assume $\mu \leq 1/2$

$$|U_i^{n+1}| \leq \mu |U_{i+1}^n| + |1-2\mu| \, |U_i^n| + \mu |U_{i-1}^n|$$

$|1-2\mu| = 1-2\mu$ positive!

$$= \underbrace{\mu |U_{i+1}^n| + (1-2\mu)|U_i^n| + \mu |U_{i-1}^n|}_{\text{and sum of coefficients } = 1}$$

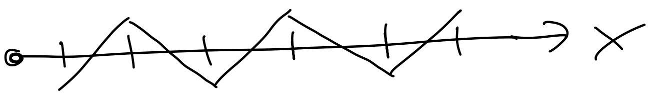
$$\leq \|U^n\|_{\infty,h} \qquad \forall i$$

$$\|U^{n+1}\|_{\infty,h} \le \|U^n\|_{\infty,h} \le \dots \le \|U^0\|_{\infty,h}$$

$\|U^{n+1}\|_{\infty,h} = E_k U^n$

"$\Rightarrow$" let fix $V_i^0 = (-1)^i \, \varepsilon$

$\boxed{\text{saw tooth}}$



$$\|U^0\|_{\infty,h} = \varepsilon$$

$$U_i^1 = \mu U_{i+1}^0 + (1-2\mu) U_i^0 + \mu U_{i-1}^0$$

$$= \left(\mu (-1)^{i+1} + (1-2\mu)(-1)^i + \mu (-1)^{i-1}\right) \varepsilon$$

$$= (-1)^i \left(1 - 4\mu\right) \varepsilon$$

$\rightarrow \quad U_i^n = (-1)^i (1-4\mu)^n \varepsilon$

$\rightarrow \quad \|U^n\|_{\infty,h} = |\underbrace{1-4\mu}_{\sim -1}|^n \varepsilon$

if $\mu > 1/2 \Rightarrow \|U^n\|_{\infty,h} \xrightarrow{n \to +\infty} +\infty$

i.e. discrete solution blows up for any $\varepsilon > 0$!

Von Neumann stability analysis:

[exact sol: $u(x,t) = \sum_{j=1}^{\infty} \hat{u}_j^0 \, e^{-\lambda t} \, \varphi_j(x)$ $\quad \lambda = j^2$

modes ($\sin jx$)

similar representation

complex unity

$$U_i^n = \sum_j a_j \, \lambda(j)^n \, e^{ij(ih)}$$

$\boxed{\text{finite sum}}$

modes that discrete scheme can carry

apply scheme to any of the modes

$$U_i^0 = e^{ij(ih)}$$

$$U_i^1 = \mu \, e^{ij(i+1)h} + (1-2\mu) e^{ijih} + \mu \, e^{ij(i-1)h}$$

$$= \left(\mu e^{ijh} + 1 - 2\mu + \mu e^{-ijh}\right) e^{ijih}$$

$$= \left(e^{ijh/2} - e^{-ijh/2}\right)^2 = -4 \sin^2\left(\frac{jh}{2}\right)$$

$$\stackrel{!}{=} \underbrace{\left(1 - 4\sin^2\left(\frac{\xi h}{2}\right)\right)}_{\lambda(\xi)} e^{i\xi i h}$$

Stability requires $|\lambda(\xi)| \le 1$

$$\therefore \quad -1 \le 1 - 4\mu \sin^2\left(\frac{\xi h}{2}\right) \le 1$$

$$\therefore \quad 0 \le \mu \underbrace{\sin^2\left(\frac{\xi h}{2}\right)}_{\le 1} \le 1/2$$

$$\Rightarrow \boxed{\mu \le 1/2}$$

as before.

Consistency: If u regular enough

$$|T_i^n| \le \frac{k}{2} M_{tt} + \frac{h^2}{12} M_{xxxx}$$

$$= \frac{k}{2}\left(M_{tt} + \frac{1}{6\mu} M_{xxxx}\right)$$

where $M_{tt} = \max_{\Omega \times [0,T]} |u_{tt}|$; $M_{xxxx} = \max |u_{xxxx}|$

$\forall\ i = 1, \dots, N_x - 1$; $n = 1, \dots, N_t$

Proof: $\dfrac{u_i^{n+1} - u_i^n}{k} = (u_t)_i^n + \dfrac{k}{2} u_{tt}(x_i, \rho_n)$ ①

$\exists\ \rho_n$

$\dfrac{u_{i+1}^n - u_i^n + u_{i-1}^n}{h^2} = (u_{xx})_i^n + \dfrac{h^2}{24} \big(u_{xxxx}(\xi_i, t_n) + u_{xxxx}(\eta_i, t_n) \big)$ ②

$\xi_i \in (x_i, x_{i+1})$ $\quad \xi_i \in (t_{i-1}, t_i)$

① − ② $= \underbrace{(u_t)_i^n - (u_{xx})_i^n}_{=0} + \dfrac{k}{2} u_{tt}(x_i, \rho_n) - \dfrac{h^2}{24}\big(u_{xxxx}(\xi_i, t_n) + u_{xxxx}(\eta_i, t_n) \big)$

$\Rightarrow O(k, h^2)$

Theorem: If $\boxed{\mu \leq 1/2}$ then EE is convergent and

$$E^n \leq T \left(\frac{k}{2} M_{tt} + \frac{h^2}{12} M_{xxxx} \right)$$

proof: exercise.

Remark: $\mu = k/h^2 \leq 1/2$ refinements should satisfies this but in principle k, h reduced independently

in particular if take $k = O(h^2)$

then method $O(k)$.

CFL condition

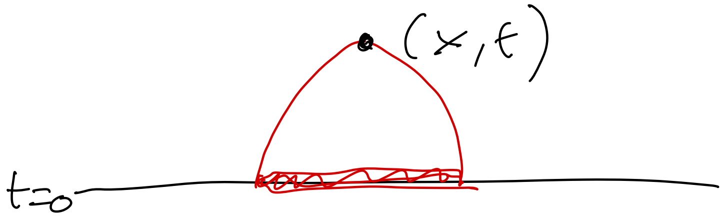
English IBM Journal 1967

((Courant, Friedrichs, Levy, 1928))

Necessary condition for convergence:
the domain of dependence (DoD) of the FD scheme must lay within the DoD of the PDE

DoD of PDE: $u_t = F(u, x, t)$ at (x, t)
the DoD is the set $X(x, t)$ of points where the initial data affects the solution

at (x,t)

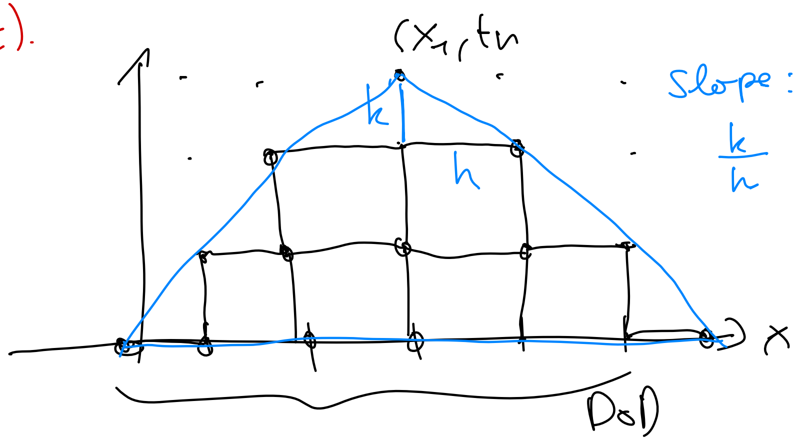


- for parabolic eq. the DoD is the whole real line $\longrightarrow$ ∞ speed of propagation

(later: hyperbolic problems, instead, are characterised by finite speed of propagation)

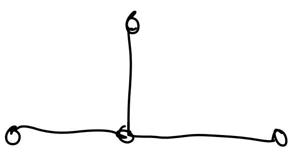
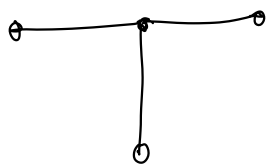
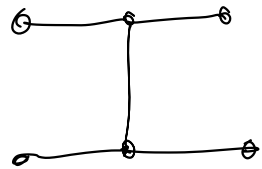
DoD for FD scheme: fix k, at (x,t) set $X_k(x,t)$ of grid points x_i such that U_i^0 is used in computation of FD solution at (x,t).

EE:



Proposition (CFL): A necessary condition for convergence of explicit FD schemes applied to parabolic PDEs is that $k = o(h)$ as $k \to 0$.

θ- method

EE	IE	CRANK-NICOLSON CN
 $n+1$, n		
$\delta^t_{k,+} U^n_i - (\delta^x_h)^2 U^n_i$	$\delta^t_{k,-} U^{n+1}_i - (\delta^x_h)^2 U^{n+1}_i$	average: $\dfrac{EE + IE}{2}$

$$\theta \in [0, 1]$$

$$\frac{U^{n+1}_i - U^n_i}{k} - (1-\theta)(\delta^x_h)^2 U^n_i - \theta (\delta^x_h)^2 U^{n+1}_i = 0$$

with $\mu = k/h^2$

$$-\mu\vartheta U_{j+1}^{n+1} + (1+2\mu\vartheta) U_j^{n+1} - \mu\vartheta U_{j-1}^{n+1}$$

$$= \mu(1-\vartheta) U_{j+1}^{n} + (1-2\mu(1-\vartheta)) U_j^{n} + \mu(1-\vartheta) U_{j-1}^{n}$$

matrix form:

$A = \text{tridiag}(-\mu\vartheta;\ 1+2\mu\vartheta;\ -\mu\vartheta)$

$B = \text{tridiag}(\mu(1-\vartheta);\ (1-2\mu(1-\vartheta));\ \mu(1-\vartheta))$

ϑ-method requires sol. of

$$\rightarrow \boxed{AU^{n+1} = BU^n}$$

unless $\vartheta = 0$ (EE) $\rightarrow A$ diag

$\Rightarrow$ explicit.

Note: (by Thomas algorithm) the solution of the tridiagonal system takes twice as many operations than EE iteration