NUMERICAL METHODS FOR PDES
Lecture 2

LT

Lemma (stability): For any $V = \{V_i\}_{i=0}^{M}$

$$|V|_{h,\infty} \leq \max\{|V_0|, |V_M|\} + \frac{1}{8} |\mathcal{L}_h V|_{h,\infty}$$

$$|V|_{h,\infty} = \max_i |V_i|$$

Proof: $\omega(x) = x - x^2 \rightarrow \begin{cases} = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 \leq \frac{1}{4} \\ \mathcal{L}_h \omega(x) = -\omega'' = 2 \end{cases}$

$$\tilde{V}_i^{\pm} = \pm V_i - \frac{1}{2} |\mathcal{L}_h V|_{h,\infty} W_i$$

$$W_i := \omega(x_i)$$

$$\mathcal{L}_h \tilde{V}_i^{\pm} = \pm \mathcal{L}_h V_i - \cancel{\frac{1}{2}} |\mathcal{L}_h V|_{h,\infty} \underbrace{\mathcal{L}_h W_i}_{\cancel{2}} \leq 0$$

by (DMP)

$$\Rightarrow \tilde{V}_i^{\pm} \leq \max\{\pm V_0, \pm V_M\} \leq \max\{|V_0|, |V_M|\}$$

$$\Rightarrow \pm V_i \leq \max\{|V_0|, |V_M|\} + \frac{1}{2} |\mathcal{L}_h V|_{h,\infty} \underbrace{W_i}_{\leq 1/4}$$

$$|V_i| \leq \quad '' \quad + \frac{1}{8} |\mathcal{L}_h V|_{h,\infty}$$

Consistency

Truncation error : $T(x) = \mathcal{L}u(x) - \mathcal{L}_h u(x)$

$$\left(f_i := f(x_i)\right) \qquad T_i := T(x_i)$$

$$= f(x) - \mathcal{L}_h u(x)$$

Lemma : $u \in C^4([0,1])$, $h > 0$, $x \in [h, 1-h]$

$$u''(x) - \delta_h^2 u(x) = \frac{h^2}{24}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right)$$

error identity

with $\xi_1 \in [x-h, x]$; $\xi_2 \in [x, x+h]$

$\rightarrow$ second order accurate

$$|u''(x) - \delta_h^2 u(x)| \leq \frac{h^2}{12} \|u^{(4)}\|_{C([0,1])}$$

error bound

Thus, $T(x) = f(x) - \mathcal{L}_h u(x)$

$$\overset{!}{=} f(x) + \delta_h^2 u(x)$$

$$\overset{!}{\underset{\text{err. identity}}{=}} \underbrace{\cancel{f(x)} + \cancel{u''(x)}}_{\text{by PDE}} - \frac{h^2}{24}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right)$$

$$\Rightarrow \quad |T(x)| \leq h^2/12 \; \|u^{(4)}\|_{C([0,1])}$$

$$\Rightarrow \quad |T(x)| \underset{h\to 0}{\longrightarrow} 0 \quad \to \text{consistent}$$

(stability + consistency $\Rightarrow$ convergence)

Theorem (error estimate ∞-norm)

$\forall\, h>0$ the unique discrete solution U satisfies

$$\max_i |\underbrace{u(x_i)-U_i}_{e_i}| \le \frac{h^2}{96} \|u^{(4)}\|_{C([0,1])}$$

$\to$ FD scheme is 2nd order accurate

<u>Proof</u>:
$$
\begin{aligned}
\mathcal{L}_h e_i &= \mathcal{L}_h u(x_i) - \mathcal{L}_h U_i \\
&\overset{\text{FD scheme}}{=} \mathcal{L}_h u(x_i) - f_i \\
&\overset{\text{PDE}}{=} \mathcal{L}_h u(x_i) - \mathcal{L} u(x_i) \\
&= T_i
\end{aligned}
$$

consistency: $|\mathcal{L}_h e_i| = |T_i| \le \dfrac{h^2}{12} \|u^{(4)}\|_C$

Stability: $|e_i| \le \max\{\underbrace{|e_0|}_{=0}, \underbrace{|e_{N+1}|}_{=0}\} + \frac{1}{8} \|\mathcal{L}_h e_i\|_{h,\infty}$

$$\max |e_i| \le \frac{1}{8} \|L_h e\|_{h,\infty} \le \frac{h^2}{96} \|u^{(4)}\|$$

__Remarks:__

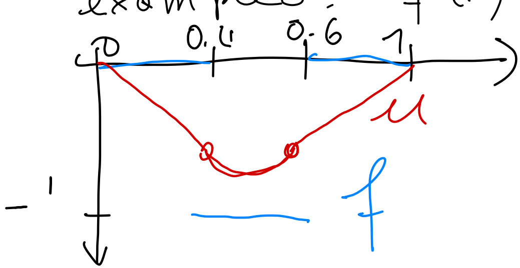
1) to apply the scheme, requires $f \in C(0,1)$ + analysis requires $u \in C^4(0,1)$
2) also we are actually only approximating the solution function at the grid points

→ Question: right functional setting

$\boxed{\text{Q sec. 3.2}}$

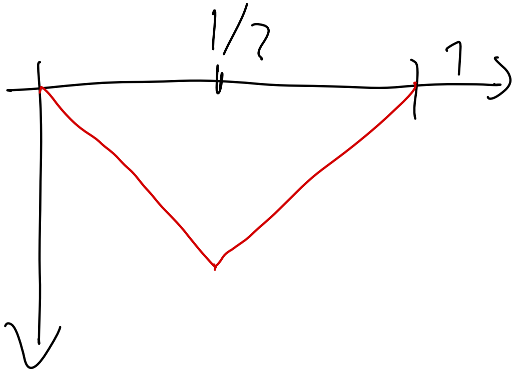
* Poisson b.v.p. models equilibrium of an elastic string
  - fixed at the end points
  - subject to transverse force f

examples: $f(x) = -\chi_{[0.4,0.6]}(x)$



$u \in C^1([0,1])$

$f(x) = -\delta_{1/2}(x)$ Dirac delta



$u \in C^0([0,1])$

Dirichlet principle: elastic string minimizes potential energy:

$$J(v) = \frac{1}{2}\int_0^1 (v')^2\,dx - \int_0^1 f v\,dx$$

note: this "energy principle" does not involve the second derivative

The minimisation problem

Find $u \in ?$: $J(u) \leq J(v)$ $\forall v \in ?$

is tackled by calculus of variations

If u minimizes, then the

"first variation" $= \frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = 0$

exercise

$= \int_0^1 u'v'\,dx - \int_0^1 f v\,dx \quad \forall v \in ?$

$\to$ need to fix the correct functional setting!

To do this (in more general setting!) start again from the PDE:

$-u'' = f \qquad \times v \in ?$

$-u''v = fv$

$-\int_0^1 u''v\,dx = \int_0^1 f v\,dx$

$\forall v$ weak problem

TEST

$\int_0 u'v'\,dx = \int_0^1 f v\,dx$

$\left(\begin{array}{l} u, v(0)=0 \\ u, v(1)=0 \end{array} \right)$

It remains to fix $V (\ni u, v)$

- $V = \{ v \in C^1([0,1]) : v(0) = 0 = v(1) \}$

weak problem makes sense <u>but</u>

1) $f = \delta_{1/2}(x) \rightarrow u \notin C^1$

2) $f \in C^0([0,1])$ $\exists!$ not guaranteed

consider $(V, (\cdot,\cdot)_1)$

$$\underbrace{(u,v)_1}_{\text{inner product}} := \int_0^1 u' v' \, dx$$

$$|v|_1 := (v,v)^{1/2}$$

V is <u>not</u> complete w.r.t. this norm require some new spaces...

Def: $L^2(0,1) = \left\{ v:(0,1)\to\mathbb{R} : \|v\|_0 := \left(\int_0^1 |v|^2 dx\right)^{1/2} < +\infty \right\}$

$$\to (u,v)_0 = \int_0^1 uv \, dx$$

$$H^1(0,1) = \left\{ v \in L^2(0,1) : v' \in L^2(0,1) \right\}$$

$$H^1_0(0,1) = \left\{ v \in H^1(0,1) : v(0) = 0 = v(1) \right\}$$

Th: $\left(H^1_0(0,1) \, ; \, (\cdot,\cdot)_1 \right)$ is Hilbert $\notin \mathcal{C}^1$

$|\cdot|_1$ is a norm for $H^1_0(0,1)$

$\|v\|_1 = \left(\|v\|_0^2 + |v|_1^2 \right)^{1/2}$ is also a norm (also for just $H^1(0,1)$)

$|v|_1^2 = \|v'\|_0^2$

Def (weak formulation):

Let $f \in V' :=$ dual of $H^1(0,1)$

hence write duality or $\langle f, v \rangle = F(v) \quad \forall v \in V := H^1_0(0,1)$

for example, if $f \in L^2(0,1)$

$$= (f, v)_0 = \int_0^1 f v$$

(WF) Find $u \in V$: $\int_0^1 u' v' = \langle f, v \rangle$

(WP)

<u>stability :</u> if $u \in V (= H^1_0(0,1))$

solves (WP), then

(energy method)

$$|u|_1^{\cancel{2}} = \int_0^1 (u')^2 = \langle f, u \rangle \le \|f\|_{V'} \cancel{|u|_1}$$

$$\Rightarrow \boxed{|u|_1 \le \|f\|_{V'}}$$

$$\left(f \in L^2 \Rightarrow \quad \|f\|_0 \right)$$

Theorem: u solves (WP) $\Longleftrightarrow$

u solves the minimization problem:

Find $u \in V$: $J(u) \leq J(v)$ $\forall v \in V$

where $J(v) := \frac{1}{2}\int_0^1 v'v' - F(v)$

$(= \langle f, v \rangle, (f, v)_0)$

Proof:

"$\Rightarrow$" $J(u+w)$ $\dots$ $\overset{\text{if } u \text{ solves (WP)}}{\geq}$ $J(u)$

"$\Leftarrow$" assume $\frac{d}{d\varepsilon} J(u+\varepsilon w)\big|_{\varepsilon=0} = 0$

$J(u+\varepsilon w) = \dots$

GALERKIN method:

- Fix a finite dim subspace

$$V_H \subseteq V$$

- Restrict (WP): Find $u_H \in V_H$:

$$\int_0^1 u_H' v_H' \, dx = \langle f, v_H \rangle \quad \forall v_H \in V_H$$

numerical methods can be set up by defining appropriate spaces V_H.