NMPDE/ATSC 2025
Lecture 13

Convection (reaction) dominated diffusion problem: (Q, Ch. 13)

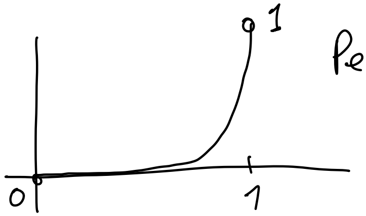
$a, b > 0$ constants

$$\begin{cases} -a u'' + \underline{b u'} = 0 & \\ u(0) = 0 \quad u(1) = 1 \end{cases} \qquad \Omega = (0,1)$$

$$Pe = \frac{b\,L}{2a} \qquad (L = 1 = |\Omega|)$$

$\to$ singularly perturbed

exact



monotone $\forall\, Pe$

$Pe \gg 1$ or $a \to 0$ $\quad u'(1) \to \infty$

perturbation of

$$\to \begin{cases} b u' = 0 \\ u = 0 \end{cases} \to u \equiv 0$$

Discretisation (by linear FEM or $\underline{\text{FD}}$) gives the scheme:

$$\begin{cases} -a \dfrac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + b \dfrac{U_{j+1} - U_{j-1}}{2h} = 0 \\ U_0 = 0 \; ; \; U_M = 1 \end{cases}$$

exercise: show that linear FEM gives same scheme

$$Pe_h := \frac{bh}{2a} \xrightarrow{h \to 0} 0 \longrightarrow Pe_h < 1$$

if $Pe_h > 1$ U is not monotone

still the scheme is convergent and monotone for h small enough

example: $\boxed{\begin{cases} a = 10^{-6} \\ b = 1 \end{cases}}$ to get $Pe_h < 1$ need $h \sim 10^{-6}$ 1D

$$\left.\begin{matrix} 2D \\ 3D \end{matrix}\right\} h^{-6} \text{ requires } \begin{matrix} 10^{12} \\ 10^{13} \end{matrix} \quad \underline{\text{DoF}}\,!$$

$\to$ need to "stabilize" the scheme...

UPWIND method

$$u'(x) \sim \frac{u(x)-u(x-h)}{h}$$

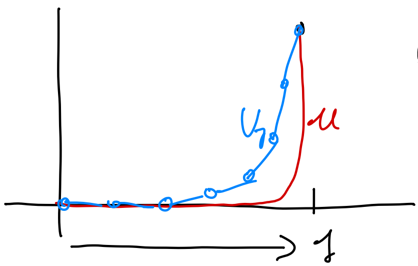
$$\downarrow$$

$$-a\,\frac{U_{j+1}-2U_j+U_{j-1}}{h^2} + b\,\frac{U_j-U_{j-1}}{h} = 0$$

$$U_j = \frac{(1+2\,Pe_h)^j-1}{(1+2\,Pe_h)^M-1}$$

monotone !
$\underline{\forall\, h}$

(exercise) $\quad |u(x_j)-U_j| = O(h)$



$Pe \gg 1$

upwind is monotone but overdiffusive

why??

$$b\,\frac{U_j-U_{j-1}}{h} = b\,\frac{U_j-\frac{1}{2}U_{j-1}-\frac{1}{2}U_{j-1}+\overbrace{\frac{1}{2}U_{j+1}-\frac{1}{2}U_{j+1}}^{=0}}{h}$$

$$= b\,\frac{U_{j+1}-U_{j-1}}{2h} - \frac{b}{2}\,h\,\frac{U_{j-1}-2U_j+U_{j+1}}{h^2}$$

$$\underbrace{b\,u'}\qquad \underbrace{-\frac{b}{2}h\,u''}$$

$\rightarrow$ inconsistency

upwind $\equiv$ centered scheme applied to

$$\underbrace{\boxed{-a\left(1+Pe_h\right)}}_{=:a_h} u'' + b u' = 0 \qquad a_h = a\left(1+\frac{bh}{2a}\right)$$

numerical / artificial diffusion

$$Pe_h^{up} = \frac{bh}{2a_h} = \frac{Pe_h}{1+Pe_h} < 1$$

In general, consider the perturbed PDE with diffusion $a_h = a\left(1+\phi(Pe_h)\right)$

upwind $\phi(t) = t$

or optimize $\rightarrow$ gives

$$\rightarrow \begin{cases} \phi(t) = t - 1 + B(2t) \\ B(t) = \begin{cases} \dfrac{t}{e^t - 1} & \text{if } t > 0 \\ 1 & t = 0 \end{cases} \end{cases} \qquad \text{(Bernoulli)}$$

then centred scheme $\mathcal{O}(h^2)$.

Gives the exponential fitting or

Scharfetter-Gummel.

Analysis of (upwind) scheme, FEM style:

instead of $\mathcal{A}(u,v) = \int_0^1 a u' v' + \int_0^1 b u' v$

use $\mathcal{A}_h(u,v) = \int_0^1 a_h u' v' + \int_0^1 b u' v$

Applying FEM results in Generalized Galerkin method

$\Rightarrow$ Analysis via Strang Lemma

$$|u-u_h|_1 \le C \inf_{v_h} \left\{ \dots + \sup_{w_h \in V_h} \frac{|\mathcal{A}(v_h, w_h) - \mathcal{A}_h(v_h, w_h)|}{\|w_h\|_1} \right.$$

(Q) $$|u-u_h|_1 \le \frac{C h^k}{a(1+\phi(Pe_h))} \|u\|_{k+1} + \frac{\phi(Pe_h)}{1+\phi(Pe_h)} \|u\|_1$$

$\mathcal{O}(h)$ upwind

$\mathcal{O}(h^2)$ S. G.

Convection-diffusion in in multi-dim.

$\Omega \subset \mathbb{R}^d$ $\quad a = a(x) \,/\, \underline{b} = \underline{b}(x)$

$$\mathcal{L} u := \begin{cases} -a\,\Delta u + \underline{b} \cdot \nabla u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$a > 0$ a.e.

conservative term $\nabla \cdot (\underline{b}\, u)$ similar as

$\nabla \cdot (\underline{b} u)$
$= \underline{b} \cdot \nabla u + \underbrace{(\nabla \cdot \underline{b})}_{\text{extra reaction term}} u$

Direct generalization of artificial diffusion method

Apply FEM to $\quad \mathcal{L} u \to \mathcal{L} u - \underbrace{h\, \bar{b}\, \Delta u}_{h\, b\, u''}$

where $\bar{b} = \|\underline{b}\|_{\infty, \Omega}$

Strong : $T(v_h) = \overbrace{A_h(u, v_h) - F(v_h)}$

$$= \bar{b}\, h \int_\Omega \nabla u \cdot \nabla v_h$$

$$|u - u_h|_1 \leq O(h^k) + h\, \bar{b}\, \|u\|_1$$

$$\longrightarrow O(h)$$

We need stronger consistency to retain the order k accuracy.

Streamline diffusion

add instead diffusion only in the direction given by $\underline{b}$

$$\rightarrow \frac{h}{\bar{b}} \int_\Omega (\underline{b} \cdot \nabla u)(\underline{b} \cdot \nabla v)$$

$$\underline{b} \cdot \nabla u = \frac{\partial u}{\partial \underline{b}} \qquad \underline{b} \cdot \nabla v = \frac{\partial v}{\partial \underline{b}}$$

Hughes–Brooks (1979) fully consistent streamline diffusion term.

$$= \sum_T \frac{h}{\bar{b}} \int_T \underbrace{(a \Delta u_h + \underline{b} \cdot \nabla u_h - f)}_{=0 \text{ for } u \text{ exact sol}} (\underline{b} \cdot \nabla v_h)$$

$$= B_h(u_h, v_h)$$

RHS

SD-FEM: Find $u_h \in V_h^k$:

$$\boxed{\mathcal{A}(u_h, v_h) + B_h(u_h, v_h) = (f, v_h) + \left(f, \frac{h}{\underline{b}}(\underline{b}\cdot\nabla u)\right)}$$

Petrov-Galerkin interpretation: SD-FEM

$\equiv$ testing with $\underbrace{v_h + \frac{h}{\underline{b}}(\underline{b}\cdot\nabla u)}_{\text{different space then } V_h^k}$

as $\underline{b}\cdot\nabla u \notin V_h^k$

$$\boxed{\tau_T := \frac{\delta\, h_T}{\underline{b}}}$$

$\delta > 0$

SD parameter

$$B_h^\tau(u_h, v_h) = \sum_T \tau_T \int_T (a\,\Delta u_h + \underline{b}\cdot\nabla u_h)(\underline{b}\cdot\nabla v_h)$$

streamline diff. term

SD norm:

$$\|v\|_{SD} = \Big(\bar{a}\, |v|_1^2 + \sum_{T\in \mathcal{T}_h} \tau_T \|\underline{b}\cdot \nabla v\|_{0,T}^2 \Big)^{1/2}$$

$\bar{a} = \|a\|_\infty$

Lemma: If $0 < \tau_T < \dfrac{h_T^2}{2\bar{a}\mu}$ $\forall T \in \mathcal{T}_h$, then

$$\underbrace{A(v_h,v_h) + B_h(v_h,v_h)}_{=: A_{SD}^{\tau}(v_h,v_h)} \geq \frac{1}{2}\|v_h\|_{SD}^2 \quad \forall v_h \in V_h^k.$$

For $\mu > 0$ some constant (from inverse ineq.).

Theorem: If $\tau_T = \begin{cases} \delta_0 h_T & Pe_h > 1 \\ \delta_1 h_T^2 & Pe_h \leq 1 \end{cases}$

for some $\delta_0, \delta_1 > 0$ large enough, then

$$\|u - u_h\|_{SD} \leq C\left(\bar{a}^{1/2} + h^{1/2}\right) h^k |u|_{k+1}$$

u_h: SD-FEM sol.

$Pe > 1 \rightarrow h^{k+1/2} |u|_{k+1}$ correct rate

given SD norm dominating

term $\sim h^{1/2} \|b \cdot \nabla (u - u_h)\|_0$

$Pe < 1 \rightarrow h^k \, |u|_{k+1}$

$\rightarrow$ SD-FEM has optimal rate of convergence !