NMPDE/ATSC 2025
Lecture 3

Videos of the lectures available at this YouTube channel:

https://www.youtube.com/playlist?list=PL2754Ml3oCY-1d1psUtDoTnC4jCn4I0w9

$$\begin{cases} -u'' = f & \text{in } (0,1) = \Omega \\ u(0) = 0 = u(1) \end{cases}$$

$$\left.\begin{array}{l} \begin{cases} -\delta_h^2 U_i = f_i := f(x_i) & i = 1, \dots, N-1 \\ U_0 = 0 \ ; \ U_N = 0 \end{cases} \\ U_i \simeq u(x_i) \\ \delta_h^2 U_i = \dfrac{U_{i+1} - 2U_i + U_{i-1}}{h^2} \end{array}\right\} \text{FD method}$$

$\hookrightarrow$ MATRIX FORM

Note: for $i=1$ $U_{i-1} = U_0 = 0$

$$A = \frac{1}{h^2} \begin{array}{r} i=1 \rightarrow \\ \\ \\ \\ i=N-1 \rightarrow \end{array} \begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ 0 & & & 1 & -2 \end{bmatrix} \ ; \quad F := \begin{bmatrix} f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

tridiagonal matrix

matrix form of FD operator

$$\rightarrow \quad A\tilde{U} = F \qquad \tilde{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_{N-1} \end{bmatrix}$$

Weak form: $u \in H^1_0(\Omega)$:

$$\text{(WP)} \quad a(u,v) = F(v) \qquad \forall v \in H^1_0(\Omega)$$

where :

$$a(u,v) = \int_\Omega u' v'$$

$$F(v) = \begin{cases} \int_\Omega f v & f \in L^2(\Omega) \\ f(v) & f \in (H^1)' \end{cases}$$

Galerkin method : restrict (WP) to a finite dim. subspace V_H :

Find $v_H \in V_H$: $a(u,v) = F(v) \quad \forall v \in V_H$

(Courant linear finite element method in 1 D

- over same mesh $x_i = ih \quad i = 0, \dots, N$

and $h = 1/N$; $I_i = [x_{i-1}, x_i]$ ← Polynomial of degree ≤ 1

$$V_h = \left\{ v \in H^1([0,1]) : v|_{I_i} \in \mathbb{P}_1(I_i),\ i = 1, \dots, N \quad v(0) = 0 = v(1) \right\}$$

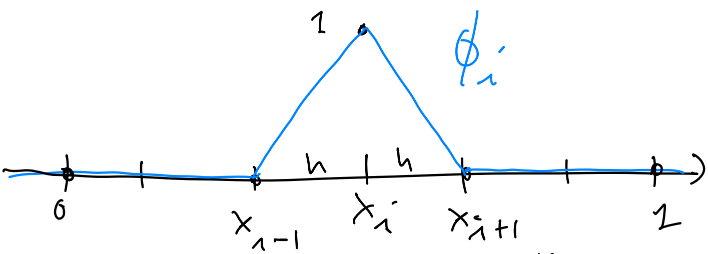
$\subseteq H^1_0([0,1])$ also $V_h \subset C^0([0,1])$

We have that $V_h = \text{Span}\{\phi_i\}_{i=1}^{M-1}$ where

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$



The solution $u_h(x) = \sum_{j=1}^{M-1} U_j \phi_j(x)$ (E)

$\{U_j\}_{j=1}^{M-1}$ "degrees of freedom" of FE function

FEM: Find $u_h \in V_h$: $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

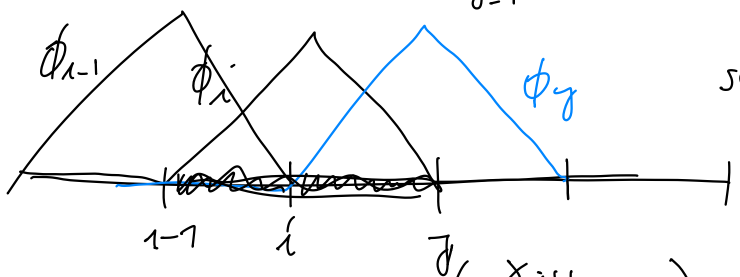
By linearity if $a(u_h, \phi_i) = F(\phi_i) \quad \forall i = 1, \dots, M-1$ then FEM is satisfied. Also, using (E),

FEM $\Longleftrightarrow$ Find $\tilde{U} = \{U_i\}_{i=1}^{M-1}$:

$$\sum_{j=1}^{M-1} U_j \int_0^1 \alpha \, \phi_j' \phi_i' = \int_0^1 f \phi_i \qquad \forall i = 1, \dots, M-1$$

(case $f \in L^2(0,1)$)

or indeed $u_h' = \sum_{j=1}^{N-1} U_j \, \phi_j'$



$\mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) \neq 0$ only if $j = i, i-1, i+1$

$$\Longleftrightarrow \left(\int_{x_{i-1}}^{x_i} \underbrace{\phi_{i-1}'}_{=-\frac{1}{h}} \, \underbrace{\phi_i'}_{=\frac{1}{h}}\right) U_{i-1} + \left(\int_{x_{i-1}}^{x_{i+1}} \underbrace{\phi_i' \, \phi_i'}_{=\left(\pm\frac{1}{h}\right)\left(\pm\frac{1}{h}\right)}\right) U_i + \left(\int_{x_i}^{x_{i+1}} \underbrace{\phi_{i+1}'}_{=\frac{1}{h}} \, \underbrace{\phi_i'}_{=-\frac{1}{h}}\right) = \int_{x_{i-1}}^{x_{i+1}} f \, \phi_i$$

$$-\frac{1}{h^{\cancel{2}}} \cdot \cancel{h} \, U_{i-1} + 2 \frac{1}{h^{\cancel{2}}} \cdot \cancel{h} - \frac{1}{h^{\cancel{2}}} \cdot \cancel{h} \, U_{i+1} = \int_{x_{i-1}}^{x_{i+1}} f \, \phi_i$$

i.e. $$\underbrace{\boxed{\frac{-U_{i-1} + 2U_i - U_{i-1}}{h^2}}}_{= -\delta_h^2 U_i} = \frac{1}{h}\left[\int_{x_{i-1}}^{x_i} f \, \phi_i + \int_{x_i}^{x_{i+1}} f \, \phi_i\right]$$

To compute the RHS integral, suppose we use the trapezoidal rule

$$\int_{x_{i-1}}^{x_i} f \, \phi_i \approx h \, \frac{f(x_i) \overbrace{\phi_i(x_i)}^{=1} + f(x_{i-1}) \overbrace{\phi_i(x_{i-1})}^{=0}}{2} = \frac{h}{2} f_i$$

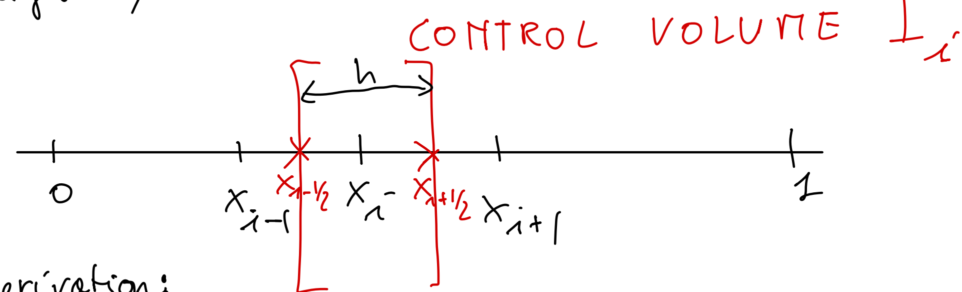
Similarly $\int_{x_i}^{x_{i+1}} f \, \phi_i \approx \frac{h}{2} f(x_i)$

$\Rightarrow$ gives $\quad -\delta_h^2 U_i = f(x_i)$

That is $\quad \boxed{\text{FEM + Trapezoidal rule for RHS} \equiv \text{FD}}$

Finite Volume

Once again, let $x_i = ih$, so get probl



FV derivation:

1) integrate your equation $-u'' = f$ on I_i:

$$= -\int_{x_{i-1/2}}^{x_{i+1/2}} u'' = \int_{x_{i-1/2}}^{x_{i+1/2}} f \approx h f_i$$

by rectangle rule

$$= -\left[u'(x_{i+1/2}) - u'(x_{i-1/2})\right]$$

2) use FD for u'

$$u'(x_{i+1/2}) \approx \frac{u(x_{i+1}) - u(x_i)}{h} = \delta_h u(x_{i+1/2})$$

$$u'(x_{i-1/2}) \approx \frac{u(x_i) - u(x_{i-1})}{h} = \delta_h u(x_{i-1/2})$$

Calling $U_i \approx u(x_i)$

$$-\frac{U_{i+1} - U_i}{h} + \frac{U_i - U_{i-1}}{h} = h f_i$$

i.e. $$-\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = f_i$$

More general two-points BVPS (LT)

Consider

diffusion, advection/convection, reaction

$$\begin{cases} -a u'' + b u' + c u = f & \Omega = (0,1) \\ u(0) = u_0 \quad u(1) = u_1 \end{cases}$$

$a = a(x)$, $b = b(x)$, $c = c(x)$ smooth enough

$a > 0$, $c \geq 0$

FD method

$x_i = ih \qquad i = 0, \dots, M$

Find $U_i \approx u(x_i)$:

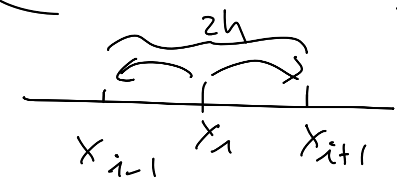
$U_0 = u_0$

$$\underbrace{-a_i \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + b_i \frac{U_{i+1} - U_{i-1}}{2h} + c_i U_i}_{=: \mathcal{L}_h U_i} = f_i \qquad \forall i = 1, \dots, M-1$$

$U_M = u_1$

$[a_i = a(x_i)$ etc.$]$

$$\left[u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h} \approx \delta_{2h} u(x_i) \right]$$



Matrix form: $L\tilde{U} = F \qquad \tilde{U} = \begin{bmatrix} U_1 \\ U_{M-1} \end{bmatrix}$

$L = A + B + C$

where $A = -\frac{1}{h^2} \begin{bmatrix} -2a_1 & a_1 & 0 \\ -a_2 & 2a_2 & a_2 \\ & \ddots & \ddots & \ddots \\ 0 \end{bmatrix} \qquad B = \frac{1}{2h} \begin{bmatrix} 0 & b_1 & 0 \\ -b_2 & 0 & b_2 \\ & \ddots & \ddots & \ddots \\ 0 \end{bmatrix}$

$$C = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_{M-1} \end{bmatrix}$$

Analysis

Consistency

$$\delta_{2h} u(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} = u'(x_i) + \frac{h^2}{12}\left[u^{(3)}(\xi) \; u^{(3)}(x)\right]$$

Taylor (exercise)

$$\Rightarrow \left| u'(x_i) - \delta_{2h} u(x_i) \right| \leq \frac{h^2}{6} \| u^{(3)} \|_C$$

Truncation error for FD method

(exercise)

$$|T_i| \leq \left(\frac{h^2}{12} \|a\|_C \|u^{(4)}\|_C + \frac{h^2}{6} \|b\|_C \|u^{(3)}\|_C \right)$$

Stability : by Discrete Max Principle DMP

[Lemma 4.1 LT]

Lemma (DMP) : Assume h is small enough so that $a_i \pm \frac{1}{2} h b_i \geq 0$ and $V = \{V_i\}$ satisfies $\mathcal{L}_h V_i \leq 0$, Then

(i) If $c=0$ then $\max_i U_i = \max\{U_0, U_M\}$

(ii) If $c \geq 0$ then $\max_i U_i = \max\{U_0, U_M, 0\}$

[Lemma 4.2 in LT]

Lemma: For any discrete function U, assuming $b=0$ we have

$$\|U\|_{h,\infty} \leq \max\{|U_0|, |U_M|\} + C|\mathcal{L}_h U|_{h,\infty}$$

and under DMP assumptions

Theorem: Let $b=0$. Then the FD solution $U = \{U_i\}$ satisfies

$$\max_i |U_i - u(x_i)| \leq C h^2 \|u^{(4)}\|_C$$

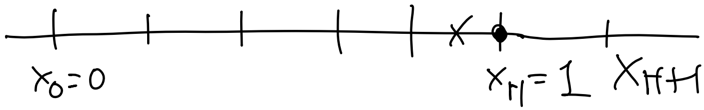
if $u \in C^4([0,1])$.

Other types of boundary conditions

No-flux (homog. Neumann):

$$u'(1)=0 \qquad (\leftarrow \text{ex } u_1)$$

$\Rightarrow u(1)$ unknown $\Rightarrow$ need 1 eq. to fix it



$u'(1) \approx$

- $\dfrac{u_M - u_{M-1}}{h}$
- higher-order one-sided FD (see notes online
- introduce ficticious node x_{M+1} and corresp. U_{M+1} and use

$$\boxed{\frac{U_{M+1}-U_{M-1}}{2h}=0} \quad 1$$

+ apply FD scheme also at x_M: 2

$$\boxed{-a_M \frac{\overset{U_{M-1}}{\overset{\parallel}{U_{M+1}}} - 2U_M + U_{M-1}}{h^2} + b_M \cancel{\frac{U_{M+1} - U_{M-1}}{2h}} + c_M U_M = f_M}$$

$$\boxed{2a_M \frac{U_M - U_{M-1}}{h^2} + c_M U_M = f_M}$$

$$\rightarrow \quad L\hat{U} = F \qquad \hat{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_{M-1} \\ U_M \end{bmatrix}$$

$$\overset{M\times M}{L} = A + B + C$$

$$A = -\frac{1}{h^2}\begin{bmatrix} -2a_1 & a_1 & & & \\ a_2 & -2a_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & 2a_M & -2a_M \end{bmatrix}$$