NMPDE/ATSC 2025
Lecture 11

Quarteroni

Chapter	Sections
1	all
2	2.1-2.5, (2.7)
3	3.1-3.4
4	4.1-4.3, 4.4.1-4.4.2, 4.5, (4.6.2)
6	6.1, 6.2
7	7.1, 7.2.1, 7.2.4
8	8.2

sections covered up to before this lecture.

The generalized Galerkin method

motivation: in practice (Q. Chap. 13)

FEM assembly process computed using quadratures.

Instead of: compute approximations

$$\mathcal{A}(u_h, v_h) \approx \mathcal{A}_h(u_h, v_h)$$

$$F(v_h) \approx F_h(v_h)$$

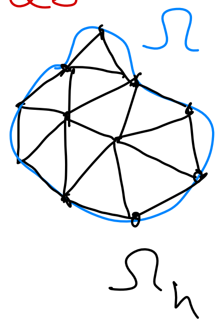
A form of VARIATIONAL CRIME

* 1. $\mathcal{A} \approx \mathcal{A}_h$, $F \approx F_h$

2. $V_h \not\subset V$ non-conforming discrete spaces

3. $\Omega_h = \bigcup_{T \in \mathcal{T}_h} T \neq \Omega$

e.g. if Ω is curved



Analysis of variational crime 1

$\Omega_h = \Omega$, $V_h = V_h^k \subset V$, consider $\mathcal{A}_h$, F_h and the generalised Galerkin Method: Find $u_h \in V_h$:

$$\mathcal{A}_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h$$

Galerkin orthogonality (= full consistency) does not hold.

Recall: Galerkin orthogonality:

$$\mathcal{A}(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

Truncation error:

$$T_h(v_h) := a_h(u, v_h) - F_h(v_h)$$

- If $T_h \equiv 0 \rightarrow$ strong (full) consistency
  $\hookrightarrow$ Céa Lemma
- otherwise require $T_h \rightarrow 0$ for consistency
  $\hookrightarrow$ Strang Lemmas

Def (uniform V_h-ellipticity):

$a_h : V_h \times V_h \longrightarrow \mathbb{R}$ is UNIFORMLY V_h-elliptic if $\exists\, \tilde{\alpha} > 0$: $\forall\, \mathcal{T}_h$

$$a_h(v_h, v_h) \geq \tilde{\alpha} \|v_h\|_V^2 \quad \forall v_h \in V_h$$

[See Ciarlet, ch. 4 / Q. Lemma 10.1, Q. Sec. 13.8.4 about Generalised Galerkin]

Theorem (1st strong Lemma)

Let $V_h \subset V$ sequence of subspaces, $\mathcal{A}_h, F_h$ discrete forms approximating $\mathcal{A}, F$.

If $F_h \in V'$, $\mathcal{A}_h$ is unif. V_h-elliptic, then

well-posed:

- $\exists!\ u_h \in V_h$ solution of the generalized Galerkin method $\mathcal{A}_h(u_h, v_h) = F_h(v_h)\quad \forall v_h \in V_h$
- $\|u_h\|_V \leq \frac{1}{\tilde{\alpha}} \|F_h\|_{V'}$

Céa

- $$\|u - u_h\|_V \leq \inf_{v_h \in V_h} \left[\left(1 + \frac{\gamma}{\tilde{\alpha}}\right) \|u - v_h\|_V + \frac{1}{\tilde{\alpha}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|\mathcal{A}(v_h, w_h) - \mathcal{A}_h(v_h, w_h)|}{\|w_h\|_V}\right] + \frac{1}{\tilde{\alpha}} \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|_V}$$

<u>Proof</u>: Well-posedness follows by Lax-Milgram applied to A_h, F_h.

$$\|u-u_h\|_V \le \boxed{\|u-v_h\|_V} + \underbrace{\|v_h-u_h\|_V}_{\tilde{w}_h \in V_h} \quad \forall v_h \in V_h$$

$$\tilde{\alpha} \underline{\|\tilde{w}_h\|_V^{\cancel{2}}} \le A_h(u_h - v_h, \tilde{w}_h)$$ by uniform V_h-ellipticity

$$\cancel{\|\tilde{w}_h\|_V} = F_h(\tilde{w}_h) - A_h(v_h, \tilde{w}_h)$$ use Gen. Galerkin

$$+ \underbrace{A(u-v_h, \tilde{w}_h) - F(\tilde{w}_h)}_{=0}$$ by cont. problem

$$+ \underbrace{A(v_h, \tilde{w}_h)}_{=0}$$

$$= A(u-v_h, \tilde{w}_h) + (A-A_h)(v_h, \tilde{w}_h)$$

$$+ (F_h - F)(\tilde{w}_h)$$

$$\le \gamma \underline{\|u-v_h\|_V \cancel{\|\tilde{w}_h\|_V}} + |(A-A_h)(v_h, \tilde{w}_h)| / \|\tilde{w}_h\|_V$$

$$\cancel{\|\tilde{w}_h\|_V} \qquad + |(F-F_h)(\tilde{w}_h)| / \|\tilde{w}_h\|_V$$

$$\|u_h - v_h\|_V \le \frac{\gamma}{\tilde{\alpha}} \|u - v_h\|_V + \frac{1}{\tilde{\alpha}} \left(\frac{|(A - A_h)(v_h, \tilde{w}_h)|}{\|\tilde{w}_h\|_V} + \frac{|(F - F_h)(\tilde{w}_h)|}{\|\tilde{w}_h\|_V} \right)$$

- take inf over v_h and sup over w_h.

<u>Example: Poisson, FEM with quadratures</u>

Let $V = H_0^1(\Omega)$, $A(u,v) = \int_\Omega A \nabla u \cdot \nabla v$, $A = (a_{ij})_{ij}$, $\Omega \to T$

$F(v) = \int_\Omega f v$

<u>Theorem</u>: Assume:
- $a_{ij} \in W^{k,\infty}$
- $f \in W^{k,q}$

$\begin{cases} q \ge 2 \\ k > \frac{d}{q} \end{cases}$,

$u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$. Let A_h, F_h computed by element wise on every $T \in \mathcal{T}_h$ using quadratures exact on $\mathbb{P}^{2k-2}(T)$.

[example: $k=1$, then need exact on constants]

Then, $|u-u_h|_1 \leq C h^k \Big(|u|_{k+1} + \sum_{i,j=1}^{d} \|a_{ij}\|_{k,\infty} \|u\|_{k+1} + \|f\|_{W^{k,q}} \Big)$

On quadrature formulas

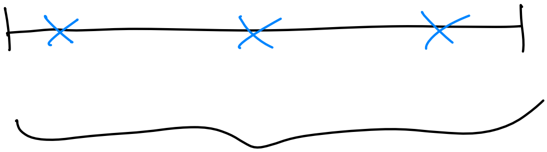
- Given T, $\int_T g(x)\,dx \approx \underbrace{\sum_{j=1}^{n_q} w_j\, g(x_j)}_{Q(g)}$

(w_j, x_j) set of quadrature weights and nodes

A quadrature rule is of order $\underline{m}$ if

$$Q(p) = \int_T p(x)\,dx \qquad \forall p \in \mathbb{P}^m(T)$$

- Gauss quadrature: maximal order of quadrature for a given number of q. points. → available of any order for the interval

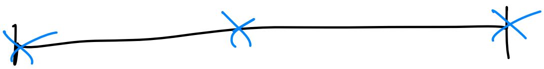


I

$$\int_I g(x)\,dx$$

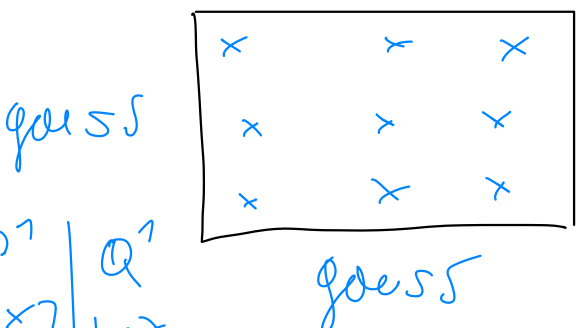
m points $\xrightarrow{\text{exact}}$ $\mathbb{P}^{2m+2}$

+ ∃ Gauss-Lobotto versions includ end points



m points $\xrightarrow{\text{exact}}$ $\mathbb{P}^{2m+1}$ (please check!)

← Over rectangle



by tensor product of 1D Gauss obtain quad. rules for rectangle

$m \times m \longrightarrow \mathbb{Q}^{2m+2}$

$\mathbb{P}^1$	$\mathbb{Q}^1$
$1, x, y$	$1, x, y, xy$

- map appropriately the rectangle into a triangle to get a (slightly wasteful) quad. rule
- Otherwise, there are ad hoc rules available (Q 8.2.1) using barycentric coordinates

FD in 2D

Start with Poisson in 2D:

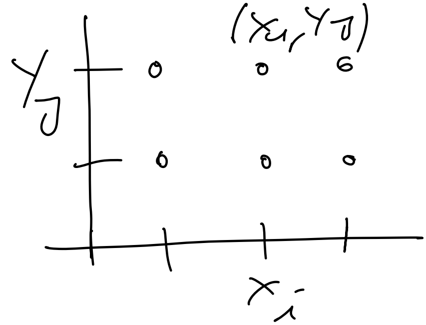
$$\begin{cases} -\Delta u = f & \Omega = (0,1)^2 \\ u = 0 & \partial\Omega \end{cases}$$

- Fix a rectangular grid:

$M_x \times M_y$: (x_i, y_j) grid points

$h_x = \frac{1}{M_x}$, $h_y = \frac{1}{M_y}$

$x_i = i\, h_x$; $y_j = j\, h_y$



$$u(x_i, y_j) \simeq U_{ij}$$

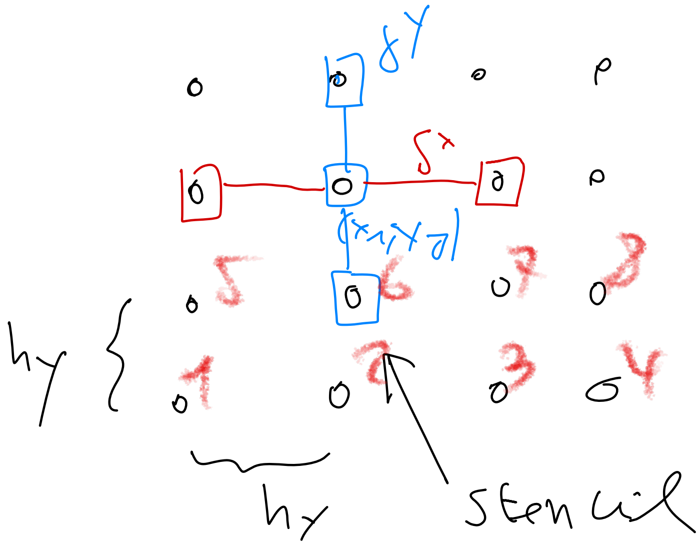
$$\Delta u = u_{xx} + u_{yy} \simeq \underbrace{\left(\delta^x_{h_x}\right)^2}_{\Uparrow} u(x,y) + \left(\delta^y_{h_y}\right)^2 u(x,y)$$

$$\left(\delta^x_{h_x}\right)^2 u(x,y) = \frac{u(x+h_x, y) - 2u(x,y) + u(x-h_x, y)}{h_x^2}$$

FD method : Find $U = (U_{ij})$:

$$\begin{cases} U_{ij} = 0 & \text{if } (x_i, y_j) \in \partial\Omega \\ \left(\delta^x_{h_x}\right)^2 U_{ij} + \left(\delta^y_{h_y}\right)^2 U_{ij} = -f_{ij} := f(x_i, y_j) \end{cases}$$

STENCIL



$\rightarrow$ every eq. has 5 nonzero entries

FD LINEAR SYSTEM

Let $\underline{U} = (U_{1,1}, U_{2,1}, \dots, U_{N_x-1,1}, U_{1,2}, \dots, U_{N_x-1,2},$

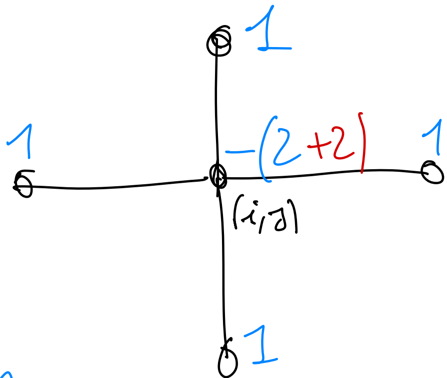
Let $h_x = h_y = h$
$N_x = N_y = N$

$\qquad \dots, U_{1,N_y-1}, \dots, U_{N_x-1,N_y-1})$

$\underline{F} = -h^2 (f_{1,1}, \dots,)$

$$A = \begin{pmatrix} B & I & & \\ I & B & I & \\ & \ddots & \ddots & \ddots \end{pmatrix} \qquad \underbrace{I = (N-1)\times(N-1)}_{\text{Identity}}$$

$$B = \begin{pmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \quad M-1 \times M-1$$



$(\delta_h^x)^2$

$(\delta_h^y)^2$

$\sim N$ bandwidth

$$\begin{pmatrix} \overbrace{-4 \quad 1 \quad 0 \cdots \quad 1}^{M-1} & & 0 \\ 1 & \ddots & \ddots \\ 0 & \ddots & \ddots \\ \vdots & & \\ 1 & \ddots & \ddots \\ 0 & & \end{pmatrix}$$

Pentadiagonal matrix