NMPDE/ATSC 2025
Lecture 9

Once again, here is the zoom link in case you need to follow remotely:

https://sissa-it.zoom.us/j/7306190508?pwd=TXhCQkFHMlgrVERqNUw2aDNxSDVUQT09

Recap from last lecture

Def : Lagrange interpolant :

$$I_h : C^0(\bar{\Omega}) \longrightarrow V_h^k$$

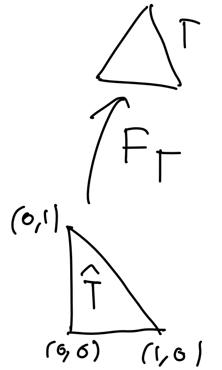
$$v \longrightarrow I_h v = \sum_j v(a_j)\, \varphi_j(x)$$

with $\{a_j\}$ the global numbering of the nodes so that $\varphi_j(a_i) = \delta_{ij}$

- $h_T = \text{diam } T \rightsquigarrow h = \max h_T$
- $\rho_T = \sup\{\text{diam}(S) : \cdot\, S \text{ is sphere},\ \cdot\, S \subset T\}$
- $\{\mathcal{T}_h\}_h$ shape regular if $\exists\, \sigma > 1 :$ $\forall h, \forall T \in \mathcal{T}_h$ we have $\dfrac{h_T}{\rho_T} \leq \sigma$

Lemma 1

$$\begin{cases} |v|_{H^m(T)} \le C \|B_T^{-1}\|^m \underline{|\det B_T|^{1/2}} |\hat{v}|_{H^m(\hat{T})} \\ |\hat{v}|_{H^m(\hat{T})} \le C \|B_T\|^m |\det B_T|^{-1/2} |v|_{H^m(T)} \end{cases}$$



$F_T v = \underline{B_T} v + b_T$

Lemma 2: We have

- $\underline{\|B_T\| \le \frac{h_T}{\hat{\rho}}}$
- $\|B_T^{-1}\| \le \frac{\hat{h}}{\rho_T}$

$\hat{l}(v) = 0 \quad \forall v \in \mathbb{P}^k(\hat{T})$

Corollary (Lemma 3 + Lemma 4):

$$|\hat{l}(\hat{v})|_{H^m(\hat{T})} \le C \|\hat{l}\|_{\mathcal{L}^m_{k+1}(\hat{T})} |\hat{v}|_{H^{k+1}(\hat{T})}$$

Theorem (local interp. error). Let $\begin{cases} k \ge 1 \\ 0 \le m \le k+1 \end{cases}$

$$\exists C > 0 : |v - I_h^k v|_{H^m(T)} \le C \frac{h_T^{k+1}}{\rho_T^m} |v|_{H^{k+1}(T)}$$

$\forall v \in C^0(T) \cap H^{k+1}(T), \forall T \in \mathcal{T}_h$.

examples: $m = 0$ $\quad \|v - I_h v\|_0 \le C h_T^{k+1} |v|_{H^{k+1}(T)}$

$m = 1$ $\quad |v - I_h v|_1 \le C h_T^{k+1}/\rho_T \, |v|_{H^{k+1}(T)}$

if shape reg.: $\rho_T \sim h_T$ $\quad h_T^k$

see later

Proof :

$$|v - I_h v|_{H^m(T)} \leq C \underbrace{\|B_T^{-1}\|^m}_{\frac{\hat{h}^m}{\rho_T^m}} |\det B_T|^{1/2} |\hat{v} - \hat{I}_h \hat{v}|_{H^m(\hat{T})} \quad \text{(Lemma 1)}$$

$$\leq C \frac{\hat{h}^m}{\rho_T^m} |\det B_T|^{1/2} \underbrace{|\hat{v} - \hat{I}_h \hat{v}|_{H^m(\hat{T})}}_{=: \hat{C} \hat{v}} \quad \text{(Lemma 2)}$$

$$\leq C \frac{\hat{h}^m}{\rho_T^m} |\det B_T|^{1/2} |\hat{v}|_{H^{k+1}(\hat{T})} \quad \text{(Corol 3+4)}$$

$\hat{h}^m = 1$

The constant C changes value at each instance but indep. of h !

(Lemma 1)

$$\leq C \frac{1}{\rho_T^m} \|B_T\|^{k+1} |v|_{H^{k+1}(\hat{T})}$$

$$\leq C \frac{h_T^{k+1}}{\rho_T^m} \frac{1}{\hat{\rho}^{k+1}} |v|_{H^{k+1}(\hat{T})}$$

$\hat{\rho} \sim 1$

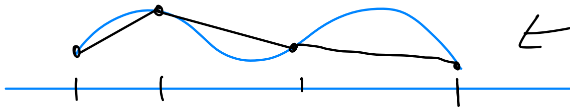
Corollary: If moreover $\mathcal{T}_h$ is shape-regular

$$|v - I_h v|_{H^m(T)} \leq C h^{k+1-m} |v|_{H^{k+1}(T)}$$

where C also depends on the shape-reg. constant σ.

proof: $\frac{h_T^{k+1}}{\rho_T^m} = h_T^{k+1-m} \cdot \underbrace{\frac{h_T^m}{\rho_T^m}}_{\leq \sigma^m} \leq \sigma^m \cdot h_T^{k+1-m}$

use the theorem !



$\leftarrow$ Interpolant is in H^1 only

Theorem (global interpolation error).

Let $m = 0, 1$, $k \geq 1$. $\mathcal{T}_h$ shape-reg w.r.t. σ

There exists $C = C(k, m, \hat{T}, \sigma)$:

$$|v - I_h v|_{H^m(\Omega)} \leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{2(k+1-m)} |v|^2_{H^{k+1}(T)} \right)^{1/2}$$

$$\forall v \in C^0(\Omega) \cap H^{k+1}(\mathcal{T}_h)$$

and

$$|v - I_h v|_{H^m(\Omega)} \leq C_I h^{k+1-m} |v|_{H^{k+1}(\Omega)}$$

$$\forall v \in C^0(\Omega) \cap H^{k+1}(\Omega)$$

Remarks:

① (crucial!) $\rightarrow$ FEM error est.

$$|u - u_h|_1 \leq \frac{\gamma}{\alpha_0} C_I h^\ell |u|_{H^\ell(\Omega)}$$

if $u \in H^{s+1}(\Omega)$ $\qquad \ell = \min(k, s+1)$

- (LT) possible to prove that for $k=1$

$$\underbrace{\|u-u_h\|_C}_{\text{supremum norm}} \le C h^2 \log(1/h) \|u\|_{C^2}$$

based on showing that $\|v - I_h v\|_C \le C h^2 \|v\|_{C^2}$

assuming that
- h small enough
- $\mathcal{T}_h$ is quasi uniform

$h_T \le h$ $\qquad\qquad \exists\, \sigma > 0: \quad h \le \sigma h_T$

FEM implementation

- based on a triangulation $\mathcal{T}_h$, polynomial degree k
- reference element $\hat{T}$, e.g.

$$(A_T)_{ij} := \boxed{\int_T a(x) \nabla\varphi_j \cdot \nabla\varphi_i} = \int_{\hat{T}} a(F_T(\hat{x}))\, J_T^{-T} \hat{\nabla}\hat{\varphi}_j \cdot J_T^{-T} \hat{\nabla}\hat{\varphi}_i \cdot \det(J_T)$$

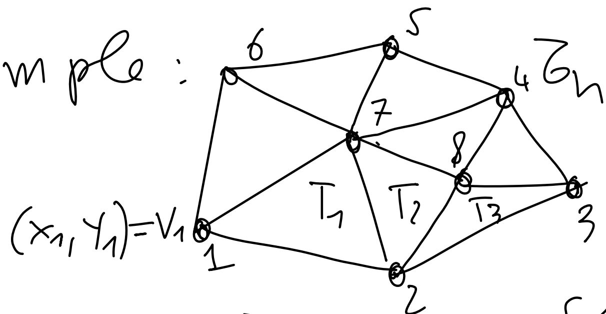
B_T

- quadratures to compute $\int_{\hat{T}}$

- ① global ordering of the FE nodes

$$\{\varphi_j\}_{j=1}^{M} \qquad M = \dim V_h^k$$

- ② local-to-global map

Example: 

, $k = 1$

locally:

1st 2nd 3rd

$$M = \begin{array}{c|c} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \end{array} \end{array} \qquad T = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 8 & 7 \\ \boxed{2 \quad 3 \quad 8} \\ & \vdots & \end{bmatrix}$$

assembly loop: over elements

$\forall T \rightarrow$ get vertices

$$A_T = \begin{pmatrix} \int_T a \nabla\varphi_1^T \cdot \nabla\varphi_1^T & & \int a \nabla\varphi_1^T \cdot \varphi_3^T \\ & \int_T a \nabla\varphi_2^T \cdot \nabla\varphi_2^T & \\ & & \int_T a \nabla\varphi_3^T \cdot \nabla\varphi_3^T \end{pmatrix} \rightarrow A = \begin{array}{c} 2 \\ 3 \\ 8 \end{array} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

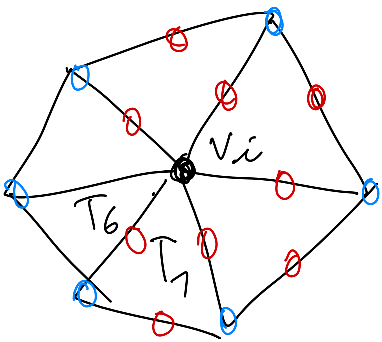
$+=$

example: $\varphi_{1}^{T}, \varphi_{2}^{T}, \varphi_{3}^{T}$

$\downarrow \quad \downarrow \quad \downarrow$

$\varphi_{2} \quad \varphi_{3} \quad \varphi_{8}$

local to global



$$A_{ii} = \int_{\Omega} a(x)\, \nabla\varphi_i \cdot \nabla\varphi_i$$

$$= \sum_{\ell=1}^{6} \underbrace{\int_{T_\ell} a(x)\, \nabla\varphi_i \cdot \nabla\varphi_i}_{A^{T}_{\ell(i)\,\ell(i)}}$$

assembly:

- $\boxed{A_{ii}} = 0$
- loop over all T

$$A_{ii} = A_{ii} + A^{T}_{\ell(i)\,\ell(i)}$$

How difficult it is solve the FE system $\boxed{AU = F}$

$\Downarrow$

<u>Def</u>: Spectral condition number of A is

$$\chi_{sp}(A) := \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = \chi_2$$

for SPD matrices
χ_2 = cond number w.r.t $\|\cdot\|_2$

$\chi(A) = \|A\|\,\|A^{-1}\|$

Rayleigh quotient:

$$R(\underline{v}) := \frac{(A\underline{v}, \underline{v})}{|\underline{v}|^2} = \frac{a(v_h, v_h)}{|\underline{v}|}$$

$\underline{v} \in \mathbb{R}^M$

if $v_h = \sum_{i=1}^{M} v_i \varphi_i$ $\quad$ $\underline{v}$ vector of DoF of v_h

Theorem (H^1-L^2 inverse ineq.):

Let $\mathcal{T}_h$ quasi-uniform, then $\exists C_{inv} > 0$

$$|v_h|_1^2 \leq C_{inv} \, h^{-2} \|v_h\|_0 \quad \forall v_h \in V_h^k$$

where $C_{inv} = C_{inv}(\sigma, \tau, k)$

Theorem: Let $\mathcal{T}_h$ quasi unif. (and shape-regular), $k \geq 1$, $\mathcal{A}$ coercive, continuous bilinear form, A corresp. FE matrix.

Assume also $\mathcal{A}$ ($\Rightarrow A$) is symmetric ($\Rightarrow A$ is SPD)

Then $\chi_{sp}(A) = \mathcal{O}(h^{-2})$

example:

$$\left\{ \begin{array}{llll} & & & \# \text{DoF} \\ 1\text{D}, & \text{sub. of } N & & N \\ 2\text{D} & \text{''} & N \times N & N^2 \\ 3\text{D} & \text{''} & N \times N \times N & N^3 \end{array} \right\} \chi_{sp} \sim h^{-2}$$