NSPDE2021 - Computational exercise 1

Feel free to utilize your favourite programming framework. Pick the exercises that are useful fro you, if any!

(1) Implement in sparse matrix form the divided difference operators $\delta_h, \delta_{h,+}, \delta_{h,-}, \delta_{h,+,2}, \delta_{h,-,2}$ applied on an interval (a, b) based on a uniform grid spacing $h = (b-a)/N$, for $N \in \mathbb{N}$. Experiment using them to approximate the derivative of $f(x) = \cos(x)$ in $[-\pi, \pi]$. Plot uniform (max) norm errors in function of either N or h and confirm the expected rates of convergence. How does the constant in the error compare with the theoretical constants?

(2) Similarly implement the second central divided difference approximation of second derivatives and use it to approximate by the FD method the Poisson problem $-au'' = f$ in (a, b) with boundary conditions $u(a) = u_a$, $u(b) = u_b$. Check the order of convergence by fixing $a = 1$ and f so that the exact solution is $u(x) = \sin(x)$ in $[0, \pi/2]$.

(3) Note that to do the above, you need a linear system solver. Research and implement the Thomas algorithm for solving three-diagonal systems and compare computational times with built in solvers. (The Thomas algorithm can be numerically unstable, but works fine if the system is diagonally dominant, as is the case for the second central divided difference matrix.)

(4) Implement the central FD method for the solution of the BVP $-au'' + bu' + cu = f$ in (a, b) with boundary conditions $u(a) = u_a$, $u(b) = u_b$. You could first set a, b, c as constants by re-using the differentiation matrices coded in (1) and (2). Then write the code for the non-constant coefficients case, using functions implementing the coefficients a, b, c, f. Pick the coefficients so that the exact solution is smooth and check that your implementation delivers the predicted rate of convergence. Experiment with less smooth solutions.

(5) Implement a solver for the problem $-\Delta u = f$ on a rectangular domain in 2 and 3 dimensions. Check numerically the rate of convergence.

(6) Do the same for the problem $u_{xx} - \frac{1}{2}u_{xy} + 2u_{yy} = f$ on a rectangular domain in 2 dimensions.

(7) Implement the compact FD method seen in the lectures for the solution of $-\Delta u = f$ on a rectangular domain in 2 dimensions. Plot together convergence plots for the standard and compact FD methods.

(8) Write a code implementing the FEM for the problem in (2) with known solution. The code should be written in the FEM spirit, that is assembling the system through a loop over the elements. Experiment using the midpoint rule, the trapezoidal rule , and exact integration (which is possible in this case), verifying that each give a different result with the trapezoidal quadrature reproducing the FD method and the exact quadrature providing a FEM solution which is exact at the nodes.